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*Gevrey Smoothing for Weak Solutions of the Fully Nonlinear Homogeneous Boltzmann and Kac Equations Without Cutoff for Maxwellian Molecules**

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Abstract

It has long been suspected that the non-cutoff Boltzmann operator has similar coercivity properties to the fractional Laplacian. This has led to the hope that the homogenous Boltzmann equation enjoys similar regularity properties to the heat equation with a fractional Laplacian. In particular, the weak solution of the fully nonlinear non-cutoff homogenous Boltzmann equation with initial datum in $L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$, i.e., finite mass, energy and entropy, should immediately become Gevrey regular for strictly positive times. We prove this conjecture for Maxwellian molecules.

Contents

1. Introduction

It has long been suspected that the non-cutoff Boltzmann operator with a singular cross section kernel has similar coercivity properties to the fractional Laplacian $(-\Delta)^\nu$, for suitable $0 < \nu < 1$. This has been made precise by ALEXANDRE, DESVILLETTES, VILLANI, and WENNBERG [3], see also the reviews by ALEXANDRE [2] and by VILLANI [49] for the idea's history. The suspicion has led to the hope that the fully nonlinear homogenous Boltzmann equation enjoys regularity properties similer to the heat equation with a fractional Laplacian given by

$$\begin{cases} \partial_t u + (-\Delta)^\nu u = 0 \\ \qquad\qquad u|_{t=0} = u_0 \in L^1(\mathbb{R}^d). \end{cases}$$

Using the Fourier transform one immediately sees that

$$\widehat{u}(t,\xi) = \mathrm{e}^{-t(2\pi|\xi|)^{2\nu}}\widehat{u_0}(\xi) \quad \text{with} \quad \widehat{u_0} \in L^\infty(\mathbb{R}^d),$$

so

$$\sup_{t>0}\sup_{\xi\in\mathbb{R}^d} \mathrm{e}^{t|\xi|^{2\nu}}|\widehat{u}(t,\xi)| \leqq \|u_0\|_{L^1(\mathbb{R}^d)} < \infty,$$

that is, the Fourier transform of the solution is extremely fast decaying for strictly positive times.

Introducing the Gevrey spaces as in Definition 1.5, it is natural to expect (see, for example, DESVILLETTES and WENNBERG [23]):

Conjecture. (Gevrey smoothing) *Any weak solution of the non-cutoff homogenous Boltzmann equation with a singular cross section kernel of order ν and with initial datum in $L^1_2(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$, i.e., finite mass, energy and entropy, belongs to the Gevrey class $G^{\frac{1}{2\nu}}(\mathbb{R}^d)$ for strictly positive times.*

The central result of our work is a proof of this conjecture for Maxwellian molecules. In particular, we prove

Theorem. *Assume that the non-cutoff Boltzman cross section has a singularity $1 + 2\nu$ with $0 < \nu < 1$ and obeys some further technical conditions, which are true in all physically relevant cases, for details see (3) and (16). Then, for initial conditions $f_0 \in L\log L \cap L^1_m$ with an integer*

$$m \geq \max\left(2, \frac{2^\nu - 1}{2(2-2^\nu)}\right),$$

any weak solution of the fully non-linear homogenous Boltzmann equation for Maxwellian molecules belongs to the Gevrey class $G^{\frac{1}{2\nu}}$ for strictly positive times.

In particular, for $\nu \leq \log(9/5)/\log(2) \simeq 0,847996$, we have $m = 2$ and the theorem does not require anything except the physically reasonable assumptions of finite mass, energy and entropy. If $\log(9/5)/\log(2) < \nu < 1$ and we assume only that $f_0 \in L\log L \cap L^1_2$, then we prove that the solution is in $G^{\frac{\log 2}{2\log(9/5)}}$, in particular, that it is ultra-analytic.

1. For a more precise formulation of our results, see Theorems 1.6, 1.9, and 1.10 for the case $m = 2$ and Theorems 3.1, 3.2, and 3.3 below.
2. We would like to stress that our results cover both the weak and strong singularity regimes, where $0 < \nu < 1/2$, respectively $1/2 \leq \nu < 1$.
3. The theorem above applies to all dimensions $d \geqq 1$. The physical case for Maxwellian molecules in dimension $d = 3$ is $\nu = 1/4$.

The main problem for establishing Gevrey regularity is that, in order to use the coercivity results of Alexandre, Desvillettes, Villani and Wennberg [3], one has to bound a non-linear and non-local commutator of the Boltzmann kernel with certain sub-Gaussian Fourier multipliers. The main ingredient in our proof is a new way of estimating this non-local and nonlinear commutator.

1.1. The Non-cutoff Boltzmann and Kac Models

We study the regularity of weak solutions of the Cauchy problem

$$\begin{cases} \partial_t f = Q(f, f) \\ f|_{t=0} = f_0 \end{cases} \tag{1}$$

for the fully nonlinear homogeneous Boltzmann and Kac equation in $d \geqq 1$ dimensions [14,28].

For $d \geqq 2$ the bilinear operator Q is given by

$$Q(g, f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b(\cos\theta) \big(g(v'_*) f(v') - g(v_*) f(v)\big) \, \mathrm{d}\sigma \mathrm{d}v_*, \tag{2}$$

that is, the Boltzmann collision operator for Maxwellian molecules with angular collision kernel b depending only on the deviation angle $\cos\theta = \sigma \cdot \frac{v-v_*}{|v-v_*|}$ for $\sigma \in \mathbb{S}^{d-1}$. Here we use the σ-representation of the collision process, in which

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \quad \text{for } \sigma \in \mathbb{S}^{d-1}.$$

By symmetry properties of the Boltzmann collision operator $Q(f, f)$, the function b can be assumed to be supported on angles $\theta \in [0, \frac{\pi}{2}]$; for otherwise (see [49]) it can be replaced by

$$\widetilde{b}(\cos\theta) = (b(\cos\theta) + b(\cos(\pi - \theta)))\, \mathbb{1}_{\{0 \leqq \theta \leqq \frac{\pi}{2}\}}.$$

We will assume that the angular collision kernel b has the non-integrable singularity

$$\sin^{d-2}\theta\, b(\cos\theta) \sim \frac{\kappa}{\theta^{1+2\nu}}, \quad \text{as } \theta \to 0^+ \tag{3}$$

for some $\kappa > 0$ and $0 < \nu < 1$, and satisfies

$$\int_0^{\pi/2} \sin^d\theta\, b(\cos\theta)\, \mathrm{d}\theta < \infty. \tag{4}$$

For inverse s-power forces (in three spatial dimensions), described by the potential $U(r) = r^{1-s}$, $s > 2$, the collision kernel is of the more general form

$$B(|v - v_*|, \cos\theta) = b(\cos\theta)|v - v_*|^\gamma, \quad \gamma = \frac{s-5}{s-1},$$

where the angular collision kernel b is locally smooth with a non-integrable singularity

$$\sin\theta\, b(\cos\theta) \sim K\theta^{-1-2\nu}, \quad \nu = \frac{1}{s-1}.$$

The case of *(physical) Maxwellian molecules* corresponds to the values $\gamma = 0$, $s = 5$, $\nu = \frac{1}{4}$.

For $d = 1$ we set

$$Q(g, f) \ = K(g, f) = \int_{\mathbb{R}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b_1(\theta) \left(f(w'_*)g(w') - f(w_*)g(w) \right) \mathrm{d}\theta \mathrm{d}w_*, \tag{5}$$

which is the Kac operator for Maxwellian molecules, and angular collision kernel $b_1 \geqq 0$. The pre- and post-collisional velocities are related by

$$\begin{pmatrix} w' \\ w'_* \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} w \\ w_* \end{pmatrix}, \quad \text{for } \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

In the original Kac model b_1 was chosen to be constant, whereas we will assume, as in [20], that b_1 is an even function and has the non-integrable singularity

$$b_1(\theta) \sim \frac{\kappa}{|\theta|^{1+2\nu}}, \quad \text{for } \theta \to 0, \tag{6}$$

with $0 < \nu < 1$ and some $\kappa > 0$, and further satisfies

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b_1(\theta) \sin^2\theta \, \mathrm{d}\theta < \infty. \tag{7}$$

Making use of symmetry properties of the collision operator $K(f, f)$, we can assume b_1 to be supported on angles $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$; for otherwise it can be replaced by its symmetrised version

$$\widetilde{b_1}(\theta) = \left(b_1(\theta) + b_1(\tfrac{\pi}{2} - \theta)\right) \mathbb{1}_{\{0 \leqq \theta \leqq \frac{\pi}{4}\}} + \left(b_1(\theta) + b_1(-\tfrac{\pi}{2} - \theta)\right) \mathbb{1}_{\{-\frac{\pi}{4} \leqq \theta \leqq 0\}}.$$

This simple observation will be very convenient for our analysis.

We will mainly work with the weighted L^p spaces, defined as

$$L^p_\alpha(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) : \langle\cdot\rangle^\alpha f \in L^p(\mathbb{R}^d) \right\},$$

$p \geqq 1$, $\alpha \in \mathbb{R}$, with norm

$$\|f\|_{L^p_\alpha(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(v)|^p \langle v\rangle^{\alpha p} \, \mathrm{d}v \right)^{1/p}, \quad \langle v\rangle := (1 + |v|^2)^{1/2}.$$

We will also use the weighted (L^2 based) Sobolev spaces

$$H^k_\ell(\mathbb{R}^d) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \langle\cdot\rangle^\ell f \in H^k(\mathbb{R}^d) \right\}, \quad k, \ell \in \mathbb{R},$$

where $H^k(\mathbb{R}^d)$ are the usual Sobolev spaces given by

$$H^k(\mathbb{R}^d) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \langle\cdot\rangle^k \hat{f} \in L^2(\mathbb{R}^d) \right\}, \quad k \in \mathbb{R}.$$

The inner product on $L^2(\mathbb{R}^d)$ is given by $\langle f, g\rangle = \int_{\mathbb{R}^d} \overline{f(v)} g(v)\, \mathrm{d}v$.

It will be assumed that the initial datum $f_0 \not\equiv 0$ is a non-negative density with finite mass, energy and entropy, which is equivalent to

$$f_0 \geqq 0, \quad f_0 \in L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d), \tag{8}$$

where

$$L \log L(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ measurable} : \|f\|_{L \log L} < \infty \right\},$$

where

$$\|f\|_{L\log L} = \int_{\mathbb{R}^d} |f(v)| \log(1 + |f(v)|)\, \mathrm{d}v,$$

and the negative of the entropy is given by $H(f) := \int_{\mathbb{R}^d} f \log f\, \mathrm{d}v$.

The space $L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$ is very natural, since we have

Lemma 1.1. *Let* $f \geqq 0$. *Then*

$$f \in L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d) \quad \Leftrightarrow \quad f \in L^1_2(\mathbb{R}^d) \text{ and } H(f) \text{ is finite.}$$

This result is well-known to experts. For the reader's convenience we will give the proof in Appendix D. The following is the precise definition of weak solutions which we use:

Definition 1.2. (Weak Solutions of the Cauchy Problem (1) [11,17,48]) Assume that the initial datum f_0 is in $L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$. $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is called a weak solution to the Cauchy problem (1), if it satisfies the following conditions:

(i) $f \geqq 0, \ f \in \mathscr{C}(\mathbb{R}_+; \mathscr{D}'(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}_+; L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d))$;

(ii) $f(0, \cdot) = f_0$;

(iii) For all $t \geqq 0$, mass is conserved, $\int_{\mathbb{R}^d} f(t, v)\, \mathrm{d}v = \int_{\mathbb{R}^d} f_0(v)\, \mathrm{d}v$, kinetic energy is decreasing, $\int_{\mathbb{R}^d} f(t, v)\, v^2\, \mathrm{d}v \leqq \int_{\mathbb{R}^d} f_0(v)\, v^2\, \mathrm{d}v$, and the entropy is increasing, that is, $H(f(t, \cdot)) \leqq H(f_0)$;

(iv) For all $\varphi \in \mathscr{C}^1(\mathbb{R}_+; \mathscr{C}^\infty_0(\mathbb{R}^d))$ one has

$$\begin{aligned} \langle f(t, \cdot), \varphi(t, v)\rangle - \langle f_0, \varphi(0, \cdot)\rangle - \int_0^t \langle f(\tau, \cdot), \partial_\tau \varphi(\tau, \cdot)\rangle\, \mathrm{d}\tau \\ = \int_0^t \langle Q(f, f)(\tau, \cdot), \varphi(\tau, \cdot)\rangle\, \mathrm{d}\tau, \quad \text{for all } t \geqq 0, \end{aligned} \tag{9}$$

where the latter expression involving Q is defined by

$$\langle Q(f,f),\varphi\rangle = \frac{1}{2}\int_{\mathbb{R}^{2d}}\int_{\mathbb{S}^{d-1}} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) f(v_*)f(v) \times \left(\varphi(v')+\varphi(v'_*)-\varphi(v)-\varphi(v_*)\right)\,\mathrm{d}\sigma\mathrm{d}v\mathrm{d}v_*,$$

for test functions $\varphi \in W^{2,\infty}(\mathbb{R}^d)$ in dimension $d \geqq 2$, and in one dimension

$$\langle Q(f,f),\varphi\rangle = \langle K(f,f),\varphi\rangle = \int_{\mathbb{R}^2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} b_1(\theta)\, g(w_*)g(w)\left(\varphi(w')-\varphi(w)\right)\,\mathrm{d}\theta\mathrm{d}w\mathrm{d}w_*$$

for test functions $\varphi \in W^{2,\infty}(\mathbb{R})$, making use of symmetry properties of the Boltzmann and Kac collision operators and cancellation effects.[1]

Collecting results from the literature, the following is known regarding the existence, uniqueness and further properties of weak solutions:

Theorem 1.3. (Arkeryd, Desvillettes, Mischler, Goudon, Villani, Wennberg) *There exists a weak solution of the Cauchy problem* (1) *in the sense of Definition* 1.2*. For* $d \geqq 2$ *momentum and energy are conserved,*

$$\int_{\mathbb{R}^d} f(t,v)\,v\,\mathrm{d}v = \int_{\mathbb{R}^d} f_0(v)\,v\,\mathrm{d}v, \quad \int_{\mathbb{R}^d} f(t,v)\,v^2\,\mathrm{d}v = \int_{\mathbb{R}^d} f_0(v)\,v^2\,\mathrm{d}v. \quad (10)$$

In the one dimensional case (Kac equation), momentum is not conserved and energy can only decrease and is conserved under the additional moment assumption $f_0 \in L^1_{2p}$ *for some* $p \geqq 2$.

Remark 1.4. $d \geqq 2$: The existence of weak solutions of the Cauchy problem (1) with initial conditions satisfying (8) for the homogeneous Boltzmann equation was first proved by ARKERYD [10,11] (see also the articles by GOUDON [27], VILLANI [48], and DESVILLETTES [19,20]). Uniqueness in this case was shown by TOSCANI and VILLANI [45], see also the review articles by MISCHLER and WENNBERG [37] (for the cut-off case) and DESVILLETTES [19].

$d = 1$: For the homogeneous non-cutoff Kac equation for Maxwellian molecules existence of weak solutions was established by DESVILLETTES [17].

1.2. Higher Regularity of Weak Solutions

It has been pointed out by several authors [2,23,49] that, for singular cross-sections, the Boltzmann operator essentially behaves like a singular integral operator with a leading term similar to a fractional Laplace operator $(-\Delta)^\nu$. In terms of compactness properties this has been noticed for the linearised Boltzmann kernel

[1] Throughout the text, whenever not explicitly mentioned, we will drop the dependence on t of a function, i.e. $f(v) := f(t,v)$ etc.

as early as in [42] and for the nonlinear Boltzmann kernel in [35,36]. Since the solutions of the heat equation with a fractional Laplacian gain a high amount of regularity for arbitrary positive times, it is natural to believe, as conjectured in [23], that weak solutions to the non-cutoff Boltzmann equation gain a certain amount of smoothness, and even analyticity, for any $t > 0$. This is in sharp contrast to the fact that in the Grad's cutoff case there cannot be any smoothing effect. Instead, regularity and singularities of the initial datum get propagated in this case, see, for example, [41].

The discussion about solutions of the heat equation with a fractional Laplacian motivates the following definition of Gevrey spaces, which give a convenient framework to describe this smoothing by interpolating between smooth and (ultra-) analytic functions.

Definition 1.5. Let $s > 0$. A function $f \in L^1(\mathbb{R}^d)$ belongs to the Gevrey class $G^s(\mathbb{R}^d)$, if there exists an $\varepsilon_0 > 0$ such that

$$e^{\varepsilon_0 \langle D_v \rangle^{1/s}} f \in L^2(\mathbb{R}^d), \quad \text{where} \quad \langle D_v \rangle = \left(1 + |D_v|^2\right)^{1/2},$$

and we use the notation $D_v = -\frac{\mathrm{i}}{2\pi}\nabla_v$. Thus, $G^1(\mathbb{R}^d)$ is the space of real analytic functions, and $G^s(\mathbb{R}^d)$ for $s \in (0, 1)$ the space of ultra-analytic functions.

Equivalently, $f \in G^s(\mathbb{R}^d)$ if $f \in \mathscr{C}^\infty(\mathbb{R}^d)$ and there exists a constant $C > 0$ such that for all $k \in \mathbb{N}_0$ one has

$$\|D^k f\|_{L^2(\mathbb{R}^d)} \leqq C^{k+1}(k!)^s,$$

where $\|D^k f\|_{L^2}^2 = \sup_{|\beta|=k} \|\partial^\beta f\|_{L^2}^2$.[2]

The first regularisation results in this direction were due to Desvillettes for the spatially homogeneous non-cutoff Kac equation [17] and the homogeneous non-cutoff Boltzmann equation for Maxwellian molecules in two dimensions [18], where $\mathscr{C}^\infty$ regularisation is proved. Later, Desvillettes and Wennberg [23] proved under rather general assumptions on the collision cross-section (excluding Maxwellian molecules, though) regularity in Schwartz space of weak solutions to the non-cutoff homogeneous Boltzmann equation. By quite different methods, using Littlewood–Paley decompositions, Alexandre and El Safadi [4] showed that the assumptions on the cross-section (3)–(4) imply that the solutions are in H^∞ for any positive time $t > 0$. By moment propagation results for Maxwellian molecules (see Truesdell [46]) this cannot be improved to regularity in Schwartz space.

For collision cross-sections corresponding to Debye–Yukawa-type interaction potentials,

$$\sin\theta\, b(\cos\theta) \sim K\theta^{-1}(\log\theta^{-1})^{\ell} \quad \text{for } \theta \to 0 \quad (\text{with some } K > 0,\ \ell > 0),$$

2 Regarding equivalency, see, for example, Theorem 4 in [33].

MORIMOTO, UKAI, XU and YANG [39] proved the same H^∞ regularising effect using suitable test functions in the weak formulation of the problem.

The question of the local existence of solutions in Gevrey spaces for Gevrey regular initial data with additional strong decay at infinity was first addressed in 1984 by UKAI [47], both in the spatially homogeneous and inhomogeneous setting.

We are interested in the Gevrey *smoothing effect*, namely that under the (physical) assumptions of finite mass, energy and entropy of the initial data, weak solutions of the homogeneous Boltzmann equation without cutoff are Gevrey functions for any strictly positive time. This question was treated in the case of the *linearised* Boltzmann equation in the homogeneous setting by MORIMOTO ET AL. [39], where they proved that, given $0 < \nu < 1$, weak solutions of the linearized Boltzmann equation belong to the space $G^{\frac{1}{\nu}}(\mathbb{R}^3)$ for any positive time. In [31], radially symmetric perturbations $g = g(|v|)$ around a global Maxwellian $\mu(v) = (2\pi)^{-\frac{3}{2}} \mathrm{e}^{-\frac{|v|^2}{2}}$, that is, for f in (1),

$$f(v) = \mu(v) + \sqrt{\mu(v)}\, g(v), \quad g(v) = g(|v|),$$

were studied by using eigenfunctions of the linearised Boltzmann operator $\mathscr{L}$, where

$$\mathscr{L} g = -\mu^{-\frac{1}{2}} Q(\mu, \mu^{\frac{1}{2}} g) - \mu^{-\frac{1}{2}} Q(\mu^{\frac{1}{2}} g, \mu).$$

In this setting, the authors obtained a Gelfand-Shilov smoothing effect, which includes Gevrey regularity.

For the non-Maxwellian Boltzmann operator, Gevrey regularity was proved under very strong unphysical decay assumptions on the initial datum in [34].

For radially symmetric solutions, the homogeneous non-cutoff Boltzmann equation for Maxwellian molecules is related to the homogeneous non-cutoff Kac equation. The non-cutoff Kac equation was introduced by DESVILLETTES in [17], where first regularity results were established, see also DESVILLETTES' review [20]. For this equation, the best available results so far are due to LEKRINE and XU [30] and GLANGETAS and NAJEME [26]: LEKRINE and XU [30] proved Gevrey regularisation of order $\frac{1}{2\alpha}$ for mild singularities $0 < \nu < \frac{1}{2}$ and all $0 < \alpha < \nu$. Strong singularities $\frac{1}{2} \leqq \nu < 1$ were treated by GLANGETAS and NAJEME [26], where they prove that for $\nu = \frac{1}{2}$ the solution becomes Gevrey regular of order $\frac{1}{2\alpha}$ for any $0 < \alpha < \frac{1}{2}$ and Gevrey regular of order 1, that is, analytic, when $\frac{1}{2} < \nu < 1$. Thus, in the critical case $\nu = \frac{1}{2}$, the result of [26] misses the analyticity of weak solutions and does not prove ultra-analyticity in the range $1/2 < \nu < 1$. Moreover, both results are obtained under the *additional* moment assumption $f_0 \in L^1_{2+2\nu}(\mathbb{R})$.

Ultra-analyticity results have previously been obtained by MORIMOTO and XU [40] for the homogeneous Landau equation in the Maxwellian molecules case and related simplified models in kinetic theory. The analysis of smoothing properties of the Landau equation is quite different from the Boltzmann and Kac equations. The Landau equation explicitly contains a second order elliptic term, which yields coercivity, and, more importantly, certain commutators with weights in Fourier space are identically zero, which simplifies the analysis tremendously, see Proposition 2.2 in [40].

For the nonlinear non-cutoff homogeneous Boltzmann equation some partial results regarding Gevrey regularisation were obtained by Morimoto and Ukai [38] including the non-Maxwellian molecules case, but under the strong additional assumptions of Maxwellian decay and smoothness of the solution. Still with these strong decay assumptions, Yin and Zhang [50,51] extended this result to a larger class of kinetic cross-sections.

The non-Maxwellian case is considerably harder, since Bobylev's identity has a much more complicated form. Assuming that the Boltzmann collision kernel can be factorised into a relative velocity (kinetic) part and an angular part,

$$B(|v - v_*|, \cos\theta) = \Phi(|v - v_*|)\, b(\cos\theta),$$

the following smoothing results are known (for the precise assumptions in each case we refer to the articles):

1. Using Littlewood–Paley theory, Alexandre and ElSafadi [5] were able to prove H^∞ smoothing of weak solutions for a regularised kinetic factor $\Phi(v) = \langle v\rangle^\gamma$.
2. Alexandre, Morimoto, Ukai, Xu, Yang [9] proved H^∞ smoothing of weak solutions for the physically relevant kinetic factor $\Phi(v) = |v|^\gamma$.
3. Chen and He [15] showed H^∞ smoothing, again in the physically relevant case $\Phi(v) = |v|^\gamma$ for the strong solutions constructed by Desvillettes and Mouhot [22]. They were also able to generalise their result to show H^∞ smoothing in the inhomomogeneous case [16].

In the spatially inhomogeneous case, the collision operator is highly degenerate, since it only acts on the velocity variable. Due to the presence of the transport term $-v \cdot \nabla_x$, one expects a transfer of regularity from the velocity variable to the space variable, and therefore some hypoelliptic smoothing effect in *both* variables. This has been highlighted in terms of a generalised uncertainty principle for kinetic equations by Alexandre, Morimoto, Ukai, Xu and Yang [7] under strong assumptions on the initial data and the solutions. For the one-dimensional inhomogeneous Kac equation, Lerner, Morimoto, Pravda-Starov and Xu obtained Gelfand-Shilov smoothing with respect to the velocity variable and Gevrey smoothing with respect to the space variable for *fluctuations around the global equilibrium* [32].

We stress that for the main result of our paper the initial datum is *only* assumed to obey the *natural* assumptions coming from physics, i.e., finiteness of mass, energy and entropy.

