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Global Well-Posedness of the Boltzmann Equation with Large Amplitude Initial Data

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Abstract

The global well-posedness of the Boltzmann equation with initial data of large amplitude has remained a long-standing open problem. In this paper, by developing a new $L^\infty_x L^1_v \cap L^\infty_{x,v}$ approach, we prove the global existence and uniqueness of mild solutions to the Boltzmann equation in the whole space or torus for a class of initial data with bounded velocity-weighted L^∞ norm under some smallness condition on the $L^1_x L^\infty_v$ norm as well as defect mass, energy and entropy so that the initial data allow large amplitude oscillations. Both the hard and soft potentials with angular cut-off are considered, and the large time behavior of solutions in the $L^\infty_{x,v}$ norm with explicit rates of convergence are also studied.

Contents

1. Introduction

In this paper, we consider the Boltzmann equation

$$F_t + v \cdot \nabla_x F = Q(F, F), \tag{1.1}$$

where $F(t, x, v) \geq 0$ is the density distribution function for the gas particles with position $x \in \Omega = \mathbb{R}^3$ or $\mathbb{T}^3$ and velocity $v \in \mathbb{R}^3$ at time $t > 0$. The Boltzmann collision term $Q(F, F)$ on the right is defined in terms of the bilinear form

$$\begin{aligned} Q(F_1, F_2) \equiv & \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta) F_1(u')F_2(v')\,\mathrm{d}\omega\mathrm{d}u \\ & - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta) F_1(u)F_2(v)\,\mathrm{d}\omega\mathrm{d}u \\ := & \, Q_+(F_1, F_2) - Q_-(F_1, F_2), \end{aligned} \tag{1.2}$$

where the relationship between the post-collison velocity (v', u') of two particles with the pre-collision velocity (v, u) is given by

$$u' = u + [(v-u)\cdot\omega]\omega, \quad v' = v - [(v-u)\cdot\omega]\omega,$$

for $\omega \in \mathbb{S}^2$, which can be determined by conservation laws of momentum and energy:

$$u' + v' = u + v, \quad |u'|^2 + |v'|^2 = |u|^2 + |v|^2.$$

The Boltzmann collision kernel $B = B(v-u, \theta)$ in (1.2) depends only on $|v-u|$ and θ with $\cos\theta = (v-u)\cdot\omega/|v-u|$. Throughout this paper, we consider both the hard and soft potentials under the Grad's angular cut-off assumption, for instance,

$$B(v-u, \theta) = |v-u|^\gamma b(\theta), \tag{1.3}$$

with

$$-3 < \gamma \leq 1, \quad 0 \leq b(\theta) \leq C|\cos\theta|$$

for a postive constant $C > 0$. We consider the Boltzmann equation (1.1) with the following initial data

$$F(t, x, v)|_{t=0} = F_0(x, v). \tag{1.4}$$

To look for a solution $F(t, x, v)$ to the Cauchy problem (1.1) and (1.4), let us take a reference global Maxwellian

$$\mu(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp\left(-\frac{|v|^2}{2}\right),$$

which is normalized to have unit density, zero bulk velocity and unit temperature. Formally, as introduced in [15], $F(t, x, v)$ satisfies the conservations laws of defect mass, momentum, energy:

$$\int_\Omega\int_{\mathbb{R}^3} (F(t,x,v) - \mu(v))\mathrm{d}v\mathrm{d}x = \int_\Omega\int_{\mathbb{R}^3} (F_0(x,v) - \mu(v))\mathrm{d}v\mathrm{d}x := M_0, \tag{1.5}$$

$$\int_{\Omega}\int_{\mathbb{R}^3} v(F(t,x,v)-\mu(v))\mathrm{d}v\mathrm{d}x = \int_{\Omega}\int_{\mathbb{R}^3} v(F_0(x,v)-\mu(v))\mathrm{d}v\mathrm{d}x := J_0, \tag{1.6}$$

$$\int_{\Omega}\int_{\mathbb{R}^3} |v|^2(F(t,x,v)-\mu(v))\mathrm{d}v\mathrm{d}x = \int_{\Omega}\int_{\mathbb{R}^3} |v|^2(F_0(x,v)-\mu(v))\mathrm{d}v\mathrm{d}x := E_0, \tag{1.7}$$

as well as the inequality of defect entropy

$$\begin{aligned}&\int_{\Omega}\int_{\mathbb{R}^3}\Big\{F(t,x,v)\ln F(t,x,v)-\mu(v)\ln\mu(v)\Big\}\mathrm{d}v\mathrm{d}x\\ &\quad\le \int_{\Omega}\int_{\mathbb{R}^3}\Big\{F_0\ln F_0-\mu(v)\ln\mu(v)\Big\}\mathrm{d}v\mathrm{d}x.\end{aligned} \tag{1.8}$$

By defining

$$\begin{aligned}\mathcal{E}(F(t)) := &\int_{\Omega}\int_{\mathbb{R}^3}\Big\{F(t,x,v)\ln F(t,x,v)-\mu\ln\mu\Big\}\mathrm{d}v\mathrm{d}x\\ &+\left[\frac{3}{2}\ln(2\pi)-1\right]M_0+\frac{1}{2}E_0,\end{aligned}$$

it follows by a direct calculation that

$$\mathcal{E}(F(t))\ge 0$$

for all $t \ge 0$. Note, in particular, that $\mathcal{E}(F_0) \ge 0$ holds true for any function $F_0(x,v)\ge 0$.

The Boltzmann equation is a fundamental model in the collisional kinetic theory, and there is an enormous literature on its well-posedness theories, cf. [5] and [27] and the references therein. Among these works we mention some only in the spatially inhomogeneous framework; for the spatially homogenous Boltzmann equation, interested readers may refer to Carleman [4] as well as the recent work [21] and references therein. For general initial data in L^∞ framework, the local existence and uniqueness was firstly investigated by Kaniel and Shinbrot [18] and the global existence was later obtained by Illner and Shinbrot [17] under additional smallness assumption on velocity weighted L^∞ norm. It is well known that for general initial data with finite mass, energy and entropy, the global existence of renormalized solutions was proved by DiPerna and Lions [7]; the uniqueness of such solutions, however, is unknown. Moreover, the convergence of a class of large amplitude solutions toward the global Maxwellian with an explicit almost exponential rate in large time was also obtained by Desvillettes and Villani [8] conditionally under some assumptions on smoothness and polynomial moment bounds of the solutions. The result has been recently improved by Gualdani et al. [12] to derive a sharp exponential time rate by developing an abstract semigroup theory for linear operators which are non-symmetric in some Banach spaces.

On the other hand, in the perturbation framework, i.e., for the case when the solution is sufficiently close to a global Maxwellian in some sense, due to the extensive study of the linearized operator (Grad [11], Ellis and Pinsky [9], and Baranger and MOUHOT [1], for instance), the well-posedness theory of the Boltzmann equation is indeed well established in different kinds of settings since the pioneering work by UKAI [24]. For instance, the energy method in smooth Sobolev spaces was developed in GUO [13] and LIU et al. [20]. Another $L^2 \cap L^\infty$ approach was found by GUO [14,15] even for treating the Boltzmann equation on a general bounded domain. Note that for the hard sphere model in the torus case, a non-symmetric energy method was also developed in [12] to obtain the asymptotic stability of solutions to the global Maxwellian with a sharp exponential time rate for initial data $F_0(x, v)$ such that $F_0 - \mu$ is small enough in $L^1_v L^\infty_x((1+|v|)^k)$ with some $k > 2$; see also the recent work [3] for an investigation of the Boltzmann equation on the bounded domain in a similar functional setting. We also refer the interested reader to [16] for the issue of the macroscopic regularity of Boltzmann equation.

We remark that in those works in the perturbation framework mentioned above, initial data are required to have small amplitude around the global Maxwellian. To the best of our knowledge, the global existence and uniqueness problem of solutions to the Boltzmann equation with initial data of large amplitude still remains open. The purpose of this paper is to develop a $L^\infty_x L^1_v \cap L^\infty_{x,v}$ method for the well-posedness theory of the Boltzmann equation when initial data are allowed to have large amplitude. Precisely speaking, we prove the global existence and uniqueness of solutions to the Boltzmann equation in the whole space or torus when

$$F_0 - \mu \in L^\infty_{x,v}((1+|v|)^\beta \mu^{-1/2})$$

with some $\beta > \max\{3, 3+\gamma\}$ satisfying an additional smallness condition that

$$\mathcal{E}(F_0) + \|F_0 - \mu\|_{L^1_x L^\infty_v(\mu^{-1/2})}$$

is small enough. In particular, initial data can have large amplitude oscillations. Note that the result is valid for the full range of both the soft and hard potentials, i.e., $-3 < \gamma \le 1$. Moreover, in the torus case, we also show that the solutions tend to the global Maxwellian with exponential convergence rates for the hard potentials and with algebraical rate for the soft potentials.

Now we begin to formulate the main results of the paper. As in [15], we define a weight function

$$w_\beta(v) := (1+|v|^2)^{\frac{\beta}{2}},$$

and look for solutions in the form

$$f(t,x,v) := \frac{F(t,x,v) - \mu(v)}{\sqrt{\mu(v)}}.$$

The Boltzmann equation (1.1) is then rewritten as

$$f_t + v \cdot \nabla_x f + Lf = \Gamma(f, f), \tag{1.9}$$

where the linearised term is given by

$$Lf = \nu(v) f - Kf = -\frac{1}{\sqrt{\mu}}\Big\{Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)\Big\},$$

with $K := K_2 - K_1$ defined (cf. [8]) as

$$(K_1 f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\sqrt{\mu(v)\mu(u)}\, f(u)\, \mathrm{d}\omega \mathrm{d}u,$$

$$\begin{aligned}(K_2 f)(v) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\sqrt{\mu(u)\mu(u')}\, f(v')\, \mathrm{d}\omega \mathrm{d}u,\\ &\quad + \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\sqrt{\mu(u)\mu(v')}\, f(u')\, \mathrm{d}\omega \mathrm{d}u,\end{aligned}$$

$$\nu(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\mu(u)\, \mathrm{d}\omega \mathrm{d}u \sim (1+|v|)^\gamma,$$

and the nonlinear term is given by

$$\begin{aligned}\Gamma(f,f) &:= \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} f)\\ &= \frac{1}{\sqrt{\mu}} Q_+(\sqrt{\mu} f, \sqrt{\mu} f) - \frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \sqrt{\mu} f)\\ &:= \Gamma_+(f,f) - \Gamma_-(f,f).\end{aligned}$$

Then, from (1.9), the mild form of the Boltzmann equation is given by

$$\begin{aligned}f(t,x,v) &= \mathrm{e}^{-\nu(v)t} f_0(x-vt,v) + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}(Kf)(s,x-v(t-s),v)\mathrm{d}s\\ &\quad + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\Gamma(f,f)(s,x-v(t-s),v)\mathrm{d}s \qquad (1.10)\end{aligned}$$

for $t \geq 0$, $x \in \Omega$, $v \in \mathbb{R}^3$.

The first result of this paper is stated as follows:

Theorem 1.1. (Global existence) *Let $\Omega = \mathbb{T}^3$ or $\mathbb{R}^3$. For given $\beta > \max\{3, 3+\gamma\}$, $\bar{M} \geq 1$, suppose the initial data F_0 satisfies $F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v) \geq 0$ and $\|w_\beta f_0\|_{L^\infty} \leq \bar{M}$. Then there is a small constant $\varepsilon_0 > 0$ depending on $\gamma, \beta, \bar{M}$ such that if*

$$\mathcal{E}(F_0) + \|f_0\|_{L^1_x L^\infty_v} \leq \varepsilon_0, \qquad (1.11)$$

the Boltzmann equation (1.1), (1.3), (1.4) has a global unique mild solution $F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v) \geq 0$ satisfying (1.5)–(1.8) and

$$\|w_\beta f(t)\|_{L^\infty} \leq \tilde{C}_1 \bar{M}^2,$$

where $\tilde{C}_1$ depends only on γ, β. Moreover, if the initial data f_0 is continuous in $(x,v) \in \Omega \times \mathbb{R}^3$, then the solution $f(t,x,v)$ is continuous in $[0,\infty) \times \Omega \times \mathbb{R}^3$.

Remark 1.2. It should be pointed out that initial data satisfying the smallness condition (1.11) are allowed to have large amplitude oscillations in spatial variable. For instance, one may take

$$F_0(x,v) = \rho_0(x)\mu = \frac{\rho_0(x)}{(2\pi)^{\frac{3}{2}}}\mathrm{e}^{-\frac{|v|^2}{2}}, \quad (x,v) \in \Omega \times \mathbb{R}^3,$$

with $\rho_0(x) \geq 0$, $\rho_0 \in L^\infty_x$, $\rho_0 - 1 \in L^1_x$ and $\rho_0 \ln \rho_0 - \rho_0 + 1 \in L^1_x$. Then, it is straightforward to verify that (1.11) holds if $\|\rho_0 \ln \rho_0 - \rho_0 + 1\|_{L^1_x} + \|\rho_0 - 1\|_{L^1_x}$ is small. Even though $\|\rho_0 \ln \rho_0 - \rho_0 + 1\|_{L^1} + \|\rho_0 - 1\|_{L^1_x}$ is required to be small, initial data are allowed to have large amplitude oscillations.

Remark 1.3. From the proof of Theorem 1.1 later on, by the same argument, the smallness condition (1.11) can be relaxed to

$$\mathcal{E}(F_0) + \sup_{(t,x)\in[t_1,\infty)\times\Omega} \int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)t} |f_0(x - vt, v)| \mathrm{d}v \leq \varepsilon_0,$$

where $t_1 := (8\tilde{C}_4[1 + \|w_\beta f_0\|_{L^\infty}])^{-1}$ is defined in Proposition 2.1 later on.

Remark 1.4. Under the assumptions of Theorem 1.1, and further let β suitably large. Let the initial data $f_0(x, v) \in C^1(\Omega\times\mathbb{R}^3)$ and $\|w_{\frac{\beta}{2}} \nabla_x f_0\|_{L^\infty} + \|w_{\frac{\beta}{2}} \nabla_v f_0\|_{L^\infty} < +\infty$, then the Boltzmann solution $f(t, x, v)$ obtained in Theorem 1.1 satisfies $f(t, x, v) \in C^1(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$ and

$$\|w_{\frac{\beta}{2}} \nabla_x f(t)\|_{L^\infty} + \|w_{\frac{\beta}{2}} \nabla_v f(t)\|_{L^\infty} \leq \exp\{C(t)\},$$

where $C(t) > 0$ is a continuous function of $t > 0$, and depends only on $\bar{M}$ and $\|w_{\frac{\beta}{2}} \nabla_x f_0\|_{L^\infty} + \|w_{\frac{\beta}{2}} \nabla_v f_0\|_{L^\infty}$. It should be pointed out that the above regularity result can be proved by using similar arguments as in the proof of Proposition 2.1 in the appendix and Gronwall inequality.

It follows immediately from Theorem 1.1 that even if initial density $\rho_0(x) := \int_{\mathbb{R}^3} F_0(x, v)\mathrm{d}v$ contains vacuum, then the macroscopic density function

$$\rho(t, x) := \int_{\mathbb{R}^3} F(t, x, v)\, \mathrm{d}v$$

must have uniformly positive lower bound in finite time. Indeed, one has

Corollary 1.5. (Positive lower bound of density) *Under the same conditions of Theorem 1.1, there exists a positive time $T_0 > 0$ such that*

$$|\rho(t, x) - 1| = \left| \int_{\mathbb{R}^3} [F(t, x, v) - \mu(v)]\mathrm{d}v \right| \leq \frac{3}{4},$$

for all $t \geq T_0$ and $x \in \Omega$.

Moreover, for the global solutions obtained in Theorem 1.1 with $\Omega = \mathbb{T}^3$, one can further obtain the explicit rates of convergence of solutions in $L^\infty_{x,v}$. Therefore, it shows that even if initial data $f_0(x, v)$ could be large in $L^\infty_{x,v}$, the solution $f(t, x, v)$ must tend to zero as time goes to infinity. In fact, one has

Theorem 1.6. (Decay estimate for hard potentials) *Let* $\Omega = \mathbb{T}^3$, $0 \leq \gamma \leq 1$, *and* $\beta > \max\{3, 3+\gamma\}$. *Assume* $(M_0, J_0, E_0) = (0, 0, 0)$, *and* $\varepsilon_0 > 0$ *sufficiently small, then there exists a positive constant* $\sigma_0 > 0$ *such that the solution* $f(t, x, v)$ *obtained in Theorem 1.1 satisfies*

$$\|w_\beta f(t)\|_{L^\infty} \leq \tilde{C}_2 \mathrm{e}^{-\sigma_0 t}, \tag{1.12}$$

for all $t \geq 0$, *where* $\tilde{C}_2 > 0$ *is a positive constant depending only on the initial data.*

Theorem 1.7. (Decay estimate for soft potentials) *Let* $\Omega = \mathbb{T}^3$, $-3 < \gamma < 0$, *and* $\beta \geq \max\{\frac{9}{2}, 4 + |\gamma|\}$. *Let* δ *be any given positive constant such that* $\delta \in (0, \frac{1}{3})$. *Assume* $(M_0, J_0, E_0) = (0, 0, 0)$, *and* $\varepsilon_0 > 0$ *is sufficiently small, then the solution* $f(t, x, v)$ *obtained in Theorem 1.1 satisfies*

$$\|f(t)\|_{L^\infty} \leq \tilde{C}_3 (1+t)^{-1-\frac{2}{|\gamma|}+\delta}, \tag{1.13}$$

for all $t \geq 0$, *where* $\tilde{C}_3 > 0$ *is a positive constant depending only on the initial data.*

Now we explain the strategy of the proof of the above main results. As mentioned before, the only global existence of large-data solutions to the Boltzmann equation is due to DiPerna and Lions [7] by the weak compactness argument, but the uniqueness of these renormalized solutions is completely open due to the lack of L^∞ a priori estimates. Indeed, it is difficult to establish the global L^∞ bound for the solutions of Boltzmann equations due to the nonlinear term $\Gamma(f, f)(t)$. In those aforementioned references [14,15,19,22,25], one usually has to estimate the nonlinear term in the following way:

$$|w_\beta(v)\Gamma(f, f)(t)| \leq C\nu(v)\|w_\beta f(t)\|^2_{L^\infty},$$

so that the smallness assumption on the L^∞-norm is necessarily required.

To remove the above smallness assumption on the L^∞-norm, we need a new idea to control the nonlinear term $\Gamma(f, f)$. For this, we firstly establish a new estimate for the nonlinear term (see Lemma 3.1 below), i.e., for $\beta \geq \frac{1}{2}$,

$$\Big|w_\beta(v)\Gamma(f, f)(t, x, v)\Big| \leq C\nu(v)\|w_\beta f(t)\|^{2-a}_{L^\infty} \cdot \Big(\int_{\mathbb{R}^3} |f(t, x, u)|\mathrm{d}u\Big)^a,$$

for some $0 < a < 1$. Secondly, under the condition (1.11), we observe that $\int_{\mathbb{R}^3} |f(t, x, u)|\mathrm{d}u$ could be small after some positive time, even if it could be initially large due to the hyperbolicity of the Boltzmann equation. This observation is the key point of this paper to control the nonlinear term $\Gamma(f, f)$. In such way, through careful analysis one can finally obtain the following uniform estimate

$$\sup_{0 \leq s \leq t} \|w_\beta f(s)\|_{L^\infty} \leq C\bar{M}^2,$$

under smallness of $\|f(t)\|_{L^\infty_x L^1_v}$ uniformly for all $t \geq t_1$ with some $t_1 > 0$. In the whole proof, we shall use only the smallness of $\mathcal{E}(F_0) + \|f_0\|_{L^1_x L^\infty_v}$ so that initial data are allowed to have large amplitude oscillations.

The paper is organized as follows. In Section 2, we introduce the local existence of solutions to the Boltzmann equation and list some properties on the kernel of linearized operator, and the detailed proofs can be found in appendix. In Section 3, we develop the $L^\infty_x L^1_v \cap L^\infty_{x,v}$ estimate to prove the main Theorem 1.1. The time-decay estimates of the Boltzmann equation on torus are established in Section 4.

