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Simple Choreographies of the Planar Newtonian N-Body Problem

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Abstract

In the N-body problem, a simple choreography is a periodic solution, where all masses chase each other on a single loop. In this paper we prove that for the planar Newtonian N-body problem with equal masses, $N \geqq 3$, there are at least $2^{N-3}+2^{[(N-3)/2]}$ different main simple choreographies. This confirms a conjecture given by Chenciner et al. (Geometry, mechanics, and dynamics. Springer, New York, pp 287–308, 2002). All the simple choreoagraphies we prove belong to the linear chain family.

1. Introduction

The Newtonian N-body problem describes the motion of N point masses under the attraction of each other according to Newton's gravitational law. When all the masses are equal, there exist periodic solutions, where all the masses travel on a single loop (it is still an open problem whether such a solution exists when the masses are unequal [9]; throughout the paper we assume all the masses are equal and that $N \geqq 3$). Such solutions usually satisfy certain symmetric constraints, as if they are dancing according to certain choreographies; this inspired Carles Simó to name them *simple choreographies*.

A trivial example of simple choreographies is the *rotating N-gon*, where the N point masses form a regular N-gon at each moment and rotate rigidly around the center of mass at a constant angular velocity. The first non-trivial simple choreography is the now famous *Figure-Eight* solution, which was discovered numerically by Moore [23], and then independently and rigorously proved by Chenciner and Montgomery [13]. This remarkable solution immediately got a lot of attention and many simple choreographies were found numerically afterwards: the *Super-Eight* solution of the 4-body problem by Gerver, as well as several families of simple choreographies for different values of N by Simó [12,26]. Some more recent numerical discoveries can be found in [20].

On the other hand, to rigorously prove the existence of there choreographies is a much harder task. Following [13], the idea is to find a simple choreography as a minimizer of the Lagrangian action functional among loops satisfying certain constraints. For Newtonian potential, as already noticed by Poincaré, when two or more bodies collide, the action functional may still be finite. As a result, the desired minimizer may contain a collision which prevents it from being a real solution. This is the main obstacle to applying variational methods to the Newtonian N-body problem.

A lot of progress has been made to overcome this difficulty since the proof of the *Figure-Eight* solution. We briefly summarize them as follow:

- *Local deformation* assuming there is an isolated collision along the minimizing path, one tries to show that after a small deformation near the isolated collision, one gets a new path with strictly smaller action value, which gives a contradiction. For the details see [8,14,19,22,27,29]. In the last three references the existence of such a local deformation is implied implicitly through Marchal's average method.
- *Level estimate* one gives a sharp lower bound estimate of the action functional among all the collision paths in the admissible class and then tries to find a test path within the admissible class such that its action value is strictly smaller than the previous lower bound estimate. The Figure-Eight solution was originally proved in [13] using this method. Results obtained using this method can also be found in [2–7] and the references therein.

Despite of the above progress, to the best of our knowledge very few simple choreographies have been rigorously proved. Besides the Figure-Eight, the Super-Eight was first proved in [18] with a rigorous numerical method and then analytically in [25]. The best result so far was obtained by FERRARIO AND TERRACINI in [14], where they proved that there is at least one Fight-Eight type simple choreography for every odd N.

One way to bypass the difficulty have is to change the potential from a Newtonian to a *strong force* (for a precise definition see the end of Section 2). For a strong force potential, the action value of a path with any collision must be infinity. It was proven in [12] that there are infinitely many distinct main simple choreographies when the potential is a strong force. By a main simple choreography, we mean one that cannot can not be derived from a given simple choreography by some orthogonal transformation of the space, a reparameterization of time, a combination of both, or just by traveling around a given one multiple times.[1] For the Newtonian potential, based on numerical discoveries, [12] made the following conjecture:

Conjecture. *For every $N \geqq 3$, there is a main simple choreography solution for the equal mass Newtonian N-body problem different from the trivial circle one (i.e., the rotating N-gon). The number of such distinct main simple choreographies grows rapidly with N.*

[1] The definition of 'main simple choreography' in [12] is slightly different from ours, where the possibility of obtaining a simple choreography from a given one by the continuation of angular momentum was also considered.

Let $[\cdot]$ denote the integer part of a real number. We confirm this conjecture by proving the following result:

Main Theorem. *For every $N \geqq 3$, there are at least $2^{N-3} + 2^{[(N-3)/2]}$ different main simple choreographies for the equal mass Newtonian N-body problem.*

We also remark here that we are not considering those solutions that are simple choreographies in rotating coordinates. In such cases, the existence of infinitely many simple choreographies has already been established by Chenciner and Féjoz in [11].

In the rest of the paper, we will only consider the planar N-body problem. For simplicity, we also assume $m_j = 1, \forall j \in \mathbf{N} := \{0, 1, \dots, N-1\}$. If we let $z = (z_j)_{j\in\mathbf{N}} \in \mathbb{C}^N$ represent the positions of the masses, then it satisfies the following equations:

$$\ddot{z}_j = \sum_{k\in\mathbf{N}\setminus\{j\}} -\frac{z_j - z_k}{|z_j - z_k|^3}, \quad j \in \mathbf{N}. \tag{1}$$

This is the Euler–Lagrange equation of the action functional

$$\mathcal{A}(z, T_1, T_2) := \int_{T_1}^{T_2} L(z, \dot{z})\, \mathrm{d}t, \quad z \in H^1\left([T_1, T_2], \mathbb{C}^N\right),$$

with $L(z, \dot{z}) := K(\dot{z}) + U(z)$, where $K(\dot{z})$ is the kinetic energy and $U(x)$ is the (negative) potential energy:

$$K(\dot{z}) := \frac{1}{2}\sum_{j\in\mathbf{N}} |\dot{z}_j|^2, \qquad U(z) := \sum_{\{j<k\}\subset\mathbf{N}} \frac{1}{|z_j - z_k|}.$$

Furthermore, we set $\mathcal{A}(z, T) := \mathcal{A}(z, 0, T)$.

By the homogeneity of the potential, given a periodic solution, one can find another with any prescribed period. Because of this, we will only consider periodic solutions with a period N.

Following [12,13], the key idea here is to impose proper symmetric constraints on the loop space, then obtain the simple choreographies as collision-free minimizers. We recall this briefly in the following.

Let $\Lambda_N = H^1(\mathbb{R}/N\mathbb{Z}, \mathbb{C}^N)$ be the space of Sobolev loops and $\hat{\mathbb{C}}^N := \{z \in \mathbb{C}^N : z_j \neq z_k, \ \forall\{j \neq k\} \subset \mathbf{N}\}$ be the space of collision-free configurations. Then $\hat{\Lambda}_N = H^1(\mathbb{R}/N\mathbb{Z}, \hat{\mathbb{C}}^N)$ is the subspace of collision-free loops. Given a finite group G, we can define its action on Λ_N as the following:

$$g\big(z(t)\big) = \big(\rho(g)z_{\sigma(g^{-1})(0)}(\tau(g^{-1})t), \dots, \rho(g)z_{\sigma(g^{-1})(N-1)}(\tau(g^{-1})t)\big), \quad \forall g \in G, \tag{2}$$

where

(1) $\tau : G \to O(2)$ representing the action of G on the time circle $\mathbb{R}/N\mathbb{Z}$,
(2) $\rho : G \to O(2)$ representing the action of G on the 2-dim Euclid space,
(3) $\sigma : G \to \mathcal{S}_N$ representing the action of G on the index set $\mathbf{N}$.

$\Lambda_N^G := \{z \in \Lambda_N : g(z(t)) = z(t), \ \forall g \in G\}$ is the space of G-equivariant loops. As all masses are equal, the action functional $\mathcal{A}$ is invariant under the above group action. By the symmetric critical principle of PALAIS [24], a critical point of $\mathcal{A}$ in $\hat{\Lambda}_N^G$ ($\hat{\Lambda}_N^G := \Lambda_N^G \cap \hat{\Lambda}_N$) is also a critical point of it in $\hat{\Lambda}_N$, and therefore a solution of Eq. (1).

Consider the cyclical group $\mathbb{Z}_N = \langle g | \ g^N = 1 \rangle$ with the action

$$\tau(g)t = t - 1, \quad \rho(g) = \text{identity}, \quad \sigma(g) = (0, 1, \dots, N-1). \tag{3}$$

Then the following holds for any $z = (z_j)_{j \in \mathbf{N}} \in \Lambda_N^{\mathbb{Z}_N}$:

$$z_j(t) = z_0(j + t), \quad \forall t \in \mathbb{R}, \quad \forall j \in \mathbf{N}. \tag{4}$$

This means any collision-free critical point of $\mathcal{A}$ in $\Lambda_N^{\mathbb{Z}_N}$ must be a simple choreography. The most obvious critical point is of course the global minimizer. However as was proved in [1], the global minimizer in $\Lambda_N^{\mathbb{Z}_N}$ is nothing but the *rotating N-gon*.

A possible way to get non-trivial simple choreographies is to consider a group G with proper action, such that it contains $\mathbb{Z}_N$ with action given in (3) as a subgroup; meanwhile the symmetric constraints already rule out the rotating N-gon. This was exactly the idea used in [13, 14].

However the real gold deposits are located on the topological constraints. As we can see, the space of collision-free loops $\hat{\Lambda}_N^{\mathbb{Z}_N}$ has infinitely many connected components. A nice way of distinguishing these components is through the braids group; for details of this see the beautiful paper by sc Montgomery [22]. In principle, it is possible to find a main choreography solution in each connected component. For a strong force potential, this was indeed proved in [12].

Meanwhile for the Newtonian potential, it is much more difficult to show that a minimizer is collision-free when topological constraints are involved. First, *local deformation* usually fails in this case, because to be able to lower the action, the possible directions of local deformation are restricted, and in many cases one ends up in a topological class different from the required one. Second, although in principle, the *level estimate* should work under topological constraints, in practice, except for some special cases, it is very difficult to give an accurate lower bound estimate of the action values of the collision paths.

To overcome the difficulty caused by the topological constraints, we propose to introduce additional *monotone constraints* (see Definition 2.1). These constraints provide further information regarding the relative positions of the masses. With such information, first, we can rule out collisions involving more than two masses, and second, when there is a binary collision, it allows us to make certain the *global deformation* of the path in order to get a new one with a strictly lower action value and to reach a contradiction. We believe that this is the first time such an idea has been used in this classic problem. The details will be given in Section 2.

In the rest of this section, a proof of the Main Theorem will be given. Let's consider the dihedral group $D_N = \langle g, h | \ g^N = h^2 = 1, \ (gh)^2 = 1 \rangle$ with the action of g defined as in (3), and h as the following:

$$\tau(h)t = -t + 1, \quad \rho(h)q = \bar{q}, \quad \sigma(h) = (0, N-1)(1, N-2) \dots (\mathfrak{n}, N-1-\mathfrak{n}), \tag{5}$$

where $q \in \mathbb{C}$, $\bar{q}$ is its conjugate, and

$$\mathfrak{n} := [(N-1)/2]. \tag{6}$$

By the above definition, $\Lambda_N^{D_N} \subset \Lambda_N^{\mathbb{Z}_N}$. However, the rotating N-gon is also contained in $\Lambda_N^{D_N}$. Hence the minimizer of $\mathcal{A}$ in $\Lambda_N^{D_N}$ is again a trivial solution. In order to get non-trivial simple choreographies, topological constraints need to be added into the problem.

First, by the symmetric constraints, $z = (z_j)_{j \in \mathbf{N}} \in \Lambda_N^{D_N}$ must satisfy (4) and

$$z_j(t) = \bar{z}_{N-1-j}(1-t), \quad \forall t \in \mathbb{R}, \ \forall j \in \mathbf{N}. \tag{7}$$

In particular,

$$z_j(0) = \bar{z}_{N-j}(0), \ \text{ if } j \in \mathbf{N}\backslash\{0\}; \quad z_j(1/2) = \bar{z}_{N-1-j}(1/2), \ \text{ if } j \in \mathbf{N}. \tag{8}$$

The symmetric constraints also imply

$$z_0(t) = \bar{z}_0(-t), \quad \forall t \in \mathbb{R}, \tag{9}$$

$$z_0(j+t) = \begin{cases} z_j(t), & \forall t \in [0, 1/2], \\ \bar{z}_{N-1-j}(1-t), & \forall t \in [1/2, 1] \end{cases} \quad \text{if } j \in \{0, 1, \ldots, \mathfrak{n}\}. \tag{10}$$

As a result, if $z \in \Lambda_N^{D_N}$ is collision-free, it must satisfy the following:

$$\operatorname{Im}(z_0(j/2)) \neq 0, \quad \forall j \in \mathbf{N}\backslash\{0\}. \tag{11}$$

Based on the above observation, we set

$$\Omega_N := \{\omega = (\omega_j)_{j \in \mathbf{N}\backslash\{0\}} : \omega_j \in \{\pm 1\}, \ \forall j \in \mathbf{N}\backslash\{0\}\}. \tag{12}$$

Definition 1.1. For any $\omega \in \Omega_N$, we say that a loop $z \in \Lambda_N^{D_N}$ satisfies the ω-topological constraints, if

$$\operatorname{Im}(z_0(j/2)) = \omega_j |\operatorname{Im}(z_0(j/2))|, \quad \forall j \in \mathbf{N}\backslash\{0\}. \tag{13}$$

Notice that if $z \in \hat{\Lambda}_N^{D_N}$, then

$$\operatorname{Im}(z_0(j/2)) \neq 0 \text{ and } \frac{\operatorname{Im}(z_0(j/2))}{|\operatorname{Im}(z_0(j/2))|} = \omega_j, \quad \forall j \in \mathbf{N}\backslash\{0\}. \tag{14}$$

Theorem 1.1. *For each $\omega \in \Omega_N$, there is at least one simple choreography $z^\omega = (z_j^\omega)_{j \in \mathbf{N}} \in \hat{\Lambda}_N^{D_N}$ satisfying (1), the ω-topological constraints and the following monotone property along the real axis:*

$$\dot{x}_0^\omega(t) > 0, \ \forall t \in (0, N/2), \ \textit{and } \dot{x}_0^\omega(0) = \dot{x}_0^\omega(N/2) = 0, \tag{15}$$

where $x_0^\omega(t) = \operatorname{Re}(z_0^\omega(t))$.

Such z^ω's will be obtained as collision-free minimizers. The detailed proof of the above theorem will be given in Section 2. For the moment, we use it to give a proof of the Main Theorem.

Proof. Although there are 2^{N-1} different ω's in Ω_N, when we try to count distinct main simple choreographies, the number is smaller.

Given an $\omega \in \Omega_N$, we first have that $z^\omega(-t)$ satisfies the $(-\omega)$-topological constraints, so we only need to count the ω's from the set $\Omega_N^+ := \{\omega \in \Omega_n : \omega_1 = 1\}$.

Second, define $\omega^* = (\omega_j^*)_{j\in\mathbf{N}\setminus\{0\}} \in \Omega_N$ by $\omega_j^* = \omega_{N-j}$. Then $-z^\omega(t+\frac{N}{2})$ satisfies the ω^*-topological constraints. To exclude those, consider the disjoint union $\Omega_N^+ = \Omega_N^{+,-} \cup \Omega_N^{+,+}$, where

$$\Omega_N^{+,-} := \{\omega \in \Omega_N^+ : \omega_{N-1} = -1\}, \quad \Omega_N^{+,+} := \{\omega \in \Omega_N^+ : \omega_{N-1} = 1\}.$$

There are 2^{N-3} different ω's in $\Omega_N^{+,+}$. However, if $\omega \in \Omega_N^{+,+}$, then $\omega^* \in \Omega_N^{+,+}$. Meanwhile, notice that there are $2^{[(N-2)/2]}$ different ω's in $\Omega_N^{+,+}$ with $\omega^* = \omega$. As a result, the number of effective ω's we can count from $\Omega_N^{+,+}$ is

$$2^{N-4} + 2^{[(N-4)/2]} = \frac{1}{2}\big(2^{N-3} + 2^{[(N-2)/2]}\big). \tag{16}$$

Now, for any $\omega \in \Omega_N^{+,-}$, we have although $\omega^* \notin \Omega_N^{+,-}$, $-\omega^*$ is, so it needs to be that excluded from our counting. If N is odd, there are $2^{[(N-2)/2]}$ different $\omega \in \Omega_N^{+,-}$ with $\omega = -\omega^*$, and if N is even, there is no such ω. This is because when N is even, $\omega = -\omega^*$ implies $\omega_{N/2} = -\omega_{N/2}$, which can never happen, as $\omega_{N/2} \in \{\pm 1\}$. As a result, the number of effective ω's we can count from $\Omega_N^{+,-}$ is

$$\begin{cases} 2^{N-4}, & \text{if } N \text{ is even,} \\ 2^{N-4} + 2^{[(N-4)/2]}, & \text{if } N \text{ is odd.} \end{cases} \tag{17}$$

By (16) and (17), there are $2^{N-3} + 2^{[(N-3)/2]}$ distinct main simple choreographies, and this finishes our proof. □

The number $2^{N-3} + 2^{[(N-3)/2]}$ obtained above is the same as the number of simple choreographies belonging to a special family called *linear chains* given in [12, Proposition 5.1]. This is not a coincidence. The family of linear chains consists of simple choreographies which look like a chain of consecutive bubbles placed along the real axis (the rotating N-gon, the Figure-Eight and the Super-Eight all belong to this family). As each $z^\omega \in \hat{\Lambda}_N^{D_N}$ obtained in Theorem 1.1 satisfies (9), (14) and (15), they belong to the family of linear chains as well.