Given $\beta > 0$ and $\alpha \in (0, 1)$ we define the Gevrey multiplier $G : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ by

$$G(t, \eta) := e^{\beta t \langle\eta\rangle^{2\alpha}}$$

and for $\Lambda > 0$ the cut-off Gevrey multiplier $G_\Lambda : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ by

$$G_\Lambda(t, \eta) := G(t, \eta)\mathbb{1}_\Lambda(|\eta|),$$

where $\mathbb{1}_\Lambda$ is the characteristic function of the interval $[0, \Lambda]$. The associated Fourier multiplication operator is denoted by $G_\Lambda(t, D_v)$,

$$(G_\Lambda(t, D_v) f)(t, v) := \int_{\mathbb{R}^d} G_\Lambda(t, \eta) \hat{f}(t, \eta)\, e^{2\pi i \eta \cdot v}\, \mathrm{d}\eta = \mathscr{F}^{-1}\Big[G_\Lambda(t, \cdot)\hat{f}(t, \cdot)\Big].$$

We use the following convention regarding the Fourier transform of a function f in this article:

$$(\mathscr{F} f)(\eta) = \hat{f}(\eta) = \int_{\mathbb{R}^d} f(v)\, e^{-2\pi i v \cdot \eta}\, \mathrm{d}v.$$

The Fourier transform of the Boltzmann operator for Maxwellian molecules has the form (Bobylev identity, [13])

$$\widehat{Q(g, f)}(\eta) = \int_{\mathbb{S}^{d-1}} b\left(\frac{\eta}{|\eta|} \cdot \sigma\right) \Big[\hat{g}(\eta^-)\hat{f}(\eta^+) - \hat{g}(0)\hat{f}(\eta)\Big]\, \mathrm{d}\sigma,$$
$$\eta^\pm = \frac{\eta \pm |\eta|\sigma}{2} \tag{11}$$

for $d \geqq 2$. There is a similar Bobylev identity for the Kac operator [17]:

$$\widehat{K(g, f)}(\eta) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} b_1(\theta) \Big[\hat{g}(\eta^-)\hat{f}(\eta^+) - \hat{g}(0)\hat{f}(\eta)\Big]\, \mathrm{d}\theta,$$
$$\eta^+ = \eta\cos\theta,\ \eta^- = \eta\sin\theta. \tag{12}$$

A simple, but in a sense important, consequence of Bobylev's identity is that, for all $d \geqq 1$,

$$P_\Lambda Q(g, f) = P_\Lambda Q(P_\Lambda g, P_\Lambda f), \tag{13}$$

where, for convenience, we put $P_\Lambda := \mathbb{1}_\Lambda(D_v)$ for the orthogonal projection onto Fourier 'modes' $|\eta| \leq \Lambda$.

Note also that, since $G_\Lambda(t, \cdot)$ has compact support in $\mathbb{R}^d_\eta$ for any $t > 0$, one has

$$G_\Lambda f,\ G_\Lambda^2 f \in L^\infty([0, T_0]; H^\infty(\mathbb{R}^d))$$

for any finite $T_0 > 0$ and $\Lambda > 0$, if $f \in L^\infty([0, T_0]; L^1(\mathbb{R}^d))$. This holds, since

$$\begin{aligned}\|G_\Lambda f\|^2_{H^s(\mathbb{R}^d_v)} &\leqq \|\hat{f}\|^2_{L^\infty(\mathbb{R}^d_\eta)} \|\langle\cdot\rangle^s G_\Lambda(t, \cdot)\|^2_{L^2(\mathbb{R}^d_\eta)} \\ &\leqq \|f\|^2_{L^1(\mathbb{R}^d_v)} \|\langle\cdot\rangle^s G_\Lambda(T_0, \cdot)\|^2_{L^2(\mathbb{R}^d_\eta)}\end{aligned}$$

for all $s \geqq 0$. These functions, due to the cut-off in Fourier space, are even analytic in a strip containing $\mathbb{R}^d_v$.

Theorem 1.6. (Gevrey smoothing I) *Assume that the cross-section b satisfies the* singularity condition (3) *and the* integrability condition (4) *for $d \geqq 2$, and for $d = 1$, b_1 satisfies the* singularity condition (6) *and the* integrability condition (7) *for some $0 < \nu < 1$. Let f be a weak solution of the Cauchy problem (1) with initial datum satisfying conditions (8). Then, for all $0 < \alpha \leqq \min\{\alpha_{2,d}, \nu\}$,*

$$f(t,\cdot) \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d) \tag{14}$$

for all $t > 0$*, where* $\alpha_{2,d} = \frac{\log[(8+d)/(4+d)]}{\log 2}$.

Remarks 1.7. (i) In numbers,

$$\alpha_{2,1} \simeq 0.847997, \quad \alpha_{2,2} \simeq 0.736966, \quad \text{and} \quad \alpha_{2,3} \simeq 0.652077.$$

This means, that under *only* physically reasonable assumptions of finite mass, energy, and entropy, weak solutions are analytic for $\nu \geqq \frac{1}{2}$ and even ultra-analytic if $\nu > \frac{1}{2}$. It is easy to see that $\alpha_{2,d}$ is decreasing in d and for $d = 6$, $\alpha_{2,6} \simeq 0.485427$, hence, for $d \geqq 6$, analyticity (respectively ultra-analyticity) does not follow from this theorem.

(ii) For the proof of Theorem 1.6 (and also 1.9 and 1.10 below) it is important that the energy of f is bounded, which brings in the technical Lemma 2.13 and its Corollary 2.14. A considerably simpler proof could be given using only that $f \in L^1_1(\mathbb{R}^d)$. In this case, $\alpha_{2,d}$ is replaced by $\alpha_{1,d} = \frac{\log[(4+d)/(2+d)]}{\log 2}$ (see also Remark 1.11 below). However, $\alpha_{1,3} < 0.4855$ in three dimensions, thus we would not be able to conclude (ultra-)analytic smoothing of weak solutions for strong singularities $\frac{1}{2} \leqq \nu < 1$.

(iii) As our theorem above shows, weak solutions of the homogenous Kac equation become Gevrey regular for strictly positive times for moderately singular collision kernels with singularity $\nu \in (0, \frac{1}{2})$, see (6) for the precise description of the singularity, for $\nu = \frac{1}{2}$ they become analytic, which improves the result of Glangetas and Najeme [26] in this critical case, and even ultra-anaytic for $\nu \in (\frac{1}{2}, 1)$.

(iv) Rotationally symmetric solutions f corresponding to rotationally symmetric initial conditions f_0 are Gevrey regular for strictly positive times under the same conditions as in the one-dimensional case $d = 1$. The proof is exactly as the proof of Theorem 3.1 with some small changes in the proof of Lemma 2.26 where the independence of the solution f on the angular coordinates can be explicitly used with the $n = 1$ version of Corollary 2.14.

Remark 1.8. Applying the same strategy as the one we developed for the proof of Theorem 1.6, we were able to show in [12] a strong smoothing effect also in the case of Debye–Yukawa type interaction potentials,

$$\sin\theta\, b(\cos\theta) \sim K\theta^{-1}(\log\theta^{-1})^{\ell} \quad \text{for } \theta \to 0 \quad (\text{with some } K > 0,\ \ell > 0).$$

Note that this singularity is much weaker than the type of singularity considered in the work at hand, which leads to a much weaker coercive term. Nevertheless, the smoothing effect we prove in [12] corresponds to exactly what one would expect from the analogy with a logarithmic heat equation $\partial_t f = -(\log(1-\Delta))^{\ell+1} f$.

As already remarked, the result of Theorem 1.6 deteriorates in the dimension. Under the same assumptions, but using quite a bit more structure of the Boltzmann operator, we can prove a dimension independent version. Its proof is considerably more involved than the proof of Theorem 1.6.

Theorem 1.9. (Gevrey smoothing II) *Let $d \geq 2$. Assume that the cross-section b satisfies the conditions of Theorem 1.6. Let f be a weak solution of the Cauchy problem (1) with initial datum satisfying conditions (8).*
Then, for all $0 < \alpha \leqq \min\{\alpha_{2,2}, \nu\}$,

$$f(t, \cdot) \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d) \tag{15}$$

for all $t > 0$, where $\alpha_{2,2} = \frac{\log(5/3)}{\log 2} \simeq 0.736966$. In particular, in contrast to Theorem 1.6, the weak solution is real analytic if $\nu = \frac{1}{2}$ and ultra-analytic if $\nu > \frac{1}{2}$ in any dimension.

If the integrability condition (4) is replaced by the slightly stronger condition that $b(\cos\theta)$ is bounded away from $\theta = 0$, that is,

$$\begin{aligned} &\text{for any } 0 < \theta_0 < \tfrac{\pi}{2} \text{ there exists } C_{\theta_0} < \infty \text{ such that} \\ &0 \leq b(\cos\theta) \leq C_{\theta_0} \text{ for all } \theta_0 \leq \theta \leq \tfrac{\pi}{2}, \end{aligned} \tag{16}$$

which is true in all physically relevant cases, we can prove an even stronger result.

Theorem 1.10. (Gevrey smoothing III) *Let $d \geq 2$. Assume that the cross-section b satisfies the conditions of Theorem 1.6 and the condition (16), that is, it is bounded away from the singularity. Let f be a weak solution of the Cauchy problem (1) with initial datum satisfying conditions (8). Then, for all $0 < \alpha \leqq \min\{\alpha_{2,1}, \nu\}$,*

$$f(t, \cdot) \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d) \tag{17}$$

for all $t > 0$, where $\alpha_{2,1} = \frac{\log(9/5)}{\log 2} \simeq 0.847997$.

Remark 1.11. (i) Since we do not rely on interpolation inequalities between Sobolev spaces, our results also include the *limiting case* $\alpha = \nu$, at least if $\nu \leqq \alpha_{2,n}$ $(n = d, 2, 1)$. This is in contrast to all previous results on smoothing properties of the Boltzmann and Kac equations.

(ii) If higher moments of the initial datum are bounded (and thus stay bounded eternally due to moment propagation results, see, for instance, VILLANI's review [49]), the results in Theorems 1.9 and 1.10 can be improved in the high singularity case, where ν is close to one. Namely, let $f_0 \in L\log L \cap L^1_m(\mathbb{R}^d)$ for some integer $m > 2$, then the constants $\alpha_{2,d}, \alpha_{2,2}$, respectively $\alpha_{2,1}$ are replaced by $\alpha_{m,n} = \frac{\log[(4m+n)/(2m+n)]}{\log 2}$ $(n = d, 2, 1)$, which are strictly increasing towards the limit $\alpha_{\infty,n} = 1$ as m becomes large. See Theorems 3.1, 3.2 and 3.3 below.

Moreover, we prove that for very strong singularities ν, we can prescribe precise conditions on the initial datum such that we have $f \in G^{\frac{1}{2\nu}}(\mathbb{R}^d)$.

Theorem 1.12. *Given $0 < \nu < 1$, there is $m(\nu)$ such that, if $m \in \mathbb{N}$ and $m \geq m(\nu)$ and $f_0 \in L\log L \cap L^1_m$, the weak solution is in $G^{\frac{1}{2\nu}}(\mathbb{R}^d)$ for all $t > 0$.*

More precisely, under the conditions of Theorem 1.6 having $m \geq \max\left(2, \frac{2^\nu-1}{2-2^\nu}\right)$ *yields Gevrey smoothing of order* $\frac{1}{2\nu}$ *and under the slightly stronger conditions of Theorem 1.10 having* $m \geq \max\left(2, \frac{2^\nu-1}{2(2-2^\nu)}\right)$ *is enough.*

Remark 1.13. The proof of this Theorem follows directly from the results of Theorems 3.1, 3.2, and 3.3 in Section 3, which extend Theorems 1.6, 1.9, and 1.10 to the case of finite moments $m \geq 2$.

The strategy of the proofs of our main results Theorems 1.6, 1.9 and 1.10 is as follows: we start with the additional assumption $f_0 \in L^2$ on the initial datum. We use the known H^∞ smoothing of the non-cutoff Boltzmann and Kac equation to allow this. This yields an L^2 reformulation of the weak formulation of the Boltzmann and Kac equations which includes suitable growing Fourier multipliers.

The inclusion of sub-Gaussian Fourier multipliers leads to a nonlocal and nonlinear commutator of the Boltzmann and Kac kernels, which turns out to be a three-linear expression in the weighted solution $\hat{f}$ on the Fourier side. In order to bound this expression with L^2 norms, one of the three terms has to be controlled pointwise, *including* a sub-Gaussian growing factor, see Proposition 2.8. The problem is that one has to control the pointwise bound with an L^2 norm, which is in general impossible. To overcome this obstacle there are several important technical steps:

(1) When working on a ball of radius Λ, we need this uniform control only on a ball of radius $\Lambda/\sqrt{2}$, which enables an inductive procedure.
(2) Using the additional a priori information that the kinetic energy is finite, or, depending on the initial condition, even higher moments are finite, we transform weighted L^2 bounds into pointwise bounds on slightly smaller balls with an additional loss of power in the weights in Fourier space. Here we rely on Kolmogorov–Landau type inequalities, see Lemma 2.17 and appendix C.
(3) Use of strict concavity of the Fourier multipliers, see Lemma 2.5, in order to compensate for this loss of power.
(4) Averaging over a codimension 2 sphere, in the proof of Theorem 1.9, which allows us to get, in any dimension, the same results as for the two dimensional Boltzmann equation.
(5) Averaging over a codimension 1 set constructed from a codimension 2 sphere and the collision angles θ away from the singularity, and using the fact that near the singularity, one of the three Fourier weights is not big due to Lemma 2.5, enables us to get, in any dimension, the same results as for the one-dimensional Kac equation under the conditions of Theorems 1.10 and 3.3.

Even though some of the auxiliary results which we use in this paper are well-known to experts in the field of Boltzmann and Kac equation, we usually give the complete proof of these results in the work at hand to keep it self-contained and make it more accessible for non-specialists.[3]

[3] Like us.

2. Gevrey Regularity and (Ultra-)Analyticity of Weak Solutions with L^2 Initial Data

In this section, we will prove the Gevrey smoothing of weak solutions with initial datum f_0 satisfying (8) and, *additionally*, $f_0 \in L^2(\mathbb{R}^d)$.

2.1. L^2-Reformulation of the Homogeneous Boltzmann Equation for Weak Solutions and Coercivity

The following is our starting point for the proof of the regularizing properties of the homogenous Boltzmann equation:

Proposition 2.1. *Let f be a weak solution of the Cauchy problem* (1) *with initial datum f_0 satisfying* (8)*, and let $T_0 > 0$. Then for all $t \in (0, T_0]$, $\beta > 0$, $\alpha \in (0, 1)$, and $\Lambda > 0$ we have $G_\Lambda f \in \mathscr{C}\big([0, T_0]; L^2(\mathbb{R}^d)\big)$ and*

$$
\begin{aligned}
&\frac{1}{2}\|G_\Lambda(t, D_v) f(t,\cdot)\|_{L^2}^2 - \frac{1}{2}\int_0^t \Big\langle f(\tau,\cdot), \Big(\partial_\tau G_\Lambda^2(\tau, D_v)\Big) f(\tau,\cdot)\Big\rangle \, \mathrm{d}\tau \\
&\quad = \frac{1}{2}\|\mathbb{1}_\Lambda(D_v) f_0\|_{L^2}^2 + \int_0^t \Big\langle Q(f,f)(\tau,\cdot), G_\Lambda^2(\tau, D_v) f(\tau,\cdot)\Big\rangle \, \mathrm{d}\tau.
\end{aligned}
\tag{18}
$$

Informally, equation (18) follows from using $\varphi(t,\cdot) := G_\Lambda^2(t, D_v) f(t,\cdot)$ in the weak formulation of the homogenous Boltzmann equation.

Recall that $G_\Lambda^2 f \in L^\infty([0, T_0]; H^\infty(\mathbb{R}^d))$ for any finite $T_0 > 0$, so it misses the required regularity in time needed to be used as a test function. The proof of Proposition 2.1 is analogous to MORIMOTO ET AL. [39], for the sake of completeness and the convenience of the reader, we prove it in appendix A.

The coercive properties of the non-cutoff Boltzmann bilinear operator which play the crucial role in the smoothing of solutions are made precise in the following sub-elliptic estimate by ALEXANDRE, DESVILLETTES, VILLANI and WENNBERG [3]. We remark that, while the proof there is given for the Boltzmann equation, it equally applies to the Kac equation.

Lemma 2.2. (Sub-elliptic Estimate, [3]) *Let $g \in L^1_2(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$, $g \geqq 0$ ($g \not\equiv 0$). Assume that the collision cross-section b satisfies* (3)–(4) *or* (6)–(7) *respectively, with $0 < \nu < 1$. Then there exists a constant $C_g > 0$ (depending only on the dimension d, the collision kernel b, $\|g\|_{L^1_2}$ and $\|g\|_{L\log L}$) and a constant $C > 0$ (depending only on d and b), such that for any $f \in H^1(\mathbb{R}^d)$ one has*

$$
-\langle Q(g,f), f\rangle \geqq C_g \|f\|_{H^\nu}^2 - C\|g\|_{L^1_2}\|f\|_{L^2}^2.
$$

Remark 2.3. As explained in, for instance [8], the constant C_g is an increasing function of $\|g\|_{L^1}$, $\|g\|_{L^1_2}^{-1}$ and $\|g\|_{L\log L}^{-1}$. In particular, if g is a weak solution of the Cauchy problem (1) with initial datum $g_0 \in L^1_2(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$, we have $\|g\|_{L^1} = \|g_0\|_{L^1}$, $\|g\|_{L^1_2} \leqq \|g_0\|_{L^1_2}$ and $\|g\|_{L\log L} \leqq \log 2\|g_0\|_{L^1} + H(g_0) + C_{\delta,d}\|g_0\|_{L^1_2}^{1-\delta}$, for small enough $\delta > 0$ (see (85)). This implies that $C_g \geqq C_{g_0}$ and thus

$$-\langle Q(g,f),f\rangle \geqq C_g\|f\|_{H^\nu}^2 - C\|g\|_{L^1_2}\|f\|_{L^2}^2 \geqq C_{g_0}\|f\|_{H^\nu}^2 - C\|g_0\|_{L^1_2}\|f\|_{L^2}^2$$

uniformly in $t \geq 0$.

Together with Proposition 2.1 the coercivity estimate Lemma 2.2 implies

Corollary 2.4. (A priori bound for weak solutions) *Let f be a weak solution of the Cauchy problem* (1) *with initial datum f_0 satisfying* (8), *and let $T_0 > 0$. Then there exist constants $\widetilde{C}_{f_0}, C_{f_0} > 0$ (depending only on the dimension d, the collision kernel b, $\|f_0\|_{L^1_2}$ and $\|f_0\|_{L\log L}$) such that for all $t \in (0, T_0]$, $\beta > 0$, $\alpha \in (0,1)$, and $\Lambda > 0$ we have*

$$\begin{aligned}\|G_\Lambda f\|_{L^2}^2 \leqq \|\mathbb{1}_\Lambda(D_v)f_0\|_{L^2}^2 &+ \int_0^t 2\left(-\widetilde{C}_{f_0}\|G_\Lambda f\|_{H^\nu}^2 + C_{f_0}\|G_\Lambda f\|_{L^2}^2\right)\mathrm{d}\tau \\ &+ \int_0^t 2\,|\langle Q(f,G_\Lambda f) - G_\Lambda Q(f,f), G_\Lambda f\rangle|\,\mathrm{d}\tau \\ &+ \int_0^t 2\beta\|G_\Lambda f\|_{H^\alpha}^2\,\mathrm{d}\tau. \qquad (19)\end{aligned}$$

Proof. We want to apply the coercivity result from Lemma 2.2 to the second integral on the right hand side of Proposition 2.1. Therefore, we write

$$\begin{aligned}\langle Q(f,f), G_\Lambda^2 f\rangle &= \langle G_\Lambda Q(f,f), G_\Lambda f\rangle \\ &= \langle Q(f,G_\Lambda f), G_\Lambda f\rangle + \langle G_\Lambda Q(f,f) - Q(f,G_\Lambda f), G_\Lambda f\rangle \\ &\leqq -\widetilde{C}_{f_0}\|G_\Lambda f\|_{H^\nu}^2 + \underbrace{C\|f_0\|_{L^1_2}}_{=:C_{f_0}}\|G_\Lambda f\|_{L^2}^2 \\ &\quad + \langle G_\Lambda Q(f,f) - Q(f,G_\Lambda f), G_\Lambda f\rangle.\end{aligned}$$

Moreover,

$$\partial_\tau G_\Lambda^2(\tau,\eta) = 2\beta\langle\eta\rangle^{2\alpha}G_\Lambda^2(t,\eta).$$

Inserting those two results into (18), we obtain

$$\begin{aligned}\|G_\Lambda f\|_{L^2}^2 \leqq \|\mathbb{1}_\Lambda(D_v)f_0\|_{L^2}^2 &+ 2\beta\int_0^t \|G_\Lambda f(\tau,\cdot)\|_{H^\alpha}^2\,\mathrm{d}\tau \\ &+ 2\int_0^t\left(-\widetilde{C}_{f_0}\|G_\Lambda f\|_{H^\nu}^2 + C_{f_0}\|G_\Lambda f\|_{L^2}^2\right)\mathrm{d}\tau \\ &+ 2\int_0^t \langle G_\Lambda Q(f,f) - Q(f,G_\Lambda f), G_\Lambda f\rangle\,\mathrm{d}\tau.\end{aligned}$$

□

The term $\langle G_\Lambda Q(f,f) - Q(f,G_\Lambda f), G_\Lambda f\rangle$ is called *commutation error*.

2.2. Bound on the Commutation Error

Next, we prove a new bound on the commutation error. An important ingredient is the following elementary observation:

Lemma 2.5. (Strict concavity bound) *Let* $\alpha \in (0, 1]$ *be fixed. The map* $0 \leqq u \mapsto \varepsilon(\alpha, u) := (1+u)^\alpha - u^\alpha$ *has the following properties:*

(i) *If* $\alpha \in (0, 1)$, *then* $\varepsilon(\alpha, \cdot)$ *is* strictly *decreasing on* $[0, \infty)$ *with* $\lim_{u\to\infty} \varepsilon(\alpha, u) = 0$.
In particular, for any $\gamma \geqq 1$ *and* $0 \leqq \gamma s^- \leqq s^+$ *one has*

$$\varepsilon\left(\alpha, \tfrac{s^+}{s^-}\right) \leq \varepsilon(\alpha, \gamma) \leq \varepsilon(\alpha, 1) = 2^\alpha - 1 < 1. \tag{20}$$

Moreover, for all $\alpha \in (0, 1)$ *and all* $u > 0$

$$\varepsilon(\alpha, u) \leq u^{\alpha-1};$$

(ii) *If* $u > 0$, *then* $\varepsilon(\cdot, u)$ *is* strictly *increasing on* $[0, 1]$;
(iii) *For all* $s^-, s^+ \geq 0$

$$(1 + s^- + s^+)^\alpha \leq \varepsilon\left(\alpha, \tfrac{s^+}{s^-}\right)(1 + s^-)^\alpha + (1 + s^+)^\alpha.$$

Proof. Since

$$\frac{\partial}{\partial u}\varepsilon(\alpha, u) = \alpha\left((1+u)^{\alpha-1} - u^{\alpha-1}\right) < 0 \quad \text{for } \alpha \in (0, 1)$$

$\varepsilon(\alpha, \cdot)$ is strictly decreasing. Furthermore, for fixed $u > 0$ we have

$$\frac{\partial}{\partial \alpha}\varepsilon(\alpha, u) = (1+u)^\alpha \log(1+u) - u^\alpha \log u > 0,$$

which shows that $\varepsilon(\cdot, u)$ is strictly increasing.

For $\alpha \in (0, 1)$ and $u \geqq 0$ we estimate

$$\varepsilon(u, \alpha) = \alpha \int_u^{1+u} r^{\alpha-1}\,\mathrm{d}r \leqq \alpha u^{\alpha-1} \leqq u^{\alpha-1}.$$

In particular, $\lim_{u\to\infty} \varepsilon(\alpha, u) = 0$. By monotonicity, the chain of inequalities (20) follows.

Let $s^-, s^+ \geq 0$. Then

$$\begin{aligned}
(1 + s^- + s^+)^\alpha &= (s^-)^\alpha\left[\left(1 + \tfrac{1+s^+}{s^-}\right)^\alpha - \left(\tfrac{1+s^+}{s^-}\right)^\alpha\right] + (1+s^+)^\alpha \\
&\leqq \varepsilon\left(\alpha, \tfrac{1+s^+}{s^-}\right)(1+s^-)^\alpha + (1+s^+)^\alpha \\
&\leqq \varepsilon\left(\alpha, \tfrac{s^+}{s^-}\right)(1+s^-)^\alpha + (1+s^+)^\alpha,
\end{aligned}$$

where we made use of the monotonicity of $\varepsilon(\alpha, \cdot)$ in the last inequality. □

Remark 2.6. The proof of Lemma 2.5 is so simple that one might wonder whether it could be of any use. In fact, it is crucial. It's usefulness is hidden in the fact that it enables us to gain a small exponent in the commutator estimates, see Proposition 2.8 and Lemma 2.10 below. Furthermore, $\varepsilon(\alpha,\gamma)$ can be made as small as we like if γ can be chosen large enough, which will be important in the proof of Theorem 1.10.

Corollary 2.7. *Let $\widetilde{G}(s) := e^{\beta t(1+s)^\alpha}$ for $s \geqq 0$, $\alpha \in (0,1]$. Then, for all $s^- + s^+ = s$ with $0 \leqq s^- \leqq s^+$,*

$$|\widetilde{G}(s) - \widetilde{G}(s^+)| \leqq 2\alpha\beta t(1+s^+)^\alpha \big(1 - \tfrac{s^+}{s}\big)\, \widetilde{G}(s^-)^{\varepsilon\left(\alpha,\frac{s^+}{s^-}\right)} \widetilde{G}(s^+)$$

with $\varepsilon(\alpha,u)$ from Lemma 2.5.

