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# *Global Resolution of the Physical Vacuum Singularity for Three-Dimensional Isentropic Inviscid Flows with Damping in Spherically Symmetric Motions*

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**Abstract**

For the gas–vacuum interface problem with physical singularity and the sound speed being $C^{1/2}$-Hölder continuous near vacuum boundaries of the isentropic compressible Euler equations with damping, the global existence of smooth solutions and the convergence to Barenblatt self-similar solutions of the corresponding porous media equation are proved in this paper for spherically symmetric motions in three dimensions; this is done by overcoming the analytical difficulties caused by the coordinate's singularity near the center of symmetry, and the physical vacuum singularity to which standard methods of symmetric hyperbolic systems do not apply. Various weights are identified to resolve the singularity near the vacuum boundary and the center of symmetry globally in time. The results obtained here contribute to the theory of global solutions to vacuum boundary problems of compressible inviscid fluids, for which the currently available results are mainly for the local-in-time well-posedness theory, and also to the theory of global smooth solutions of dissipative hyperbolic systems which fail to be strictly hyperbolic.

## 1. Introduction

Vacuum boundary problems with physical singularity in compressible fluids have received much attention recently (cf. [11,13,14,23,24,31–34,46–50,62,64]), due to their physical importance and their mathematical challenges. Significant progress has been made on the local well-posedness theory (cf. [11,13,14,32,34, 49]). However, much less is known on the global existence and long time dynamics of solutions, which are of fundamental importance in both physics and nonlinear partial differential equations. This is the main issue we address in this work for the spherically symmetric motions of three-dimensional isentropic compressible inviscid flows with damping. The vacuum boundary with physical singularity (we call it the *physical vacuum* for short) arises in many physical situations naturally, for example, in the study of the evolution and structure of gaseous stars (cf. [5,15]), for which vacuum boundaries are natural boundaries. Another situation in which the physical vacuum plays an important role is the gas–vacuum interface problem of compressible isentropic Euler equations with damping (cf. [46–48,62,64]). In three dimensions, this problem reads as

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \quad \text{in } \Omega(t), \tag{1.1a}$$
$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} p(\rho) = -\rho \mathbf{u} \quad \text{in } \Omega(t), \tag{1.1b}$$
$$\rho > 0 \quad \text{in } \Omega(t), \tag{1.1c}$$
$$\rho = 0 \quad \text{on } \Gamma(t) := \partial\Omega(t), \tag{1.1d}$$
$$\mathcal{V}(\Gamma(t)) = \mathbf{u} \cdot \mathbf{n}, \tag{1.1e}$$
$$(\rho, \mathbf{u}) = (\rho_0, \mathbf{u}_0) \quad \text{on } \Omega := \Omega(0). \tag{1.1f}$$

Here $(\mathbf{x}, t) \in \mathbb{R}^3 \times [0, \infty)$, $\rho$, $\mathbf{u}$, and $p$ denote, respectively, the space and time variables, density, velocity and pressure; $\Omega(t) \subset \mathbb{R}^3$, $\Gamma(t)$, $\mathcal{V}(\Gamma(t))$ and $\mathbf{n}$ represent, respectively, the changing volume occupied by a gas at time $t$, the moving interface of fluids and vacuum states, the normal velocity of $\Gamma(t)$ and the exterior unit normal vector to $\Gamma(t)$. We consider a polytropic gas; the equation of state is given by

$$p(\rho) = \rho^\gamma \ \text{ for } \ \gamma > 1. \tag{1.2}$$

Equations (1.1a) and (1.1b) describe the balance laws of mass and momentum, respectively; conditions (1.1c) and (1.1d) state that $\Gamma(t)$ is the interface to be investigated; (1.1e) indicates that the interface moves with the normal component of the fluid velocity; and (1.1f) is the initial conditions for the density, velocity and domain.

Let $c(\rho) = \sqrt{p'(\rho)} = \sqrt{\gamma \rho^{\gamma-1}}$ be the sound speed, and the condition

$$-\infty < \nabla_{\mathbf{n}} \left( c^2(\rho) \right) < 0 \ \text{ on } \ \Gamma(t) \tag{1.3}$$

defines a *physical vacuum* boundary (cf. [11,14,34,46–48]). This yields from (1.1b) that

$$D_t \mathbf{u} + \frac{1}{\gamma - 1} \nabla_{\mathbf{x}} \left( c^2(\rho) \right) = -\mathbf{u},$$

where $D_t \mathbf{u} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u}$ is the acceleration. For a physical vacuum boundary, the normal acceleration of the boundary $\Gamma(t)$ is finite.

The physical vacuum that the sound speed is $C^{1/2}$-Hölder continuous but not $C^1$-continuous near vacuum boundaries makes it extremely challenging in the study of the well-poseness of isentropic Euler equations, since standard methods of symmetric hyperbolic systems developed by FRIEDRICHS–KATO–LAX (cf. [20,35,39]) do not apply. Indeed, for the Cauchy problem of isentropic Euler equations with damping in three dimensions with initial data being small perturbations of constant states $(\underline{\rho}, 0)$ away from vacuum (i.e., $\underline{\rho} > 0$), the transformation $\xi = 2(\gamma - 1)^{-1}(c(\rho) - c(\underline{\rho}))$ was used in [54] to symmetrize the system so that energy methods could be employed to establish the global existence of smooth solutions in the function space $(\xi, u) \in C\left([0,\infty);\ H^3(\mathbb{R}^3)\right) \cap C^1\left([0,\infty);\ H^2(\mathbb{R}^3)\right)$ and the decay estimates. (See also [9,19,53,57] for further results in the same spirit.) For the Cauchy problems of isentropic Euler equations and Euler–Poisson equations with initial data containing vacuum (e.g., initial densities have compact supports), the local existence of smooth solutions were established in [51,52] in the function space $(w, u) \in C\left([0,T^*);\ H^3(\mathbb{R}^3)\right) \cap C^1\left([0,T^*);\ H^2(\mathbb{R}^3)\right)$ for some finite time $T^*$, where $w = 2(\gamma-1)^{-1}c(\rho)$ was used to symmetrize the systems so that standard methods of symmetric hyperbolic systems could work. However, since the Sobolev space $H^3(\mathbb{R}^3)$ can be embedded into $C^1(\mathbb{R}^3)$, the local existence theory obtained in [51,52] cannot apply to the case of a physical vacuum for which the sound speed is only $C^{1/2}$- but not $C^1$-continuous near vacuum states.

To capture a physical vacuum, one studies the gas–vacuum interface problem for compressible inviscid flows with special attention to the behavior (1.3) near vacuum boundaries. This is a challenging problem even for the local-in-time existence theory, since system (1.1) is a degenerate and characteristic hyperbolic system which violates the uniform Kreiss–Lopatinskii condition (cf. [36]) due to resonant wave speeds at vacuum boundaries. As realized in [14,34] for physical vacuums, the appearance of the density functions as coefficients in a nonlinear wave equation which governs the dynamics of the divergence of the velocity of the gas, and weighted estimates, show that this wave equation loses derivatives with respect to the non-degenerate case of a compressible liquid, wherein the density takes the value of a strictly positive constant on moving boundaries (cf. [43]). Also, characteristic speeds of the compressible isentropic Euler equations become singular with infinite spatial derivatives at vacuum boundaries due to (1.3). Therefore, the physical singularity (1.3) creates rather severe difficulties in analyzing the regularity near vacuum boundaries. Recently, important progress has been made in the local-in-time well-posedness theory for the compressible Euler equations (cf. [11,13,14,32,34]). On the other hand, this poses great challenges to extend the local-in-time existence theory to a global one of smooth solutions. In the local well-posedness theory mentioned above, suitable weights are found to resolve the singularity near vacuum boundaries. To obtain the global-in-time regularity, one has to find the resolution of the singularity near vacuum boundaries uniform in time.

In order to understand physical vacuum phenomena for problem (1.1), a family of explicit solutions with spherical symmetry was constructed in [46] to capture the behavior (1.3) using the ansatz as follows:

$$\Omega(t) = B_{R(t)}(\mathbf{0}),\ c^2(\mathbf{x},t) = c^2(r,t) = e(t) - b(t)r^2 \quad \text{and} \quad \mathbf{u}(\mathbf{x},t) = (\mathbf{x}/r)u(r,t), \tag{1.4}$$

where $r = |\mathbf{x}|$, $R(t) = \sqrt{e(t)/b(t)}$ and $u(r,t) = a(t)r$. In [46], a system of ordinary differential equations for $(e, b, a)(t)$ was derived with $e(t), b(t) > 0$ for $t \geqq 0$ by substituting (1.4) into (1.1a) and (1.1b), and this family of explicit solutions was proved to be time-asymptotically equivalent to the Barenblatt self-similar solution (cf. [4]) with the same total mass. The Barenblatt solution solves the porous media equation

$$\rho_t = \Delta p(\rho), \tag{1.5}$$

when (1.1b) is simplified to Darcy's law:

$$\nabla_{\mathbf{x}} p(\rho) = -\rho \mathbf{u}. \tag{1.6}$$

(The equivalence can be seen formally by the rescaling $\mathbf{x}' = \varepsilon \mathbf{x}, t' = \varepsilon^2 t, \mathbf{u}' = \mathbf{u}/\varepsilon$.) The Barenblatt self-similar solution (cf. [4]) with a finite mass $M > 0$, which is spherically symmetric, is given by

$$\bar{\rho}(\mathbf{x}, t) = \bar{\rho}(r, t) = (1+t)^{-\frac{3}{3\gamma-1}} \left(A - B(1+t)^{-\frac{2}{3\gamma-1}} r^2\right)^{\frac{1}{\gamma-1}} \quad \text{with} \quad r = |\mathbf{x}|, \tag{1.7}$$

where

$$B = \frac{\gamma - 1}{2\gamma(3\gamma - 1)} \quad \text{and}$$

$$(\gamma A)^{\frac{3\gamma-1}{2(\gamma-1)}} = M \gamma^{\frac{1}{\gamma-1}} (\gamma B)^{\frac{3}{2}} \left(\int_0^1 y^2 \left(1 - y^2\right)^{\frac{1}{\gamma-1}} dy\right)^{-1}. \tag{1.8}$$

Clearly,

$$\int_0^{\bar{R}(t)} r^2 \bar{\rho}(r, t)\, \mathrm{d}r = M \quad \text{for} \quad t \geqq 0, \quad \text{where} \quad \bar{R}(t) = \sqrt{A/B}(1+t)^{1/(3\gamma-1)}. \tag{1.9}$$

The corresponding Barenblatt velocity $\bar{\mathbf{u}}$ is defined by $\bar{\mathbf{u}}(\mathbf{x}, t) = (\mathbf{x}/r)\bar{u}(r, t)$ in the region $\{(r, t) : 0 \leqq r \leqq \bar{R}(t),\ t > 0\}$, where

$$\bar{u}(r, t) = -\frac{p(\bar{\rho})_r}{\bar{\rho}} = \frac{r}{(3\gamma - 1)(1+t)} \quad \text{satisfying} \quad \bar{u}(0, t) = 0 \quad \text{and}$$
$$\dot{\bar{R}}(t) = \bar{u}\left(\bar{R}(t), t\right).$$

So, $(\bar{\rho}, \bar{\mathbf{u}})$ defined in the region $\{(r, t) : 0 \leqq r \leqq \bar{R}(t),\ t > 0\}$ solves (1.5) and (1.6). The vacuum boundary $r = \bar{R}(t)$ of Barenblatt's solution is clearly physical. This is one of major motivations to study the physical vacuum boundary problem of compressible Euler equations with damping. Indeed, the Barenblatt solution of (1.5) and (1.6) can be obtained by the same ansatz as (1.4): $\bar{c}^2(\mathbf{x}, t) = \bar{e}(t) - \bar{b}(t) r^2$ and $\mathbf{u}(\mathbf{x}, t) = \bar{a}(t)\mathbf{x}$. Substituting this into (1.5), (1.6) and (1.9) with $\bar{R}(t) = \sqrt{\bar{e}(t)/\bar{b}(t)}$ gives

$$\bar{e}(t) = \gamma A(1+t)^{-3(\gamma-1)/(3\gamma-1)}, \ \bar{b}(t) = \gamma B(1+t)^{-1} \quad \text{and}$$
$$\bar{a}(t) = (3\gamma - 1)^{-1}(1+t)^{-1},$$

where $A$ and $B$ are determined by (1.8). It was proved in [46] the time-asymptotic equivalence:

$$(a,\ b,\ e)(t) = (\bar{a},\ \bar{b},\ \bar{e})(t) + O(1)(1+t)^{-1}\ln(1+t) \quad \text{as} \quad t \to \infty.$$

A question was raised in [46] whether this equivalence is still true for general solutions to problem (1.1). Luo and the author (cf. [50]) studied this problem in one-dimensional case and proved the global smooth solutions and the convergence to Barenblatt solutions as time goes to infinity. However, it is well known that the situation is more complicated and difficult in multiple dimensions. In a broader context, there has been a relatively satisfactory understanding of hyperbolic systems in one dimension, but the understanding is extremely poor in multiple dimensions. Compressible Euler equations with damping fall into the class of hyperbolic systems with lower-order dissipations. Although lower-order dissipations such as damping or relaxation can have certain smoothing effects, most available results on the global existence of smooth solutions are for *strictly* hyperbolic systems or systems endowed with *strict* convex entropy (cf. [25–27,41,45,54,57,66–68]). Indeed, the isentropic compressible Euler equations with damping are strictly hyperbolic or endowed with strict convex entropy only away from vacuum states. To the best knowledge of the author, in the presence of vacuum states, for any kind of time evolutionary problems such as Cauchy problems, initial-boundary problems, or free-boundary problems, there has been no result of the global existence of smooth solutions to compressible Euler equations with damping in multiple dimensions, even for the spherically symmetric case. The purpose of this paper also serves as a step towards the resolution of this problem.

For the isentropic Euler equations (with or without damping) with data containing a vacuum, the available results of the global existence of solutions are for $L^\infty$-*weak solutions* or *weak solutions with finite energy* via the approach of compensated compactness (cf. [6–8,16–18,40,44]), except the recent result in [50]. For $L^\infty$-weak solutions to the Cauchy problem of the one-dimensional compressible Euler equations with damping, the $L^p$-convergence to Barenblatt solutions of the porous media equations was given in [28] with $p = 2$ if $1 < \gamma \leqq 2$ and $p = \gamma$ if $\gamma > 2$ and in [29] with $p = 1$, respectively, using entropy-type estimates for the solution itself without deriving estimates for derivatives. However, it seems difficult to adopt the approach of [28,29], which depends on the uniform $L^\infty$-bound of solutions crucially, in the case of spherically symmetric motions in multiple dimensions. Indeed, there have been no uniform $L^\infty$-bounds available for spherically symmetric weak solutions in multiple dimensions obtained via compensated compactness in [8] and [40], respectively, for a cut-off domain excluding the origin and a domain containing the origin. (The $L^\infty$-bound of weak solutions obtained in [8] depends on time and may become unbounded as time goes to infinity; the case is worse for that of weak solutions obtained in [40].) Even for spherically symmetric weak solutions away from the vacuum in a cut-off domain excluding the origin, the uniform $L^\infty$-bound is only available for a nonlinear model problem of compressible Euler equations for which all the waves move at constant speeds obtained via the Glimm scheme in [63]. These show the subtlety in the study of spherically symmetric solutions of compressible Euler equations in multiple dimensions. Moreover, interfaces separating gases and a vacuum cannot be traced in the framework of $L^\infty$-weak solutions. The aim of this work is to understand the behavior and long time dynamics of physical vacuum boundaries for spherically symmetric motions in a domain containing the origin in three dimensions, for which obtaining the global-in-time regularity of solutions is essential to the well-definiteness and realization of the evolution of the vacuum boundary.

In between the one-dimensional theory in [50] and the general multi-dimensional theory which we will pursue in the future, we study spherically symmetric solutions of (1.1) with the motivation that the Barenblatt solution posses the same symmetry, and it is expected that spherically symmetric solutions will provide insights on the local and long time behaviors of solutions to the general three-dimensional problem (1.1). Locally, at each point $\mathbf{x}$ in $\Omega(t)$, it might be plausible to rotate a solution in all possible ways about $\mathbf{x}$ and average all rotations in the spirit of spherical mean. In the long term, for a general three-dimensional problem, it is expected that the geometry of the boundary becomes more and more symmetric due to the dissipation of damping which dissipates the total energy. For this purpose, we seek solutions with symmetry to problem (1.1) of the form:

$$\Omega(t) = B_{R(t)}(\mathbf{0}), \quad \rho(\mathbf{x}, t) = \rho(r, t), \quad \mathbf{u}(\mathbf{x}, t) = (\mathbf{x}/r)u(r, t) \;\text{ with }\; r = |\mathbf{x}|.$$

Then problem (1.1) reduces to

$$(r^2\rho)_t + (r^2\rho u)_r = 0 \qquad \text{in } (0, R(t)), \tag{1.10a}$$
$$\rho(u_t + uu_r) + p_r = -\rho u \qquad \text{in } (0, R(t)), \tag{1.10b}$$
$$\rho > 0 \qquad \text{in } [0, R(t)), \tag{1.10c}$$
$$\rho\,(R(t), t) = 0, \quad u(0, t) = 0, \tag{1.10d}$$
$$\dot{R}(t) = u(R(t), t) \;\text{ with }\; R(0) = R_0, \tag{1.10e}$$
$$(\rho, u)(r, t = 0) = (\rho_0, u_0)\,(r) \qquad \text{on } (0, R_0), \tag{1.10f}$$

so that $R(t)$ is the radius of the domain occupied by the gas at time $t$ and $r = R(t)$ represents the vacuum boundary which issues from $r = R_0$ and moves with the fluid velocity.

In the spherically symmetric setting, the physical vacuum boundary condition (1.3) reduces to $-\infty < (c^2)_r < 0$ in a small neighborhood of the boundary. To capture this singularity, the initial domain is taken to be a ball $\{0 \leqq r \leqq R_0\}$ and the initial density is assumed to satisfy

$$\rho_0(r) > 0 \text{ for } 0 \leqq r < R_0,\; \rho_0(R_0) = 0 \text{ and } -\infty < \left(\rho_0^{\gamma-1}\right)_r < 0 \text{ at } r = R_0. \tag{1.11}$$

We require that the initial total mass is the same as that of the Barenblatt solution, that is,

$$\int_0^{R_0} r^2\rho_0(r)\,\mathrm{d}r = \int_0^{\bar{R}(0)} r^2\bar{\rho}_0(r)\,\mathrm{d}r = M. \tag{1.12}$$

The conservation law of mass, (1.10a), and (1.9), give

$$\int_0^{R(t)} r^2\rho(r, t)\,\mathrm{d}r = \int_0^{R_0} r^2\rho_0(r)\,\mathrm{d}r = M = \int_0^{\bar{R}(t)} r^2\bar{\rho}(r, t)\,\mathrm{d}r \;\text{ for }\; t \geqq 0.$$

In the present work, we prove the global existence of smooth solutions to problem (1.10) when initial data are small spherically symmetric perturbations of Barenblatt solutions and they have the same total masses. Moreover, we obtain the pointwise convergence with a rate of density which gives the detailed behavior of the density, the convergence rate of velocity in supreme norm and the precise expanding rate of physical vacuum boundaries. The results obtained in this article also prove the nonlinear asymptotic stability of both the Barenblatt solution and the explicit solution (1.4) in the setting of physical vacuum boundary problems.

The key idea in obtaining the global-in-time higher-order regularity of solutions to problem (1.10) is to construct nonlinear weighted functionals, and to perform nonlinear weighted estimates and elliptic estimates. To obtain the global-in-time regularity, the decay estimates are essential, which are achieved in the present work by introducing suitable weights involving both space and time variables to quantify the behavior of solutions both near the vacuum boundary and origin, and in large time. There is a distinction between the weights constructed here and those for the local-in-time well-posedness theory (cf. [11,13,14,32,34,49]) where only spatial weights are needed. Besides this, since the Barenblatt solution to the porous media equation does not solve (1.10) exactly and an error appears, we introduce a new higher-order correction with which the nonlinear weighted energy estimates and elliptic estimates can be performed. Compared with the one-dimensional case studied in [50], much more obstacles appear for solving problem (1.10). Besides the difficulty of strong degeneracy of the equations at vacuum states, the coordinates singularity at the origin which carries the true three-dimensional nature is another one. Indeed, the difficulty of the coordinates singularity at the origin was avoided in many previous studies of spherically symmetric motions in multiple dimensions for compressible fluids due to the challenge of how to resolve this singularity. In this paper, suitable weights are constructed carefully to resolve the coordinates singularity. As an intermediate step passing from one-dimensional theory in [50] to the general three-dimensional problem, (1.1), we believe the ideas and estimates including the nonlinear weighted functionals and pointwise decay estimates developed in this paper will contribute to a understanding of the behavior of solutions to problem (1.1).

There has been a recent explosion of interest in the analysis of free-boundary problems for both compressible and incompressible inviscid fluids. (As for viscous flows, there have been many results on the free-boundary Navier-Stokes equations which cause quite different difficulties in analyses from those for inviscid flows, so we do not discuss the works in that regime here.) For incompressible inviscid flows, one may refer to [2,3,10,12,38,42,55,58,59,69] for the local-in-time theory; while the global-in-time theory is rather recent which is for both two-dimensional and three-dimensional water wave problems of irrotational flows (cf. [21,22,30, 60,61]). For compressible inviscid flows, besides the aforementioned results on vacuum boundary problems, the local-in-time existence and uniqueness for the three-dimensional compressible Euler equations modeling a liquid rather than a gas were established in [43] by using Lagrangian variables combined with Nash-Moser iteration to construct solutions. (For a compressible liquid, the density is assumed to be a strictly positive constant on the moving boundary. As such, the compressible liquid provides a strictly hyperbolic, but characteristic, system.) An alternative proof for the existence of a compressible liquid was given in [56], employing a strategy based on symmetric hyperbolic systems combined with Nash-Moser iteration. From the above discussions, one may see that the current available theories of free-boundary problems for invscid flows, in particular for compressible invscid flows, are mainly on local-in-time solutions. The results obtained in this paper are among the first ones on the global solutions of free-boundary problems for compressible inviscid fluids in the presence of vacuum states.