Notations. Throughout this paper, C denotes a generic positive constant which may depend on γ, β and vary from line to line. $C_a, C_b, \ldots$ denote the generic positive constants depending on $a,\ b, \ldots$, respectively, which also may vary from line to line. $\|\cdot\|_{L^2}$ denotes the standard $L^2(\Omega\times\mathbb{R}^3_v)$-norm, and $\|\cdot\|_{L^\infty}$ denotes the $L^\infty(\Omega\times\mathbb{R}^3_v)$-norm.

2. Preliminaries

As mentioned before, Kaniel–Shinbrot [18] investigated the local existence and uniqueness of solutions to the Boltzmann equation for large initial data around vacuum. Though, to prove Theorem 1.1, we need to figure out more quantitative properties of the local existence regarding the lifespan of the local L^∞ solution in terms of the L^∞ bound of initial data. Therefore, we would give a representation of the local existence and uniqueness of solutions to the Boltzmann equation applicable for the global L^∞ estimates in our own setting. The proof of the following result will be given in the appendix.

Proposition 2.1. (Local existence) *Let* $\Omega = \mathbb{T}^3$ *or* $\mathbb{R}^3$, $-3 < \gamma \le 1$, $\beta > 3$, $F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v) \ge 0$ *and* $\|wf_0\|_{L^\infty} < \infty$, *then there exists a positive time*

$$t_1 := (8\tilde{C}_4[1 + \|w_\beta f_0\|_{L^\infty}])^{-1} > 0, \tag{2.1}$$

such that the Boltzmann equation (1.1), (1.3), (1.4) *has a unique solution* $F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v) \ge 0$ *satisfying*

$$\sup_{0\le t\le t_1} \|w_\beta f(t)\|_{L^\infty} \le 2\|w_\beta f_0\|_{L^\infty},$$

where the positive constant $\tilde{C}_4 \ge 1$ *depending only on* γ, β. *Moreover, the conservations of defect mass, momentum, energy* (1.5)–(1.7) *as well as the additional defect entropy inequality* (1.8) *hold. Finally, if initial data* f_0 *are continuous, then the solution* $f(t,x,v)$ *is continuous in* $[0,t_1]\times\Omega\times\mathbb{R}^3$.

For later use, we list the following result on the operator K, whose proof will be given in the appendix. Interested readers may also refer to [2,10] for more details.

Lemma 2.2. *For* $-3 < \gamma \le 1$, *the following Grad's estimates hold*

$$K_1 f(v) = \int_{\mathbb{R}^3} k_1(v,\eta) f(\eta)\,\mathrm{d}\eta, \quad K_2 f(v) = \int_{\mathbb{R}^3} k_2(v,\eta) f(\eta)\,\mathrm{d}v,$$

where $k_1(v,\eta)$ and $k_2(v,\eta)$ satisfy

$$0 \le k_1(v,\eta) = c_1|v-\eta|^{\gamma} e^{-\frac{|v|^2}{4}} e^{-\frac{|\eta|^2}{4}},$$

and

$$0 \le k_2(v,\eta) \le \frac{C_\gamma}{|v-\eta|^{\frac{3-\gamma}{2}}} e^{-\frac{|v-\eta|^2}{8}} e^{-\frac{||v|^2-|\eta|^2|^2}{8|v-\eta|^2}}, \tag{2.2}$$

where $c_1 > 0$ is a given constant, and C_γ is a constant depending only on γ.

Remark 2.3. Note that the upper bound in (2.2) is not optimal, but it is enough for the use of the later proof. Moreover, we will not make any effort on the optimal estimates related to K in order to show Theorem 1.1.

From Lemma 2.2, one has that

$$Kf = \int_{\mathbb{R}^3} k(v,\eta) f(\eta)\, d\eta := \int_{\mathbb{R}^3} \Big\{k_2(v,\eta) - k_1(v,\eta)\Big\} f(\eta)\, d\eta,$$

with

$$|k(v,\eta)| \le c_1|v-\eta|^{\gamma} e^{-\frac{|v|^2}{4}} e^{-\frac{|\eta|^2}{4}} + \frac{C_\gamma}{|v-\eta|^{\frac{3-\gamma}{2}}} e^{-\frac{|v-\eta|^2}{8}} e^{-\frac{||v|^2-|\eta|^2|^2}{8|v-\eta|^2}}. \tag{2.3}$$

By the same calculations as in [2,10], it is straightforward to check that for $\alpha \ge 0$,

$$\int_{\mathbb{R}^3} \left| k(v,\eta) \cdot \frac{w_\alpha(v)}{w_\alpha(\eta)} \right| d\eta \le C_\gamma (1+|v|)^{-1}. \tag{2.4}$$

In order to deal with difficulties in the case of the soft potentials, as in [23] we introduce a smooth cutoff function $0 \le \chi_m \le 1$ with $0 < m \le 1$ such that

$$\chi_m(s) = 1 \ \text{ for } s \le m, \quad \chi_m(s) = 0 \ \text{ for } s \ge 2m.$$

Then we define

$$\begin{aligned}
(K^m g)(v) = &\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\chi_m(|v-u|)\sqrt{\mu(u)\mu(u')} f(v')\, d\omega du \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\chi_m(|v-u|)\sqrt{\mu(u)\mu(v')} f(u')\, d\omega du \\
&- \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\chi_m(|v-u|)\sqrt{\mu(v)\mu(u)} f(u)\, d\omega du \\
:= &K_2^m f(v) - K_1^m f(v),
\end{aligned}$$

and

$$K^c = K - K^m. \tag{2.5}$$

The following result on K^m and K^c can be regarded as a refined version of [23, Lemma 1], and its proof can be found in the appendix:

Lemma 2.4. *Let $-3 < \gamma \le 1$, then it holds that*

$$|(K^m g)(v)| \le Cm^{3+\gamma} \mathrm{e}^{-\frac{|v|^2}{10}} \|g\|_{L^\infty}, \tag{2.6}$$

and

$$(K^c g)(v) = \int_{\mathbb{R}^3} l(v,\eta) g(\eta)\,\mathrm{d}\eta.$$

Here the kernel $l(v,\eta)$ satisfies that for $0 \le a \le 1$,

$$\begin{aligned}|l(v,\eta)| \le{}& \frac{C_\gamma m^{a(\gamma-1)}}{|v-\eta|^{1+\frac{(1-a)}{2}(1-\gamma)}} \frac{1}{(1+|v|+|\eta|)^{a(1-\gamma)}} \mathrm{e}^{-\frac{|v-\eta|^2}{10}} \mathrm{e}^{-\frac{||v|^2-|\eta|^2|^2}{16|v-\eta|^2}} \\ &+ C|v-\eta|^\gamma \mathrm{e}^{-\frac{|v|^2}{4}} \mathrm{e}^{-\frac{|\eta|^2}{4}},\end{aligned} \tag{2.7}$$

where C_γ is a constant depending only on γ. It is worth to point out that C_γ is uniform in $a \in [0, 1]$.

Since the constant C_γ in (2.7) does not depend on $a \in [0, 1]$, we have the following estimates on $l(v,\eta)$ from Lemma 2.4 by taking $a = 1$ and $a = 0$, respectively:

Lemma 2.5. *Let $-3 < \gamma \le 1$, both the following two bounds on $l(l, v)$ hold:*

$$|l(v,\eta)| \le \frac{C_\gamma m^{\gamma-1}}{|v-\eta|(1+|v|+|\eta|)^{1-\gamma}} \mathrm{e}^{-\frac{|v-\eta|^2}{10}} \mathrm{e}^{-\frac{||v|^2-|\eta|^2|^2}{16|v-\eta|^2}} + C|v-\eta|^\gamma \mathrm{e}^{-\frac{|v|^2}{4}} \mathrm{e}^{-\frac{|\eta|^2}{4}}, \tag{2.8}$$

and

$$|l(v,\eta)| \le \frac{C_\gamma}{|v-\eta|^{\frac{3-\gamma}{2}}} \mathrm{e}^{-\frac{|v-\eta|^2}{10}} \mathrm{e}^{-\frac{||v|^2-|\eta|^2|^2}{16|v-\eta|^2}} + C|v-\eta|^\gamma \mathrm{e}^{-\frac{|v|^2}{4}} \mathrm{e}^{-\frac{|\eta|^2}{4}}. \tag{2.9}$$

Moreover, it holds that

$$\int_{\mathbb{R}^3} \left| l(v,\eta) \cdot \frac{w_\alpha(v)}{w_\alpha(\eta)} \right| \mathrm{d}\eta \le C_\gamma m^{\gamma-1} (1+|v|)^{\gamma-2} + C_\gamma \mathrm{e}^{-\frac{|v|^2}{4}} \le C(\gamma) m^{\gamma-1} \frac{\nu(v)}{(1+|v|)^2}, \tag{2.10}$$

and

$$\int_{\mathbb{R}^3} \left| l(v,\eta) \cdot \frac{w_\alpha(v)}{w_\alpha(\eta)} \right| \mathrm{d}\eta \le C_\gamma (1+|v|)^{-1} + C_\gamma \mathrm{e}^{-\frac{|v|^2}{4}} \le C_\gamma (1+|v|)^{-1}, \tag{2.11}$$

where $\alpha \ge 0$ is an arbitrary positive constant.

Remark 2.6. Indeed, the estimate (2.9) and (2.11) are the same as the ones in (2.3) and (2.4). On the other hand, the estimates (2.8) and (2.10) imply that one can get more decay with respect to v, but at the cost of growth with respect to the parameter $\frac{1}{m}$. All these properties will be used later.

Motivated by GUO [15], we have the following lemma which will be used later.

Lemma 2.7. [15] *Let $F(t,x,v)$ satisfy* (1.5), (1.7) *and the additional defect entropy inequality* (1.8), *then it holds that*

$$\begin{aligned}&\int_\Omega\int_{\mathbb{R}^3}\frac{|F(t,x,v)-\mu(v)|^2}{4\mu(v)}I_{\{|F(t,x,v)-\mu(v)|\le\mu(v)\}}\mathrm{d}v\mathrm{d}x\\&\quad+\int_\Omega\int_{\mathbb{R}^3}\frac{1}{4}|F(t,x,v)-\mu(v)|I_{\{|F(t,x,v)-\mu(v)|\ge\mu(v)\}}\mathrm{d}v\mathrm{d}x\\&\quad\le\int_\Omega\int_{\mathbb{R}^3}F_0\ln F_0-\mu\ln\mu\mathrm{d}v\mathrm{d}x+\left[\frac{3}{2}\ln(2\pi)-1\right]M_0+\frac{1}{2}E_0=\mathcal{E}(F_0).\end{aligned}\tag{2.12}$$

Proof. By Taylor expansion, we have

$$F(t)\ln F(t)-\mu\ln\mu=(1+\ln\mu)[F(t)-\mu]+\frac{1}{2\tilde{F}}|F(t)-\mu|^2,$$

where $\tilde{F}$ is between $F(t)$ and μ. Noting $1+\ln\mu=-[\frac{3}{2}\ln(2\pi)-1]-\frac{1}{2}|v|^2$, we have

$$\begin{aligned}&\int_\Omega\int_{\mathbb{R}^3}\frac{1}{2\tilde{F}}|F(t)-\mu|^2\mathrm{d}v\mathrm{d}x\\&\quad=\int_\Omega\int_{\mathbb{R}^3}[F(t)\ln F(t)-\mu\ln\mu]\mathrm{d}v\mathrm{d}x\\&\qquad+\left[\frac{3}{2}\ln(2\pi)-1\right]\int_\Omega\int_{\mathbb{R}^3}[F(t)-\mu]\mathrm{d}v\mathrm{d}x\\&\qquad+\frac{1}{2}\int_\Omega\int_{\mathbb{R}^3}|v|^2[F(t)-\mu]\mathrm{d}v\mathrm{d}x\\&\quad\le\int_\Omega\int_{\mathbb{R}^3}[F_0\ln F_0-\mu\ln\mu]\mathrm{d}v\mathrm{d}x+\left[\frac{3}{2}\ln(2\pi)-1\right]M_0+\frac{1}{2}E_0,\end{aligned}\tag{2.13}$$

where we have used (1.5), (1.7) and (1.8) in the last inequality. We note that $|F-\mu|\ge\mu$ yields that $F\ge 2\mu$ or $F=0$, thus we have

$$\frac{|F-\mu|}{\tilde{F}}\ge\frac{1}{2},$$

which, together with (2.13), yields (2.12). Therefore, the proof of this lemma is completed. □

3. Global Estimates

In order to prove the global existence of solutions to the Boltzmann equation, it suffices to get uniform estimates on solutions since one has already obtained in Proposition 2.1 the local existence of unique solutions to the Boltzmann equation with possibly large initial data. In this section, we devote ourselves to establish the global uniform estimate for the obtained solutions to the Boltzmann equation.

3.1. Weighted L^∞-Estimate

Define

$$h(t,x,v) := w_\beta(v) f(t,x,v).$$

Multiplying (1.9) by $w_\beta(v)$, one gets that

$$h_t + v\cdot\nabla_x h + \nu(v)h - K_{w_\beta}h = \Gamma_{w_\beta}(h,h), \tag{3.1}$$

where

$$\begin{aligned}(K_{w_\beta}h) = w_\beta(v)\Big(K\frac{h}{w_\beta}\Big)(v) &= w_\beta(v)\Big(K^m\frac{h}{w_\beta}\Big)(v) + w_\beta(v)\Big(K^c\frac{h}{w_\beta}\Big)(v)\\ &:= K^m_{w_\beta}h + K^c_{w_\beta}h,\end{aligned}$$

and

$$\begin{aligned}\Gamma_{w_\beta}(h,h) &:= w_\beta(v)\Gamma\left(\frac{h}{w_\beta},\frac{h}{w_\beta}\right)\\ &= w_\beta(v)\Gamma_+\left(\frac{h}{w_\beta},\frac{h}{w_\beta}\right) - w_\beta(v)\Gamma_-\left(\frac{h}{w_\beta},\frac{h}{w_\beta}\right)\\ &\equiv w_\beta(v)\Gamma(f,f) = w_\beta(v)\Gamma_+(f,f) - w_\beta(v)\Gamma_-(f,f).\end{aligned} \tag{3.2}$$

Then the mild solution of (3.1) can be written as

$$\begin{aligned}h(t,x,v) = &\,\mathrm{e}^{-\nu(v)t}h_0(x-vt,v) + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\left(K^m_{w_\beta}h\right)(s,x-v(t-s),v)\,\mathrm{d}s\\ &+ \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\left(K^c_{w_\beta}h\right)(s,x-v(t-s),v)\,\mathrm{d}s\\ &+ \int_0^t \mathrm{e}^{-\nu(v)(t-s)}(\Gamma_{w_\beta}(h,h))(s,x-v(t-s),v)\,\mathrm{d}s.\end{aligned} \tag{3.3}$$

Firstly, we give estimates on the nonlinear term $\Gamma(f,f)$.

Lemma 3.1. *Let $-3<\gamma\le 1$. For $\alpha\ge 0$, it holds that*

$$\begin{cases}|w_\alpha(v)\Gamma_-(f,f)(s,y,v)| \le C_\gamma \nu(v)\|w_\alpha f(s)\|_{L^\infty}\cdot\|f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\cdot\left(\int_{\mathbb{R}^3}|f(s,y,u)|\,\mathrm{d}u\right)^{\frac{p-1}{5p}},\\ |w_\alpha(v)\Gamma_+(f,f)(s,y,v)| \le C_\gamma \nu(v)\|w_\alpha f(s)\|_{L^\infty}\cdot\|w_{\frac12} f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\cdot\left(\int_{\mathbb{R}^3}|f(s,y,u)|\,\mathrm{d}u\right)^{\frac{p-1}{5p}},\end{cases} \tag{3.4}$$

where $p>1$ is defined in (3.6).

Proof. It is noted that

$$\Big|w_\alpha(v)\Gamma_-(f,f)(s,y,v)\Big| \le C\|w_\alpha f(s)\|_{L^\infty}\int_{\mathbb{R}^3}|v-u|^\gamma\sqrt{\mu(u)}|f(s,y,u)|\,\mathrm{d}u. \tag{3.5}$$

To estimate the integration term on the RHS of (3.5), we choose

$$p = 1+\frac{3+\gamma}{4(9-\gamma)}\ \text{ for } -3<\gamma\le 1, \tag{3.6}$$

which yields that

$$1<p\le\frac{9}{8},\quad p_* := \frac{p}{p-1}\ge 9,\quad p\frac{\gamma-3}{2}>-3,\quad p\gamma>-3. \tag{3.7}$$

Then it follows from (3.5), (3.7) and Hölder inequality that

$$\begin{aligned}
&\Big|w_\alpha(v)\Gamma_-(f,f)(s,y,v)\Big|\\
&\le C\|w_\alpha f(s)\|_{L^\infty}\Big(\int_{\mathbb{R}^3}|v-u|^{p\gamma}\sqrt{\mu(u)}\mathrm{d}u\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^3}\sqrt{\mu(u)}|f(s,y,u)|^{\frac{p}{p-1}}\mathrm{d}u\Big)^{1-\frac{1}{p}}\\
&\le C_\gamma\nu(v)\|w_\alpha f(s)\|_{L^\infty}\Big(\int_{\mathbb{R}^3}|f(s,y,u)|^{\frac{5p}{p-1}}\mathrm{d}u\Big)^{\frac{p-1}{5p}}\\
&\le C_\gamma\nu(v)\|w_\alpha f(s)\|_{L^\infty}\|f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\Big(\int_{\mathbb{R}^3}|f(s,y,u)|\mathrm{d}u\Big)^{\frac{p-1}{5p}}.
\end{aligned} \tag{3.8}$$

Next, we consider the gain term which needs much more care. We note that

$$|v|^2\le|u'|^2+|v'|^2,$$

which yields

$$\text{either }\ \frac{1}{2}|v|^2\le|u'|^2\ \text{ or }\ \frac{1}{2}|v|^2\le|v'|^2.$$

Hence one obtains that

$$\begin{aligned}
&\Big|w_\alpha(v)\Gamma_+(f,f)(s,y,v)\Big| = \Big|\frac{w_\alpha(v)}{\sqrt{\mu(v)}}Q_+(\sqrt{\mu}f,\sqrt{\mu}f)(s,y,v)\Big|\\
&\le w_\alpha(v)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)}\Big|f(s,y,u')f(s,y,v')\Big|I_{\{\frac{1}{2}|v|^2\le|u'|^2\}}\mathrm{d}u\mathrm{d}\omega\\
&\quad+w_\alpha(v)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)}\Big|f(s,y,u')f(s,y,v')\Big|I_{\{\frac{1}{2}|v|^2\le|v'|^2\}}\mathrm{d}u\mathrm{d}\omega\\
&\le C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)}\Big|w_\alpha(u')f(s,y,u')f(s,y,v')\Big|\mathrm{d}u\mathrm{d}\omega\\
&\quad+C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)}\Big|f(s,y,u')w_\alpha(v')f(s,y,v')\Big|\mathrm{d}u\mathrm{d}\omega\\
&= I_1+I_2.
\end{aligned} \tag{3.9}$$

To estimate I_1, as in [2,10], we use the change of variables

$$z = u - v, \quad z_{\shortparallel} = [z \cdot \omega]\omega, \quad z_\perp = z - z_{\shortparallel}, \quad \eta = v + z_{\shortparallel}. \tag{3.10}$$

Then it holds that

$$u' = v + z_\perp, \quad v' = v + z_{\shortparallel}, \tag{3.11}$$

and

$$B(v-u,\theta) \le C|z_{\shortparallel}|(|z_{\shortparallel}| + |z_\perp|)^{\gamma-1}. \tag{3.12}$$