For each ω, the number of bubbles of the corresponding z^ω has a minimum determined by ω. For example, if $\omega_j = 1$, $\forall j \in \mathbf{N}\setminus\{0\}$, then the minimum is 1. In fact, we know in this case that z^ω must be the rotating N-gon, which consists of exactly one bubble. However, in general, we don't know if the number of bubbles of z^ω is exactly this minimum.

Our paper is organized as follows in Section 2, we give a proof of Theorem 1.1; in Section 3, we consider simple choreographies with extra symmetries; in the last section, the Appendix, the proof of a technical lemma will be given.

Notations the following notation rules will apply through out the paper:

- i will always represent $\sqrt{-1}$;
- Given a path $z(t)$ of the N-body, then $z_j(t)$ is the corresponding path of m_j, for any $j \in \mathbf{N}$ with $z_j(t) = x_j(t) + iy_j(t)$ and $x_j(t), y_j(t) \in \mathbb{R}$;
- If, instead of $z(t)$, a path of the N-body is denoted by $\tilde{z}(t), z^{\omega}(t), z^{\varepsilon}(t) \ldots$, then the corresponding changes will be made on $z_j(t), x_j(t)$ and $y_j(t)$;
- Given any two non-negative integers j_0, j_1:

$$\{j_0, \ldots, j_1\} := \begin{cases} \{j \in \mathbb{Z} : \ j_0 \leqq j \leqq j_1\}, & \text{if } j_0 \leqq j_1, \\ \emptyset, & \text{if } j_0 > j_1; \end{cases}$$

- $C_1, C_2, \ldots,$ representing positive constants, vary from proof to proof.

2. Proof of Theorem 1.1

When there is no confusion, in this section $\Lambda_N^{D_N}$, $\hat{\Lambda}_N^{D_N}$ will simply be written as Λ, $\hat{\Lambda}$, respectively.

Due to the symmetric constraints, a loop $z \in \Lambda$ is entirely determined by $z(t), t \in [0, 1/2]$, so it is enough to focus on the *fundamental domain*: $[0, 1/2]$. In the rest of the paper, when we try to define a loop $z \in \Lambda$, only $z(t), t \in [0, 1/2]$ will be given explicitly (the rest will follow from the symmetric constraints). Furthermore, in this section, we set $\mathcal{A}(z) := \mathcal{A}_K(z) + \mathcal{A}_U(z)$, where

$$\mathcal{A}_K(z) := \int_0^{1/2} K(\dot{z})\,\mathrm{d}t, \quad \mathcal{A}_U(z) := \int_0^{1/2} U(z)\,\mathrm{d}t.$$

First we will introduce the *monotone constraints*.

Definition 2.1. We say that a loop $z \in \Lambda$ satisfies the monotone constraints if

$$x_0(j/2) \leqq x_0(j/2 + t) \leqq x_0(j/2 + 1/2), \qquad \forall t \in [0, 1/2], \ \forall j \in \mathbf{N}$$

and that is satisfies the strictly monotone constraints if

$$x_0(t_1) < x_0(t_2), \quad \text{for any } 0 \leqq t_1 < t_2 \leqq N/2.$$

Let Λ^+ be the subset of all loops in Λ which satisfy the monotone constraints and the following inequalities:

$$x_0(0) \leqq 0 \leqq x_0(N/2). \tag{18}$$

For technical reasons, we will not fix the center of mass at the origin in our proof; this could cause a lack of coercivity during the minimizing process. The above inequality was introduced to overcome this.

By (7), (8) and (10), we have that $z \in \Lambda$, satisfying the monotone constraints, is equivalent to the following:

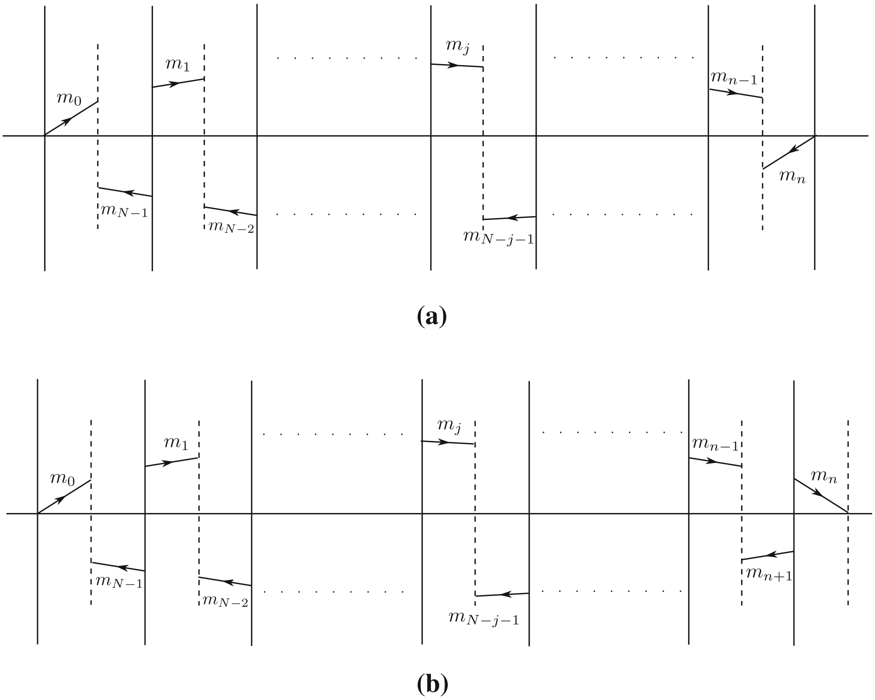

Fig. 1. **a** $N = 2n$, $\mathfrak{n} = n - 1$, **b** $N = 2n$, $\mathfrak{n} = n$

$$\begin{cases} x_j(0) \leqq x_j(t) \leqq x_j(1/2), & \text{if } j \in \{0, \dots, \mathfrak{n}\}, \\ x_j(0) \geqq x_j(t) \geqq x_j(1/2), & \text{if } j \in \mathbf{N} \backslash \{0, \dots, \mathfrak{n}\}, \end{cases} \quad \forall t \in [0, 1/2]; \tag{19}$$

$$\begin{aligned} &x_0(0) \leqq x_0(1/2) = x_{N-1}(1/2) \leqq x_{N-1}(0) = x_1(0) \leqq \cdots \\ &\cdots \leqq x_{n+1}(0) = x_{n-1}(0) \leqq x_{n-1}(1/2) = x_n(1/2) \leqq x_n(0), \quad \text{if } N = 2n; \end{aligned} \tag{20}$$

$$\begin{aligned} &x_0(0) \leqq x_0(1/2) = x_{N-1}(1/2) \leqq x_{N-1}(0) = x_1(0) \leqq \cdots \\ &\cdots \leqq x_{n-1}(1/2) = x_{n+1}(1/2) \leqq x_{n+1}(0) = x_n(0) \leqq x_n(1/2), \quad \text{if } N = 2n + 1, \end{aligned} \tag{21}$$

For illuminating pictures, see Fig. 1. Roughly speaking, the masses are located on the solid vertical lines when $t = 0$; on the dashed vertical lines when $t = 1/2$. Furthermore, between $t = 0$ and $t = 1/2$, each mass is confined within the unique vertical strip bounded by the neighboring vertical lines. Whether the masses are above or below the real axis on each vertical line is determined by the ω-topological constraints. The pictures in Fig. 1 correspond to $\omega \in \Omega_N$ with $\omega_j = 1$, for any $j \in \mathbf{N} \backslash \{0\}$.

To put the topological constraints back into the problem, for any $\omega \in \Omega_N$, we define Λ^+_ω as the subset of loops in Λ^+ which satisfy the ω-topological constraints. Correspondingly, we set $\hat{\Lambda}^+ := \Lambda^+ \cap \hat{\Lambda}$, $\hat{\Lambda}^+_\omega := \Lambda^+_\omega \cap \hat{\Lambda}$.

Proposition 2.1. *For each $\omega \in \Omega_N$, there is a $z^\omega \in \Lambda^+_\omega$ which is a minimizer of the action functional $\mathcal{A}$ in Λ^+_ω.*

Proof. It is well known that the action functional $\mathcal{A}$ is weakly lower-semi continuous with respect to the Sobolev norm H^1. Meanwhile, Λ_ω^+ is weakly closed with respect to the same norm. Therefore we just need to show that $\mathcal{A}$ is coercive in Λ_ω^+.

Choose a sequence of loops $\{z^k \in \Lambda_\omega^+\}$ with $\|z^k\|_{H^1}$ going to infinity; as k goes to infinity, it is enough to show that $\mathcal{A}(z^k, N)$ goes to infinity as well.

Recall that $z_0^k(t) = x_0^k(t) + i y_0^k(t)$. By (9), $y_0^k(0) = 0$, so then

$$\left|y_0^k(t_0)\right| = \left|y_0^k(t_0) - y_0^k(0)\right| \leqq \int_0^N \left|\dot{y}_0^k(t)\right| \mathrm{d}t, \quad \forall t_0 \in [0, N).$$

By Cauchy-Schwartz inequality,

$$|y_0^k(t_0)|^2 \leqq \left(\int_0^N \left|\dot{y}_0^k(t)\right| \mathrm{d}t\right)^2 \leqq N \int_0^N \left|\dot{y}_0^k(t)\right|^2 \mathrm{d}t, \quad \forall t_0 \in [0, N).$$

Hence,

$$\int_0^N \left|y_0^k(t)\right|^2 \mathrm{d}t \leqq N^2 \int_0^N \left|\dot{y}_0^k(t)\right|^2 \mathrm{d}t.$$

Meanwhile, by (18), there is always a $t_k \in [0, N/2]$ such that $x_0^k(t_k) = 0$. Then, by computations similar to these above,

$$\int_0^N \left|x_0^k(t)\right|^2 \mathrm{d}t \leqq N^2 \int_0^N \left|\dot{x}_0^k(t)\right|^2 \mathrm{d}t.$$

As a result,

$$\int_0^N \left|z_0^k(t)\right|^2 \mathrm{d}t \leqq N^2 \int_0^N \left|\dot{z}_0^k(t)\right|^2 \mathrm{d}t. \tag{22}$$

Because $z^k \in \Lambda_\omega^+$, (4) is satisfied. Then

$$\left\|z^k\right\|_{H^1}^2 = \int_0^N \sum_{j \in \mathbf{N}} \left(\left|z_j^k(t)\right|^2 + \left|\dot{z}_j^k(t)\right|^2\right) \mathrm{d}t = N \int_0^N \left|z_0^k(t)\right|^2 + \left|\dot{z}_0^k(t)\right|^2 \mathrm{d}t. \tag{23}$$

Notice that the action functional satisfies

$$\mathcal{A}\left(z^k, N\right) \geqq \frac{1}{2} \int_0^N \sum_{j \in \mathbf{N}} \left|\dot{z}_j^k(t)\right|^2 \mathrm{d}t = \frac{N}{2} \int_0^N \left|\dot{z}_0^k(t)\right|^2 \mathrm{d}t. \tag{24}$$

Together (22), (23) and (24) imply

$$\mathcal{A}\left(z^k, N\right) \geqq \frac{1}{2\left(N^2 + 1\right)} \left\|z^k\right\|_{H^1}^2.$$

As a result, $\mathcal{A}(z^k, N)$ goes to infinity as k goes to infinity. □

We need to prove that the action minimizer z^ω is collision-free. However, because of the monotone constraints, this does not necessarily mean that it is a solution of Eq. (1). In addition, we also need to show that it satisfies the *strictly monotone constraints*. As this property will be useful in showing that z^ω is collision-free, we will prove it first.

Lemma 2.1. *For any $\omega \in \Omega_N$, if $z^\omega \in \Lambda^+_\omega$ is an action minimizer of $\mathcal{A}$ in Λ^+_ω, then*

$$x_0^\omega(t_1) \leqq x_0^\omega(t_2), \quad \textit{for any } 0 \leqq t_1 < t_2 \leqq N/2.$$

Proof. For simplicity, let $z = z^\omega$. As $z \in \Lambda^+_\omega$, by the definition of monotone constraints, it is enough to show that, for any $0 \leqq t_1 \leqq t_2 \leqq 1/2$,

$$x_0(j/2 + t_1) \leqq x_0(j/2 + t_2), \quad \forall j \in \mathbf{N}.$$

By a contradiction argument, let's assume that there is a $j_0 \in \mathbf{N}$ and $0 \leqq t_1 < t_2 \leqq 1/2$, such that

$$x_0(j_0/2 + s_1) > x_0(j_0/2 + s_2), \quad \text{for any } t_1 \leqq s_1 < s_2 \leqq t_2.$$

By (10), depending on whether j_0 is even or odd, this is equivalent to

$$\text{for any } t_1 \leqq s_1 < s_2 \leqq t_2, \begin{cases} x_k(s_1) > x_k(s_2), & \text{if } j_0 = 2k, \\ x_{N-k}(1/2 - s_1) > x_{N-k}(1/2 - s_2), & \text{if } j_0 = 2k - 1. \end{cases}$$

Let $\varepsilon = x_0(j_0/2 + s_1) - x_0(j_0/2 + s_2) > 0$. We discuss this in two corresponding cases.

Case 1: j_0 is even ($j_0 = 2k$). We define a new loop $z^\varepsilon \in \Lambda^+_\omega$ as follows:

$$z_k^\varepsilon(t) = \begin{cases} z_k(t) - 2\varepsilon, & \forall t \in [0, t_1], \\ \big(2x_k(t_2) - x_k(t)\big) + i y_k(t), & \forall t \in [t_1, t_2], \\ z_k(t), & \forall t \in [t_2, 1/2], \end{cases}$$

$$z_j^\varepsilon(t) = \begin{cases} z_j(t), & \text{if } j \in \{k+1, \ldots, N-1-k\}, \\ z_j(t) - 2\varepsilon, & \text{if } j \in \mathbf{N}\backslash\{k, \ldots, N-1-k\}, \end{cases} \quad \forall t \in [0, 1/2].$$

By the above definition of z^ε, $\mathcal{A}_K(z^\varepsilon) = \mathcal{A}_K(z)$. Meanwhile, by the monotone constraints, $\mathcal{A}_U(z^\varepsilon) < \mathcal{A}_U(z)$. Therefore $\mathcal{A}(z^\varepsilon) < \mathcal{A}(z)$, which is absurd.

Case 2: j_0 is odd ($j_0 = 2k - 1$). Similarly, we define a new loop $z^\varepsilon \in \Lambda^+_\omega$ as follows:

$$z_{N-k}^\varepsilon(t) = \begin{cases} z_{N-k}(t), & \forall t \in [0, 1/2 - t_2], \\ \big(2x_{N-k}(1/2 - t_2) - x_{N-k}(t)\big) + i y_{N-k}(t), & \forall t \in [1/2 - t_2, 1/2 - t_1], \\ z_{N-k}(t) - 2\varepsilon, & \forall t \in [1/2 - t_1, 1/2], \end{cases}$$

$$z_j^\varepsilon(t) = \begin{cases} z_j(t), & \text{if } j \in \{k, \ldots, N-1-k\}, \\ z_j(t) - 2\varepsilon, & \text{if } j \in \mathbf{N}\backslash\{k, \ldots, N-k\}, \end{cases} \quad \forall t \in [0, 1/2].$$

The rest is the same as in *Case 1*. □

Next we need to exclude the degenerate case, where the masses always stay on a single vertical line.

Lemma 2.2. *For any $\omega \in \Omega_N$, if $z^\omega \in \Lambda_\omega^+$ is an action minimizer of $\mathcal{A}$ in Λ_ω^+, then $x_0^\omega(N/2) - x_0^\omega(0) > 0$.*

Our approach is to show that in the degenerate case, z^ω becomes a collinear solution that contains at least one isolated collision. Then, by local deformations near the isolated collision, we can find another loop from Λ_ω^+ whose action value is strictly smaller than z^ω's, which gives us a contradiction. The proof will be given in the Appendix.