## 2. Reformulation of the Problem and Main Results

### *2.1. Fix the Domain and Lagrangian Variables*

In this subsection, we adopt the the Lagrangian particle trajectory formulation as used in [11,13,14,31,34] to reduce the original free-boundary problem (1.10) to an initial boundary value problem. We make the initial domain of the Barenblatt solution, $\big(0,\, \bar R(0)\big)$, as the reference domain and define a diffeomorphism $\eta_0 : \big(0,\, \bar R(0)\big) \to (0, R_0)$ by

$$\int_0^{\eta_0(r)} r^2 \rho_0(r)\,\mathrm{d}r = \int_0^r r^2 \bar\rho_0(r)\,\mathrm{d}r \ \text{ for } \ r \in \big(0,\, \bar R(0)\big),$$

where $\bar\rho_0(r) := \bar\rho(r, 0)$ is the initial density of the Barenblatt solution. Clearly,

$$\eta_0^2(r)\rho_0(\eta_0(r))\eta_{0r}(r) = r^2\bar\rho_0(r) \ \text{ for } \ r \in \big(0,\, \bar R(0)\big). \tag{2.1}$$

Due to (1.11), (1.7) and the fact that the total mass of the Barenblatt solution is the same as that of $\rho_0$, (1.12), the diffeomorphism $\eta_0$ is well defined. For simplicity of presentation, set $\mathcal{I} := (0,\ \bar R(0)) = (0,\ \sqrt{A/B})$. To fix the boundary, we transform system (1.10) into Lagrangian variables. For $r \in \mathcal{I}$, we define the Lagrangian variable $\eta(r, t)$ by

$$\eta_t(r, t) = u(\eta(r, t), t) \ \text{ for } \ t > 0 \ \text{ and } \ \eta(r, 0) = \eta_0(r), \tag{2.2}$$

and set the Lagrangian density and velocity by

$$f(r, t) = \rho(\eta(r, t), t) \ \text{ and } \ v(r, t) = u(\eta(r, t), t). \tag{2.3}$$

Then the Lagrangian version of system (1.10) can be written on the reference domain $\mathcal{I}$ as

$$(\eta^2 f)_t + r^2 f v_r/\eta_r = 0 \qquad \text{in } \mathcal{I} \times (0, \infty), \tag{2.4a}$$

$$f v_t + (f^\gamma)_r/\eta_r = -f v \qquad \text{in } \mathcal{I} \times (0, \infty), \tag{2.4b}$$

$$v(0, t) = 0 \qquad \text{on } (0, \infty), \tag{2.4c}$$

$$(f, v) = (\rho_0(\eta_0), u_0(\eta_0)) \qquad \text{on } \mathcal{I} \times \{t = 0\}. \tag{2.4d}$$

It should be noted that we need $\eta_r(r, t) > 0$ for $r \in \mathcal{I}$ and $t \geqq 0$ to make the Lagrangian transformation sensible, which will be verified in (3.3). Indeed, $\eta_r > 0$

implies $\eta(r,t) > 0$ for $r \in \mathcal{I}$ and $t \geqq 0$, due to the boundary condition that the center of the symmetry does not move, $v(0,t) = 0$. The map $\eta(\cdot, t)$ defined above can be extended to $\bar{\mathcal{I}} = [0, \sqrt{A/B}]$. In the setting, the vacuum boundaries for problem (1.10) are given by

$$R(t) = \eta\left(\bar{R}(0),\ t\right) = \eta\left(\sqrt{A/B},\ t\right) \quad \text{for } t \geqq 0. \tag{2.5}$$

It follows from solving (2.4a) and using (2.1) that

$$f(r,t)\eta^2(r,t)\eta_r(r,t) = \rho_0(\eta_0(r))\eta_0^2(r)\eta_{0r}(r) = r^2\bar{\rho}_0(r), \quad r \in \mathcal{I}. \tag{2.6}$$

Thus, the initial density of the Barenblatt solution, $\bar{\rho}_0$, can be viewed as a parameter and system (2.4) can be rewritten as

$$\bar{\rho}_0\eta_{tt} + \bar{\rho}_0\eta_t + \left(\frac{\eta}{r}\right)^2\left[\left(\frac{r^2}{\eta^2}\frac{\bar{\rho}_0}{\eta_r}\right)^\gamma\right]_r = 0 \qquad \text{in } \mathcal{I} \times (0,\infty), \tag{2.7a}$$

$$\eta(0,t) = 0, \qquad \text{on } (0,\infty), \tag{2.7b}$$

$$(\eta, \eta_t) = (\eta_0, u_0(\eta_0)) \qquad \text{on } \mathcal{I} \times (0,\infty). \tag{2.7c}$$

## *2.2. Ansatz*

Define the Lagrangian variable $\bar{\eta}(r,t)$ for the Barenblatt flow in $\bar{\mathcal{I}}$ by

$$\partial_t\bar{\eta}(r,t) = \bar{u}(\bar{\eta}(r,t),t) = \frac{\bar{\eta}(r,t)}{(3\gamma-1)(1+t)} \quad \text{for } t > 0 \text{ and } \bar{\eta}(r,0) = r, \tag{2.8}$$

so that

$$\bar{\eta}(r,t) = r\,(1+t)^{1/(3\gamma-1)} \quad \text{for } (r,t) \in \bar{\mathcal{I}} \times [0,\infty) \tag{2.9}$$

and

$$\bar{\rho}_0\bar{\eta}_t + \left(\frac{\bar{\eta}}{r}\right)^2\left[\left(\frac{r^2}{\bar{\eta}^2}\frac{\bar{\rho}_0}{\bar{\eta}_r}\right)^\gamma\right]_r = 0 \text{ in } \mathcal{I} \times (0,\infty).$$

Since $\bar{\eta}$ does not solve (2.7a) exactly, we introduce a correction $h(t)$ which is a solution of the following initial value problem of ordinary differential equations:

$$\begin{aligned} &h_{tt} + h_t - (\bar{\eta}_r + h)^{2-3\gamma}/(3\gamma-1) + \bar{\eta}_{rtt} + \bar{\eta}_{rt} = 0, \quad t > 0, \\ &h(t=0) = h_t(t=0) = 0. \end{aligned} \tag{2.10}$$

(Notice that $\bar{\eta}_r$, $\bar{\eta}_{rt}$ and $\bar{\eta}_{rtt}$ are independent of $r$.) The new ansatz is then given by

$$\tilde{\eta}(r,t) := \bar{\eta}(r,t) + rh(t), \tag{2.11}$$

so that

$$\bar{\rho}_0\tilde{\eta}_{tt} + \bar{\rho}_0\tilde{\eta}_t + \left(\frac{\tilde{\eta}}{r}\right)^2\left[\left(\frac{r^2}{\tilde{\eta}^2}\frac{\bar{\rho}_0}{\tilde{\eta}_r}\right)^\gamma\right]_r = 0 \text{ in } \mathcal{I} \times (0,\infty). \tag{2.12}$$

It should be noted that $\tilde{\eta}_r$ is independent of $r$. We will prove in the "Appendix" that $\tilde{\eta}$ behaves similar to $\bar{\eta}$, that is, there exist positive constants $K$ and $C(n)$ independent of time $t$ such that for all $t \geqq 0$,

$$(1+t)^{1/(3\gamma-1)} \leqq \tilde{\eta}_r(t) \leqq K\,(1+t)^{1/(3\gamma-1)}, \quad \tilde{\eta}_{rt} \geqq 0, \tag{2.13a}$$

$$\left|\frac{\mathrm{d}^k\tilde{\eta}_r(t)}{\mathrm{d}t^k}\right| \leqq C(n)\,(1+t)^{\frac{1}{3\gamma-1}-k}, \quad k=1,2,\cdots,n. \tag{2.13b}$$

Moreover, there exists a certain constant $C$ independent of $t$ such that

$$\begin{aligned} &0 \leqq h(t) \leqq C(1+t)^{\frac{1}{3\gamma-1}-1}\ln(1+t) \ \text{ and} \\ &|h_t(t)| \leqq C(1+t)^{\frac{1}{3\gamma-1}-2}\ln(1+t), \quad t \geqq 0. \end{aligned} \tag{2.14}$$

The proof of (2.14) will also be given in the "Appendix".

### 2.3. Main Results

To state the main theorem, we write equation (2.7a) in a perturbation form around the Barenblatt solution. Let $\zeta(r,t) := \eta(r,t)/r - \tilde{\eta}(r,t)/r$. Thus,

$$\eta(r,t) = \tilde{\eta}(r,t) + r\zeta(r,t) \ \text{ and } \ \eta_r(r,t) = \tilde{\eta}_r(t) + \zeta(r,t) + r\zeta_r(r,t). \tag{2.15}$$

It follows from (2.7a) and (2.12) that

$$\begin{aligned} &r\bar{\rho}_0\zeta_{tt} + r\bar{\rho}_0\zeta_t + (\tilde{\eta}_r+\zeta)^2\left[\bar{\rho}_0^{\gamma}\,(\tilde{\eta}_r+\zeta)^{-2\gamma}\,(\tilde{\eta}_r+\zeta+r\zeta_r)^{-\gamma}\right]_r - \tilde{\eta}_r^{2-3\gamma} \\ &\left(\bar{\rho}_0^{\gamma}\right)_r = 0, \end{aligned} \tag{2.16}$$

Denote $\alpha := 1/(\gamma-1)$, $\ l := 3 + \min\{m \in \mathbb{N}: \ m > \alpha\} = 4 + [\alpha]$. For $j = 0, \cdots, l$ and $i = 0, \cdots, l-j$, we set

$$\begin{aligned} &\mathcal{E}_j(t) := (1+t)^{2j} \\ &\int_{\mathcal{I}}\left[r^4\bar{\rho}_0\left(\partial_t^j\zeta\right)^2 + r^2\bar{\rho}_0^{\gamma}\left|\partial_t^j\,(\zeta, r\zeta_r)\right|^2 + (1+t)r^4\bar{\rho}_0\left(\partial_t^j\zeta_t\right)^2\right]\mathrm{d}r, \\ &\mathcal{E}_{j,i}(t) := (1+t)^{2j} \\ &\int_{\mathcal{I}}\left[r^2\bar{\rho}_0^{1+(i-1)(\gamma-1)}\left(\partial_t^j\partial_r^i\zeta\right)^2 + r^4\bar{\rho}_0^{1+(i+1)(\gamma-1)}\left(\partial_t^j\partial_r^{i+1}\zeta\right)^2\right]\mathrm{d}r. \end{aligned}$$

The higher-order norm is defined by

$$\mathcal{E}(t) := \sum_{j=0}^{l}\left(\mathcal{E}_j(t) + \sum_{i=1}^{l-j}\mathcal{E}_{j,i}(t)\right).$$

It will be proved in Lemma 3.7 that

$$\sup_{r\in\mathcal{I}}\left\{\sum_{j=0}^{2}(1+t)^{2j}\left|\partial_t^j\zeta(r,t)\right|^2 + \sum_{j=0}^{1}(1+t)^{2j}\left|\partial_t^j\zeta_r(r,t)\right|^2\right\} \leqq C\mathcal{E}(t)$$

for some constant $C$ independent of $t$. So the boundedness of $\mathcal{E}(t)$ gives the uniform boundedness and decay of the perturbation $\zeta$ and its derivatives. In what follows, we state our main result.

**Theorem 2.1.** *Suppose that* (1.12) *holds. There exists a constant* $\bar{\delta} > 0$ *such that if* $\mathcal{E}(0) \leqq \bar{\delta}$, *then the problem* (2.7) *admits a global unique smooth solution in* $\mathcal{I} \times [0, \infty)$ *satisfying for all* $t \geqq 0$,

$$\mathcal{E}(t) \leqq C\mathcal{E}(0)$$

*and*

$$\begin{aligned}\sup_{r\in\mathcal{I}} \Bigg\{ &\sum_{j=0}^{2}(1+t)^{2j}\left|\partial_t^j \zeta(r,t)\right|^2 + \sum_{j=0}^{1}(1+t)^{2j}\left|\partial_t^j \zeta_r(r,t)\right|^2 + \sum_{i+j\leqq l-2,\ 2i+j\geqq 3}(1+t)^{2j} \\ &\times\left|\bar{\rho}_0^{\frac{(\gamma-1)(2i+j-3)}{2}}\partial_t^j\partial_r^i\zeta(r,t)\right|^2 + \sum_{i+j=l-1}(1+t)^{2j}\left|r\bar{\rho}_0^{\frac{(\gamma-1)(2i+j-3)}{2}}\partial_t^j\partial_r^i\zeta(r,t)\right|^2 \\ &+ \sum_{i+j=l}(1+t)^{2j}\left|r^2\bar{\rho}_0^{\frac{(\gamma-1)(2i+j-3)}{2}}\partial_t^j\partial_r^i\zeta(r,t)\right|^2\Bigg\} \leqq C\mathcal{E}(0),\end{aligned} \tag{2.17}$$

*where* $C$ *is a positive constant independent of* $t$.

It should be noticed that the time derivatives involved in the initial higher-order energy norm, $\mathcal{E}(0)$, can be determined via the equation by the initial data $\rho_0$ and $u_0$ (see [13,49] for instance).

As a corollary of Theorem 2.1, we have the following theorem for solutions to the original vacuum boundary problem (1.10).

**Theorem 2.2.** *Suppose that* (1.12) *holds. There exists a constant* $\bar{\delta} > 0$ *such that if* $\mathcal{E}(0) \leqq \bar{\delta}$, *then the problem* (1.10) *admits a global unique smooth solution* $(\rho(\eta,t), u(\eta,t), R(t))$ *for* $t \in [0,\infty)$ *satisfying*

$$\begin{aligned}|\rho(\eta(r,t),t) - \bar{\rho}(\bar{\eta}(r,t),t)| \leqq & C\left(A - Br^2\right)^{\frac{1}{\gamma-1}}(1+t)^{-\frac{4}{3\gamma-1}} \\ &\times\left(\sqrt{\mathcal{E}(0)} + (1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\ln(1+t)\right),\end{aligned} \tag{2.18}$$

$$\begin{aligned}|u(\eta(r,t),t) - \bar{u}(\bar{\eta}(r,t),t)| \leqq & Cr(1+t)^{-1} \\ &\left(\sqrt{\mathcal{E}(0)} + (1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\ln(1+t)\right),\end{aligned} \tag{2.19}$$

*for all* $r \in \mathcal{I}$ *and* $t \geqq 0$; *and for all* $t \geqq 0$,

$$c_1(1+t)^{\frac{1}{3\gamma-1}} \leqq R(t) \leqq c_2(1+t)^{\frac{1}{3\gamma-1}}, \tag{2.20}$$

$$\left|\frac{\mathrm{d}^k R(t)}{\mathrm{d}t^k}\right| \leqq C(1+t)^{\frac{1}{3\gamma-1}-k}, \quad k=1,2,3, \tag{2.21}$$

$$c_3(1+t)^{-\frac{3\gamma-2}{3\gamma-1}} \leqq \left|\left(\rho^{\gamma-1}\right)_\eta(\eta,t)\right| \leqq c_4(1+t)^{-\frac{3\gamma-2}{3\gamma-1}} \text{ when } \frac{1}{2}R(t) \leqq \eta \leqq R(t). \tag{2.22}$$

*Here $C$, $c_1$, $c_2$, $c_3$ and $c_4$ are positive constants independent of $t$.*

The pointwise behavior of the density and velocity for the vacuum boundary problem (1.10) to that of the Barenblatt solution are given by (2.18) and (2.19), respectively. It is also shown in (2.18) that the difference of density to problem (1.10) and the corresponding Barenblatt density decays at the rate of $(1+t)^{-4/(\gamma+1)}$ in $L^\infty$, while the density of the Barenblatt solution, $\bar{\rho}$, decays at the rate of $(1+t)^{-3/(\gamma+1)}$ in $L^\infty$ [see (1.7)]. (2.20) gives the precise expanding rate of the vacuum boundaries of the problem (1.10), which is the same as that for the Barenblatt solution shown in (1.9). Furthermore, it verifies in (2.22) that the vacuum boundary $R(t)$ is physical at any finite time.

## 3. Proof of Theorem 2.1

The proof is based on the local existence of smooth solutions (cf. [13,32,49]) and continuation arguments. The uniqueness of the smooth solutions can be obtained as in Section 11 of [49]. In order to prove the global existence of smooth solutions, we need to obtain the uniform-in-time *a priori* estimates on any given time interval $[0, T]$ satisfying $\sup_{t\in[0,T]} \mathcal{E}(t) < \infty$. For this purpose, we use a bootstrap argument by making the following *a priori* assumption: Let $\zeta$ be a smooth solution to (2.16) on $[0, T]$, there exists a suitably small fixed positive number $\varepsilon_0 \in (0, 1)$ independent of $t$ such that for $t \in [0, T]$,

$$\sum_{j=0}^{2}(1+t)^{2j}\left\|\partial_t^j\zeta(\cdot,t)\right\|_{L^\infty}^2+\sum_{j=0}^{1}(1+t)^{2j}\left\|\partial_t^j\zeta_r(\cdot,t)\right\|_{L^\infty}^2 + \sum_{i+j\leqq l-2,\ 2i+j\geqq 3}(1+t)^{2j}\times\left\|\bar{\rho}_0^{\frac{(\gamma-1)(2i+j-3)}{2}}\partial_t^j\partial_r^i\zeta(\cdot,t)\right\|_{L^\infty}^2 + \sum_{i+j=l-1}(1+t)^{2j}\left\|r\bar{\rho}_0^{\frac{(\gamma-1)(2i+j-3)}{2}}\partial_t^j\partial_r^i\zeta(\cdot,t)\right\|_{L^\infty}^2 \leqq \varepsilon_0^2. \tag{3.1}$$

This, in particular, implies, noting (2.13), that for $0 \leqq \theta_1, \theta_2 \leqq 1$,

$$\frac{1}{2}(1+t)^{\frac{1}{3\gamma-1}} \leqq (\tilde{\eta}_r+\theta_1\zeta+\theta_2 r\zeta_r)(r,t) \leqq 2K(1+t)^{\frac{1}{3\gamma-1}},\quad (r,t)\in\mathcal{I}\times[0,T]. \tag{3.2}$$

Moreover, it follows from (2.15) and (3.2) that

$$\frac{1}{2}(1+t)^{\frac{1}{3\gamma-1}} \leqq \eta_r(r,t),\ r^{-1}\eta(r,t) \leqq 2K(1+t)^{\frac{1}{3\gamma-1}},\quad (r,t)\in\mathcal{I}\times[0,T]. \tag{3.3}$$

Here $K$ is the positive constant appearing in (2.13a).

Under this *a priori* assumption, we prove in Section 3.2 the following elliptic estimates: $\mathcal{E}_{j,i}(t) \leqq C\sum_{\iota=0}^{i+j}\mathcal{E}_\iota(t)$, when $j \geqq 0$, $i \geqq 1$, $i+j \leqq l$, where $C$

is a positive constant independent of $t$. With the *a priori* assumption and elliptic estimates, we show in Section 3.3 the following nonlinear weighted energy estimate: for some positive constant $C$ independent of $t$, $\mathcal{E}_j(t) \leqq C\sum_{\iota=0}^{j}\mathcal{E}_\iota(0)$, $j = 0, 1, \ldots, l$. Finally, the *a priori* assumption (3.1) can be verified in Section 3.4 by proving

$$\sum_{j=0}^{2}(1+t)^{2j}\left\|\partial_t^j\zeta(\cdot,t)\right\|_{L^\infty}^2+\sum_{j=0}^{1}(1+t)^{2j}\left\|\partial_t^j\zeta_r(\cdot,t)\right\|_{L^\infty}^2$$
$$+\sum_{i+j\leqq l-2,\ 2i+j\geqq 3}(1+t)^{2j}\left\|\bar{\rho}_0^{\frac{(\gamma-1)(2i+j-3)}{2}}\partial_t^j\partial_r^i\zeta(\cdot,t)\right\|_{L^\infty}^2$$
$$+\sum_{i+j=l-1}(1+t)^{2j}\left\|r\bar{\rho}_0^{\frac{(\gamma-1)(2i+j-3)}{2}}\partial_t^j\partial_r^i\zeta(\cdot,t)\right\|_{L^\infty}^2$$
$$+\sum_{i+j=l}(1+t)^{2j}\left\|r^2\bar{\rho}_0^{\frac{(\gamma-1)(2i+j-3)}{2}}\partial_t^j\partial_r^i\zeta(\cdot,t)\right\|_{L^\infty}^2\leqq C\mathcal{E}(t)$$

for some positive constant $C$ independent of $t$. This closes the whole bootstrap argument for small initial perturbations and completes the proof of Theorem 2.1.

### *3.1. Preliminaries*

In this subsection, we list some embedding estimates for weighted Sobolev spaces which will be used later and introduce some notations to simplify the presentation. For any bounded interval $I$, set $\mathrm{d}(r) = dist(r, \partial I)$. For any $a > 0$ and nonnegative integer $b$, the weighted Sobolev space $H^{a,b}(I)$ is given by

$$H^{a,b}(I) := \left\{d^{a/2}F \in L^2(I):\ \int_I d^a|\partial_r^k F|^2\mathrm{d}r < \infty,\ \ 0 \leqq k \leqq b\right\}$$

with the norm $\|F\|_{H^{a,b}(I)}^2 := \sum_{k=0}^{b}\int_I d^a|\partial_r^k F|^2\mathrm{d}r$. Then for $b \geqq a/2$, it holds the following *embedding of weighted Sobolev spaces* (cf. [37]): $H^{a,b}(I) \hookrightarrow H^{b-a/2}(I)$ with the estimate

$$\|F\|_{H^{b-a/2}(I)} \leqq C\|F\|_{H^{a,b}(I)} \tag{3.4}$$

for some positive constant $C$ depending on $a$, $b$ and $I$.

The following general version of the *Hardy inequality* whose proof can be found in [37] will also be used often in this paper. Let $k > 1$ be a given real number and $F$ be a function satisfying $\int_0^\delta r^k\left(F^2 + F_r^2\right)\mathrm{d}r < \infty$, where $\delta$ is a positive constant; then it holds that $\int_0^\delta r^{k-2}F^2\mathrm{d}r \leqq C(\delta, k)\int_0^\delta r^k\left(F^2 + F_r^2\right)\mathrm{d}r$, where $C(\delta, k)$ is a constant depending only on $\delta$ and $k$. As a consequence, one has

$$\int_{\sqrt{A/(4B)}}^{\sqrt{A/B}}\left(\sqrt{A/B}-r\right)^{k-2}F^2\mathrm{d}r \leqq C\int_{\sqrt{A/(4B)}}^{\sqrt{A/B}}\left(\sqrt{A/B}-r\right)^{k}\left(F^2+F_r^2\right)\mathrm{d}r, \tag{3.5}$$

where $C$ is a constant depending on $A$, $B$ and $k$.

**Notations:**

(1) Throughout the rest of paper, $C$ will denote a positive constant which only depend on the parameters of the problem, $\gamma$ and $M$, but does not depend on the data. They are referred as universal and can change from one inequality to another one. Also we use $C(\beta)$ to denote a certain positive constant depending on quantity $\beta$.

(2) We will employ the notation $a \lesssim b$ to denote $a \leqq Cb$, $a \sim b$ to denote $C^{-1}b \leqq a \leqq Cb$ and $a \gtrsim b$ to denote $a \geqq C^{-1}b$, where $C$ is the universal constant as defined above.

(3) In the rest of the paper, we will use the notations $\int =: \int_{\mathcal{I}}$, $\|\cdot\| =: \|\cdot\|_{L^2(\mathcal{I})}$ and $\|\cdot\|_{L^\infty} =: \|\cdot\|_{L^\infty(\mathcal{I})}$.

(4) We set $\sigma(r) := \bar{\rho}_0^{\gamma-1}(r) = A - Br^2$, $r \in \mathcal{I}$. Then $\mathcal{E}_j$ and $\mathcal{E}_{j,i}$ can be rewritten as

$$\mathcal{E}_j(t) = (1+t)^{2j} \int \left[ r^4\sigma^\alpha \left(\partial_t^j \zeta\right)^2 + r^2\sigma^{\alpha+1} \left|\partial_t^j (\zeta, r\zeta_r)\right|^2 \right. \\ \left. +(1+t)r^4\sigma^\alpha \left(\partial_t^j \zeta_t\right)^2 \right] (r,t)\mathrm{d}r,$$

$$\mathcal{E}_{j,i}(t) = (1+t)^{2j} \int \left[ r^2\sigma^{\alpha+i-1} \left(\partial_t^j \partial_r^i \zeta\right)^2 + r^4\sigma^{\alpha+i+1} \left(\partial_t^j \partial_r^{i+1} \zeta\right)^2 \right] (r,t)\mathrm{d}r.$$

(5) We set $\mathcal{I}_o := \left(0, \sqrt{A/(4B)}\right)$ and $\mathcal{I}_b := \left(\sqrt{A/(4B)}, \sqrt{A/B}\right)$. Then $\mathcal{I} = \mathcal{I}_o \cup \mathcal{I}_b$. Moreover, it gives from the Hardy inequality (3.5) that for $k > 1$,

$$\int_{\mathcal{I}_b} \sigma^{k-2}(r)F^2\mathrm{d}r \leqq C(A,B,k) \int_{\mathcal{I}_b} \sigma^k(r) \left(F^2 + F_r^2\right)\mathrm{d}r, \tag{3.6}$$

provided that the right-hand side of (3.6) is finite.