Hence it follows from (3.6), (3.7), (3.10) and (3.11) that

$$\begin{aligned}
I_1 &= C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\sqrt{\mu(u)}\Big|w_\alpha(u')f(s,y,u')f(s,y,v')\Big|\mathrm{d}u\mathrm{d}\omega \\
&\le C\|w_\alpha f(s)\|_{L^\infty}\Big(\int_{\mathbb{R}^3}|v-u|^{p\gamma}\sqrt{\mu(u)}\mathrm{d}u\Big)^{\frac{1}{p}} \\
&\quad \Big(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\sqrt{\mu(u)}|f(s,y,v')|^{\frac{p}{p-1}}\mathrm{d}u\mathrm{d}\omega\Big)^{1-\frac{1}{p}} \\
&\le C_\gamma \nu(v)\|w_\alpha f(s)\|_{L^\infty}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathrm{e}^{-\frac{|v+z_{\shortparallel}+z_\perp|^2}{4}}|f(s,y,v+z_{\shortparallel})|^{\frac{p}{p-1}}\mathrm{d}z\mathrm{d}\omega\Big)^{1-\frac{1}{p}} \\
&\le C_\gamma \nu(v)\|w_\alpha f(s)\|_{L^\infty}\Big(\int_{\mathbb{R}^3}\int_{z_\perp}\frac{1}{|\eta-v|^2}\mathrm{e}^{-\frac{|\eta+z_\perp|^2}{4}}|f(s,y,\eta)|^{\frac{p}{p-1}}\mathrm{d}z_\perp\mathrm{d}\eta\Big)^{1-\frac{1}{p}} \\
&\le C_\gamma \nu(v)\|w_\alpha f(s)\|_{L^\infty}\Big(\int_{\mathbb{R}^3}\frac{(1+|\eta|)^{-4}}{|\eta-v|^{\frac{5}{2}}}\mathrm{d}\eta\Big)^{\frac{4}{5}\left(1-\frac{1}{p}\right)} \\
&\quad \Big(\int_{\mathbb{R}^3}(1+|\eta|)^{16}|f(s,y,\eta)|^{\frac{5p}{p-1}}\mathrm{d}\eta\Big)^{\frac{p-1}{5p}} \\
&\le C_\gamma \nu(v)\|w_\alpha f(s)\|_{L^\infty}\|(1+|\eta|)^{16\frac{p-1}{4p+1}}f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\Big(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\Big)^{\frac{p-1}{5p}} \\
&\le C_\gamma \nu(v)\|w_\alpha f(s)\|_{L^\infty}\|w_{\frac{1}{2}}f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\Big(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\Big)^{\frac{p-1}{5p}},
\end{aligned}$$

where in the last inequality, we have used the fact that $\frac{16(p-1)}{4p+1} \le \frac{1}{2}$. On the other hand, it is noted that by a rotation, one obtains the interchange of v' and u', and then I_2 can be changed to a form similar to I_1. Hence, for I_2, one can also obtain the same estimate as above. Thus one can get that

$$\begin{aligned}
&\Big|w_\alpha(v)\Gamma_+(f,f)(s,y,v)\Big| \\
&\quad \le C_\gamma \nu(v)\|w_\alpha f(s)\|_{L^\infty}\|w_{\frac{1}{2}}f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\Big(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\Big)^{\frac{p-1}{5p}}.
\end{aligned} \tag{3.13}$$

Then (3.4) follows from (3.8) and (3.13). The proof of Lemma 3.1 is completed. □

Lemma 3.2. *Let $\beta > 3$ and $-3 < \gamma \leq 1$, then it holds that*

$$\sup_{0\leq s\leq t} \|h(s)\|_{L^\infty} \leq C_1\Big\{\|h_0\|_{L^\infty} + \|h_0\|_{L^\infty}^2 + \sqrt{\mathcal{E}(F_0)} + \mathcal{E}(F_0)\Big\} + C_1 \sup_{t_1\leq s\leq t,\ y\in\Omega} \left\{\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}} \Big(\int_{\mathbb{R}^3} |f(s,y,\eta)|\mathrm{d}\eta\Big)^{\frac{p-1}{5p}}\right\}, \tag{3.14}$$

where the positive constant $C_1 \geq 1$ depends only on γ, β, the lifespan $t_1 > 0$ is defined in (2.1)*, and $p > 1$ is defined in* (3.6)*.*

Proof. It follows from (3.3) that

$$\begin{aligned}
|h(t,x,v)| &\leq \mathrm{e}^{-\nu(v)t}\|h_0\|_{L^\infty} + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\Big|\Big(K_{w_\beta}^m h\Big)(s, x-v(t-s), v)\Big|\mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\Big|\Big(K_{w_\beta}^c h\Big)(s, x-v(t-s), v)\Big|\mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\Big|(\Gamma_{w_\beta}(h,h))(s, x-v(t-s), v)\Big|\mathrm{d}s \\
&= \mathrm{e}^{-\nu(v)t}\|h_0\|_{L^\infty} + J_1 + J_2 + J_3.
\end{aligned}$$

Using (2.6), one gets that

$$\begin{aligned}
J_1 &= \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\Big|w_\beta(v)\Big(K^m f\Big)(s, x-v(t-s), v)\Big|\mathrm{d}s \\
&\leq Cm^{3+\gamma}\int_0^t \mathrm{e}^{-\nu(v)(t-s)} w_\beta(v)\mathrm{e}^{-\frac{|v|^2}{10}}\|f(s)\|_{L^\infty}\mathrm{d}s \\
&\leq Cm^{3+\gamma}\mathrm{e}^{-\frac{|v|^2}{20}}\sup_{0\leq s\leq t}\|f(s)\|_{L^\infty}.
\end{aligned} \tag{3.15}$$

For J_3, it follows from (3.2) and (3.4) that for $\beta \geq 1/2$,

$$\begin{aligned}
J_3 &= \int_0^t \mathrm{e}^{-\nu(v)(t-s)}|\Gamma_{w_\beta}(h,h)(s, x-v(t-s), v)|\mathrm{d}s \\
&\leq C\sup_{0\leq s\leq t,\ y\in\Omega}\left\{\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}}\Big(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\Big)^{\frac{p-1}{5p}}\right\}.
\end{aligned} \tag{3.16}$$

Let $l_{w_\beta}(v, v')$ be the corresponding kernel associated with $K_{w_\beta}^c$, then we have that

$$l_{w_\beta}(v,v') = l(v,v')\frac{w_\beta(v)}{w_\beta(v')},$$

which together with (2.10) and (2.11), yield that

$$\int_{\mathbb{R}^3}\Big|l_{w_\beta}(v,v')\Big|\mathrm{d}v' \leq C_\gamma m^{\gamma-1}\frac{\nu(v)}{(1+|v|)^2} \text{ and } \int_{\mathbb{R}^3}\Big|l_{w_\beta}(v,v')\Big|\mathrm{d}v' \leq C_\gamma(1+|v|)^{-1}. \tag{3.17}$$

For J_2, denoting $\tilde{x} := x - v(t-s)$, we note that

$$J_2 \le \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |l_{w_\beta}(v, v')h(s, \tilde{x}, v')| \mathrm{d}v' \mathrm{d}s.$$

To bound the above term, similar as in [15,26], we use (3.3) again to get that

$$\begin{aligned} J_2 \le\ & \|h_0\|_{L^\infty} \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |l_{w_\beta}(v, v')| \mathrm{d}v' \mathrm{d}s \\ & + \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |l_{w_\beta}(v, v')| \\ & \times \int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} |\left(K_{w_\beta}^m h\right)(\tau, \tilde{x} - v'(s-\tau), v')| \mathrm{d}\tau \mathrm{d}v' \mathrm{d}s \\ & + \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |l_{w_\beta}(v, v')| \\ & \times \int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} |(\Gamma_{w_\beta}(h, h))(\tau, \tilde{x} - v'(s-\tau), v')| \mathrm{d}\tau \mathrm{d}v' \mathrm{d}s \\ & + \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |l_{w_\beta}(v, v') l_{w_\beta}(v', v'')| \\ & \times \int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} |h(\tau, \tilde{x} - v'(s-\tau), v'')| \mathrm{d}\tau \mathrm{d}v' \mathrm{d}v'' \mathrm{d}s \\ :=\ & J_{21} + J_{22} + J_{23} + J_{24}. \end{aligned} \tag{3.18}$$

It follows from (3.16) and (3.17) that, for $\beta \ge \frac{1}{2}$,

$$\begin{aligned} J_{21} + J_{23} \le\ & C \left\{ \|h_0\|_{L^\infty} + \sup_{0 \le s \le t,\ y \in \Omega} \left(\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}} \left(\int_{\mathbb{R}^3} |f(s, y, \eta)| \mathrm{d}\eta \right)^{\frac{p-1}{5p}} \right) \right\} \\ & \times \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |l_{w_\beta}(v, v')| \mathrm{d}v' \mathrm{d}s \\ \le\ & C m^{\gamma-1} \left\{ \|h_0\|_{L^\infty} + \sup_{0 \le s \le t,\ y \in \Omega} \left(\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}} \left(\int_{\mathbb{R}^3} |f(s, y, \eta)| \mathrm{d}\eta \right)^{\frac{p-1}{5p}} \right) \right\}. \end{aligned}$$

For J_{22}, using (3.15), one has that

$$\begin{aligned} J_{22} \le\ & C m^{3+\gamma} \sup_{0 \le s \le t} \|f(s)\|_{L^\infty} \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |l_{w_\beta}(v, v')| \mathrm{e}^{-\frac{1}{20}|v'|^2} \mathrm{d}v' \mathrm{d}s \\ \le\ & C m^{3+\gamma} \sup_{0 \le s \le t} \|f(s)\|_{L^\infty}, \end{aligned}$$

where we have used the fact that

$$\int_{\mathbb{R}^3} |l_{w_\beta}(v, v')| \mathrm{e}^{-\frac{1}{20}|v'|^2} \mathrm{d}v' \le C \mathrm{e}^{-\frac{|v|^2}{100}},$$

which follows from (2.9) and similar arguments as in [10].

We now concentrate on the last term J_{24} on the RHS of (3.18). As in [15], we divide it into the following several cases.

Case 1. For $|v| \geq N$, it follows from (3.17) that

$$\int_{\mathbb{R}^3} \left| l_{w_\beta}(v, v') \right| dv' \leq C_m \frac{\nu(v)}{N^2},$$

which yields immediately that

$$\begin{aligned} J_{24} \leq C_m & \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty} \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |l_{w_\beta}(v, v')| \\ & \times \int_0^s e^{-\nu(v')(s-\tau)} \frac{\nu(v')}{1+|v'|^2} d\tau dv' ds \\ \leq & \frac{C_m}{N^2} \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty}. \end{aligned}$$

Case 2. For either $|v| \leq N,\ |v'| \geq 2N$ or $|v'| \leq 2N,\ |v''| \geq 3N$. It is noted that we have either $|v - v'| \geq N$ or $|v' - v''| \geq N$, and either one of the following is valid

$$\begin{aligned} |l_{w_\beta}(v, v')| &\leq C e^{-\frac{N^2}{20}} \left| l_{w_\beta}(v, v') e^{\frac{|v-v'|^2}{20}} \right|, \\ |l_{w_\beta}(v', v'')| &\leq C e^{-\frac{N^2}{20}} \left| l_{w_\beta}(v', v'') e^{\frac{|v'-v''|^2}{20}} \right|. \end{aligned} \tag{3.19}$$

From (2.8), a direct calculation shows that

$$\begin{aligned} \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, v') e^{\frac{|v-v'|^2}{20}} \right| dv' &\leq C_m \frac{\nu(v)}{(1+|v|)^2}, \\ \int_{\mathbb{R}^3} \left| l_{w_\beta}(v', v'') e^{\frac{|v'-v''|^2}{20}} \right| dv'' &\leq C_m \frac{\nu(v')}{(1+|v'|)^2}. \end{aligned} \tag{3.20}$$

Then it follows from (3.19)–(3.20) that

$$\begin{aligned} & \int_0^t e^{-\nu(v)(t-s)} \Big\{ \int_{|v|\leq N, |v'|\geq 2N} + \int_{|v'|\leq 2N, |v''|\geq 3N} \Big\} |l_{w_\beta}(v, v') l_{w_\beta}(v', v'')| \\ & \quad \times \int_0^s e^{-\nu(v')(s-\tau)} |h(\tau, \tilde{x} - v'(s-\tau), v'')| d\tau dv' dv'' ds \\ & \leq C_m e^{-\frac{N^2}{20}} \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty}. \end{aligned}$$

Case 3. $|v| \leq N,\ |v'| \leq 2N,\ |v''| \leq 3N$. This is the last remaining case. It is noted that

$$\begin{aligned} & \int_0^t e^{-\nu(v)(t-s)} \int_{|v'|\leq 2N, |v''|\leq 3N} |l_{w_\beta}(v, v') l_{w_\beta}(v', v'')| \\ & \quad \times \int_0^s e^{-\nu(v')(s-\tau)} |h(\tau, \tilde{x} - v'(s-\tau), v'')| d\tau dv' dv'' ds \\ & \leq \int_0^t e^{-\nu(v)(t-s)} \int_{|v'|\leq 2N, |v''|\leq 3N} |l_{w_\beta}(v, v') l_{w_\beta}(v', v'')| \end{aligned}$$

$$\times \int_{s-\lambda}^{s} e^{-\nu(v')(s-\tau)}|h(\tau, \tilde{x}-v'(s-\tau), v'')|d\tau dv' dv'' ds$$
$$+\int_0^t e^{-\nu(v)(t-s)} \int_{|v'|\le 2N, |v''|\le 3N} |l_{w_\beta}(v, v') l_{w_\beta}(v', v'')|$$
$$\times \int_0^{s-\lambda} e^{-\nu(v')(s-\tau)}|h(\tau, \tilde{x}-v'(s-\tau), v'')|d\tau dv' dv'' ds. \tag{3.21}$$

Using (3.17), we can bound the first term on the RHS of (3.21) by

$$C_m \lambda \sup_{0\le s\le t} \|h(s)\|_{L^\infty} \int_0^t e^{-\nu(v)(t-s)}\nu(v) ds \le C_m \lambda \sup_{0\le s\le t} \|h(s)\|_{L^\infty}. \tag{3.22}$$

Now we estimate the second term on the RHS of (3.21). Since $l_{w_\beta}(v, v')$ has a possible singularity of $\frac{1}{|v-v'|}$, we choose a smooth compact support function $l_N(v, v')$ such that

$$\sup_{|p|\le 3N} \int_{|v'|\le 3N} \Big| l_{w_\beta}(p, v') - l_N(p, v')\Big| dv' \le \frac{C_m}{N^7}. \tag{3.23}$$

Noting

$$l_{w_\beta}(v, v') l_{w_\beta}(v', v'') = \Big(l_{w_\beta}(v, v') - l_N(v, v')\Big) l_{w_\beta}(v', v'') + \Big(l_{w_\beta}(v', v'') - l_N(v', v'')\Big) l_N(v, v') + l_N(v, v') l_N(v', v''), \tag{3.24}$$

and then using (3.23) and (3.24), we can bound the second term on the RHS of (3.21) by

$$\frac{C_m}{N^7} \sup_{0\le s\le t} \|h(s)\|_{L^\infty} \int_0^t e^{-\nu(v)(t-s)} \int_0^{s-\lambda} e^{-\nu(v')(s-\tau)} d\tau ds$$
$$\times \left\{ \sup_{|v'|\le 2N} \int_{|v''|\le 3N} |l_{w_\beta}(v', v'')| dv'' + \sup_{|v|\le N} \int_{|v'|\le 2N} |l_N(v, v')| dv' \right\}$$
$$+\int_0^t e^{-\nu(v)(t-s)} \int_{|v'|\le 2N, |v''|\le 3N} |l_N(v, v') l_N(v', v'')|$$
$$\times \int_0^{s-\lambda} e^{-\nu(v')(s-\tau)}|h(\tau, \tilde{x}-v'(s-\tau), v'')|d\tau dv' dv'' ds$$
$$\le \frac{C_m}{N} \sup_{0\le s\le t} \|h(s)\|_{L^\infty}$$
$$+ C_{N,m} \int_0^t \int_0^{s-\lambda} e^{-c_N(t-s)} e^{-c_N(s-\tau)}$$
$$\times \int_{|v'|\le 2N, |v''|\le 3N} |h(\tau, \tilde{x}-v'(s-\tau), v'')| dv' dv'' d\tau ds, \tag{3.25}$$

where we have used the facts that $l_N(v, v')l_N(v', v'')$ is bounded and

$$\nu(v) \geq c_N \ \text{ for } |v| \leq N, \ \text{ and } \ \nu(v') \geq c_N \ \text{ for } |v'| \leq 2N.$$

It follows from (2.12) and Hölder inequality that

$$\begin{aligned}
&\int_{|v'|\leq 2N, |v''|\leq 3N} |h(\tau, \tilde{x} - v'(s-\tau), v'')| \mathrm{d}v' \mathrm{d}v'' \\
&\quad = C_{N,m} \int_{|v'|\leq 2N, |v''|\leq 3N} \\
&\qquad \frac{|F(\tau, \tilde{x} - v'(s-\tau), v'') - \mu(v'')|}{\sqrt{\mu(v'')}} I_{\{|F(\tau, \tilde{x}-v'(s-\tau), v'')-\mu(v'')|\leq \mu(v'')\}} \mathrm{d}v' \mathrm{d}v'' \\
&\quad + C_{N,m} \int_{|v'|\leq 2N, |v''|\leq 3N} |F(\tau, \tilde{x} - v'(s-\tau), v'') - \mu(v'')| \\
&\qquad \times I_{\{|F(\tau, \tilde{x}-v'(s-\tau), v'')-\mu(v'')|\geq \mu(v'')\}} \mathrm{d}v' \mathrm{d}v'' \\
&\leq C_{N,m} \frac{1 + (s-\tau)^{\frac{3}{2}}}{(s-\tau)^{\frac{3}{2}}} \left\{ \int_\Omega \int_{|v''|\leq 3N} \frac{|F(\tau, y, v'') - \mu(v'')|^2}{\mu(v'')} \right. \\
&\qquad \left. \times I_{\{|F(\tau, y, v'')-\mu(v'')|\leq \mu(v'')\}} \mathrm{d}v'' \mathrm{d}y \right\}^{\frac{1}{2}} \\
&\quad + C_{N,m} \frac{1 + (s-\tau)^3}{(s-\tau)^3} \\
&\qquad \times \int_\Omega \int_{|v''|\leq 3N} |F(\tau, y, v'') - \mu(v'')| I_{\{|F(\tau, y, v'')-\mu(v'')|\geq \mu(v'')\}} \mathrm{d}v'' \mathrm{d}y \\
&\leq C_{N,m} \frac{1 + (s-\tau)^{\frac{3}{2}}}{(s-\tau)^{\frac{3}{2}}} \sqrt{\mathcal{E}(F_0)} + C_{N,m} \frac{1 + (s-\tau)^3}{(s-\tau)^3} \mathcal{E}(F_0),
\end{aligned} \tag{3.26}$$

where we have made a change of variable $y = \tilde{x} - v'(s - \tau)$. From (3.26), we can bound the second term on the RHS of (3.25) as follows:

$$\begin{aligned}
&C_{N,m} \int_0^t \int_0^{s-\lambda} \mathrm{e}^{-c_N(t-s)} \mathrm{e}^{-c_N(s-\tau)} \\
&\quad \times \int_{|v'|\leq 2N, |v''|\leq 3N} |h(\tau, \tilde{x} - v'(s-\tau), v'')| \mathrm{d}v' \mathrm{d}v'' \mathrm{d}\tau \mathrm{d}s \\
&\leq C_{N,m} \lambda^{-\frac{3}{2}} \sqrt{\mathcal{E}(F_0)} + C_{N,m} \lambda^{-3} \mathcal{E}(F_0).
\end{aligned} \tag{3.27}$$

Combining (3.27), (3.25), (3.22) and (3.21), one gets that

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{|v'|\leq 2N, |v''|\leq 3N} |l_{w_\beta}(v, v') l_{w_\beta}(v', v'')| \\
&\quad \times \int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} |h(\tau, \tilde{x} - v'(s-\tau), v'')| \mathrm{d}\tau \mathrm{d}v' \mathrm{d}v'' \mathrm{d}s
\end{aligned}$$

$$\leq C_m\left(\lambda+\frac{1}{N}\right)\sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}+C_{N,m}\lambda^{-\frac{3}{2}}\sqrt{\mathcal{E}(F_0)}+C_{N,m}\lambda^{-3}\mathcal{E}(F_0).$$