With the above two lemmas, the fact that z^ω satisfies the strictly monotone constraints will be established by the following:

Lemma 2.3. *When $N = 2n$, for any $\omega \in \Omega_N$, if $z^\omega \in \Lambda_\omega^+$ is an action minimizer of $\mathcal{A}$ in Λ_ω^+, then $x_0^\omega(t_1) < x_0^\omega(t_2)$ for any $0 \leqq t_1 < t_2 \leqq N/2$.*

Proof. For simplicity, let $z = z^\omega$. By Lemma 2.1, it is enough to show that

$$x_0(j/2 + t_1) \neq x_0(j/2 + t_2), \quad \text{for any } 0 \leqq t_1 < t_2 \leqq 1/2, \quad \forall j \in \mathbf{N}.$$

Let's assume that there exists a $j_0 \in \mathbf{N}$, and that $0 \leqq t_1 < t_2 \leqq 1/2$ such that

$$x_0(j_0/2 + t) = x_0(j_0/2 + t_1), \quad \forall t \in [t_1, t_2]. \tag{25}$$

By Lemma 2.2, there exist t_0 and $\delta_1 > 0$ small enough such that

$$x_n(t) - x_0(t) \geqq \delta_1, \quad \forall t \in [0, t_0]. \tag{26}$$

Meanwhile, we can always find a $\delta_2 > 0$ small enough such that

$$|z_n(t) - z_0(t)| \leqq \delta_2^{-1}, \quad \forall t \in [0, 1/2]. \tag{27}$$

Depending on the value of j_0, two different cases will be considered.

Case 1: If j_0 is even ($j_0 = 2k$), then (25) implies

$$x_k(t) = x_k(t_1), \quad \forall t \in [t_1, t_2].$$

Choosing a $\varepsilon > 0$ small enough, we define a new loop $z^\varepsilon \in \Lambda_\omega^+$ as follows:

$$z_k^\varepsilon(t) = \begin{cases} z_k(t) - \varepsilon, & \forall t \in [0, t_1], \\ z_k(t) - \frac{t_2 - t}{t_2 - t_1}\varepsilon, & \forall t \in [t_1, t_2], \\ z_k(t), & \forall t \in [t_2, 1/2]; \end{cases} \tag{28}$$

$$z_j^\varepsilon(t) = \begin{cases} z_j(t), & \text{if } j \in \{k+1, \dots, N-1-k\}, \\ z_j(t) - \varepsilon, & \text{if } j \in \mathbf{N} \backslash \{k, \dots, N-1-k\}, \end{cases} \quad \forall t \in [0, 1/2]. \tag{29}$$

By the above definition of z^ε and (25), we have

$$\mathcal{A}_K(\dot{z}^\varepsilon) - \mathcal{A}_k(\dot{z}) = \frac{1}{2}\int_{t_1}^{t_2} |\dot{z}^\varepsilon(t)|^2 - |\dot{z}|^2 \, dt = \frac{1}{2}\int_{t_1}^{t_2} \frac{\varepsilon^2}{(t_2 - t_1)^2} \, dt = \frac{\varepsilon^2}{2(t_2 - t_1)}. \tag{30}$$

This shows the change in kinetic energy. For potential energy, notice that

$$\left|z_j^\varepsilon(t) - z_l^\varepsilon(t)\right| \geqq \left|z_j(t) - z_l(t)\right|, \quad \forall t \in [0, 1/2], \ \forall \{j \neq l\} \subset \mathbf{N}. \tag{31}$$

When $k > 0$, by the definition of z^ε and (26), for any $t \in [0, t_0]$, we have

$$\begin{aligned} \left|z_n^\varepsilon(t) - z_0^\varepsilon(t)\right|^2 &= |x_n(t) - x_0(t) + \varepsilon + i(y_n(t) - y_0(t))|^2 \\ &= |z_n(t) - z_0(t)|^2 + 2(x_n(t) - x_0(t))\varepsilon + \varepsilon^2 \\ &\geqq |z_n(t) - z_0(t)|^2 + 2\delta_1\varepsilon + \varepsilon^2. \end{aligned}$$

Together with (27), for any $t \in [0, t_0]$,

$$\begin{aligned} &\left|z_n^\varepsilon(t) - z_0^\varepsilon(t)\right|^{-1} - |z_n(t) - z_0(t)|^{-1} \\ &\leqq \frac{1}{|z_n(t) - z_0(t)|}\left(\left(1 + \frac{2\delta_1\varepsilon}{|z_n(t) - z_0(t)|^2} + \frac{\varepsilon^2}{|z_n(t) - z_0(t)|^2}\right)^{-\frac{1}{2}} - 1\right) \\ &\leqq -\frac{\delta_1\varepsilon}{|z_n(t) - z_0(t)|^3} + o(\varepsilon) \leqq -\delta_1\delta_2^3\varepsilon + o(\varepsilon). \end{aligned}$$

Combine this with (31) to get

$$\begin{aligned} \mathcal{A}_U(z^\varepsilon) - \mathcal{A}_U(z) &\leqq \int_0^{t_0} |z_n^\varepsilon(t) - z_0^\varepsilon(t)|^{-1} - |z_n(t) - z_0(t)|^{-1}\, \mathrm{d}t \\ &\leqq \int_0^{t_0} -\delta_1\delta_2^3\varepsilon + o(\varepsilon)\, \mathrm{d}t = -C_1(\delta_1, \delta_2, t_0)\varepsilon + o(\varepsilon). \end{aligned} \tag{32}$$

When $k = 0$, if $t_0 < t_1$, the above estimates for potential energy will still hold. However if $t_0 > t_1$, then things are slightly different. Assuming that $k = 0$ and $t_0 > t_1$, by (26), for any $t \in [t_1, t_3]$, where $t_3 = \min\{t_0, t_2\}$, we have

$$\begin{aligned} \left|z_n^\varepsilon(t) - z_0^\varepsilon(t)\right|^2 &= \left|x_n(t) - x_0(t) + \frac{t_2 - t}{t_2 - t_1}\varepsilon + i(y_n(t) - y_0(t))\right|^2 \\ &= |z_n(t) - z_0(t)|^2 + 2(x_n(t) - x_0(t))\frac{t_2 - t}{t_2 - t_1}\varepsilon + \left(\frac{t_2 - t}{t_2 - t_1}\right)^2\varepsilon^2 \\ &\geqq |z_n(t) - z_0(t)|^2 + 2\delta_1\varepsilon\frac{t_2 - t}{t_2 - t_1} + o(\varepsilon). \end{aligned}$$

Combining the above with (27), we get, for any $t \in [t_1, t_3]$,

$$\begin{aligned} &|z_n^\varepsilon(t) - z_0^\varepsilon(t)|^{-1} - |z_n(t) - z_0(t)|^{-1} \\ &\leqq \frac{1}{|z_n(t) - z_0(t)|}\left(\left(1 + \frac{2\delta_1\varepsilon\frac{t_2-t}{t_2-t_1}}{|z_n(t) - z_0(t)|^2}\right)^{-\frac{1}{2}} - 1\right) \\ &\leqq -\frac{\delta_1\varepsilon\frac{t_2-t}{t_2-t_1}}{|z_n(t) - z_0(t)|^3} + o(\varepsilon) \leqq -\delta_1\delta_3^3\varepsilon\frac{t_2 - t}{t_2 - t_1} + o(\varepsilon). \end{aligned}$$

As a result,

$$\begin{aligned}\mathcal{A}_U(z^\varepsilon) - \mathcal{A}_U(z) &\leqq \int_{t_1}^{t_3} |z_n^\varepsilon(t) - z_0^\varepsilon(t)|^{-1} - |z_n(t) - z_0(t)|^{-1}\,\mathrm{d}t \\ &\leqq \int_{t_1}^{t_3} -\delta_1\delta_3^3\varepsilon\frac{t_2 - t}{t_2 - t_1} + o(\varepsilon)\,\mathrm{d}t \leqq -C_2(\delta_1, \delta_2, t_0, t_1, t_2)\varepsilon + o(\varepsilon).\end{aligned} \tag{33}$$

Together, (30), (32) and (33) show that, for ε small enough,

$$\mathcal{A}(z^\varepsilon) - \mathcal{A}(z) \leqq -C_3(\delta_1, \delta_2, t_0, t_1, t_2)\varepsilon + \frac{\varepsilon^2}{2(t_2 - t_1)} + o(\varepsilon) < 0,$$

which is a contradiction. This finishes our proof of *Case 1*.

Case 2: If j_0 is odd ($j_0 = 2k - 1$), then (25) implies

$$x_{N-k}(t) = x_{N-k}(1/2 - t_1), \quad \forall t \in [1/2 - t_2, 1/2 - t_1].$$

For $\varepsilon > 0$ small enough, we define a new loop $z^\varepsilon \in \Lambda_\omega^+$ as follows:

$$z_{N-k}^\varepsilon(t) = \begin{cases} z_{N-k}(t) + \varepsilon, & \forall t \in [0, 1/2 - t_2], \\ z_{N-k}(t) + \frac{1/2 - t_1 - t}{t_2 - t_1}\varepsilon, & \forall t \in [1/2 - t_2, 1/2 - t_1], \\ z_{N-k}(t), & \forall t \in [1/2 - t_1, 1/2]; \end{cases} \tag{34}$$

$$z_j^\varepsilon(t) = \begin{cases} z_j(t) + \varepsilon, & \text{if } j \in \{k, \ldots, N-1-k\}, \\ z_j(t), & \text{if } j \in \mathbf{N}\backslash\{k, \ldots, N-k\}, \end{cases} \quad \forall t \in [0, 1/2]. \tag{35}$$

By estimates similar to these for *Case 1*, we can show that $\mathcal{A}(z^\varepsilon) - \mathcal{A}(z) < 0$ for ε small enough, which is a contradiction. □

Lemma 2.4. *When $N = 2n+1$, for any $\omega \in \Omega_N$, if $z^\omega \in \Lambda_\omega^+$ is an action minimizer of $\mathcal{A}$ in Λ_ω^+, then $x_0^\omega(t_1) < x_0^\omega(t_2)$ for any $0 \leqq t_1 < t_2 \leqq N/2$.*

Proof. Let $z = z^\omega$. As with the proof of Lemma 2.3, by a contradiction argument, let's assume that (25) holds for some $j_0 \in \mathbf{N}$ and $0 \leqq t_1 < t_2 \leqq 1/2$.

Notice that, by Lemmas 2.1 and 2.2,

$$\max\{x_0(n) - x_0(0),\ x_0(n + 1/2) - x_0(1/2)\} > 0.$$

Hence there exist t_0 and $\delta_1 > 0$ such that one of the following must hold:

$$x_n(t) - x_0(t) \geqq \delta_1, \quad \forall t \in [0, t_0]; \tag{36}$$

$$x_n(t) - x_0(t) \geqq \delta_1, \quad \forall t \in [1/2 - t_0, 1/2]. \tag{37}$$

First, let's assume that (36) holds. Again we will consider two different cases depending on the value of j_0 (to separate them from the two cases considered in the proof of Lemma 2.3, we will count them as *Case 3* and *Case 4*).

Case 3: j_0 is odd ($j_0 = 2k - 1$). Let $z^\varepsilon \in \Lambda_\omega^+$ be defined as in *Case 2*, then, as with case 2, a contradiction can reached.

Case 4: j_0 is even ($j_0 = 2k$). In this case $0 \leqq k \leqq n$. Let $z^\varepsilon \in \Lambda_\omega^+$ be defined as in *Case 1*; when $k < n$, a contradiction can be reached by the same argument given there. For when $k = n$, more needs to be said. First, the set $\{k+1, \dots, N-1-k\}$ in (29) is empty. By what we have proven so far in *Case 4* and in *Case 3*, there is a $\delta_3 > 0$, such that

$$x_n(t) - x_0(t) \geqq \delta_3, \quad \forall t \in [0, 1/2]. \tag{38}$$

With (27) still holding, by computations similar to those given in *Case 1*,

$$\begin{aligned} \mathcal{A}_U(z^\varepsilon) - \mathcal{A}_U(z) &\leqq \int_{t_1}^{t_2} |z_n^\varepsilon(t) - z_0^\varepsilon(t)|^{-1} - |z_n(t) - z_0(t)|^{-1} \, \mathrm{d}t \\ &\leqq -C_4(t_1, t_2, \delta_1, \delta_3)\varepsilon + o(\varepsilon). \end{aligned}$$

Since the change in kinetic energy is still given by (30), we get $\mathcal{A}(z^\varepsilon) - \mathcal{A}(z) < 0$ for ε small enough, which is absurd. This finishes our proof of *Case 4*.

Now let's assume that (37) holds. The proof is almost the same as above, so we will not repeat the details here. □

Let z^ω be an action minimizer of $\mathcal{A}$ in Λ_ω^+ (for simplicity, in the rest of this section, we set $z = z^\omega$). By Lemmas 2.3 and 2.4, this satisfies the strictly monotone constraints. As a result, if z is collision-free, it must be a solution of (1). Notice that Lemmas 2.3 and 2.4 already imply that $z(t)$ and $t \in (0, 1/2)$ must be collision-free, and the only possible collisions are binary collisions at $t = 0$ or $t = 1/2$ between certain pairs of masses determined by the symmetric constraints. To be precise, when $t = 0$, a binary collision can only happen between the pairs of masses with the indices

$$\{1, N-1\}, \{2, N-2\}, \dots, \{\mathfrak{n}, N-\mathfrak{n}\};$$

when $t = 1/2$, it is between the following pairs:

$$\{0, N-1\}, \{1, N-2\}, \dots, \{\mathfrak{n}, N-\mathfrak{n}-1\}.$$

When $N = 2n+1$, $\mathfrak{n} = N - \mathfrak{n} - 1 = n$, so in this case, there is no collision between $m_\mathfrak{n}$ and $m_{N-\mathfrak{n}-1}$ at $t = 1/2$.

In the following, we will show that none of the above binary collisions can exist. First, let's assume $z_j(0) = z_{N-j}(0)$ for some $j \in \{1, \dots, \mathfrak{n}\}$. Notice that $y_j(0) = y_{N-j}(0) = 0$ as $z_j(0) = \bar{z}_{N-j}(0)$. Without loss of generality, we may further assume that $z_j(0) = z_{N-j}(0) = 0$.

For any $t \in [0, 1/2]$ and $k \in \{j, N-j\}$, we set

$$\hat{z}(t) := \hat{x}(t) + i\hat{y}(t) := (z_j(t) + z_{N-j}(t))/2, \tag{39}$$
$$w_k(t) := u_k(t) + iv_k(t) := z_k(t) - \hat{z}(t). \tag{40}$$

Here $\hat{z}(t)$ is the center of mass of m_j and m_{N-j}, and $w_k(t)$ is the relative position of m_k with respect to $\hat{z}$. Putting $w_k(t)$ in polar coordinates,

$$w_k(t) = \rho_k(t) e^{i\theta_k(t)}, \tag{41}$$

and we have the following two results:

Proposition 2.2. *For any $k \in \{j, N-j\}$, when $t > 0$ is small enough,*

$$\rho_k(t) = C_1 t^{\frac{2}{3}} + o\left(t^{\frac{2}{3}}\right), \quad \dot{\rho}_k(t) = C_2 t^{-\frac{1}{3}} + o\left(t^{-\frac{1}{3}}\right).$$

This is the well-known Sundman's estimate; for a proof see [14, (6.25)].

Proposition 2.3. *For any $k \in \{j, N-j\}$, there exist finite θ_k^+ satisfying the following:*

$$\lim_{t \to 0^+} \theta_k(t) = \theta_k^+, \quad \lim_{t \to 0^+} \dot{\theta}_k(t) = 0, \quad \theta_{N-j}^+ = \theta_j^+ + \pi. \tag{42}$$

The above proposition is also well-known, and shows that m_j and m_{N-j} approach the binary collision from two directions that are definitely in opposition to each other (a proof can be found in [29, Proposition 4.2]).