### *3.2. Elliptic Estimates*

In this subsection, we prove the following elliptic estimates:

**Proposition 3.1.** *Suppose that* (3.1) *holds for suitably small positive number* $\varepsilon_0 \in (0,1)$. *Then it holds that for* $t \in [0,T]$,

$$\mathcal{E}_{j,i}(t) \lesssim \sum_{\iota=0}^{i+j} \mathcal{E}_\iota(t) \quad \text{when} \quad j \geqq 0, \quad i \geqq 1, \quad i+j \leqq l. \tag{3.7}$$

The proof of this proposition consists of Lemmas 3.2 and 3.3.

**3.2.1. Lower-Order Elliptic Estimates** Dividing equation (2.16) by $\bar{\rho}_0$, one has

$$r\zeta_{tt} + r\zeta_t + \sigma(\tilde{\eta}_r + \zeta)^2 \left[ (\tilde{\eta}_r + \zeta)^{-2\gamma} (\tilde{\eta}_r + \zeta + r\zeta_r)^{-\gamma} \right]_r \\ + \frac{\gamma}{\gamma-1}\sigma_r \left[ (\tilde{\eta}_r + \zeta)^{2-2\gamma} (\tilde{\eta}_r + \zeta + r\zeta_r)^{-\gamma} - \tilde{\eta}_r^{2-3\gamma} \right] = 0.$$

Note that

$$
\begin{aligned}
&(\tilde{\eta}_r+\zeta)^2\left[(\tilde{\eta}_r+\zeta)^{-2\gamma}(\tilde{\eta}_r+\zeta+r\zeta_r)^{-\gamma}\right]_r\\
&\quad=-\gamma\tilde{\eta}_r^{1-3\gamma}(4\zeta_r+r\zeta_{rr})+\mathfrak{J}_1,\\
&(\tilde{\eta}_r+\zeta)^{2-2\gamma}(\tilde{\eta}_r+\zeta+r\zeta_r)^{-\gamma}-\tilde{\eta}_r^{2-3\gamma}\\
&\quad=-\gamma\tilde{\eta}_r^{1-3\gamma}(r\zeta_r)+(2-3\gamma)\tilde{\eta}_r^{1-3\gamma}\zeta+\mathfrak{J}_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{J}_1:=&-2\gamma\left[(\tilde{\eta}_r+\zeta)^{1-2\gamma}(\tilde{\eta}_r+\zeta+r\zeta_r)^{-\gamma}-\tilde{\eta}_r^{1-3\gamma}\right]\zeta_r\\
&-\gamma\left[(\tilde{\eta}_r+\zeta)^{2-2\gamma}(\tilde{\eta}_r+\zeta+r\zeta_r)^{-\gamma-1}-\tilde{\eta}_r^{1-3\gamma}\right](2\zeta_r+r\zeta_{rr}),\\
\mathfrak{J}_2:=&(\tilde{\eta}_r+\zeta)^{2-2\gamma}(\tilde{\eta}_r+\zeta+r\zeta_r)^{-\gamma}-\tilde{\eta}_r^{2-3\gamma}\\
&+\gamma\tilde{\eta}_r^{1-3\gamma}(r\zeta_r)-(2-3\gamma)\tilde{\eta}_r^{1-3\gamma}\zeta.
\end{aligned}\tag{3.8}
$$

Then,

$$
\begin{aligned}
\gamma\tilde{\eta}_r^{1-3\gamma}\left[r\sigma\zeta_{rr}+4\sigma\zeta_r+\frac{\gamma}{\gamma-1}r\sigma_r\zeta_r\right]=&\,r\zeta_{tt}+r\zeta_t+\frac{\gamma(2-3\gamma)}{\gamma-1}\sigma_r\tilde{\eta}_r^{1-3\gamma}\zeta\\
&+\sigma\mathfrak{J}_1+\frac{\gamma}{\gamma-1}\sigma_r\mathfrak{J}_2.
\end{aligned}\tag{3.9}
$$

**Lemma 3.2.** *Assume that (3.1) holds for suitably small positive number* $\varepsilon_0\in(0,1)$. *Then,*

$$
\mathcal{E}_{0,1}(t)\lesssim\mathcal{E}_0(t)+\mathcal{E}_1(t),\quad 0\leqq t\leqq T.
$$

**Proof.** Multiply equation (3.9) by $\tilde{\eta}_r^{3\gamma-1}r\sigma^{\alpha/2}$ and square the spatial $L^2$-norm of the product to obtain

$$
\begin{aligned}
&\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}+4r\sigma^{1+\frac{\alpha}{2}}\zeta_r+(1+\alpha)r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2\\
&\quad\lesssim\mathcal{E}_1+(1+t)^2\left(\left\|r\sigma^{1+\frac{\alpha}{2}}\mathfrak{J}_1\right\|^2+\left\|r\sigma^{\frac{\alpha}{2}}\sigma_r\mathfrak{J}_2\right\|^2\right)+\left\|r\sigma^{\frac{\alpha}{2}}\zeta\right\|^2
\end{aligned}\tag{3.10}
$$

where we have used (2.13) and the definition of $\mathcal{E}_1$. It follows from the Taylor expansion, (3.2) and (3.1) that

$$
\begin{aligned}
|\mathfrak{J}_1|&\lesssim(1+t)^{-\frac{3\gamma}{3\gamma-1}}(|r\zeta_r|+|\zeta|)(|r\zeta_{rr}|+|\zeta_r|)\lesssim(1+t)^{-\frac{3\gamma}{3\gamma-1}}\varepsilon_0(|r\zeta_{rr}|+|\zeta_r|),\\
|\mathfrak{J}_2|&\lesssim(1+t)^{-\frac{3\gamma}{3\gamma-1}}\left(|r\zeta_r|^2+|\zeta|^2\right)\lesssim(1+t)^{-\frac{3\gamma}{3\gamma-1}}\varepsilon_0(|r\zeta_r|+|\zeta|).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(1+t)^2\left(\left\|r\sigma^{1+\frac{\alpha}{2}}\mathfrak{J}_1\right\|^2+\left\|r\sigma^{\frac{\alpha}{2}}\sigma_r\mathfrak{J}_2\right\|^2\right)\\
&\quad\lesssim\varepsilon_0^2\left(\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2+\left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2+\left\|r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2+\left\|r\sigma^{\frac{\alpha}{2}}\sigma_r\zeta\right\|^2\right).
\end{aligned}\tag{3.11}
$$

Note that

$$\left\|r\sigma^{\frac{\alpha}{2}}\zeta\right\|^2 = \int_{\mathcal{I}_o} r^2\sigma^{\alpha}\zeta^2 \mathrm{d}r + \int_{\mathcal{I}_b} r^2\sigma^{\alpha}\zeta^2 \mathrm{d}r \lesssim \int_{\mathcal{I}_o} r^2\sigma^{1+\alpha}\zeta^2 \mathrm{d}r + \int_{\mathcal{I}_b} r^4\sigma^{\alpha}\zeta^2 \mathrm{d}r \lesssim \mathcal{E}_0. \tag{3.12}$$

Then, it is yielded from (3.10), (3.11) and (3.12) that

$$\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr} + 4r\sigma^{1+\frac{\alpha}{2}}\zeta_r + (1+\alpha)\, r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2 \lesssim \mathcal{E}_0 + \mathcal{E}_1 + \varepsilon_0^2\left(\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2 + \left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2 + \left\|r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2\right). \tag{3.13}$$

In what follows, we analyze the left-hand side of (3.13), which can be expanded as

$$\begin{aligned}&\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr} + 4r\sigma^{1+\frac{\alpha}{2}}\zeta_r + (1+\alpha)\, r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2 \\ &= \left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2 + 16\left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2 + (1+\alpha)^2\left\|r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2 \\ &\quad + \int\left[4r^3\sigma^{2+\alpha}+(1+\alpha)r^4\sigma^{1+\alpha}\sigma_r\right]\left(\zeta_r^2\right)_r \mathrm{d}r + 8(1+\alpha)\int r^3\sigma^{1+\alpha}\sigma_r\zeta_r^2 \mathrm{d}r.\end{aligned} \tag{3.14}$$

With the help of the integration by parts and the fact $\sigma_r = -2Br$, one has

$$\begin{aligned}&\int\left[4r^3\sigma^{2+\alpha} + (1+\alpha)r^4\sigma^{1+\alpha}\sigma_r\right]\left(\zeta_r^2\right)_r \mathrm{d}r \\ &\geqq -12\int r^2\sigma^{2+\alpha}\zeta_r^2 \mathrm{d}r - (1+\alpha)^2\int r^4\sigma^{\alpha}\sigma_r^2\zeta_r^2 \mathrm{d}r - C\int r^4\sigma^{1+\alpha}\zeta_r^2 \mathrm{d}r.\end{aligned}$$

Substitute this into (3.14) and use $\sigma_r = -2Br$ to give

$$\begin{aligned}&\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr} + 4r\sigma^{1+\frac{\alpha}{2}}\zeta_r + (1+\alpha)\, r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2 \\ &\geqq \left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2 + 4\left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2 - C\int r^4\sigma^{1+\alpha}\zeta_r^2 \mathrm{d}r.\end{aligned}$$

In view of (3.13), we then see that

$$\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2 + 4\left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2 \lesssim \mathcal{E}_0 + \mathcal{E}_1 + \varepsilon_0^2\left(\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2 + \left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2 + \left\|r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2\right). \tag{3.15}$$

On the other hand, it follows from (3.13) and (3.15) that

$$\left\|(1+\alpha)\, r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2 \lesssim \mathcal{E}_0 + \mathcal{E}_1 + \varepsilon_0^2\left(\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2 + \left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2 + \left\|r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2\right).$$

This, together with (3.15), gives

$$\begin{aligned}&\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2+\left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2+\left\|r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2\\&\lesssim \mathcal{E}_0+\mathcal{E}_1+\varepsilon_0^2\left(\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2+\left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2+\left\|r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2\right),\end{aligned}\tag{3.16}$$

which implies, with the aid of the smallness of $\varepsilon_0$, that

$$\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2+\left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2+\left\|r^2\sigma^{\frac{\alpha}{2}}\sigma_r\zeta_r\right\|^2\lesssim \mathcal{E}_0+\mathcal{E}_1.$$

In view of $\sigma_r=-2Br$, we then see that

$$\left\|r^2\sigma^{1+\frac{\alpha}{2}}\zeta_{rr}\right\|^2+\left\|r\sigma^{1+\frac{\alpha}{2}}\zeta_r\right\|^2+\left\|r^3\sigma^{\frac{\alpha}{2}}\zeta_r\right\|^2\lesssim \mathcal{E}_0+\mathcal{E}_1,$$

which implies

$$\left\|r\sigma^{\frac{\alpha}{2}}\zeta_r\right\|^2\lesssim\int_{\mathcal{I}_o}r^2\sigma^{2+\alpha}\zeta_r^2\mathrm{d}r+\int_{\mathcal{I}_b}r^6\sigma^{\alpha}\zeta_r^2\mathrm{d}r\lesssim \mathcal{E}_0+\mathcal{E}_1.\tag{3.17}$$

This finishes the proof of Lemma 3.2. □

**3.2.2. Higher-Order Elliptic Estimates** For $i\geqq 1$ and $j\geqq 0$, it yields from $\partial_t^j\partial_r^{i-1}$(3.9) and $\sigma_r=-2Br$ that

$$\begin{aligned}&\gamma\tilde{\eta}_r^{1-3\gamma}\left[r\sigma\partial_t^j\partial_r^{i+1}\zeta+(i+3)\sigma\partial_t^j\partial_r^i\zeta+(\alpha+i)\,r\sigma_r\partial_t^j\partial_r^i\zeta\right]\\&\quad=r\partial_t^{j+2}\partial_r^{i-1}\zeta+r\partial_t^{j+1}\partial_r^{i-1}\zeta+\mathfrak{P}_1+\mathfrak{P}_2,\end{aligned}\tag{3.18}$$

where

$$\begin{aligned}\mathfrak{P}_1:=&-\gamma\sum_{\iota=1}^{j}\left[\partial_t^\iota\left(\tilde{\eta}_r^{1-3\gamma}\right)\right]\partial_t^{j-\iota}\left[r\sigma\partial_r^{i+1}\zeta+(i+3)\sigma\partial_r^i\zeta+(\alpha+i)\,r\sigma_r\partial_r^i\zeta\right]\\&-\gamma\partial_t^j\left\{\tilde{\eta}_r^{1-3\gamma}\left[\sum_{\iota=2}^{i-1}C_{i-1}^\iota\left[\partial_r^\iota(r\sigma)\right]\partial_r^{i+1-\iota}\zeta+4\sum_{\iota=1}^{i-1}C_{i-1}^\iota\left(\partial_r^\iota\sigma\right)\partial_r^{i-\iota}\zeta\right.\right.\\&\left.\left.+(\alpha+1)\sum_{\iota=1}^{i-1}C_{i-1}^\iota\left[\partial_r^\iota\left(r\sigma_r\right)\right]\partial_r^{i-\iota}\zeta\right]\right\}+(i-1)\partial_r^{i-2}\left(\partial_t^{j+2}\zeta+\partial_t^{j+1}\zeta\right)\\&-\frac{2\gamma(2-3\gamma)B}{\gamma-1}\partial_t^j\left[\tilde{\eta}_r^{1-3\gamma}\left(r\partial_r^{i-1}\zeta+(i-1)\partial_r^{i-2}\zeta\right)\right],\end{aligned}\tag{3.19}$$

$$\mathfrak{P}_2:=\partial_r^{i-1}\left(\sigma\partial_t^j\mathfrak{J}_1\right)+(1+\alpha)\partial_r^{i-1}\left(\sigma_r\partial_t^j\mathfrak{J}_2\right).\tag{3.20}$$

(Recall that $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are defined in (3.8).) Here and thereafter $C_m^j$ is used to denote the binomial coefficients for $0\leqq j\leqq m$, $C_m^j=\frac{m!}{j!(m-j)!}$ and summations $\sum_{\iota=1}^{i-1}$ and $\sum_{\iota=2}^{i-1}$ should be understood as zero when $i=1$ and $i=1,2$, respectively.

Multiply equation (3.18) by $\tilde{\eta}_r^{3\gamma-1} r\sigma^{(\alpha+i-1)/2}$, square the spatial $L^2$-norm of the product and use (2.13) to give

$$\begin{aligned}
&\left\| r^2\sigma^{\frac{\alpha+i+1}{2}}\partial_t^j\partial_r^{i+1}\zeta + (i+3)r\sigma^{\frac{\alpha+i+1}{2}}\partial_t^j\partial_r^i\zeta + (\alpha+i)\, r^2\sigma^{\frac{\alpha+i-1}{2}}\sigma_r\partial_t^j\partial_r^i\zeta \right\|^2 \\
&\lesssim (1+t)^2\left( \left\| r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+2}\partial_r^{i-1}\zeta \right\|^2 + \left\| r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1}\partial_r^{i-1}\zeta \right\|^2 \right) \\
&\quad + (1+t)^2\left( \left\| r\sigma^{\frac{\alpha+i-1}{2}}\mathfrak{P}_1 \right\|^2 + \left\| r\sigma^{\frac{\alpha+i-1}{2}}\mathfrak{P}_2 \right\|^2 \right).
\end{aligned}$$

Similar to the derivation of (3.16) and (3.17), we can then obtain

$$\begin{aligned}
(1+t)^{-2j}\mathcal{E}_{j,i}(t) &= \left\| r^2\sigma^{\frac{\alpha+i+1}{2}}\partial_t^j\partial_r^{i+1}\zeta \right\|^2 + \left\| r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^j\partial_r^i\zeta \right\|^2 \\
&\lesssim \left\| r^2\sigma^{\frac{\alpha+i}{2}}\partial_t^j\partial_r^i\zeta \right\|^2 + (1+t)^2\left( \left\| r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+2}\partial_r^{i-1}\zeta \right\|^2 \right. \\
&\quad \left. + \left\| r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1}\partial_r^{i-1}\zeta \right\|^2 \right) \\
&\quad + (1+t)^2\left( \left\| r\sigma^{\frac{\alpha+i-1}{2}}\mathfrak{P}_1 \right\|^2 + \left\| r\sigma^{\frac{\alpha+i-1}{2}}\mathfrak{P}_2 \right\|^2 \right). \qquad (3.21)
\end{aligned}$$

We will use this estimate to prove the following lemma by the mathematical induction:

**Lemma 3.3.** *Assume that (3.1) holds for suitably small positive number $\varepsilon_0 \in (0, 1)$. Then for $j \geqq 0$, $i \geqq 1$ and $2 \leqq i + j \leqq l$,*

$$\mathcal{E}_{j,i}(t) \lesssim \sum_{\iota=0}^{i+j} \mathcal{E}_\iota(t), \quad t \in [0, T]. \qquad (3.22)$$

**Proof.** We use the induction for $i + j$ to prove this lemma. As shown in Lemma 3.2 we know that (3.22) holds for $i + j = 1$. For $1 \leqq k \leqq l - 1$, we make the induction hypothesis that (3.22) holds for all $j \geqq 0$, $i \geqq 1$ and $i + j \leqq k$, that is,

$$\mathcal{E}_{j,i}(t) \lesssim \sum_{\iota=0}^{i+j} \mathcal{E}_\iota(t), \quad j \geqq 0, \quad i \geqq 1, \quad i + j \leqq k. \qquad (3.23)$$

It then suffices to prove (3.22) for $j \geqq 0$, $i \geqq 1$ and $i + j = k + 1$. (Indeed, there exists an order of $(i, j)$ for the proof. For example, when $i + j = k + 1$ we will bound $\mathcal{E}_{k+1-\iota,\iota}$ from $\iota = 1$ to $k + 1$ step by step.)

Before going to the estimate, we notice a fact that $\mathcal{E}_{j,0} \lesssim \mathcal{E}_j$ for $j = 0, \ldots, l$. Indeed, it follows from (3.6) that

$$\int_{\mathcal{I}_b} \sigma^{\alpha-1}\left(\partial_t^j \zeta\right)^2 \mathrm{d}r \lesssim \int_{\mathcal{I}_b} \sigma^{\alpha+1}\left[r^2\left(\partial_t^j \zeta\right)^2 + r^4\left(\partial_t^j \zeta_r\right)^2\right]\mathrm{d}r \leqq (1+t)^{-2j}\mathcal{E}_j(t),$$

which implies for $j = 0, 1, \cdots, l$,

$$\begin{aligned}\mathcal{E}_{j,0}(t) &= (1+t)^{2j}\int\left[r^2\sigma^{\alpha-1}\left(\partial_t^j \zeta\right)^2 + r^4\sigma^{\alpha+1}\left(\partial_t^j \zeta_r\right)^2\right](r,t)\mathrm{d}r\\ &\lesssim (1+t)^{2j}\left[\int_{\mathcal{I}_o} r^2\sigma^{\alpha+1}\left(\partial_t^j \zeta\right)^2(r,t)\mathrm{d}r + \int_{\mathcal{I}_b}\sigma^{\alpha-1}\left(\partial_t^j \zeta\right)^2(r,t)\mathrm{d}r\right]\\ &\quad + \mathcal{E}_j(t) \lesssim \mathcal{E}_j(t).\end{aligned} \tag{3.24}$$

This, together with the induction hypothesis (3.23), gives

$$\mathcal{E}_{j,i}(t) \lesssim \sum_{\iota=0}^{i+j}\mathcal{E}_\iota(t), \quad j \geqq 0, \quad i \geqq 0, \quad i + j \leqq k. \tag{3.25}$$

In what follows, we assume $j \geqq 0$, $i \geqq 1$ and $i + j = k + 1 \leqq l$. First, We estimate $\mathfrak{P}_1$ and $\mathfrak{P}_2$ given by (3.19) and (3.20), respectively. For $\mathfrak{P}_1$, it follows from (2.13) and $\sigma_r = -2Br$ that

$$\begin{aligned}|\mathfrak{P}_1| \lesssim& \sum_{\iota=1}^{j}(1+t)^{-1-\iota}\left(\left|rw\partial_t^{j-\iota}\partial_r^{i+1}\zeta\right| + \left|\partial_t^{j-\iota}\partial_r^{i}\zeta\right|\right)\\ &+ \sum_{\iota=0}^{j}\sum_{m=1}^{i-1}(1+t)^{-1-\iota}\left|\partial_t^{j-\iota}\partial_r^{m}\zeta\right| + (i-1)\\ &\times\left(\left|\partial_t^{j+2}\partial_r^{i-2}\zeta\right| + \left|\partial_t^{j+1}\partial_r^{i-2}\zeta\right| + \sum_{\iota=0}^{j}(1+t)^{-1-\iota}\left|\partial_t^{j-\iota}\partial_r^{i-2}\zeta\right|\right)\\ &+ \sum_{\iota=0}^{j}(1+t)^{-1-\iota}\left|r\partial_t^{j-\iota}\partial_r^{i-1}\zeta\right|,\end{aligned}$$

which implies

$$\begin{aligned}\left\|r\sigma^{\frac{\alpha+i-1}{2}}\mathfrak{P}_1\right\|^2 \lesssim& \sum_{\iota=1}^{j}(1+t)^{-2-2\iota}\left(\left\|r^2\sigma^{\frac{\alpha+i+1}{2}}\partial_t^{j-\iota}\partial_r^{i+1}\zeta\right\|^2 + \left\|r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j-\iota}\partial_r^{i}\zeta\right\|^2\right)\\ &+ \sum_{\iota=0}^{j}(1+t)^{-2-2\iota}\left(\sum_{m=1}^{i-1}\left\|r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j-\iota}\partial_r^{m}\zeta\right\|^2 + \left\|r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j-\iota}\partial_r^{i-1}\zeta\right\|^2\right)\\ &+ (i-1)^2\left(\sum_{\iota=j+1}^{j+2}\left\|r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{\iota}\partial_r^{i-2}\zeta\right\|^2 + \sum_{\iota=0}^{j}(1+t)^{-2-2\iota}\left\|r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j-\iota}\partial_r^{i-2}\zeta\right\|^2\right).\end{aligned}$$

So,

$$
\begin{aligned}
&\left\| r\sigma^{\frac{\alpha+i-1}{2}} \mathfrak{P}_1 \right\|^2 \\
&\lesssim \begin{cases} (1+t)^{-2-2j} \left( \sum_{\iota=0}^{j-1} \mathcal{E}_{\iota,1} + \sum_{\iota=0}^{j} \mathcal{E}_{\iota} \right)(t), & i=1, \\ (1+t)^{-2-2j} \left( \sum_{\iota=0}^{j-1} \mathcal{E}_{\iota,i} + \sum_{\iota=0}^{j} \sum_{m=1}^{i-1} \mathcal{E}_{\iota,m} + \sum_{\iota=0}^{j+2} \mathcal{E}_{\iota,i-2} \right)(t), & i \geqq 2. \end{cases}
\end{aligned}
\tag{3.26}
$$