Proof. Since $s^+ \leqq s$ and $\alpha \in (0,1]$,

$$\begin{aligned}|\widetilde{G}(s) - \widetilde{G}(s^+)| &\leqq \int_{s^+}^{s} \left|\frac{\mathrm{d}}{\mathrm{d}r}\widetilde{G}(r)\right| \mathrm{d}r = \alpha\beta t \int_{s^+}^{s} (1+r)^{\alpha-1}\widetilde{G}(r)\,\mathrm{d}r \\ &\leqq \alpha\beta t(1+s^+)^{\alpha-1}(s-s^+)\widetilde{G}(s).\end{aligned}$$

In addition, since $s \leqq 2s^+$,

$$\frac{s-s^+}{1+s^+} = \left(1 - \frac{s^+}{s}\right)\frac{s}{1+s^+} \leqq 2\left(1 - \frac{s^+}{s}\right).$$

Moreover, since $s = s^+ + s^-$, the strict concavity Lemma 2.5 gives

$$\widetilde{G}(s) \leqq \widetilde{G}(s^-)^{\varepsilon\left(\alpha,\frac{s^+}{s^-}\right)} \widetilde{G}(s^+),$$

which completes the proof. □

Proposition 2.8. (Bound on Commutation Error) *Let f be a weak solution of the Cauchy problem (1) with initial datum f_0 satisfying (8). Recall $\varepsilon(\alpha,u) = (1+u)^\alpha - u^\alpha$. Then for all $t \in (0,T_0]$, $\beta > 0$, $\alpha \in (0,1)$, and $\Lambda > 0$ we have*

$$\begin{aligned}&|\langle Q(f, G_\Lambda f) - G_\Lambda Q(f,f), G_\Lambda f\rangle| \\ &\quad \leq 2\alpha\beta t \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(\frac{\eta}{|\eta|}\cdot\sigma\right)\left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right) G(\eta^-)^{\varepsilon(\alpha,|\eta^+|^2/|\eta^-|^2)} |\hat{f}(\eta^-)| \\ &\qquad \times G_\Lambda(\eta^+)|\hat{f}(\eta^+)|\, G_\Lambda(\eta)|\hat{f}(\eta)|\, \langle\eta^+\rangle^{2\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\eta, \end{aligned} \tag{21}$$

for $d \geqq 2$, and

$$\begin{aligned}&|\langle Q(f, G_\Lambda f) - G_\Lambda Q(f,f), G_\Lambda f\rangle| \\ &\quad \leq 2\alpha\beta t \int_{\mathbb{R}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} b_1(\theta)\sin^2\theta\, G(\eta^-)^{\varepsilon(\alpha,|\eta^+|^2/|\eta^-|^2)} |\hat{f}(\eta^-)| \\ &\qquad \times G_\Lambda(\eta^+)|\hat{f}(\eta^+)|\, G_\Lambda(\eta)|\hat{f}(\eta)|\, \langle\eta^+\rangle^{2\alpha}\,\mathrm{d}\theta\,\mathrm{d}\eta, \end{aligned} \tag{22}$$

in the one-dimensional case.

Remark 2.9. If the weight G was growing *polynomially*, the term $G(\eta^-)$ in the integral (21), respectively (22), would be replaced by 1. In this case, the "bad terms" which contain η^- can simply be bounded by $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1} = \|f_0\|_{L^1}$ and the rest can be bounded nicely in terms of $\|G_\Lambda \hat{f}\|_{L^2}$ and $\|G_\Lambda \hat{f}\|_{H^\alpha}$, see the discussion in appendix B.

If the weight G is exponential, the estimate of the terms containing η^- in (21), respectively (22), is an additional challenge and the methods we devised in order to control this term in the commutation error is probably the most important new contribution of this work.

Proof of Proposition 2.8. We start with $d \geqq 2$. By Bobylev's identity, one has

$$
\begin{aligned}
|\langle Q(f, G_\Lambda f) - G_\Lambda Q(f,f), G_\Lambda f\rangle| &= \left|\langle \mathscr{F}[Q(f, G_\Lambda f) - G_\Lambda Q(f,f)], \mathscr{F}[G_\Lambda f]\rangle_{L^2}\right| \\
&\leqq \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(\frac{\eta}{|\eta|}\cdot\sigma\right) G_\Lambda(\eta)|\hat{f}(\eta)|\,|\hat{f}(\eta^-)|\,|\hat{f}(\eta^+)||G_\Lambda(\eta^+) - G_\Lambda(\eta)|\,\mathrm{d}\sigma\,\mathrm{d}\eta \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(\frac{\eta}{|\eta|}\cdot\sigma\right) G_\Lambda(\eta)|\hat{f}(\eta)|\,|\hat{f}(\eta^-)|\,|\hat{f}(\eta^+)||G(\eta^+) - G(\eta)|\,\mathrm{d}\sigma\,\mathrm{d}\eta,
\end{aligned}
$$

where the latter equality follows from the fact that G_Λ is supported on the ball $\{|\eta| \leqq \Lambda\}$ and $|\eta^+| \leqq |\eta|$.

To estimate $|G(\eta^+) - G(\eta)|$, we use Corollary 2.7 with $s := |\eta|^2$ and, accordingly, $s^\pm = |\eta^\pm|^2$. Notice that

$$
|\eta^\pm|^2 = \frac{|\eta|^2}{2}\left(1 \pm \frac{\eta}{|\eta|}\cdot\sigma\right), \quad |\eta|^2 = |\eta^+|^2 + |\eta^-|^2,
$$

and, writing $\cos\theta = \frac{\eta\cdot\sigma}{|\eta|}$, we also have

$$
|\eta^+|^2 = |\eta|^2\cos^2\tfrac{\theta}{2}, \quad |\eta^-|^2 = |\eta|^2\sin^2\tfrac{\theta}{2}.
$$

Since b is supported on angles in $[0, \pi/2]$, one sees $0 \leqq |\eta^-|^2 \leqq \frac{1}{2}|\eta|^2$ and $\frac{1}{2}|\eta|^2 \leqq |\eta^+|^2 \leqq |\eta|^2$. Therefore, $s^- \leqq \frac{s}{2} \leqq s^+ \leqq s$ and $s = s^+ + s^-$.

It follows that for all $\eta \in \mathbb{R}^d$ with $|\eta| \leqq \Lambda$, noting that $|\eta^+| \leqq |\eta| \leqq \Lambda$,

$$
|G(\eta) - G(\eta^+)| \leqq 2\alpha\beta t\langle\eta^+\rangle^{2\alpha}\left(1 - \tfrac{|\eta^+|^2}{|\eta|^2}\right) G(\eta^-)^{\varepsilon(\alpha, |\eta^+|^2/|\eta^-|^2)} G_\Lambda(\eta^+), \quad (23)
$$

which finishes the proof in dimension $d \geq 2$.

For the Kac model we remark that the above proof depends only on $|\eta^-| \leq |\eta^+| \leqq |\eta|$ and $|\eta^-|^2 + |\eta^+|^2 = |\eta|^2$, hence $|\eta^-|^2 \leq |\eta|^2/2$, and the strict concavity Lemma 2.5 and the Corollary 2.7. Since, by symmetry, we assume that b_1 is supported in $[-\pi/4, \pi/4]$, the same bounds for η^- and η^+ hold in dimension one and the above proof can be literally translated, with obvious changes in notation, to the Kac equation. □

The bound on the commutation error in Proposition 2.8 is a trilinear expression in the weak solution f. In order to close the a priori bound from Corollary 2.4 in L^2, one of the terms has to be controlled *uniformly* in η. Seemingly impossible with the growing weights, it is exactly at this place where the gain of the small exponent

$\varepsilon(\alpha, |\eta^+|^2/|\eta^-|^2) \leq \varepsilon(\alpha, 1) < 1$ in the $G(\eta^-)$ term in (21) and (22) allows us to proceed with this strategy. This gain of the small exponent is new and enabled by the strict concavity bound of Lemma 2.5 and its Corollary 2.7 and it is crucial for our inductive approach for controlling the commutation error.

The change of variables is a standard computation used earlier, for instance in [3,39]. We repeat it for the convenience of the reader and, more importantly, since some care has to be exercised in view of the strategy of our inductive setup for controlling the commutation error.

Lemma 2.10. *The inequality*

$$|\langle Q(f, G_\Lambda f) - G_\Lambda Q(f, f), G_\Lambda f\rangle| \leq I_{d,\Lambda} + I^+_{d,\Lambda}$$

holds, where, for $d \geq 2$

$$\begin{aligned} I_{d,\Lambda} =&\alpha\beta t \int_{\mathbb{R}^d} \left(\int_0^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta)} \sin^d\theta\, b(\cos\theta)\, G(\eta^-)^{\varepsilon\left(\alpha, \cot^2\frac{\theta}{2}\right)} |\hat{f}(\eta^-)| \right. \\ &\left. \times \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\, \mathrm{d}\omega\, \mathrm{d}\theta \right) |G_\Lambda(\eta) \hat{f}(\eta)|^2 \langle\eta\rangle^{2\alpha}\, \mathrm{d}\eta. \end{aligned} \tag{24}$$

Here the vector η^- *is expressed as a function of* η *and* σ*, that is,*

$$\eta^- = \eta^-(\eta, \sigma) = \frac{1}{2}(\eta - |\eta|\sigma) = |\eta| \sin^2(\tfrac{\theta}{2}) \frac{\eta}{|\eta|} - |\eta| \sin(\tfrac{\theta}{2}) \cos(\tfrac{\theta}{2})\, \omega \tag{25}$$

and σ *is is a vector on the unit sphere given by*

$$\sigma = \sigma(\theta, \omega) = \cos(\theta) \frac{\eta}{|\eta|} + \sin(\theta)\, \omega \tag{26}$$

with polar angle $\theta \in [0, \pi/2]$ *with respect to the north pole in the* η *direction,* $\omega \in \mathbb{S}^{d-2}(\eta) := \{\widetilde{\omega} \in \mathbb{R}^d : \widetilde{\omega} \perp \eta, |\widetilde{\omega}| = 1\}$*, the* $d-2$ *sphere in* $\mathbb{R}^d$ *orthogonal to the* η *direction, and* $\mathrm{d}\omega$ *the canonical measure on* $\mathbb{S}^{d-2}$:

$$\begin{aligned} I^+_{d,\Lambda} =& 2^d \alpha\beta t \int_{\mathbb{R}^d} \left(\int_0^{\frac{\pi}{4}} \int_{\mathbb{S}^{d-2}(\eta^+)} \sin^d\vartheta\, b\left(\cos 2\vartheta\right)\ G(\eta^-)^{\varepsilon\left(\alpha, \cot^2\vartheta\right)} |\hat{f}(\eta^-)| \right. \\ &\left. \times \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\, \mathrm{d}\omega\, \mathrm{d}\vartheta \right) |G_\Lambda(\eta^+) \hat{f}(\eta^+)|^2 \langle\eta^+\rangle^{2\alpha}\, \mathrm{d}\eta^+ \end{aligned} \tag{27}$$

where now the vector η^- *is expressed as a function of* η^+ *and* σ*, that is,*

$$\eta^- = \eta^-(\eta^+, \sigma) = \eta^+ - |\eta^+| \left(\frac{\eta^+ \cdot \sigma}{|\eta^+|} \right)^{-1} \sigma = -|\eta^+| \tan(\vartheta)\, \omega, \tag{28}$$

where σ *is now a vector on the unit sphere with north pole in the* η^+ *direction given by*

$$\sigma = \sigma(\vartheta, \omega) = \cos(\vartheta) \frac{\eta^+}{|\eta^+|} + \sin(\vartheta)\, \omega \tag{29}$$

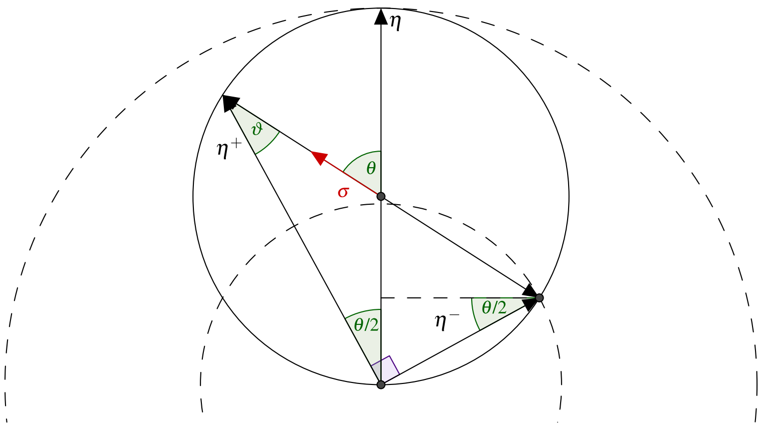

Fig. 1. Geometry of the collision process in Fourier space

with polar angle $\vartheta \in [0, \pi/4]$ and $\omega \in \mathbb{S}^{d-2}(\eta^+)$, the $(d-2)$-sphere in $\mathbb{R}^d$ orthogonal to the η^+ direction. If $d = 2$ we set $\mathbb{S}^0 := \emptyset$ in this context.

For $d = 1$ we have

$$I_{1,\Lambda} = \alpha\beta t \int_{\mathbb{R}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta b_1(\theta)\, G(\eta^-)^{\varepsilon\left(\alpha,\cot^2\frac{\theta}{2}\right)} |\hat{f}(\eta^-)|\, \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\,\mathrm{d}\theta \times |G_\Lambda(\eta)\hat{f}(\eta)|^2 \langle\eta\rangle^{2\alpha}\,\mathrm{d}\eta,$$

$$I^+_{1,\Lambda} = \sqrt{2}\alpha\beta t \int_{\mathbb{R}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta b_1(\theta)\, G(\eta^-)^{\varepsilon\left(\alpha,\cot^2\frac{\theta}{2}\right)} |\hat{f}(\eta^-)|\, \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\,\mathrm{d}\theta \times |G_\Lambda(\eta^+)\hat{f}(\eta^+)|^2 \langle\eta^+\rangle^{2\alpha}\,\mathrm{d}\eta^+,$$

where in the first case $\eta^- = \eta^-(\eta, \theta) = \eta\sin\theta$ and in the second case

$$\eta^- = \eta^-(\eta^+, \theta) = \eta^+\tan\theta$$

and there is no need to distinguish between the θ and ϑ parametrization.

Remark 2.11. In the η, respectively η^+, integrals above η^- and σ are always the same vectors expressed in different parametrizations. We therefore have the relation $\vartheta = \theta/2$, see Figure 1 for the geometry of the collision process in Fourier space.

Remark 2.12. From the bounds given in Lemma 2.10 one might already see that, in order to bound the commutation error by some multiple of $\|G_\Lambda f\|^2_{H^\alpha(\mathbb{R}^d)}$, one has to control integrals of the form

$$\sup_{|\eta|\le\Lambda} \int_0^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta)} \sin^d\theta b(\cos\theta)\, G^{\varepsilon\left(\alpha,\cot^2\frac{\theta}{2}\right)}(\eta^-)\, |\hat{f}(\eta^-)|\, \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\,\mathrm{d}\omega\,\mathrm{d}\theta,$$

with the parametrisation (25) for η^-, and similarly for (27) and the corresponding integrals in the one dimensional case. Due to the characteristic function in η^-, this uniform control is not needed on the full ball of radius Λ, but only on a strictly smaller one, giving rise to an *induction-over-length-scales* type of argument.

Proof of Lemma 2.10. Let $d \geq 2$. Using the elementary estimate

$$|G_\Lambda(\eta)\hat{f}(\eta)|\,|G_\Lambda(\eta^+)\hat{f}(\eta^+)| \leqq \frac{1}{2}\left(|G_\Lambda(\eta)\hat{f}(\eta)|^2 + |G_\Lambda(\eta^+)\hat{f}(\eta^+)|^2\right)$$

in the bound (21) gives

$$|\langle Q(f, G_\Lambda f) - G_\Lambda Q(f, f), G_\Lambda f\rangle| \leq \widetilde{I}_{d,\Lambda} + \widetilde{I}^+_{d,\Lambda}$$

with

$$\widetilde{I}_{d,\Lambda} = \alpha\beta t \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(\frac{\eta}{|\eta|}\cdot\sigma\right)\left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right) G(\eta^-)^{\varepsilon(\alpha,|\eta^+|^2/|\eta^-|^2)}|\hat{f}(\eta^-)| \\ \times \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\,|G_\Lambda(\eta)\hat{f}(\eta)|^2\,\langle\eta^+\rangle^{2\alpha}\,\mathrm{d}\sigma\mathrm{d}\eta,$$

and

$$\widetilde{I}^+_{d,\Lambda} = \alpha\beta t \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(\frac{\eta}{|\eta|}\cdot\sigma\right)\left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right) G(\eta^-)^{\varepsilon(\alpha,|\eta^+|^2/|\eta^-|^2)}|\hat{f}(\eta^-)| \\ \times \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\,|G_\Lambda(\eta^+)\hat{f}(\eta^+)|^2\,\langle\eta^+\rangle^{2\alpha}\,\mathrm{d}\sigma\mathrm{d}\eta.$$

First we consider $\widetilde{I}_{d,\Lambda}$. Writing σ in a parametrization where the north pole is in the η direction, one has

$$\sigma = \cos\theta\frac{\eta}{|\eta|} + \sin\theta\,\omega,$$

where $\cos\theta = \frac{\eta\cdot\sigma}{|\eta|} \geq 0$ and ω is a unit vector orthogonal to η, that is, $\omega \in \mathbb{S}^{d-2}(\eta)$. Due to the support condition on b one has $\cos\theta \geq 0$, that is, σ is restricted to the northern hemisphere $\theta \in [0, \pi/2]$. In this parametization one has $\mathrm{d}\sigma = \sin^{d-2}\theta\mathrm{d}\theta\mathrm{d}\omega$. From the definition of $\eta^\pm$ one sees

$$\eta^\pm = \frac{1}{2}(\eta \pm |\eta|\sigma) = \frac{|\eta|}{2}(1 \pm \cos\theta)\frac{\eta}{|\eta|} \pm \frac{|\eta|}{2}\sin(\theta)\,\omega,$$

so

$$\eta^+ = |\eta|\cos^2(\tfrac{\theta}{2})\frac{\eta}{|\eta|} + |\eta|\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})\,\omega.$$

In particular,

$$|\eta^+| = |\eta|\cos\frac{\theta}{2}, \quad \text{and} \quad 1 - \frac{|\eta^+|^2}{|\eta|^2} = 1 - \cos^2\frac{\theta}{2} = \sin^2\frac{\theta}{2}.$$

Moreover,

$$\eta^- = |\eta|\sin^2\frac{\theta}{2}\frac{\eta}{|\eta|} - |\eta|\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\omega, \quad \text{and} \quad |\eta^-| = |\eta|\sin\frac{\theta}{2},$$

so

$$\frac{|\eta^+|^2}{|\eta^-|^2} = \frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \cot^2\frac{\theta}{2}.$$

After this preparation, using also $\langle\eta^+\rangle^{2\alpha} \leqq \langle\eta\rangle^{2\alpha}$ and $\sin\frac{\theta}{2} \leqq \sin\theta$ for $\theta \in [0, \frac{\pi}{2}]$, the inequality $\widetilde{I}_{d,\Lambda} \leq I_{d,\Lambda}$ is immediate. The inclusion of the additional factor $\mathbb{1}_\Lambda(|\eta|) = \mathbb{1}_{\sin\frac{\theta}{2}\Lambda}(|\eta^-|) \leqq \mathbb{1}_{\Lambda/\sqrt{2}}(|\eta^-|)$ seems artificial for the moment, but will be convenient to keep track of the fact that η^- is always restricted to a ball of radius $\frac{\Lambda}{\sqrt{2}}$.

Concerning $\widetilde{I}_{d,\Lambda}$, we want to implement a change of variables from η to η^+. As a function of η and σ, $\eta^+ = \frac{1}{2}(\eta - |\eta|\sigma)$. Thus

$$\left|\frac{\partial\eta^+}{\partial\eta}\right| = \left|\frac{1}{2}\left(\mathbb{1} + \frac{\eta}{|\eta|}\otimes\sigma\right)\right| = \frac{1}{2^d}\left(1 + \frac{\eta}{|\eta|}\cdot\sigma\right) \geqq \frac{1}{2^d},$$

since $\eta\cdot\sigma \geqq 0$ and the second equality is an application of Sylvester's determinant theorem. Therefore, the Jacobian of the transformation from η to η^+ can be bounded by

$$\left|\frac{\partial\eta}{\partial\eta^+}\right| = \left|\frac{\partial\eta^+}{\partial\eta}\right|^{-1} \leq 2^d.$$

In addition,

$$|\eta^+|^2 = \frac{|\eta|^2}{2}\left(1 + \frac{\eta\cdot\sigma}{|\eta|}\right) \quad \text{and} \quad \eta^+\cdot\sigma = \frac{|\eta|}{2}\left(1 + \frac{\eta\cdot\sigma}{|\eta|}\right) = \frac{|\eta^+|^2}{|\eta|},$$

which implies

$$\frac{\eta^+\cdot\sigma}{|\eta^+|} = \frac{|\eta^+|}{|\eta|} \quad \text{and} \quad \frac{\eta\cdot\sigma}{|\eta|} = 2\frac{|\eta^+|^2}{|\eta|^2} - 1 = 2\left(\frac{\eta^+\cdot\sigma}{|\eta^+|}\right)^2 - 1.$$

Moreover, from the definition of $\eta^\pm$, one sees

$$\eta = 2\eta^+ - |\eta|\sigma,$$

so

$$\eta^- = \eta^+ - |\eta|\sigma = \eta^+ - |\eta^+|\left(\frac{\eta^+\cdot\sigma}{|\eta^+|}\right)^{-1}\sigma.$$

Therefore, taking care of the domain of integration,

$$\widetilde{I}_d^+ \leqq 2^d \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(2\left(\frac{\eta^+\cdot\sigma}{|\eta^+|}\right)^2 - 1\right)\left(1 - \left(\frac{\eta^+\cdot\sigma}{|\eta^+|}\right)^2\right)\mathbb{1}_{\frac{\eta^+\cdot\sigma}{|\eta^+|}\Lambda}(|\eta^+|)$$
$$\times\, G^{\varepsilon(\alpha,|\eta^+|^2/|\eta^-|^2)}(\eta^-)|\hat{f}(\eta^-)|\,|G_\Lambda(\eta^+)\hat{f}(\eta^+)|^2\langle\eta^+\rangle^{2\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\eta^+.$$

Introducing spherical coordinates with north pole in the η^+ direction, one has

$$\sigma = \sigma(\vartheta, \omega) = \cos(\vartheta)\frac{\eta^+}{|\eta^+|} + \sin(\vartheta)\,\omega$$

where now $\cos\vartheta = \frac{\eta^+\cdot\sigma}{|\eta^+|}$. From Figure 1 one sees $\vartheta = \frac{\theta}{2} \in [0, \pi/4]$. In this parametrisation one has

$$\eta^- = \eta^+ - \frac{|\eta^+|}{\cos\vartheta}\sigma = -|\eta^+|\tan(\vartheta)\,\omega,$$

and again $\mathrm{d}\sigma = \sin^{d-2}\vartheta\,\mathrm{d}\vartheta\mathrm{d}\omega$. Thus

$$\begin{aligned}\widetilde{I}_d^+ \leqq 2^d \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-2}}\int_0^{\frac{\pi}{4}} b\,(\cos 2\vartheta)\,\sin^d\vartheta\; G^{\varepsilon\left(\alpha,\cot^2\vartheta\right)}(\eta^-)|\hat{f}(\eta^-)|\,\mathbb{1}_{(\cos\vartheta)\Lambda}(|\eta^+|)\\ \times |G(\eta^+)\hat{f}(\eta^+)|^2\langle\eta^+\rangle^{2\alpha}\,\mathrm{d}\vartheta\,\mathrm{d}\omega\,\mathrm{d}\eta^+.\end{aligned}$$

Since $|\eta^-| = |\eta^+|\tan\vartheta$, we obtain $\mathbb{1}_{(\cos\vartheta)\Lambda}(|\eta^+|) = \mathbb{1}_{(\sin\vartheta)\Lambda}(|\eta^-|) \leqq \mathbb{1}_{\Lambda/\sqrt{2}}(|\eta^-|)$, because $\vartheta \in [0, \pi/4]$. Hence $\widetilde{I}_{d,\Lambda}^+ \leq I_{d,\Lambda}^+$.

The proof in the $d = 1$ case is completely analogous. □

2.3. Extracting Pointwise Information from Local L^2 Bounds.

Lemma 2.13. *Let $m \geqq 2$ and $h \in W^{m,\infty}(\mathbb{R})$ and $q \geqq \frac{1}{m}$. Then there exists a constant $L_m < \infty$ depending only on $q, m, \|h\|_{L^\infty(\mathbb{R})}$ and $\|h^{(m)}\|_{L^\infty(\mathbb{R})}$ such that*

$$|h(r)|^q \leqq L_m \int_{\Omega_r} |h(\xi)|^{q-\frac{1}{m}}\,\mathrm{d}\xi \quad \textit{for all } r \in \mathbb{R},$$

where $\Omega_r = [r, r+2]$ if $r \geqq 0$ and $\Omega_r = [r-2, r]$ if $r < 0$.