Therefore, collecting all the above estimates, we have established that for any $\lambda>0$ and large $N\geq 1$,

$$\begin{aligned}\sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}\leq C_m\|h_0\|_{L^\infty}&+C\left(m^{3+\gamma}+C_m\lambda+\frac{C_m}{N}\right)\cdot\sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}\\&+C_m\sup_{0\leq s\leq t,\ y\in\Omega}\left\{\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}\\&+C_{N,m}\lambda^{-\frac{3}{2}}\sqrt{\mathcal{E}(F_0)}+C_{N,m}\lambda^{-3}\mathcal{E}(F_0).\end{aligned}$$

Noting $3+\gamma>0$, first choosing m small, then λ small, and finally letting N be sufficiently large so that $C\left(m^{3+\gamma}+C_m\lambda+\frac{C_m}{N}\right)\leq\frac{1}{2}$, one obtains that for $\beta\geq\frac{1}{2}$,

$$\begin{aligned}\sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}\leq C&\left\{\|h_0\|_{L^\infty}+\sqrt{\mathcal{E}(F_0)}+\mathcal{E}(F_0)\right\}\\&+C\sup_{0\leq s\leq t,\ y\in\Omega}\left\{\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}.\end{aligned}\tag{3.28}$$

Using Proposition 2.1, one has that for $\beta>3$,

$$\begin{aligned}&\sup_{0\leq s\leq t_1,\ y\in\Omega}\left\{\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}\\&\quad\leq C\sup_{0\leq s\leq t_1}\|h(s)\|_{L^\infty}^2\leq C\|h_0\|_{L^\infty}^2.\end{aligned}\tag{3.29}$$

Substituting (3.29) into (3.28), one gets that for $\beta>3$,

$$\begin{aligned}\sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}\leq C_1&\left\{\|h_0\|_{L^\infty}+\|h_0\|_{L^\infty}^2+\sqrt{\mathcal{E}(F_0)}+\mathcal{E}(F_0)\right\}\\&+C_1\sup_{t_1\leq s\leq t,\ y\in\Omega}\left\{\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\},\end{aligned}$$

which yields immediately (3.14), where the positive constant C_1 depends only on γ, β. Thus the proof of Lemma 3.2 is completed. □

3.2. $L_x^\infty L_v^1$ *Estimate*

In this subsection, we will concentrate on the estimate of

$$\int_{\mathbb{R}^3}|f(t,x,v)|\mathrm{d}v.$$

If $\mathcal{E}(F_0)+\|f_0\|_{L_x^1L_v^\infty}$ is small, due to the hyperbolicity of the Boltzmann equation one should be able to show that $\int_{\mathbb{R}^3}|f(t,x,v)|\mathrm{d}v$ is small for $t\geq t_1$, even though it could be initially large, i.e., $\int_{\mathbb{R}^3}|f_0(x,v)|\mathrm{d}v$ is large. Indeed, we have the following lemma which plays a key role in this paper.

Lemma 3.3. *Let* $-3 < \gamma \leq 1$ *and* $\beta > \max\{3, 3+\gamma\}$*, then it holds that*

$$\begin{aligned}\int_{\mathbb{R}^3} |f(t,x,v)|\mathrm{d}v \leq& \int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)t}|f_0(x-vt,v)|\mathrm{d}v\\&+ C_N\lambda^{-\frac{3}{2}}\sqrt{\mathcal{E}(F_0)} + C_N\lambda^{-3}\mathcal{E}(F_0)\\&+ C\Big(m^{3+\gamma} + C_m\Big[\lambda + \frac{1}{N} + \frac{1}{N^{\beta-3}}\Big]\Big)\\&\times \sup_{0\leq s\leq t}\Big\{\|h(s)\|_{L^\infty} + \|h(s)\|_{L^\infty}^2\Big\}\\&+ C_N\lambda^{-3}\Big(\sqrt{\mathcal{E}(F_0)} + \mathcal{E}(F_0)\Big)^{1-\frac{1}{p}} \cdot \sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}^{1+\frac{1}{p}},\end{aligned} \tag{3.30}$$

where $\lambda > 0, m > 0$ *and* $N \geq 1$ *are to be chosen later. Recall that* $p > 1$ *is defined in* (3.6).

Proof. It follows from (1.10) and (2.5) that

$$\begin{aligned}&\int_{\mathbb{R}^3} |f(t,x,v)|\mathrm{d}v\\&\leq \int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)t}|f_0(x-vt,v)|\mathrm{d}v\\&\quad+ \int_0^t\int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)(t-s)}\Big|(K^m f)(s,x-v(t-s),v)\Big|\mathrm{d}v\mathrm{d}s\\&\quad+ \int_0^t\int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)(t-s)}\Big|(K^c f)(s,x-v(t-s),v)\Big|\mathrm{d}v\mathrm{d}s\\&\quad+ \int_0^t\int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)(t-s)}\Big|\Gamma(f,f)(s,x-v(t-s),v)\Big|\mathrm{d}v\mathrm{d}s\\&:= \int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)t}|f_0(x-vt,v)|\mathrm{d}v + H_1 + H_2 + H_3.\end{aligned} \tag{3.31}$$

For H_1, it follows from (2.6) that

$$\begin{aligned}H_1 &\leq Cm^{3+\gamma}\sup_{0\leq s\leq t}\|f(s)\|_{L^\infty}\int_0^t\int_{\mathbb{R}^3}\mathrm{e}^{-\nu(v)(t-s)}\mathrm{e}^{-\frac{|v|^2}{10}}\mathrm{d}v\mathrm{d}s\\&\leq Cm^{3+\gamma}\sup_{0\leq s\leq t}\|f(s)\|_{L^\infty}.\end{aligned} \tag{3.32}$$

For H_2, we notice that

$$\begin{aligned}H_2 =& \int_{t-\lambda}^t\int_{\mathbb{R}^3}\mathrm{e}^{-\nu(v)(t-s)}\Big|\int_{\mathbb{R}^3_{v'}} l(v,v')f(s,x-v(t-s),v')\mathrm{d}v'\Big|\mathrm{d}v\mathrm{d}s\\&+ \int_0^{t-\lambda}\int_{\mathbb{R}^3}\mathrm{e}^{-\nu(v)(t-s)}\Big|\int_{\mathbb{R}^3_{v'}} l(v,v')f(s,x-v(t-s),v')\mathrm{d}v'\Big|\mathrm{d}v\mathrm{d}s\\:=& H_{21} + H_{22}.\end{aligned} \tag{3.33}$$

It is straightforward to obtain that for $\beta > 2$,

$$H_{21} \le \sup_{0\le s\le t} \|h(s)\|_{L^\infty} \int_{t-\lambda}^{t} \int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)(t-s)} (1+|v|)^{-\beta} \int_{\mathbb{R}^3_{v'}} \Big| l_{w_\beta}(v,v') \Big| \mathrm{d}v' \mathrm{d}v \mathrm{d}s$$
$$\le C\lambda \sup_{0\le s\le t} \|h(s)\|_{L^\infty},$$

where we have used (3.17) in the last inequality. For the term H_{22}, one notices that

$$H_{22} = \int_0^{t-\lambda} \int_{|v|\ge N} \mathrm{e}^{-\nu(v)(t-s)} \Big| \int_{\mathbb{R}^3_{v'}} l(v,v') f(s, x - v(t-s), v') \mathrm{d}v' \Big| \mathrm{d}v \mathrm{d}s$$
$$+ \int_0^{t-\lambda} \int_{|v|\le N} \mathrm{e}^{-\nu(v)(t-s)} \Big| \int_{|v'|\ge 2N} l(v,v') f(s, x - v(t-s), v') \mathrm{d}v' \Big| \mathrm{d}v \mathrm{d}s$$
$$+ \int_0^{t-\lambda} \int_{|v|\le N} \mathrm{e}^{-\nu(v)(t-s)} \Big| \int_{|v'|\le 2N} l(v,v') f(s, x - v(t-s), v') \mathrm{d}v' \Big| \mathrm{d}v \mathrm{d}s$$
$$:= H_{221} + H_{222} + H_{223}.$$

It follows from (3.17) and (3.20) that for $\beta > 2$,

$$H_{221} \le \sup_{0\le s\le t} \|h(s)\|_{L^\infty} \cdot \int_0^{t-\lambda} \int_{|v|\ge N} \mathrm{e}^{-\nu(v)(t-s)} w_\beta(v)^{-1} \Big| \int_{\mathbb{R}^3_{v'}} l_{w_\beta}(v,v') \mathrm{d}v' \Big| \mathrm{d}v \mathrm{d}s$$
$$\le C_m \sup_{0\le s\le t} \|h(s)\|_{L^\infty} \int_0^{t-\lambda} \int_{|v|\ge N} \mathrm{e}^{-\nu(v)(t-s)} (1+|v|)^{-2-\beta} \nu(v) \mathrm{d}v \mathrm{d}s$$
$$\le \frac{C_m}{N^{\beta-1}} \sup_{0\le s\le t} \|h(s)\|_{L^\infty} \le \frac{C_m}{N} \sup_{0\le s\le t} \|h(s)\|_{L^\infty},$$

and

$$H_{222} \le \mathrm{e}^{-\frac{N^2}{20}} \sup_{0\le s\le t} \|h(s)\|_{L^\infty}$$
$$\times \int_0^{t-\lambda} \int_{|v|\le N} \mathrm{e}^{-\nu(v)(t-s)} w_\beta(v)^{-1} \Big| \int_{|v'|\ge 2N} l_{w_\beta}(v,v') \mathrm{e}^{\frac{|v-v'|^2}{20}} \mathrm{d}v' \Big| \mathrm{d}v \mathrm{d}s$$
$$\le C_m \mathrm{e}^{-\frac{N^2}{20}} \sup_{0\le s\le t} \|h(s)\|_{L^\infty}.$$

Since $l_{w_\beta}(v,v')$ has a possible singularity of $\frac{1}{|v-v'|}$, as before we choose the smooth compact support function $l_N(v,v')$ satisfying (3.23). Then it follows from (3.26) and (3.23) that

$$H_{223} \le \int_0^{t-\lambda} \int_{|v|\le N} \mathrm{e}^{-\nu(v)(t-s)} w_\beta(v)^{-1}$$
$$\times \int_{|v'|\le 2N} |l_{w_\beta}(v,v') - l_N(v,v')| \cdot |h(s, x - v(t-s), v')| \mathrm{d}v' \mathrm{d}v \mathrm{d}s$$
$$+ \int_0^{t-\lambda} \int_{|v|\le N} \mathrm{e}^{-\nu(v)(t-s)} \int_{|v'|\le 2N} \Big| l_N(v,v') h(s, x - v(t-s), v') \Big| \mathrm{d}v' \mathrm{d}v \mathrm{d}s$$

$$
\begin{aligned}
&\le \frac{C_m}{N}\sup_{0\le s\le t}\|h(s)\|_{L^\infty}\\
&\quad + C_N\int_0^{t-\lambda}\mathrm{e}^{-c_N(t-s)}\int_{|v|\le N,|v'|\le 2N}\Big|h(s,x-v(t-s),v')\Big|\mathrm{d}v'\mathrm{d}v\mathrm{d}s\\
&\le \frac{C_m}{N}\sup_{0\le s\le t}\|h(s)\|_{L^\infty} + C_N\lambda^{-\frac{3}{2}}\sqrt{\mathcal{E}(F_0)} + C_N\lambda^{-3}\mathcal{E}(F_0). \qquad (3.34)
\end{aligned}
$$

Hence, combining (3.33)–(3.34), one obtains that for $\beta > 2$,

$$
H_2 \le C_m\left(\lambda+\frac{1}{N}\right)\sup_{0\le s\le t}\|h(s)\|_{L^\infty} + C_N\lambda^{-\frac{3}{2}}\sqrt{\mathcal{E}(F_0)} + C_N\lambda^{-3}\mathcal{E}(F_0). \quad (3.35)
$$

Next we estimate H_3. Firstly, we note that

$$
\begin{aligned}
H_3 &\le \int_0^t\int_{\mathbb{R}^3}\mathrm{e}^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\mathrm{e}^{-\frac{|u|^2}{4}}\\
&\quad\times\Big|f(s,x-v(t-s),u)f(s,x-v(t-s),v)\Big|\mathrm{d}u\mathrm{d}\omega\mathrm{d}v\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\mathrm{e}^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\mathrm{e}^{-\frac{|u|^2}{4}}\\
&\quad\times\Big|f(s,x-v(t-s),u')f(s,x-v(t-s),v')\Big|\mathrm{d}u\mathrm{d}\omega\mathrm{d}v\mathrm{d}s\\
&:= H_{31}+H_{32}. \qquad (3.36)
\end{aligned}
$$

For H_{31}, one has that for $\beta > \max\{3, 3+\gamma\}$,

$$
\begin{aligned}
H_{31} &\le C\int_{t-\lambda}^t\int_{\mathbb{R}^3}\mathrm{e}^{-\nu(v)(t-s)}\nu(v)w_\beta(v)^{-1}\|h(s)\|_{L^\infty}^2\mathrm{d}v\mathrm{d}s\\
&\quad + C\int_0^{t-\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\|h(s)\|_{L^\infty}\mathrm{e}^{-\nu(v)(t-s)}\\
&\quad\times w_\beta(v)^{-1}|v-u|^\gamma\mathrm{e}^{-\frac{|u|^2}{4}}|f(s,x-v(t-s),u)|\mathrm{d}u\mathrm{d}v\mathrm{d}s\\
&\le C\lambda\sup_{0\le s\le t}\|h(s)\|_{L^\infty}^2 + C\int_0^{t-\lambda}\left\{\int_{|v|\ge N}\int_{\mathbb{R}^3_u}+\int_{\mathbb{R}^3_v}\int_{|u|\ge N}\right\}\{\cdots\}\mathrm{d}u\mathrm{d}v\mathrm{d}s\\
&\quad + C\int_0^{t-\lambda}\int_{|v|\le N}\int_{|u|\le N}\{\cdots\}\mathrm{d}u\mathrm{d}v\mathrm{d}s\\
&\le C\left(\lambda+\frac{1}{N^{\beta-3}}\right)\sup_{0\le s\le t}\|h(s)\|_{L^\infty}^2 + C\int_0^{t-\lambda}\int_{|v|\le N}\int_{|u|\le N}\{\cdots\}\mathrm{d}u\mathrm{d}v\mathrm{d}s.
\end{aligned}
$$
(3.37)

To estimate the last term on the RHS of above, for $p > 1$ defined in (3.6), it follows from the Hölder inequality that

$$
\begin{aligned}
&C\int_0^{t-\lambda}\int_{|v|\le N}\int_{|u|\le N}\{\cdots\}\mathrm{d}u\mathrm{d}v\mathrm{d}s\\
&\le C\int_0^{t-\lambda}\mathrm{e}^{-c_N(t-s)}\|h(s)\|_{L^\infty}\int_{|v|\le N}\int_{|u|\le N}\\
&\quad w_\beta(v)^{-1}|v-u|^\gamma\mathrm{e}^{-\frac{|u|^2}{4}}|f(s,x-v(t-s),u)|\mathrm{d}u\mathrm{d}v\mathrm{d}s\\
&\le C_N\int_0^{t-\lambda}\mathrm{e}^{-c_N(t-s)}\|h(s)\|_{L^\infty}\\
&\quad\times\left(\int_{|v|\le N}\int_{|u|\le N}|f(s,x-v(t-s),u)|^{\frac{p}{p-1}}\mathrm{d}u\mathrm{d}v\right)^{1-\frac{1}{p}}\mathrm{d}s\\
&\le C_N\lambda^{-3}\Big(\sqrt{\mathcal{E}(F_0)}+\mathcal{E}(F_0)\Big)^{1-\frac{1}{p}}\sup_{0\le s\le t}\|h(s)\|_{L^\infty}^{1+\frac{1}{p}},
\end{aligned}
\tag{3.38}
$$

where in the last inequality we have used the following fact that

$$
\begin{aligned}
&\int_{|v|\le N,|u|\le 3N}|f(s,x-v(t-s),u)|\mathrm{d}u\mathrm{d}v\\
&\quad\le C_N\frac{1+(t-s)^{\frac{3}{2}}}{(t-s)^{\frac{3}{2}}}\sqrt{\mathcal{E}(F_0)}+C_N\frac{1+(t-s)^3}{(t-s)^3}\mathcal{E}(F_0).
\end{aligned}
\tag{3.39}
$$

Hence, from (3.37)–(3.38), one obtains, for $\beta>\max\{3,3+\gamma\}$, that

$$
\begin{aligned}
H_{31}&\le C\left(\lambda+\frac{1}{N^{\beta-3}}\right)\sup_{0\le s\le t}\|h(s)\|_{L^\infty}^2\\
&\quad+C_N\lambda^{-3}\Big(\sqrt{\mathcal{E}(F_0)}+\mathcal{E}(F_0)\Big)^{1-\frac{1}{p}}\sup_{0\le s\le t}\|h(s)\|_{L^\infty}^{1+\frac{1}{p}}.
\end{aligned}
\tag{3.40}
$$

For H_{32}, we notice, for $\beta>\max\{3,3+\gamma\}$, that

$$
\begin{aligned}
H_{32}&\le C\int_{t-\lambda}^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{e}^{-\nu(v)(t-s)}|v-u|^\gamma w_\beta(v)^{-1}\mathrm{e}^{-\frac{|u|^2}{4}}\mathrm{d}u\mathrm{d}v\mathrm{d}s\cdot\sup_{0\le s\le t}\|h(s)\|_{L^\infty}^2\\
&\quad+\int_0^{t-\lambda}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\mathrm{e}^{-\nu(v)(t-s)}w_\beta(v)^{-1}\\
&\quad\times\mathrm{e}^{-\frac{|u|^2}{4}}\|h(s)\|_{L^\infty}|h(s,x-v(t-s),v')|\mathrm{d}u\mathrm{d}\omega\mathrm{d}v\mathrm{d}s\\
&\le C\lambda\sup_{0\le s\le t}\|h(s)\|_{L^\infty}^2\\
&\quad+\int_0^{t-\lambda}\left\{\int_{|v|\ge N}\int_{\mathbb{R}^3_u}\int_{\mathbb{S}^2}+\int_{\mathbb{R}^3_v}\int_{|u|\ge N}\int_{\mathbb{S}^2}\right\}\{\cdots\}\mathrm{d}u\mathrm{d}\omega\mathrm{d}v\mathrm{d}s\\
&\quad+\int_0^{t-\lambda}\int_{|v|\le N}\int_{|u|\le N}\int_{\mathbb{S}^2}\{\cdots\}\mathrm{d}u\mathrm{d}\omega\mathrm{d}v\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\left(\lambda+\frac{1}{N^{\beta-3}}\right)\sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}^2\\
&\quad+\int_0^{t-\lambda}\int_{|v|\leq N}\int_{|u|\leq N}\int_{\mathbb{S}^2}\{\cdots\}\mathrm{d}u\mathrm{d}\omega\mathrm{d}v\mathrm{d}s.
\end{aligned}
$$