By the strictly monotone constraints, $x_j(t) - x_{N-j}(t) > 0$, $\forall t \in (0, 1/2]$. Hence,

$$x_{N-j}(t) - \hat{x}(t) < 0 < x_j(t) - \hat{x}(t), \quad \forall t \in (0, 1/2].$$

As a result, we may assume

$$\theta_j(t) \in (-\pi/2, \pi/2), \quad \theta_{N-j}(t) \in (\pi/2, 3\pi/2), \quad \forall t \in (0, 1/2],$$

and this means

$$\theta_j^+ \in [-\pi/2, \pi/2], \quad \theta_{N-j}^+ \in [\pi/2, 3\pi/2].$$

Depending on the values of θ_j^+, two types of deformations will be applied to z to get a contradiction. To make sure that the loop we obtained after the deformation is still contained in Λ_ω^+, we need to know the precise value of ω_{2j}. Without loss of generality, let's assume $\omega_{2j} = 1$ in the following:

Lemma 2.5. *If $z_j(0) = z_{N-j}(0)$ and $\theta_j^+ \in (-\pi/2, \pi/2]$, then there is a $\tilde{z} \in \Lambda_\omega^+$ with $\mathcal{A}(\tilde{z}) < \mathcal{A}(z)$.*

To prove the above lemma, we need the following local deformation result near an isolated binary collision:

Proposition 2.4. *If $z_j(0) = z_{N-j}(0)$ and $\theta_j^+ \in (-\pi/2, \pi/2]$, then for $\varepsilon > 0$ and $t_0 = t_0(\varepsilon) > 0$ small enough, there is an $z^\varepsilon \in H^1([0, 1/2], \mathbb{C}^N)$ (a local deformation of z near $t = 0$) satisfying $\mathcal{A}(z^\varepsilon) < \mathcal{A}(z)$ and the following:*

(a) For any $l \in \mathbf{N} \backslash \{j, N-j\}$, $z_l^\varepsilon(t) = z_l(t)$, $\forall t \in [0, 1/2]$;
(b) For any $k \in \{j, N-j\}$,

$$\begin{cases} z_k^\varepsilon(t) = z_k(t), & \text{when } t \in [t_0, 1/2]; \\ |z_k^\varepsilon(t) - z_k(t)| \leqq \varepsilon, & \text{when } t \in [0, t_0]; \end{cases}$$

(c) $z^\varepsilon(t)$, $t \in (0, 1/2)$, is collision-free;
(d) $z_k^\varepsilon(0) \neq z_l^\varepsilon(0)$, for any $k \in \{j, N-j\}$ and $l \in \mathbf{N} \backslash \{j, N-j\}$;

(e) $z_j^\varepsilon(0) \neq z_{N-j}^\varepsilon(0)$*, in particular*

$$z_j^\varepsilon(0) = z_j(0) + i\varepsilon, \quad z_{N-j}^\varepsilon(0) = z_{N-j}(0) - i\varepsilon.$$

Remark 2.1. A proof of the above proposition can be found in [29, Proposition 4.3]. The proof essentially relies on the following element of the Kepler problem: the parabolic collision-ejection solution connecting two different points with the same distance to the origin has an action value strictly smaller than the direct and indirect Keplerian arcs joining them (with the same transfer time). This result was attributed to Marchal in [10]. A proof can be found in [16,28].

Proof (Lemma 2.5). By Proposition 2.4, after a local deformation of z near the isolated collision $z_j(0) = z_{N-j}(0)$, we get a new path $z^\varepsilon \in H^1([0, 1/2], \mathbb{C}^N)$ with $\mathcal{A}(z^\varepsilon) < \mathcal{A}(z)$. After applying the action of the dihedral group D_N defined before, we get a loop which will be denoted by z^ε. Notice that, as a loop, z^ε is contained in Λ and satisfies the ω-topological constraints. However, it is not so clear whether it is also contained in Λ_ω^+, as the monotone constraints may be violated after the local deformation. Nevertheless we will show that, by further modification of z^ε, we can always get a $\tilde{z} \in \Lambda_\omega^+$ satisfying $\mathcal{A}(\tilde{z}) \leqq \mathcal{A}(z^\varepsilon) < \mathcal{A}(z)$.

Recall that $z_j(0) = z_{N-j}(0) = 0$. By Proposition 2.4(e), $x_j^\varepsilon(0) = x_{N-j}^\varepsilon(0) = 0$. Letting t_0 be given as in Proposition 2.4, we define δ_1, δ_2 as follows:

$$\delta_1 = -\min\{x_j^\varepsilon(t) : t \in [0, t_0]\}, \quad \delta_2 = \max\{x_{N-j}^\varepsilon(t) : t \in [0, t_0]\}.$$

Then $\delta_1 \geqq 0$, as is δ_2. Furthermore, let

$$t_1 = \min\{t \in [0, t_0] : x_j^\varepsilon(t) = -\delta_1\}, \quad t_2 = \min\{t \in [0, t_0] : x_{N-j}^\varepsilon(t) = \delta_2\}$$

and

$$\mathbb{T}_1 = \{t \in [0, t_1] : x_j^\varepsilon(t) < 0\}, \quad \mathbb{T}_2 = \{t \in [0, t_2] : x_{N-j}^\varepsilon(t) > 0\}.$$

Now we define a new path $\tilde{z}(t) = (\tilde{z}_k(t))_{k \in \mathbf{N}}$ as follows:

$$\tilde{z}_j(t) = \begin{cases} z_j^\varepsilon(t) & \text{if } t \in [0, t_1] \backslash \mathbb{T}_1, \\ -\bar{z^\varepsilon}_j(t) & \text{if } t \in \mathbb{T}_1, \\ z_j^\varepsilon(t) + 2\delta_1, & \text{if } t \in [t_1, 1/2], \end{cases}$$

$$\tilde{z}_{N-j}(t) = \begin{cases} z_{N-j}^\varepsilon(t), & \text{if } t \in [0, t_2] \backslash \mathbb{T}_2, \\ -\bar{z^\varepsilon}_{N-j}(t), & \text{if } t \in \mathbb{T}_2, \\ z_{N-j}^\varepsilon(t) - 2\delta_2, & \text{if } t \in [t_2, 1/2], \end{cases}$$

$$\tilde{z}_k(t) = \begin{cases} z_k^\varepsilon(t) + 2\delta_1, & \text{if } k \in \{j+1, \ldots, N-j-1\}, \\ z_k^\varepsilon(t) - 2\delta_2, & \text{if } k \in \mathbf{N} \backslash \{j, \ldots, N-j\}, \end{cases} \quad \forall t \in [0, 1/2].$$

We point out that for any $k \in \{j, N-j\}$, if $\delta_k = 0$, then $t_k = 0$ and $\mathbb{T}_k = \emptyset$. In particular, when $\delta_1 = \delta_2 = 0$, then $\tilde{z} = z^\varepsilon$. See Fig. 2 for illuminating illustrations of $\tilde{z}_j$ (or $\tilde{z}_{N-j}$) for when $\delta_1 > 0$ (or $\delta_2 > 0$).

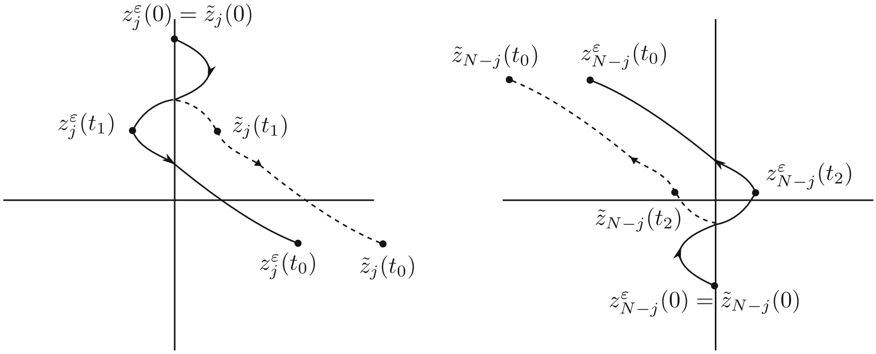

Fig. 2. Deformation of the paths of m_j and m_{N-j}

For $\tilde{z}$ defined as above, the monotone constraints are satisfied and $\tilde{z} \in \Lambda_\omega^+$. Furthermore, $\mathcal{A}(\tilde{z}) \leqq \mathcal{A}(z^\varepsilon) < \mathcal{A}(z)$, as $\mathcal{A}_K(\dot{\tilde{z}}) = \mathcal{A}_K(\dot{z}^\varepsilon)$ and $\mathcal{A}_U(\tilde{z}) \leqq \mathcal{A}_U(z^\varepsilon)$, where the estimate on potential energy follows from

$$|\tilde{z}_k(t) - \tilde{z}_l(t)| \geqq |z_k^\varepsilon(t) - z_l^\varepsilon(t)|, \quad \forall t \in [0, 1/2], \quad \forall \{k < l\} \subset \mathbf{N}.$$

□

We point out that Lemma 2.5 does not apply when $\theta_j^+ = -\pi/2$. By Gordon's result on the Kepler problem (see [17]), an argument regarding the nature of local deformation can never rule out the binary collision in this case; some type of global estimate has to be involved, which generally is hard to do. The advantage of our approach is that the necessary global estimate can be obtained based on the monotone constraints. This estimate will be given in the next lemma.

Lemma 2.6. *If $z_j(0) = z_{N-j}(0)$ and $\theta_j^+ = -\pi/2$, then for $\varepsilon_1 > 0$ small enough, there is a $z^{\varepsilon_1} \in \Lambda_\omega^+$ with $\mathcal{A}(z^{\varepsilon_1}) < \mathcal{A}(z)$.*

Proof. For $\varepsilon_1 > 0$ small enough, define a new loop $z^{\varepsilon_1} \in \Lambda_\omega^+$ as follows

$$z_j^{\varepsilon_1}(t) = \begin{cases} z_j(t) + t(2\varepsilon_1 - t), & \forall t \in [0, \varepsilon_1], \\ z_j(t) + \varepsilon_1^2, & \forall t \in [\varepsilon_1, 1/2], \end{cases}$$

$$z_{N-j}^{\varepsilon_1} = \begin{cases} z_{N-j}(t) - t(2\varepsilon_1 - t), & \forall t \in [0, \varepsilon_1], \\ z_{N-j}(t) - \varepsilon_1^2, & \forall t \in [\varepsilon_1, 1/2], \end{cases}$$

$$z_k^{\varepsilon_1}(t) = \begin{cases} z_k(t) + \varepsilon_1^2, & \text{if } k \in \{j+1, \dots, N-j-1\}, \\ z_k(t) - \varepsilon_1^2, & \text{if } k \in \mathbf{N} \backslash \{j, \dots, N-j\}, \end{cases} \quad \forall t \in [0, 1/2].$$

We claim $\mathcal{A}(z^{\varepsilon_1}) < \mathcal{A}(z)$ for ε_1 small enough. The above definition immediately implies

$$\mathcal{A}_K(\dot{z}^{\varepsilon_1}) > \mathcal{A}_K(\dot{z}), \quad \mathcal{A}_U(\dot{z}^{\varepsilon_1}) < \mathcal{A}_U(z).$$

However, to get the desired result, the above estimates have to be improved.

For $\mathcal{A}_K$, we need the estimates of $\dot{x}_j(t)$ and $\dot{x}_{N-j}(t)$ when t goes to zero. Let $\hat{z}$ and w_k, as well as $k \in \{j, N-j\}$, be defined as in (39) and (40). Then $\dot{x}_k(t) = \dot{u}_k(t) + \dot{\hat{x}}(t)$. Meanwhile, by (41),

$$\dot{u}_k = \dot{\rho}_k \cos\theta_k - \rho_k \dot{\theta}_k \sin\theta_k, \quad \forall k \in \{j, N-j\}.$$

As $\theta_j^+ = \lim_{t\to 0^+} \theta_j(t) = -\pi/2$, by Propositions 2.2 and 2.3, we get that

$$|\dot{u}_j(t)| \leqq C_1 t^{\frac{2}{3}} \quad \text{for } t > 0 \text{ small enough.} \tag{43}$$

Similarly, since $\theta_{N-j}^+ = \theta_j^+ \pi = \pi/2$, we have that

$$|\dot{u}_{N-j}(t)| \leqq C_2 t^{\frac{2}{3}} \quad \text{for } t > 0 \text{ small enough.} \tag{44}$$

Notice that although m_j and m_{N-j} collide at $t = 0$, there the center of mass $\hat{z}(t)$ is C^2 at $t = 0$, and can we claim that

$$\dot{\hat{x}}(0) = 0.$$

Otherwise, let's assume that $\dot{\hat{x}}(0) < 0$. Then, by (43),

$$\dot{x}_j(t) = \dot{u}_j(t) + \dot{\hat{x}}(t) < 0, \text{ for } t > 0 \text{ small enough.}$$

This, however, violates the monotone constraints. Similarly, if $\dot{\hat{x}}(0) > 0$, then

$$\dot{x}_{N-j}(t) = \dot{u}_{N-j}(t) + \dot{\hat{x}}(t) > 0, \text{ for } t > 0 \text{ small enough,}$$

which is again in contradiction to the monotone constraints. This proves our claim. As a result, we have

$$|\dot{\hat{x}}(t)| \leqq C_3 t \text{ for } t > 0 \text{ small enough.} \tag{45}$$

Combining this with (43) and (44) we get

$$|\dot{x}_k(t)| \leqq C_4 t^{\frac{2}{3}} \text{ for } t > 0 \text{ small enough, } \forall k \in \{j, N-j\}. \tag{46}$$

By the definition of z^{ε_1} and (46), we have

$$\begin{aligned} \mathcal{A}_K(z^{\varepsilon_1}) - \mathcal{A}_K(z) &= \frac{1}{2}\int_0^{\varepsilon_1} \sum_{k\in\{j,N-j\}} \left(|\dot{z}_k^{\varepsilon_1}| - |\dot{z}_k|^2\right) \mathrm{d}t \\ &= \int_0^{\varepsilon_1} 4(\varepsilon_1 - t)^2 + 2(\varepsilon_1 - t)\dot{x}_j(t) - 2(\varepsilon_1 - t)\dot{x}_{N-j}(t)\, \mathrm{d}t \\ &\leqq 4\int_0^{\varepsilon_1} (\varepsilon_1 - t)^2 + C_4 t^{\frac{2}{3}}(\varepsilon_1 - t)\, \mathrm{d}t \leqq C_5 \varepsilon_1^{\frac{8}{3}} + o(\varepsilon_1^{\frac{8}{3}}). \end{aligned} \tag{47}$$

Now we will estimate the change in potential energy. By Lemmas 2.3 and 2.4, we have this $x_j(1/2) - x_{N-j}(1/2) > 0$. Then for any $\delta > 0$ small enough, there exist positive constants C_6 and C_7 (both independent of ε_1) such that

$$x_j(t) - x_{N-j}(t) \geqq C_6, \quad |z_j(t) - z_{N-j}(t)|^{-1} \geqq C_7, \quad \forall t \in [1/2-\delta, 1/2]. \tag{48}$$

Meanwhile for any $t \in [1/2-\delta, 1/2]$,

$$|z_j^{\varepsilon_1}(t) - z_{N-j}^{\varepsilon_1}(t)|^{-1} = \left(|z_j(t) - z_{N-j}(t)|^2 + 4\varepsilon_1^2(x_j(t) - x_{N-j}(t)) + 4\varepsilon_1^4\right)^{-\frac{1}{2}}.$$

By (48) and a simple computation, we get, for any $t \in [1/2-\delta, 1/2]$,

$$\frac{1}{|z_j^{\varepsilon_1}(t) - z_{N-j}^{\varepsilon_1}(t)|} - \frac{1}{|z_j(t) - z_{N-j}(t)|} \leqq \frac{-2(x_j(t) - x_{N-j}(t))}{|z_j(t) - z_{N-j}(t)|^3}\varepsilon_1^2 + o(\varepsilon^2).$$

Notice that the definition of z^{ε_1} implies

$$|z_k^{\varepsilon_1}(t) - z_l^{\varepsilon_1}(t)| \geqq |z_k(t) - z_l(t)|, \quad \forall t \in [0, 1/2], \quad \forall \{k \neq l\} \subset \mathbf{N}. \tag{49}$$

As a result,

$$\mathcal{A}_U(z^{\varepsilon_1}) - \mathcal{A}_U(z) \leqq \int_{\frac{1}{2}-\delta}^{\frac{1}{2}} \frac{1}{|z_j^{\varepsilon_1} - z_{N-j}^{\varepsilon_1}|} - \frac{1}{|z_j - z_{N-j}|}\,\mathrm{d}t \leqq -C_8\delta\varepsilon_1^2.$$

Combining this with our estimate on $\mathcal{A}_K$ obtained earlier, we get

$$\mathcal{A}(z^{\varepsilon_1}) - \mathcal{A}(z) \leqq -C_8\delta\varepsilon^2 + C_5\varepsilon_1^{\frac{8}{3}} < 0$$

for ε_1 small enough, as C_5, C_8 are independent of ε_1. □

By Lemmas 2.5 and 2.6, we have proven that there is no collision between m_j and m_{N-j} at $t=0$, for any $j \in \{1, \ldots, \mathfrak{n}\}$. Similarly, it can be proven that there is no collision between m_j and m_{N-1-j} at $t = 1/2$ for any $j \in \{0, \ldots, \mathfrak{n}\}$; we will not repeat it here. As a result, we have proven the following:

Proposition 2.5. *For any $\omega \in \Omega_N$, the action functional $\mathcal{A}$ has at least one minimizer $z^\omega \in \Lambda_\omega^+$. Furthermore, every action minimizer z^ω satisfies the strictly monotone constraints and is a collision-free solution of* (1).

However, we still haven't proven (15) in Theorem 1.1. This will be demonstrated by the next lemma. Notice that this is not necessarily true, even when the strictly monotone constraints are satisfied.