Looking into the proof of Lemma 2.13, it is clear that its $m = 1$ version also holds, even with a much simpler proof. Before actually going into the proof, we state an important consequence of it, which will enable us to get pointwise decay estimates on a function once suitable L^2 norms are bounded.

For $m \in \mathbb{N}$ define $\|D^m f\|_{L^\infty(\mathbb{R}^d)} := \sup_{\omega\in\mathbb{S}^{d-1}} \|(\omega\cdot\nabla)^m f\|_{L^\infty(\mathbb{R}^d)}$. Notice that this norm is invariant under rotations of the function f.

Corollary 2.14. *Let $H \in \mathscr{C}^m(\mathbb{R}^n)$. Then there exists a constant $L_{m,n} < \infty$ (depending only on $m, n, \|H\|_{L^\infty(\mathbb{R}^n)}$ and, $\|D^m H\|_{L^\infty(\mathbb{R}^n)}$) such that*

$$|H(x)| \leqq L_{m,n}\left(\int_{Q_x} |H(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{m}{2m+n}},$$

where Q_x is a cube in $\mathbb{R}^n$ of side length 2, with x being one of the corners, such that it is oriented away from x in the sense that $x\cdot(\xi - x) \geqq 0$ for all $\xi \in Q_x$.

Remark 2.15. The constant $L_{m,n}$ in Corollary 2.14 is invariant under rotations of the function H. This will be convenient for its application in Sections 2.5 and 2.6.

Proof. We apply Lemma 2.13 iteratively in each coordinate direction to obtain

$$
\begin{aligned}
&|H(x_1, x_2, \dots, x_n)|^{2+\frac{n}{m}} \\
&\quad \leqq L_m^{(1)} \int_{\Omega_{x_1}} |H(\xi_1, x_2, \dots, x_d)|^{2+\frac{n-1}{m}} \, \mathrm{d}\xi_1 \\
&\quad \leqq L_m^{(1)} L_m^{(2)} \int_{\Omega_{x_1}} \int_{\Omega_{x_2}} |H(\xi_1, \xi_2, x_3 \dots, x_d)|^{2+\frac{n-2}{m}} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \\
&\quad \leqq L_m^{(1)} \cdots L_m^{(n)} \int_{\Omega_{x_1}} \cdots \int_{\Omega_{x_d}} |H(\xi_1, \dots, \xi_d)|^2 \, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n.
\end{aligned}
$$

The constants $L_m^{(i)}$, $i = 1, \dots, n$, only depend on m,

$$
\|H(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)\|_{L^\infty(\mathbb{R})} \leqq \|H\|_{L^\infty(\mathbb{R}^n)}
$$

and

$$
\|\partial_i^m H(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)\|_{L^\infty(\mathbb{R})} \leqq \|D^m H\|_{L^\infty(\mathbb{R}^n)}.
$$

Setting $L_{m,n} = \prod_{i=1}^n L_m^{(i)}$ yields the stated inequality with $Q_x = \Omega_{x_1} \times \cdots \times \Omega_{x_n}$. □

Remark 2.16. It is worth noticing that the exponent in Corollary 2.14 is decreasing in the dimension and increasing in m.

For the proof of Lemma 2.13 we need the following interpolation result between L^∞ norms of derivatives of a function.

Lemma 2.17. (Kolmogorov–Landau inequality on the unit interval) *Let $m \geqq 2$ be an integer. There exists a constant $C_m > 0$ such that for all $w \in W^{m,\infty}([0, 1])$,*

$$
\|w^{(k)}\|_{L^\infty([0,1])} \leqq C_m \left(\frac{\|w\|_{L^\infty([0,1])}}{u^k} + u^{m-k} \|w^{(m)}\|_{L^\infty([0,1])} \right),
$$
$$
k = 1, \dots, m-1,
$$

for all $0 < u \leqq 1$.

Proof. The result dates back to LANDAU and KOLMOGOROV, who proved it on $\mathbb{R}$ and $\mathbb{R}^+$. A proof of the inequality on a finite interval can be found in the book by DEVORE and LORENTZ [24] (pp. 37–39), but for the reader's convenience we also give a short proof in Appendix C. □

For us, the important consequence we are going to make use of is

Corollary 2.18. *Let $C_m > 0$ be the constant from Lemma 2.17. Then for all $w \in W^{m,\infty}([0, 1])$,*

$$
\|w^{(k)}\|_{L^\infty([0,1])} \leqq 2C_m \|w\|_{L^\infty([0,1])}^{1-k/m} \max \left\{ \|w\|_{L^\infty([0,1])}^{k/m}, \|w^{(m)}\|_{L^\infty([0,1])}^{k/m} \right\}, \quad (30)
$$

$k = 1, \dots, m-1$.

Proof. If $\|w^{(m)}\|_{L^\infty([0,1])} \leqq \|w\|_{L^\infty([0,1])}$, we choose $u = 1$ in the bound from Lemma 2.17, which gives

$$\|w^{(k)}\|_{L^\infty([0,1])} \leqq 2C_m \|w\|_{L^\infty([0,1])}.$$

In this case, and if $\|w^{(m)}\|_{L^\infty([0,1])} \geqq \|w\|_{L^\infty([0,1])}$, we can choose

$$u = \|w\|_{L^\infty([0,1])}^{1/m} \|w^{(m)}\|_{L^\infty([0,1])}^{-1/m} \leqq 1$$

to obtain

$$\|w^{(k)}\|_{L^\infty([0,1])} \leqq 2C_m \|w\|_{L^\infty([0,1])}^{1-k/m} \|w^{(m)}\|_{L^\infty([0,1])}^{k/m}.$$

Together this proves (30). □

We can now turn to the

Proof of Lemma 2.13. Assume without loss of generality that $r \geqq 0$, so that $\Omega_r = [r, r+2]$. By the Sobolev embedding theorem h is continuous and we let r^* be a point in Ω_r where $|h|$ attains its maximum. We can assume that $r^* \in [r, r+1]$ and set $\langle h\rangle_{r^*} := \int_{r^*}^{r^*+1} h(\xi)\,\mathrm{d}\xi$ (otherwise we use $\langle h\rangle_{r^*} := \int_{r^*-1}^{r^*} h(\xi)\,\mathrm{d}\xi$). Then for some $p \geqq 1$ we have

$$\begin{aligned} |h(r^*)|^p - \left|\langle h^p\rangle_{r^*}\right| &\leqq \int_{r^*}^{r^*+1} |h^p(r^*) - h^p(\xi)|\,\mathrm{d}\xi \\ &= \int_0^1 |h^p(r^*) - h^p(r^*+\zeta)|\,\mathrm{d}\zeta. \end{aligned}$$

By a fundamental theorem of calculus, for any $\zeta \in [0, 1]$, the integrand can be bounded by

$$\begin{aligned} |h^p(r^*) - h^p(r^*+\zeta)| &\leqq p \int_0^1 |h(r^*+s\zeta)|^{p-1} |h'(r^*+s\zeta)|\zeta\,\mathrm{d}s \\ &\leqq p \sup_{s\in[0,1]} |h'(r^*+s\zeta)| \int_0^1 |h(r^*+s\zeta)|^{p-1}\zeta\,\mathrm{d}s. \end{aligned}$$

We now use that

$$\begin{aligned} \sup_{s\in[0,1]} |h'(r^*+s\zeta)| &= \sup_{x\in[0,\zeta]} |h'(r^*+x)| \\ &\leqq \sup_{x\in[0,1]} |h'(r^*+x)| = \|h'(r^*+\cdot)\|_{L^\infty([0,1])} \end{aligned}$$

and apply the Kolmogorov–Landau inequality for the first derivative in its multiplicative form Corollary 2.18 to the function $[0, 1] \ni x \mapsto h(r^*+x) \in W^{m,\infty}([0, 1])$ to obtain

$$\begin{aligned} \|h'(r^*+\cdot)\|_{L^\infty([0,1])} &\leqq 2C_m \|h(r^*+\cdot)\|_{L^\infty([0,1])}^{1-1/m} \\ &\quad\times \max\left\{\|h(r^*+\cdot)\|_{L^\infty([0,1])}^{1/m}, \|h^{(m)}(r^*+\cdot)\|_{L^\infty([0,1])}^{1/m}\right\} \\ &\leqq 2C_m |h(r^*)|^{1-1/m} \max\left\{\|h\|_{L^\infty(\mathbb{R})}^{1/m}, \|h^{(m)}\|_{L^\infty(\mathbb{R})}^{1/m}\right\}. \end{aligned}$$

It follows that

$$|h(r^*)|^p - \left|\langle h^p\rangle_{r^*}\right| \leqq 2pC_m|h(r^*)|^{1-1/m} \max\left\{\|h\|_{L^\infty(\mathbb{R})}^{1/m}, \|h^{(m)}\|_{L^\infty(\mathbb{R})}^{1/m}\right\} \times \int_0^1\int_0^1 |h(r^*+s\zeta)|^{p-1}\zeta \,\mathrm{d}s\,\mathrm{d}\zeta.$$

The latter integral can be further estimated by

$$\begin{aligned}\int_0^1\int_0^1 |h(r^*+s\zeta)|^{p-1}\zeta\,\mathrm{d}s\,\mathrm{d}\zeta &= \int_0^1\int_0^\zeta |h(r^*+x)|^{p-1}\,\mathrm{d}x\,\mathrm{d}\zeta \\ &\leqq \int_0^1\int_0^1 |h(r^*+x)|^{p-1}\,\mathrm{d}\zeta\,\mathrm{d}x = \int_0^1 |h(r^*+x)|^{p-1}\,\mathrm{d}x \\ &= \int_{r^*}^{r^*+1} |h(\xi)|^{p-1}\,\mathrm{d}\xi \leqq \int_{\Omega_r} |h(\xi)|^{p-1}\,\mathrm{d}\xi.\end{aligned}$$

Using

$$\begin{aligned}\left|\langle h^p\rangle_{r^*}\right| &\leqq \int_{r^*}^{r^*+1} |h(\xi)|^p\,\mathrm{d}\xi \leqq \|h\|_{L^\infty(\Omega_r)}\int_{\Omega_r}|h(\xi)|^{p-1}\,\mathrm{d}\xi \\ &\leqq |h(r^*)|^{1-1/m}\|h\|_{L^\infty(\mathbb{R})}^{1/m}\int_{\Omega_r}|h(\xi)|^{p-1}\,\mathrm{d}\xi,\end{aligned}$$

we get

$$|h(r^*)|^p \leqq L_m|h(r^*)|^{1-1/m}\int_{\Omega_r}|h(\xi)|^{p-1}\,\mathrm{d}\xi$$

with $L_m = 2pC_m \max\left\{\|h\|_{L^\infty(\mathbb{R})}^{1/m}, \|h^{(m)}\|_{L^\infty(\mathbb{R})}^{1/m}\right\} + \|h\|_{L^\infty(\mathbb{R})}^{1/m}$, and therefore

$$|h(r^*)|^{p-1+1/m} \leqq L_m\int_{\Omega_r}|h(\xi)|^{p-1}\,\mathrm{d}\xi.$$

Choosing $q := p-1+1/m \geqq 1/m$ then yields

$$|h(r)|^q \leqq |h(r^*)|^q \leqq L_m\int_{\Omega_r}|h(\xi)|^{q-1/m}\,\mathrm{d}\xi,$$

which is the claimed inequality. □

2.4. Gevrey Smoothing of Weak Solutions for L^2 Initial Data: Part I

Equipped with Corollary 2.14 we can construct an inductive scheme based upon a uniform bound on $G(\eta^-)^{\varepsilon(\alpha,1)}|\hat{f}(\eta^-)|$. As already remarked, this result will depend on the dimension, and will actually deteriorate quickly as dimension increases. Nevertheless it leads to strong regularity properties of weak solutions in the physically relevant cases.

Theorem 2.19. *Assume that the initial datum* f_0 *satisfies* $f_0 \geqq 0$, $f_0 \in L\log L(\mathbb{R}^d) \cap L^1_m(\mathbb{R}^d)$ *for some* $m \geqq 2$*, and, in addition,* $f_0 \in L^2(\mathbb{R}^d)$*. Further assume that the cross-section b satisfies the* singularity condition (3) *and the* integrability condition (4) *for* $d \geqq 2$*, and for* $d = 1$*,* b_1 *satisfies the* singularity condition (6) *and the* integrability condition (7) *for some* $0 < \nu < 1$*. Let f be a weak solution of the Cauchy problem* (1) *with initial datum* f_0*. Set* $\alpha_{m,d} := \log\left(\frac{4m+d}{2m+d}\right)/\log 2$*. Then, for all* $0 < \alpha \leqq \min\{\alpha_{m,d}, \nu\}$ *and* $T_0 > 0$*, there exists* $\beta > 0$*, such that for all* $t \in [0, T_0]$

$$e^{\beta t \langle D_v\rangle^{2\alpha}} f(t,\cdot) \in L^2(\mathbb{R}^d), \tag{31}$$

that is, $f \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d)$ *for all* $t \in (0, T_0]$.

By decreasing β, if necessary, one even has a uniform bound;

Corollary 2.20. *Let* $T_0 > 0$*. Under the same conditions as in Theorem* 2.19 *there exit* $\beta > 0$ *and* $M_1 < \infty$ *such that*

$$\sup_{0\le t\le T_0} \sup_{\eta\in\mathbb{R}^d} e^{\beta t\langle\eta\rangle^{2\alpha}} |\hat{f}(t,\eta)| \le M_1. \tag{32}$$

Remark 2.21. (i) For strong singularities, the restriction on the Gevrey class originates in the bound on the commutation error, with the best value in $d = 1$ dimension. The aim of part II below will be to recover the two-dimensional result in *any dimension* $d \geqq 2$. Under slightly stronger assumptions on the angular cross-section, which still covers all physically relevant cases, we can get the one-dimensional result in any dimension $d \geqq 1$, see part III.

(ii) In dimensions $d = 1, 2, 3$ and $m = 2$, corresponding to initial data with finite energy, we have $\alpha_{2,d} = \log\left(\frac{8+d}{4+d}\right)/\log 2 \geqq \log\left(\frac{11}{7}\right)/\log 2 \simeq 0.652077$. This means that for $\nu = \frac{1}{2}$ the weak solution gets analytic and even ultra-analytic for $\nu > \frac{1}{2}$.

(iii) In the case of physical Maxwellian molecules, where $\nu = \frac{1}{4}$, in three dimensions and with initial datum having finite mass, energy and entropy, we obtain Gevrey $G^2(\mathbb{R}^3)$ regularity.

(iv) Even though the range of α in Theorem 2.19 above deteriorates as the dimension increases, it only fails to cover (ultra-)analyticity results in dimensions $d \geqq 6$. Theorems 2.29 and 2.34 below yield results uniformly in the dimension.

We will prove Theorem 2.19 inductively over suitable length scales $\Lambda_N \to \infty$ as $N \to \infty$ in Fourier space. To prepare for this, we fix some $M < \infty, 0 < T_0 < \infty$ and introduce

Definition 2.22. (Hypothesis $\mathrm{Hyp1}_\Lambda(M)$) Let $M \geqq 0$. Then for all $0 \le t \le T_0$

$$\sup_{|\zeta|\leqq\Lambda} G(t,\zeta)^{\varepsilon(\alpha,1)} |\hat{f}(t,\zeta)| \leqq M. \tag{33}$$

Remark 2.23. Recall that $G(t,\zeta) = e^{\beta t \langle \zeta \rangle^\alpha}$, that is, it depends on α, β, and t, and also f is a time dependent function, even though we suppress this dependence in our notation. Thus $\mathrm{Hyp1}_\Lambda(M)$ also depends on the parameters in $G(t,\zeta)$ and on M and T_0, which, for simplicity, we do not emphasise in our notation. We will later fix some $T_0 > 0$ and a suitable large enough M. The main reason why this is possible is that, since $\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1} = \|f_0\|_{L^1} < \infty$, for any $\Lambda, \beta, T_0 > 0$ the hypothesis $\mathrm{Hyp1}_\Lambda(M)$ is true for large enough M and even any $M > \|f_0\|_{L^1}$ is possible by choosing $\beta > 0$ small enough.

A first step into the inductive proof is the following:

Lemma 2.24. *Let $\alpha \leqq \nu$ and define $c_{b,d} := |\mathbb{S}^{d-2}| \int_0^{\frac{\pi}{2}} \sin^d\theta\, b(\cos\theta)\,\mathrm{d}\theta$ for $d \geqq 3$, $c_{b,2} := \int_0^{\frac{\pi}{2}} \sin^2\theta\, b(\cos\theta)\,\mathrm{d}\theta$, $c_{b,1} := \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta\, b_1(\theta)\,\mathrm{d}\theta$, which are finite by the integrability assumptions (4) and (7), and let $\beta \le \frac{\tilde{C}_{f_0}}{(1+2^{d-1})\,c_{b,d}\alpha T_0 M+1}$. Then, for any weak solution of the homogenous Boltzmann equation,*

$$\mathrm{Hyp1}_\Lambda(M) \quad \Rightarrow \quad \|G_{\sqrt{2}\Lambda} f\|_{L^2(\mathbb{R}^d)} \le \|\mathbb{1}_{\sqrt{2}\Lambda}(D_v) f_0\|_{L^2(\mathbb{R}^d)}\, e^{C_{f_0} T_0} \tag{34}$$

for all $0 \le t \le T_0$.

Remark 2.25. The main point of this lemma is that the right hand side of (34) does not depend on M. This is crucial for our analysis and might seem a bit surprising, at first. It is achieved by making β small enough.

Proof. Let $d \ge 2$. Since $\cot^2\frac{\theta}{2} \geqq 1$ for $\theta \in [0, \frac{\pi}{2}]$ and $\cot^2\vartheta \geqq 1$ for $\vartheta \in [0, \frac{\pi}{4}]$, we can bound $\varepsilon(\alpha, \cot^2\frac{\theta}{2})$ and $\varepsilon(\alpha, \cot^2\vartheta)$ by $\varepsilon(\alpha, 1)$ in the integrals $I_{d,\sqrt{2}\Lambda}$ and $I^+_{d,\sqrt{2}\Lambda}$ from Lemma 2.10.

Assume $\mathrm{Hyp1}_\Lambda(M)$ holds. Then

$$G(t,\zeta)^{\varepsilon(\alpha,1)}|\hat{f}(t,\zeta)| \leqq M \quad \text{for all} \quad |\zeta| \leqq \Lambda.$$

In particular, the terms containing η^- in $I_{d,\sqrt{2}\Lambda}$ and $I^+_{d,\sqrt{2}\Lambda}$ can be bounded by M. Thus, these integrals can now be further estimated by

$$\begin{aligned} I_{d,\sqrt{2}\Lambda} &\leqq \alpha\beta t\, M\, |\mathbb{S}^{d-2}| \int_0^{\frac{\pi}{2}} \sin^d\theta\, b(\cos\theta)\,\mathrm{d}\theta \int_{\mathbb{R}^d} |G_{\sqrt{2}\Lambda}(\eta)\hat{f}(\eta)|^2 \langle\eta\rangle^{2\alpha}\,\mathrm{d}\eta \\ &= \alpha\beta t\, M\, c_{b,d} \|G_{\sqrt{2}\Lambda} f\|^2_{H^\alpha(\mathbb{R}^d)} \end{aligned}$$

and,

$$\begin{aligned} I^+_{d,\sqrt{2}\Lambda} &\leqq 2^d \alpha\beta t\, M\, |\mathbb{S}^{d-2}| \int_0^{\frac{\pi}{4}} \sin^d\vartheta\, b\,(\cos 2\vartheta)\,\mathrm{d}\vartheta \\ &\quad \times \int_{\mathbb{R}^d} |G(\eta^+)\hat{f}(\eta^+)|^2 \langle\eta^+\rangle^{2\alpha}\,\mathrm{d}\eta^+. \end{aligned}$$

In the ϑ integral, we bound $\sin\vartheta \leqq \sin(2\vartheta)$ to obtain

$$I^{+}_{d,\sqrt{2}\Lambda} \leqq 2^{d-1}\alpha\beta t\, M\, c_{b,d}\|G_{\sqrt{2}\Lambda} f\|^2_{H^\alpha(\mathbb{R}^d)}.$$

By Lemma 2.10, the commutation error corresponding to the weight $G_{\sqrt{2}\Lambda}$ is thus bounded by

$$\begin{aligned}\Big|\Big\langle Q(f, G_{\sqrt{2}\Lambda} f) - G_{\sqrt{2}\Lambda} Q(f,f), G_{\sqrt{2}\Lambda} f\Big\rangle\Big| &\leqq I_{d,\sqrt{2}\Lambda} + I^{+}_{d,\sqrt{2}\Lambda}\\ &\leqq (1+2^{d-1})\,\alpha\beta t\, M\, c_{b,d}\|G_{\sqrt{2}\Lambda} f\|^2_{H^\alpha(\mathbb{R}^d)}.\end{aligned} \tag{35}$$

With Corollary 2.4 we then have

$$\begin{aligned}\|G_{\sqrt{2}\Lambda} f\|^2_{L^2(\mathbb{R}^d)} \leqq \|\mathbb{1}_{\sqrt{2}\Lambda}(D_v) f_0\|^2_{L^2} &+ \int_0^t 2C_{f_0}\|G_{\sqrt{2}\Lambda} f\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau\\ &+ \int_0^t 2\Big(- \widetilde{C}_{f_0}\|G_{\sqrt{2}\Lambda} f\|^2_{H^\nu(\mathbb{R}^d)}\\ &\quad + \Big((1+2^{d-1})\,\alpha\beta t\, M\, c_{b,d} + \beta\Big)\,\|G_{\sqrt{2}\Lambda} f\|^2_{H^\alpha(\mathbb{R}^d)}\Big)\,\mathrm{d}\tau.\end{aligned}$$

Since $\alpha \leqq \nu$ and $\beta \leq \frac{\widetilde{C}_{f_0}}{(1+2^{d-1})c_{b,d}\,\alpha T_0 M+1}$, this implies

$$\|G_{\sqrt{2}\Lambda} f\|^2_{L^2(\mathbb{R}^d)} \leqq \|\mathbb{1}_{\sqrt{2}\Lambda}(D_v) f_0\|^2_{L^2(\mathbb{R}^d)} + \int_0^t 2C_{f_0}\|G_{\sqrt{2}\Lambda} f\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau,$$

and with Gronwall's inequality,

$$\|G_{\sqrt{2}\Lambda} f\|^2_{L^2(\mathbb{R}^d)} \leqq \|\mathbb{1}_{\sqrt{2}\Lambda}(D_v) f_0\|^2_{L^2(\mathbb{R}^d)}\, e^{2C_{f_0}T_0} \tag{36}$$

follows.

For $d = 1$, we note that, with the obvious change in notation, the above proof literally translates to the Kac equation. □

The second ingredient gives a uniform bound in terms of a weighted L^2 norm and some a priori uniform bound on some higher derivative of $\hat{f}$.

Lemma 2.26. *Assume that there exist finite constants A_m and B, such that*

$$\|f(t,\cdot)\|_{L^1_m} \leqq A_m, \quad \text{and} \quad \|(G_{\sqrt{2}\Lambda} f)(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leq B \tag{37}$$

for some integer $m \geqq 2$ and for all $0 \leq t \leq T_0$. Set

$$\widetilde{\Lambda} := \frac{1+\sqrt{2}}{2}\Lambda \tag{38}$$

and assume furthermore that

$$\Lambda \geq \Lambda_0 := \frac{4\sqrt{d}}{\sqrt{2}-1}. \tag{39}$$

Then for all $|\eta| \leq \widetilde{\Lambda}$,

$$|\hat{f}(t,\eta)| \leqq K_1\, G(t,\eta)^{-\frac{2m}{2m+d}} \quad \text{for all} \quad 0 \leqq t \leqq T_0, \tag{40}$$

with a constant K_1 depending only on the dimension d, m, A_m, and B.

Remark 2.27. The exponent $\frac{2m}{2m+d}$ in equation (40) comes from Corollary 2.14, choosing $n = d$. This is responsible for our definition of $\alpha_{m,d}$, since then $\varepsilon\left(\alpha_{m,d}, 1\right) = \frac{2m}{2m+d}$.

Remark 2.28. The assumptions of Lemma 2.26 are quite natural: since the Boltzmann equation conserves mass and kinetic energy does not increase, we have the a priori estimate

$$\|f(t,\cdot)\|_{L^1_2(\mathbb{R}^d)} \leqq \|f_0\|_{L^1_2(\mathbb{R}^d)} =: A_2,$$

and due to the known results on moment propagation for the homogeneous Boltzmann equation in the Maxwellian molecules case, we have

$$f_0 \in L^1_m(\mathbb{R}^d) \quad \Longrightarrow \quad f(t,\cdot) \in L^1_m(\mathbb{R}^d) \text{ uniformly in } t \geq 0$$

for any $m > 2$ in addition to assumptions (8).[4]

The importance of Lemma 2.26 is that it effectively converts a local L^2 bound on suitable balls into a *pointwise bound* on slightly smaller balls.