To estimate the last term on the RHS of above, we utilize the changing of variables (3.10), (3.11) and (3.12) to obtain that

$$
\begin{aligned}
&\int_0^{t-\lambda}\int_{|v|\leq N}\int_{|u|\leq N}\int_{\mathbb{S}^2}\{\cdots\}\mathrm{d}u\mathrm{d}\omega\mathrm{d}v\mathrm{d}s\\
&\quad\leq C\int_0^{t-\lambda}\mathrm{e}^{-c_N(t-s)}\|h(s)\|_{L^\infty}\int_{|v|\leq N}\int_{|z|\leq 2N}\int_{\mathbb{S}^2}w_\beta(v)^{-1}\\
&\qquad\frac{|z_{\shortparallel}|}{(|z_\perp|+|z_{\shortparallel}|)^{1-\gamma}}\mathrm{e}^{-\frac{|v+z_{\shortparallel}+z_\perp|^2}{4}}\Big|h(s,x-v(t-s),v+z_{\shortparallel})\Big|\mathrm{d}z\mathrm{d}\omega\mathrm{d}v\mathrm{d}s\\
&\leq C\int_0^{t-\lambda}\mathrm{e}^{-c_N(t-s)}\|h(s)\|_{L^\infty}\int_{|v|\leq N}\int_{|\eta|\leq 3N}\int_{z_\perp}w_\beta(v)^{-1}\frac{|z_\perp|^{\frac{\gamma-1}{2}}}{|\eta-v|^{\frac{3-\gamma}{2}}}\mathrm{e}^{-\frac{|\eta+z_\perp|^2}{4}}\\
&\quad\times\Big|h(s,x-v(t-s),\eta)\Big|\mathrm{d}z_\perp\mathrm{d}\eta\mathrm{d}v\mathrm{d}s\\
&\quad\leq C_N\int_0^{t-\lambda}\mathrm{e}^{-c_N(t-s)}\|h(s)\|_{L^\infty}^{1+\frac{1}{p}}\\
&\qquad\times\left(\int_{|v|\leq N}\int_{|\eta|\leq 3N}|f(s,x-v(t-s),\eta)|\mathrm{d}\eta\mathrm{d}v\right)^{1-\frac{1}{p}}\mathrm{d}s\\
&\quad\leq C_N\lambda^{-3}\Big(\sqrt{\mathcal{E}(F_0)}+\mathcal{E}(F_0)\Big)^{1-\frac{1}{p}}\cdot\sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}^{1+\frac{1}{p}},
\end{aligned}
\tag{3.41}
$$

where we have used (3.39) in the last inequality and recall that $p>1$ is defined in (3.6). Hence, combining (3.36) and (3.40)–(3.41), one obtains that for $\beta>\max\{3,3+\gamma\}$,

$$
\begin{aligned}
H_3&\leq C\Big(\lambda+\frac{1}{N^{\beta-3}}\Big)\sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}^2\\
&\quad+C_N\lambda^{-3}\Big(\sqrt{\mathcal{E}(F_0)}+\mathcal{E}(F_0)\Big)^{1-\frac{1}{p}}\cdot\sup_{0\leq s\leq t}\|h(s)\|_{L^\infty}^{1+\frac{1}{p}}.
\end{aligned}
\tag{3.42}
$$

Submitting (3.32), (3.35) and (3.42) into (3.31), one proves (3.30) for $\beta>\max\{3,3+\gamma\}$. Hence the proof of Lemma 3.3 is completed. □

3.3. Global Existence and Uniqueness

Now we are in a position to give the

Proof of Theorem 1.1. Let $\beta>\max\{3,3+\gamma\}$. In terms of (3.14), we make the *a priori* assumption

$$\|h(t)\|_{L^\infty} \le 2A_0 := 2C_1\Big\{2\bar{M}^2 + \sqrt{\mathcal{E}(F_0)} + \mathcal{E}(F_0)\Big\}, \tag{3.43}$$

where the positive constant $C_1 \ge 1$ is defined in Lemma 3.2. Then it follows from Lemma 3.2 and the a priori assumption (3.43) that

$$\|h(t)\|_{L^\infty} \le A_0 + C_1(2A_0)^{\frac{9p+1}{5p}} \cdot \sup_{t_1\le s\le t,\ y\in\Omega} \Big(\int_{\mathbb{R}^3} |f(s,y,\eta)|\mathrm{d}\eta\Big)^{\frac{p-1}{5p}}. \tag{3.44}$$

To estimate the second term on the RHS of (3.44), we first notice that for $\Omega = \mathbb{R}^3$ and $t \ge t_1$,

$$\int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)t}|f_0(x-vt,v)|\mathrm{d}v \le t_1^{-3}\|f_0\|_{L^1_xL^\infty_v} \le C\bar{M}^3\|f_0\|_{L^1_xL^\infty_v}. \tag{3.45}$$

For $\Omega = \mathbb{T}^3$ and $t \ge t_1$, it holds that

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{e}^{-\nu(v)t}|f_0(x-vt,v)|\mathrm{d}v &\le \Big(\int_{|v|\ge\tilde{N}} + \int_{|v|\le\tilde{N}}\Big)|f_0(x-vt,v)|\mathrm{d}v\\ &\le C\|w_\beta f_0\|_{L^\infty}^{\frac{3}{\beta}}\|f_0\|_{L^1_xL^\infty_v}^{1-\frac{3}{\beta}} + \frac{C}{t_1^3}\|f_0\|_{L^1_xL^\infty_v}\\ &\le C\bar{M}^{\frac{3}{\beta}}\|f_0\|_{L^1_xL^\infty_v}^{1-\frac{3}{\beta}} + C\bar{M}^3\|f_0\|_{L^1_xL^\infty_v},\end{aligned} \tag{3.46}$$

where we have chosen $\tilde{N} = \|w_\beta f_0\|_{L^\infty}^{\frac{1}{\beta}}\|f_0\|_{L^1_xL^\infty}^{-\frac{1}{\beta}}$. Then it follows from Lemma 3.3, (3.45), (3.46) and the a priori assumption (3.43) that

$$\begin{aligned}&\sup_{t_1\le s\le t, y\in\mathbb{R}^3}\int_{\mathbb{R}^3}|f(t,y,v)|\mathrm{d}v\\ &\le \begin{cases} C\bar{M}^3\|f_0\|_{L^1_xL^\infty_v}, \text{ for } \Omega=\mathbb{R}^3\\ C\bar{M}^{\frac{3}{\beta}}\|f_0\|_{L^1_xL^\infty}^{1-\frac{3}{\beta}} + C\bar{M}^3\|f_0\|_{L^1_xL^\infty_v} \text{ for } \Omega=\mathbb{T}^3\end{cases} + C_N\lambda^{-\frac{3}{2}}\sqrt{\mathcal{E}(F_0)} + C_N\lambda^{-3}\mathcal{E}(F_0)\\ &\quad + C\Big\{m^{3+\gamma} + C_m\Big[\lambda + \frac{1}{N} + \frac{1}{N^{\beta-3}}\Big]\Big\}(2A_0)^2 + C_N\lambda^{-3}\Big(\sqrt{\mathcal{E}(F_0)} + \mathcal{E}(F_0)\Big)^{1-\frac{1}{p}}(2A_0)^{1+\frac{1}{p}}.\end{aligned} \tag{3.47}$$

Note $\beta > \max\{3, 3+\gamma\}$ and $p > 1$. One can firstly choose λ sufficiently small, then $N \ge 1$ large enough, and finally let $\mathcal{E}(F_0) + \|f_0\|_{L^1_xL^\infty_v} \le \varepsilon_0$ with ε_0 small depending only on β, γ and $\bar{M}$, such that

$$4C_1A_0^{\frac{4p+1}{5p}} \cdot \sup_{t_1\le s\le t,\ y\in\Omega}\Big(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\Big)^{\frac{p-1}{5p}} \le \frac{3}{4}, \tag{3.48}$$

which together with (3.44), yield immediately that

$$\|h(t)\|_{L^\infty} \le \frac{7}{4}A_0, \tag{3.49}$$

for all $t \ge 0$. Hence we have closed the a priori assumption (3.43). Therefore the proof of Theorem 1.1 is completed. □

3.4. Positive Lower Bound of Density

At the end of this section, we give the proof of Corollary 1.5. Noting $C_1 \geq 1$ and $A_0 \geq 1$, it follows from (3.48) that for $t \geq t_1$,

$$\int_{\mathbb{R}^3} |f(t, x, v)| \mathrm{d}v \leq \frac{3}{4}. \tag{3.50}$$

Then, using (3.50), it is straightforward to get that

$$|\rho(t, x) - 1| = \left| \int_{\mathbb{R}^3} [F(t, x, v) - \mu(v)] \mathrm{d}v \right| \leq \int_{\mathbb{R}^3} |f(t, x, v)| \mathrm{d}v \leq \frac{3}{4},$$

which yields immediately that initial vacuum of the density function should disappear for $t \geq T_0 := t_1$. Therefore the proof of Corollary 1.5 is completed. □

4. Time-Decay Estimates in Torus

In this section, we consider the time-decay estimates for the global solutions obtained in Theorem 1.1. Let $\Omega = \mathbb{T}^3$, we consider the following linearized Boltzmann equation

$$\zeta_t + v \cdot \nabla_x \zeta + \nu(v)\zeta - K\zeta = 0, \quad \zeta(0, x, v) = \zeta_0(x, v). \tag{4.1}$$

Denoting the semigroup of (4.1) by $S(t)$, it holds that

$$\zeta(t) = S(t)\zeta_0.$$

Let $\zeta(t, x, v)$ be the solution of the linearized equation (4.1), and denote

$$\xi(t, x, v) = w_\beta(v)\zeta(t, x, v).$$

Then it follows from (4.1) that

$$\xi_t + v \cdot \nabla_x \xi + \nu(v)\xi - K_{w_\beta}\xi = 0, \quad \xi(0, x, v) = \xi_0(x, v). \tag{4.2}$$

For later use, we denote the semigroup of (4.2) by $U(t)$, and write the solution as

$$\xi(t) = U(t)\xi_0.$$

4.1. Case of Hard Potentials

In this subsection, we consider the decay estimate for hard potentials on torus. The following proposition is a starting point for further geting the exponential decay in L^∞ norm.

Proposition 4.1. [19] *Let $0 \leq \gamma \leq 1$, $\Omega = \mathbb{T}^3$. Let $\zeta(t, x, v)$ be any solution to the linearized Boltzmann equation (4.1) and satisfies the conservations of mass (1.5), momentum (1.6) and energy (1.7) with $(M_0, J_0, E_0) = (0, 0, 0) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$. Then there exists positive constants $\sigma > 0$ and $C > 0$ such that*

$$\|S(t)\zeta_0\|_{L^2} = \|\zeta(t)\|_{L^2} \leq C\mathrm{e}^{-\sigma t}\|\zeta_0\|_{L^2},$$

for all $t \geq 0$.

Utilizing Proposition 4.1, we can obtain the following L^∞ decay estimate for the linearized Boltzmann equation.

Lemma 4.2. *Let* $0 \leq \gamma \leq 1$, $\Omega = \mathbb{T}^3$. *Let* $\zeta(t, x, v)$ *be any solution to the linear Boltzmann equation* (4.1) *and satisfies the conservations of mass* (1.5), *momentum* (1.6) *and energy* (1.7) *with* $(M_0, J_0, E_0) = (0, 0, 0) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$. *Then there exists positive constants* $0 < \sigma_1 \leq \sigma$ *and* $C > 0$ *such that*

$$\|U(t)\xi_0\|_{L^\infty} = \|\xi(t)\|_{L^\infty} \leq C\mathrm{e}^{-\sigma_1 t}\|w_\beta\zeta_0\|_{L^\infty} \text{ for } t \geq 0. \tag{4.3}$$

Proof. Notice that via Lemma 19 in [14], we only need to prove that there exist $\sigma_2 > 0$, $T_1 > 0$ and C_{T_1} such that

$$\|\xi(T_1)\|_{L^\infty} \leq \mathrm{e}^{-\sigma_2 T_1}\|\xi_0\|_{L^\infty} + C_{T_1}\int_0^{T_1} \|\zeta(s)\|_{L^2}\mathrm{d}s.$$

The rest of the proof is similar to KIM [19]; see also GUO [14]. Indeed, our case is simpler than [14,19] since the the characteristic lines in case without forcing are straight lines. Here we omit the details for brevity of presentations. □

Based on the above preparations, we utilize Lemma 4.2 to prove Theorem 1.6.

Proof of Theorem 1.6. Using the semigroup $U(t)$ for the weighted linearized Boltzmann equation (4.2), by the Duhamel Principle, we have the solution formula for the nonlinear weighted Boltzmann equation (3.1) as

$$h(t) = U(t)h_0 + \int_0^t U(t-s)\Big\{w_\beta\Gamma(f, f)(s)\Big\}\mathrm{d}s.$$

Then it follows from (4.3) that

$$\|h(t)\|_{L^\infty} \leq C\mathrm{e}^{-\sigma_1 t}\|h_0\|_{L^\infty} + \Big\|\int_0^t U(t-s)\Big\{w_\beta\Gamma(f, f)(s)\Big\}\mathrm{d}s\Big\|_{L^\infty}. \tag{4.4}$$

To bound the last term on the RHS of (4.4), we notice that

$$\begin{aligned}\int_0^t U(t-s)\Big\{w_\beta\Gamma(f, f)(s)\Big\}\mathrm{d}s &= \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\Big\{w_\beta\Gamma(f, f)(s)\Big\}\mathrm{d}s \\ &+ \int_0^t\int_s^t \mathrm{e}^{-\nu(v)(t-s_1)}K_{w_\beta}\Big\{U(s_1-s)w_\beta\Gamma(f, f)(s)\Big\}\mathrm{d}s_1\mathrm{d}s.\end{aligned} \tag{4.5}$$

For the first term on the RHS of (4.5), it follows from (3.4) that

$$\begin{aligned}&\Big|\int_0^t \mathrm{e}^{-\nu(v)(t-s)}\Big\{w_\beta(v)\Gamma(f, f)(s)\Big\}\mathrm{d}s\Big| \\ &\leq C\int_0^t \mathrm{e}^{-\nu(v)(t-s)}\nu(v)\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}}\sup_{y\in\Omega}\Big(\int_{\mathbb{R}^3}|f(s, y, \eta)|\mathrm{d}\eta\Big)^{\frac{p-1}{5p}}\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^t \mathrm{e}^{-\nu(v)(t-s)}\nu(v)\mathrm{e}^{-\frac{\sigma_1}{2}s}\\
&\quad\times \sup_{0\le s\le t,\, y\in\Omega}\left\{\left[\mathrm{e}^{\frac{\sigma_1}{2}s}\|h(s)\|_{L^\infty}\right]\cdot\|h(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}\mathrm{d}s\\
&\le C\mathrm{e}^{-\frac{\sigma_1}{2}t}\sup_{0\le s\le t,\, y\in\Omega}\left\{\left[\mathrm{e}^{\frac{\sigma_1}{2}s}\|h(s)\|_{L^\infty}\right]\cdot\|h(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}.
\end{aligned}
\tag{4.6}
$$

To estimate the second term on the RHS of (4.5), as in [14] we define a new semigroup $\tilde{U}(t)$ such that it solves

$$
\left\{\partial_t + v\cdot\nabla_x + \nu(v) - K_{\tilde{w}}\right\}\{\tilde{U}(t)\tilde{h}_0\} = 0,\quad \tilde{U}(0)\tilde{h}_0 = \tilde{h}_0,
$$

with $\tilde{w}(v) = \frac{w_\beta(v)}{\sqrt{1+|v|^2}}$. A direct calculation shows that

$$
\sqrt{1+|v|^2}\tilde{U}(t),
$$

also solves the original weighted linearized Boltzmann equation (4.2). Then the uniqueness in L^∞ class with $\tilde{h}_0 = \frac{h_0}{\sqrt{1+|v|^2}}$ yields that

$$
U(t)h_0 \equiv \sqrt{1+|v|^2}\tilde{U}(t)\left\{\frac{h_0}{\sqrt{1+|v|^2}}\right\}.
$$

Here we point out that (4.3) also holds for semigroup $\tilde{U}(t)$. Then it follows from (4.3) and (3.4) that

$$
\begin{aligned}
&\left|\int_0^t\int_s^t \mathrm{e}^{-\nu(v)(t-s_1)}K_{w_\beta}\left\{U(s_1-s)w_\beta\Gamma(f,f)(s)\right\}\mathrm{d}s_1\mathrm{d}s\right| \le \int_0^t\int_s^t \mathrm{e}^{-\nu_0(t-s_1)}\\
&\quad\times\left|\int_{\mathbb{R}^3_{v'}} k_{w_\beta}(v,v')\sqrt{1+|v'|^2}\left\{\tilde{U}(s_1-s)\frac{w_\beta}{\sqrt{1+|v|^2}}\Gamma(f,f)(s)\right\}\mathrm{d}v'\right|\mathrm{d}s_1\mathrm{d}s\\
&\le \int_0^t\int_s^t \mathrm{e}^{-\nu_0(t-s_1)}\left|\int_{\mathbb{R}^3_{v'}} k_{w_\beta}(v,v')\sqrt{1+|v'|^2}\mathrm{d}v'\right|\\
&\quad\times\left\|\left\{\tilde{U}(s_1-s)\frac{w_\beta}{\sqrt{1+|v|^2}}\Gamma(f,f)(s)\right\}\right\|_{L^\infty}\mathrm{d}s_1\mathrm{d}s\\
&\le \int_0^t\int_s^t \mathrm{e}^{-\nu_0(t-s_1)}\mathrm{e}^{-\sigma_1(s_1-s)}\left\|\frac{w_\beta}{\sqrt{1+|v|^2}}\Gamma(f,f)(s)\right\|_{L^\infty}\mathrm{d}s_1\mathrm{d}s\\
&\le C\int_0^t\int_s^t \mathrm{e}^{-\nu_0(t-s_1)}\mathrm{e}^{-\sigma_1(s_1-s)}\left\|\frac{\nu(v)}{\sqrt{1+|v|^2}}\right\|_{L^\infty}\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}}
\end{aligned}
$$

$$\times \sup_{y\in\Omega}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\mathrm{d}s_1\mathrm{d}s$$
$$\leq C\mathrm{e}^{-\frac{\sigma_1}{2}t}\sup_{0\leq s\leq t,y\in\Omega}\left\{\left[\mathrm{e}^{\frac{\sigma_1}{2}s}\|h(s)\|_{L^\infty}\right]\cdot\|h(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}. \tag{4.7}$$

Combining (4.4)–(4.6) and (4.7) and using (3.49), one obtains that

$$\begin{aligned}
&\sup_{0\leq s\leq t}\left\{\mathrm{e}^{\frac{\sigma_1}{2}s}\|h(s)\|_{L^\infty}\right\}\\
&\leq C\|h_0\|_{L^\infty}+C\sup_{0\leq s\leq t,y\in\Omega}\left\{\left[\mathrm{e}^{\frac{\sigma_1}{2}s}\|h(s)\|_{L^\infty}\right]\right.\\
&\quad\left.\times\ \|h(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}\\
&\leq C\left\{\|h_0\|_{L^\infty}+\sup_{0\leq s\leq 1}\|h(s)\|_{L^\infty}^2\right\}\\
&\quad+C\sup_{1\leq s\leq t}\left[\mathrm{e}^{\frac{\sigma_1}{2}s}\|h(s)\|_{L^\infty}\right]\cdot\sup_{1\leq s\leq t,y\in\Omega}\left\{\|h(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}\\
&\leq C_2\bar{M}^4+C_2\sup_{1\leq s\leq t}\left[\mathrm{e}^{\frac{\sigma_1}{2}s}\|h(s)\|_{L^\infty}\right]\\
&\quad\times\sup_{1\leq s\leq t,y\in\Omega}\left\{\|h(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}.
\end{aligned}\tag{4.8}$$

Then, using (3.47) and similar arguments as in (3.48), if ε_0 is small enough, one can obtain that

$$C_2\sup_{1\leq s\leq t,y\in\Omega}\left\{\|h(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}\leq\frac{1}{2}. \tag{4.9}$$

Substituting (4.9) into (4.8), one gets that

$$\mathrm{e}^{\frac{\sigma_1}{2}t}\|h(t)\|_{L^\infty}\leq 2C_2\bar{M}^4,\quad\forall\, t\geq 0. \tag{4.10}$$

Finally, choosing

$$\sigma_0=\frac{\sigma_1}{2}\quad\text{and}\quad\tilde{C}_2=2C_2\bar{M}^4,$$

we then obtain (1.12) from (4.10). Therefore the proof of Theorem 1.6 is completed. □

4.2. Case of Soft Potentials

In this subsection, we consider the decay estimates for soft potentials on torus. Firstly, we define the Fourier transformation as

$$\hat{\zeta}(t,k,v) = \int_\Omega \mathrm{e}^{-ik\cdot x}\zeta(t,x,v)\mathrm{d}x, \quad k \in \mathbb{Z}^3.$$

Then the have the following estimate, whose proof can be found in [6,22].