For $\mathfrak{P}_2$, it follows from (2.13), (3.1), (3.2) and $\sigma_r = -2Br$ that

$$
\begin{aligned}
|\mathfrak{P}_2| &\lesssim \sum_{n=0}^{j} \sum_{m=0}^{i-1} K_{nm} \left( \left| \partial_t^{j-n} \partial_r^{i-1-m} (\sigma r \zeta_{rr}) \right| + \left| \partial_t^{j-n} \partial_r^{i-1-m} (\sigma \zeta_r) \right| \right. \\
&\qquad \left. + \left| \partial_t^{j-n} \partial_r^{i-1-m} (\sigma_r r \zeta_r) \right| + \left| \partial_t^{j-n} \partial_r^{i-1-m} (\sigma_r \zeta) \right| \right) \\
&\lesssim \sum_{n=0}^{j} \sum_{m=0}^{i-1} K_{nm} \left( \left| \sigma r \partial_t^{j-n} \partial_r^{i-m+1} \zeta \right| + \sum_{\iota=0}^{i-m} \left| \partial_t^{j-n} \partial_r^{\iota} \zeta \right| \right) =: \sum_{n=0}^{j} \sum_{m=0}^{i-1} \mathfrak{P}_{2nm},
\end{aligned}
$$

where

$$
\begin{aligned}
&K_{00} = \varepsilon_0 (1+t)^{-1-\frac{1}{3\gamma-1}}; \; K_{10} = \varepsilon_0 (1+t)^{-2-\frac{1}{3\gamma-1}}, \\
&K_{01} = (1+t)^{-1-\frac{1}{3\gamma-1}} \left( \varepsilon_0 + |r \partial_r^2 \zeta| \right); \\
&K_{20} = \varepsilon_0 (1+t)^{-3-\frac{1}{3\gamma-1}} + (1+t)^{-1-\frac{1}{3\gamma-1}} \left| r \partial_t^2 \partial_r \zeta \right|, \\
&K_{11} = (1+t)^{-2-\frac{1}{3\gamma-1}} \left( \varepsilon_0 + \left| r \partial_r^2 \zeta \right| \right) + (1+t)^{-1-\frac{1}{3\gamma-1}} \left| r \partial_t \partial_r^2 \zeta \right|, \\
&K_{02} = (1+t)^{-1-\frac{1}{3\gamma-1}} \left( \left| \partial_r^2 \zeta \right| + \left| r \partial_r^3 \zeta \right| \right) + (1+t)^{-1-\frac{2}{3\gamma-1}} \left( \varepsilon_0^2 + \left| r \partial_r^2 \zeta \right|^2 \right).
\end{aligned}
$$

We do not list here $K_{nm}$ for $n+m \geqq 3$ since we can use the same method to estimate $\mathfrak{P}_{2nm}$ for $n+m \geqq 3$ as that for $n+m \leqq 2$. Easily, $\mathfrak{P}_{200}$ and $\mathfrak{P}_{210}$ can be bounded by

$$
\begin{aligned}
\left\| r\sigma^{\frac{\alpha+i-1}{2}} \mathfrak{P}_{200} \right\|^2 &\lesssim \varepsilon_0^2 (1+t)^{-2} \left( \left\| r^2 \sigma^{\frac{\alpha+i+1}{2}} \partial_t^j \partial_r^{i+1} \zeta \right\|^2 + \sum_{\iota=0}^{i} \left\| r\sigma^{\frac{\alpha+i-1}{2}} \partial_t^j \partial_r^{\iota} \zeta \right\|^2 \right) \\
&\lesssim \varepsilon_0^2 (1+t)^{-2-2j} \left( \mathcal{E}_{j,i} + \sum_{\iota=0}^{i-1} \mathcal{E}_{j,\iota} \right)(t), \\
\left\| r\sigma^{\frac{\alpha+i-1}{2}} \mathfrak{P}_{210} \right\|^2 &\lesssim \varepsilon_0^2 (1+t)^{-4} \left( \left\| r^2 \sigma^{\frac{\alpha+i+1}{2}} \partial_t^{j-1} \partial_r^{i+1} \zeta \right\|^2 + \sum_{\iota=0}^{i} \left\| r\sigma^{\frac{\alpha+i-1}{2}} \partial_t^{j-1} \partial_r^{\iota} \zeta \right\|^2 \right) \\
&\lesssim \varepsilon_0^2 (1+t)^{-2-2j} \sum_{\iota=0}^{i} \mathcal{E}_{j-1,\iota}(t).
\end{aligned}
$$

For $\mathfrak{P}_{201}$, we use (3.1) to get $|\sigma^{1/2}\partial_r^2\zeta| \lesssim \varepsilon_0$ and then obtain

$$\left\| r\sigma^{\frac{\alpha+i-1}{2}} \mathfrak{P}_{201} \right\|^2 \lesssim \varepsilon_0^2 (1+t)^{-2} \left( \left\| r^2 \sigma^{\frac{\alpha+i}{2}} \partial_t^j \partial_r^i \zeta \right\|^2 + \sum_{\iota=0}^{i-1} \left\| r\sigma^{\frac{\alpha+i-2}{2}} \partial_t^j \partial_r^\iota \zeta \right\|^2 \right)$$
$$\lesssim \varepsilon_0^2 (1+t)^{-2-2j} \sum_{\iota=0}^{i-1} \mathcal{E}_{j,\iota}(t),$$

For $\mathfrak{P}_{220}$, we use (3.1) again to get $|r\sigma^{1/2}\partial_t^2\partial_r\zeta| \lesssim \varepsilon_0(1+t)^{-2}$ and then achieve

$$\left\| r\sigma^{\frac{\alpha+i-1}{2}} \mathfrak{P}_{220} \right\|^2 \lesssim \varepsilon_0^2 (1+t)^{-6}$$
$$\left( \left\| r^2 \sigma^{\frac{\alpha+i}{2}} \partial_t^{j-2} \partial_r^{i+1} \zeta \right\|^2 + \sum_{\iota=0}^{i} \left\| r\sigma^{\frac{\alpha+i-2}{2}} \partial_t^{j-2} \partial_r^\iota \zeta \right\|^2 \right)$$
$$\lesssim \varepsilon_0^2 (1+t)^{-2-2j} \sum_{\iota=0}^{i+1} \mathcal{E}_{j-2,\iota}(t),$$

because it can be derived from (3.6) that

$$\left\| r\sigma^{\frac{\alpha+i-2}{2}} \partial_t^{j-2} \partial_r^i \zeta \right\|^2 = \int_{\mathcal{I}_o} r^2 \sigma^{\alpha+i-2} \left| \partial_t^{j-2} \partial_r^i \zeta \right|^2 \mathrm{d}r + \int_{\mathcal{I}_b} r^2 \sigma^{\alpha+i-2} \left| \partial_t^{j-2} \partial_r^i \zeta \right|^2 \mathrm{d}r$$
$$\lesssim \int_{\mathcal{I}_o} r^2 \sigma^{\alpha+i-1} \left| \partial_t^{j-2} \partial_r^i \zeta \right|^2 \mathrm{d}r + \int_{\mathcal{I}_b} \sigma^{\alpha+i} \left( r^2 \left| \partial_t^{j-2} \partial_r^i \zeta \right|^2 + r^2 \left| \partial_t^{j-2} \partial_r^{i+1} \zeta \right|^2 \right) \mathrm{d}r$$
$$\lesssim \int r^2 \sigma^{\alpha+i-1} \left| \partial_t^{j-2} \partial_r^i \zeta \right|^2 dr + \int r^2 \sigma^{\alpha+i} \left| \partial_t^{j-2} \partial_r^{i+1} \zeta \right|^2 \mathrm{d}r.$$

Similar to the estimate for $\mathfrak{P}_{220}$, we can obtain

$$\left\| r\sigma^{\frac{\alpha+i-1}{2}} \mathfrak{P}_{211} \right\|^2 \lesssim \varepsilon_0^2 (1+t)^{-4}$$
$$\left( \left\| r^2 \sigma^{\frac{\alpha+i-1}{2}} \partial_t^{j-1} \partial_r^i \zeta \right\|^2 + \sum_{\iota=0}^{i-1} \left\| r\sigma^{\frac{\alpha+i-3}{2}} \partial_t^{j-1} \partial_r^\iota \zeta \right\|^2 \right)$$
$$\lesssim \varepsilon_0^2 (1+t)^{-2-2j} \sum_{\iota=0}^{i} \mathcal{E}_{j-1,\iota}(t),$$

$$\left\| r\sigma^{\frac{\alpha+i-1}{2}} \mathfrak{P}_{202} \right\|^2 \lesssim \varepsilon_0^2 (1+t)^{-2} \left( \left\| r^2 \sigma^{\frac{\alpha+i-2}{2}} \partial_t^j \partial_r^{i-1} \zeta \right\|^2 + \sum_{\iota=0}^{i-2} \left\| r\sigma^{\frac{\alpha+i-4}{2}} \partial_t^j \partial_r^\iota \zeta \right\|^2 \right)$$
$$\lesssim \varepsilon_0^2 (1+t)^{-2-2j} \sum_{\iota=0}^{i-1} \mathcal{E}_{j,\iota}(t).$$

It should be noted that $\mathfrak{P}_{211}$ and $\mathfrak{P}_{202}$ appear when $i \geqq 2$ and $i \geqq 3$, respectively. This ensures the application of the Hardy inequality (3.6). Other cases can be done similarly, since the leading term of $K_{nm}$ is

$$\sum_{q=0}^{n}(1+t)^{-1-\frac{1}{3\gamma-1}-q}\left(\left|r\partial_t^{n-q}\partial_r^{m+1}\zeta\right|+\left|\partial_t^{n-q}\partial_r^{m}\zeta\right|\right)$$

and

$$\begin{aligned}&\sum_{n=0}^{j}\sum_{m=0}^{i-1}\sum_{q=0}^{n}(1+t)^{-2-2q}\left\|r\sigma^{\frac{\alpha+i-1}{2}}\left(\left|r\partial_t^{n-q}\partial_r^{m+1}\zeta\right|\right.\right.\\&\left.\left.+\left|\partial_t^{n-q}\partial_r^{m}\zeta\right|\right)\left(\left|\sigma r\partial_t^{j-n}\partial_r^{i-m+1}\zeta\right|\sum_{\iota=0}^{i-m}\left|\partial_t^{j-n}\partial_r^{\iota}\zeta\right|\right)\right\|^2\\&\lesssim\varepsilon_0^2(1+t)^{-2-2j}\left(\mathcal{E}_{j,i}+\sum_{0\leqq\iota\leqq j,\ p\geqq 0,\ \iota+p\leqq i+j-1}\mathcal{E}_{\iota,p}\right)(t).\end{aligned}\tag{3.27}$$

(Estimate (3.27) will be verified in the "Appendix".) Now, we may conclude that

$$\left\|r\sigma^{\frac{\alpha+i-1}{2}}\mathfrak{P}_2\right\|^2\lesssim\varepsilon_0^2(1+t)^{-2-2j}\left(\mathcal{E}_{j,i}+\sum\nolimits_{0\leqq\iota\leqq j,\ p\geqq 0,\ \iota+p\leqq i+j-1}\mathcal{E}_{\iota,p}\right)(t).\tag{3.28}$$

Substitute (3.26) and (3.28) into (3.21) gives, for suitably small $\varepsilon_0$, that

$$\mathcal{E}_{j,i}(t)\lesssim\begin{cases}\mathcal{E}_j(t)+\mathcal{E}_{j+1}(t)\\[1ex]+\displaystyle\sum_{\iota\geqq 0,\ p\geqq 0,\ \iota+p\leqq j}\mathcal{E}_{\iota,p}(t)+\sum_{\iota=0}^{j}\mathcal{E}_\iota(t), & i=1,\\[1ex]\mathcal{E}_{j,i-1}(t)+\mathcal{E}_{j+2,i-2}(t)+\mathcal{E}_{j+1,i-2}(t)\\[1ex]+\displaystyle\sum_{0\leqq\iota\leqq j,\ p\geqq 0,\ \iota+p\leqq i+j-1}\mathcal{E}_{\iota,p}(t), & i\geqq 2.\end{cases}\tag{3.29}$$

Now, we use estimate (3.25), derived from the induction hypothesis (3.23), and (3.29) to show that (3.22) holds for $i+j=k+1$. First, choosing $j=k$ and $i=1$ in (3.29) gives

$$\mathcal{E}_{k,1}(t)\lesssim\sum_{\iota=0}^{k+1}\mathcal{E}_\iota(t)+\sum_{\iota\geqq 0,\ p\geqq 0,\ \iota+p\leqq k}\mathcal{E}_{\iota,p}(t)\lesssim\sum_{\iota=0}^{k+1}\mathcal{E}_\iota(t).\tag{3.30}$$

We choose $j=k-1$ and $i=2$ in (3.29) and use (3.24)–(3.25) to show

$$\begin{aligned}\mathcal{E}_{k-1,2}(t)&\lesssim\mathcal{E}_{k-1,1}(t)+\mathcal{E}_{k+1,0}(t)+\mathcal{E}_{k,0}(t)\\&\quad+\sum_{0\leqq\iota\leqq k-1,\ p\geqq 0,\ \iota+p\leqq k}\mathcal{E}_{\iota,p}(t)\lesssim\sum_{\iota=0}^{k+1}\mathcal{E}_\iota(t).\end{aligned}$$

For $\mathcal{E}_{k-2,3}$, it follows from (3.29), (3.25) and (3.30) to obtain

$$\mathcal{E}_{k-2,3}(t) \lesssim \mathcal{E}_{k-2,2}(t) + \mathcal{E}_{k,1}(t) + \mathcal{E}_{k-1,1}(t) + \sum_{0 \leqq \iota \leqq k-2,\ p \geqq 0,\ \iota + p \leqq k} \mathcal{E}_{\iota,p}(t) \lesssim \sum_{\iota=0}^{k+1} \mathcal{E}_{\iota}(t).$$

The other cases can be handled similarly. So we have proved (3.22) when $i + j = k + 1$. This finishes the proof of Lemma 3.3. □

*3.3. Nonlinear Weighted Energy Estimates*

In this section, we show that the weighted energy $\mathcal{E}_j(t)$ can be bounded by the initial date for all $t \in [0, T]$.

**Proposition 3.4.** *Suppose that* (3.1) *holds for suitably small positive number* $\varepsilon_0 \in (0, 1)$. *Then it holds that for* $t \in [0, T]$,

$$\mathcal{E}_j(t) \lesssim \sum_{\iota=0}^{j} \mathcal{E}_{\iota}(0), \quad j = 0, 1, \ldots, l. \tag{3.31}$$

The proof of this proposition consists of Lemmas 3.5 and 3.6.

#### 3.3.1. Basic Energy Estimates

**Lemma 3.5.** *Assume that* (3.1) *holds for suitably small positive number* $\varepsilon_0 \in (0, 1)$. *Then,*

$$\begin{aligned} &\mathcal{E}_0(t) + \int_0^t \int \left[(1+s)^{-1} r^2 \bar{\rho}_0^{\gamma} \left(\zeta^2 + (r\zeta_r)^2\right) + (1+s) r^4 \bar{\rho}_0 \zeta_s^2\right] \mathrm{d}r \mathrm{d}s \\ &\lesssim \mathcal{E}_0(0), \quad t \in [0, T]. \end{aligned} \tag{3.32}$$

**Proof.** Multiplying (2.16) by $r^3 \zeta_t$, and integrating the product with respect to the spatial variable, we obtain, using the integration by parts, that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} r^4 \bar{\rho}_0 \zeta_t^2 \mathrm{d}r + \int r^4 \bar{\rho}_0 \zeta_t^2 \mathrm{d}r + \int \bar{\rho}_0^{\gamma} \mathfrak{L}_1 \mathrm{d}r = 0, \tag{3.33}$$

where

$$\begin{aligned} \mathfrak{L}_1 :=& -(\tilde{\eta}_r + \zeta)^{-2\gamma} (\tilde{\eta}_r + \zeta + r\zeta_r)^{-\gamma} \left[r^3 (\tilde{\eta}_r + \zeta)^2 \zeta_t\right]_r + \tilde{\eta}_r^{2-3\gamma} \left(r^3 \zeta_t\right)_r \\ =:& -\mathfrak{L}_{11} + \mathfrak{L}_{12}. \end{aligned}$$

For $\mathfrak{L}_{11}$, note that

$$\left[r^3 (\tilde{\eta}_r + \zeta)^2 \zeta_t\right]_r = 2r^2 (\tilde{\eta}_r + \zeta)(\tilde{\eta}_r + \zeta + r\zeta_r) \zeta_t + r^2 (\tilde{\eta}_r + \zeta)^2 (\zeta + r\zeta_r)_t,$$

thus,

$$\begin{aligned}\mathfrak{L}_{11} =& \frac{r^2}{1-\gamma}\left[(\tilde\eta_r+\zeta)^{2-2\gamma}(\tilde\eta_r+\zeta+r\zeta_r)^{1-\gamma}\right]_t\\ &-r^2\Big[2(\tilde\eta_r+\zeta)^{1-2\gamma}(\tilde\eta_r+\zeta+r\zeta_r)^{1-\gamma}\\ &+(\tilde\eta_r+\zeta)^{2-2\gamma}(\tilde\eta_r+\zeta+r\zeta_r)^{-\gamma}\Big]\tilde\eta_{rt}.\end{aligned}$$

Clearly, $\mathfrak{L}_{12}$ can be rewritten as

$$\begin{aligned}\mathfrak{L}_{12} =& r^2(3\zeta+r\zeta_r)_t\,\tilde\eta_r^{2-3\gamma} = r^2\left[(3\zeta+r\zeta_r)\,\tilde\eta_r^{2-3\gamma}\right]_t\\ &-(2-3\gamma)r^2(3\zeta+r\zeta_r)\,\tilde\eta_r^{1-3\gamma}\,\tilde\eta_{rt}.\end{aligned}$$

Substitute these calculations into (3.33) to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac12 r^4\bar\rho_0\zeta_t^2+r^2\bar\rho_0^\gamma\widetilde{\mathfrak{E}}_0\right)\mathrm{d}r+\int r^4\bar\rho_0\zeta_t^2\mathrm{d}r+\int r^2\bar\rho_0^\gamma\tilde\eta_{rt}\mathfrak{F}\mathrm{d}r=0, \quad (3.34)$$

where

$$\begin{aligned}\widetilde{\mathfrak{E}}_0 :=& \frac{1}{\gamma-1}\Big[(\tilde\eta_r+\zeta)^{2-2\gamma}(\tilde\eta_r+\zeta+r\zeta_r)^{1-\gamma}-\tilde\eta_r^{3-3\gamma}\\ &+(\gamma-1)(3\zeta+r\zeta_r)\,\tilde\eta_r^{2-3\gamma}\Big],\\ \mathfrak{F} :=& 2(\tilde\eta_r+\zeta)^{1-2\gamma}(\tilde\eta_r+\zeta+r\zeta_r)^{1-\gamma}+(\tilde\eta_r+\zeta)^{2-2\gamma}(\tilde\eta_r+\zeta+r\zeta_r)^{-\gamma}\\ &-3\tilde\eta_r^{2-3\gamma}-(2-3\gamma)(3\zeta+r\zeta_r)\,\tilde\eta_r^{1-3\gamma}.\end{aligned}$$

It follows from the Taylor expansion, the smallness of $\zeta$ and $r\zeta_r$ which is a consequence of (3.1), and (2.13) that

$$\begin{aligned}\widetilde{\mathfrak{E}}_0 =& \tilde\eta_r^{1-3\gamma}\left[\frac32(3\gamma-2)\zeta^2+(3\gamma-2)\zeta r\zeta_r+\frac{\gamma}{2}(r\zeta_r)^2\right]\\ &+O(1)\tilde\eta_r^{-3\gamma}(|\zeta|+|r\zeta_r|)\left(\zeta^2+(r\zeta_r)^2\right)\\ &\sim \tilde\eta_r^{1-3\gamma}\left(\zeta^2+(r\zeta_r)^2\right)\sim(1+t)^{-1}\left(\zeta^2+(r\zeta_r)^2\right),\\ \mathfrak{F} \geqq& (3\gamma-1)\tilde\eta_r^{-3\gamma}\left[\frac32(3\gamma-2)\zeta^2+(3\gamma-2)\zeta r\zeta_r+\frac{\gamma}{2}(r\zeta_r)^2\right]\\ &-C\tilde\eta_r^{-3\gamma-1}(|\zeta|+|r\zeta_r|)\left(\zeta^2+(r\zeta_r)^2\right)\\ \geqq& (1+t)^{-\frac{3\gamma}{3\gamma-1}}\left(\zeta^2+(r\zeta_r)^2\right)\geqq 0.\end{aligned}$$

Here and thereafter the notation $O(1)$ represents a finite number could be positive or negative. We then have, by integrating (3.34) with respect to the temporal variable, that

$$\int\left(\frac12 r^4\bar\rho_0\zeta_t^2+r^2\bar\rho_0^\gamma\widetilde{\mathfrak{E}}_0\right)(r,s)\mathrm{d}r\Big|_{s=0}^{t}+\int_0^t\int r^4\bar\rho_0\zeta_s^2\mathrm{d}r\mathrm{d}s\leqq 0$$

and

$$\int \left[ r^4 \bar{\rho}_0 \zeta_t^2 + (1+t)^{-1} r^2 \bar{\rho}_0^{\gamma} \left( \zeta^2 + (r\zeta_r)^2 \right) \right] (r,t) \mathrm{d}r + \int_0^t \int r^4 \bar{\rho}_0 \zeta_s^2 \mathrm{d}r \mathrm{d}s$$
$$\lesssim \int \left[ r^4 \bar{\rho}_0 \zeta_t^2 + r^2 \bar{\rho}_0^{\gamma} \left( \zeta^2 + (r\zeta_r)^2 \right) \right] (r,0) \mathrm{d}r. \tag{3.35}$$

Multiplying (2.16) by $r^3\zeta$, and integrating the product with respect to the spatial variable, we have, using the integration by parts, that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int r^4 \bar{\rho}_0 \left( \frac{1}{2}\zeta^2 + \zeta\zeta_t \right) \mathrm{d}r + \int \bar{\rho}_0^{\gamma} \mathfrak{L}_2 \mathrm{d}r = \int r^4 \bar{\rho}_0 \zeta_t^2 \mathrm{d}r, \tag{3.36}$$

where

$$\mathfrak{L}_2 := -\left(\tilde{\eta}_r + \zeta\right)^{-2\gamma} \left(\tilde{\eta}_r + \zeta + r\zeta_r\right)^{-\gamma} \left[ r^3 \left(\tilde{\eta}_r + \zeta\right)^2 \zeta \right]_r + \tilde{\eta}_r^{2-3\gamma} \left( r^3 \zeta \right)_r,$$

which can be rewritten as

$$\begin{aligned} \mathfrak{L}_2 = & r^2 \Big[ 3\tilde{\eta}_r^{2-3\gamma} - 2\left(\tilde{\eta}_r + \zeta\right)^{1-2\gamma} \left(\tilde{\eta}_r + \zeta + r\zeta_r\right)^{1-\gamma} \\ & - \left(\tilde{\eta}_r + \zeta\right)^{2-2\gamma} \left(\tilde{\eta}_r + \zeta + r\zeta_r\right)^{-\gamma} \Big] \zeta \\ & + r^2 \left[ \tilde{\eta}_r^{2-3\gamma} - \left(\tilde{\eta}_r + \zeta\right)^{2-2\gamma} \left(\tilde{\eta}_r + \zeta + r\zeta_r\right)^{-\gamma} \right] r\zeta_r. \end{aligned}$$