Proof of Lemma 2.26. By the Riemann–Lebesgue lemma, the function $\hat{f}$ has continuous and bounded derivatives of order up to m. Since for any multi-index $\alpha \in \mathbb{N}_0^d$ one has $\partial^\alpha \hat{f} = (-2\pi \mathrm{i})^{|\alpha|}\widehat{v^\alpha f}$, we obtain the bound

$$\begin{aligned}
\|D^m \hat{f}(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} &= \sup_{\omega\in\mathbb{S}^{d-1}} \|(\omega\cdot\nabla)^m \hat{f}(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \\
&\leqq \sup_{\omega\in\mathbb{S}^{d-1}} \sup_{\eta\in\mathbb{R}^d} \sum_{|\alpha|=m} \binom{m}{\alpha} |\omega^\alpha|\,|\partial^\alpha \hat{f}(\eta)| \\
&\leqq (2\pi)^m \sup_{\omega\in\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \sum_{|\alpha|=m} \binom{m}{\alpha} |\omega^\alpha v^\alpha|\, f(v)\,\mathrm{d}v \\
&\leqq (2\pi)^m \sup_{\omega\in\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} (\omega\cdot v)^m\, f(v)\,\mathrm{d}v \\
&\leqq (2\pi)^m \int_{\mathbb{R}^d} |v|^m\, f(v)\,\mathrm{d}v \\
&\leqq (2\pi)^m \|f(t,\cdot)\|_{L^1_m(\mathbb{R}^d)} \leqq (2\pi)^m A_m
\end{aligned}$$

Of course, also $\|\hat{f}\|_{L^\infty(\mathbb{R}^d)} \leqq \|f\|_{L^1(\mathbb{R}^d)} \leqq A_m$.

Let $\eta \in \mathbb{R}^d$ such that $|\eta| \leqq \widetilde{\Lambda}$. By Corollary 2.14 applied to the function $\hat{f}$, there is a constant $L_{m,d}$ that depends only on d, m, and A_m such that

$$|\hat{f}(\eta)| \leqq L_{m,d} \left(\int_{Q_\eta} |\hat{f}(\zeta)|^2\,\mathrm{d}\zeta \right)^{\frac{m}{2m+d}},$$

[4] For more on moment propagation see, for instance, VILLANI's review ([49] pp. 73ff) and references therein.

where Q_η is the cube of side length 2 at η, such that all sides are oriented away from the origin. The definitions of $\widetilde{\Lambda}$ and Λ_0 guarantee by Pythagoras' theorem, that, for $|\eta| \leqq \widetilde{\Lambda}$, Q_η always stays inside the ball around the origin with radius $\sqrt{2}\Lambda$. Since the orientation of Q_η is such that η is the point closest to the origin and the weight G is radial and increasing, we have

$$\begin{aligned}|\hat{f}(\eta)| &\leqq L_{m,d}\left(G(\eta)^{-2}\int_{Q_\eta} G(\zeta)^2|\hat{f}(\zeta)|^2\,\mathrm{d}\zeta\right)^{\frac{m}{2m+d}}\\ &\leqq L_{m,d}\,G(\eta)^{-\frac{2m}{2m+d}}\left(\int_{\{|\eta|\leqq\sqrt{2}\Lambda\}} G(\zeta)^2|\hat{f}(\zeta)|^2\,\mathrm{d}\zeta\right)^{\frac{m}{2m+d}}\\ &\leqq L_{m,d}B^{\frac{2m}{2m+d}}\,G(\eta)^{-\frac{2m}{2m+d}}.\end{aligned}$$

Setting $K_1 := L_{m,d}B^{\frac{2m}{2m+d}}$ yields the claimed inequality. □

Proof of Theorem 2.19. By Lemma 2.24, 2.26, and Remark 2.28, a suitable choice for A_m, B, and the length scales Λ_N is

$$\begin{aligned}B &:= \|f_0\|_{L^2(\mathbb{R}^d)}e^{C_{f_0}T_0},\\ A_m &:= \sup_{t\geqq 0}\|f(t,\cdot)\|_{L^1_m(\mathbb{R}^d)} < \infty,\end{aligned}$$

and

$$\Lambda_N := \frac{\Lambda_{N-1}+\sqrt{2}\Lambda_{N-1}}{2} = \frac{1+\sqrt{2}}{2}\Lambda_{N-1} = \left(\frac{1+\sqrt{2}}{2}\right)^N \Lambda_0$$

with Λ_0 from (39).

Furthermore, we set

$$M_1 := \max\{2A_m+1, K_1\},$$

with the constant K_1 from equation (40).

For the start of the induction, we need $\mathrm{Hyp1}_{\Lambda_0}(M_1)$ to be true. Since

$$\sup_{0\leq t\leq T_0}\sup_{|\eta|\leq\Lambda_0} G(\eta)^{\varepsilon(\alpha,1)}|\hat{f}(\eta)| \leq e^{\varepsilon(\alpha,1)\beta T_0(1+\Lambda_0^2)^\alpha}A_m,$$

and from our choice of M_1, there exists $\beta_0 > 0$ such that $\mathrm{Hyp1}_{\Lambda_0}(M_1)$ is true for all $0 \leq \beta \leq \beta_0$.

Now, we choose

$$\beta = \min\left(\beta_0, \frac{\tilde{C}_{f_0}}{(1+2^{d-1})c_{b,d}\,\alpha T_0 M_1 + 1}\right).$$

With this choice the conditions of Lemmas 2.24 and 2.26 are fulfilled and $\mathrm{Hyp1}_{\Lambda_0}(M_1)$ is true.

For the induction step assume that $\mathrm{Hyp1}_{\Lambda_N}(M_1)$ is true. Then Lemma 2.24 gives

$$\|G_{\sqrt{2}\Lambda_N} f\|_{L^2(\mathbb{R}^d)} \le \|\mathbb{1}_{\sqrt{2}\Lambda_N}(D_v) f_0\|_{L^2(\mathbb{R}^d)}\, e^{C_{f_0} T_0} \leqq B.$$

Note that $\varepsilon(\alpha, 1) \le \frac{2m}{2m+d}$, since $\alpha \leqq \min\left\{\alpha_{m,d}, \nu\right\}$, see Remark 2.27. In addition, $\Lambda_{N+1} = \widetilde{\Lambda}_N$, so Lemma 2.26 shows

$$\sup_{|\eta| \le \Lambda_{N+1}} G(\eta)^{\varepsilon(\alpha,1)} |\hat{f}(\eta)| \le K_1 \leqq M_1,$$

that is, $\mathrm{Hyp1}_{\Lambda_{N+1}}(M_1)$ is true. By induction, it is true for all $N \in \mathbb{N}$. Invoking Lemma 2.24 again, we also have

$$\|G_{\sqrt{2}\Lambda_N} f\|_{L^2(\mathbb{R}^d)} \leqq B$$

for all $N \in \mathbb{N}$ and passing to the limit $N \to \infty$, we see $\|Gf\|_{L^2(\mathbb{R}^d)} \leqq B$, which concludes the proof of the theorem. □

Proof of Corollary 2.20. The proof of Theorem 2.19 showed that given $T_0 > 0$ there exists $M_1 > 0$ and $\beta > 0$ such that $\mathrm{Hyp1}_{\Lambda_N}(M_1)$ is true for all $N \in \mathbb{N}$. This clearly implies (32).

2.5. Gevrey Smoothing of Weak Solutions for L^2 Initial Data: Part II

The results of Part I are best in one dimension and give the correct smoothing in terms of the Gevrey class for ν not too close to one, more precisely $\nu \le \alpha_{m,d}$. In order to improve this in higher dimensions $d \geqq 2$ and for a larger range of singularities $0 < \nu < 1$, the commutator estimates have to be refined. We have

Theorem 2.29. *Let $d \ge 3$. Assume that the initial datum f_0 satisfies $f_0 \geqq 0$, $f_0 \in L\log L(\mathbb{R}^d) \cap L^1_m(\mathbb{R}^d)$ for some $m \geqq 2$, and, in addition, $f_0 \in L^2(\mathbb{R}^d)$. Further assume that the cross-section b satisfies the* singularity condition (3) *and the* integrability condition (4) *for some $0 < \nu < 1$. Let f be a weak solution of the Cauchy problem (1) with initial datum f_0, then for all $0 < \alpha \leqq \min\left\{\alpha_{m,2}, \nu\right\}$ and $T_0 > 0$, there exists $\beta > 0$, such that for all $t \in [0, T_0]$*

$$e^{\beta t \langle D_v\rangle^{2\alpha}} f(t,\cdot) \in L^2(\mathbb{R}^d), \tag{41}$$

that is, $f \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d)$ for all $t \in (0, T_0]$.

In particular, the weak solution is real analytic if $\nu = \frac{1}{2}$ and ultra-analytic if $\nu > \frac{1}{2}$.

The beauty of this theorem is that, in contrast to Theorem 2.19, its result does not deteriorate as dimension increases. We also have a corollary similar to Corollary 2.20, however with a weaker conclusion. Moreover, it is *not* uniform in the time $t \ge 0$ but only holds on finite, but arbitrary, time intervals $[0, T_0]$.

Corollary 2.30. *Under the same assumptions as in Theorem 2.29, for any weak solution f of the Cauchy problem* (1) *and any* $0 < T_0 < \infty$ *there exists* $\widetilde{\beta} > 0$ *and* $M < \infty$ *such that*

$$\sup_{0\le t\le T_0}\ \sup_{\eta\in\mathbb{R}^d} e^{\widetilde{\beta} t\langle\eta\rangle^{2\alpha}}\,|\hat{f}(t,\eta)| \le M. \tag{42}$$

The proof of Theorem 2.29 is again based on an induction over length scales in Fourier space. Having a close look at the integrals $I_{d,\Lambda}$ and $I^+_{d,\Lambda}$ from Lemma 2.10 and using that $\varepsilon(\alpha,\gamma)$ is decreasing in γ, one sees that it should be enough to bound expressions of the form

$$\int_{\mathbb{S}^{d-2}(\eta)} G(\eta^-)^{\varepsilon(\alpha,1)}|\hat{f}(\eta^-)|\mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\,\mathrm{d}\omega$$

and

$$\int_{\mathbb{S}^{d-2}(\eta^+)} G(\eta^-)^{\varepsilon(\alpha,1)}|\hat{f}(\eta^-)|\mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\,\mathrm{d}\omega$$

uniformly in η and θ, respectively η^+ and ϑ, with the parametrization (25), respectively (28), that is, instead of having to use the purely pointwise estimates expressed in the hypothesis $\mathrm{Hyp1}_\Lambda$ from the previous section, one can take advantage of averaging over codimension 2 spheres first. This motivates

Definition 2.31. (Hypothesis $\mathrm{Hyp2}_\Lambda$(M)) Let $M \geqq 0$ be finite. Then for all $0 \le t \le T_0$,

$$\sup_{\zeta\in\mathbb{R}^d\setminus\{0\}}\ \sup_{(z,\rho)\in A_\Lambda} \int_{\mathbb{S}^{d-2}(\zeta)} G\left(t, z\tfrac{\zeta}{|\zeta|}-\rho\omega\right)^{\varepsilon(\alpha,1)} \left|\hat{f}\left(t, z\tfrac{\zeta}{|\zeta|}-\rho\omega\right)\right| \mathrm{d}\omega \leqq M, \tag{43}$$

where $A_\Lambda = \{(z,\rho)\in\mathbb{R}^2 : 0 \leqq z \leqq \rho, z^2+\rho^2 \leqq \Lambda^2\}$ and $\mathbb{S}^{d-2}(\zeta) = \{\omega\in\mathbb{R}^d : \omega\perp\zeta, |\omega| = 1\}$.

Again, we have

Lemma 2.32. *Let* $\alpha \leqq \nu$, *define* $c_{b,d,2} = \int_0^{\frac{\pi}{2}} \sin^d\theta\, b(\cos\theta)\,\mathrm{d}\theta$ (*which is finite by the integrability assumption* (4)), *and let* $\beta \le \frac{\tilde{C}_{f_0}}{(1+2^{d-1})c_{b,d,2}\alpha T_0 M+1}$. *Then, for any weak solution of the homogenous Boltzmann equation,*

$$\mathrm{Hyp2}_\Lambda(M) \quad\Rightarrow\quad \|G_{\sqrt{2}\Lambda} f\|_{L^2(\mathbb{R}^d)} \le \|\mathbb{1}_{\sqrt{2}\Lambda}(D_v) f_0\|_{L^2(\mathbb{R}^d)}\, e^{C_{f_0}T_0} \tag{44}$$

for all $0 \le t \le T_0$.

Proof. Using the monotonicity of $\varepsilon(\alpha, \gamma)$ in γ and (24) one sees

$$I_{d,\sqrt{2}\Lambda} \le \alpha\beta t \int_{\mathbb{R}^d} \left(\int_0^{\frac{\pi}{2}} \left(\int_{\mathbb{S}^{d-2}(\eta)} G(\eta^-)^{\varepsilon(\alpha,1)} \, |\hat{f}(\eta^-)| \, \mathbb{1}_\Lambda(|\eta^-|) \, \mathrm{d}\omega \right) \right.$$
$$\left. \times \sin^d \theta \, b(\cos\theta) \, \mathrm{d}\theta \right) |G_{\sqrt{2}\Lambda}(\eta) \hat{f}(\eta)|^2 \, \langle \eta \rangle^{2\alpha} \, \mathrm{d}\eta$$

where $\eta^- = \eta^-(\eta, \theta, \omega)$ is expressed via the parametrization (25). For $\sigma = (\theta, \omega) \in [0, \frac{\pi}{2}] \times \mathbb{S}^{d-2}$, one has $\eta^- = |\eta| \sin^2 \frac{\theta}{2} \frac{\eta}{|\eta|} + |\eta| \sin\frac{\theta}{2} \cos\frac{\theta}{2} \, \omega$ and if $|\eta| \le \sqrt{2}\Lambda$, then $|\eta^-| \le \Lambda$. Identifying $z = |\eta| \sin^2 \frac{\theta}{2}$ and $\rho = |\eta| \sin\frac{\theta}{2} \cos\frac{\theta}{2}$, and the direction of ζ with the direction of η, hypothesis $\mathrm{Hyp2}_\Lambda(M)$ clearly implies

$$\sup_{|\eta| \le \sqrt{2}\Lambda} \sup_{\theta \in [0,\pi/2]} \int_{\mathbb{S}^{d-2}(\eta)} G(\eta^-)^{\varepsilon(\alpha,1)} \, |\hat{f}(\eta^-)| \, \mathbb{1}_\Lambda(|\eta^-|) \, \mathrm{d}\omega \le M.$$

It follows that

$$\begin{aligned} I_{d,\sqrt{2}\Lambda} &\leqq \alpha\beta t \, M \int_{\mathbb{R}^d} \int_0^{\frac{\pi}{2}} \sin^d \theta b(\cos\theta) \, \mathrm{d}\theta \, |G_{\sqrt{2}\Lambda}(\eta) \hat{f}(\eta)|^2 \, \langle \eta \rangle \, \mathrm{d}\eta \\ &= \alpha\beta t \, M \, c_{b,d,2} \| G_{\sqrt{2}\Lambda} f \|^2_{H^\alpha(\mathbb{R}^d)}. \end{aligned}$$

Similarly one has

$$I^+_{d,\sqrt{2}\Lambda} \le 2^d \alpha\beta t \int_{\mathbb{R}^d} \left(\int_0^{\frac{\pi}{4}} \left(\int_{\mathbb{S}^{d-2}(\eta^+)} G(\eta^-)^{\varepsilon(\alpha,1)} \, |\hat{f}(\eta^-)| \, \mathbb{1}_\Lambda(|\eta^-|) \, \mathrm{d}\omega \right) \right.$$
$$\left. \times \sin^d \vartheta \, b(\cos 2\vartheta) \, \mathrm{d}\vartheta \right) |G_{\sqrt{2}\Lambda}(\eta^+) \hat{f}(\eta^+)|^2 \, \langle \eta^+ \rangle^{2\alpha} \, \mathrm{d}\eta^+,$$

where $\eta^- = \eta^-(\eta, \vartheta, \omega)$ is expressed via the parametrization (28). The vectors η^- and η^+ are orthogonal and we have $\eta^- = -|\eta^+| \tan\vartheta \, \omega$ for $(\vartheta, \omega) \in [0, \frac{\pi}{4}] \times \mathbb{S}^{d-2}(\eta^+)$.

Setting $z = 0$ and $\rho = |\eta^+| \tan\vartheta$ we have $\rho = |\eta^-| \le \Lambda$ in the ϑ and η^+ integrals above. Thus $\mathrm{Hyp2}_\Lambda(M)$ again implies

$$\sup_{|\eta^+| \le \sqrt{2}\Lambda} \sup_{\vartheta \in [0,\pi/4]} \int_{\mathbb{S}^{d-2}(\eta^+)} G(\eta^-)^{\varepsilon(\alpha,1)} \, |\hat{f}(\eta^-)| \, \mathbb{1}_\Lambda(|\eta^-|) \, \mathrm{d}\omega \le M.$$

Hence,

$$\begin{aligned} I^+_{d,\sqrt{2}\Lambda} &\leqq 2^d \alpha\beta t \, M \int_0^{\frac{\pi}{2}} \sin^d \theta b(\cos\theta) \, \mathrm{d}\theta \int_{\mathbb{R}^d} |G_{\sqrt{2}\Lambda}(\eta^+) \hat{f}(\eta^+)|^2 \, \langle \eta^+ \rangle \, \mathrm{d}\eta^+ \\ &\leqq 2^{d-1} \alpha\beta t \, M \, c_{b,d,2} \| G_{\sqrt{2}\Lambda} f \|^2_{H^\alpha(\mathbb{R}^d)}. \end{aligned}$$

The rest of the proof is the same as in the proof of Lemma 2.24. □

To close the induction process, we next show

Lemma 2.33. *Let $\beta \leqq \frac{1}{T_0}$. Assume that there exist finite constants A_m and B, such that*

$$\|f(t,\cdot)\|_{L^1_m} \leqq A_m, \quad \textit{and} \quad \|(G_{\sqrt{2}\Lambda} f)(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leq B \tag{45}$$

for some integer $m \geqq 2$ and for all $0 \leq t \leq T_0$.

Set $\widetilde{\Lambda} := \frac{1+\sqrt{2}}{2}\Lambda$ and assume that

$$\Lambda \geq \Lambda_0 := \frac{4\sqrt{2}}{\sqrt{2}-1}. \tag{46}$$

Then for all $\zeta \in \mathbb{R}^d \setminus \{0\}$ and $0 \leqq z \leqq \rho$ with $\rho^2 + z^2 \leqq \widetilde{\Lambda}^2$ one has

$$\int_{\mathbb{S}^{d-2}(\zeta)} \left| \hat{f}\left(t, z\tfrac{\zeta}{|\zeta|} + \rho\omega\right)\right| \mathrm{d}\omega \leqq K_2\, \widetilde{G}(t, z^2+\rho^2)^{-\frac{2m}{2m+2}} \quad \textit{for all } 0 \leq t \leq T_0$$

with a constant K_2 depending only on d, m, A_m, and B. Recall that $\widetilde{G}(t,s) = e^{\beta t(1+s)^\alpha}$.

Proof. Fix $0 < t \leq T_0$, $\zeta \in \mathbb{R}^d \setminus \{0\}$, and set $F(\rho, z) := \hat{f}(t, z\frac{\zeta}{|\zeta|} + \rho\omega)$, where we drop, for simplicity, the dependence on the time t in our notation for F. Then, since $\|f(t,\cdot)\|_{L^1_m} \leq A_m$ one has $\hat{f}(t,\cdot) \in \mathscr{C}^m(\mathbb{R}^d)$ and thus also $F \in \mathscr{C}^m(\mathbb{R}^2)$ with $\|F\|_{L^\infty} \leqq A_m$ $\|\partial_\rho^m F\|_{L^\infty} \leqq (2\pi)^m A_m$, and $\|\partial_z^m F\|_{L^\infty} \leqq (2\pi)^m A_m$ and Corollary 2.14 applied to F yields

$$\left|\hat{f}\left(z\tfrac{\zeta}{|\zeta|} + \rho\omega\right)\right| \leqq L_{m,2}\left(\int_\rho^{\rho+2}\int_z^{z+2} \left|\hat{f}\left(x\tfrac{\zeta}{|\zeta|} + y\omega\right)\right|^2 \mathrm{d}x\mathrm{d}y\right)^{\frac{m}{2m+2}}, \tag{47}$$

where we also dropped the dependence of $\hat{f}$ on the time variable t. Furthermore, we will drop the time dependence of G and $\widetilde{G}$ in the following, that is, $G(\xi)$ and $\widetilde{G}(s)$ will stand for $G(t,\xi)$, respectively $\widetilde{G}(t,s)$.

To recover the L^2 norm of $G_{\sqrt{2}\Lambda} f$ in the right hand side of (47) we now need to take care of three things:

(i) Multiply with a suitable power of the radially increasing weight G;
(ii) Integrate over the missing $d-2$ directions, which will be taken care of by integrating over $\mathbb{S}^{d-2}(\zeta)$ and taking into account additional factors to get the d-dimensional Lebesgue measure;
(iii) Ensure that the region of integration $[\rho, \rho+2] \times [z, z+2] \times \mathbb{S}^{d-2}(\zeta)$ stays inside a ball of radius $\sqrt{2}\Lambda$ uniformly in the direction of ζ. This we control by choosing Λ_0 large enough (a simple geometric consideration shows that Λ_0 from the statement of Lemma 2.33 works) and restricting ρ and z by $\rho^2 + z^2 \leqq \widetilde{\Lambda}^2$.

Let $z, \rho \geq 0$. In the region of integration in (47), the point $\rho\omega + z\frac{\eta}{|\eta|}$ is closest to the origin in $\mathbb{R}^d$, and since the weight G is radially increasing, we get

$$\left|\hat{f}\left(z\tfrac{\zeta}{|\zeta|} + \rho\omega\right)\right| \leqq L_{m,2}\widetilde{G}\left(z^2+\rho^2\right)^{-\frac{2m}{2m+2}} \left(\int_\rho^{\rho+2}\int_z^{z+2} G\left(x\tfrac{\zeta}{|\zeta|} + y\omega\right)^2 \left|\hat{f}\left(x\tfrac{\zeta}{|\zeta|} + y\omega\right)\right|^2 \mathrm{d}x\mathrm{d}y\right)^{\frac{m}{2m+2}}. \tag{48}$$

Assume that $z^2+\rho^2 \leq \widetilde{\Lambda}^2$. Then the integration of inequality (48) over $\mathbb{S}^{d-2}(\zeta)$ yields, with an application of Jensen's inequality ($t \mapsto t^{\frac{m}{2m+2}}$ is concave!),

$$\int_{\mathbb{S}^{d-2}(\zeta)} \left|\hat{f}\left(z\tfrac{\zeta}{|\zeta|}+\rho\omega\right)\right| \mathrm{d}\omega \leqq L_{m,2}|\mathbb{S}^{d-2}|^{\frac{m+2}{2m+2}}\, \widetilde{G}\left(z^2+\rho^2\right)^{-\frac{2m}{2m+2}}$$
$$\times\left(\int_{\mathbb{S}^{d-2}(\zeta)}\int_{\rho}^{\rho+2}\int_{z}^{z+2} G_{\sqrt{2}\Lambda}\left(x\tfrac{\zeta}{|\zeta|}+y\omega\right)^2 \left|\hat{f}\left(x\tfrac{\eta}{|\eta|}+y\omega\right)\right|^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\omega\right)^{\frac{m}{2m+2}}.$$

Now assume additionally $0 \leqq z \leqq \rho$ and $\Lambda_0^2 \leqq \rho^2+z^2 \leqq \widetilde{\Lambda}^2$. Since $0 \leqq z \leqq \rho$ we have $\Lambda_0^2 \leqq z^2+\rho^2 \leqq 2\rho^2$ and therefore

$$\int_{\mathbb{S}^{d-2}(\zeta)}\int_{\rho}^{\rho+2}\int_{z}^{z+2} G_{\sqrt{2}\Lambda}\left(x\tfrac{\zeta}{|\zeta|}+y\omega\right)^2 \left|\hat{f}\left(x\tfrac{\zeta}{|\zeta|}+y\omega\right)\right|^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\omega$$
$$\leqq 2^{\frac{d-2}{2}}\Lambda_0^{2-d}\int_{\mathbb{S}^{d-2}(\zeta)}\int_{\rho}^{\rho+2}\int_{z}^{z+2} G_{\sqrt{2}\Lambda}\left(x\tfrac{\zeta}{|\zeta|}+y\omega\right)^2 \left|\hat{f}\left(x\tfrac{\zeta}{|\zeta|}+y\omega\right)\right|^2$$
$$\times\, y^{d-2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\omega$$
$$\leqq 2^{\frac{d-2}{2}}\Lambda_0^{2-d}\|G_{\sqrt{2}\Lambda} f\|^2_{L^2(\mathbb{R}^d)},$$

since $y^{d-2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\omega$ is the d-dimensional Lebesgue measure in the cylindrical coordinates $(x, y\omega)$ with $x \in \mathbb{R}$, $y > 0$, $\omega \in \mathbb{S}^{d-2}(\zeta)$ along the cylinder with axis ζ. So with the assumption $\|G_{\sqrt{2}\Lambda} f\|_{L^2(\mathbb{R}^d)} \leqq B$ we obtain

$$\int_{\mathbb{S}^{d-2}(\zeta)} \left|\hat{f}\left(t, z\tfrac{\zeta}{|\zeta|}+\rho\omega\right)\right| \mathrm{d}\omega$$
$$\leqq L_{m,2}|\mathbb{S}^{d-2}|^{\frac{m+2}{2m+2}}\left(2^{\frac{d-2}{2}}\Lambda_0^{2-d}B^2\right)^{\frac{m}{2m+2}}\widetilde{G}\left(t, z^2+\rho^2\right)^{-\frac{2m}{2m+2}}.$$

In the case $z^2+\rho^2 \leqq \Lambda_0^2$ we have $\widetilde{G}(t, z^2+\rho^2)^{-1}\, e^{\beta t(1+\Lambda_0^2)^\alpha} \geqq 1$ and we can simply bound

$$\int_{\mathbb{S}^{d-2}(\zeta)} \left|\hat{f}\left(t, z\tfrac{\zeta}{|\zeta|}+\rho\omega\right)\right| \mathrm{d}\omega$$
$$\leq \widetilde{G}\left(t, z^2+\rho^2\right)^{-\frac{2m}{2m+2}} e^{\frac{2m}{2m+2}\beta t(1+\Lambda_0^2)^\alpha}|\mathbb{S}^{d-2}|\,\|\hat{f}(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}$$
$$\leq A_m|\mathbb{S}^{d-2}|e^{1+\Lambda_0^2}\widetilde{G}\left(t, z^2+\rho^2\right)^{-\frac{2m}{2m+2}}$$

since $\beta \leq 1/T_0$, by assumption. So choosing

$$K_2 := \max\left(L_{m,2}|\mathbb{S}^{d-2}|^{\frac{m+2}{2m+2}}\left(2^{\frac{d-2}{2}}\Lambda_0^{2-d}B^2\right)^{\frac{m}{2m+2}}, A_m|\mathbb{S}^{d-2}|e^{1+\Lambda_0^2}\right)$$

finishes the proof of the lemma. □

Now we have all the ingredients for the inductive

Proof of Theorem 2.29. By Lemmata 2.32 and 2.33 a suitable choice for A_m and B is

$$B := \|f_0\|_{L^2(\mathbb{R}^d)} e^{C_{f_0} T_0},$$
$$A_m := \sup_{t \geqq 0} \|f(t, \cdot)\|_{L^1_m(\mathbb{R}^d)} < \infty.$$

Note that the finiteness of A_m is guaranteed since $f_0 \in L^1_m(\mathbb{R}^d)$, see Remark 2.28. We further choose the length scales Λ_N to be

$$\Lambda_N := \frac{\Lambda_{N-1} + \sqrt{2}\Lambda_{N-1}}{2} = \frac{1+\sqrt{2}}{2}\Lambda_{N-1} = \left(\frac{1+\sqrt{2}}{2}\right)^N \Lambda_0$$

with Λ_0 now from (46), and we set

$$M_2 := \max\left\{2|\mathbb{S}^{d-2}|A_m + 1, K_2\right\}$$

with the constant K_2 from Lemma 2.33.