Proposition 4.3. [6] *Let $-3 < \gamma < 0$, and let $d \geq 0$, $r > 0$ be given constants. Let $\zeta(t,x,v)$ be any solution to the linearized Boltzmann equation (4.1) and satisfies the conservations of mass (1.5), momentum (1.6) and energy (1.7) with $(M_0, J_0, E_0) = (0,0,0) \in \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}$. Then the following estimate holds*

$$\|\sqrt{\nu(\cdot)}^{-d}\hat{\zeta}(t,k,\cdot)\|^2_{L^2_v} \leq C(1+t)^{-r}\|\sqrt{\nu(\cdot)}^{-d-r_+}\hat{\zeta}_0(k,\cdot)\|^2_{L^2_v},$$

for all $t \geq 0$ and $k \in \mathbb{Z}^3$, where r_+ denotes the arbitrary constant which is strictly greater than r.

Using Proposition 4.3 and Plancherel theorem, we have the following L^2 decay estimate.

Lemma 4.4. *Under the assumptions of Proposition 4.3, the following estimate holds*

$$\|\sqrt{\nu}^{-d}S(t)\zeta_0\|^2_{L^2} = \|\sqrt{\nu}^{-d}\zeta(t)\|^2_{L^2} \leq C(1+t)^{-r}\|\sqrt{\nu}^{-d-r_+}\zeta_0\|^2_{L^2}, \tag{4.11}$$

for all $t \geq 0$.

We have the following L^∞-decay estimate for the solutions to the linearized Boltzmann equation.

Lemma 4.5. *Under the assumptions of Proposition 4.3, it holds that*

$$\|S(t)\zeta_0\|_{L^\infty} = \|\zeta(t)\|_{L^\infty} \leq C(1+t)^{-r}\|w_{2+|\gamma|r}\zeta_0\|_{L^\infty}, \tag{4.12}$$

for any given $r \in (0, 1+\frac{2}{|\gamma|})$.

Proof. It is noted that

$$\begin{aligned}\zeta(t,x,v) = \mathrm{e}^{-\nu(v)t}\zeta_0(x-vt,v) + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}K^m\zeta(s,x-v(t-s),v)\mathrm{d}s \\ + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}K^c\zeta(s,x-v(t-s),v)\mathrm{d}s,\end{aligned} \tag{4.13}$$

which yields immediately that

$$
\begin{aligned}
|\zeta(t,x,v)| &\le \mathrm{e}^{-\nu(v)t}|\zeta_0(x-vt,v)| + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}|K^m\zeta(s,x-v(t-s),v)|\mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}|K^c\zeta(s,x-v(t-s),v)|\mathrm{d}s \\
&:= L_1 + L_2 + L_3,
\end{aligned}
$$

where $m > 0$ is a small constant to be chosen later.

Firstly, it is easy to get that

$$
L_1 \le C(1+t)^{-r}\|\nu^{-r}\zeta_0\|_{L^\infty}. \tag{4.14}
$$

It follows from (2.6) that

$$
\begin{aligned}
L_2 &\le Cm^{3+\gamma}\int_0^t \mathrm{e}^{-\nu(v)(t-s)}\mathrm{e}^{-\frac{|v|^2}{10}}\|\zeta(s)\|_{L^\infty}\mathrm{d}s \\
&\le Cm^{3+\gamma}\mathrm{e}^{-\frac{|v|^2}{20}}\cdot\int_0^t (1+t-s)^{-1-r}\|\zeta(s)\|_{L^\infty}\mathrm{d}s \\
&\le Cm^{3+\gamma}\mathrm{e}^{-\frac{|v|^2}{20}}\sup_{0\le s\le t}\Big\{(1+s)^r\|\zeta(s)\|_{L^\infty}\Big\}\cdot\int_0^t (1+t-s)^{-1-r}(1+s)^{-r}\mathrm{d}s \\
&\le Cm^{3+\gamma}\mathrm{e}^{-\frac{|v|^2}{20}}(1+t)^{-r}\sup_{0\le s\le t}\Big\{(1+s)^r\|\zeta(s)\|_{L^\infty}\Big\}.
\end{aligned} \tag{4.15}
$$

To bound L_3, we use (4.13) again to get that

$$
\begin{aligned}
L_3 &\le \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\int_{\mathbb{R}^3_{v'}}|l(v,v')|\mathrm{e}^{-\nu(v')s}|\zeta_0(x-v(t-s)-v's,v')|\mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\int_{\mathbb{R}^3_{v'}}|l(v,v')|\Big\{\int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} \\
&\quad \times |K^m\zeta(\tau,x-v(t-s)-v'(s-\tau),v')|\mathrm{d}\tau\Big\}\mathrm{d}v'\mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{-\nu(v)(t-s)}\int_{\mathbb{R}^3_{v'}}\int_{\mathbb{R}^3_{v''}}|l(v,v')l(v',v'')|\int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} \\
&\quad \times |\zeta(\tau,x-v(t-s)-v'(s-\tau),v'')|\mathrm{d}\tau\mathrm{d}v''\mathrm{d}v'\mathrm{d}s \\
&:= L_{31}+L_{32}+L_{33}.
\end{aligned} \tag{4.16}
$$

For L_{31}, it follows from (2.10) that

$$
\begin{aligned}
L_{31} &\le C\|\nu^{-r}\zeta_0\|_{L^\infty}\int_0^t \mathrm{e}^{-\nu(v)(t-s)}(1+s)^{-r}\cdot\int_{\mathbb{R}_{v'}}|l(v,v')|\mathrm{d}v'\mathrm{d}s \\
&\le C_m\|\nu^{-r}\zeta_0\|_{L^\infty}\int_0^t \mathrm{e}^{-\nu(v)(t-s)}(1+s)^{-r}\cdot\nu(v)^{1+\frac{2}{|\gamma|}}\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le C_m \|\nu^{-r}\zeta_0\|_{L^\infty} \int_0^t (1+t-s)^{-1-\frac{2}{|\gamma|}} (1+s)^{-r} \mathrm{d}s \\
&\le C_m (1+t)^{-r} \|\nu^{-r}\zeta_0\|_{L^\infty}.
\end{aligned}
$$

Using (2.6) and (2.11), one can obtain that

$$
\begin{aligned}
L_{32} \le{}& Cm^{3+\gamma} \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{\mathbb{R}^3_{v'}} |l(v,v')| \int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} \\
&\quad \mathrm{e}^{-\frac{|v'|^2}{10}} \|\zeta(\tau)\|_{L^\infty} \mathrm{d}\tau \mathrm{d}v' \mathrm{d}s \\
\le{}& Cm^{3+\gamma} \sup_{0\le\tau\le t} \Big\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\Big\} \int_0^t (1+t-s)^{-1-r} \\
&\times \int_0^s (1+s-\tau)^{-1-r} (1+\tau)^{-r} \mathrm{d}\tau \mathrm{d}s \\
\le{}& Cm^{3+\gamma} (1+t)^{-r} \sup_{0\le\tau\le t} \Big\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\Big\}.
\end{aligned}
\tag{4.17}
$$

Now we concentrate on the term L_{33}. As before, we divide it into several cases.

Case 1. For $|v| \ge N$, then it follows from (2.10) that

$$
\begin{aligned}
L_{33} \le{}& C_m \sup_{0\le\tau\le t} \Big\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\Big\} \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \frac{\nu(v)}{(1+|v|)^2} \\
&\times \int_0^s (1+s-\tau)^{-1-\frac{2}{|\gamma|}} (1+\tau)^{-r} \mathrm{d}\tau \mathrm{d}s \\
\le{}& \frac{C_m}{N^{\delta|\gamma|}} \sup_{0\le\tau\le t} \Big\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\Big\} \int_0^t (1+t-s)^{-1-\frac{2}{|\gamma|}+\delta} \\
&\times \int_0^s (1+s-\tau)^{-1-\frac{2}{|\gamma|}} (1+\tau)^{-r} \mathrm{d}\tau \mathrm{d}s,
\end{aligned}
\tag{4.18}
$$

where $\delta > 0$ is a small positive constant such that

$$
0 < r \le 1 + \frac{2}{|\gamma|} - \delta \;\text{ and }\; 1 + \frac{2}{\gamma} - \delta > 1. \tag{4.19}
$$

It is noted that such $\delta > 0$ must exist since $r < 1 + \frac{2}{|\gamma|}$. Then, from (4.19) and a direct calculation, one can get that

$$
\int_0^t (1+t-s)^{-1-\frac{2}{|\gamma|}+\delta} \int_0^s (1+s-\tau)^{-1-\frac{2}{|\gamma|}} (1+\tau)^{-r} \mathrm{d}\tau \mathrm{d}s \le C(1+t)^{-r},
$$

which together with (4.18), yield that

$$
L_{33} \le \frac{C_m}{N^{\delta|\gamma|}} (1+t)^{-r} \cdot \sup_{0\le\tau\le t} \Big\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\Big\}. \tag{4.20}
$$

Case 2. For either $|v| \leq N,\ |v'| \geq 2N$ or $|v'| \leq 2N,\ |v''| \geq 3N$. It is noted that we have either $|v - v'| \geq N$ or $|v' - v''| \geq N$. Then it follows from (3.20) that

$$\begin{aligned}
L_{33} \leq {} & \mathrm{e}^{-\frac{|N|^2}{20}} \sup_{0\leq\tau\leq t} \left\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\right\} \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{\mathbb{R}^3_{v'}} \left| l(v,v')\mathrm{e}^{\frac{|v-v'|^2}{20}} \right| \mathrm{d}v' \\
& \times \int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} (1+\tau)^{-r} \int_{\mathbb{R}^3_{v''}} \left| l(v',v'')\mathrm{e}^{\frac{|v'-v''|^2}{20}} \right| \mathrm{d}v''\mathrm{d}\tau\mathrm{d}s \\
\leq {} & C_m \mathrm{e}^{-\frac{N^2}{20}} (1+t)^{-r} \sup_{0\leq\tau\leq t} \left\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\right\}. \qquad (4.21)
\end{aligned}$$

Case 3. $|v| \leq N,\ |v'| \leq 2N,\ |v''| \leq 3N$. This is the last remaining case. Firstly, we note that

$$\begin{aligned}
& \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{|v'|\leq 2N, |v''|\leq 3N} |l(v,v')l(v',v'')| \\
& \quad \times \int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} |\zeta(\tau, x - v(t-s) - v'(s-\tau), v'')| \mathrm{d}\tau\mathrm{d}v'\mathrm{d}v''\mathrm{d}s \\
& \leq \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{|v'|\leq 2N, |v''|\leq 3N} |l(v,v')l(v',v'')| \\
& \quad \times \int_{s-\lambda}^s \mathrm{e}^{-\nu(v')(s-\tau)} |\zeta(\tau, x - v(t-s) - v'(s-\tau), v'')| \mathrm{d}\tau\mathrm{d}v'\mathrm{d}v''\mathrm{d}s \\
& \quad + \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{|v'|\leq 2N, |v''|\leq 3N} |l(v,v')l(v',v'')| \\
& \quad \times \int_0^{s-\lambda} \mathrm{e}^{-\nu(v')(s-\tau)} |\zeta(\tau, x - v(t-s) - v'(s-\tau), v'')| \mathrm{d}\tau\mathrm{d}v'\mathrm{d}v''\mathrm{d}s. \qquad (4.22)
\end{aligned}$$

We can bound the first term on the RHS of (4.22) by

$$\begin{aligned}
& C_m\lambda \sup_{0\leq\tau\leq t} \left\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\right\} \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \\
& \quad \times \int_{|v'|\leq 2N, |v''|\leq 3N} |l(v,v')l(v',v'')|(1+s)^{-r}\mathrm{d}v'\mathrm{d}v''\mathrm{d}s \\
& \quad \leq C_m\lambda \sup_{0\leq\tau\leq t} \left\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\right\} \int_0^t \mathrm{e}^{-\nu(v)(t-s)} \frac{\nu(v)^2}{(1+|v|)^4} (1+s)^{-r}\mathrm{d}s \\
& \quad \leq C_m\lambda(1+t)^{-r} \sup_{0\leq\tau\leq t} \left\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\right\}.
\end{aligned}$$

Now we shall estimate the second term on the RHS of (4.22). Since $l(v, v')$ has singularity of $|v - v'|^{-1}$, as before, we can choose a smooth compact support function $\tilde{l}_N(v, v')$ such that

$$\sup_{|p|\leq 3N} \int_{|v'|\leq 3N} \left| l(p,v') - \tilde{l}_N(p,v') \right| \mathrm{d}v' \leq C_m N^{2\gamma-5}. \qquad (4.23)$$

Noting

$$l(v,v')l(v',v'') = \Big(l(v,v') - \tilde{l}_N(v,v')\Big)l(v',v'') \\ + \Big(l(v',v'') - \tilde{l}_N(v',v'')\Big)\tilde{l}_N(v,v') + \tilde{l}_N(v,v')\tilde{l}_N(v',v''), \tag{4.24}$$

and then using (2.11), (4.23) and (4.24), we can bound the second term on the RHS of (4.22) by

$$\begin{aligned} &C_m N^{2\gamma-5} \sup_{0\le\tau\le t}\Big\{(1+\tau)^{-r}\|\zeta(\tau)\|_{L^\infty}\Big\}\int_0^t \mathrm{e}^{-cN^\gamma(t-s)} \\ &\quad\times\int_0^{s-\lambda}\mathrm{e}^{-cN^\gamma(s-\tau)}(1+\tau)^{-r}\mathrm{d}\tau\mathrm{d}s \\ &\quad+\int_0^t \mathrm{e}^{-cN^\gamma(t-s)}\int_{|v'|\le 2N,|v''|\le 3N}|\tilde{l}_N(v,v')\tilde{l}_N(v',v'')| \\ &\quad\times\int_0^{s-\lambda}\mathrm{e}^{-cN^\gamma(s-\tau)}|\zeta(\tau,x-v(t-s)-v'(s-\tau),v'')|\mathrm{d}\tau\mathrm{d}v'\mathrm{d}v''\mathrm{d}s \\ &\le \frac{C_m}{N}(1+t)^{-r}\sup_{0\le s\le t}\Big\{(1+\tau)^{-r}\|\zeta(\tau)\|_{L^\infty}\Big\} \\ &\quad+C_{N,m}\int_0^t\int_0^{s-\lambda}\mathrm{e}^{-cN^\gamma(t-s)}\mathrm{e}^{-cN^\gamma(s-\tau)} \\ &\quad\times\int_{|v'|\le 2N,|v''|\le 3N}|\zeta(\tau,x-v(t-s)-v'(s-\tau),v'')|\mathrm{d}v'\mathrm{d}v''\mathrm{d}\tau\mathrm{d}s, \end{aligned}$$

where we have used the facts that $\tilde{l}_N(v,v')\tilde{l}_N(v',v'')$ is bounded and

$$\nu(v)\ge cN^\gamma \quad \text{for } |v|\le N, \quad \text{and} \quad \nu(v')\ge cN^\gamma \quad \text{for } |v'|\le 2N.$$

As in Section 4, using the changing of variables, one obtains that

$$\begin{aligned} &C_{N,m}\int_0^t\int_0^{s-\lambda}\mathrm{e}^{-cN^\gamma(t-s)}\mathrm{e}^{-cN^\gamma(s-\tau)} \\ &\quad\times\int_{|v'|\le 2N,|v''|\le 3N}|\zeta(\tau,x-v(t-s)-v'(s-\tau),v'')|\mathrm{d}v'\mathrm{d}v''\mathrm{d}\tau\mathrm{d}s \\ &\le C_{N,m,\lambda}\int_0^t\int_0^{s-\lambda}\mathrm{e}^{-cN^\gamma(t-s)}\mathrm{e}^{-cN^\gamma(s-\tau)}\|\zeta(\tau)\|_{L^2}\mathrm{d}\tau \\ &\le C_{N,m,\lambda}(1+t)^{-r}\sup_{0\le\tau\le t}\Big\{(1+\tau)^r\|\zeta(\tau)\|_{L^2}\Big\}\le C_{N,m}(1+t)^{-r}\|\nu^{-r_+}\zeta_0\|_{L^2} \\ &\le C_{N,m,\lambda}(1+t)^{-r}\|w_{2+|\gamma|r}\zeta_0\|_{L^\infty}, \end{aligned} \tag{4.25}$$

where we have used (4.11) with $d=0$. Thus, combining (4.22)–(4.25), one gets that

$$\int_0^t \mathrm{e}^{-\nu(v)(t-s)} \int_{|v'|\leq 2N, |v''|\leq 3N} |l(v, v')l(v', v'')| \\ \times \int_0^s \mathrm{e}^{-\nu(v')(s-\tau)} |\zeta(\tau, x - v(t-s) - v'(s-\tau), v'')| \mathrm{d}\tau \mathrm{d}v' \mathrm{d}v'' \mathrm{d}s \\ \leq C_m \Big(\lambda + \frac{1}{N}\Big)(1+t)^{-r} \cdot \sup_{0\leq\tau\leq t} \Big\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\Big\} \\ + C_{N,m,\lambda}(1+t)^{-r} \cdot \|w_{2+|\gamma|r}\zeta_0\|_{L^\infty}. \tag{4.26}$$

Therefore, it follows from (4.20), (4.21) and (4.26) that

$$L_{33} \leq C_m \left(\lambda + \frac{1}{N} + \frac{1}{N^{\delta|\gamma|}}\right)(1+t)^{-r} \cdot \sup_{0\leq\tau\leq t} \Big\{(1+\tau)^r \|\zeta(\tau)\|_{L^\infty}\Big\} \\ + C_{N,m,\lambda}(1+t)^{-r} \cdot \|w_{2+|\gamma|r}\zeta_0\|_{L^\infty},$$

where, together with (4.14), (4.15) and (4.16)–(4.17), it yields that

$$\sup_{0\leq s\leq t} \Big\{(1+s)^r \|\zeta(s)\|_{L^\infty}\Big\} \\ \leq C\Big\{m^{3+\gamma} + C_m \left(\lambda + \frac{1}{N} + \frac{1}{N^{\delta|\gamma|}}\right)\Big\} \sup_{0\leq\tau\leq t} \Big\{(1+s)^r \|\zeta(s)\|_{L^\infty}\Big\} \\ + C_{N,m,\lambda}\|w_{2+|\gamma|r}\zeta_0\|_{L^\infty}.$$

Note $-3 < \gamma < 0$. By first choosing m small, then λ small, and finally letting N sufficiently large so that $C\Big\{m^{3+\gamma} + C_m(\lambda + \frac{1}{N} + \frac{1}{N^{\delta|\gamma|}})\Big\} \leq \frac{1}{2}$, one obtains that

$$\|\zeta(t)\|_{L^\infty} \leq C(1+t)^{-r}\|w_{2+|\gamma|r}\zeta_0\|_{L^\infty},$$

for all $t \geq 0$. This yields immediately (4.12). Thus we complete the proof of this lemma. □

Based on the above preparations, we now use Lemma 4.5 to give the

Proof of Theorem 1.7. Using the semigroup $S(t)$ for the linearized Boltzmann equation (4.1), by the Duhamel Principle, we have the solution formula for the nonlinear Boltzmann equation (1.9) as

$$f(t) = S(t)f_0 + \int_0^t S(t-s)\Big\{\Gamma(f, f)(s)\Big\}\mathrm{d}s.$$

From now on, we take $r := 1 + \frac{2}{|\gamma|} - \delta > 1 + \frac{1}{|\gamma|}$ with δ being an arbitary small positive constant such that $0 < \delta \leq \frac{1}{3}$. Then it follows from (4.12) that

$$\|f(t)\|_{L^\infty} \leq C(1+r)^{-r}\|w_{2+|\gamma|r}f_0\|_{L^\infty} \\ + C\int_0^t (1+t-s)^{-r}\left\|w_{2+|\gamma|r}\Big\{\Gamma(f, f)(s)\Big\}\right\|_{L^\infty}\mathrm{d}s$$

$$\le C(1+r)^{-r}\|w_{4+|\gamma|}f_0\|_{L^\infty}$$
$$+C\int_0^t(1+t-s)^{-r}\left\|w_{4+|\gamma|}\Big\{\Gamma(f,f)(s)\Big\}\right\|_{L^\infty}\mathrm{d}s. \quad (4.27)$$