Again, we use the Taylor expansion, (3.2) and (3.1) to obtain

$$\begin{aligned} \mathfrak{L}_2 \gtrsim & r^2 (1+t)^{-1} \\ & \left[ 3(3\gamma-2)\zeta^2 + 2(3\gamma-2)\zeta r\zeta_r + \gamma (r\zeta_r)^2 - C\varepsilon_0 \left( \zeta^2 + (r\zeta_r)^2 \right) \right] \\ \gtrsim & r^2 (1+t)^{-1} \left( \zeta^2 + (r\zeta_r)^2 \right), \end{aligned}$$

provide that $\varepsilon_0$ is suitably small. It then follows from (3.36), the Cauchy inequality and (3.35) that

$$\int \left( r^4 \bar{\rho}_0 \zeta^2 \right)(r,t) \mathrm{d}r + \int_0^t \int (1+s)^{-1} r^2 \bar{\rho}_0^{\gamma} \left( \zeta^2 + (r\zeta_r)^2 \right) \mathrm{d}r \mathrm{d}s$$
$$\lesssim \int \left( r^4 \bar{\rho}_0 \left( \zeta^2 + \zeta_t^2 \right) \right)(r,0) \mathrm{d}r + \int \left( r^4 \bar{\rho}_0 \zeta_t^2 \right)(r,t) \mathrm{d}r + \int_0^t \int r^4 \bar{\rho}_0 \zeta_s^2 \mathrm{d}r \mathrm{d}s$$
$$\lesssim \int \left[ r^4 \bar{\rho}_0 \left( \zeta^2 + \zeta_t^2 \right) + r^2 \bar{\rho}_0^{\gamma} \left( \zeta^2 + (r\zeta_r)^2 \right) \right](r,0) \mathrm{d}r = \mathcal{E}_0(0). \tag{3.37}$$

Next, we show the time decay of the energy norm. Multiply equation (3.34) by (1+t) and integrate the product with respect to the temporal variable to get

$$(1+t) \int \left( \frac{1}{2} r^4 \bar{\rho}_0 \zeta_t^2 + r^2 \bar{\rho}_0^{\gamma} \widetilde{\mathfrak{E}}_0 \right)(r,t) \mathrm{d}r + \int_0^t (1+s) \int r^4 \bar{\rho}_0 \zeta_s^2 \mathrm{d}r \mathrm{d}s$$
$$\leqq \int \left( \frac{1}{2} r^4 \bar{\rho}_0 \zeta_t^2 + r^2 \bar{\rho}_0^{\gamma} \widetilde{\mathfrak{E}}_0 \right)(r,0) \mathrm{d}r + \int_0^t \int \left( \frac{1}{2} r^4 \bar{\rho}_0 \zeta_s^2 + \widetilde{\mathfrak{E}}_0 \right) \mathrm{d}r \mathrm{d}s$$
$$\lesssim \int \left[ r^4 \bar{\rho}_0 \left( \zeta^2 + \zeta_t^2 \right) + r^2 \bar{\rho}_0^{\gamma} \left( \zeta^2 + (r\zeta_r)^2 \right) \right](r,0) \mathrm{d}r = \mathcal{E}_0(0),$$

where estimates (3.35) and (3.37) have been used to derive the last inequality. This means

$$\int \left[(1+t)r^4\bar\rho_0\zeta_t^2 + r^2\bar\rho_0^\gamma\left(\zeta^2+(r\zeta_r)^2\right)\right](r,t)\mathrm{d}r \\ + \int_0^t (1+s)\int r^4\bar\rho_0\zeta_s^2\mathrm{d}r\mathrm{d}s \lesssim \mathcal{E}_0(0),$$

which, together with (3.37), gives (3.32). This finishes the proof of Lemma 3.5. □

**3.3.2. Higher-Order Energy Estimates** Equation (2.16) reads

$$r\bar\rho_0\zeta_{tt} + r\bar\rho_0\zeta_t + \left[\bar\rho_0^\gamma\left(\tilde\eta_r+\zeta\right)^{2-2\gamma}\left(\tilde\eta_r+\zeta+r\zeta_r\right)^{-\gamma}\right]_r - \tilde\eta_r^{2-3\gamma}\left(\bar\rho_0^\gamma\right)_r \\ -2\bar\rho_0^\gamma\left(\tilde\eta_r+\zeta\right)^{1-2\gamma}\left(\tilde\eta_r+\zeta+r\zeta_r\right)^{-\gamma}\zeta_r = 0.$$

Let $k \geqq 1$ be an integer and take the $k$-th time derivative of the equation above. One has

$$r\bar\rho_0\partial_t^k\zeta_{tt} + r\bar\rho_0\partial_t^k\zeta_t + \left[\bar\rho_0^\gamma\left(w_1\partial_t^k\zeta + w_2 r\partial_t^k\zeta_r + K_1\right)\right]_r \\ +\bar\rho_0^\gamma\left[(3w_2-w_1)\partial_t^k\zeta_r + K_2\right] \\ -2\bar\rho_0^\gamma\left(w_3\zeta_r\partial_t^k\zeta + K_3\right) + \partial_t^{k-1}\left\{\bar\rho_0^\gamma\tilde\eta_{rt}\left[w_1-(2-3\gamma)\tilde\eta_r^{1-3\gamma}\right]\right\}_r \\ -2\bar\rho_0^\gamma\partial_t^{k-1}\left(\tilde\eta_{rt}w_3\zeta_r\right) = 0. \tag{3.38}$$

Here

$$\begin{aligned} w_1 =&(2-2\gamma)\left(\tilde\eta_r+\zeta\right)^{1-2\gamma}\left(\tilde\eta_r+\zeta+r\zeta_r\right)^{-\gamma} \\ &-\gamma\left(\tilde\eta_r+\zeta\right)^{2-2\gamma}\left(\tilde\eta_r+\zeta+r\zeta_r\right)^{-\gamma-1}, \\ w_2 =&-\gamma\left(\tilde\eta_r+\zeta\right)^{2-2\gamma}\left(\tilde\eta_r+\zeta+r\zeta_r\right)^{-\gamma-1}, \\ w_3 =&(1-2\gamma)\left(\tilde\eta_r+\zeta\right)^{-2\gamma}\left(\tilde\eta_r+\zeta+r\zeta_r\right)^{-\gamma} \\ &-\gamma\left(\tilde\eta_r+\zeta\right)^{1-2\gamma}\left(\tilde\eta_r+\zeta+r\zeta_r\right)^{-\gamma-1}, \end{aligned}$$

and

$$K_1 =\partial_t^{k-1}\left(w_1\zeta_t + w_2 r\zeta_{tr}\right) - \left(w_1\partial_t^k\zeta + w_2 r\partial_t^k\zeta_r\right),$$
$$K_2 =\partial_t^{k-1}\left[(3w_2-w_1)\zeta_{tr}\right] - (3w_2-w_1)\partial_t^k\zeta_r,\ K_3 = \partial_t^{k-1}(w_3\zeta_r\zeta_t) - w_3\zeta_r\partial_t^k\zeta.$$

It should be noted that $K_1$, $K_2$ and $K_3$ contain lower-order terms involving $\partial_t^\iota(\zeta,\zeta_r)$ with $\iota = 0,\cdots,k-1$; and $w_1$, $w_2$ and $w_3$ can be expanded, according to the Taylor expansion and the smallness of $\zeta$ and $r\zeta_r$ which is a consequence of (3.1), as follows

$$\begin{aligned} w_1 &= (2-3\gamma)\tilde\eta_r^{1-3\gamma} + (3\gamma-1)\tilde\eta_r^{-3\gamma}\left[(3\gamma-2)\zeta+\gamma r\zeta_r\right] + \bar w_1 \\ w_2 &= -\gamma\tilde\eta_r^{1-3\gamma} + \gamma\tilde\eta_r^{-3\gamma}\left[(3\gamma-1)\zeta + (\gamma+1)r\zeta_r\right] + \bar w_2, \\ w_3 &= (1-3\gamma)\tilde\eta_r^{-3\gamma} + \bar w_3. \end{aligned} \tag{3.39}$$

Here $\bar{w}_i$ satisfies

$$|\bar{w}_1| + |\bar{w}_2| \lesssim \tilde{\eta}_r^{-3\gamma-1}\left(|\zeta|^2 + |r\zeta_r|^2\right), \quad \text{and} \quad |\bar{w}_3| \lesssim \tilde{\eta}_r^{-3\gamma-1}\left(|\zeta| + |r\zeta_r|\right). \tag{3.40}$$

In particular, $K_1 = K_2 = K_3 = 0$ when $k = 1$.

**Lemma 3.6.** *Assume that* (3.1) *holds for suitably small positive number* $\varepsilon_0 \in (0, 1)$. *Then for all* $j = 1, \ldots, l$, *and* $t \in [0, T]$

$$\mathcal{E}_j(t)+\int_0^t \int \left[(1+s)^{2j-1} r^2 \bar{\rho}_0^{\gamma} \left|\partial_s^j (\zeta, r\zeta_r)\right|^2 + (1+s)^{2j+1} r^4 \bar{\rho}_0 \left(\partial_s^j \zeta_s\right)^2\right] \mathrm{d}r\mathrm{d}s \lesssim \sum_{\iota=0}^{j} \mathcal{E}_\iota(0). \tag{3.41}$$

**Proof.** We use induction to prove (3.41). As shown in Lemma 3.5, we know that (3.41) holds for $j = 0$. For $1 \leqq k \leqq l$, we make the induction hypothesis that (3.41) holds for all $j = 0, \ldots, k-1$, that is, for all $j = 0, \ldots, k-1$,

$$\mathcal{E}_j(t)+\int_0^t \int \left[(1+s)^{2j-1} r^2 \bar{\rho}_0^{\gamma} \left|\partial_s^j (\zeta, r\zeta_r)\right|^2 + (1+s)^{2j+1} r^4 \bar{\rho}_0 \left(\partial_s^j \zeta_s\right)^2\right] \mathrm{d}r\mathrm{d}s \lesssim \sum_{\iota=0}^{j} \mathcal{E}_\iota(0). \tag{3.42}$$

It suffices to prove (3.41) holds for $j = k$ under the induction hypothesis (3.42).

*Step 1*. In this step, we prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int \frac{1}{2} r^4 \bar{\rho}_0 \left(\partial_t^k \zeta_t\right)^2 \mathrm{d}r + \overline{\mathfrak{E}}_k\right] + \int r^4 \bar{\rho}_0 \left(\partial_t^k \zeta_t\right)^2 \mathrm{d}r \lesssim (\varepsilon_0 + \delta)(1+t)^{-2k-2}\mathcal{E}_k(t) + \left(\varepsilon_0 + \delta^{-1}\right)(1+t)^{-2k-2} \sum_{\iota=0}^{k-1} \mathcal{E}_\iota(t) \tag{3.43}$$

for any positive number $\delta > 0$ which will be specified later, where $\overline{\mathfrak{E}}_k := \int r^2 \bar{\rho}_0^{\gamma} \widetilde{\mathfrak{E}}_k \mathrm{d}r + M_2$. Here $\widetilde{\mathfrak{E}}_k$ and $M_2$ are defined by (3.49) and (3.47), respectively. Moreover, we show that $\overline{\mathfrak{E}}_k$ satisfies the following estimates:

$$\overline{\mathfrak{E}}_k \geqq C^{-1}(1+t)^{-1} \int r^2 \bar{\rho}_0^{\gamma} \left|\partial_t^k (\zeta, r\zeta_r)\right|^2 \mathrm{d}r - C(1+t)^{-2k-1} \sum_{\iota=0}^{k-1} \mathcal{E}_\iota(t), \tag{3.44}$$

$$\overline{\mathfrak{E}}_k \lesssim (1+t)^{-1} \int r^2 \bar{\rho}_0^{\gamma} \left|\partial_t^k (\zeta, r\zeta_r)\right|^2 \mathrm{d}r + (1+t)^{-2k-1} \sum_{\iota=0}^{k-1} \mathcal{E}_\iota(t). \tag{3.45}$$

We start with integrating the production of (3.38) and $r^3\partial_t^k\zeta_t$ with respect to the spatial variable which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}r^4\bar{\rho}_0\left(\partial_t^k\zeta_t\right)^2\mathrm{d}r+\int r^4\bar{\rho}_0\left(\partial_t^k\zeta_t\right)^2\mathrm{d}r+N_1+N_2=0, \tag{3.46}$$

where

$$\begin{aligned}N_1:=&-\int\bar{\rho}_0^\gamma\left(w_1\partial_t^k\zeta+w_2r\partial_t^k\zeta_r\right)\left(r^3\partial_t^k\zeta_t\right)_r\mathrm{d}r\\&+\int r^2\bar{\rho}_0^\gamma(3w_2-w_1)\left(r\partial_t^k\zeta_r\right)\partial_t^k\zeta_t\mathrm{d}r\\&-2\int r^2\bar{\rho}_0^\gamma w_3\,(r\zeta_r)\left(\partial_t^k\zeta\right)\partial_t^k\zeta_t\mathrm{d}r,\\N_2:=&-\int\bar{\rho}_0^\gamma\left\{K_1+\partial_t^{k-1}\left[\tilde{\eta}_{rt}\left(w_1-(2-3\gamma)\tilde{\eta}_r^{1-3\gamma}\right)\right]\right\}\left(r^3\partial_t^k\zeta_t\right)_r\mathrm{d}r\\&+\int r^2\bar{\rho}_0^\gamma\left(\partial_t^k\zeta_t\right)\left[r(K_2-2K_3)-2\partial_t^{k-1}\left(\tilde{\eta}_{rt}w_3r\zeta_r\right)\right]\mathrm{d}r.\end{aligned}$$

Note that $N_1$ and $N_2$ can be rewritten as

$$\begin{aligned}N_1=&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int r^2\bar{\rho}_0^\gamma\Big[(3w_1+2w_3r\zeta_r)\left(\partial_t^k\zeta\right)^2\\&+2w_1\left(\partial_t^k\zeta\right)r\partial_t^k\zeta_r+w_2\left(r\partial_t^k\zeta_r\right)^2\Big]\mathrm{d}r+\widetilde{N}_1,\\N_2=&\frac{\mathrm{d}}{\mathrm{d}t}M_2+\int\bar{\rho}_0^\gamma\left\{K_{1t}+\partial_t^k\left[\tilde{\eta}_{rt}\left(w_1-(2-3\gamma)\tilde{\eta}_r^{1-3\gamma}\right)\right]\right\}\left(r^3\partial_t^k\zeta\right)_r\mathrm{d}r\\&-\int r^2\bar{\rho}_0^\gamma\left(\partial_t^k\zeta\right)\left[r(K_2-2K_3)_t-2\partial_t^k\left(\tilde{\eta}_{rt}w_3r\zeta_r\right)\right]\mathrm{d}r=:\frac{\mathrm{d}}{\mathrm{d}t}M_2+\widetilde{N}_2,\end{aligned}$$

where

$$\begin{aligned}\widetilde{N}_1:=&\frac{1}{2}\int r^2\bar{\rho}_0^\gamma\left[(3w_1+2w_3r\zeta_r)_t\left(\partial_t^k\zeta\right)^2+2w_{1t}\left(\partial_t^k\zeta\right)r\partial_t^k\zeta_r+w_{2t}\left(r\partial_t^k\zeta_r\right)^2\right]\mathrm{d}r,\\M_2:=&-\int\bar{\rho}_0^\gamma\left\{K_1+\partial_t^{k-1}\left[\tilde{\eta}_{rt}\left(w_1-(2-3\gamma)\tilde{\eta}_r^{1-3\gamma}\right)\right]\right\}\left(r^3\partial_t^k\zeta\right)_r\mathrm{d}r\\&+\int r^2\bar{\rho}_0^\gamma\left(\partial_t^k\zeta\right)\left[r(K_2-2K_3)-2\partial_t^{k-1}\left(\tilde{\eta}_{rt}w_3r\zeta_r\right)\right]\mathrm{d}r.\end{aligned} \tag{3.47}$$

It then follows from equation (3.46) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int\left(\frac{1}{2}r^4\bar{\rho}_0\left(\partial_t^k\zeta_t\right)^2+r^2\bar{\rho}_0^\gamma\widetilde{\mathfrak{E}}_k\right)\mathrm{d}r+M_2\right]+\int r^4\bar{\rho}_0\left(\partial_t^k\zeta_t\right)^2\mathrm{d}r\\&\quad=-\widetilde{N}_1-\widetilde{N}_2,\end{aligned} \tag{3.48}$$

where

$$\widetilde{\mathfrak{E}}_k:=-\frac{1}{2}\left[(3w_1+2w_3r\zeta_r)\left(\partial_t^k\zeta\right)^2+2w_1\left(\partial_t^k\zeta\right)r\partial_t^k\zeta_r+w_2\left(r\partial_t^k\zeta_r\right)^2\right], \tag{3.49}$$

which satisfies

$$\widetilde{\mathfrak{E}}_k = \tilde{\eta}_r^{1-3\gamma}\left[\frac{3}{2}(3\gamma-2)\left(\partial_t^k\zeta\right)^2 + (3\gamma-2)\left(\partial_t^k\zeta\right)r\partial_t^k\zeta_r + \frac{\gamma}{2}\left(r\partial_t^k\zeta_r\right)^2\right]$$
$$+O(1)\tilde{\eta}_r^{-3\gamma}\left(|\zeta|+|r\zeta_r|\right)\left(\left(\partial_t^k\zeta\right)^2+\left(r\partial_t^k\zeta_r\right)^2\right)$$
$$\sim \tilde{\eta}_r^{1-3\gamma}\left[\left(\partial_t^k\zeta\right)^2+\left(r\partial_t^k\zeta_r\right)^2\right] \sim (1+t)^{-1}\left[\left(\partial_t^k\zeta\right)^2+\left(r\partial_t^k\zeta_r\right)^2\right]. \quad (3.50)$$

Here we have used (3.39), (2.13) and the smallness of $\zeta$ and $r\zeta_r$ which is a consequence of (3.1) to derive the above equivalence. We will show later that $M_2$ can be bounded by the integral of $\widetilde{\mathfrak{E}}_k$ and lower-order terms, see (3.65).

In what follows, we analyze the terms on the right-hand side of (3.48). Clearly, $-\widetilde{N}_1$ can be bounded by

$$-\widetilde{N}_1 \leqq (1-3\gamma)\int r^2\bar{\rho}_0^{\gamma}\tilde{\eta}_r^{-3\gamma}\tilde{\eta}_{rt}\left[\left(\frac{9}{2}\gamma-3\right)\left(\partial_t^k\zeta\right)^2+(3\gamma-2)\left(\partial_t^k\zeta\right)\left(r\partial_t^k\zeta_r\right)\right.$$
$$\left.+\frac{\gamma}{2}\left(r\partial_t^k\zeta_r\right)^2\right]\mathrm{d}r + C\int r^2\bar{\rho}_0^{\gamma}\tilde{\eta}_r^{-3\gamma}\left[\tilde{\eta}_r^{-1}\tilde{\eta}_{rt}\left(|\zeta|+|r\zeta_r|\right)\right.$$
$$\left.+\left(1+\tilde{\eta}_r^{-1}\left(|\zeta|+|r\zeta_r|\right)\right)\left(|\zeta_t|+|r\zeta_{rt}|\right)\right]\left(\left(\partial_t^k\zeta\right)^2+\left(r\partial_t^k\zeta_r\right)^2\right)\mathrm{d}r.$$

It should be noted that the first integral on the right-hand side of the inequality above is non-positive due to $\tilde{\eta}_{rt} \geqq 0$. Thus, we have by use of (2.13) and (3.1) that

$$-\widetilde{N}_1 \lesssim \varepsilon_0(1+t)^{-2-\frac{1}{3\gamma-1}}\int r^2\bar{\rho}_0^{\gamma}\left(\left(\partial_t^k\zeta\right)^2+\left(r\partial_t^k\zeta_r\right)^2\right)\mathrm{d}r. \quad (3.51)$$

To control $\widetilde{N}_2$, we may rewrite it as

$$\widetilde{N}_2 = \int r^2\bar{\rho}_0^{\gamma}\left\{\left(3\partial_t^k\zeta+r\partial_t^k\zeta_r\right)\partial_t^k\left[\tilde{\eta}_{rt}\left(w_1-(2-3\gamma)\tilde{\eta}_r^{1-3\gamma}\right)\right]\right.$$
$$\left.+2\left(\partial_t^k\zeta\right)\partial_t^k\left(\tilde{\eta}_{rt}w_3r\zeta_r\right)\right\}\mathrm{d}r$$
$$+\int r^2\bar{\rho}_0^{\gamma}\left[K_{1t}\left(3\partial_t^k\zeta+r\partial_t^k\zeta_r\right)-r(K_2-2K_3)_t\left(\partial_t^k\zeta\right)\right]\mathrm{d}r =: \widetilde{N}_{21}+\widetilde{N}_{22}.$$

For $\widetilde{N}_{21}$, note that

$$\partial_t^k\left[\tilde{\eta}_{rt}\left(w_1-(2-3\gamma)\tilde{\eta}_r^{1-3\gamma}\right)\right] = (3\gamma-1)\tilde{\eta}_{rt}\tilde{\eta}_r^{-3\gamma}\left((3\gamma-2)\partial_t^k\zeta+\gamma r\partial_t^k\zeta_r\right)$$
$$+O(1)\sum_{\iota=1}^{k}\left|\partial_t^{\iota}\left(\tilde{\eta}_{rt}\tilde{\eta}_r^{-3\gamma}\right)\right|\left|\partial_t^{k-\iota}\left(\zeta,r\zeta_r\right)\right|+\partial_t^k\left(\tilde{\eta}_{rt}\bar{w}_1\right),$$

$$\partial_t^k\left(\tilde{\eta}_{rt}w_3r\zeta_r\right) = (1-3\gamma)\tilde{\eta}_{rt}\tilde{\eta}_r^{-3\gamma}\left(r\partial_t^k\zeta_r\right)+O(1)\sum_{\iota=1}^{k}\left|\partial_t^{\iota}\left(\tilde{\eta}_{rt}\tilde{\eta}_r^{-3\gamma}\right)\right|\left|r\partial_t^{k-\iota}\zeta_r\right|$$
$$+\partial_t^k\left(\tilde{\eta}_{rt}\bar{w}_3r\zeta_r\right).$$

Thus,

$$-\widetilde{N}_{21} \leqq (1-3\gamma)\int r^2\bar{\rho}_0^{\gamma}\tilde{\eta}_r^{-3\gamma}\tilde{\eta}_{rt}\left[(9\gamma-6)\left(\partial_t^k\zeta\right)^2+(6\gamma-4)\left(\partial_t^k\zeta\right)\left(r\partial_t^k\zeta_r\right)\right.$$
$$\left.+\gamma\left(r\partial_t^k\zeta_r\right)^2\right]\mathrm{d}r + C\int r^2\bar{\rho}_0^{\gamma}\left(\left|\partial_t^k\zeta\right|+\left|r\partial_t^k\zeta_r\right|\right)$$
$$\times\left[\sum_{\iota=1}^{k}\left|\partial_t^{\iota}\left(\tilde{\eta}_{rt}\tilde{\eta}_r^{-3\gamma}\right)\right|\left|\partial_t^{k-\iota}(\zeta, r\zeta_r)\right|+\left|\partial_t^k(\tilde{\eta}_{rt}\bar{w}_1, \tilde{\eta}_{rt}\bar{w}_3 r\zeta_r)\right|\right]\mathrm{d}r. \tag{3.52}$$