Lemma 2.7. *For any $\omega \in \Omega_N$, if $z^\omega \in \Lambda_\omega^+$ is a minimizer of $\mathcal{A}$ in Λ_ω^+, then it satisfies* (15).

Proof. For simplicity, let $z = z^\omega$. We give the details for $N = 2n$ (for $N = 2n+1$, it can be proven similarly). By (10), this is equivalent to proving the following:

$$\dot{x}_0(0) = \dot{x}_{\mathfrak{n}+1}(0) = 0, \tag{50}$$

$$\begin{cases} \dot{x}_0(t) > 0, & \forall t \in (0, 1/2], \\ \dot{x}_j(t) > 0, & \forall t \in [0, 1/2],\ \forall j \in \{1, \ldots, \mathfrak{n}\}, \end{cases} \tag{51}$$

$$\begin{cases} \dot{x}_{\mathfrak{n}+1}(t) < 0, & \forall t \in (0, 1/2], \\ \dot{x}_j(t) < 0, & \forall t \in [0, 1/2],\ \forall j \in \{\mathfrak{n}+2, \ldots, N-1\}. \end{cases} \tag{52}$$

Since z is a collision-free minimizer of $\mathcal{A}$ in Λ_ω^+, $z_0(t)$ and $z_{\mathfrak{n}+1}(t)$ must be perpendicular to the real axis at $t = 0$, and (50) follows immediately.

Now let's prove (51) (the proof of (52) is similarly); by a contradiction argument, we assume that it does not hold. Then, by the strictly monotone constraints, $\dot{x}_j(t_0) = 0$ for some $t_0 \in [0, 1/2]$, and $j \in \{0, \ldots, \mathfrak{n}\}$ (if $t_0 = 0$, $j \neq 0$). Depending on the value of t_0, three different cases will be considered.

Case 1: $t_0 = 0$. For $\varepsilon_1 > 0$ small enough, let $z^{\varepsilon_1} \in \Lambda_\omega^+$ be the same path defined in the proof of Lemma 2.6. Then

$$\mathcal{A}_U(z^{\varepsilon_1}) - \mathcal{A}_U(z) \leqq -C_1\varepsilon_1^2. \tag{53}$$

Meanwhile, as $\dot{x}_j(0) = \dot{x}_{N-j}(0) = 0$ (by the symmetric constraints, $\dot{x}_{N-j}(0) = -\dot{x}_j(0)$), we have

$$|\dot{x}_k(t)| \leqq C_2 t \text{ for } t > 0 \text{ small enough}, \quad \forall k \in \{j, N-j\}.$$

This, in fact, is a better estimate than (46). Then, by a computation similar to (47), we get

$$\mathcal{A}_K(z^{\varepsilon_1}) - \mathcal{A}_K(z) \leqq C_3\varepsilon_1^3. \tag{54}$$

As a result, $\mathcal{A}(z^{\varepsilon_1}) - \mathcal{A}(z) < 0$ for $\varepsilon_1 > 0$ small enough, which is a contradiction. This finishes our proof of *Case 1*.

For the remaining two cases, estimates similar to those above will give us contradictions as well. We just give the definition of $z^{\varepsilon_1} \in \Lambda_\omega^+$ and omit the details.

Case 2: $t_0 = 1/2$. Let $z^{\varepsilon_1} \in \Lambda_\omega^+$ be defined as follows:

$$z_j^{\varepsilon_1}(t) = \begin{cases} z_j^{\varepsilon_1}(t) - \varepsilon_1^2, & \forall t \in [0, 1/2 - \varepsilon_1], \\ z_j^{\varepsilon_1}(t) - (1/2 - t)(2\varepsilon_1 - (1/2 - t)), & \forall t \in [1/2 - \varepsilon_1, 1/2], \end{cases}$$

$$z_{N-j-1}^{\varepsilon_1}(t) = \begin{cases} z_{N-j-1}^{\varepsilon_1}(t) + \varepsilon_1^2, & \forall t \in [0, 1/2 - \varepsilon_1], \\ z_{N-j-1}^{\varepsilon_1}(t) + (1/2 - t)(2\varepsilon_1 - (1/2 - t)), & \forall t \in [1/2 - \varepsilon_1, 1/2], \end{cases}$$

$$z_k^{\varepsilon_1}(t) = \begin{cases} z_k^{\varepsilon_1}(t) + \varepsilon_1^2, & \text{if } k \in \{j+1, \ldots, N-j-2\}, \\ z_k^{\varepsilon_1}(t) - \varepsilon_1^2, & \text{if } k \in \mathbf{N}\backslash\{j, \ldots, N-j-1\}, \end{cases} \quad \forall t \in [0, 1/2].$$

Case 3: $t_0 \in (0, 1/2)$. Let $z^{\varepsilon_1} \in \Lambda_\omega^+$ be defined as follows:

$$z_j^{\varepsilon_1}(t) = \begin{cases} z_j^{\varepsilon_1}(t) - \varepsilon_1^2, & \forall t \in [0, t_0 - \varepsilon_1], \\ z_j^{\varepsilon_1}(t) + (t - t_0)(2\varepsilon_1 - |t - t_0|), & \forall t \in [t_0 - \varepsilon_1, t_0 + \varepsilon_1], \\ z_j^{\varepsilon_1}(t) + \varepsilon_1^2, & \forall t \in [t_0 + \varepsilon_1, 1/2], \end{cases}$$

$$z_k^{\varepsilon_1}(t) = \begin{cases} z_k^{\varepsilon_1}(t) + \varepsilon_1^2, & \text{if } k \in \{j+1, \ldots, N-j-1\}, \\ z_k^{\varepsilon_1}(t) - \varepsilon_1^2, & \text{if } k \in \mathbf{N}\backslash\{j, \ldots, N-j-1\}, \end{cases} \quad \forall t \in [0, 1/2].$$

□

We finish this section with a remark about the existence of these simple choreographies in the general homogeneous potentials

$$U_\alpha(z) = \sum_{\{j<k\}\subset \mathbf{N}} \frac{1}{|z_j - z_k|^\alpha}, \quad \alpha > 0.$$

Such a potential is called a *strong force* if $\alpha \geqq 2$; a *weak force* if $0 < \alpha < 2$. The Newtonian potential corresponds to $\alpha = 1$.

All the results proved in this section hold for U_α with $\alpha \geqq 1$, but not when $0 < \alpha < 1$. The reason for this is that when $\alpha < 1$, Proposition 2.4 only holds for $\theta_j^+ \in (-\pi/2 + \beta(\alpha), \pi/2]$, for some $0 < \beta(\alpha) < \pi$.

3. Simple Choreographies with Additional Symmetries

As we mentioned, all of the simple choreographies that were proven in Section 2 belong to the linear chain family, as are the Figure-Eight solution and the Super-Eight solution. We would like to compare our results with those two. Recall that there are several different figure eight type simple choreographies, depending on the symmetric constraints, see [8,14]. Here, by the Figure-Eight solution, we only mean the one proved by Chenciner and Montgomery in [13].

Example 3.1. The Figure-Eight solution, z^e, is a collision-free minimizer of the action functional $\mathcal{A}$ in $\Lambda_3^{D_6}$ with the action of the dihedral group D_6 defined as

$$\tau(g)t = t - 1/2, \quad \rho(g)q = -\bar{q}, \quad \sigma(g) = (0, 1, 2)^2,$$
$$\tau(h)t = -t + 1, \quad \rho(h)q = \bar{q}, \quad \sigma(h) = (0, 2).$$

The corresponding loop $z_0^e \in H^1(\mathbb{R}/3\mathbb{Z}, \mathbb{C})$ is in the shape of a figure eight symmetric with respect to the origin and the real and imaginary axes. Furthermore, it was proved in [15,18] that each lobe of the eight is convex.

Example 3.2. The Super-Eight solution z^s belongs to $\Lambda_4^{D_4\times\mathbb{Z}_2}$, where D_4 is the dihedral group with the action defined as in Section 1, and $\mathbb{Z}_2 = \langle f | f^2 = 1\rangle$ with the action defined as follows:

$$\tau(f)t = t, \quad \rho(f)q = -q, \quad \sigma(f) = (0, 2)(1, 3).$$

However, the action minimizer of $\mathcal{A}$ in $\Lambda_4^{D_4\times\mathbb{Z}_2}$ is not z^s, but the rotating 4-gon.

Thus, z^s is an action minimizer among all $z \in \Lambda_4^{D_4\times\mathbb{Z}_2}$ satisfying the following topological constraints:

$$y_0(j/2) = \omega_j |y_2(j/2)|, \quad \forall j \in \{1, 2, 3\},$$

where $\omega = (\omega_1, \omega_2, \omega_3) = (1, -1, 1)$.

This was proved analytically by Shibayama in [25], who also confirmed numerically that it should look like a super eight.

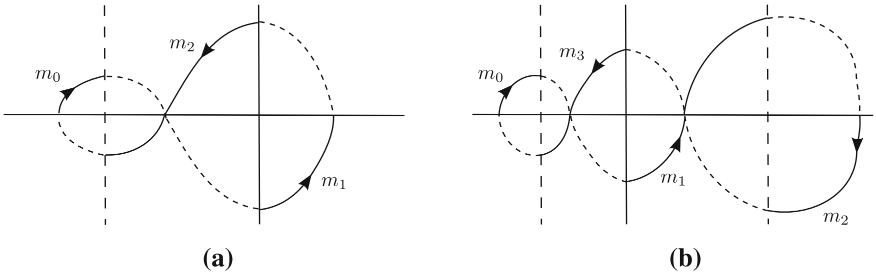

Fig. 3. **a** $N = 3, \omega = (1, -1)$, **b** $N = 4, \omega = (1, -1, 1)$

Meanwhile, for any $\omega \in \Omega_N$, let $z^\omega \in \Lambda_N^{D_N}$ be a simple choreography obtained by Theorem 1.1.

Example 3.3. When $N = 3$ and $\omega = (\omega_1, \omega_2) = (-1, 1)$, an illuminating picture of z^ω can be found in Fig. 3a, where the solid curves represent the motion of the masses between $t = 0$ and $t = 1/2$. Such a z^ω also looks like a figure eight. The symmetric constraints only imply that the loop z_0 is symmetric with respect to the real axis. However, unlike the Figure-Eight solution, it is not clear whether the loop is also symmetric with respect to the imaginary axis and the origin.

Example 3.4. When $N = 4$ and $\omega = (\omega_1, \omega_2, \omega_3) = (1, -1, 1)$, an illuminating picture of z^ω can be found in Fig. 3b, where the solid curves represent the motion of the masses between $t = 0$ and $t = 1/2$. Such a z^ω looks like a super eight that is symmetric with respect to the real axis, but unlike the Super-Eight solution, we don't know if it is also symmetric with respect to the imaginary axis and the origin.

The similarities and differences between the above examples inspire us to introduce the symmetric groups $H_N := D_N \times \mathbb{Z}_2$ and the define the actions as follows: if $N = 2n$,

$$\begin{aligned}
&\tau((g,1))t = t-1, \quad \rho((g,1))q = q, \quad \sigma((g,1)) = (0,\dots,2n-1),\\
&\tau((h,1))t = 1-t, \quad \rho((h,1))q = \bar{q}, \quad \sigma((h,1)) = (0,2n-1)\dots(n-1,n),\\
&\tau((1,f))t = t, \quad \rho((1,f))q = -q, \quad \sigma((1,f)) = (0,n)\dots(n-1,2n-1);
\end{aligned} \tag{55}$$

and if $N = 2n + 1$,

$$\begin{aligned}
&\tau((g,f))t = t-1/2, \quad \rho((g,f))q = -\bar{q}, \quad \sigma((g,f)) = (0,\dots,2n)^{n+1},\\
&\tau((h,1))t = 1-t, \quad \rho((h,1))q = \bar{q}, \quad \sigma((h,1)) = (0,2n)\dots(n-1,n+1).
\end{aligned} \tag{56}$$

Notice that when $N = 2n + 1$,

$$H_{2n+1} \cong D_{4n+2}\left\langle (g,f),(h,1) | (g,f)^{4n+2} = (h,1)^2 = 1, \big((g,f)(h,1)\big)^2 = 1 \right\rangle.$$

In particular, $(g, 1) = (g, f)^{2n+2}$ and

$$\tau((g,1))t = t-(n+1), \quad \rho((g,1))q = q, \quad \sigma((g,1)) = \sigma((g,f)),$$

As a result, for any N, the action of D_N induced from H_N is the same as the action defined in Section 1.

Let $\Lambda_N^{H_N,+}$ be the subset of all loops in $\Lambda_N^{H_N}$ satisfying the monotone constraints given in Definition 2.1. For each $\omega \in \Omega_N$ with

$$|\omega_j - \omega_{N-j}| = \begin{cases} 2, & \text{if } N = 2n+1, \\ 0, & \text{if } N = 2n, \end{cases} \quad \forall j \in \{1, \dots, \mathfrak{n}\}, \tag{57}$$

let $\Lambda_{N,\omega}^{H_N,+}$ be the subset of all loops in $\Lambda_N^{H_N,+}$ satisfying the ω-topological constraints. Then we have the following result:

Theorem 3.1. *For each $\omega \in \Omega_N$ satisfying (57), the action functional $\mathcal{A}$ has at least one minimizer $z^\omega \in \Lambda_{N,\omega}^{H_N,+}$. Furthermore, z^ω is a collision-free simple choreography of Eq. (1) satisfying all the properties in Theorem 1.1 and the following:*

$$z_0^\omega(t) = \begin{cases} -z_0^\omega(N/2 - t), & \textit{if } N = 2n+1, \\ -\bar{z}_0^\omega(N/2 - t), & \textit{if } N = 2n, \end{cases} \quad \forall t \in \mathbb{R}/N\mathbb{Z}.$$

Remark 3.1. (i) The above theorem can be proven by the same argument used in the proof of Theorem 1.1. However, during the deformation one needs to make sure that the additional symmetric constraints are satisfied afterwards. Similar problems, and the detailed arguments there to appertaining, can be found in another paper by the author [30].

(ii) Condition (57) is implied by the symmetric constraints of H_N. For those ω's satisfying this condition, it will be interesting to see whether the simple choreographies obtained by Theorems 1.1 and 3.1 are actually the same. Similar questions were asked by Chenciner regarding the figure eight type solutions in [8].

(iii) When $N = 3$, $H_3 \cong D_6$ and the action of H_3 defined above are identical to that given in Example 3.1. Similarly, when $N = 4$, the action of H_4 is the same as the one given in Example 3.2. Hence, for N and ω, given in Examples 3.3 and 3.4, the simple choreographies obtained by Theorem 3.1 are exactly the Figure-Eight solution and the Super-Eight solution.

As we mentioned, in [25] the actual shape of the Super-Eight solution was confirmed numerically. In the remainder of this section, we prove a result that *almost* confirms the shape of the Super-Eight solution without any numerical result.