To estimate the last term on the RHS of (4.27), we note, from (3.4), that

$$\begin{aligned}
&\left|(1+|v|)^{4+|\gamma|}\Big\{\Gamma(f,f)(s,x-v(t-s),v)\Big\}\right|\\
&\le C\|w_4 f(s)\|_{L^\infty}\|w_{\frac12}f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\sup_{y\in\Omega}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\\
&\le C\|w_\beta f(s)\|_{L^\infty}^{\frac{4}{\beta}}\|f(s)\|_{L^\infty}^{1-\frac{4}{\beta}}\|w_\beta f(s)\|_{L^\infty}^{\frac{1}{2\beta}\frac{4p+1}{5p}}\|f(s)\|_{L^\infty}^{\frac{4p+1}{5p}\left(1-\frac{1}{2\beta}\right)}\\
&\quad\times\sup_{y\in\Omega}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\\
&\le C\|f(s)\|_{L^\infty}^{1-\frac{4}{\beta}+\frac{4p+1}{5p}\left(1-\frac{1}{2\beta}\right)}\|w_\beta f(s)\|_{L^\infty}^{\frac{4}{\beta}+\frac{4p+1}{5p}\frac{1}{2\beta}}\sup_{y\in\Omega}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\\
&\le C\|f(s)\|_{L^\infty}\|w_\beta f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\sup_{y\in\Omega3}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}},
\end{aligned}\quad (4.28)$$

where we have used $\frac{4p+1}{5p}(1-\frac{1}{4\beta})-\frac{4}{\beta}\ge 0$ due to $\beta\ge\frac92$ and $1<p\le\frac87$. Then it follows from (4.28) that

$$\begin{aligned}
&C\int_0^t(1+t-s)^{-r}\left\|(1+|v|)^{4+|\gamma|}\Big\{\Gamma(f,f)(s)\Big\}\right\|_{L^\infty}\mathrm{d}s\\
&\le C\int_0^t(1+t-s)^{-r}(1+s)^{-r}\\
&\quad\times\sup_{0\le s\le t,y\in\Omega}\left\{\Big[(1+s)^r\|f(s)\|_{L^\infty}\Big]\|w_\beta f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}\mathrm{d}s\\
&\le C(1+t)^{-r}\\
&\quad\times\sup_{0\le s\le t,y\in\Omega}\left\{\Big[(1+s)^r\|f(s)\|_{L^\infty}\Big]\|w_\beta f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\},
\end{aligned}$$

which together with (3.49) and (4.27), yield that for $\beta\ge\max\{\frac92,4+|\gamma|\}$,

$$\begin{aligned}
&\sup_{0\le s\le t}\Big\{(1+s)^r\|f(s)\|_{L^\infty}\Big\}\\
&\le C\|w_\beta f_0\|_{L^\infty}\\
&\quad+\sup_{0\le s\le t,y\in\Omega}\left\{\Big[(1+s)^r\|f(s)\|_{L^\infty}\Big]\|w_\beta f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}
\end{aligned}$$

$$\leq C_3\bar{M}^4 + C_3 \sup_{1\leq s\leq t}\Big[(1+s)^r\|f(s)\|_{L^\infty}\Big]$$
$$\times \sup_{1\leq s\leq t, y\in\Omega}\left\{\|w_\beta f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\}. \tag{4.29}$$

Then, using (3.47) and similar arguments as in (3.48), if ε_0 is small enough, one can obtain that

$$C_3 \sup_{1\leq s\leq t, y\in\Omega}\left\{\|f(s)\|_{L^\infty}^{\frac{4p+1}{5p}}\left(\int_{\mathbb{R}^3}|f(s,y,\eta)|\mathrm{d}\eta\right)^{\frac{p-1}{5p}}\right\} \leq \frac{1}{2}. \tag{4.30}$$

Substituting (4.30) into (4.29), one proves that for $\beta > \max\{\frac{9}{2}, 4+|\gamma|\}$,

$$\|f(t)\|_{L^\infty} \leq 2C_3\bar{M}^4(1+t)^{-r}, \quad \forall t\geq 0. \tag{4.31}$$

Taking

$$\tilde{C}_3 = 2C_3\bar{M}^4,$$

then we obtain (1.13) from (4.31). Therefore the proof of Theorem 1.7 is completed. □

5. Appendix

5.1. Estimates on K

In this subsection, we give the proof of some lemmas in Section 2 for completeness.

Proof of Lemma 2.2. The estimate on $k_1(v,\eta)$ follows from a direct calculation. We will mainly focus on $K_2(v,\eta)$. It follows from [2,10] that

$$0 \leq k_2(v,\eta) = \frac{c_2}{|v-\eta|^2}\mathrm{e}^{-\frac{|v-\eta|^2}{8}}\mathrm{e}^{-\frac{||v|^2-|\eta|^2|^2}{8|v-\eta|^2}}\int_{\mathbb{R}^2}B^*(|\eta-v|,|z_\perp|)\mathrm{e}^{-\frac{|z_\perp+\zeta_\perp|^2}{2}}\mathrm{d}z_\perp, \tag{5.1}$$

and $B^*(|\eta-v|,|z_\perp|)$ satisfies

$$B^*(|\eta-v|,|z_\perp|) \leq C\frac{|\eta-v|}{\left(|\eta-v|^2+|z_\perp|^2\right)^{\frac{1-\gamma}{2}}}, \tag{5.2}$$

where

$$z = u-v, \quad z_{\shortparallel} = [z\cdot\omega]\omega, \quad z_\perp = z - z_{\shortparallel}, \quad \eta = v + z_{\shortparallel}.$$

Then, substituting (5.2) into (5.1), one obtains that

$$
\begin{aligned}
k_2(v,\eta) &\le \frac{c_2}{|v-\eta|} e^{-\frac{|v-\eta|^2}{8}} e^{-\frac{||v|^2-|\eta|^2|^2}{8|v-\eta|^2}} \int_{\mathbb{R}^2} \left(|\eta-v|^2+|z_\perp|^2\right)^{\frac{\gamma-1}{2}} e^{-\frac{|z_\perp+\zeta_\perp|^2}{2}} dz_\perp \\
&\le \frac{c_2}{|v-\eta|^{\frac{3-\gamma}{2}}} e^{-\frac{|v-\eta|^2}{8}} e^{-\frac{||v|^2-|\eta|^2|^2}{8|v-\eta|^2}} \int_{\mathbb{R}^2} |z_\perp|^{\frac{\gamma-1}{2}} e^{-\frac{|z_\perp+\zeta_\perp|^2}{2}} dz_\perp \\
&\le \frac{C_\gamma}{|v-\eta|^{\frac{3-\gamma}{2}}} e^{-\frac{|v-\eta|^2}{8}} e^{-\frac{||v|^2-|\eta|^2|^2}{8|v-\eta|^2}}.
\end{aligned}
$$

Thus the proof of Lemma 2.2 is completed. □

Proof Lemma 2.4. Firstly, it is straightforward to prove (2.6) due to the fact that

$$
e^{-\frac{|u|^2}{4}} \le C e^{-\frac{|v|^2}{8}} \quad \text{for } |v-u| \le 2m.
$$

Next, we shall prove (2.7). It is noted that

$$
\begin{aligned}
\left(k_1-k_1^m\right)(v,\eta) &= c|v-\eta|^\gamma [1-\chi_m(|v-u|)] e^{-\frac{|v|^2}{4}} e^{-\frac{|\eta|^2}{4}} \\
&\le c|v-\eta|^\gamma e^{-\frac{|v|^2}{4}} e^{-\frac{|\eta|^2}{4}}.
\end{aligned} \tag{5.3}
$$

From [10], we have that

$$
k_2(v,\eta) \le \frac{C}{|\eta-v|} e^{-\frac{|\eta-v|^2}{8}} e^{-\frac{1}{2}|\zeta_{\shortparallel}|^2} \int_{\mathbb{R}^2} \frac{1}{\left(|\eta-v|^2+|z_\perp|^2\right)^{\frac{1-\gamma}{2}}} e^{-\frac{|\zeta_\perp+z_\perp|^2}{2}} dz_\perp, \tag{5.4}
$$

where

$$
z = u - v, \quad z_{\shortparallel} = [z\cdot\omega]\omega, \quad z_\perp = z - z_{\shortparallel}, \quad \eta = v + z_{\shortparallel} \tag{5.5}
$$

$$
\zeta := \frac{1}{2}(v+\eta) = v + \frac{1}{2} z_{\shortparallel}, \quad \zeta_{\shortparallel} = [\zeta\cdot\omega]\omega. \tag{5.6}
$$

Denote

$$
\tilde{\chi}_m(s) = 1 - \chi_m(s) \quad \text{for } s \ge 0.
$$

Then, it follows from (5.4) that for $0 \le a \le 1$,

$$
\begin{aligned}
&\left(k_2-k_2^m\right)(v,\eta) \\
&\le \frac{1}{|\eta-v|} e^{-\frac{|\eta-v|^2}{8}} e^{-\frac{1}{2}|\zeta_{\shortparallel}|^2} \\
&\quad \times \int_{\mathbb{R}^2} \tilde{\chi}_m\left(\sqrt{|\eta-v|^2+|z_\perp|^2}\right) \left(|\eta-v|^2+|z_\perp|^2\right)^{\frac{\gamma-1}{2}} e^{-\frac{|\zeta_\perp+z_\perp|^2}{2}} dz_\perp \\
&\le \frac{1}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} e^{-\frac{|\eta-v|^2}{8}} e^{-\frac{1}{2}|\zeta_{\shortparallel}|^2}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{\mathbb{R}^2} \tilde{\chi}_m\left(\sqrt{|\eta-v|^2+|z_\perp|^2}\right) \frac{1}{\left(1+|\eta-v|^2+|z_\perp|^2\right)^{a\frac{1-\gamma}{2}}}\\
&\times \frac{\left(1+|\eta-v|^2+|z_\perp|^2\right)^{a\frac{1-\gamma}{2}}}{\left(|\eta-v|^2+|z_\perp|^2\right)^{a\frac{1-\gamma}{2}}} \cdot |z_\perp|^{(1-a)\frac{\gamma-1}{2}} \mathrm{e}^{-\frac{|\zeta_\perp+z_\perp|^2}{2}} \mathrm{d}z_\perp\\
\leq{}& C \frac{1}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{1}{2}|\zeta_{\shortparallel}|^2}\\
&\times \int_{\mathbb{R}^2} \tilde{\chi}_m\left(\sqrt{|\eta-v|^2+|z_\perp|^2}\right) \frac{1}{\left(1+|\eta-v|^2+|z_\perp|^2\right)^{a\frac{1-\gamma}{2}}}\\
&\times \frac{\left(1+|\eta-v|^2+|z_\perp|^2\right)^{a\frac{1-\gamma}{2}}}{\left(m^2+|\eta-v|^2+|z_\perp|^2\right)^{a\frac{1-\gamma}{2}}} \cdot |z_\perp|^{(1-a)\frac{\gamma-1}{2}} \mathrm{e}^{-\frac{|\zeta_\perp+z_\perp|^2}{2}} \mathrm{d}z_\perp\\
\leq{}& \frac{m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{1}{2}|\zeta_{\shortparallel}|^2}\\
&\times \int_{\mathbb{R}^2} \frac{|z_\perp|^{(1-a)\frac{\gamma-1}{2}}}{\left(1+|\eta-v|^2+|z_\perp|^2\right)^{a\frac{1-\gamma}{2}}} \mathrm{e}^{-\frac{|\zeta_\perp+z_\perp|^2}{2}} \mathrm{d}z_\perp\\
\leq{}& \frac{m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{1}{2}|\zeta_{\shortparallel}|^2}\\
&\times \int_{\mathbb{R}^2} \frac{|\zeta_\perp-z_\perp|^{(1-a)\frac{\gamma-1}{2}}}{\left(1+|\eta-v|^2+|\zeta_\perp-z_\perp|^2\right)^{a\frac{1-\gamma}{2}}} \mathrm{e}^{-\frac{|z_\perp|^2}{2}} \mathrm{d}z_\perp,
\end{aligned}
$$

where in the last inequality we have made a change of variable $z_\perp+\zeta_\perp \to z_\perp$. Now we estimate the RHS of above term. Following [23], we split it into two cases.

Case 1: For $|z_\perp| \geq \frac{1}{2}|\zeta_\perp|$, it holds that

$$
\begin{aligned}
&\left(k_2-k_2^m\right)(v,\eta)\\
&\leq \frac{m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{1}{2}|\zeta_{\shortparallel}|^2} \int_{\mathbb{R}^2} |\zeta_\perp-z_\perp|^{(1-a)\frac{\gamma-1}{2}} \mathrm{e}^{-\frac{|z_\perp|^2}{4}} \mathrm{e}^{-\frac{|\zeta_\perp|^2}{16}} \mathrm{d}z_\perp\\
&\leq \frac{m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{1}{4}|\zeta_{\shortparallel}|^2} \int_{\mathbb{R}^2} |\zeta_\perp-z_\perp|^{(1-a)\frac{\gamma-1}{2}} \mathrm{e}^{-\frac{|z_\perp|^2}{4}} \mathrm{e}^{-\frac{|\zeta|^2}{16}} \mathrm{d}z_\perp\\
&\leq \frac{m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{10}} \mathrm{e}^{-\frac{1}{4}|\zeta_{\shortparallel}|^2}\\
&\quad\times \int_{\mathbb{R}^2} |\zeta_\perp-z_\perp|^{(1-a)\frac{\gamma-1}{2}} \mathrm{e}^{-\frac{|z_\perp|^2}{4}} \mathrm{e}^{-\frac{1}{40}|\eta-v|^2-\frac{|\zeta|^2}{16}} \mathrm{d}z_\perp.
\end{aligned}
$$

It follows from (5.5) and (5.6) that

$$
|\eta-v|^2+|\zeta|^2 = |\eta-v|^2+\frac{1}{4}|\eta+v|^2 \geq \frac{1}{4}(|\eta|^2+|v|^2).
$$

This yields that

$$
\begin{aligned}
&\left(k_2 - k_2^m\right)(v,\eta) \\
&\quad \le \frac{m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{10}} \mathrm{e}^{-\frac{1}{4}|\zeta_{\shortparallel}|^2} \mathrm{e}^{-\frac{1}{40}(|\eta|^2+|v|^2)} \\
&\qquad \times \int_{\mathbb{R}^2} |\zeta_\perp - z_\perp|^{(1-a)\frac{\gamma-1}{2}} \mathrm{e}^{-\frac{|z_\perp|^2}{4}} \mathrm{d}z_\perp \\
&\quad \le \frac{C_\gamma m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{10}} \mathrm{e}^{-\frac{||v|^2-|\eta|^2|^2}{16|v-\eta|^2}} \mathrm{e}^{-\frac{1}{40}(|\eta|^2+|v|^2)},
\end{aligned}
$$

where in the last inequality we have used the following fact

$$
\begin{aligned}
|\zeta_{\shortparallel}|^2 &:= |[\zeta\cdot\omega]\omega|^2 = |[\zeta\cdot\omega]|^2 = \left|\zeta\cdot\frac{z_{\shortparallel}}{|z_{\shortparallel}|}\right|^2 \\
&= \frac{1}{4}\left|(\eta+v)\cdot\frac{(\eta-v)}{|\eta-v|}\right|^2 = \frac{1}{4}\frac{(|\eta|^2-|v|^2)^2}{|\eta-v|^2}.
\end{aligned} \tag{5.7}
$$

Case 2: For $|z_\perp| \le \frac{1}{2}|\zeta_\perp|$, it holds that

$$
\begin{aligned}
&\left(k_2 - k_2^m\right)(v,\eta) \\
&\quad \le \frac{C m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{1}{2}|\zeta_{\shortparallel}|^2} \\
&\qquad \times \int_{\mathbb{R}^2} \frac{|\zeta_\perp - z_\perp|^{(1-a)\frac{\gamma-1}{2}}}{\left(1+|\eta-v|^2+\frac{1}{4}|\zeta_\perp|^2\right)^{a\frac{1-\gamma}{2}}} \mathrm{e}^{-\frac{|z_\perp|^2}{2}} \mathrm{d}z_\perp \\
&\quad \le \frac{C m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{1}{4}|\zeta_{\shortparallel}|^2} \\
&\qquad \times \int_{\mathbb{R}^2} \frac{|\zeta_\perp - z_\perp|^{(1-a)\frac{\gamma-1}{2}}}{\left(1+|\eta-v|^2+|\zeta_{\shortparallel}|^2+|\zeta_\perp|^2\right)^{a\frac{1-\gamma}{2}}} \mathrm{e}^{-\frac{|z_\perp|^2}{2}} \mathrm{d}z_\perp \\
&\quad \le \frac{C m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{1}{4}|\zeta_{\shortparallel}|^2} \\
&\qquad \times \int_{\mathbb{R}^2} \frac{|\zeta_\perp - z_\perp|^{(1-a)\frac{\gamma-1}{2}}}{\left(1+|\eta-v|^2+|\zeta|^2\right)^{a\frac{1-\gamma}{2}}} \mathrm{e}^{-\frac{|z_\perp|^2}{2}} \mathrm{d}z_\perp \\
&\quad \le \frac{C m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \frac{1}{(1+|v|^2+|\eta|^2)^{a\frac{1-\gamma}{2}}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{1}{4}|\zeta_{\shortparallel}|^2} \\
&\qquad \times \int_{\mathbb{R}^2} |\zeta_\perp - z_\perp|^{(1-a)\frac{\gamma-1}{2}} \mathrm{e}^{-\frac{|z_\perp|^2}{2}} \mathrm{d}z_\perp \\
&\quad \le \frac{C_\gamma m^{a(\gamma-1)}}{|\eta-v|^{1+\frac{(1-a)}{2}(1-\gamma)}} \frac{1}{(1+|v|+|\eta|)^{a(1-\gamma)}} \mathrm{e}^{-\frac{|\eta-v|^2}{8}} \mathrm{e}^{-\frac{||v|^2-|\eta|^2|^2}{16|v-\eta|^2}},
\end{aligned} \tag{5.8}
$$

where we have used (5.7) in the last inequality. Combining (5.3)–(5.8), one proves (2.7). Therefore, the proof of Lemma 2.4 is completed. □

5.2. Local-in-Time Existence

In the following, we consider the local existence of unique solutions to the Boltzmann equation (1.1) with large initial data in L^∞-norm.

Proof of Proposition 2.1. To prove the local existence for the Boltzmann equation, we consider the iteration that for $n = 0, 1, 2, \dots$,

$$F_t^{n+1} + v \cdot \nabla_x F^{n+1} + F^{n+1} \cdot \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u,\theta) F^n(t,x,u) \mathrm{d}\omega \mathrm{d}u = Q_+(F^n, F^n) \tag{5.9}$$

with

$$F^{n+1}(t,x,v)\Big|_{t=0} = F_0(x,v) \geq 0, \quad \text{and} \quad F^0(t,x,v) = \mu(v). \tag{5.10}$$

Denote

$$f^{n+1} = \frac{F^{n+1} - \mu}{\sqrt{\mu}}.$$

Then (5.9) can be written equivalently as

$$\begin{aligned} &f_t^{n+1} + v \cdot \nabla_x f^{n+1} + f^{n+1} \cdot \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u,\theta) \Big\{ \mu(u) + \sqrt{\mu} f^n(t,x,u) \Big\} \mathrm{d}\omega \mathrm{d}u \\ &= K f^n + \frac{1}{\sqrt{\mu}} Q_+(\sqrt{\mu} f^n, \sqrt{\mu} f^n) \end{aligned} \tag{5.11}$$

with $n = 0, 1, 2, \dots$ and

$$f^{n+1}(t,x,v)\Big|_{t=0} = f_0(x,v) \quad \text{and} \quad f^0(t,x,v) = 0. \tag{5.12}$$

It is a normal procedure to solve the approximated problems (5.9)–(5.10) (or equivalently (5.11)–(5.12)) since they are linear at each step and the angular cutoff assumption is posed. Then we get an approximation sequence F^{n+1}, $n = 0, 1, 2, \dots$.