For the start of the induction, we need $\mathrm{Hyp2}_{\Lambda_0}(M_2)$ to be true. Since

$$\sup_{0\le t\le T_0} \sup_{\zeta\in\mathbb{R}^d\setminus\{0\}} \sup_{(z,\rho)\in A_{\Lambda_0}} \int_{\mathbb{S}^{d-2}(\zeta)} G\left(t, z\tfrac{\zeta}{|\zeta|} - \rho\omega\right)^{\varepsilon(\alpha,1)} \left|\hat{f}\left(t, z\tfrac{\zeta}{|\zeta|} - \rho\omega\right)\right| \mathrm{d}\omega$$
$$\le |\mathbb{S}^{d-2}| e^{\beta T_0 (1+\Lambda_0^2)^\alpha} A_m$$

and from our choice of M_2 there exists $\beta_0 > 0$ such that $\mathrm{Hyp2}_{\Lambda_0}(M_2)$ is true for all $0 \le \beta \le \beta_0$.

Now, we choose

$$\beta = \min\left(\beta_0, T_0^{-1}, \frac{\tilde{C}_{f_0}}{(1+2^{d-1})c_{b,d,2}\,\alpha T_0 M_2 + 1}\right).$$

With this choice the conditions of Lemmas 2.32 and 2.33 are fulfilled and $\mathrm{Hyp2}_{\Lambda_0}(M_2)$ is true.

For the induction step assume that $\mathrm{Hyp2}_{\Lambda_N}(M_2)$ is true. Then Lemma 2.32 gives

$$\|G_{\sqrt{2}\Lambda_N} f\|_{L^2(\mathbb{R}^d)} \le \|\mathbb{1}_{\sqrt{2}\Lambda_N}(D_v) f_0\|_{L^2(\mathbb{R}^d)}\, e^{C_{f_0} T_0} \le B$$

and then, since $\varepsilon(\alpha, 1) \le \frac{2m}{2m+2}$ by our choice of α, and $\Lambda_{N+1} = \widetilde{\Lambda}_N$, Lemma 2.33 shows that $\mathrm{Hyp2}_{\Lambda_{N+1}}(M_2)$ is true, so by induction, it is true for all $N \in \mathbb{N}$. Invoking Lemma 2.32 again, we also have

$$\|G_{\sqrt{2}\Lambda_N} f\|_{L^2(\mathbb{R}^d)} \leqq B$$

for all $N \in \mathbb{N}$ and letting $N \to \infty$, we see $\|Gf\|_{L^2(\mathbb{R}^d)} \leqq B$, which concludes the proof of Theorem 2.29. □

Proof of Corollary 2.30. Theorem 2.29 shows that $Gf \in L^2(\mathbb{R}^d)$ for all $0 \leqq t \leqq T_0$. applying Corollary 2.14 with $n = d$ to $\hat{f}$ yields

$$
\begin{aligned}
|\hat{f}(\eta)| &\leqq L_{m,d} G(\eta)^{-\frac{2m}{2m+d}} \left(\int_{Q_\eta} G(\zeta)^2 |\hat{f}(\zeta)|^2 \, \mathrm{d}\zeta \right)^{\frac{m}{2m+d}} \\
&\leqq L_{m,d} \|Gf\|_{L^2(\mathbb{R}^d)}^{\frac{2m}{2m+d}} G(\eta)^{-\frac{2m}{2m+d}},
\end{aligned}
$$

where we also used that the Fourier multiplier is radially increasing. This proves the uniform bound (42) with $\widetilde{\beta} = \beta \frac{2m}{2m+d}$. □

2.6. *Gevrey Smoothing of Weak Solutions for L^2 Initial Data: Part III*

Under the slightly stronger assumption that the angular collision cross-section b is bounded away from the singularity, we can state our theorem about Gevrey regularisation in its strongest form.

Theorem 2.34. *Assume that the initial datum f_0 satisfies $f_0 \geqq 0$, $f_0 \in L\log L(\mathbb{R}^d) \cap L^1_m(\mathbb{R}^d)$ for some $m \geqq 2$, and, in addition, $f_0 \in L^2(\mathbb{R}^d)$. Further assume that the cross-section b in dimensions $d \geqq 2$ satisfies the* singularity condition (3) *for some $0 < \nu < 1$ and the* boundedness condition (16). *Let f be a weak solution of the Cauchy problem (1) with initial datum f_0, then for all $0 < \alpha \leqq \min\{\alpha_{m,1}, \nu\}$ and all $T_0 > 0$, there exists $\beta > 0$, such that for all $t \in [0, T_0]$*

$$
e^{\beta t \langle D_v \rangle^{2\alpha}} f(t, \cdot) \in L^2(\mathbb{R}^d), \tag{49}
$$

that is, $f \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d)$ for all $t \in (0, T_0]$.

In particular, the weak solution is real analytic if $\nu = \frac{1}{2}$ and ultra-analytic if $\nu > \frac{1}{2}$.

Remark 2.35. Thus, under a slightly stronger assumption on b than in Theorem 2.19, which we stress are nevertheless fulfilled in any physically reasonable cases, we can prove the same regularity in *any dimension* as can be obtained for radially symmetric solutions of the homogenous Boltzmann equation.

Corollary 2.36. *Under the same assumptions as in Theorem 2.34, for any weak solution f of the Cauchy problem (1) and any $0 < T_0 < \infty$ there exists $\beta > 0$ and $M < \infty$ such that*

$$
\sup_{0 \leq t \leq T_0} \sup_{\eta \in \mathbb{R}^d} e^{\beta t \langle \eta \rangle^{2\alpha}} |\hat{f}(t, \eta)| \leq M. \tag{50}
$$

Proof. Given Theorem 2.34, the proof of Corollary 2.36 is the same as the proof of Corollary 2.30. □

The proof of Theorem 2.34 shows the delicate interplay between the angular singularity of the collision kernel, the strict concavity of the Gevrey weights, and the use of averages of the weak solution in Fourier space, together with our inductive procedure, which has proved to be successful in Theorems 2.19 and 2.29. Again, the main work is to bound the expressions $I_{d,\Lambda}$ and $I^+_{d,\Lambda}$ from Lemma 2.10. Before we start the proof of Theorem 2.34, we start with some preparations. It is clear that we only have to prove Theorem 2.34 in dimension $d \geq 2$ and for singularities $\nu > \alpha_{2,m}$, since otherwise the result is already contained in Theorems 2.19 and 2.29.

Looking at the integral $I_{d,\Lambda}$ from Lemma 2.10, one has

$$I_{d,\Lambda} = \alpha\beta t \int_{\mathbb{R}^d} \left(\int_0^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta)} \sin^d \theta b(\cos\theta)\, G(\eta^-)^{\varepsilon\left(\alpha, \cot^2 \frac{\theta}{2}\right)} |\hat{f}(\eta^-)| \right.$$
$$\left. \times \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-|)\, \mathrm{d}\omega\, \mathrm{d}\theta \right) |G_\Lambda(\eta)\hat{f}(\eta)|^2 \langle\eta\rangle^{2\alpha}\, \mathrm{d}\eta,$$

where we use the parametrization (25) for $\eta^- = \eta^-(\eta, \theta, \omega)$. Splitting the θ integral above at a point $\theta_0 \in (0, \frac{\pi}{2})$ and using the monotonicity of the cotangent on $[0, \frac{\pi}{2}]$ and of $\varepsilon(\alpha, \gamma)$ in γ one sees

$$I_{d,\Lambda} \leq I_{d,\Lambda,1} + I_{d,\Lambda,2}$$

with

$$I_{d,\Lambda,1} := \alpha\beta T_0 \, \|G_\Lambda f\|^2_{H^\alpha(\mathbb{R}^d)} \int_0^{\theta_0} \sin^d \theta\, b(\cos\theta)\, \mathrm{d}\theta$$
$$\times \left(\sup_{0<\theta\leq\frac{\pi}{2}} \sup_{0<|\eta|\leq\Lambda} \int_{\mathbb{S}^{d-2}(\eta)} G(\eta^-(\eta,\theta,\omega))^{\varepsilon\left(\alpha, \cot^2 \frac{\theta_0}{2}\right)} \right.$$
$$\left. \times |\hat{f}(\eta^-(\eta,\theta,\omega))| \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-(\eta,\theta,\omega)|)\, \mathrm{d}\omega \right) \tag{51}$$

and

$$I_{d,\Lambda,2} := C_{\theta_0}\alpha\beta T_0 \, \|G_\Lambda f\|^2_{H^\alpha(\mathbb{R}^d)}$$
$$\times \left(\sup_{0<|\eta|\leq\Lambda} \int_{\theta_0}^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta)} G(\eta^-(\eta,\theta,\omega))^{\varepsilon(\alpha,1)} |\hat{f}(\eta^-(\eta,\theta,\omega))| \right.$$
$$\left. \times \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-(\eta,\theta,\omega)|)\, \mathrm{d}\omega\, \mathrm{d}\theta \right), \tag{52}$$

where C_{θ_0} is an upper bound for $b(\cos\theta)$ on $[\theta_0, \frac{\pi}{2}]$. Now we choose $\theta_0 > 0$ so small that

$$\varepsilon\left(\alpha, \cot^2 \frac{\theta_0}{2}\right) \leq \varepsilon(\alpha_{2,m}, 1) = \frac{2m}{2m+2}$$

and note that from Corollary 2.30, since $\nu > \alpha_{2,m}$, there exists a finite M_2 such that

$$\sup_{0<\theta\leq\frac{\pi}{2}} \sup_{0<|\eta|\leq\Lambda} \int_{\mathbb{S}^{d-2}(\eta)} G(\eta^-(\eta,\theta,\omega))^{\varepsilon(\alpha_{2,m},1)} |\hat{f}(\eta^-(\eta,\theta,\omega))|$$
$$\times \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-(\eta,\theta,\omega)|)\, \mathrm{d}\omega \leq M_2 < \infty.$$

So from (51) we get the bound

$$I_{d,\Lambda,1} \leq \alpha\beta T_0 M_2 c_{b,d,2} \|G_\Lambda f\|^2_{H^\alpha(\mathbb{R}^d)} \tag{53}$$

where the finiteness of $c_{b,d,2}$ follows from the singularity condition and the boundedness of $b(\cos\theta)$ away from $\theta = 0$.

For the integral $I^+_{d,\Lambda}$ from Lemma 2.10, a completely analogous reasoning as above shows for small enough ϑ_0 such that $\varepsilon(\alpha, \cot^2\vartheta) \leq \varepsilon(\alpha_{2,m}, 1)$ we also have

$$I^+_{d,\Lambda} \leq I^+_{d,\Lambda,1} + I^+_{d,\Lambda,2}$$

with

$$I^+_{d,\Lambda,1} \leq 2^{d-1}\alpha\beta T_0 M_2 c_{b,d,2} \|G_\Lambda f\|^2_{H^\alpha(\mathbb{R}^d)} \tag{54}$$

and

$$\begin{aligned} I^+_{d,\Lambda,2} := {} & 2^d C_{\vartheta_0}\alpha\beta T_0 \, \|G_\Lambda f\|^2_{H^\alpha(\mathbb{R}^d)} \\ & \times \Bigg(\sup_{0<|\eta^+|\leq\Lambda} \int_{\vartheta_0}^{\frac{\pi}{4}} \int_{\mathbb{S}^{d-2}(\eta^+)} G(\eta^-(\eta^+, \vartheta, \omega))^{\varepsilon(\alpha,1)} \\ & \times |\hat{f}(\eta^-(\eta^+, \vartheta, \omega))| \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-(\eta^+, \vartheta, \omega)|) \, \mathrm{d}\omega \, \mathrm{d}\vartheta \Bigg), \end{aligned} \tag{55}$$

where we use the parametrization (28) for $\eta^- = \eta^-(\eta^+, \vartheta, \omega)$ and where C_{ϑ_0} is an upper bound for $b(\cos(2\vartheta))$ on $[\vartheta_0, \frac{\pi}{4}]$.

Recall that we always assume $\alpha \leq \alpha_{1,m}$, so $\varepsilon(\alpha, 1) \leq \varepsilon(\alpha_{1,m}, 1) = \frac{2m}{2m+1}$. Thus we see that in order to set up our inductive procedure for controlling $I_{d,\Lambda}$ and $I^+_{d,\Lambda}$ it is natural to introduce

Definition 2.37 (Hypothesis $\mathrm{Hyp3}_\Lambda(M)$). Let $M \geq 0$ be finite, $0 < \theta_0, \vartheta_0 < \frac{\pi}{4}$, $T_0 > 0$, and $m \geq 2$ an integer. Then for all $0 \leq t \leq T_0$ one has

$$\begin{aligned} \sup_{|\eta|\leq\sqrt{2}\Lambda} \int_{\theta_0}^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta)} & G\left(t, \eta^-(\eta, \theta, \omega)\right)^{\frac{2m}{2m+1}} \left|\hat{f}\left(\eta^-(\eta, \theta, \omega)\right)\right| \\ & \times \mathbb{1}_\Lambda(|\eta^-(\eta, \theta, \omega)|) \, \mathrm{d}\omega \, \mathrm{d}\theta \leqq M, \end{aligned} \tag{56}$$

where we use the parametrization given in (25) for η^-, and

$$\begin{aligned} \sup_{|\eta^+|\leq\sqrt{2}\Lambda} \int_{\vartheta_0}^{\frac{\pi}{4}} \int_{\mathbb{S}^{d-2}(\eta^+)} & G\left(t, \eta^-(\eta^+, \vartheta, \omega)\right)^{\frac{2m}{2m+1}} \left|\hat{f}\left(\eta^-(\eta^+, \vartheta, \omega)\right)\right| \\ & \times \mathbb{1}_\Lambda(|\eta^-(\eta^+, \vartheta, \omega)|) \, \mathrm{d}\omega \, \mathrm{d}\vartheta \leqq M, \end{aligned} \tag{57}$$

where we use the parametrization given in (28) for η^-.

For the induction proof of Theorem 2.34, we again start with

Lemma 2.38. *Let $M \geq 0$, $T_0 > 0$, $m \geq 2$ an integer, $\alpha_{m,2} < \nu < 1$, $0 < \alpha \leqq \nu$ and recall $c_{b,d,2} = \int_0^{\frac{\pi}{2}} \sin^d \theta b(\cos\theta)\,\mathrm{d}\theta$ (which is finite by the singularity assumption (4) and the boundedness assumption (16)). Let M_2 be from Corollary 2.30 and $\beta \leq \frac{\tilde{C}_{f_0}}{\alpha T_0[(1+2^{d-1})c_{b,d,2}M_2+(C_{\theta_0}+2^d C_{\vartheta_0})M]+1}$. Then for any weak solution of the homogenous Boltzmann equation,*

$$\mathrm{Hyp3}_\Lambda(M) \quad \Rightarrow \quad \|G_{\sqrt{2}\Lambda} f\|_{L^2(\mathbb{R}^d)} \leq \|\mathbb{1}_{\sqrt{2}\Lambda}(D_v) f_0\|_{L^2(\mathbb{R}^d)}\, e^{C_{f_0} T_0} \tag{58}$$

for all $0 \leq t \leq T_0$.

Proof. Given Lemma 2.10 and the above discussion with the bounds in (53), (54) and using the hypothesis $\mathrm{Hyp3}_\Lambda$ for the terms in (52) and (55), one sees that the commutation error on the level $\sqrt{2}\Lambda$ is bounded by

$$\begin{aligned}
&\left|\left\langle Q(f, G_{\sqrt{2}\Lambda} f) - G_{\sqrt{2}\Lambda} Q(f,f), G_{\sqrt{2}\Lambda} f\right\rangle\right| \\
&\leqq I_{d,\sqrt{2}\Lambda} + I^+_{d,\sqrt{2}\Lambda} \\
&\leq (1+2^{d-1})\alpha\beta T_0 M_2 c_{b,d,2} \|G_\Lambda f\|^2_{H^\alpha(\mathbb{R}^d)} \\
&\quad + (C_{\theta_0} + 2^d C_{\vartheta_0})\alpha\beta T_0 M \|G_\Lambda f\|^2_{H^\alpha(\mathbb{R}^d)}.
\end{aligned}$$

Given this bound on the commutation error, the rest of the proof is the same as in the proof of Lemma 2.24. □

To close the induction step we also need a suitable version of Lemma 2.33 but before we prove this we need a preparatory Lemma.

Lemma 2.39. *Let $H : \mathbb{R}^d \to \mathbb{R}_+$ be a locally integrable function and let $\eta, \eta_+ \in \mathbb{R}^d$ with $|\eta|, |\eta^+| \geq \Lambda_0 > 0$, $0 < \theta_0 \leq \frac{\pi}{2}$, and $0 < \vartheta_0 \leq \frac{\pi}{8}$. Then with the parametrization $\eta^- = \eta^-(\eta, \theta, \omega)$ given in (25) one has*

$$\begin{aligned}
&\int_{\theta_0}^{\frac{\pi}{2}} \int_0^2 H\left(\eta^-(\eta,\theta,\omega) + z\tfrac{\eta}{|\eta|}\right) \mathrm{d}z\,\mathrm{d}\theta \\
&\leqq \frac{2}{\Lambda_0 \cos\theta_0} \int_{\Lambda_0 \sin^2\frac{\theta_0}{2}}^{\frac{|\eta|}{2}+2} \int_{\Lambda_0 \sin\theta_0}^{\frac{|\eta|}{2}} H\left(x\tfrac{\eta}{|\eta|} - y\omega\right) \mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

for any unit vector ω orthogonal to η.

Moreover, with the parametrization $\eta^- = \eta^-(\eta^+, \theta, \omega)$ given in (28) one has, for any $\widetilde{\Lambda} \geq \frac{1+\sqrt{2}}{2}\Lambda_0$,

$$\begin{aligned}
&\int_{\vartheta_0}^{\frac{\pi}{4}} \int_0^2 H\left(\eta^-(\eta^+,\vartheta,\omega) + z\tfrac{\eta}{|\eta|}\right) \mathbb{1}_{\frac{\widetilde{\Lambda}}{\sqrt{2}}}(|\eta^-(\eta^+,\vartheta,\omega)|)\,\mathrm{d}z\,\mathrm{d}\vartheta \\
&\leq \frac{1}{2\Lambda_0} \int_0^2 \int_{\Lambda_0 \tan\vartheta_0}^{\frac{\widetilde{\Lambda}}{\sqrt{2}}} H\left(x\tfrac{\eta}{|\eta|} - y\omega\right) \mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Remark 2.40. The restriction $\vartheta_0 \le \frac{\pi}{8}$ is only for convenience, to ensure that $\Lambda_0 \tan\vartheta_0 \le \frac{\widetilde{\Lambda}}{\sqrt{2}}$.

Proof. Fix η as required and ω orthogonal to it. We want to have a map $\Phi_1 : (\theta, z) \mapsto \Phi_1(\theta, z) = (x, y)$ such that

$$\eta^-(\eta, \theta, \omega) + z\tfrac{\eta}{|\eta|} = x\tfrac{\eta}{|\eta|} - y\omega.$$

From the parametrization (25) we read off

$$x = |\eta| \sin^2\frac{\theta}{2} + z \quad \text{and} \quad y = \frac{|\eta|}{2}\sin\theta$$

and we can compute the Jacobian going from the (θ, z) variables to (x, y) as

$$\left|\frac{\partial(x, y)}{\partial(\theta, z)}\right| = |\det D\Phi_1| = \frac{|\eta|}{2}\cos\theta \ge \frac{|\eta|}{2}\cos\theta_0.$$

Since $|\eta| \ge \Lambda_0$, $\theta \in [\theta_0, \frac{\pi}{2}]$, and $0 \le z \le 2$, we have $\Lambda_0 \sin^2\frac{\theta_0}{2} \le x \le |\eta|\sin^2\frac{\pi}{4} = \frac{\eta}{2}$ and $\frac{\Lambda_0}{2}\sin\theta_0 \le y \le \frac{\eta}{2}$. Thus, doing a change of variables $(\theta, z) = \Phi_1^{-1}(x, y)$ in the integral we can bound

$$\int_{\theta_0}^{\frac{\pi}{2}}\int_0^2 H\left(\eta^-(\eta, \theta, \omega) + z\tfrac{\eta}{|\eta|}\right) \mathrm{d}z\, \mathrm{d}\theta$$
$$\le \frac{2}{\Lambda_0\cos\theta_0}\int_{\Lambda_0\sin^2\frac{\theta_0}{2}}^{\frac{|\eta|}{2}+2}\int_{\Lambda_0\sin\theta_0}^{\frac{|\eta|}{2}} H\left(x\tfrac{\eta}{|\eta|} + y\omega\right) \mathrm{d}y\, \mathrm{d}x,$$

since the map Φ_1 is a nice diffeomorphism.

For the second bound the calculation is, in fact, a bit easier, one just has to take care that $|\eta^-|$ cannot be too large, which is taken into account by the factor $\mathbb{1}_\Lambda(|\eta^-|)$. We now want a map $\Phi_2 : (\theta, z) \mapsto \Phi_2(\theta, z) = (x, y)$ such that

$$\eta^-(\eta^+, \vartheta, \omega) + z\tfrac{\eta^+}{|\eta^+|} = x\tfrac{\eta^+}{|\eta^+|} - y\omega.$$

From the parametrization (25) we read off

$$x = z \quad \text{and} \quad y = |\eta^-| = |\eta^+|\tan\vartheta$$

and the Jacobian going from the (ϑ, z) variables to (x, y) is simply

$$\left|\frac{\partial(x, y)}{\partial(\vartheta, z)}\right| = |\det D\Phi_2| = 2|\eta^+| \ge 2\Lambda_0.$$

We certainly have $0 \le x \le 2$ and also $\Lambda_0\tan\vartheta_0 \le y$. Since $y = |\eta^-|$, we also have the restriction $y \le \Lambda$. So the proof of the second inequality follows similar to the proof of first one. □

Finally, we can state and prove the second step in our inductive procedure.

Lemma 2.41. *Let $\beta \leqq \frac{1}{T_0}$. Asssume that there exist finite constants A_m and B, such that*

$$\|f(t,\cdot)\|_{L^1_m} \leqq A_m, \quad \text{and} \quad \|(G_{\sqrt{2}\Lambda} f)(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leq B \tag{59}$$

for some integer $m \geqq 2$ and for all $0 \leq t \leq T_0$.