For $\widetilde{N}_{22}$, note that

$$K_{1t} =(k-1)\left(w_{1t}\partial_t^k\zeta + w_{2t}r\partial_t^k\zeta_r\right)+O(1)\sum_{\iota=2}^{k}\left|\partial_t^{\iota}(w_1, w_2)\right|\left|\partial_t^{k+1-\iota}(\zeta, r\zeta_r)\right|,$$

$$rK_{2t} =(k-1)(3w_2-w_1)_t\left(r\partial_t^k\zeta_r\right)+O(1)\sum_{\iota=2}^{k}\left|\partial_t^{\iota}(w_1, w_2)\right|\left|\partial_t^{k+1-\iota}(r\zeta_r)\right|,$$

$$rK_{3t} =(k-1)(w_3 r\zeta_r)_t\left(\partial_t^k\zeta\right)+O(1)\sum_{\iota=2}^{k}\left|\partial_t^{\iota}(w_3 r\zeta_r)\right|\left|\partial_t^{k+1-\iota}\zeta\right|.$$

Thus,

$$\widetilde{N}_{22} \geqq 2(k-1)\widetilde{N}_1 - C\int r^2\bar{\rho}_0^{\gamma}\left(\left|\partial_t^k\zeta\right|+\left|r\partial_t^k\zeta_r\right|\right)$$
$$\times\sum_{\iota=2}^{k}\left|\partial_t^{\iota}(w_1, w_2, w_3 r\zeta_r)\right|\left|\partial_t^{k+1-\iota}(\zeta, r\zeta_r)\right|\mathrm{d}r. \tag{3.53}$$

In a similar way to dealing with $\widetilde{N}_1$ shown in (3.51), we have, with the aid of (3.52) and (3.53), that

$$-\widetilde{N}_2 \lesssim \varepsilon_0(1+t)^{-2-\frac{1}{3\gamma-1}}\int r^2\bar{\rho}_0^{\gamma}\left(\left(\partial_t^k\zeta\right)^2+\left(r\partial_t^k\zeta_r\right)^2\right)\mathrm{d}r$$
$$+\int r^2\bar{\rho}_0^{\gamma}\left(\left|\partial_t^k\zeta\right|+\left|r\partial_t^k\zeta_r\right|\right)Q\mathrm{d}r, \tag{3.54}$$

where

$$Q=\sum_{\iota=1}^{k}\left|\partial_t^{\iota}\left(\tilde{\eta}_{rt}\tilde{\eta}_r^{-3\gamma}\right)\right|\left|\partial_t^{k-\iota}(\zeta, r\zeta_r)\right|+\left|\partial_t^k(\tilde{\eta}_{rt}\bar{w}_1, \tilde{\eta}_{rt}\bar{w}_3 r\zeta_r)\right|$$
$$+\sum_{\iota=2}^{k}\left|\partial_t^{\iota}(w_1, w_2, w_3 r\zeta_r)\right|\left|\partial_t^{k+1-\iota}(\zeta, r\zeta_r)\right|. \tag{3.55}$$

Therefore, it is produced from (3.48), (3.51) and (3.54) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int\left(\frac{1}{2}r^4\bar{\rho}_0\left(\partial_t^k\zeta_t\right)^2+r^2\bar{\rho}_0^{\gamma}\widetilde{\mathfrak{E}}_k\right)\mathrm{d}r+M_2\right]+\int r^4\bar{\rho}_0\left(\partial_t^k\zeta_t\right)^2\mathrm{d}r$$
$$\lesssim\varepsilon_0(1+t)^{-2-\frac{1}{3\gamma-1}}\int r^2\bar{\rho}_0^{\gamma}\left|\partial_t^k(\zeta,r\zeta_r)\right|^2\mathrm{d}r+\int r^2\bar{\rho}_0^{\gamma}\left|\partial_t^k(\zeta,r\zeta_r)\right|Q\mathrm{d}r. \tag{3.56}$$

We are to bound the last term on the right-hand side of (3.56). It follows from (2.13) and (3.1) that

$$Q\lesssim\varepsilon_0(1+t)^{-2-\frac{1}{3\gamma-1}}\left|\partial_t^k(\zeta,r\zeta_r)\right|+\widetilde{Q}, \tag{3.57}$$

where

$$\widetilde{Q}:=\sum_{\iota=1}^{k}(1+t)^{-2-\iota}\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|+(1+t)^{-1-\frac{1}{3\gamma-1}}\left|\partial_t^2(\zeta,r\zeta_r)\right|\left|\partial_t^{k-1}(\zeta,r\zeta_r)\right|$$
$$+(1+t)^{-2-\frac{1}{3\gamma-1}}\left|\partial_t^2(\zeta,r\zeta_r)\right|\left|\partial_t^{k-2}(\zeta,r\zeta_r)\right|$$
$$+(1+t)^{-1-\frac{1}{3\gamma-1}}\left|\partial_t^3(\zeta,r\zeta_r)\right|\left|\partial_t^{k-2}(\zeta,r\zeta_r)\right|$$
$$+\Big[(1+t)^{-1-\frac{1}{3\gamma-1}}\left|\partial_t^4(\zeta,r\zeta_r)\right|+(1+t)^{-2-\frac{1}{3\gamma-1}}\left|\partial_t^3(\zeta,r\zeta_r)\right|$$
$$+(1+t)^{-3-\frac{1}{3\gamma-1}}$$
$$\times\left|\partial_t^2(\zeta,r\zeta_r)\right|+(1+t)^{-1-\frac{2}{3\gamma-1}}\left|\partial_t^2(\zeta,r\zeta_r)\right|^2\Big]\left|\partial_t^{k-3}(\zeta,r\zeta_r)\right|+\text{l.o.t.}. \tag{3.58}$$

Here and thereafter the notation l.o.t. is used to represent the lower-order terms involving $\partial_t^{\iota}(\zeta,r\zeta_r)$ with $\iota=2,\cdots,k-4$. It should be noticed that the second term on the right-hand side of (3.58) only appears as $k-1\geqq 2$, the third term as $k-2\geqq 2$, the fourth term as $k-2\geqq 3$, and so on. Clearly, we use (3.1) again to obtain

$$\widetilde{Q}\lesssim\sum_{\iota=1}^{k}(1+t)^{-2-\iota}\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|+\varepsilon_0\sigma^{-\frac{1}{2}}(1+t)^{-3-\frac{1}{3\gamma-1}}\left|\partial_t^{k-1}(\zeta,r\zeta_r)\right|$$
$$+\varepsilon_0\sigma^{-1}(1+t)^{-4-\frac{1}{3\gamma-1}}\left|\partial_t^{k-2}(\zeta,r\zeta_r)\right|$$
$$+\varepsilon_0\sigma^{-\frac{3}{2}}(1+t)^{-5-\frac{1}{3\gamma-1}}\left|\partial_t^{k-3}(\zeta,r\zeta_r)\right|+\text{l.o.t.},$$

if $k\geqq 7$. Similarly, we can bound l.o.t. and achieve

$$\widetilde{Q}\lesssim\sum_{\iota=1}^{k}(1+t)^{-2-\iota}\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|$$
$$+\varepsilon_0\sum_{\iota=1}^{[(k-1)/2]}\sigma^{-\frac{\iota}{2}}(1+t)^{-2-\iota-\frac{1}{3\gamma-1}}\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|,$$

which implies

$$\int r^2\bar{\rho}_0^{\gamma}\left|\partial_t^k(\zeta,r\zeta_r)\right|\widetilde{\mathcal{Q}}\mathrm{d}r \lesssim \sum_{\iota=1}^{k}(1+t)^{-2-\iota}\int r^2\bar{\rho}_0^{\gamma}\left|\partial_t^k(\zeta,r\zeta_r)\right|\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|\mathrm{d}r$$
$$+\varepsilon_0\sum_{\iota=1}^{[(k-1)/2]}(1+t)^{-2-\iota}\int r^2\bar{\rho}_0^{\gamma}\sigma^{-\frac{\iota}{2}}\left|\partial_t^k(\zeta,r\zeta_r)\right|\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|\mathrm{d}r =: \widetilde{\mathcal{Q}}_1+\widetilde{\mathcal{Q}}_2. \tag{3.59}$$

Easily, it follows from the Cauchy inequality that for any $\delta > 0$,

$$\widetilde{\mathcal{Q}}_1 \lesssim \delta(1+t)^{-2-2k}\mathcal{E}_k(t)+\delta^{-1}(1+t)^{-2-2k}\sum_{\iota=0}^{k-1}\mathcal{E}_\iota(t), \tag{3.60}$$

$$\widetilde{\mathcal{Q}}_2 \lesssim \varepsilon_0(1+t)^{-2}\int r^2\bar{\rho}_0^{\gamma}\left|\partial_t^k(\zeta,r\zeta_r)\right|^2\mathrm{d}r$$
$$+\varepsilon_0\sum_{\iota=1}^{[(k-1)/2]}(1+t)^{-2-2\iota}\int r^2\bar{\rho}_0^{\gamma}\sigma^{-\iota}\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|^2\mathrm{d}r. \tag{3.61}$$

In view of the Hardy inequality (3.6), we see that for $\iota = 1,\cdots,[(k-1)/2]$,

$$\int_{\mathcal{I}_b}\sigma^{\alpha+1-\iota}\left|\partial_t^{k-\iota}(\zeta,\zeta_r)\right|^2\mathrm{d}r \lesssim \int_{\mathcal{I}_b}\sigma^{\alpha+3-\iota}\left|\partial_t^{k-\iota}(\zeta,\zeta_r,\zeta_{rr})\right|^2\mathrm{d}r \lesssim \cdots$$
$$\lesssim \sum_{i=0}^{\iota+1}\int_{\mathcal{I}_b}\sigma^{\alpha+1+\iota}\left|\partial_t^{k-\iota}\partial_r^i\zeta\right|^2\mathrm{d}r \lesssim \sum_{i=0}^{\iota+1}\int_{\mathcal{I}_b}r^4\sigma^{\alpha+1+\iota}\left|\partial_t^{k-\iota}\partial_r^i\zeta\right|^2\mathrm{d}r$$
$$\lesssim (1+t)^{2\iota-2k}\left(\mathcal{E}_{k-\iota}+\sum_{i=1}^{\iota}\mathcal{E}_{k-\iota,i}\right)(t) \lesssim (1+t)^{2\iota-2k}\sum_{\iota=0}^{k}\mathcal{E}_\iota(t),$$

due to $\alpha+1-\iota \geqq \alpha-[([\alpha]+1)/2] \geqq 0$ for $k \leqq l$, which ensures the application of the Hardy inequality. Here the last inequality follows from the elliptic estimate (3.7). Thus, we can obtain for $\iota = 1,\ldots,[(k-1)/2]$,

$$\int r^2\bar{\rho}_0^{\gamma}\sigma^{-\iota}\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|^2\mathrm{d}r \lesssim \int_{\mathcal{I}_o}r^2\sigma^{\alpha+1}\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|^2\mathrm{d}r$$
$$+\int_{\mathcal{I}_b}\sigma^{\alpha+1-\iota}\left|\partial_t^{k-\iota}(\zeta,\zeta_r)\right|^2\mathrm{d}r$$
$$\lesssim (1+t)^{2\iota-2k}\sum_{\iota=0}^{k}\mathcal{E}_\iota(t). \tag{3.62}$$

This, together with (3.61), implies

$$\widetilde{\mathcal{Q}}_2 \lesssim \varepsilon_0(1+t)^{-2-2k}\sum_{\iota=0}^{k}\mathcal{E}_\iota(t). \tag{3.63}$$

So, it yields from (3.57), (3.59), (3.60) and (3.63) that for $\delta > 0$,

$$\int r^2 \bar{\rho}_0^{\gamma} \left| \partial_t^k (\zeta, r\zeta_r) \right| Q \mathrm{d}r \lesssim (1+t)^{-2k-2} \times \left[ (\varepsilon_0 + \delta)\mathcal{E}_k(t) + \left(\varepsilon_0 + \delta^{-1}\right) \sum_{\iota=0}^{k-1} \mathcal{E}_\iota(t) \right].$$

Substitute this into (3.56) to give (3.43).

To prove (3.44) and (3.45), we adopt a similar but much easier way to dealing with $\widetilde{N}_2$ as shown in (3.54) to show

$$|M_2| \lesssim \int r^2 \bar{\rho}_0^{\gamma} \left( \left| \partial_t^k \zeta \right| + \left| r \partial_t^k \zeta_r \right| \right) P \mathrm{d}r, \tag{3.64}$$

where

$$P = \sum_{\iota=0}^{k-1} \left| \partial_t^{\iota} \left( \tilde{\eta}_{rt} \tilde{\eta}_r^{-3\gamma} \right) \right| \left| \partial_t^{k-1-\iota} (\zeta, r\zeta_r) \right| + \left| \partial_t^{k-1} (\tilde{\eta}_{rt} \bar{w}_1, \tilde{\eta}_{rt} \bar{w}_3 r \zeta_r) \right| + \sum_{\iota=1}^{k-1} \left| \partial_t^{\iota} (w_1, w_2, w_3 r \zeta_r) \right| \left| \partial_t^{k-\iota} (\zeta, r\zeta_r) \right|.$$

In view of (2.13) and (3.1), we have

$$\begin{aligned} P \lesssim & \sum_{\iota=0}^{k-1} (1+t)^{-2-\iota} \left| \partial_t^{k-1-\iota} (\zeta, r\zeta_r) \right| \\ & + \left| \partial_t^{k-2} (\zeta, r\zeta_r) \right| (1+t)^{-1-\frac{1}{3\gamma-1}} \left| \partial_t^2 (\zeta, r\zeta_r) \right| \\ & + \left| \partial_t^{k-3} (\zeta, r\zeta_r) \right| \left[ (1+t)^{-1-\frac{1}{3\gamma-1}} \left| \partial_t^3 (\zeta, r\zeta_r) \right| \right. \\ & \left. + (1+t)^{-2-\frac{1}{3\gamma-1}} \left| \partial_t^2 (\zeta, r\zeta_r) \right| \right] + \text{l.o.t.}, \end{aligned}$$

which implies

$$\begin{aligned} P \lesssim & \sum_{\iota=0}^{k-1} (1+t)^{-2-\iota} \left| \partial_t^{k-1-\iota} (\zeta, r\zeta_r) \right| \\ & + \varepsilon_0 \sum_{\iota=2}^{[k/2]} \sigma^{\frac{1-\iota}{2}} (1+t)^{-1-\iota-\frac{1}{3\gamma-1}} \left| \partial_t^{k-\iota} (\zeta, r\zeta_r) \right|. \end{aligned}$$

Similar to the derivation of (3.62), we can use the Hardy inequality (3.6) and elliptic estimate (3.7) to obtain

$$\begin{aligned}\int r^2\bar{\rho}_0^{\gamma}P^2\mathrm{d}r &\lesssim (1+t)^{-2-2k}\sum_{\iota=0}^{k-1}\mathcal{E}_\iota(t)\\ &\quad+\varepsilon_0^2\sum_{\iota=2}^{[k/2]}(1+t)^{-2-2\iota}\int r^2\bar{\rho}_0^{\gamma}\sigma^{1-\iota}\left|\partial_t^{k-\iota}(\zeta,r\zeta_r)\right|^2\mathrm{d}r\\ &\lesssim (1+t)^{-2-2k}\sum_{\iota=0}^{k-1}\mathcal{E}_\iota(t).\end{aligned}$$

It then gives from the Cauchy inequality and (3.64) that for any $\delta>0$,

$$|M_2|\lesssim \delta(1+t)^{-1}\int r^2\bar{\rho}_0^{\gamma}\left|\partial_t^k(\zeta,r\zeta_r)\right|^2\mathrm{d}r+\delta^{-1}(1+t)^{-1-2k}\sum_{\iota=0}^{k-1}\mathcal{E}_\iota(t). \tag{3.65}$$

This, together with (3.50), proves (3.44) (by choosing suitably small $\delta$) and (3.45).

*Step 2*. To control the fist term on the right-hand side of (3.43), we will prove that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}_k+\int\left[(1+t)^{-1}r^2\bar{\rho}_0^{\gamma}\left|\partial_t^k(\zeta,r\zeta_r)\right|^2+r^4\bar{\rho}_0\left(\partial_t^k\zeta_t\right)^2\right]\mathrm{d}r\\ &\lesssim (1+t)^{-1-2k}\sum_{\iota=0}^{k-1}(1+t)^{2\iota}\int\left[r^2\bar{\rho}_0^{\gamma}\left|\partial_t^\iota(\zeta,r\zeta_r)\right|^2+(1+t)r^4\bar{\rho}_0\left(\partial_t^\iota\zeta_t\right)^2\right]\mathrm{d}r,\end{aligned} \tag{3.66}$$

where

$$\mathfrak{E}_k:=\int r^4\bar{\rho}_0\left[\left(\partial_t^k\zeta_t\right)^2+\left(\partial_t^k\zeta\right)\partial_t^k\zeta_t+\frac{1}{2}\left(\partial_t^k\zeta\right)^2\right]\mathrm{d}r+2\overline{\mathfrak{E}}_k.$$

We start with integrating the product of (3.38) and $r^3\partial_t^k\zeta$ with respect to $r$ to give

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int r^4\bar{\rho}_0\left(\left(\partial_t^k\zeta\right)\partial_t^k\zeta_t+\frac{1}{2}\left(\partial_t^k\zeta\right)^2\right)\mathrm{d}r\\ &\quad-\int r^4\bar{\rho}_0\left(\partial_t^k\zeta_t\right)^2\mathrm{d}r+M_1+M_2=0,\end{aligned} \tag{3.67}$$

where $M_2$ is defined in (3.47), and

$$\begin{aligned}M_1=&-\int\bar{\rho}_0^{\gamma}\left(w_1\partial_t^k\zeta+w_2r\partial_t^k\zeta_r\right)\left(r^3\partial_t^k\zeta\right)_r\mathrm{d}r\\ &+\int r^2\bar{\rho}_0^{\gamma}(3w_2-w_1)\left(r\partial_t^k\zeta_r\right)\partial_t^k\zeta\mathrm{d}r\\ &-2\int r^2\bar{\rho}_0^{\gamma}w_3(r\zeta_r)\left(\partial_t^k\zeta\right)^2\mathrm{d}r.\end{aligned}$$

A direct calculation shows that $M_1$ is positive and can be bounded from below as follows

$$\begin{aligned} M_1 \geqq & \int r^2 \bar{\rho}_0^\gamma \tilde{\eta}_r^{1-3\gamma} \left\{ (9\gamma-6)\left(\partial_t^k \zeta\right)^2 + (6\gamma-4)\left(\partial_t^k \zeta\right)\left(r\partial_t^k \zeta_r\right) + \gamma\left(r\partial_t^k \zeta_r\right)^2 \right. \\ & \left. - C\left(|\zeta| + |r\zeta_r|\right)\left[\left(\partial_t^k \zeta\right)^2 + \left(r\partial_t^k \zeta_r\right)^2\right]\right\} \mathrm{d}r \\ \gtrsim & \int r^2 \bar{\rho}_0^\gamma \tilde{\eta}_r^{1-3\gamma} \left[\left(\partial_t^k \zeta\right)^2 + \left(r\partial_t^k \zeta_r\right)^2\right] \mathrm{d}r \\ \gtrsim & (1+t)^{-1} \int r^2 \bar{\rho}_0^\gamma \left[\left(\partial_t^k \zeta\right)^2 + \left(r\partial_t^k \zeta_r\right)^2\right] \mathrm{d}r, \end{aligned}$$

due to (3.39), the smallness of $\zeta_r$ and $r\zeta_r$ and (2.13). We then obtain, by making a summation of $2 \times$ (3.43) and (3.67), that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \mathfrak{E}_k + \int r^4 \bar{\rho}_0 \left(\partial_t^k \zeta_t\right)^2 \mathrm{d}r + (1+t)^{-1} \int r^2 \bar{\rho}_0^\gamma \left[\left(\partial_t^k \zeta\right)^2 + \left(r\partial_t^k \zeta_r\right)^2\right] \mathrm{d}r \\ & \lesssim \delta(1+t)^{-1} \int r^2 \bar{\rho}_0^\gamma \left|\partial_t^k (\zeta, r\zeta_r)\right|^2 \mathrm{d}r + (\varepsilon_0 + \delta)(1+t)^{-2k-2} \mathcal{E}_k(t) \\ & \quad + \delta^{-1}(1+t)^{-1-2k} \sum_{\iota=0}^{k-1} \mathcal{E}_\iota(t) + \left(\varepsilon_0 + \delta^{-1}\right)(1+t)^{-2k-2} \sum_{\iota=0}^{k-1} \mathcal{E}_\iota(t), \quad (3.68) \end{aligned}$$

because of (3.65). Notice from the Hardy inequality (3.6) that for $j = 0, 1, \ldots, l$,

$$\begin{aligned} & \int r^4 \bar{\rho}_0 \left(\partial_t^j \zeta\right)^2 \mathrm{d}r \lesssim \int_{\mathcal{I}_o} r^2 \sigma^{\alpha+1} \left(\partial_t^j \zeta\right)^2 \mathrm{d}r + \int_{\mathcal{I}_b} \sigma^{\alpha+2} \left[\left(\partial_t^j \zeta\right)^2 + \left(\partial_t^j \zeta_r\right)^2\right] \mathrm{d}r \\ & \lesssim \int r^2 \sigma^{\alpha+1} \left(\partial_t^j \zeta\right)^2 \mathrm{d}r + \int_{\mathcal{I}_b} \sigma^{\alpha+1} \left[r^2 \left(\partial_t^j \zeta\right)^2 + r^4 \left(\partial_t^j \zeta_r\right)^2\right] \mathrm{d}r \\ & \lesssim \int r^2 \bar{\rho}_0^\gamma \left|\partial_t^j (\zeta, r\zeta_r)\right|^2 \mathrm{d}r. \end{aligned}$$

Thus,

$$\begin{aligned} & \mathcal{E}_j(t) \lesssim (1+t)^{2j} \int \left[r^2 \bar{\rho}_0^\gamma \left|\partial_t^j (\zeta, r\zeta_r)\right|^2 + (1+t) r^4 \bar{\rho}_0 \left(\partial_t^j \zeta_t\right)^2\right] \mathrm{d}r, \\ & \quad j = 0, \cdots, l. \quad (3.69) \end{aligned}$$

This finishes the proof of (3.66), by using (3.68) and (3.69), choosing suitably small $\delta$ and noting the smallness of $\varepsilon_0$. Moreover, it follows from (3.44) and (3.45) that

$$\begin{aligned} \mathfrak{E}_k \geqq & C^{-1} \int r^4 \bar{\rho}_0 \left|\partial_t^k (\zeta, \zeta_t)\right|^2 \mathrm{d}r + C^{-1}(1+t)^{-1} \int r^2 \bar{\rho}_0^\gamma \left|\partial_t^k (\zeta, r\zeta_r)\right|^2 \mathrm{d}r \\ & - C(1+t)^{-2k-1} \sum_{\iota=0}^{k-1} \mathcal{E}_\iota(t), \quad (3.70) \\ \mathfrak{E}_k \lesssim & \int r^4 \bar{\rho}_0 \left|\partial_t^k (\zeta, \zeta_t)\right|^2 \mathrm{d}r \end{aligned}$$

$$+ (1+t)^{-1} \int r^2 \bar{\rho}_0^{\gamma} \left| \partial_t^k (\zeta, r\zeta_r) \right|^2 \mathrm{d}r + (1+t)^{-2k-1} \sum_{\iota=0}^{k-1} \mathcal{E}_\iota(t). \quad (3.71)$$