Proposition 3.1. *Let z^ω be the simple choreography obtained in Theorem 3.1 with $N = 4$ and $\omega = (-1, 1, -1)$. Then, between $t = 0$ and $t = 1/2$, $z_1^\omega(t)$, as well as $z_3^\omega(t)$, has exactly one transversal intersection with the real axis; $z_0^\omega(t)$, as well as $z_2^\omega(t)$, has at least one and at most two transversal intersections with the real axis.*

Proof. For simplicity, let $z = z^\omega$. The four masses always form a parallelogram:

$$z_3(t) = -z_1(t), \quad z_0(t) = z_2(t), \quad \forall t \in \mathbb{R}.$$

As a result, the kinetic and negative potential energy only depend on z_1 and z_2:

$$K(\dot{z}) = K(\dot{z}_1, \dot{z}_2) = |\dot{z}_1|^2 + |\dot{z}_2|^2;$$
$$U(z) = U(z_1, z_2) = \frac{1}{2|z_1|} + \frac{1}{2|z_2|} + \frac{2}{|z_1 - z_2|} + \frac{2}{|z_1 + z_2|}.$$

By Theorem 3.1,

$$y_1(0) < 0 < y_1(1/2), \quad y_2(0) = 0 > y_2(1/2), \tag{58}$$
$$0 = x_1(0) < x_1(t) < x_1(1/2) = x_2(1/2) < x_2(t) < x_2(0), \quad \forall t \in (0, 1/2). \tag{59}$$

The key to our proof is the following feature of the parallelogram 4-body problem that we borrowed from [25]:

$$U(z_1, z_2) > U(\bar{z}_1, z_2) = U(z_1, \bar{z}_2), \quad \text{if } z_1, z_2 \text{ belongs to the same quadrant.} \tag{60}$$

By (58), $z_1(t)$ has at least one transversal intersection with the real axis. We claim that this is the only one. Otherwise, let's assume that $0 < t_0 < t_1 < t_2 < 1/2$ are the three earliest moments that a transversal intersection occurs. We define two subsets $\mathbb{T}_1$, $\mathbb{T}_2$ of $[0, 1/2]$ as follows:

$$\begin{aligned}
&\text{if } y_2(t_1) \leqq 0,\\
&\qquad \mathbb{T}_1 := \{t \in [t_1, 1/2] : y_1(t) < 0\}, \quad \mathbb{T}_2 := \{t \in [t_1, 1/2] : y_2(t) > 0\};\\
&\text{if } y_2(t_1) > 0, \ y_2(t_0) \leqq 0,\\
&\qquad \mathbb{T}_1 := \{t \in [t_0, 1/2] : y_1(t) < 0\}, \quad \mathbb{T}_2 := \{t \in [t_0, 1/2] : y_2(t) > 0\};\\
&\text{if } y_2(t_1) > 0, \ y_2(t_0) > 0,\\
&\qquad \mathbb{T}_1 := \{t \in [t_0, t_1] : y_1(t) > 0\}, \quad \mathbb{T}_2 := \{t \in [t_0, t_1] : y_2(t) < 0\},
\end{aligned}$$

and a new path $\tilde{z}(t) = (-\tilde{z}_2(t), \tilde{z}_1(t), \tilde{z}_2(t), -\tilde{z}_1(t))$, as follows (see Fig. 4):

$$\tilde{z}_j(t) = \begin{cases} \bar{z}_j(t) & \text{if } t \in \mathbb{T}_j; \\ z_j(t) & \text{if } t \in [0, 1/2]\backslash\mathbb{T}_j, \end{cases} \qquad \forall j \in \{1, 2\}.$$

Obviously, $\int_0^{1/2} K(\dot{z})\,\mathrm{d}t = \int_0^{1/2} K(\dot{\tilde{z}})\,\mathrm{d}t$. Meanwhile, (60) implies that

$$\int_0^{1/2} U(z_1, z_2)\,\mathrm{d}t \geqq \int_0^{1/2} U(\tilde{z}_1, \tilde{z}_2)\,\mathrm{d}t.$$

Hence, $\mathcal{A}(\tilde{z}) \leqq \mathcal{A}(z)$. This means that $\tilde{z}$ must be a collision-free minimizer and then a smooth solution of (1). However, under our assumption, and the way that is defined, $\tilde{z}$ cannot be smooth, which is a contradiction. This proves the first half of the proposition. For the second half, as $z_2(t)$ already has an transversal intersection with the real axis at $t = 0$, a similar argument as to that above shows that $z_2(t)$ can have at most one transversal intersection with the real axis besides the one at $t = 0$. □

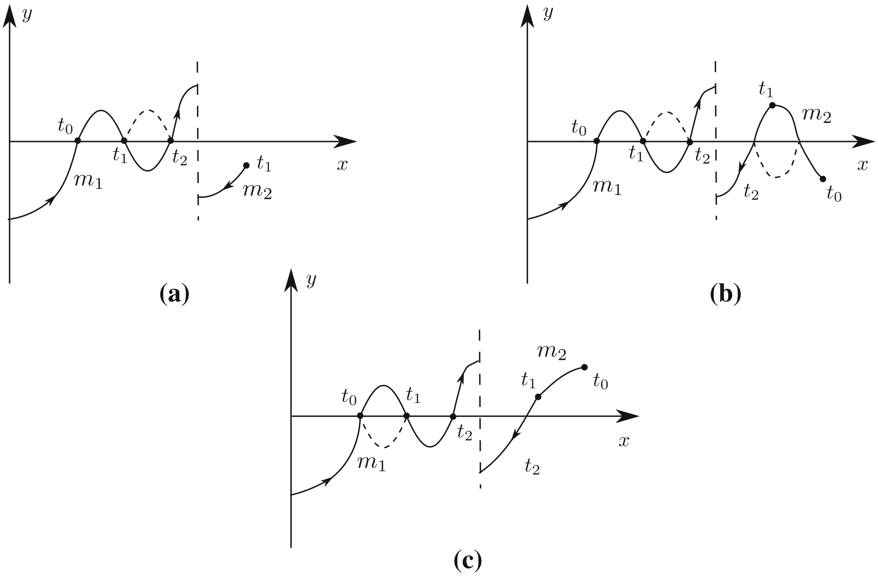

Fig. 4. **a** $y_2(t_1) \leqq 0$, **b** $y_2(t_1) > 0$, $y_2(t_2) \leqq 0$, **c** $y_2(t_1) > 0$, $y_2(t_2) > 0$

4. Appendix

We give a proof of Lemma 2.2 in this section. Given a $z \in H^1([T_1, T_2], \mathbb{C}^N)$, we say it is a generalized solution of (1) if $\Delta(z) := \{t \in [T_1, T_2] : z(t) \notin \hat{\mathbb{C}}^N\}$ has measure zero and $z(t)$ is a C^2 solution of (1) in $[T_1, T_2]\backslash\Delta(z)$. Assuming $t_0 \in \Delta(z)$, then there is a subset of indices $\mathbf{I} \subset \mathbf{N}$, such that

$$z_j(t_0) = z_k(t_0), \forall\{j \neq k\} \subset \mathbf{I}, \quad z_j(t_0) \neq z_k(t_0), \forall j \in \mathbf{I}, \ \forall k \in \mathbf{N}\backslash\mathbf{I}.$$

If there is a $\delta > 0$ such that $((t_0 - \delta, t_0 + \delta) \cap [T_1, T_2])\backslash\{t_0\} \subset [T_1, T_2]\backslash\Delta(z)$, then t_0 is an isolated (**I**-cluster) collision moment and $z(t_0)$ will be called an isolated (**I**-cluster) collision.

Furthermore, let the Lagrange functional and the corresponding action functional of the **I**-body sub-system be defined as

$$L_{\mathbf{I}}(z, \dot{z}) := K_{\mathbf{I}}(\dot{z}) + U_{\mathbf{I}}(z), \quad \mathcal{A}_{\mathbf{I}}(z, T) := \int_0^T L_{\mathbf{I}}(z, \dot{z})\, \mathrm{d}t,$$
$$K_{\mathbf{I}}(\dot{z}) := \frac{1}{2}\sum_{j\in\mathbf{I}} |\dot{z}_j|^2, \quad U_{\mathbf{I}}(z) := \sum_{\{j<k\}\in\mathbf{I}} \frac{1}{|z_j - z_k|}.$$

We define $\mathfrak{T} \subset \{0, \pm 1\}^N$ as

$$\mathfrak{T} := \big\{\tau = (\tau_j)_{j\in\mathbf{N}} |\ \tau_j \neq \tau_k, \text{ for some } \{j \neq k\} \subset \mathbf{I} \text{ and } \tau_l = 0, \ \forall l \in \mathbf{N}\backslash\mathbf{I}\big\}. \tag{61}$$

Then we have the following three local deformation lemmas:

Lemma 4.1. *Given a generalized solution* $z \in H^1([T_1, T_2], (i\mathbb{R})^N)$*, if* $t_0 = T_1$ *is an isolated* $\mathbf{I}$*-cluster collision moment, then for* $\varepsilon_0 > 0$ *small enough and* $\tau \in \mathfrak{T}$*, there exists a* $h \in H^1([t_0, t_0 + \delta], \mathbb{R})$ *and a path* $z^{\varepsilon_0} = (z_j^{\varepsilon_0})_{j \in \mathbf{N}} \in H^1([t_0, t_0 + \delta], \mathbb{C}^N)$*, with* $z_j^{\varepsilon_0}(t) = z_j(t) + \varepsilon_0 h(t)\tau_j$*,* $t \in [t_0, t_0 + \delta]$*,* $\forall j \in \mathbf{N}$*, satisfying* $\mathcal{A}(z^{\varepsilon_0}, t_0, t_0 + S) < \mathcal{A}(z, t_0, t_0 + \delta)$ *and the following:*

(a) $h(t) = 1$*,* $\forall t \in [t_0, t_0 + \delta_1]$*, for some* $0 < \delta_1 = \delta_1(\varepsilon_0) < \delta$ *small enough;*
(b) $h(t) = 0$*,* $\forall t \in [t_0 + \delta_2, t_0 + \delta]$*, for some* $\delta_1 < \delta_2 = \delta_2(\varepsilon_0) < \delta$ *small enough;*
(c) $h(t)$ *is decreasing for* $t \in [t_0 + \delta_1, t_0 + \delta_2]$*.*

Lemma 4.2. *Given a generalized solution* $z \in H^1([T_1, T_2], (i\mathbb{R})^N)$*, if* $t_0 = T_2$ *is an isolated* $\mathbf{I}$*-cluster collision moment, then for* $\varepsilon_0 > 0$ *small enough and* $\tau \in \mathfrak{T}$*, there exists a* $h \in H^1([t_0 - \delta, t_0], \mathbb{R})$ *and a path* $z^{\varepsilon_0} = (z_j^{\varepsilon_0})_{j \in \mathbf{N}} \in H^1([t_0 - \delta, t_0], \mathbb{C}^N)$*, with* $z_j^{\varepsilon_0}(t) = z_j(t) + \varepsilon_0 h(t)\tau_i$*,* $t \in [t_0 - \delta, t_0]$ *and* $\forall j \in \mathbf{N}$*, satisfying* $\mathcal{A}(z^{\varepsilon_0}, t_0 - \delta, t_0) < \mathcal{A}(z, t_0 - \delta, t_0)$ *and the following:*

(a) $h(t) = 1$*,* $\forall t \in [t_0 - \delta_1, t_0]$*, for some* $0 < \delta_1 = \delta_1(\varepsilon_0) < \delta$ *small enough;*
(b) $h(t) = 0$*,* $\forall t \in [t_0 - \delta, t_0 - \delta_2]$*, for some* $\delta_1 < \delta_2 = \delta_2(\varepsilon_0) < \delta$ *small enough;*
(c) $h(t)$ *is increasing for* $t \in [t_0 - \delta_2, t_0 - \delta_1]$*.*

While the above two lemmas can be used on isolated collision moments at the boundary moments of a fundamental domain, the next one is for the interior moments.

Lemma 4.3. *Given a generalized solution* $z \in H^1([T_1, T_2], (i\mathbb{R})^N)$*, if* $t_0 \in (T_1, T_2)$ *is an isolated* $\mathbf{I}$*-cluster collision moment, then for* $\varepsilon_0 > 0$ *small enough and* $\tau \in \mathfrak{T}$*, there exists a* $h \in H^1([t_0 - \delta, t_0 + \delta], \mathbb{R})$ *and a path* $z^{\varepsilon_0} = (z_j^{\varepsilon_0})_{j \in \mathbf{N}} \in H^1([t_0 - \delta, t_0 + \delta], \mathbb{C}^N)$*, with* $z_i^{\varepsilon_0}(t) = z_i(t) + i\varepsilon_0 h(t)\tau_i$*,* $t \in [t_0 - \delta, t_0 + \delta]$ *and* $\forall j \in \mathbf{N}$*, satisfying* $\mathcal{A}(z^{\varepsilon_0}, t_0 - \delta, t_0 + \delta) < \mathcal{A}(z, t_0 - \delta, t_0 + \delta)$ *and the following:*

(a) $h(t) = 1$*,* $\forall t \in [t_0 - \delta_1, t_0 + \delta_1]$*, for some* $0 < \delta_1 = \delta_1(\varepsilon_0) < \delta$ *small enough;*
(b) $h(t) = 0$*,* $\forall t \in [t_0 - \delta, t_0 - \delta_2] \cup [t_0 + \delta_2, t_0 + \delta]$*, for some* $\delta_1 < \delta_2 = \delta_2(\varepsilon_0) < \delta$ *small enough;*
(c) $h(t)$ *is increasing for* $t \in [t_0 - \delta_2, t_0 - \delta_1]$*;*
(d) $h(t)$ *is decreasing for* $t \in [t_0 + \delta_1, t_0 + \delta_2]$*.*

Remark 4.1. Although we only consider the planar problem here, for the general N-body problem in $\mathbb{R}^d$ with $d \geqq 2$, $z(t)$ is a solution with an isolated collision, satisfying $z_j(t) \in V, \forall t$ and $\forall j \in \mathbf{N}$, where V is a $d-1$ dimension linear subspace of $\mathbb{R}^d$. When we only try to deform the path $z(t)$ along the directions orthogonal to V, then results similar to those at Lemmas 4.1–4.3 can be proven following the approach given below.

To prove the above lemmas, we combine a local deformation technique introduced by MONTGOMERY in [21] and a blow-up technique by Terracini (see [14,27,29]). The main difference is that in our setting the configurations or τ's that serve as the directions of deformation need not be a *centered configuration* (see Definition 4.1), while in the above references this condition needs to be satisfied. In general, without this condition, the action value of the deformed path may not be strictly smaller than the original one. In our setting, however, all the masses will be traveling on a single vertical line, and we are only making deformations along directions that are orthogonal to this vertical line, so the desired results will still hold.

We will only give a proof of Lemma 4.1 (Lemma 4.2 can be proven similarly after reversing the time parameter, and Lemma 4.3 follows directly once we have the previous two lemmas). Let z be a generalized solution satisfying all the conditions given in Lemma 4.1. For simplicity, let's assume $t_0 = T_1 = 0$. Then we have

Definition 4.1. We say that w is an $\mathbf{I}$-configuration if $w = (w_j)_{j\in\mathbf{I}} \in \mathbb{C}^{|\mathbf{I}|}$, and that is a centered $\mathbf{I}$-configuration if $\sum_{j\in\mathbf{I}} w_j = 0$.

Let $\hat{z}(t) = \hat{x}(t) + i\hat{y}(t) = \frac{1}{|\mathbf{I}|}\sum_{j\in\mathbf{I}} z_j(t)$ be the center of mass of the $\mathbf{I}$-body sub-system. Since $x_j(t) \equiv 0$, $\forall j \in \mathbf{I}$, we have the same for $\hat{x}(t)$. As a result, we define $iv(t) = i(v_j(t))_{j\in\mathbf{I}}$ with

$$v_j(t) = y_j(t) - \hat{y}(t), \quad \forall j \in \mathbf{I}.$$

Each $iv_j(t)$ represents the relative position of m_j, $j \in \mathbf{I}$ with respect to the center of mass of the $\mathbf{I}$-body sub-system.

Definition 4.2. For any $0 < \lambda < 1$, we define $z^\lambda \in H^1([0, S/\lambda], \mathbb{C}^N)$, $z^\lambda(t) = \lambda^{-\frac{2}{3}} z(\lambda t)$ as the λ-blow-up of z, and $v^\lambda \in H^1([0, S/\lambda], \mathbb{R}^{|\mathbf{I}|})$, $v^\lambda(t) = \lambda^{-\frac{2}{3}} v(\lambda t)$ as the λ-blow-up of v.

Let $\sigma(t) = \frac{v(t)}{|v(t)|}$ be the normalization of $v(t)$, where $|v(t)| = \left(\sum_{j\in\mathbf{I}} |v_j(t)|^2\right)^{\frac{1}{2}}$. It is well known that as $\{t_n\} \searrow 0$, the limit of $i\sigma(t_n)$, if it exists, is a central configuration of the $\mathbf{I}$-body problem; for a proof of this see [14]. Since the space of all normalized $\mathbf{I}$-configurations is compact, we can always find that a sequence of positive numbers $\{\lambda_n\} \searrow 0$ with the following limits exist:

$$\lim_{n\to\infty} i\sigma(\lambda_n) = i\tilde{\sigma} = i(\tilde{\sigma}_j)_{j\in\mathbf{I}}, \quad \text{with } \tilde{\sigma}_j \in \mathbb{R}.$$

Then $i\tilde{\sigma}$ is a $\mathbf{I}$-central configuration, and $i\tilde{v}(t)$, $t \in [0, +\infty)$, given below, is a parabolic homothetic solution associated with $i\tilde{\sigma}$:

$$i\tilde{v}(t) = i(\tilde{v}_j(t))_{j\in\mathbf{I}}, \text{ where } \tilde{v}_j(t) = (\kappa t)^{\frac{2}{3}}\tilde{\sigma}_j, \quad \forall j \in \mathbf{I}, \tag{62}$$

where κ is a positive constant determined by $i\tilde{\sigma}$.