Firstly, we consider the positivity of F^{n+1}. It is noted that

$$\begin{aligned} F^{n+1}(t,x,v) = &\, \mathrm{e}^{-\int_0^t g^n(\tau, x - v(t-\tau), v) \mathrm{d}\tau} \cdot F_0(x - vt, v) \\ &+ \int_0^t \mathrm{e}^{-\int_s^t g^n(\tau, x - v(t-\tau), v) \mathrm{d}\tau} \cdot Q_+(F^n, F^n)(s, x - v(t-s), v) \mathrm{d}s, \end{aligned} \tag{5.13}$$

with

$$g^n(\tau, y, v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)F^n(\tau, y, u)\mathrm{d}u\mathrm{d}\omega$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\Big\{\mu(u) + \sqrt{\mu} f^n(\tau, y, u)\Big\}\mathrm{d}\omega\mathrm{d}u.$$

By induction on n, we can prove that if $F^n \geq 0$, then it holds that

$$g^n(s, x - v(t-s), v) \geq 0 \ \text{ and } \ Q_+(F^n, F^n)(s, x - v(t-s), v) \geq 0,$$

which together with (5.13) yield that

$$F^{n+1}(t, x, v) \geq \mathrm{e}^{-\int_0^t g^n(\tau, x - v(t-\tau), v)\mathrm{d}\tau} \cdot F_0(x - vt, v) \geq 0.$$

Therefore, we have proved the positivity of the approximation sequences, i.e., $F^{n+1} \geq 0,\ n = 0, 1, \ldots$.

Next, we consider the uniform estimate for the approximation sequence. And it is more convenient to use the equivalent form f^{n+1}. Then, it follows from (5.11) that

$$\begin{aligned} &f^{n+1}(t, x, v)\\ &\quad = \mathrm{e}^{-\int_0^t g^n(\tau, x - v(t-\tau), v)\mathrm{d}\tau} \cdot f_0(x - vt, v)\\ &\qquad + \int_0^t \mathrm{e}^{-\int_s^t g^n(\tau, x - v(t-\tau), v)\mathrm{d}\tau} \cdot (Kf^n)(s, x - v(t-s), v)\mathrm{d}s\\ &\qquad + \int_0^t \mathrm{e}^{-\int_s^t g^n(\tau, x - v(t-\tau), v)\mathrm{d}\tau}\\ &\qquad \times \frac{1}{\sqrt{\mu(v)}} Q_+(\sqrt{\mu} f^n, \sqrt{\mu} f^n)(s, x - v(t-s), v)\mathrm{d}s, \end{aligned} \tag{5.14}$$

which yields that

$$\begin{aligned} |w_\beta(v) f^{n+1}(t, x, v)| \leq \|w_\beta(v) f_0\|_{L^\infty} &+ \int_0^t \Big|w_\beta(v)(Kf^n)(s, x - v(t-s), v)\Big|\mathrm{d}s\\ &+ \int_0^t \frac{w_\beta(v)}{\sqrt{\mu(v)}}\Big|Q_+(\sqrt{\mu} f^n, \sqrt{\mu} f^n)(s, x - v(t-s), v)\Big|\mathrm{d}s. \end{aligned} \tag{5.15}$$

It follows from (2.4) that

$$\begin{aligned} &\int_0^t \Big|w_\beta(v)(Kf^n)(s, x - v(t-s), v)\Big|\mathrm{d}s\\ &\quad \leq \int_0^t \|w_\beta f^n(s)\|_{L^\infty}\mathrm{d}s \int_{\mathbb{R}^3} w_\beta(v)|k(v, \eta)|w_\beta(\eta)^{-1}\mathrm{d}\eta\\ &\quad \leq C_\gamma \int_0^t \|w_\beta f^n(s)\|_{L^\infty}\mathrm{d}s. \end{aligned} \tag{5.16}$$

To estimate the last term on the RHS of (5.15), by similar arguments as in (3.9), one gets that

$$
\begin{aligned}
&\frac{w_\beta(v)}{\sqrt{\mu(v)}}\Big|Q_+(\sqrt{\mu}f^n,\sqrt{\mu}f^n)(s,x-v(t-s),v)\Big| \\
&\quad\le C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)}\Big|w_\beta(u')f^n(s,x-v(t-s),u') \\
&\quad\quad f^n(s,x-v(t-s),v')\Big|\mathrm{d}u\mathrm{d}\omega \\
&\quad\quad+C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)}\Big|f^n(s,x-v(t-s),u')w_\beta(v') \\
&\quad\quad f^n(s,x-v(t-s),v')\Big|\mathrm{d}u\mathrm{d}\omega, \\
&\quad:= I_1+I_2.
\end{aligned}
\tag{5.17}
$$

It follows from the change of variables (3.10)–(3.12) that for $\beta>3$,

$$
\begin{aligned}
I_1&\le C\|w_\beta f^n(s)\|_{L^\infty} \\
&\quad\times\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\frac{|z_{\shortparallel}|}{(|z_{\shortparallel}|+|z_\perp|)^{1-\gamma}}\mathrm{e}^{-\frac{|v+z|^2}{4}}\Big|f^n(s,x-v(t-s),v+z_{\shortparallel})\Big|\mathrm{d}z_\perp d|z_{\shortparallel}|\mathrm{d}\omega \\
&\le C\|w_\beta f^n(s)\|_{L^\infty}\int_{\mathbb{R}^3}\int_{z_\perp}\frac{|z_\perp|^{\frac{\gamma-1}{2}}}{|\eta-v|^{\frac{3-\gamma}{2}}}\mathrm{e}^{-\frac{|\eta+z_\perp|^2}{4}}\Big|f^n(s,x-v(t-s),\eta)\Big|\mathrm{d}z_\perp\mathrm{d}\eta \\
&\le C_\gamma\|w_\beta f^n(s)\|_{L^\infty}\int_{\mathbb{R}^3}\frac{1}{|\eta-v|^{\frac{3-\gamma}{2}}}\Big|f^n(s,x-v(t-s),\eta)\Big|\mathrm{d}\eta \\
&\le C_\gamma\|w_\beta f^n(s)\|_{L^\infty}^2\int_{\mathbb{R}^3}\frac{(1+|\eta|)^{-\beta}}{|\eta-v|^{\frac{3-\gamma}{2}}}\mathrm{d}\eta\le C(\gamma,\beta)\|w_\beta f^n(s)\|_{L^\infty}^2.
\end{aligned}
\tag{5.18}
$$

It is noted that by a rotation, one obtains the interchange of v' and u'. Then, one can change I_2 to a similar form as I_1. Thus, by similar arguments as above, one can obtain that for $\beta>3$,

$$
I_2\le C(\gamma,\beta)\|w_\beta f^n(s)\|_{L^\infty}^2,
$$

which together with (5.18) and (5.17) yields that for $\beta>3$,

$$
\begin{aligned}
&\int_0^t\frac{w_\beta(v)}{\sqrt{\mu(v)}}\Big|Q_+(\sqrt{\mu}f^n,\sqrt{\mu}f^n)(s,x-v(t-s),v)\Big|\mathrm{d}s \\
&\quad\le C(\gamma,\beta)\int_0^t\|w_\beta f^n(s)\|_{L^\infty}^2\mathrm{d}s.
\end{aligned}
\tag{5.19}
$$

Substituting (5.19) and (5.16) into (5.15), one obtains that for $\beta>3$,

$$
\begin{aligned}
&\|w_\beta(v)f^{n+1}(t)\|_{L^\infty}\le\|w_\beta f_0\|_{L^\infty} \\
&\quad+C_4t\Big\{\sup_{0\le s\le t}\|w_\beta f^n(s)\|_{L^\infty}+\sup_{0\le s\le t}\|w_\beta f^n(s)\|_{L^\infty}^2\Big\},
\end{aligned}
\tag{5.20}
$$

where the positive constant $C_4 \geq 1$ depends only on γ, β. By induction on n, we can prove that if

$$\sup_{0\leq s\leq t_1} \|w_\beta f^n(s)\|_{L^\infty} \leq 2\|w_\beta f_0\|_{L^\infty}, \quad t_1 = \Big(8C_4[1+\|w_\beta f_0\|_{L^\infty}]\Big)^{-1}, \tag{5.21}$$

then it follows from (5.20) and (5.21) that for $\beta > 3$,

$$\sup_{0\leq s\leq t_1} \|w_\beta f^{n+1}(s)\|_{L^\infty} \leq 2\|w_\beta f_0\|_{L^\infty} \text{ with } t_1 = \Big(8C_4[1+\|w_\beta f_0\|_{L^\infty}]\Big)^{-1}, \tag{5.22}$$

for all $n \geq 0$.

Now we prove that f^{n+1}, $n = 0, 1, 2, \ldots$ is a Cauchy sequence. It follows from (5.14) that

$$\begin{aligned}
&\Big|\sqrt{w_\beta(v)}(f^{n+2}-f^{n+1})(t,x,v)]\Big| \\
&\leq |\sqrt{w_\beta(v)}f_0(x-vt,v)| \cdot \int_0^t \Big|(g^{n+1}-g^n)(\tau, x-v(t-\tau), v)\Big| \mathrm{d}\tau \\
&\quad + \int_0^t |\sqrt{w_\beta(v)}Kf^{n+1}(s, x-v(t-s), v)| \\
&\quad \times \int_s^t \Big|(g^{n+1}-g^n)(\tau, x-v(t-\tau), v)\Big| \mathrm{d}\tau\mathrm{d}s \\
&\quad + \int_0^t \frac{\sqrt{w_\beta(v)}}{\sqrt{\mu(v)}} |Q_+(\sqrt{\mu}f^{n+1}, \sqrt{\mu}f^{n+1})(s, x-v(t-s), v)| \\
&\quad \times \int_s^t \Big|(g^{n+1}-g^n)(\tau, x-v(t-\tau), v)\Big| \mathrm{d}\tau\mathrm{d}s \\
&\quad + \int_0^t |\sqrt{w_\beta(v)}(Kf^{n+1}-Kf^n)(s, x-v(t-s), v)| \mathrm{d}s \\
&\quad + \int_0^t \frac{\sqrt{w_\beta(v)}}{\sqrt{\mu(v)}} \Big|\Big(Q_+(\sqrt{\mu}f^{n+1}, \sqrt{\mu}f^{n+1}) \\
&\quad - Q_+(\sqrt{\mu}f^n, \sqrt{\mu}f^n)\Big)(s, x-v(t-s), v)\Big| \mathrm{d}s \\
&:= I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned} \tag{5.23}$$

A direct calculation shows that

$$\Big|(g^{n+1}-g^n)(\tau, y, u)\Big| \leq C\nu(v)\|(f^{n+1}-f^n)(\tau)\|_{L^\infty},$$

which, together with (5.16) and (5.19), yields that, for $0 \leq t \leq t_1$,

$$\begin{aligned}
I_3 + I_4 + I_5 \leq C\Big\{&\|\nu\sqrt{w_\beta}f_0\|_{L^\infty} \\
&+ \int_0^t \Big|\sqrt{w_\beta(v)}\nu(v)Kf^{n+1}(s, x-v(t-s), v)\Big| \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \frac{\sqrt{w_\beta(v)}\nu(v)}{\sqrt{\mu(v)}} \Big| Q_+(\sqrt{\mu} f^{n+1}, \sqrt{\mu} f^{n+1})(s, x - v(t-s), v) \Big| \mathrm{d}s \Big\} \\
&\times \int_0^t \|(f^{n+1} - f^n)(\tau)\|_{L^\infty} \mathrm{d}\tau \\
&\leq Ct \Big\{ \|w_\beta f_0\|_{L^\infty} + t \sup_{0 \leq s \leq t} \|w_\beta f^{n+1}(s)\|_{L^\infty} + t \sup_{0 \leq s \leq t} \|w_\beta f^{n+1}(s)\|_{L^\infty}^2 \Big\} \\
&\times \sup_{0 \leq \tau \leq t} \|(f^{n+1} - f^n)(\tau)\|_{L^\infty} \\
&\leq Ct \|w_\beta f_0\|_{L^\infty} \cdot \sup_{0 \leq \tau \leq t} \|(f^{n+1} - f^n)(\tau)\|_{L^\infty},
\end{aligned}
\tag{5.24}
$$

with $\beta > 3$, where we have used the uniform estimate (5.22) in the last inequality. By the same arguments as (5.16), one can obtains that

$$
I_6 \leq Ct \sup_{0 \leq \tau \leq t} \|\sqrt{w_\beta}(f^{n+1} - f^n)(\tau)\|_{L^\infty}. \tag{5.25}
$$

To prove I_7, we note that

$$
\begin{aligned}
&\Big| \Big(Q_+(\sqrt{\mu} f^{n+1}, \sqrt{\mu} f^{n+1}) - Q_+(\sqrt{\mu} f^n, \sqrt{\mu} f^n) \Big)(s, x - v(t-s), v) \Big| \\
&\leq \Big| Q_+(\sqrt{\mu} f^{n+1}, \sqrt{\mu}(f^{n+1} - f^n))(s, x - v(t-s), v) \Big| \\
&\quad + \Big| Q_+(\sqrt{\mu}(f^{n+1} - f^n), \sqrt{\mu} f^n)(s, x - v(t-s), v) \Big|.
\end{aligned}
\tag{5.26}
$$

Denoting $y = x - v(t-s)$, by similar arguments as in (5.17), we have that

$$
\begin{aligned}
&\frac{\sqrt{w_\beta(v)}}{\sqrt{\mu(v)}} \Big| Q_+(\sqrt{\mu} f^{n+1}, \sqrt{\mu}(f^{n+1} - f^n))(s, x - v(t-s), v) \Big| \\
&\quad \leq C \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \theta) \sqrt{\mu(u)} |f^{n+1}(s, y, u')| \\
&\qquad \times \Big| \sqrt{w_\beta(v')}(f^{n+1} - f^n)(s, y, v') \Big| \mathrm{d}u \mathrm{d}\omega \\
&\qquad + C \frac{\nu(v)}{\sqrt{w_\beta(v)}} \|w_\beta f^{n+1}(s)\|_{L^\infty} \cdot \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty}.
\end{aligned}
\tag{5.27}
$$

To estimate the first term on the RHS of (5.27), by a rotation, we interchange v' and u', then using the same arguments as in (3.10)–(5.18), one can obtain that

$$
\begin{aligned}
&C \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \theta) \sqrt{\mu(u)} |f^{n+1}(s, y, u')| \\
&\quad \times \Big| \sqrt{w_\beta(v')}(f^{n+1} - f^n)(s, y, v') \Big| \mathrm{d}u \mathrm{d}\omega \\
&\quad \leq C \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty}
\end{aligned}
$$

$$\times \int_{\mathbb{R}^3}\int_{z_\perp} \frac{|z_\perp|^{\frac{\gamma-1}{2}}}{|\eta - v|^{\frac{3-\gamma}{2}}} \mathrm{e}^{-\frac{|\eta+z_\perp|^2}{4}} \Big| f^{n+1}(s, y, \eta)\Big| \mathrm{d}z_\perp \mathrm{d}\eta$$
$$\leq C\|w_\beta f^{n+1}(s)\|_{L^\infty} \cdot \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty},$$

with $\beta > 3$, which together with (5.27) yield that

$$\frac{\sqrt{w_\beta(v)}}{\sqrt{\mu(v)}} \Big| Q_+(\sqrt{\mu} f^{n+1}, \sqrt{\mu}(f^{n+1} - f^n))(s, x - v(t-s), v)\Big|$$
$$\leq C\|w_\beta f^{n+1}(s)\|_{L^\infty} \cdot \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty}, \tag{5.28}$$

for $\beta > 3$. Similarly, one can gets that for $\beta > 3$,

$$\frac{\sqrt{w_\beta(v)}}{\sqrt{\mu(v)}} \Big| Q_+(\sqrt{\mu}(f^{n+1} - f^n), \sqrt{\mu} f^n)(s, x - v(t-s), v)\Big|$$
$$\leq C\|w_\beta f^n(s)\|_{L^\infty} \cdot \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty}. \tag{5.29}$$

Then it follows from (5.26), (5.28) and (5.29) that for $0 \leq t \leq t_1$,

$$I_7 \leq Ct \sup_{0\leq s\leq t} \Big\{ \|w_\beta f^{n+1}(s)\|_{L^\infty} + \|w_\beta f^n(s)\|_{L^\infty} \Big\}$$
$$\times \sup_{0\leq s\leq t} \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty}$$
$$\leq Ct\|w_\beta f_0\|_{L^\infty} \sup_{0\leq s\leq t} \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty}, \tag{5.30}$$

with $\beta > 3$, where we have used the uniform estimate (5.22) in the last inequality.

Substituting (5.24), (5.25) and (5.30) into (5.23), one obtains that for $0 \leq t \leq t_1$,

$$\sup_{0\leq s\leq T_1} \|\sqrt{w_\beta}(f^{n+2} - f^{m+1})(s)\|$$
$$\leq Ct_1(1 + \|w_\beta f_0\|_{L^\infty}) \cdot \sup_{0\leq s\leq T_1} \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty}$$
$$\leq \frac{C}{8C_4} \sup_{0\leq s\leq t_1} \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty}$$
$$\leq \frac{1}{2} \sup_{0\leq s\leq t_1} \|\sqrt{w_\beta}(f^{n+1} - f^n)(s)\|_{L^\infty},$$

where we have chosen C_4 suitably large such that $\frac{C}{8C_4} \leq \frac{1}{2}$. Thus, by induction on n, it is direct to obtain that

$$\sup_{0\leq s\leq t_1} \|\sqrt{w_\beta}(f^{n+2} - f^{n+1})(s)\| \leq 2^{-n-1}\|\sqrt{w_\beta}(f^1 - f^0)\|_{L^\infty} \leq 2^{-n}\|w_\beta f_0\|_{L^\infty},$$

which yields immediately that f^{n+1}, $n = 0, 1, 2, \ldots$ is a Cauchy sequence. Therefore, there exists a limit f such that

$$\sup_{0\leq s\leq t_1} \|\sqrt{w_\beta}(f^n - f)(s)\|_{L^\infty} \to 0 \text{ as } n \to +\infty.$$

The limit function f is indeed a mild solution to the Boltzmann equation (1.1) and (1.4). It follows from (5.22) that

$$\sup_{0\le t\le t_1} \|w_\beta f(t)\|_{L^\infty} \le 2\|w_\beta f_0\|_{L^\infty}.$$

Now we consider the uniqueness. Let $\tilde{f}(t,x,v)$ be another solution of the Boltzmann equation (1.1) and (1.4) with the bound $\sup_{0\le t\le t_1} \|w_\beta \tilde{f}(t)\|_{L^\infty} < +\infty$, by similar arguments as (5.23)–(5.30), it is directly obtained that

$$\begin{aligned}&\|\sqrt{w_\beta}(f-\tilde{f})(t)\|_{L^\infty}\\ &\quad\le C(1+\|w_\beta f\|_{L^\infty}+\|w_\beta \tilde{f}\|_{L^\infty})\cdot\int_0^t \|\sqrt{w_\beta}(f-\tilde{f})(s)\|_{L^\infty}\mathrm{d}s,\end{aligned}$$

which together with the Gronwall inequality yields the uniqueness, i.e., $f=\tilde{f}$.

Multiplying (5.9) by 1, v, $|v|^2$ and F^{m+1}, integrating by parts and then taking the limit $m\to+\infty$, one can directly obtain (1.5)–(1.8).

Finally, if F_0 (or equivalent f_0) is continuous, it is direct to check that $F^{n+1}(t,x,v)$ (or equivalent $f^{n+1}(t,x,v)$) is continuous in $[0,\infty)\times\Omega\times\mathbb{R}^3$. The continuous of $f(t,x,v)$ is an immediate consequence of $\sup_{0\le s\le t_1}\|(f^{n+1}-f)(s)\|_{L^\infty}\to 0$ as $n\to+\infty$. Therefore the proof of Proposition 2.1 is completed. □

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