*Step 3*. In this step, we show the time decay of the norm. We integrate (3.66) and use the induction hypothesis (3.42) to show, noting (3.70) and (3.71), that

$$\begin{aligned} &\int \left[ r^4 \bar{\rho}_0 \left| \partial_t^k (\zeta, \zeta_t) \right|^2 + (1+t)^{-1} r^2 \bar{\rho}_0^{\gamma} \left| \partial_t^k (\zeta, r\zeta_r) \right|^2 \right] (r,t) \mathrm{d}r \\ &\quad + \int_0^t \int \left[ (1+s)^{-1} r^2 \bar{\rho}_0^{\gamma} \left| \partial_s^k (\zeta, r\zeta_r) \right|^2 + r^4 \bar{\rho}_0 \left( \partial_s^k \zeta_s \right)^2 \right] \mathrm{d}r \mathrm{d}s \\ &\lesssim \sum_{\iota=0}^{k} \mathcal{E}_\iota(0) + \sum_{\iota=0}^{k-1} \int_0^t (1+s)^{2\iota-1-2k} \int \left[ r^2 \bar{\rho}_0^{\gamma} \left| \partial_s^\iota (\zeta, r\zeta_r) \right|^2 \right. \\ &\quad \left. + (1+s) r^4 \bar{\rho}_0 \left( \partial_s^\iota \zeta_s \right)^2 \right] \mathrm{d}r \lesssim \sum_{\iota=0}^{k} \mathcal{E}_\iota(0). \end{aligned}$$

Multiply (3.66) by $(1+t)^p$ and integrate the product with respect to the temporal variable from $p=1$ to $p=2k$ step by step to get

$$\begin{aligned} &(1+t)^{2k} \int \left[ r^4 \bar{\rho}_0 \left| \partial_t^k (\zeta, \zeta_t) \right|^2 + (1+t)^{-1} r^2 \bar{\rho}_0^{\gamma} \left| \partial_t^k (\zeta, r\zeta_r) \right|^2 \right] (r,t) \mathrm{d}r \\ &\quad + \int_0^t (1+s)^{2k} \int \left[ (1+s)^{-1} r^2 \bar{\rho}_0^{\gamma} \left| \partial_s^k (\zeta, r\zeta_r) \right|^2 + r^4 \bar{\rho}_0 \left( \partial_s^k \zeta_s \right)^2 \right] \mathrm{d}r \mathrm{d}s \\ &\lesssim \sum_{\iota=0}^{k} \mathcal{E}_\iota(0) + \sum_{\iota=0}^{k-1} \int_0^t (1+s)^{2\iota-1} \int \left[ r^2 \bar{\rho}_0^{\gamma} \right. \\ &\quad \left. \left| \partial_s^\iota (\zeta, r\zeta_r) \right|^2 + (1+s) r^4 \bar{\rho}_0 \left( \partial_s^\iota \zeta_s \right)^2 \right] \mathrm{d}r \\ &\lesssim \sum_{\iota=0}^{k} \mathcal{E}_\iota(0). \end{aligned} \quad (3.72)$$

With this estimate at hand, we finally integrate $(1+t)^{2k+1}$(3.43) with respect to the temporal variable and use (3.69), (3.42) and (3.72) to show

$$\begin{aligned} &(1+t)^{2k} \int \left[ (1+t) r^4 \bar{\rho}_0 \left| \partial_t^k \zeta_t \right|^2 + r^2 \bar{\rho}_0^{\gamma} \left| \partial_t^k (\zeta, r\zeta_r) \right|^2 \right] (r,t) \mathrm{d}r \\ &\quad + \int_0^t (1+s)^{2k+1} \int r^4 \bar{\rho}_0 \left( \partial_s^k \zeta_s \right)^2 \mathrm{d}r \mathrm{d}s \lesssim \sum_{\iota=0}^{k} \mathcal{E}_\iota(0) \\ &\quad + \sum_{\iota=0}^{k} \int_0^t (1+s)^{2\iota-1} \int \left[ r^2 \bar{\rho}_0^{\gamma} \left| \partial_s^\iota (\zeta, r\zeta_r) \right|^2 + (1+s) r^4 \bar{\rho}_0 \left( \partial_s^\iota \zeta_s \right)^2 \right] \mathrm{d}r \\ &\lesssim \sum_{\iota=0}^{k} \mathcal{E}_\iota(0). \end{aligned} \quad (3.73)$$

It finally follows from (3.72) and (3.73) that

$$\mathcal{E}_k(t) + \int_0^t \int (1+s)^{2k-1} \left[ r^2 \bar{\rho}_0^{\gamma} \left| \partial_s^k (\zeta, r\zeta_r) \right|^2 + (1+s)^2 r^4 \bar{\rho}_0 \left( \partial_s^k \zeta_s \right)^2 \right] \mathrm{d}r\mathrm{d}s$$
$$\lesssim \sum_{\iota=0}^{k} \mathcal{E}_\iota(0).$$

This completes the proof of Lemma 3.6. □

### *3.4. Verification of the a Priori Assumption*

In this subsection, we prove the following lemma.

**Lemma 3.7.** *Suppose that $\mathcal{E}(t)$ is finite, then it holds that*

$$\sum_{j=0}^{2} (1+t)^{2j} \left\| \partial_t^j \zeta(\cdot, t) \right\|_{L^\infty}^2 + \sum_{j=0}^{1} (1+t)^{2j} \left\| \partial_t^j \zeta_r(\cdot, t) \right\|_{L^\infty}^2$$
$$+ \sum_{i+j \leqq l-2,\ 2i+j \geqq 3} (1+t)^{2j}$$
$$\times \left\| \sigma^{\frac{2i+j-3}{2}} \partial_t^j \partial_r^i \zeta(\cdot, t) \right\|_{L^\infty}^2 + \sum_{i+j=l-1} (1+t)^{2j} \left\| r\sigma^{\frac{2i+j-3}{2}} \partial_t^j \partial_r^i \zeta(\cdot, t) \right\|_{L^\infty}^2$$
$$+ \sum_{i+j=l} (1+t)^{2j} \left\| r^2 \sigma^{\frac{2i+j-3}{2}} \partial_t^j \partial_r^i \zeta(\cdot, t) \right\|_{L^\infty}^2 \lesssim \mathcal{E}(t). \tag{3.74}$$

Once this lemma is proved, the a priori assumption (3.1) is then verified and the proof of Theorem 2.1 is finished, since it follows from the elliptic estimate (3.7) and the nonlinear weighted energy estimate (3.31) that

$$\mathcal{E}(t) \lesssim \mathcal{E}(0), \quad t \in [0, T].$$

**Proof.** The proof consists of two steps. In Step 1, we derive the $L^\infty$-bounds away from the boundary, that is,

$$\sum_{i+j \leqq l-2} \left\| \partial_t^j \partial_r^i \zeta \right\|_{L^\infty(\mathcal{I}_o)}^2 + \sum_{i+j=l-1} \left\| r\partial_t^j \partial_r^i \zeta \right\|_{L^\infty(\mathcal{I}_o)}^2$$
$$+ \sum_{i+j=l-2} \left\| r^2 \partial_t^j \partial_r^i \zeta \right\|_{L^\infty(\mathcal{I}_o)}^2 \lesssim (1+t)^{-2j} \mathcal{E}(t). \tag{3.75}$$

Away from the origin, we show in Step 2 the following $L^\infty$-estimates:

$$\sum_{j=0}^{3} (1+t)^{2j} \left\| \partial_t^j \zeta \right\|_{L^\infty(\mathcal{I}_b)}^2 + \sum_{j=0}^{1} (1+t)^{2j} \left\| \partial_t^j \zeta_r \right\|_{L^\infty(\mathcal{I}_b)}^2 \lesssim \mathcal{E}(t), \tag{3.76}$$

$$\left\| \sigma^{\frac{2i+j-3}{2}} \partial_t^j \partial_r^i \zeta \right\|_{L^\infty(\mathcal{I}_b)}^2 \lesssim (1+t)^{-2j} \mathcal{E}(t) \ \text{ when } \ 2i+j \geqq 4. \tag{3.77}$$

We obtain (3.74) by using (3.75)–(3.77) and noting the facts $l \geqq 4$ and $\mathcal{I} = \mathcal{I}_o \cup \mathcal{I}_b$. It suffices to show (3.75)–(3.77).

To this end, we first notice some facts. It follows from (3.24) that $\mathcal{E}_{j,0} \lesssim \mathcal{E}_j$ for $j = 0, \cdots, l$, which implies

$$\sum_{j=0}^{l} \left( \mathcal{E}_j(t) + \sum_{i=0}^{l-j} \mathcal{E}_{j,i}(t) \right) \lesssim \mathcal{E}(t). \tag{3.78}$$

The following embedding (cf. [1]): $H^{1/2+\delta}(\mathcal{I}) \hookrightarrow L^\infty(\mathcal{I})$ with the estimate

$$\|F\|_{L^\infty(\mathcal{I})} \leqq C(\delta)\|F\|_{H^{1/2+\delta}(\mathcal{I})}, \tag{3.79}$$

for $\delta > 0$ will be used in the rest of the proof.

*Step 1* (away from the boundary). It follows from (3.78) that for $j = 0, 1, \ldots, l$,

$$\begin{aligned} &(1+t)^{2j} \left[ \sum_{i=0}^{l-j} \int_{\mathcal{I}_o} r^2 \left( \partial_t^j \partial_r^i \zeta \right)^2 \mathrm{d}r + \int_{\mathcal{I}_o} r^4 \left( \partial_t^j \partial_r^{l-j+1} \zeta \right)^2 \mathrm{d}r \right] \\ &\lesssim \sum_{i=0}^{l-j} \mathcal{E}_{j,i}(t) \leqq \mathcal{E}(t), \end{aligned} \tag{3.80}$$

which implies, using (3.4), that for $j = 0, 1, \ldots, l-1$,

$$\left\| \partial_t^j \zeta \right\|_{H^{l-j-1}(\mathcal{I}_o)}^2 \lesssim \left\| \partial_t^j \zeta \right\|_{H^{2,l-j}(\mathcal{I}_o)}^2 \leqq \sum_{i=0}^{l-j} \int_{\mathcal{I}_o} r^2 \left( \partial_t^j \partial_r^i \zeta \right)^2 \mathrm{d}r \leqq (1+t)^{-2j} \mathcal{E}(t). \tag{3.81}$$

In view of (3.79) and (3.81), we see that for $j = 0, 1, \ldots, l-2$,

$$\begin{aligned} &\sum_{i=0}^{l-j-2} \left\| \partial_t^j \partial_r^i \zeta \right\|_{L^\infty(\mathcal{I}_o)}^2 \lesssim \sum_{i=0}^{l-j-2} \left\| \partial_t^j \partial_r^i \zeta \right\|_{H^1(\mathcal{I}_o)}^2 \lesssim \\ &\left\| \partial_t^j \zeta \right\|_{H^{l-j-1}(\mathcal{I}_o)}^2 \lesssim (1+t)^{-2j} \mathcal{E}(t). \end{aligned} \tag{3.82}$$

It gives from (3.79), (3.80) and (3.81) that

$$\begin{aligned} &\left\| r \partial_t^j \partial_r^{l-j-1} \zeta \right\|_{L^\infty(\mathcal{I}_o)}^2 \lesssim \left\| r \partial_t^j \partial_r^{l-j-1} \zeta \right\|_{H^1(\mathcal{I}_o)}^2 \lesssim (1+t)^{-2j} \mathcal{E}(t), \\ &\quad j = 0, 1, \cdots, l-1, \end{aligned} \tag{3.83}$$

$$\left\| r^2 \partial_t^j \partial_r^{l-j} \zeta \right\|_{L^\infty(\mathcal{I}_o)}^2 \lesssim \left\| r^2 \partial_t^j \partial_r^{l-j} \zeta \right\|_{H^1(\mathcal{I}_o)}^2 \lesssim (1+t)^{-2j} \mathcal{E}(t), \quad j = 0, 1, \ldots, l. \tag{3.84}$$

So that we can derive (3.75) from (3.82)–(3.84).

*Step 2* (away from the origin). We set

$$d_b(r) := dist(r, \partial \mathcal{I}_b) \leqq \sqrt{A/B} - r \lesssim \sigma(r), \quad r \in \mathcal{I}_b. \tag{3.85}$$

It follows from (3.4) and (3.85) that for $j \leqq 5+[\alpha]-\alpha$,

$$\begin{aligned}
&\left\|\partial_t^j \zeta\right\|^2_{H^{\frac{5-j+[\alpha]-\alpha}{2}}(\mathcal{I}_b)} = \left\|\partial_t^j \zeta\right\|^2_{H^{l-j+1-\frac{l-j+1+\alpha}{2}}(\mathcal{I}_b)} \lesssim \left\|\partial_t^j \zeta\right\|^2_{H^{l-j+1+\alpha,l-j+1}(\mathcal{I}_b)} \\
&= \sum_{k=0}^{l-j+1} \int_{\mathcal{I}_b} d_b^{\alpha+1+l-j}(r)|\partial_r^k \partial_t^j \zeta|^2 \mathrm{d}r \lesssim \sum_{k=0}^{l-j+1} \int_{\mathcal{I}_b} \sigma^{\alpha+1+l-j}|\partial_r^k \partial_t^j \zeta|^2 \mathrm{d}r \\
&\lesssim \sum_{k=0}^{l-j+1} \int_{\mathcal{I}_b} r^4 \sigma^{\alpha+k}|\partial_r^k \partial_t^j \zeta|^2 \mathrm{d}r \\
&\leqq (1+t)^{-2j}\left(\mathcal{E}_j(t)+\sum_{k=1}^{l-j}\mathcal{E}_{j,k}(t)\right) \leqq (1+t)^{-2j}\mathcal{E}(t).
\end{aligned}$$

This, together with (3.79), gives (3.76).

To prove (3.77), we denote $\psi := \sigma^{\frac{2i+j-3}{2}}\partial_t^j \partial_r^i \zeta$. In what follows, we assume $2i+j \geqq 4$ and $i+j \leqq l$ and show that

$$\|\psi\|^2_{L^\infty(\mathcal{I}_b)} \lesssim (1+t)^{-2j}\mathcal{E}(t). \tag{3.86}$$

The estimate (3.86) will be proved by separating the cases when $\alpha$ is or is not an integer.

*Case 1* ($\alpha \neq [\alpha]$). When $\alpha$ is not an integer, we choose $\sigma^{2(l-i-j)+\alpha-[\alpha]}$ as the spatial weight. A simple calculation yields

$$\begin{aligned}
&|\partial_r \psi| \lesssim \left|\sigma^{\frac{2i+j-3}{2}}\partial_t^j \partial_r^{i+1}\zeta\right| + \left|\sigma^{\frac{2i+j-3}{2}-1}\partial_t^j \partial_r^i \zeta\right|, \\
&\left|\partial_r^2 \psi\right| \lesssim \left|\sigma^{\frac{2i+j-3}{2}}\partial_t^j \partial_r^{i+2}\zeta\right| + \left|\sigma^{\frac{2i+j-3}{2}-1}\partial_t^j \partial_r^{i+1}\zeta\right| + \left|\sigma^{\frac{2i+j-3}{2}-2}\partial_t^j \partial_r^i \zeta\right|, \\
&\cdots\cdots \\
&\left|\partial_r^k \psi\right| \lesssim \sum_{p=0}^{k}\left|\sigma^{\frac{2i+j-3}{2}-p}\partial_t^j \partial_r^{i+k-p}\zeta\right| \quad \text{for } k=1,2,\ldots,l+1-j-i.
\end{aligned} \tag{3.87}$$

It follows from (3.87) that for $1 \leqq k \leqq l+1-i-j$,

$$\begin{aligned}
&\int_{\mathcal{I}_b} \sigma^{2(l-i-j)+\alpha-[\alpha]}\left|\partial_r^k \psi\right|^2 \mathrm{d}r \lesssim \sum_{p=0}^{k}\int_{\mathcal{I}_b}\sigma^{\alpha+l-j+1-2p}\left|\partial_t^j \partial_r^{i+k-p}\zeta\right|^2 \mathrm{d}r \\
&\lesssim \int_{\mathcal{I}_b}\sigma^{l-i-j+1-k}\sum_{p=0}^{1}\sigma^{\alpha+i+k-2p}\left|\partial_t^j \partial_r^{i+k-p}\zeta\right|^2 \mathrm{d}r \\
&\quad + \sum_{p=2}^{k}\int_{\mathcal{I}_b}\sigma^{\alpha+l-j+1-2p}\left|\partial_t^j \partial_r^{i+k-p}\zeta\right|^2 \mathrm{d}r \\
&\lesssim \sum_{p=0}^{1}\int_{\mathcal{I}_b} r^4 \sigma^{\alpha+i+k-2p}\left|\partial_t^j \partial_r^{i+k-p}\zeta\right|^2 \mathrm{d}r
\end{aligned}$$

$$+\sum_{p=2}^{k}\int_{\mathcal{I}_b}\sigma^{\alpha+l-j+1-2p}\left|\partial_t^j\partial_r^{i+k-p}\zeta\right|^2\mathrm{d}r$$
$$\lesssim(1+t)^{-2j}\mathcal{E}_{j,i+k-1}+\sum_{p=2}^{k}\int_{\mathcal{I}_b}\sigma^{\alpha+l-j+1-2p}\left|\partial_t^j\partial_r^{i+k-p}\zeta\right|^2\mathrm{d}r.$$

To bound the 2nd term on the right-hand side of the inequality above, notice that

$$\begin{aligned}\alpha+l-j+1-2p&=2(l+1-i-j-k)+2(k-p)\\&+(\alpha-[\alpha])+(2i+j-4)-1>-1\end{aligned}\tag{3.88}$$

for $p\in[2,k]$, due to $\alpha>[\alpha]$ and $2i+j\geqq 4$. We then have, with the aid of the Hardy inequality (3.6), that for $p\in[2,k]$,

$$\begin{aligned}&\int_{\mathcal{I}_b}\sigma^{\alpha+l-j+1-2p}\left|\partial_t^j\partial_r^{i+k-p}\zeta\right|^2\mathrm{d}r\\
&\lesssim\int_{\mathcal{I}_b}\sigma^{\alpha+l-j+1-2p+2}\sum_{\iota=0}^{1}\left|\partial_t^j\partial_r^{i+k-p+\iota}\zeta\right|^2\mathrm{d}r\lesssim\cdots\\
&\lesssim\int_{\mathcal{I}_b}\sigma^{\alpha+l-j+1}\sum_{\iota=0}^{p}\left|\partial_t^j\partial_r^{i+k-p+\iota}\zeta\right|^2\mathrm{d}r\\
&=\sum_{\iota=0}^{p}\int_{\mathcal{I}_b}\sigma^{(l+1-i-j-k)+(p-\iota)}\sigma^{\alpha+i+k-p+\iota}\left|\partial_t^j\partial_r^{i+k-p+\iota}\zeta\right|^2\mathrm{d}r\\
&\lesssim\sum_{\iota=0}^{p}\int_{\mathcal{I}_b}r^4\sigma^{\alpha+i+k-p+\iota}\left|\partial_t^j\partial_r^{i+k-p+\iota}\zeta\right|^2\mathrm{d}r\leqq\sum_{\iota=i+k-p}^{i+k-1}(1+t)^{-2j}\mathcal{E}_{j,\iota}.\end{aligned}$$

That yields, for $1\leqq k\leqq l+1-i-j$,

$$\begin{aligned}&\int_{\mathcal{I}_b}\sigma^{2(l-i-j)+\alpha-[\alpha]}\left|\partial_r^k\psi\right|^2\mathrm{d}r\lesssim(1+t)^{-2j}\mathcal{E}_{j,i+k-1}\\
&+\sum_{p=2}^{k}\sum_{\iota=i+k-p}^{i+k-1}(1+t)^{-2j}\mathcal{E}_{j,\iota}\lesssim(1+t)^{-2j}\sum_{\iota=i}^{i+k-1}\mathcal{E}_{j,\iota}.\end{aligned}$$

Therefore, it follows from (3.85) and (3.78) that

$$\begin{aligned}&\|\psi\|^2_{H^{2(l-i-j)+\alpha-[\alpha],\ l+1-i-j}(\mathcal{I}_b)}=\sum_{k=0}^{l+1-i-j}\int_{\mathcal{I}_b}d_b^{2(l-i-j)+\alpha-[\alpha]}\left|\partial_r^k\psi\right|^2\mathrm{d}r\\
&\lesssim\sum_{k=0}^{l+1-i-j}\int_{\mathcal{I}_b}\sigma^{2(l-i-j)+\alpha-[\alpha]}\left|\partial_r^k\psi\right|^2\mathrm{d}r\lesssim\int_{\mathcal{I}_b}\sigma^{\alpha+l-j+1}\left|\partial_t^j\partial_r^i\zeta\right|^2\mathrm{d}r\\
&+(1+t)^{-2j}\sum_{\iota=i}^{l-j}\mathcal{E}_{j,\iota}\end{aligned}$$

$$\lesssim \int_{\mathcal{I}_b} r^4 \sigma^{\alpha+i+1} \left| \partial_t^j \partial_r^i \zeta \right|^2 \mathrm{d}r + (1+t)^{-2j} \sum_{\iota=i}^{l-j} \mathcal{E}_{j,\iota}$$
$$\lesssim (1+t)^{-2j} \sum_{\iota=i}^{l-j} \mathcal{E}_{j,\iota} \leqq (1+t)^{-2j} \mathcal{E}(t).$$

When $\alpha$ is not an integer, $\alpha - [\alpha] \in (0, 1)$. So, it follows from (3.79) and (3.4) that

$$\|\psi\|^2_{L^\infty(\mathcal{I}_b)} \lesssim \|\psi\|^2_{H^{1-\frac{\alpha-[\alpha]}{2}}(\mathcal{I}_b)} \lesssim \|\psi\|^2_{H^{2(l-i-j)+\alpha-[\alpha],\ l+1-i-j}(\mathcal{I}_b)} \lesssim (1+t)^{-2j} \mathcal{E}(t). \tag{3.89}$$

*Case 2* ($\alpha = [\alpha]$). In this case $\alpha$ is an integer, we choose $\sigma^{2(l-i-j)+1/2}$ as the spatial weight. As shown in Case 1, we have for $1 \leqq k \leqq l+1-i-j$,

$$\int_{\mathcal{I}_b} \sigma^{2(l-i-j)+1/2} \left| \partial_r^k \psi \right|^2 \mathrm{d}r \lesssim (1+t)^{-2j} \mathcal{E}_{j,i+k-1}$$
$$+ \sum_{p=2}^{k} \int_{\mathcal{I}_b} \sigma^{\alpha+l-j+1-2p+1/2} \left| \partial_t^j \partial_r^{i+k-p} \zeta \right|^2 \mathrm{d}r.$$

Note that for $1 \leqq k \leqq l+1-i-j$ and $2 \leqq p \leqq k$,

$$\alpha + l - j + 1 - 2p + \frac{1}{2} = 2(l+1-i-j-k) + 2(k-p)$$
$$+ (2i + j - 4) - \frac{1}{2} \geqq -\frac{1}{2}.$$

We can then use the Hardy inequality (3.6) to obtain

$$\int_{\mathcal{I}_b} \sigma^{2(l-i-j)+1/2} \left| \partial_r^k \psi \right|^2 \mathrm{d}r \lesssim (1+t)^{-2j} \sum_{\iota=i}^{i+k-1} \mathcal{E}_{j,\iota},\ k = 1, 2, \ldots, l-j+1-i,$$

which, together with (3.85) and (3.78), implies that

$$\|\psi\|^2_{H^{2(l-i-j)+1/2,\ l+1-i-j}(\mathcal{I}_b)} \lesssim (1+t)^{-2j} \sum_{\iota=i}^{l-j} \mathcal{E}_{j,\iota} \leqq (1+t)^{-2j} \mathcal{E}(t).$$

Therefore, it follows from (3.79) and (3.4) that

$$\|\psi\|^2_{L^\infty(\mathcal{I}_b)} \lesssim \|\psi\|^2_{H^{3/4}(\mathcal{I}_b)} \lesssim \|\psi\|^2_{H^{2(l-i-j)+1/2,\ l+1-i-j}(\mathcal{I}_b)} \lesssim (1+t)^{-2j} \mathcal{E}(t). \tag{3.90}$$

In view of (3.89) and (3.90), we obtain (3.86) or equivalently (3.77). □

## 4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2. First, it follows from (2.3), (2.6), (1.7), (2.9), (2.2) and (2.8) that for $(r, t) \in \mathcal{I} \times [0, \infty)$,

$$\rho(\eta(r,t),t) - \bar{\rho}(\bar{\eta}(r,t),t) = \frac{r^2\bar{\rho}_0(r)}{\eta^2(r,t)\eta_r(r,t)} - \frac{r^2\bar{\rho}_0(r)}{\bar{\eta}^2(r,t)\bar{\eta}_r(r,t)},$$
$$u(\eta(r,t),t) - \bar{u}(\bar{\eta}(r,t),t) = \eta_t(r,t) - \bar{\eta}_t(r,t).$$

Then, we have, using (2.15), (2.11), (2.9), (3.2), (2.14) and (2.17) that

$$|\rho(\eta(r,t),t) - \bar{\rho}(\bar{\eta}(r,t),t)| \lesssim (A - Br^2)^{\frac{1}{\gamma-1}}(1+t)^{-\frac{4}{3\gamma-1}} \times \left[\sqrt{\mathcal{E}(0)} + (1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\ln(1+t)\right],$$
$$|u(\eta(r,t),t) - \bar{u}(\bar{\eta}(r,t),t)| \lesssim r(1+t)^{-1}\left[\sqrt{\mathcal{E}(0)} + (1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\ln(1+t)\right].$$

This gives the proof of (2.18) and (2.19).