Set $\widetilde{\Lambda} := \frac{1+\sqrt{2}}{2}\Lambda$ and assume that

$$\Lambda \geq \Lambda_0 := 3. \tag{60}$$

Then there exist a finite K_3, depending only on d, m, A_m, and B such that $\mathrm{Hyp3}_{\widetilde{\Lambda}}(K_3)$ *is true.*

Proof. Fix $0 < t \leq T_0$, a direction $\eta \in \mathbb{R}^d \setminus \{0\}$, and define the function

$$z \mapsto F(z) := \hat{f}\left(t, \eta^- + z\frac{\eta}{|\eta|}\right)$$

of the single real variable z, where we think of η^- as given in the η-parametrization (25) for some θ and $\omega \in \mathbb{S}^{d-2}(\eta)$, and where we drop, for simplicity, the dependence on the time t in our notation for F and f. Then, since $\|f(t,\cdot)\|_{L^1_m} \leq A_m$ one has $\hat{f}(t,\cdot) \in \mathscr{C}^m(\mathbb{R}^d)$ and thus also $F \in \mathscr{C}^m(\mathbb{R})$ with $\|F\|_{L^\infty} \leqq A_m$, $\|\partial_z^m F\|_{L^\infty} \leqq (2\pi)^m A_m$, and Corollary 2.14 applied to F now gives

$$|\hat{f}(\eta^-)| \leq L_{m,1}\left(\int_0^2 |\hat{f}(\eta^- + z\tfrac{\eta}{|\eta|})|^2\, \mathrm{d}z\right)^{\frac{m}{2m+2}}.$$

We multiply this with the radially increasing weight G to get

$$G(\eta^-)^{\frac{2m}{2m+1}} |\hat{f}(\eta^-)| \leqq L_{m,1}\left(\int_0^2 |G(\eta^- + z\tfrac{\eta}{|\eta|})\hat{f}(\eta^- + z\tfrac{\eta}{|\eta|})|^2\, \mathrm{d}z\right)^{\frac{m}{2m+2}}.$$

Integrating this with respect to ω and θ, where we think of $\eta^- = \eta^-(\eta,\theta,\omega)$ in the parametrization (25), and using Jensen's inequality for concave functions, one gets

$$\begin{aligned}
&\int_{\theta_0}^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta)} G(\eta^-)^{\frac{2m}{2m+1}} |\hat{f}(\eta^-)|\, \mathrm{d}\theta\, \mathrm{d}\omega \\
&\leqq L_{m,1}(\tfrac{\pi}{2})^{\frac{m+1}{2m+1}} |\mathbb{S}^{d-2}|^{\frac{m+1}{2m+1}} \\
&\quad \times \left(\int_{\theta_0}^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta)} \int_0^2 |G(\eta^- + z\tfrac{\eta}{|\eta|})\hat{f}(\eta^- + z\tfrac{\eta}{|\eta|})|^2\, \mathrm{d}z\, \mathrm{d}\theta\, \mathrm{d}\omega\right)^{\frac{m}{2m+1}}.
\end{aligned} \tag{61}$$

Now assume that $|\eta| \geq \Lambda_0$. Because of the first part of Lemma 2.39, we can further bound

$$(61) \leq L_{m,1}(\tfrac{\pi}{2})^{\frac{m+1}{2m+1}} |\mathbb{S}^{d-2}|^{\frac{m+1}{2m+1}} \left(\frac{2}{\Lambda_0 \cos\theta_0}\right)^{\frac{m}{2m+1}}$$
$$\times \left(\int_{\mathbb{S}^{d-2}(\eta)} \int_{\Lambda_0 \sin^2\frac{\theta_0}{2}}^{\frac{|\eta|}{2}+2} \int_{\Lambda_0 \sin\theta_0}^{\frac{|\eta|}{2}} |G(x\tfrac{\eta}{|\eta|} - y\omega)\hat{f}(x\tfrac{\eta}{|\eta|} - y\omega)|^2 \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\omega\right)^{\frac{m}{2m+1}}$$
$$\leq L_{m,1}(\tfrac{\pi}{2})^{\frac{m+1}{2m+1}} |\mathbb{S}^{d-2}|^{\frac{m+1}{2m+1}} \left(\frac{2}{\Lambda_0 \cos\theta_0}\right)^{\frac{m}{2m+1}} (\Lambda_0 \sin\theta_0)^{2-d}$$
$$\times \left(\int_{\mathbb{S}^{d-2}(\eta)} \int_{\Lambda_0 \sin^2\frac{\theta_0}{2}}^{\frac{|\eta|}{2}+2} \int_{\Lambda_0 \sin\theta_0}^{\frac{|\eta|}{2}} |G(x\tfrac{\eta}{|\eta|} - y\omega)\hat{f}(x\tfrac{\eta}{|\eta|} - y\omega)|^2 \, y^{d-2}\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\omega\right)^{\frac{m}{2m+1}}.$$

Again, the integration measure $y^{d-2}\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\omega$ is d-dimensional Lebesgue measure in the cylindrical coordinates $(x, y\omega)$ with respect to the cylinder in the η direction. One checks that the condition $\Lambda \geq \Lambda_0 \geq 3$ ensures that

$$(\widetilde{\Lambda}/2+2)^2 + (\widetilde{\Lambda}/2) \leq (\sqrt{2}\Lambda)^2,$$

so since $|\eta| \leqq \widetilde{\Lambda}$, we can extend the integration above to a ball of radius $\sqrt{2}\Lambda$ to get

$$(61) \leq L_{m,1}(\tfrac{\pi}{2})^{\frac{m+1}{2m+1}} |\mathbb{S}^{d-2}|^{\frac{m+1}{2m+1}} \left(\frac{2}{\Lambda_0 \cos\theta_0}\right)^{\frac{m}{2m+1}} (\Lambda_0 \sin\theta_0)^{2-d} \|G_{\sqrt{2}\Lambda} f\|_{L^2(\mathbb{R}^d)}^{\frac{2m}{2m+1}}$$
$$\leq L_{m,1}(\tfrac{\pi}{2})^{\frac{m+1}{2m+1}} |\mathbb{S}^{d-2}|^{\frac{m+1}{2m+1}} \left(\frac{2}{\Lambda_0 \cos\theta_0}\right)^{\frac{m}{2m+1}} (\Lambda_0 \sin\theta_0)^{2-d} B^{\frac{2m}{2m+1}}. \quad (62)$$

If $|\eta| \leq \Lambda_0$ we simply bound

$$\int_{\theta_0}^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta)} G(\eta^-)^{\frac{2m}{2m+1}} |\hat{f}(\eta^-)| \,\mathrm{d}\theta\,\mathrm{d}\omega \leq \|\hat{f}\|_{L^\infty} \frac{\pi}{2} |\mathbb{S}^{d-2}| e^{\beta T_0(1+\Lambda_0^2/2)}$$
$$\leq A_m \frac{\pi}{2} |\mathbb{S}^{d-2}| e^{1+\Lambda_0^2/2}. \quad (63)$$

Concerning the bound in the second half of Hyp3$_{\widetilde{\Lambda}}$, a completely analogous calculation as the one above, using the second half of Lemma 2.39 gives for $\lambda_0 \leq |\eta^+| \leq \widetilde{\Lambda}$,

$$\int_{\vartheta_0}^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta^+)} G\left(t, \eta^-(\eta^+, \vartheta, \omega)\right)^{\frac{2m}{2m+1}} \left|\hat{f}\left(\eta^-(\eta^+, \vartheta, \omega)\right)\right|$$
$$\times \mathbb{1}_{\frac{\Lambda}{\sqrt{2}}}(|\eta^-(\eta^+, \vartheta, \omega)|)\,\mathrm{d}\omega\,\mathrm{d}\vartheta$$
$$\leqq L_{m,1}(\tfrac{\pi}{2})^{\frac{m+1}{2m+1}} |\mathbb{S}^{d-2}|^{\frac{m+1}{2m+1}} \left(\frac{1}{2\Lambda_0}\right)^{\frac{m}{2m+1}} (\Lambda_0 \tan\vartheta_0)^{2-d}$$
$$\times \left(\int_{\mathbb{S}^{d-2}(\eta^+)} \int_0^2 \int_0^{\frac{\widetilde{\Lambda}}{\sqrt{2}}} |G(x\tfrac{\eta}{|\eta|} - y\omega)\hat{f}(x\tfrac{\eta}{|\eta|} - y\omega)|^2 \, y^{d-2}\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\omega\right)^{\frac{m}{2m+1}}. \quad (64)$$

By our choice of $\widetilde{\Lambda}$ and Λ_0, we always have $2^2 + (\widetilde{\Lambda}/2)^2 \le (\sqrt{2}\Lambda)^2$, so we can extend the integration above to the whole ball $|\eta^+| \le \sqrt{2}\Lambda$ to see

$$(64) \le L_{m,1}(\tfrac{\pi}{2})^{\frac{m+1}{2m+1}} |\mathbb{S}^{d-2}|^{\frac{m+1}{2m+1}} \left(\frac{1}{2\Lambda_0}\right)^{\frac{m}{2m+1}} (\Lambda_0 \tan \vartheta_0)^{2-d} \|G_{\sqrt{2}\Lambda} f\|_{L^2(\mathbb{R}^d)}^{\frac{2m}{2m+1}}$$
$$\le L_{m,1}(\tfrac{\pi}{2})^{\frac{m+1}{2m+1}} |\mathbb{S}^{d-2}|^{\frac{m+1}{2m+1}} \left(\frac{1}{2\Lambda_0}\right)^{\frac{m}{2m+1}} (\Lambda_0 \tan \vartheta_0)^{2-d} B^{\frac{2m}{2m+1}}. \tag{65}$$

If $|\eta^+| \le \Lambda_0$ we simply bound as above

$$\int_{\vartheta_0}^{\frac{\pi}{4}} \int_{\mathbb{S}^{d-2}(\eta^+)} G(\eta^-)^{\frac{2m}{2m+1}} |\hat{f}(\eta^-)| \, \mathrm{d}\vartheta \, \mathrm{d}\omega \le A_m \frac{\pi}{4} |\mathbb{S}^{d-2}| e^{1+\Lambda_0^2}. \tag{66}$$

Now we set K_3 equal to the maximum of the constants in (62), (63), (65), (66). With this choice, K_3 depends only on d, m, A_m, and B and $\mathrm{Hyp3}_{\widetilde{\Lambda}}(K_3)$ is true. □

Proof of Theorem 2.34. In view of Lemmata 2.38 and 2.41 a suitable choice for A_m and B is

$$B := \|f_0\|_{L^2(\mathbb{R}^d)} e^{C_{f_0} T_0}, \quad A_m := \sup_{t \geqq 0} \|f(t, \cdot)\|_{L^1_m(\mathbb{R}^d)}.$$

The finiteness of A_m is guaranteed since $f_0 \in L^1_m(\mathbb{R}^d)$, see Remark 2.28. We again choose the length scales Λ_N to be

$$\Lambda_N := \frac{\Lambda_{N-1} + \sqrt{2}\Lambda_{N-1}}{2} = \frac{1+\sqrt{2}}{2}\Lambda_{N-1} = \left(\frac{1+\sqrt{2}}{2}\right)^N \Lambda_0$$

with $\Lambda_0 = 3$, see (60), and we set

$$M_3 := \max\left\{2|\mathbb{S}^{d-2}|A_m + 1, K_3\right\},$$

with the constant K_3 from Lemma 2.41. Since

$$\sup_{0\le t\le T_0} \sup_{|\eta|\le\sqrt{2}\Lambda} \int_{\theta_0}^{\frac{\pi}{2}} \int_{\mathbb{S}^{d-2}(\eta)} G\left(t, \eta^-(\eta, \theta\omega)\right)^{\frac{2m}{2m+1}} \left|\hat{f}\left(\eta^-(\eta, \theta\omega)\right)\right| \mathrm{d}\omega \, \mathrm{d}\theta$$
$$\le \frac{\pi}{2} |\mathbb{S}^{d-2}| e^{\frac{2m}{2m+1}\beta T_0 (1+\Lambda_0^2)^\alpha} A_m,$$

and similarly for the η^+ term, it follows from our choice of M_3 that there exists $\beta_0 > 0$ such that $\mathrm{Hyp3}_{\Lambda_0}(M_3)$ is true for all $0 \le \beta \le \beta_0$.

Now, we pick

$$\beta = \min\left(\beta_0, T_0^{-1}, \frac{\tilde{C}_{f_0}}{\alpha T_0 \left[\left(1+2^{d-1}\right) c_{b,d,2} M_2 + (C_{\theta_0} + 2^d C_{v_0}) M\right] + 1}\right).$$

with the constant M_2 from Corollary 2.30, so that the conditions of Lemma 2.38 and 2.41 are fulfilled.

For the induction step assume that $\mathrm{Hyp3}_{\Lambda_N}(M_3)$ is true. Lemma 2.38 then implies

$$\|G_{\sqrt{2}\Lambda_N} f\|_{L^2(\mathbb{R}^d)} \leq \|\mathbb{1}_{\sqrt{2}\Lambda_N}(D_v) f_0\|_{L^2(\mathbb{R}^d)} \, e^{C_{f_0} T_0} = B,$$

and Lemma 2.41 shows that $\mathrm{Hyp3}_{\Lambda_{N+1}}(M_3)$ is true.

It follows that $\mathrm{Hyp3}_{\Lambda_N}(M_3)$ is true for all $N \in \mathbb{N}$, and therefore also

$$\|G_{\sqrt{2}\Lambda_N} f\|_{L^2(\mathbb{R}^d)} \leqq B$$

for all $N \in \mathbb{N}$ In particular, letting $N \to \infty$, we see that $\|Gf\|_{L^2(\mathbb{R}^d)} \leqq B$, which concludes the proof of Theorem 2.34. □

3. Removing the L^2 Constraint: Gevrey Regularity and (Ultra-)Analyticity of Weak Solutions

In this section we will give the proofs of Theorems 1.6, 1.9, and 1.10 in a slightly more general form. More precisely, we will prove

Theorem 3.1. (Gevrey smoothing I) *Assume that the cross-section b satisfies the* singularity condition (3) *and the* integrability condition (4) *for $d \geqq 2$, and for $d = 1$, b_1 satisfies the* singularity condition (6) *and the* integrability condition (7) *for some $0 < \nu < 1$. Let f be a weak solution of the Cauchy problem* (1) *with initial datum $f_0 \geq 0$ and $f_0 \in L^1_m(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$ for some integer $m \geq 2$. Then, for all $0 < \alpha \leqq \min\{\alpha_{m,d}, \nu\}$,*

$$f(t, \cdot) \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d) \tag{67}$$

for all $t > 0$, where $\alpha_{m,d} = \frac{\log[(4m+d)/(2m+d)]}{\log 2}$.

Theorem 3.2. (Gevrey smoothing II) *Let $d \geq 2$. Assume that the cross-section b satisfies the conditions of Theorem 1.6. Let f be a weak solution of the Cauchy problem* (1) *with initial datum $f_0 \geq 0$ and $f_0 \in L^1_m(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d)$ for some integer $m \geq 2$. Then, for all $0 < \alpha \leqq \min\{\alpha_{m,2}, \nu\}$,*

$$f(t, \cdot) \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d) \tag{68}$$

for all $t > 0$, where $\alpha_{m,2} = \frac{\log[(4m+2)/(2m+2)]}{\log 2}$. In particular, the weak solution is real analytic if $\nu = \frac{1}{2}$ and ultra-analytic if $\nu > \frac{1}{2}$ in any dimension.

If the integrability condition (4) is replaced by the slightly stronger condition (16), which is true in all physically relevant cases, we can prove the stronger result

Theorem 3.3. (Gevrey smoothing III) *Let $d \geq 2$. Assume that the cross-section b satisfies the conditions of Theorem 1.6 and the condition* (16)*, that is, they are bounded away from the singularity. Let f be a weak solution of the Cauchy problem*

(1) *with initial datum* $f_0 \geq 0$ *and* $f_0 \in L^1_m(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$ *for some integer* $m \geq 2$*. Then, for all* $0 < \alpha \leqq \min\left\{\alpha_{m,1}, \nu\right\}$*,*

$$f(t,\cdot) \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d) \tag{69}$$

for all $t > 0$*, where* $\alpha_{m,1} = \frac{\log[(4m+1)/(2m+1)]}{\log 2}$.

We even have the uniform bound

Corollary 3.4. *Under the same assumptions as in Theorem 3.1 (or 3.2, respectively 3.3), for any weak solution* f *of the Cauchy problem (1) with initial datum* $f_0 \geq 0$ *and* $f_0 \in L^1_m(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$ *for some integer* $m \geq 2$ *and for any* $0 < \alpha \leqq \min\{\alpha_{d,m}, \nu\}$ *(or any* $0 < \alpha \leqq \min\{\alpha_{m,2}, \nu\}$*, respectively* $0 < \alpha \leqq \min\{\alpha_{m,1}, \nu\}$*) there exist constants* $0 < K, C < \infty$ *such that*

$$\sup_{0\leq t<\infty} \sup_{\eta\in\mathbb{R}^d} e^{K\min(t,1)\langle\eta\rangle^{2\alpha}} |\hat{f}(t,\eta)| \leqq C. \tag{70}$$

Proof of Theorems 3.1 through 3.3. In the case where the initial condition f_0 obeys $f_0 \geq 0$ and $f_0 \in L^1_m(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$ for some integer $m \geq 2$, but is not necessarily in $L^2(\mathbb{R}^d)$, we use the known H^∞ smoothing of the Boltzmann [4,23,39] and the Kac equation [30] in a mild way (see also Appendix B): for $\tau > 0$ one has $f(\tau,\cdot) \in L^2(\mathbb{R}^d)$ and using this as a new initial condition in Theorems 1.6 through 1.10, and noting that T_0 in those theorems is arbitrary, this implies that $f(t,\cdot) \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d)$ for $t > 0$.[5] □

Proof of Corollary 3.4. Using known results about propagation of Gevrey regularity by Desvillettes, Furioli, and Terraneo [21] for the non-cutoff homogeneous Boltzmann and Kac equation for Maxwellian molecules, the bounds from Corollary 2.20 through 2.36 extend to all times. □

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[5] A H^∞ smoothing effect for the homogeneous non-cutoff Kac equation was first proved by L. Desvillettes [17], but under the stronger assumption that all polynomial moments of the initial datum f_0 are bounded, i.e. $f_0 \in L^1_k(\mathbb{R}) \cap L\log L(\mathbb{R})$ for all $k \in \mathbb{N}$.

A L^2 Type Reformulation of the Boltzmann and Kac Equations

A reformulation of the weak form (9) of the Boltzmann and Kac equations is derived. We want to choose a suitable test function φ in terms of the weak solution f itself in the weak formulation of the Cauchy problem (1). We use $\varphi(t,\cdot) := G_\Lambda^2(t, D_v) f(t,\cdot)$ and since this involves a hard cut-off in Fourier space, we automatically have high regularity of $\varphi(t, v)$ in the velocity variable, the question is to have $\mathscr{C}^1$ regularity in the time variable. For this we follow the strategy by MORIMOTO ET AL. [39].

Proposition A.1. *Let f be a weak solution of the Cauchy problem* (1) *with initial datum f_0 satisfying* (8)*, and let $T_0 > 0$. Then for all $t \in (0, T_0]$, $\beta > 0$, $\alpha \in (0, 1)$, and $\Lambda > 0$ we have $G_\Lambda f \in \mathscr{C}\left([0, T_0]; L^2(\mathbb{R}^d)\right)$ and*

$$\begin{aligned} &\frac{1}{2}\|G_\Lambda(t, D_v) f(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{2}\int_0^t \left\langle f(\tau,\cdot), \left(\partial_t G^2_\Lambda(\tau, D_v)\right) f(\tau,\cdot)\right\rangle \mathrm{d}\tau \\ &= \frac{1}{2}\|\mathbb{1}_\Lambda(D_v) f_0\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \left\langle Q(f, f)(\tau,\cdot), G^2_\Lambda(\tau, D_v) f(\tau,\cdot)\right\rangle \mathrm{d}\tau. \end{aligned} \tag{71}$$

To ensure that we can use $G_\Lambda^2 f$ as a test function in the weak formulation of the Boltzmann equation, we need the following bilinear estimate on $Q(g, f)$, which is a special case of a larger class of functional inequalities by ALEXANDRE [1,2,6].

Lemma A.2. (Functional Estimate on Collision Operator) *Assume that the angular collision cross-section b satisfies assumptions* (3)–(4) *or* (6)–(7)*, respectively. Then for any $k > \frac{d+4}{2}$ there exists a constant $C > 0$ such that*

$$\|Q(g, f)\|_{H^{-k}(\mathbb{R}^d)} \leqq C\|g\|_{L^1_2(\mathbb{R}^d)}\|f\|_{L^1_2(\mathbb{R}^d)}. \tag{72}$$

Proof. This is a direct consequence of Theorem 7.4 in ALEXANDRE's review [2]: under the assumptions on b, for any $m \in \mathbb{R}$ there exists a constant $\widetilde{C} > 0$ such that [6]

$$\|Q(g, f)\|_{H^{-m}(\mathbb{R}^d)} \leqq \widetilde{C}\|g\|_{L^1_{2\nu}(\mathbb{R}^d)}\|f\|_{H^{-m+2\nu}_{2\nu}(\mathbb{R}^d)}.$$

Since $L^1(\mathbb{R}^d) \subset H^{-s}(\mathbb{R}^d)$ for any $s > \frac{d}{2}$, we obtain for $k > \frac{d+4}{2}$ and $\nu \in (0, 1)$,

$$\begin{aligned} \|f\|_{H^{-k+2\nu}_{2\nu}(\mathbb{R}^d)} &= \|\langle\cdot\rangle^{2\nu} f\|_{H^{-k+2\nu}(\mathbb{R}^d)} \leqq C\|\langle\cdot\rangle^{2\nu} f\|_{L^1(\mathbb{R}^d)} \\ &\leqq c\|\langle\cdot\rangle^2 f\|_{L^1(\mathbb{R}^d)} = c\|f\|_{L^1_2(\mathbb{R}^d)}, \end{aligned}$$

i.e., $L^1_2(\mathbb{R}^d) \subset H^{-k+2\nu}_{2\nu}(\mathbb{R}^d)$ for any $k > \frac{d+4}{2}$ and $\nu \in (0, 1)$. Therefore,

$$\|Q(g, f)\|_{H^{-k}(\mathbb{R}^d)} \leqq \widetilde{C}\|g\|_{L^1_{2\nu}(\mathbb{R}^d)}\|f\|_{H^{-k+2\nu}_{2\nu}(\mathbb{R}^d)} \leqq C\|g\|_{L^1_2(\mathbb{R}^d)}\|f\|_{L^1_2(\mathbb{R}^d)}.$$

[6] This result is proved in [2] for $d = 3$, but the proof depends only on assumption (3) and general properties of Littlewood–Paley decompositions and holds in any dimension $d \geq 1$.

Lemma A.2 implies that for $f, g \in L^1_2(\mathbb{R}^d)$, $\langle Q(g, f), h\rangle$ is well-defined for all $h \in H^k(\mathbb{R}^d)$, $k > \frac{d+4}{2}$, and one has $\langle Q(g, f), h\rangle = \langle \widehat{Q(g, f)}, \widehat{h}\,\rangle_{L^2}$.