The parabolic homothetic solution $i\tilde{v}(t)$ is related to the isolated collision solution $z(t)$ through the blow-up's $v^{\lambda_n}(t)$ in the following way (for a proof see [14, (7.4)]):

Proposition 4.1. *For any $T > 0$, the sequences $\{v^{\lambda_n}\}$ and $\{\frac{dv^{\lambda_n}}{dt}\}$ converge to $\tilde{v}$ and its derivative $\dot{\tilde{v}}$ correspondingly. Furthermore, the convergences are uniform on $[0, T]$ and on compact subsets of $(0, T]$ correspondingly.*

A local deformation lemma of the above parabolic homothetic solution will be given first. The key to its proof is the following function first introduced by MONTGOMERY in [21]: for any $T > \varepsilon > 0$, let $f \in C([0, T], \mathbb{R})$ be defined as

$$f(t) = \begin{cases} 1, & \text{if } t \in [0, \varepsilon^{\frac{3}{2}}], \\ 1 + (\varepsilon^{\frac{3}{2}} - t)/\varepsilon, & \text{if } t \in [\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}} + \varepsilon], \\ 0, & \text{if } t \in [\varepsilon^{\frac{3}{2}} + \varepsilon, T]. \end{cases} \tag{63}$$

For any $\tau \in \mathfrak{T}$ and $\varepsilon > 0$ small enough, define $\tilde{v}^{\varepsilon}(t) = (\tilde{v}^{\varepsilon}_j(t))_{j \in \mathbf{I}}$ as

$$\tilde{v}^{\varepsilon}_j(t) = i\tilde{v}_j(t) + \varepsilon f(t)\tau_j, \quad t \in [0, +\infty). \tag{64}$$

Lemma 4.4. *For any $T > 0$, we have that $\mathcal{A}_{\mathbf{I}}(\tilde{v}^{\varepsilon}, T) < \mathcal{A}_{\mathbf{I}}(i\tilde{v}, T)$, for $\varepsilon > 0$ small enough.*

Proof. By the definition of $\tilde{v}^{\varepsilon}$ and $f(t)$, we have

$$\begin{aligned} \mathcal{A}_{\mathbf{I}}(\tilde{v}^{\varepsilon}, T) - \mathcal{A}_{\mathbf{I}}(i\tilde{v}, T) &= \int_{\varepsilon^{\frac{3}{2}}}^{\varepsilon^{\frac{3}{2}}+\varepsilon} K_{\mathbf{I}}(\tilde{v}^{\varepsilon}) - K_{\mathbf{I}}(i\tilde{v}) + \int_0^{\varepsilon^{\frac{3}{2}}} U_{\mathbf{I}}(\tilde{v}^{\varepsilon}) - U_{\mathbf{I}}(i\tilde{v}) \\ &\quad + \int_{\varepsilon^{\frac{3}{2}}}^{\varepsilon^{\frac{3}{2}}+\varepsilon} U_{\mathbf{I}}(\tilde{v}^{\varepsilon}) - U_{\mathbf{I}}(i\tilde{v}) \\ &:= A_1 + A_2 + A_3. \end{aligned}$$

We estimate each A_j separately in that which follows.

For A_1, notice that $|\dot{\tilde{v}}^{\varepsilon}_j|^2 = |\varepsilon \dot{f} \tau_j|^2 + |\dot{\tilde{v}}_j|^2$, $\forall j \in \mathbf{I}$. Then, by the definition of $f(t)$,

$$A_1 = \frac{1}{2} \sum_{j \in \mathbf{I}} \int_{\varepsilon^{\frac{3}{2}}}^{\varepsilon^{\frac{3}{2}}+\varepsilon} \tau_j^2 \, \mathrm{d}t = \frac{\varepsilon}{2} \sum_{j \in \mathbf{I}} \tau_j^2 = C_1 \varepsilon. \tag{65}$$

C_1 is a positive constant, as $\tau_j \neq 0$ for some $j \in \mathbf{I}$.

To estimate A_2, we introduce a new time parameter $s = t^{\frac{2}{3}}/\varepsilon$, then

$$\begin{aligned} A_2 &= \sum_{\{j<k\} \subset \mathbf{I}} \int_0^{\varepsilon^{\frac{3}{2}}} |\varepsilon(\tau_j - \tau_k) + i(\tilde{v}_j(t) - \tilde{v}_k(t))|^{-1} - |\tilde{v}_j(t) - \tilde{v}_k(t)|^{-1} \, \mathrm{d}t \\ &= \sum_{\{j<k\} \subset \mathbf{I}} \int_0^{\varepsilon^{\frac{3}{2}}} \left(\varepsilon^2(\tau_j - \tau_k)^2 + (\kappa t)^{\frac{4}{3}}(\tilde{\sigma}_j - \tilde{\sigma}_k)^2\right)^{-\frac{1}{2}} - |(\kappa t)^{\frac{2}{3}}(\tilde{\sigma}_j - \tilde{\sigma}_k)|^{-1} \, \mathrm{d}t \\ &= \frac{3\varepsilon^{\frac{1}{2}}}{2} \sum_{\{j<k\} \subset \mathbf{I}} \int_0^1 \Big\{\left((\tau_j - \tau_k)^2 + \kappa^{\frac{4}{3}} s^2 (\tilde{\sigma}_j - \tilde{\sigma}_k)^2\right)^{-\frac{1}{2}} \\ &\quad - |\kappa^{\frac{2}{3}} s(\tilde{\sigma}_j - \tilde{\sigma}_k)|^{-1}\Big\} s^{\frac{1}{2}} \, \mathrm{d}s \\ &= -C_2 \varepsilon^{\frac{1}{2}}. \end{aligned}$$

The last equality holds for some positive constant C_2, as $\tau_j \neq \tau_k$ for some $j \neq k \in \mathbf{I}$.

For A_3, notice that for any $\{j < k\} \subset \mathbf{I}$,

$$U_{j,k}(\tilde{v}^{\varepsilon}(t)) - U_{j,k}(i\tilde{v}(t)) \leqq 0, \quad \forall t \in [\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}} + \varepsilon].$$

Then $A_3 \leqq 0$. Combining the above estimates, we get, for $\varepsilon > 0$ small enough,

$$\mathcal{A}_{\mathbf{I}}(\tilde{v}^{\varepsilon}, T) - \mathcal{A}_{\mathbf{I}}(i\tilde{v}, T) \leqq C_1\varepsilon - C_2\varepsilon^{\frac{1}{2}} < 0.$$

□

Lemma 4.5. *For any $\tau \in \mathfrak{T}$ and $g(t) \in H^1([0,T],\mathbb{R})$, where $g(t)$ is C^1 in a neighborhood of T, we define $\phi = (\phi_j)_{j\in\mathbf{I}} \in H^1([0,T],\mathbb{R}^{|\mathbf{I}|})$ as $\phi_j(t) = g(t)\tau_j$, $\forall t \in [0,T]$, $\forall j \in \mathbf{I}$, and $\{\psi_n \in H^1([0,T],\mathbb{C})\}_{n\in\mathbb{Z}^+}$ as*

$$\psi_n(t) = \begin{cases} i\tilde{v}(t) - i\tilde{v}^{\lambda_n}(t) & \text{if } t \in \left[0, T - \frac{1}{N_n}\right], \\ N_n(T-t)\big(i\tilde{v}(t) - i\tilde{v}^{\lambda_n}(t)\big) & \text{if } t \in \left[T - \frac{1}{N_n}, T\right], \end{cases} \tag{66}$$

where $\{N_n\}$ is a sequence of positive integers going to infinity, then

$$\lim_{n\to\infty} \mathcal{A}(z^{\lambda_n} + \phi + \psi_n, T) - \mathcal{A}(z^{\lambda_n}, T) = \mathcal{A}_{\mathbf{I}}(\phi + i\tilde{v}, T) - \mathcal{A}_{\mathbf{I}}(i\tilde{v}, T).$$

Proof. For when $\phi(t)$ is a centered $\mathbf{I}$-configuration, the above result was proved in [14, (7.9)]. In this proof the condition was used to show that

$$L_{\mathbf{I}}(iv^{\lambda} + \phi) - L_{\mathbf{I}}(iv^{\lambda}) = L_{\mathbf{I}}(z^{\lambda} + \phi) - L_{\mathbf{I}}(z^{\lambda}), \quad \forall\lambda > 0. \tag{67}$$

In the following, we show that (67) still holds, even when $\phi(t)$ is not centered. It is enough to demonstrate this for $\lambda = 1$; the others are the same.

Recall that $iv_j(t) = z_j(t) - \hat{z}(t) = iy_j(t) - i\hat{y}(t)$, $\forall j \in \mathbf{I}$. Hence,

$$U_{\mathbf{I}}(iv(t)) = U_{\mathbf{I}}(z), \quad U_{\mathbf{I}}(\phi(t) + iv(t)) = U_{\mathbf{I}}(\phi(t) + z(t)), \tag{68}$$

$$\begin{aligned} 2K_{\mathbf{I}}(\dot{z}) &= \sum_{j\in\mathbf{I}} |\dot{z}_j|^2 = \sum_{j\in\mathbf{I}} |iy_j|^2 = \sum_{j\in\mathbf{I}} |i\dot{v}_j + i\dot{\hat{y}}|^2 \\ &= \sum_{j\in\mathbf{I}} |\dot{v}_j|^2 + |\mathbf{I}||\dot{\hat{y}}|^2 = 2K_{\mathbf{I}}(i\dot{v}) + |\mathbf{I}||\dot{\hat{y}}|^2. \end{aligned} \tag{69}$$

Meanwhile, by the fact that $\phi_j(t) \in \mathbb{R}$, for any $j \in \mathbf{I}$, we have

$$\begin{aligned} 2K_{\mathbf{I}}(z + \phi) &= \sum_{j\in\mathbf{I}} |i\dot{v}_j + i\dot{\hat{y}} + \dot{\phi}_j|^2 = \sum_{j\in\mathbf{I}} |i\dot{v}_j + i\dot{\hat{y}}|^2 + \sum_{j\in\mathbf{I}} |\dot{\phi}_j|^2 \\ &= \sum_{j\in\mathbf{I}} |\dot{v}_j|^2 + \sum_{j\in\mathbf{I}} |\dot{\phi}_j|^2 + |\mathbf{I}||\dot{\hat{y}}|^2 = \sum_{j\in\mathbf{I}} |i\dot{v}_j + \dot{\phi}_j|^2 + |\mathbf{I}||\dot{\hat{y}}|^2 \\ &= 2K_{\mathbf{I}}(i\dot{v} + \dot{\phi}) + |\mathbf{I}||\dot{\hat{y}}|^2. \end{aligned} \tag{70}$$

Combining (68), (69) and (70), we get

$$L_{\mathbf{I}}(z+\phi)-L_{\mathbf{I}}(iv+\phi)=L_{\mathbf{I}}(z)-L_{\mathbf{I}}(iv)=\frac{1}{2}|\mathbf{I}||\dot{\hat{y}}|^2.$$

This establishes (67). The rest of the proof is exactly the same as [14, (7.9)]. We will not repeat the details here. □

With the above results, we can now give a proof of Lemma 4.1.

Proof (Lemma 4.1). Without loss of generality, we set $t_0 = 0$. Now we choose $T \in (0, \delta)$. Letting $f(t)$ and $\{\lambda_n\}_{n\in\mathbb{Z}^+} \searrow 0$ be defined as above, for $\varepsilon > 0$ small, we set $\phi(t) = (\phi_j(t))_{j\in\mathbf{I}}$ as $\phi_j(t) = \varepsilon f(t)\tau_j$, $\forall t \in [0, T]$, $\forall j \in \mathbf{I}$. By Lemma 4.5, there is a sequence of functions $\{\psi_n\}_{n\in\mathbb{Z}^+}$ defined by (66) satisfying

$$\lim_{n\to\infty} \mathcal{A}(z^{\lambda_n}+\phi+\psi_n, T)-\mathcal{A}(z^{\lambda_n}, T)=\mathcal{A}_{\mathbf{I}}(i\tilde{v}+\phi, T)-\mathcal{A}_{\mathbf{I}}(\tilde{v}, T).$$

For each n, define a $\tilde{z}^{\lambda_n} \in H^1([0, \delta/\lambda_n], \mathbb{C}^N)$ as follows:

$$\tilde{z}^{\lambda_n}(t)=\begin{cases} z^{\lambda_n}(t) & \text{if } t\in[T,\delta/\lambda_n],\\ z^{\lambda_n}(t)+\phi(t)+\psi(t) & \text{if } t\in[0,T].\end{cases}$$

By the definition of $\tilde{z}^{\lambda_n}$ and Lemma 4.5, we have

$$\lim_{n\to\infty} \mathcal{A}(\tilde{z}^{\lambda_n}, \delta/\lambda_n)-\mathcal{A}(z^{\lambda_n}, \delta/\lambda_n)=\mathcal{A}_{\mathbf{I}}(i\tilde{v}+\phi, T)-\mathcal{A}_{\mathbf{I}}(i\tilde{v}, T).$$

For any $j \in \mathbf{I}$, by (64), we have $\tilde{v}^{\varepsilon}_j(t) = i\tilde{v}_j(t) + \varepsilon f(t)\tau_j = i\tilde{v}_j(t) + \phi_j(t)$. By Lemma 4.4,

$$\lim_{n\to\infty} \mathcal{A}(\tilde{z}^{\lambda_n}, \delta/\lambda_n)-\mathcal{A}(z^{\lambda_n}, \delta/\lambda_n)=\mathcal{A}_{\mathbf{I}}(\tilde{v}^{\varepsilon}, T)-\mathcal{A}_{\mathbf{I}}(i\tilde{v}, T)<0,$$

so for n large enough,

$$\mathcal{A}(\tilde{z}^{\lambda_n}, \delta/\lambda_n)<\mathcal{A}(z^{\lambda_n}, \delta/\lambda_n). \tag{71}$$

Now define $\tilde{z}_n \in H^1([0, \delta], \mathbb{C}^N)$ as $\tilde{z}_n(t) = \lambda_n^{\frac{2}{3}}\tilde{z}^{\lambda_n}(t/\lambda_n)$. By (71), a straightforward computation shows that $\mathcal{A}(\tilde{z}_n, \delta) < \mathcal{A}(z, \delta)$, for n large enough. Notice that for any $t \in [\lambda_n T, \delta]$, we have that $\tilde{z}_n(t) = z(t)$, and for any $t \in [0, \lambda_n T]$,

$$\tilde{z}_n(t)=z(t)+\lambda_n^{\frac{2}{3}}\big(\phi(t/\lambda_n)+\psi(t/\lambda_n)\big)=z(t)+\lambda_n^{\frac{2}{3}}\varepsilon f(t/\lambda_n)\tau+\lambda_n^{\frac{2}{3}}\psi(t/\lambda_n).$$

By the definitions of f and ψ, for n large enough, $\tilde{z}_n$ satisfies all of the conditions that are required for z^{ε_0}. This finishes our proof. □

The following result will be needed in our proof of Lemma 2.2:

Lemma 4.6. *For any $\omega \in \Omega_N$, let $z \in \Lambda^+_\omega \cap H^1(\mathbb{R}/N\mathbb{Z}, (i\mathbb{R})^N)$ be a generalized solution of (1), and if $t_0 \in (0, 1/2)$ is an isolated $\mathbf{I}$-cluster collision moment, then there is a $\tilde{z} \in \Lambda^+_\omega$ satisfying $\mathcal{A}(\tilde{z}, 1/2) < \mathcal{A}(z, 1/2)$.*

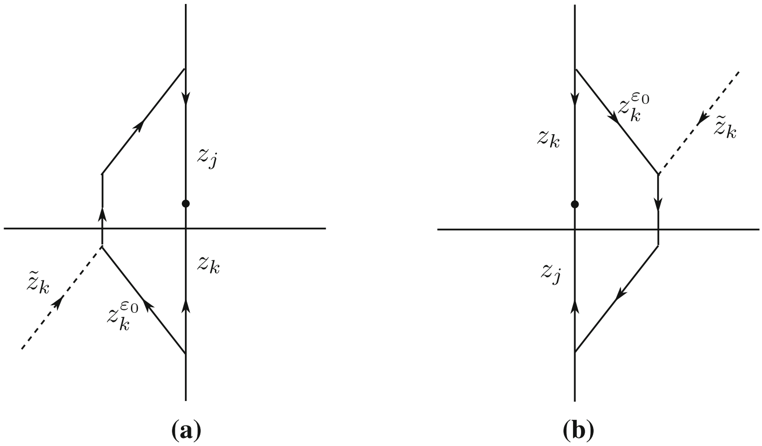

Fig. 5. Deformation of the path of m_k

Proof. If there is a $k \in \mathbf{I} \cap \{0, \ldots, \mathfrak{n}\} \neq \emptyset$, choose a $\tau \in \mathfrak{T}$ satisfying

$$\tau_k = -1, \text{ and } \tau_j = 0, \ \forall j \in \mathbf{N}\backslash\{k\}.$$

A new path $z^{\varepsilon_0} \in H^1([0, 1/2], \mathbb{C}^N)$ with $\mathcal{A}(z^{\varepsilon_0}, 1/2) < \mathcal{A}(z, 1/2)$ can be obtained by applying Lemma 4.3 with the above τ. Since only the path of m_k was deformed in a small neighborhood of t_0 and $t_0 \in (0, 1/2)$, z^{ε_0} still satisfies the symmetric and topological constraints, but not the monotone constraints. Because of this we make a further deformation of z^{ε_0} as follows to get a new path $\tilde{z} \in H^1([0, 1/2], \mathbb{C}^N)$:

$$\tilde{z}_k(t) = \begin{cases} 2x_k^{\varepsilon_0}(t_0) - x_k^{\varepsilon_0}(t) + iy_k^{\varepsilon_0}(t), & \forall t \in [0, t_0], \\ z_k^{\varepsilon_0}(t), & \forall t \in [t_0, 1/2], \end{cases}$$

$$\tilde{z}_j(t) = \begin{cases} z_j^{\varepsilon_0}(t), & \text{if } j \in \{k+1, \ldots, N-1-k\}, \\ z_j^{\varepsilon_0}(t) + \tilde{x}_k(0), & \text{if } j \in \mathbf{N}\backslash\{k, \ldots, N-1-k\}, \end{cases} \quad \forall t \in [0, 1/2].$$

Figure 5a shows how the path of m_k was deformed in the above process. It is not hard to see that $\tilde{z}$ satisfies the monotone constraints and belongs to Λ_ω^+. Meanwhile, the above deformation preserves the kinetic energy and does not increase the potential energy, so $\mathcal{A}(\tilde{z}, 1/2) \leqq \mathcal{A}(z^{\varepsilon_0}, 1/2) < \mathcal{A}(z, 1/2)$.