For the boundary behavior, it follows from (2.5), (2.15), (2.11) and (2.9) that

$$\begin{aligned} R(t) =& \eta\left(\sqrt{A/B}, t\right) = (\tilde{\eta} + r\zeta)\left(\sqrt{A/B}, t\right) = (\bar{\eta} + rh + r\zeta)\left(\sqrt{A/B}, t\right) \\ =& [r(\bar{\eta}_r + h + \zeta)]\left(\sqrt{A/B}, t\right) = \sqrt{A/B}\left[(1+t)^{1/(3\gamma-1)} + h(t) + \zeta(t)\right] \end{aligned}$$

which, together with (2.17) and (2.14), gives that

$$R(t) \geqq \sqrt{A/B}\left[(1+t)^{\frac{1}{3\gamma-1}} - C\sqrt{\mathcal{E}(0)}\right],$$
$$R(t) \leqq \sqrt{A/B}\left[(1+t)^{\frac{1}{3\gamma-1}} + C(1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\ln(1+t) + C\sqrt{\mathcal{E}(0)}\right].$$

Thus, (2.20) follows from the smallness of $\mathcal{E}(0)$. Notice that for $k = 1, 2, 3$,

$$\frac{\mathrm{d}^k R(t)}{\mathrm{d}t^k} = \partial_t^k \tilde{\eta}\left(\sqrt{A/B}, t\right) + \left(r\partial_t^k \zeta\right)\left(\sqrt{A/B}, t\right).$$

Therefore, (2.21) follows from (2.13) and (2.17).

We are to verify the physical vacuum condition, (2.22). It follows from (2.3), (2.6), (2.15) that

$$\begin{aligned} \left(\rho^{\gamma-1}\right)_\eta(\eta,t) =& \frac{\left(f^{\gamma-1}\right)_r(r,t)}{\eta_r(r,t)} = \frac{1}{\eta_r}\left[\bar{\rho}_0^{\gamma-1}\left(\frac{r^2}{\eta^2\eta_r}\right)^{\gamma-1}\right]_r \\ =& (1-\gamma)\bar{\rho}_0^{\gamma-1}\left[2\left(\frac{\eta}{r}\right)^{1-2\gamma}\eta_r^{-\gamma}\zeta_r + \left(\frac{\eta}{r}\right)^{2-2\gamma}\eta_r^{-\gamma-1}(2\zeta_r + r\zeta_{rr})\right] \\ & - 2Br\left(\frac{\eta}{r}\right)^{2-2\gamma}\eta_r^{-\gamma}, \end{aligned}$$

which implies, with the aid of (3.3) and (2.17), that

$$\left|\left(\rho^{\gamma-1}\right)_\eta(\eta,t)\right| \lesssim (1+t)^{-1}\sqrt{\mathcal{E}(0)} + r(1+t)^{-1+\frac{1}{3\gamma-1}}, \tag{4.1}$$

$$\begin{aligned}\left|\left(\rho^{\gamma-1}\right)_\eta(\eta,t)\right| &\geqq 2Br\left(\frac{\eta}{r}\right)^{2-2\gamma}\eta_r^{-\gamma} - C(1+t)^{-1}\sqrt{\mathcal{E}(0)} \\ &\geqq C^{-1}r(1+t)^{-1+\frac{1}{3\gamma-1}} - C(1+t)^{-1}\sqrt{\mathcal{E}(0)}.\end{aligned} \tag{4.2}$$

In view of (3.3), we see that $\eta(r,t) \sim (1+t)^{1/(3\gamma-1)}r$, which, together with (2.20), (4.1) and (4.2), gives for $R(t)/2 \leqq \eta \leqq R(t)$,

$$\begin{aligned}C^{-1}(1+t)^{-\frac{3\gamma-2}{3\gamma-1}} - C(1+t)^{-1}\sqrt{\mathcal{E}(0)} &\leqq \left|\left(\rho^{\gamma-1}\right)_\eta(\eta,t)\right| \\ &\lesssim (1+t)^{-1}\sqrt{\mathcal{E}(0)} + (1+t)^{-\frac{3\gamma-2}{3\gamma-1}}.\end{aligned}$$

Thus, (2.22) follows from the smallness of $\mathcal{E}(0)$. This finishes the proof of Theorem 2.2. □

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## Appendix

**Proof of (2.13a).** We may write (2.10) as the following system:

$$\begin{aligned}&h_t = z, \quad z_t = -z - \left[\bar{\eta}_r^{2-3\gamma} - (\bar{\eta}_r + h)^{2-3\gamma}\right]/(3\gamma-1) - \bar{\eta}_{rtt}, \\ &(h,z)(t=0) = (0,0).\end{aligned} \tag{A-1}$$

Recalling that $\bar{\eta}_r(t) = (1+t)^{1/(3\gamma-1)}$, thus $\bar{\eta}_{rtt} < 0$. A simple phase plane analysis shows that there exist $0 < t_0 < t_1 < t_2$ such that, starting from $(h,z) = (0,0)$ at $t = 0$, $h$ and $z$ increases in the interval $[0, t_0]$ and $z$ reaches its positive maxima at $t_0$; in the interval $[t_0, t_1]$, $h$ keeps increasing and reaches its maxima at $t_1$, $z$ decreases from its positive maxima to 0; in the interval $[t_1, t_2]$, both $h$ and $z$ decrease, and $z$ reaches its negative minima at $t_2$; in the interval $[t_2, \infty)$, $h$ decreases and $z$ increases, and $(h,z) \to (0,0)$ as $t \to \infty$. This can be summarized as follows:

$$\begin{aligned}&z(t)\uparrow_0, \quad h(t)\uparrow_0, \quad t\in[0,t_0]; \quad z(t)\downarrow_0, \quad h(t)\uparrow, \quad t\in[t_0,t_1]; \\ &z(t)\downarrow^0, \quad h(t)\downarrow, \quad t\in[t_1,t_2]; \quad z(t)\uparrow^0, \quad h(t)\downarrow_0, \quad t\in[t_2,\infty).\end{aligned}$$

We have from the above analysis that there exists a finite constant $C = C(\gamma, M)$ such that

$$0 \leqq h(t) \leqq C \quad \text{for} \quad t \geqq 0. \tag{A-2}$$

In view of (2.9) and (2.11), we then see that $(1+t)^{1/(3\gamma-1)} \leqq \tilde{\eta}_r(t) \leqq K(1+t)^{1/(3\gamma-1)}$. On the other hand, equation (2.10) can be rewritten as

$$\begin{aligned} &\tilde{\eta}_{rtt} + \tilde{\eta}_{rt} - \tilde{\eta}_r^{2-3\gamma}/(3\gamma-1) = 0, \quad t > 0, \\ &\tilde{\eta}_r(t=0) = 1, \quad \tilde{\eta}_{rt}(t=0) = 1/(3\gamma-1). \end{aligned} \tag{A-3}$$

Then, we have by solving (A-3), that

$$\tilde{\eta}_{rt}(t) = \frac{1}{3\gamma-1}e^{-t} + \frac{1}{3\gamma-1}\int_0^t e^{-(t-s)}\tilde{\eta}_r^{2-3\gamma}(s)\mathrm{d}s \geqq 0. \tag{A-4}$$

□

**Proof of (2.13b).** We use mathematical induction to prove (2.13b). First, it follows from (A-4) that

$$\begin{aligned} (3\gamma-1)\tilde{\eta}_{rt}(t) &= e^{-t} + \int_0^{t/2} e^{-(t-s)}\tilde{\eta}_r^{2-3\gamma}(s)\mathrm{d}s + \int_{t/2}^t e^{-(t-s)}\tilde{\eta}_r^{2-3\gamma}(s)\mathrm{d}s \\ &\leqq e^{-t} + e^{-t/2}\int_0^{t/2}(1+s)^{\frac{2-3\gamma}{3\gamma-1}}\mathrm{d}s + (1+\ t/2)^{\frac{2-3\gamma}{3\gamma-1}}\int_{t/2}^t e^{-(t-s)}\mathrm{d}s \\ &\leqq e^{-t} + Ce^{-t/2}(1+t/2)^{\frac{1}{3\gamma-1}} + (1+t/2)^{\frac{2-3\gamma}{3\gamma-1}} \leqq C(1+t)^{\frac{2-3\gamma}{3\gamma-1}}, \quad t \geqq 0, \end{aligned} \tag{A-5}$$

for some constant $C$ independent of $t$. This proves (2.13b) when $k=1$. Suppose that (2.13b) holds for all $k = 1, 2, \cdots, m-1$, that is,

$$\left|\frac{d^k\tilde{\eta}_r(t)}{\mathrm{d}t^k}\right| \leqq C(m)(1+t)^{\frac{1}{3\gamma-1}-k}, \quad k = 1, 2, \ldots, m-1. \tag{A-6}$$

It suffices to prove that (2.13b) holds for $k=m$. We derive from (A-3) that for $m = 1, \ldots, k$,

$$\frac{\mathrm{d}^{m+1}}{\mathrm{d}t^{m+1}}\tilde{\eta}_r(t) + \frac{\mathrm{d}^m}{\mathrm{d}t^m}\tilde{\eta}_r(t) - \frac{1}{3\gamma-1}\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\tilde{\eta}_r^{2-3\gamma}(t) = 0, \quad t \geqq 0,$$

so that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\tilde{\eta}_r(t) = e^{-t}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\tilde{\eta}_r(0) + \frac{1}{3\gamma-1}\int_0^t e^{-(t-s)}\frac{d^{m-1}\tilde{\eta}_r^{2-3\gamma}}{\mathrm{d}s^{m-1}}(s)\mathrm{d}s, \quad t \geqq 0, \tag{A-7}$$

where $(\mathrm{d}^m/\mathrm{d}t^m)\tilde{\eta}_r(0)$ is finite, which can be determined by the equation inductively. In view of (2.13a) and (A-6), we see that

$$\begin{aligned} &\left|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\eta}_r^{2-3\gamma}(t)\right| \lesssim \left|\tilde{\eta}_r^{1-3\gamma}(t)\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\eta}_r(t)\right| \lesssim (1+t)^{\frac{1}{3\gamma-1}-2}, \\ &\qquad \cdots\cdots \\ &\left|\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\tilde{\eta}_r^{2-3\gamma}(t)\right| \leqq C(\gamma, m)(1+t)^{\frac{1}{3\gamma-1}-m}. \end{aligned} \tag{A-8}$$

Similar to deriving (A-5), we can obtain, noting (A-7) and (A-8), that

$$\left|\frac{\mathrm{d}^m \tilde{\eta}_r(t)}{\mathrm{d}t^m}\right| \leqq C(\gamma, m)\,(1+t)^{\frac{1}{3\gamma-1}-m}\,.$$

□

**Proof of (2.14).** We may write the equation for $h$, (2.10), as

$$h_t + \frac{1}{3\gamma-1}(1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\left[1-\left(1+(1+t)^{-\frac{1}{3\gamma-1}}h\right)^{2-3\gamma}\right] = -\tilde{\eta}_{rtt}, \quad t>0. \tag{A-9}$$

Notice that

$$\left(1+(1+t)^{-\frac{1}{3\gamma-1}}h\right)^{2-3\gamma} \leqq 1+(2-3\gamma)(1+t)^{-\frac{1}{3\gamma-1}}h + \frac{(2-3\gamma)(1-3\gamma)}{2}(1+t)^{-\frac{2}{3\gamma-1}}h^2,$$

due to the fact that $h \geqq 0$. We then obtain, in view of (2.13b), that

$$h_t + \frac{3\gamma-2}{3\gamma-1}(1+t)^{-1}h \leqq \frac{3\gamma-2}{2}(1+t)^{-\frac{3\gamma}{3\gamma-1}}h^2 + C(1+t)^{\frac{1}{3\gamma-1}-2},$$

So

$$h(t) \leqq C(1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\int_0^t \left((1+s)^{-\frac{2}{3\gamma-1}}h^2(s) + (1+s)^{-1}\right)\mathrm{d}s. \tag{A-10}$$

We use an iteration to prove (2.14). First, since $h$ is bounded due to (A-2), we have

$$h(t) \leqq C(1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\int_0^t (1+s)^{-\frac{2}{3\gamma-1}}\mathrm{d}s \leqq C(1+t)^{-\frac{1}{3\gamma-1}}. \tag{A-11}$$

Substituting this into (A-10), we obtain

$$h(t) \leqq C(1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\int_0^t \left((1+s)^{-\frac{4}{3\gamma-1}} + (1+s)^{-1}\right)\mathrm{d}s,$$

which implies $h(t) \leqq C(1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\ln(1+t)$ if $\gamma \leqq 5/3$, $h(t) \leqq C(1+t)^{-\frac{3}{3\gamma-1}}$ if $\gamma > 5/3$. If $\gamma \leqq 5/3$, then the first part of (2.14) has been proved. If $\gamma > 5/3$, we repeat this procedure and obtain

$$h(t) \leqq C(1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\int_0^t \left((1+s)^{-\frac{8}{3\gamma-1}} + (1+s)^{-1}\right)\mathrm{d}s,$$

which implies $h(t) \leqq C(1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\ln(1+t)$ if $\gamma \leqq 3$, $h(t) \leqq C(1+t)^{-\frac{7}{3\gamma-1}}$ if $\gamma > 3$. For general $\gamma$, we repeat this procedure $k$ times to obtain $h(t) \leqq C(1+t)^{-\frac{3\gamma-2}{3\gamma-1}}\ln(1+t)$. This, together with (A-2), proves the first part of (2.14), which in turn implies the second part of (2.14), by virtue of (A-9) and (2.13b). □

**Proof of** (3.27)**.** Recall that $j \geqq 0$, $i \geqq 1$ and $i + j \leqq l$. Let $n \in [0, j]$, $m \in [0, i-1]$ and $q \in [0, n]$ be integers. Denote

$$\mathcal{H} := \Big\| r\sigma^{\frac{\alpha+i-1}{2}} \left( \left| r\partial_t^{n-q}\partial_r^{m+1}\zeta \right| + \left| \partial_t^{n-q}\partial_r^{m}\zeta \right| \right) \left( \left| \sigma r\partial_t^{j-n}\partial_r^{i-m+1}\zeta \right| + \sum_{\iota=0}^{i-m} \left| \partial_t^{j-n}\partial_r^{\iota}\zeta \right| \right) \Big\|^2 .$$

*Case 1*. Assume $2n + 4m \geqq 2i + j + q$. We first note that

$$\alpha + (2m+n) - (i+j) + 2 \geqq \alpha - \frac{j}{2} + \frac{q}{2} + 2 \geqq \alpha - \frac{l-1}{2} + 2 \geqq 0, \tag{A-12}$$

$$i + j - (n+m) \leqq l - 2. \tag{A-13}$$

(Indeed, if $i + j - (n+m) = l$, then $i + j = l$ and $n + m = 0$, so that it is a contradiction due to $0 = 4(n+m) \geqq 2n + 4m \geqq 2i + j + q \geqq i + j = l$; if $i + j - (n+m) = l - 1$, then $i + j = l - 1$ and $n + m = 0$ or $i + j = l$ and $n + m = 1$, so that it is also a contradiction because of $0 = 4(n+m) \geqq 2n + 4m \geqq 2i + j + q \geqq i + j = l - 1 > 0$ or $4 = 4(n+m) \geqq 2n + 4m \geqq 2i + j + q \geqq i + i + j \geqq 1 + l = 5 + [\alpha] \geqq 5$. So, (A-13) holds.)
When $2i + j \leqq 2m + n + 3$, it follows from (3.1) and (A-13) that

$$\begin{aligned} \mathcal{H} &\lesssim \varepsilon_0^2 (1+t)^{2n-2j} \Big\| r\sigma^{\frac{\alpha+i-1}{2}} \left( \left| r\partial_t^{n-q}\partial_r^{m+1}\zeta \right| + \left| \partial_t^{n-q}\partial_r^{m}\zeta \right| \right) \Big\|^2 \\ &\lesssim \varepsilon_0^2 (1+t)^{2q-2j} \left( \mathcal{E}_{n-q,m+1} + \mathcal{E}_{n-q,m} \right). \end{aligned} \tag{A-14}$$

When $2i + j \geqq 2m + n + 4$, it follows from (3.1) and (A-13) that

$$\begin{aligned} \mathcal{H} &\lesssim \varepsilon_0^2 (1+t)^{2n-2j} \Big\| r\sigma^{\frac{\alpha+i-1}{2} - \frac{j+2i-(n+2m)-3}{2}} \left( \left| r\partial_t^{n-q}\partial_r^{m+1}\zeta \right| + \left| \partial_t^{n-q}\partial_r^{m}\zeta \right| \right) \Big\|^2 \\ &= \varepsilon_0^2 (1+t)^{2n-2j} \Big\| r\sigma^{\frac{\alpha+m+(n+m-(i+j)+2)}{2}} \left( \left| r\partial_t^{n-q}\partial_r^{m+1}\zeta \right| + \left| \partial_t^{n-q}\partial_r^{m}\zeta \right| \right) \Big\|^2, \end{aligned}$$

which implies for $n + m - (i+j) + 2 \geqq 0$ that

$$\begin{aligned} \mathcal{H} &\lesssim \varepsilon_0^2 (1+t)^{2n-2j} \Big\| r\sigma^{\frac{\alpha+m}{2}} \left( \left| r\partial_t^{n-q}\partial_r^{m+1}\zeta \right| + \left| \partial_t^{n-q}\partial_r^{m}\zeta \right| \right) \Big\|^2 \\ &\lesssim \varepsilon_0^2 (1+t)^{2q-2j} \left( \mathcal{E}_{n-q,m+1} + \mathcal{E}_{n-q,m} \right), \end{aligned} \tag{A-15}$$

and for $n + m - (i+j) + 2 \leqq -1$, that

$$\begin{aligned} \mathcal{H} &\lesssim \varepsilon_0^2 (1+t)^{2n-2j} \\ &\Big( \Big\| r^2\partial_t^{n-q}\partial_r^{m+1}\zeta \Big\|^2_{L^2(\mathcal{I}_o)} + \Big\| r\partial_t^{n-q}\partial_r^{m}\zeta \Big\|^2_{L^2(\mathcal{I}_o)} + \Big\| \sigma^{\frac{\alpha+m+(n+m-(i+j)+2)}{2}} \\ &\times \left( \left| \partial_t^{n-q}\partial_r^{m+1}\zeta \right| + \left| \partial_t^{n-q}\partial_r^{m}\zeta \right| \right) \Big\|^2_{L^2(\mathcal{I}_b)} \Big) \lesssim \varepsilon_0^2 (1+t)^{2q-2j} \sum_{h=m}^{i+j-n+q-1} \mathcal{E}_{n-q,h}. \end{aligned} \tag{A-16}$$

Here we have used (A-12) and the Hardy inequality (3.6) to derive

$$\begin{aligned}
&\left\| \sigma^{\frac{\alpha+m+(n+m-(i+j)+2)}{2}} \left( \left| \partial_t^{n-q} \partial_r^{m+1} \zeta \right| + \left| \partial_t^{n-q} \partial_r^{m} \zeta \right| \right) \right\|_{L^2(\mathcal{I}_b)}^2 \\
&\lesssim \sum_{h=0}^{1} \left\| \sigma^{\frac{\alpha+m+(n+m-(i+j)+2)}{2}} \partial_t^{n-q} \partial_r^{m+h} \zeta \right\|_{L^2(\mathcal{I}_b)}^2 \\
&\lesssim \sum_{h=0}^{2} \left\| \sigma^{\frac{\alpha+m+(n+m-(i+j)+2)+2}{2}} \partial_t^{n-q} \partial_r^{m+h} \zeta \right\|_{L^2(\mathcal{I}_b)}^2 \\
&\lesssim \cdots \lesssim \sum_{h=0}^{i+j-(n+m)+q} \left\| \sigma^{\frac{\alpha+m+(n+m-(i+j)+2)+2(i+j-(n+m)+q-1)}{2}} \partial_t^{n-q} \partial_r^{m+h} \zeta \right\|_{L^2(\mathcal{I}_b)}^2 \\
&\lesssim \sum_{h=0}^{i+j-(n+m)+q} \left\| r^2 \sigma^{\frac{\alpha+i+j-n+q}{2}} \partial_t^{n-q} \partial_r^{m+h} \zeta \right\|_{L^2(\mathcal{I}_b)}^2 \\
&\lesssim (1+t)^{2q-2n} \sum_{h=m}^{i+j-n+q-1} \mathcal{E}_{n-q,h},
\end{aligned}$$

which implies (A-16). Therefore, we have from (A-14), (A-15) and (A-16) that

$$\mathcal{H} \lesssim \varepsilon_0^2 (1+t)^{2q-2j} \left( \mathcal{E}_{n-q,m} + \mathcal{E}_{n-q,m+1} + \sum_{h=m}^{i+j-n+q-1} \mathcal{E}_{n-q,h} \right). \quad \text{(A-17)}$$

*Case 2*. Assume $2n+4m < 2i+j+q$. In this case, we can use a means similar to the way in which we dealt with case 1 to obtain $\mathcal{H} \lesssim \varepsilon_0^2 (1+t)^{2q-2j} \sum_{h=0}^{i+n-j} \mathcal{E}_{j-n,h}(t)$. This, together with (A-17), gives (3.27). □

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