If there is a $k \in \mathbf{I} \cap \{\mathfrak{n}+1, \ldots, N-1\} \neq \emptyset$, then we choose a $\tau \in \mathfrak{T}$ with

$$\tau_k = 1, \text{ and } \tau_j = 0, \ \forall j \in \mathbf{N}\backslash\{k\}.$$

The rest of the proof is exactly the same as above, except $\tilde{z}_j$, $j \neq k$ should be defined as follows:

$$\tilde{z}_j(t) = \begin{cases} z_j^{\varepsilon_0}(t), & j \in \mathbf{N}\backslash\{N-k, \ldots, k\}, \\ z_j^{\varepsilon_0}(t) + \tilde{x}_k(0), & j \in \{N-k, \ldots, k-1\}, \end{cases} \quad \forall t \in [0, 1/2];$$

see Fig. 5b for an illuminating picture. □

Now we are ready to prove Lemma 2.2.

Proof (Lemma 2.2). For simplicity, let $z = z^{\omega}$. By a contradiction argument, let's assume that $x_0(N/2) - x_0(0) = 0$. By (18), $x_0(t) \equiv 0$, $\forall t \in \mathbb{R}$. This means that the masses stay on the imaginary axis all the time.

Since z is a minimizer of the action functional $\mathcal{A}$ in $\Lambda^+_{\omega} \cap H^1(\mathbb{R}/N\mathbb{Z}, (i\mathbb{R})^N)$, it is a generalized solution of (1) (see [14]). We will show that such a z has at least one isolated collision and cannot be a minimizer.

First let's assume $N = 2n$. Then $z_0(0) = z_n(0) = 0$, and z has an $\mathbf{I}$-cluster collision with $\{0, n\} \subset \mathbf{I}$, at $t = 0$. If such a collision is isolated, then choose a $\tau \in \mathfrak{T}$ with

$$\tau_0 = -1, \ \tau_n = 1 \text{ and } \tau_j = 0, \quad \forall j \in \mathbf{N}\backslash\{0, n\}. \tag{72}$$

By Lemma 4.1, there is a $z^{\varepsilon_0} \in H^1([0, 1/2], \mathbb{C}^N)$ which is a local deformation of z satisfying $\mathcal{A}(z^{\varepsilon_0}, 1/2) < \mathcal{A}(z, 1/2)$ for $\varepsilon_0 > 0$ small enough. Since only the paths of m_0 and m_n were deformed, by the properties listed in Lemma 4.1, z^{ε_0} satisfies the monotone constraints and as a loop is contained in Λ^+_{ω}. This gives us a contradiction. If 0 is not an isolated collison moment, then there must be an isolated collision moment $t_0 \in (0, 1/2)$ close to 0 (see [14, Section 5]) and a contradiction can be reached by Lemma 4.6.

For the rest, we assume that $N = 2n+1$. The precise value of ω_1 will be needed in the what follows. Without loss of generality, let's assume $\omega_1 = 1$. Then

$$y_0(0) = 0, \quad y_0(1/2) \geqq 0. \tag{73}$$

Meanwhile, if $\omega_j = 1$, $\forall j \in \mathbf{N}\backslash\{0, 1\}$, and then

$$y_n(0) \geqq 0, \quad y_n(1/2) = 0. \tag{74}$$

On the other hand, if there is a $1 \leqq j_0 \leqq N-2$ such that $w_j = 1$, $\forall j \in \{1, \dots, j_0\}$ and $w_{j_0+1} = -1$, then, by (10),

$$\begin{cases} y_{j_0/2}(0) \geqq 0, \ y_{j_0/2}(1/2) \leqq 0, & \text{if } j_0 \text{ is even,} \\ y_{N-\frac{j_0+1}{2}}(0) \geqq 0, \ y_{N-\frac{j_0+1}{2}}(1/2) \leqq 0, & \text{if } j_0 \text{ is odd.} \end{cases} \tag{75}$$

As $x_j(t) \equiv 0$, $\forall j \in \mathbf{N}$, (73), (74) and (75) show that there must be a $k \in \mathbf{N}\backslash\{0\}$ and $t_0 \in [0, 1/2]$ such that $z_0(t_0) = z_k(t_0)$. In other words, there must be a $\mathbf{I}$-cluster collision at $t = t_0$, with $\{0, k\} \subset \mathbf{I}$.

First, when $t_0 \in (0, 1/2)$, we have that either t_0 or a moment close enough must be an isolated collision moment; in this case Lemma 4.6 gives us a contradiction.

Hence we only need to consider the cases with $t_0 \in \{0, 1/2\}$. Furthermore we may always assume that t_0 is isolated, as otherwise there is an isolated collision moment in $(0, 1/2)$ and again Lemma 4.6 gives a contradiction. Depending on the value of k and t_0, four different cases need to be considered.

As with the above, the local deformation lemmas will be used to find a new path $z^{\varepsilon_0} \in H^1([0, 1/2], \mathbb{C}^N)$ with $\mathcal{A}(z^{\varepsilon_0}, 1/2) < \mathcal{A}(z, 1/2)$ for $\varepsilon_0 > 0$ small enough. However z^{ε_0} may not satisfy the monotone constraints, which prevents it from being a member of Λ^+_{ω}. Fortunately, after some proper modification, if needed, we can

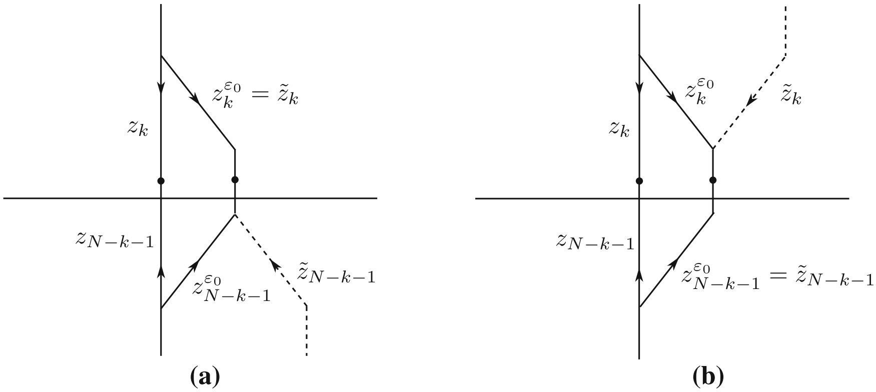

Fig. 6. **a** Case 2, **b** Case 4

always get a $\tilde{z} \in \Lambda_\omega^+$ with $\mathcal{A}(\tilde{z}, 1/2) \leqq \mathcal{A}(z^{\varepsilon_0}, 1/2) < \mathcal{A}(z, 1/2)$, which gives us a contradiction.

Case 1: $k \in \{1, \dots, n\}$ and $t_0 = 0$. By (8), we have that $z_0(0) = z_k(0) = z_{N-k}(0)$, so $\{0, k, N-k\} \subset \mathbf{I}$. Choose a $\tau \in \mathfrak{T}$ with

$$\tau_0 = -1 \ \text{ and } \ \tau_j = 0, \ \forall j \in \mathbf{N} \backslash \{0\}.$$

By Lemma 4.1, for $\varepsilon_0 > 0$ small enough, there is a $z^{\varepsilon_0} \in H^1([0, 1/2], \mathbb{C}^N)$ which is a local deformation of z satisfying $\mathcal{A}(z^{\varepsilon_0}, 1/2) < \mathcal{A}(z, 1/2)$. Since only the path of m_0 is deformed, with the properties listed in Lemma 4.1, we have $z^{\varepsilon_0} \in \Lambda_\omega^+$. We set $\tilde{z} = z^{\varepsilon_0}$.

Case 2: $k \in \{1, \dots, n\}$ and $t_0 = 1/2$. By (8), $z_0(1/2) = z_k(1/2) = z_{N-k-1}(1/2)$, so $\{0, k, N-k-1\} \subset \mathbf{I}$. Choose a $\tau \in \mathfrak{T}$ with

$$\tau_k = \tau_{N-k-1} = 1 \ \text{ and } \ \tau_j = 0, \forall j \in \mathbf{N} \backslash \{k, N-k-1\}.$$

By Lemma 4.2, there is a $z^{\varepsilon_0} \in H^1([0, 1/2], \mathbb{C}^N)$ which is a local deformation of z satisfying $\mathcal{A}(z^{\varepsilon_0}, 1/2) < \mathcal{A}(z, 1/2)$. Here, only the paths of m_k and m_{N-k-1} are deformed. However $x_{N-k-1}^{\varepsilon_0}(0) < x_{N-k-1}^{\varepsilon_0}(1/2)$ violates (19), so we define a new path $\tilde{z}(t)$, $t \in [0, 1/2]$ as follows:

$$\tilde{z}_j(t) = \begin{cases} 2x_j^{\varepsilon_0}(1/2) - x_j^{\varepsilon_0}(t) + i y_j^{\varepsilon_0}(t), & \text{if } j = N-k-1, \\ z_j^{\varepsilon_0}(t) + 2x_{N-k-1}^{\varepsilon_0}(1/2), & \text{if } j \in \{k+1, \dots, N-k-2\}, \\ z_j^{\varepsilon_0}(t), & \text{if } j \in \mathbf{N} \backslash \{k+1, \dots, N-k-1\}. \end{cases}$$

Figure 6a shows how the paths of m_k and m_{N-k-1} were defomred in the above process. By the above definition, $\mathcal{A}(\tilde{z}, 1/2) \leqq \mathcal{A}(z^{\varepsilon_0}, 1/2)$. Furthermore, this satisfies the monotone constraints and as a loop is contained in Λ_ω^+.

Case 3: $k \in \{n+1, \dots, N-1\}$ and $t_0 = 0$. By (8), $z_0(0) = z_k(0) = z_{N-k}(0)$, so $\{0, k, N-k\} \subset \mathbf{I}$. The rest follows from using the same τ and a similar argument as to that of *Case 1*.

Case 4: $k \in \{n+1, \ldots, N-1\}$ and $t_0 = 1/2$. The rest follows from using the same τ and a similar argument as to that of *Case 2*, the only difference being that $\tilde{z}(t)$, $t \in [0, 1/2]$, has to be defined as follows (see Fig. 6b):

$$\tilde{z}_j(t) = \begin{cases} 2x_j^{\varepsilon_0}(1/2) - x_j^{\varepsilon_0}(t) + iy_j^{\varepsilon_0}(t), & \text{if } j = k, \\ z_j^{\varepsilon_0}(t) + 2z_k^{\varepsilon_0}(1/2), & \text{if } j \in \{N-k, \ldots, k-1\}, \\ z_j^{\varepsilon_0}(t), & \text{if } j \in \mathbf{N}\backslash\{N-k, \ldots, k\}. \end{cases}$$

This finishes our proof for $N = 2n+1$, as well as the entire lemma. □

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References

1. BARUTELLO, V., TERRACINI, S.: Action minimizing orbits in the n-body problem with simple choreography constraint. *Nonlinearity*, **17**(6), 2015–2039, 2004
2. CHEN, K.-C.: Action-minimizing orbits in the parallelogram four-body problem with equal masses. *Arch. Ration. Mech. Anal.*, **158**(4), 293–318, 2001
3. CHEN, K.-C.: Binary decompositions for planar N-body problems and symmetric periodic solutions. *Arch. Ration. Mech. Anal.*, **170**(3), 247–276, 2003
4. CHEN, K.-C.: Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses. *Ann. Math. (2)*, **167**(2), 325–348, 2008
5. CHEN, K.-C.: Keplerian action functional, convex optimization, and an application to the four-body problem, 2013
6. CHEN, K.-C., LIN, Y.-C.: On action-minimizing retrograde and prograde orbits of the three-body problem. *Commun. Math. Phys.*, **291**(2), 403–441, 2009
7. CHEN, K.-C., OUYANG, T. and XIA, Z.: Action-minimizing periodic and quasi-periodic solutions in the n-body problem. *Math. Res. Lett.*, **19**(2), 483–497, 2012
8. CHENCINER, A.: Action minimizing solutions of the Newtonian n-body problem: from homology to symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*. Higher Ed. Press, Beijing, 279–294, 2002
9. CHENCINER, A.: Are there perverse choreographies? *New Advances in Celestial Mechanics and Hamiltonian Systems*, 63–76. Kluwer/Plenum, New York, 2004
10. CHENCINER, A.: Symmetries and "simple" solutions of the classical n-body problem. *XIVth International Congress on Mathematical Physics*, pages 4–20. World Sci. Publ., Hackensack, NJ, 2005
11. CHENCINER, A., FÉJOZ, J.: Unchained polygons and the N-body problem. *Regul. Chaotic Dyn.*, **14**(1), 64–115, 2009
12. CHENCINER, A., GERVER, J., MONTGOMERY, R., SIMÓ, C.: Simple choreographic motions of N bodies: a preliminary study. *Geometry, mechanics, and dynamics*. Springer, New York, 287–308, 2002
13. CHENCINER, A., MONTGOMERY, R.: A remarkable periodic solution of the three-body problem in the case of equal masses. *Ann. Math. (2)*, **152**(3):881–901, 2000
14. FERRARIO, D.L., TERRACINI, S.: On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.*, **155**(2), 305–362, 2004

15. Fujiwara, T., Montgomery, R.: Convexity of the figure eight solution to the three-body problem. *Pac. J. Math.*, **219**(2), 271–283, 2005
16. Fusco, G., Gronchi, G.F., Negrini, P.: Platonic polyhedra, topological constraints and periodic solutions of the classical N-body problem. *Invent. Math.*, **185**(2), 283–332, 2011
17. Gordon, W. B.: A minimizing property of Keplerian orbits. *Amer. J. Math.*, **99**(5), 961–971, 1977
18. Kapela, T., Zgliczyński, P.: The existence of simple choreographies for the N-body problem—a computer-assisted proof. *Nonlinearity*, **16**(6), 1899–1918, 2003
19. Marchal, C.: How the method of minimization of action avoids singularities. *Celest. Mech. Dynam. Astronom.*, 83(1-4), 325–353, 2002. Modern celestial mechanics: from theory to applications (Rome, 2001)
20. Montaldi, J., Steckles, K.: Classification of symmetry groups for planar n-body choreographies. *Forum Math. Sigma*, 1:e5, 55, 2013
21. Montgomery, R.: Figure 8s with three bodies. count.ucsc.edu/rmont/papers/fig8total.pdf.
22. Montgomery, R.: The N-body problem, the braid group, and action-minimizing periodic solutions. *Nonlinearity*, **11**(2), 363–376, 1998
23. Moore, C.: Braids in classical dynamics. *Phys. Rev. Lett.*, **70**(24), 3675–3679, 1993
24. Palais, R.S.: The principle of symmetric criticality. *Commun. Math. Phys.*, **69**(1), 19–30, 1979
25. Shibayama, M.: Variational proof of the existence of the super-eight orbit in the four-body problem. *Arch. Ration. Mech. Anal.*, **214**(1), 77–98, 2014
26. Simó, C.: New families of solutions in N-body problems. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.* Birkhäuser, Basel, 101–115, 2001
27. Venturelli, A.: *Application de la Minimisation de L'action au Problème des N Corps dans le plan et dans L'espace,*. PhD thesis, Université Denis Diderot in Paris, 2002
28. Yu, G.: Periodic Solutions of the Planar N-Center Problem with topological constraints. *Discrete Contin. Dyn. Syst.*, **36**(9), 5131–5162, 2016
29. Yu, G.: Shape Space Figure-8 Solution of Three Body Problem with Two Equal Masses. accepted by *Nonlinearity*, arXiv:1507.02892, 2016
30. Yu, G.: Spatial double choreographies of the Newtonian $2n$-body problem. arXiv:1608.07956, 2016

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