Proof of Proposition A.1. Choosing a constant in time test function $\varphi(t, \cdot) = \psi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ in the weak formulation (9) yields

$$\int_{\mathbb{R}^d} f(t, v)\psi(v)\,\mathrm{d}v - \int_{\mathbb{R}^d} f(s, v)\psi(v)\,\mathrm{d}v = \int_s^t \langle Q(f, f)(\tau, \cdot), \psi\rangle\,\mathrm{d}\tau,$$

for $0 \leqq s \leqq t \leqq T_0$ for all $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ (this was already remarked by Villani [48] as an equivalent formulation of (9)). By means of (72) this equality can be extended to test functions $\psi \in H^k$ for $k > \frac{d+4}{2}$, in particular one can choose $\psi = G_\Lambda^2 f(t, \cdot)$ and $\psi = G_\Lambda^2 f(s, \cdot)$, which, taking the sum of both resulting equations, yields

$$\begin{aligned}
&\|G_\Lambda f(t, \cdot)\|^2_{L^2(\mathbb{R}^d)} - \|G_\Lambda f(s, \cdot)\|^2_{L^2(\mathbb{R}^d)} \\
&\quad = \Big\langle f(t, \cdot), G_\Lambda^2 f(t, \cdot)\Big\rangle - \Big\langle f(s, \cdot), G_\Lambda^2 f(s, \cdot)\Big\rangle \\
&\quad = \Big\langle f(t, \cdot), \Big(G_\Lambda^2(t, D_v) - G_\Lambda^2(s, D_v)\Big) f(s, \cdot)\Big\rangle \\
&\qquad + \int_s^t \Big\langle Q(f, f)(\tau, \cdot), G_\Lambda^2 f(t, \cdot) + G_\Lambda^2 f(s, \cdot)\Big\rangle\,\mathrm{d}\tau.
\end{aligned} \tag{73}$$

Using Plancherel, the first term on the right hand side of (73) can be estimated by

$$\begin{aligned}
&\Big|\Big\langle f(t, \cdot), \Big(G_\Lambda^2(t, D_v) - G_\Lambda^2(s, D_v)\Big) f(s, \cdot)\Big\rangle\Big| \\
&\quad = \Big|\Big\langle \hat{f}(t, \cdot), \Big(G_\Lambda^2(t, \cdot) - G_\Lambda^2(s, \cdot)\Big) \hat{f}(s, \cdot)\Big\rangle\Big| \\
&\quad \leqq \int_{\mathbb{R}^d} |\hat{f}(t, \eta)|\,|G_\Lambda^2(t, \eta) - G_\Lambda^2(s, \eta)|\,|\hat{f}(s, \eta)|\,\mathrm{d}\eta \\
&\quad \leqq |t - s| \int_{\mathbb{R}^d} 2\beta\langle\eta\rangle^{2\alpha} G_\Lambda^2(t, \eta)\,\mathrm{d}\eta\,\|f(t, \cdot)\|_{L^1(\mathbb{R}^d)}\|f(s, \cdot)\|_{L^1(\mathbb{R}^d)} \\
&\quad \leqq C_{\Lambda, T_0}|t - s|\,\|f_0\|^2_{L^1(\mathbb{R}^d)},
\end{aligned}$$

and, using that the terms involving the collision operator can, for any $k > \frac{d+4}{2}$ (compare (72)), be bounded by

$$\begin{aligned}
&|\langle Q(f, f)(\tau, \cdot), G_\Lambda^2 f(t, \cdot)\rangle| \\
&\quad \leqq \|Q(f, f)(\tau, \cdot)\|_{H^{-k}(\mathbb{R}^d)}\|G_\Lambda^2 f(t, \cdot)\|_{H^k(\mathbb{R}^d)} \\
&\quad \leqq C\|f\|^2_{L^1_2(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}\langle\eta\rangle^{2k} G_\Lambda^4(t, \eta)|\hat{f}(t, \eta)|^2\,\mathrm{d}\eta\right)^{1/2} \\
&\quad \leqq C\|f\|^2_{L^1_2(\mathbb{R}^d)}\|f(t, \cdot)\|_{L^1(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}\langle\eta\rangle^{2k} G_\Lambda^4(T_0, \eta)\,\mathrm{d}\eta\right)^{1/2} \\
&\quad \leqq C'_{\Lambda, T_0}\|f_0\|^2_{L^1_2(\mathbb{R}^d)}\|f_0\|_{L^1(\mathbb{R}^d)}
\end{aligned}$$

for any $t \in [0, T_0]$, yielding

$$\left| \int_s^t \langle Q(f, f)(\tau, \cdot), G_\Lambda^2 f(t, \cdot) + G_\Lambda^2 f(s, \cdot) \rangle \, \mathrm{d}\tau \right| \leqq 2C'_{\Lambda, T_0} |t - s| \, \|f_0\|^2_{L^1_2(\mathbb{R}^d)} \|f_0\|_{L^1(\mathbb{R}^d)}.$$

Plugging the latter two bounds into (73) shows that $G_\Lambda f \in \mathscr{C}([0, T_0]; L^2(\mathbb{R}^d))$, in fact, the map $[0, T_0] \ni t \mapsto \|G_\Lambda f(t, \cdot)\|_{L^2(\mathbb{R}^d)}$ is even Lipschitz continuous. For any test function $\varphi \in \mathscr{C}^1(\mathbb{R}^+; \mathscr{C}_0^\infty(\mathbb{R}^d))$ the term involving the partial derivative $\partial_t \varphi$ in the weak formulation (9) can be rewritten as

$$\int_0^t \langle f(\tau, \cdot), \partial_\tau \varphi(\tau, \cdot) \rangle \, \mathrm{d}\tau = \lim_{h \to 0} \int_0^t \left\langle f(\tau, \cdot) + f(\tau + h, \cdot), \frac{\varphi(\tau + h, \cdot) - \varphi(\tau, \cdot)}{2h} \right\rangle \mathrm{d}\tau,$$

since $f \in \mathscr{C}(\mathbb{R}^+; \mathscr{D}'(\mathbb{R}^d))$. The integral on the right hand side is well-defined even for $\varphi \in L^\infty([0, T_0]; W^{2,\infty}(\mathbb{R}^d))$, in particular for $\varphi = G_\Lambda^2 f$, yielding

$$\begin{aligned}
&\int_0^t \left\langle f(\tau, \cdot) + f(\tau + h, \cdot), \frac{\varphi(\tau + h, \cdot) - \varphi(\tau, \cdot)}{2h} \right\rangle \mathrm{d}\tau \\
&= \int_0^t \left\langle f(\tau, \cdot) + f(\tau + h, \cdot), \frac{G_\Lambda^2 f(\tau + h, \cdot) - G_\Lambda^2 f(\tau, \cdot)}{2h} \right\rangle \mathrm{d}\tau \\
&= \frac{1}{2h} \int_0^t \left(\|G_\Lambda f(\tau + h, \cdot)\|^2_{L^2} - \|G_\Lambda f(\tau, \cdot)\|^2_{L^2} \right) \mathrm{d}\tau \\
&\quad + \int_0^t \left\langle f(\tau, \cdot), \frac{G_\Lambda^2(\tau + h, D_v) - G_\Lambda^2(\tau, D_v)}{2h} f(\tau + h, \cdot) \right\rangle \mathrm{d}\tau.
\end{aligned}$$

Using $G_\Lambda f \in \mathscr{C}([0, T_0]; L^2(\mathbb{R}^d))$ it follows that

$$\begin{aligned}
&\frac{1}{2h} \int_0^t \left(\|G_\Lambda f(\tau + h, \cdot)\|^2_{L^2(\mathbb{R}^d)} - \|G_\Lambda f(\tau, \cdot)\|^2_{L^2(\mathbb{R}^d)} \right) \mathrm{d}\tau \\
&= \frac{1}{2h} \int_t^{t+h} \|G_\Lambda f(\tau, \cdot)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}\tau - \frac{1}{2h} \int_0^h \|G_\Lambda f(\tau, \cdot)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}\tau \\
&\xrightarrow{h \to 0} \frac{1}{2} \|G_\Lambda f(t, \cdot)\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{2} \|G_\Lambda f(0, \cdot)\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where $\|G_\Lambda f(0, \cdot)\|_{L^2(\mathbb{R}^d)} = \|\mathbb{1}_\Lambda(D_v) f_0\|_{L^2(\mathbb{R}^d)}$. For the second integral, an application of dominated convergence gives

$$\begin{aligned}
&\lim_{h \to 0} \int_0^t \left\langle f(\tau, \cdot), \frac{G_\Lambda^2(\tau + h, D_v) - G_\Lambda^2(\tau, D_v)}{2h} f(\tau + h, \cdot) \right\rangle \mathrm{d}\tau \\
&= \frac{1}{2} \int_0^t \left\langle f(\tau, \cdot), \left(\partial_\tau G_\Lambda^2 \right) (\tau, D_v) f(\tau, \cdot) \right\rangle \mathrm{d}\tau.
\end{aligned}$$

Putting everything together, we thus have proved equation (71), i.e.

$$\frac{1}{2}\|G_\Lambda f\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{2}\|\mathbb{1}_\Lambda(D_v) f_0\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\int_0^t \left\langle f(\tau,\cdot), \left(\partial_\tau G^2_\Lambda\right)(\tau, D_v) f(\tau,\cdot)\right\rangle \mathrm{d}\tau + \int_0^t \left\langle Q(f,f), G^2_\Lambda f\right\rangle \mathrm{d}\tau.$$

□

B H^∞ Smoothing of the Boltzmann an Kac Equations

We follow the strategy as in our proof of Gevrey regularity, with several simplifications. Of course, we *do not* assume that f_0 is square integrable! We have

Theorem B.1. (H^∞ smoothing for the homogeneous Boltzmann and Kac equation) *Assume that the cross-section b satisfies* (3)–(4) *for $d \geqq 2$, respectively* (6)–(7) *for $d = 1$, with $0 < \nu < 1$. Let f be a weak solution of the Cauchy problem* (1) *with initial datum satisfying conditions* (8)*. Then*

$$f(t,\cdot) \in H^\infty(\mathbb{R}^d) \tag{74}$$

for all $t > 0$.

The proof is known, at least for the three dimensional Boltzmann equation see [39], we give a proof for the convenience of the reader. Again, one has to use suitable time-dependent Fourier multipliers. Note that for $f_0 \in L^1(\mathbb{R}^d)$ one has

$$\|f_0\|_{H^{-\gamma}(\mathbb{R}^d)} \leq C_{d,\gamma}\|f_0\|_{L^1(\mathbb{R}^d)}$$

with $C_{d,\gamma} = \left(\int_{\mathbb{R}^d} \langle\eta\rangle^{-\gamma}\,\mathrm{d}\eta\right)^{1/2}$ which is finite for all $\gamma > d/2$. We choose $\gamma = d$, for convenience, and

$$M_\Lambda(t,\eta) := \langle\eta\rangle^{-d} e^{\beta t \log\langle\eta\rangle}\mathbb{1}_\Lambda(|\eta|)$$

as a multiplier. Then

$$\begin{aligned}\sup_{\Lambda>0}\|M_\Lambda(0,D_v) f_0\|_{L^2(\mathbb{R}^d)} &= \|M_\infty(0,\cdot)\hat{f}_0\|_{L^2(\mathbb{R}^d)} \\ &= \|f_0\|_{H^{-d}(\mathbb{R}^d)} \leq C_{d,d}\|f_0\|_{L^1(\mathbb{R}^d)}\end{aligned}$$

The proof of Proposition A.1 carries over and we have

$$\begin{aligned}&\frac{1}{2}\|M_\Lambda(t,D_v) f(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{2}\int_0^t \left\langle f(\tau,\cdot), \left(\partial_\tau M^2_\Lambda(\tau,D_v)\right) f(\tau,\cdot)\right\rangle \mathrm{d}\tau \\ &= \frac{1}{2}\|M_\Lambda(0,D_v) f_0\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \left\langle Q(f,f)(\tau,\cdot), M^2_\Lambda(\tau,D_v) f(\tau,\cdot)\right\rangle \mathrm{d}\tau,\end{aligned} \tag{75}$$

and as in the proof of Corollary 2.4, we have

$$\begin{aligned}\langle Q(f,f), M_\Lambda^2 f\rangle &= \langle Q(f, M_\Lambda f), M_\Lambda f\rangle + \langle M_\Lambda Q(f,f) - Q(f, M_\Lambda f), M_\Lambda f\rangle \\ &\leqq -\widetilde{C}_{f_0}\|M_\Lambda f\|_{H^\nu}^2 + C_{f_0}\|M_\Lambda f\|_{L^2}^2 + \langle M_\Lambda Q(f,f) - Q(f, M_\Lambda f), M_\Lambda f\rangle.\end{aligned} \tag{76}$$

The replacement of Proposition 2.8 is

Proposition B.2. *The commutation error is bounded by*

$$\begin{aligned}&|\langle M_\Lambda Q(f,f) - Q(f, M_\Lambda f), M_\Lambda f\rangle| \\ &\quad\leqq (1+2^{d-1}) c_{b,d} \|f\|_{L^1} \left(\frac{d}{2} + \frac{\beta t}{2} 2^{\beta t/2}\right) \|M_\Lambda f\|_{L^2}^2\end{aligned} \tag{77}$$

with the constant $c_{b,d}$ from Lemma 2.24.

Remark B.3. Of course, for any weak solution f of the Boltzmann and Kac equations,

$$\|f\|_{L^1} = \|f(t,\cdot)\|_{L^1} = \|f_0\|_{L^1}.$$

The fact that the commutator is bounded in terms of the L^2 norm of $M_\Lambda f$ makes the proof of H^∞ smoothing for the Boltzmann and Kac equations much simpler than the proof of Gevrey regularity.

Proof. As in the proof of Proposition 2.8, Bobylev's formula shows

$$\begin{aligned}&|\langle M_\Lambda Q(f,f) - Q(f, M_\Lambda f), M_\Lambda f\rangle| \\ &\quad\leq \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(\frac{\eta}{|\eta|}\cdot\sigma\right) M_\Lambda(\eta)|\hat{f}(\eta)||\hat{f}(\eta^-)||\hat{f}(\eta^+)| \\ &\qquad\qquad \times |M_\Lambda(t,\eta) - M_\Lambda(t,\eta^+)|\,\mathrm{d}\sigma\,\mathrm{d}\eta \\ &\quad\leq \|\hat{f}\|_{L^\infty}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(\frac{\eta}{|\eta|}\cdot\sigma\right) M_\Lambda(\eta)|\hat{f}(\eta)||\hat{f}(\eta^+)| \\ &\qquad\qquad \times |M_\Lambda(t,\eta) - M_\Lambda(t,\eta^+)|\,\mathrm{d}\sigma\,\mathrm{d}\eta,\end{aligned} \tag{78}$$

where, as before, $\eta^\pm = \frac{1}{2}(\eta \pm |\eta|\sigma)$. To bound $|M_\Lambda(\eta) - M_\Lambda(\eta^+)|$, we let $s := |\eta|^2$ and $s^+ = |\eta^+|^2$. Recall that $|\eta^+|^2 = \frac{|\eta|^2}{2}(1 + \frac{\eta}{|\eta|}\cdot\sigma)$ and

$$1 - \frac{s^+}{s} = 1 - \frac{|\eta^+|^2}{|\eta|^2} = \frac{1}{2}\left(1 - \frac{\eta}{|\eta|}\cdot\sigma\right).$$

Again, because of the support condition on the collision kernel $b(\cos\theta)$, we have $\frac{s}{2} \leqq s^+ \leqq s$. Set $\widetilde{M}(s) := (1+s)^{-d/2} e^{\frac{\beta t}{2}\log(1+s)}$. Then, for $|\eta| \leq \Lambda$,

$$\begin{aligned}M_\Lambda(\eta) - M_\Lambda(\eta^+) &= \widetilde{M}(s) - \widetilde{M}(s^+) \\ &= (1+s)^{-d/2} e^{\frac{\beta t}{2}\log(1+s)} - (1+s^+)^{-d/2} e^{\frac{\beta t}{2}\log(1+s^+)} \\ &= (1+s)^{-d/2}\left(e^{\frac{\beta t}{2}\log(1+s)} - e^{\frac{\beta t}{2}\log(1+s^+)}\right) \\ &\quad + \left((1+s)^{-d/2} - (1+s^+)^{-d/2}\right) e^{\frac{\beta t}{2}\log(1+s^+)}.\end{aligned} \tag{79}$$

Since $s \leq 2s^+$, we have $(1+s^+)^{-1} \leq 2(1+s)^{-1}$. Hence

$$\begin{aligned}\left|(1+s)^{-d/2} - (1+s^+)^{-d/2}\right| &= \frac{d}{2}\int_{s^+}^{s} (1+r)^{-d/2-1}\,\mathrm{d}r \\ &\leq \frac{d}{2}(1+s^+)^{-d/2-1}(s-s^+) \\ &\leq d(1+s^+)^{-d/2}\Big(1-\frac{s^+}{s}\Big).\end{aligned}$$

In addition, $\log(1+s) \leq \log(2(1+s^+)) = \log 2 + \log(1+s^+)$. So

$$\begin{aligned}&\left|e^{\frac{\beta t}{2}\log(1+s)} - e^{\frac{\beta t}{2}\log(1+s^+)}\right| \leq \frac{\beta t}{2}\int_{s^+}^{s}\frac{1}{1+r}e^{\frac{\beta t}{2}\log(1+r)}\,\mathrm{d}r \\ &\leq \frac{\beta t}{2}\frac{s}{1+s^+}e^{\frac{\beta t}{2}\log(1+s)}\Big(1-\frac{s^+}{s}\Big) \leq \beta t 2^{\frac{\beta t}{2}} e^{\frac{\beta t}{2}\log(1+s^+)}\Big(1-\frac{s^+}{s}\Big).\end{aligned}$$

Also $\log(1+s) \leq \log(2(1+s^+)) = \log 2 + \log(1+s^+)$. These bounds together with (79) show

$$\left|M_\Lambda(\eta) - M_\Lambda(\eta^+)\right| \leq \left(d + \beta t\, 2^{\frac{\beta t}{2}}\right)\left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right) M_\Lambda(\eta^+)$$

for all $|\eta| \leq \Lambda$. Since the integration in (78) is only over $|\eta| \leq \Lambda$, plugging this together with $\|\hat{f}\|_{L^\infty} \leqq \|f\|_{L^1}$ into (78) yields

$$\begin{aligned}&|\langle M_\Lambda Q(f,f) - Q(f, M_\Lambda f), M_\Lambda f\rangle| \\ &\quad\leq \|f\|_{L^1}\left(d + \beta t\, 2^{\frac{\beta t}{2}}\right)\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(\frac{\eta}{|\eta|}\cdot\sigma\right)\left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right) \\ &\qquad\qquad \times M_\Lambda(\eta)|\hat{f}(\eta)|\, M_\Lambda(\eta^+)|\hat{f}(\eta^+)|\,\mathrm{d}\sigma\,\mathrm{d}\eta.\end{aligned}$$

Noting again

$$M_\Lambda(\eta)|\hat{f}(\eta)|\, M_\Lambda(\eta^+)|\hat{f}(\eta^+)| \leqq \frac{1}{2}\left((M_\Lambda(\eta)|\hat{f}(\eta)|)^2 + (M_\Lambda(\eta^+)|\hat{f}(\eta^+)|)^2\right)$$

and performing the same change of variables for the integral containing η^+ as in the proof of Lemma 2.10 finishes the proof of equation (77). □

Now we can finish the

Proof of Theorem B.1. Using (75), (76), Proposition B.2, and

$$\partial_\tau M_\Lambda(\tau,\eta)^2 = 2\beta\log\langle\eta\rangle\, M_\Lambda(\tau,\eta)^2,$$

one sees that

$$\|M_\Lambda(t, D_v)f(t,\cdot)\|_{L^2}^2 \le \|f_0\|_{H^{-d}}^2 + 2C_{f_0}\int_0^t \|M_\Lambda(\tau, D_v)f(\tau,\cdot)\|_{L^2}^2\,\mathrm{d}\tau$$
$$+ \int_0^t \Big\langle M_\Lambda(\tau, D_v)f(\tau,\cdot), \Big(\beta\log\langle D_v\rangle - 2\widetilde{C}_{f_0}\langle D_v\rangle^{2\nu}\Big)M_\Lambda(\tau, D_v)f(\tau,\cdot)\Big\rangle\,\mathrm{d}\tau$$
$$+ (1+2^{d-1})c_{b,d}\|f_0\|_{L^1}\int_0^t \left(\frac{d}{2} + \frac{\beta\tau}{2}2^{\frac{\beta\tau}{2}}\right)\|M_\Lambda(\tau, D_v)f(\tau,\cdot)\|_{L^2}^2.$$

Setting

$$\begin{aligned} A(\beta,\tau) &:= \sup_{\eta\in\mathbb{R}^d}\Big(\beta\log\langle\eta\rangle - 2\widetilde{C}_{f_0}\langle\eta\rangle^{2\nu}\Big) + 2C_{f_0} \\ &\quad + (1+2^{d-1})c_{b,d}\|f_0\|_{L^1}\left(\frac{d}{2} + \frac{\beta\tau}{2}2^{\frac{\beta\tau}{2}}\right) \\ &= \frac{\beta}{2\nu}\left[\log\left(\frac{\beta}{4\nu\widetilde{C}_{f_0}}\right) - 1\right] + 2C_{f_0} \\ &\quad + (1+2^{d-1})c_{b,d}\|f_0\|_{L^1}\left(\frac{d}{2} + \frac{\beta\tau}{2}2^{\frac{\beta\tau}{2}}\right), \end{aligned}$$

the above can be bounded by

$$\|M_\Lambda(t, D_v)f(t,\cdot)\|_{L^2}^2 \le \|f_0\|_{H^{-d}}^2 + \int_0^t A(\beta,\tau)\|M_\Lambda(\tau, D_v)f(\tau,\cdot)\|_{L^2}^2\,\mathrm{d}\tau,$$

and from Gronwall's lemma we get

$$\|M_\Lambda(t, D_v)f(t,\cdot)\|_{L^2}^2 \le \|f_0\|_{H^{-d}}^2 \exp\left(\int_0^t A(\beta,\tau)\,\mathrm{d}\tau\right).$$

Letting $\Lambda \to \infty$, one sees that

$$\|f(t,\cdot)\|_{H^{\beta t-d}}^2 = \|M_\infty(t, D_v)f(t,\cdot)\|_{L^2}^2 \le \|f_0\|_{H^{-d}}^2 \exp\left(\int_0^t A(\beta,\tau)\,\mathrm{d}\tau\right).$$

that is, $f(t,\cdot) \in H^{\beta t-d}(\mathbb{R}^d)$. Now let $\beta \to \infty$ to see that $f(t,\cdot) \in H^\infty(\mathbb{R}^d)$ for any $t > 0$. $\square$

Remark B.4. Setting $\beta = \frac{\gamma+d}{t}$, one sees that $\|f(t,\cdot)\|_{H^\gamma(\mathbb{R}^d)} \lesssim t^{-\frac{\gamma+d}{4\nu}}$, so the H^γ norms, in particular the L^2 norm, of $f(t,\cdot)$ blow up at most polynomially as $t \to 0$.

C The Kolmogorov–Landau Inequality

In this section we give a short proof of

Lemma C.1. (Kolmogorov–Landau inequality on the unit interval) *Let $m \geqq 2$ be an integer. There exists a constant $C_m > 0$ such that for all $w \in W^{m,\infty}([0,1])$,*

$$\|w^{(k)}\|_{L^\infty([0,1])} \leqq C_m \left(\frac{\|w\|_{L^\infty([0,1])}}{u^k} + u^{m-k} \|w^{(m)}\|_{L^\infty([0,1])} \right), \quad k = 1, \ldots, m-1,$$

for all $0 < u \leqq 1$.

For the convenience of the reader, we give a short proof. The following argument is in part borrowed from R. A. DeVore and G. G. Lorentz's book [24] (pp. 37–39).

Proof. Since $w \in W^{m,\infty}([0,1])$, it has absolutely continuous derivatives of order up to $m-1$ and essentially bounded m^{th} derivative.
Let $x \in [0, \frac{1}{2}]$ and $h \in [0, \frac{1}{2}]$. Then, by Taylor's theorem,

$$w(x+h) = w(x) + \sum_{j=1}^{m-1} \frac{h^j}{j!} w^{(j)}(x) + R_m(x,h)$$

with the remainder $R_m(x,h) = \int_0^h \frac{(h-t)^{m-1}}{(m-1)!} w^{(m)}(x+t)\,\mathrm{d}t$, which can be bounded by

$$|R_m(x,h)| \leqq \|w^{(m)}\|_{L^\infty([0,1])} \int_0^h \frac{(h-t)^{m-1}}{(m-1)!}\,\mathrm{d}t = \frac{h^m}{m!} \|w^{(m)}\|_{L^\infty([0,1])}.$$

Choosing $m-1$ real numbers $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{m-1} \leqq 1$ we obtain for $h \in [0, \frac{1}{2}]$ the system of equations

$$\sum_{j=1}^{m-1} \lambda_s^j \frac{h^j}{j!} w^{(j)}(x) = w(x+\lambda_s h) - w(x) - R_m(x, \lambda_s h) \quad \text{for } s = 1, \cdots, m-1. \tag{80}$$

Setting

$$V = \begin{pmatrix} \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} \\ \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} \\ \vdots & & \ddots & \vdots \\ \lambda_{m-1} & \lambda_{m-1}^2 & \cdots & \lambda_{m-1}^{m-1} \end{pmatrix}, \quad \mathbf{w}(x) = \begin{pmatrix} h w'(x) \\ \frac{h^2}{2} w''(x) \\ \vdots \\ \frac{h^{m-1}}{(m-1)!} w^{(m-1)}(x) \end{pmatrix},$$

$$\mathbf{b}(x) = \begin{pmatrix} w(x+\lambda_1 h) - w(x) - R_m(x, \lambda_1 h) \\ w(x+\lambda_2 h) - w(x) - R_m(x, \lambda_2 h) \\ \vdots \\ w(x+\lambda_{m-1} h) - w(x) - R_m(x, \lambda_{m-1} h) \end{pmatrix},$$

we have $V\mathbf{w}(x) = \mathbf{b}(x)$. Since the Vandermonde determinant

$$\det V = \prod_{i=1}^{m-1} \lambda_i \prod_{1 \leqq j < l \leqq m-1} (\lambda_l - \lambda_j) \neq 0,$$

V is invertible and we obtain $\mathbf{w}(x) = V^{-1}\mathbf{b}(x)$ and therefore

$$\left|\frac{h^k}{k!} w^{(k)}(x)\right| \leqq \|\mathbf{w}(x)\| \leqq \|V^{-1}\| \, \|\mathbf{b}(x)\|, \tag{81}$$

where $\|\cdot\|$ is any norm on $\mathbb{R}^{m-1}$, respectively the induced operator norm on the space of $(m-1)\times(m-1)$ real matrices. Choosing for concreteness the ℓ^1 norm on $\mathbb{R}^{m-1}$, we have

$$\begin{aligned}\|\mathbf{b}(x)\| &= \sum_{s=1}^{m-1} |w(x+\lambda_s h) - w(x) - R_m(x, \lambda_s h)| \\ &\leqq (m-1)\left(2\|w\|_{L^\infty([0,1])} + \frac{h^m}{m!}\|w^{(m)}\|_{L^\infty([0,1])}\right).\end{aligned}$$

While for our application the size of $\|V^{-1}\|$ is of no importance, one can even explicitly calculate it: The inverse of the Vandermonde matrix V is explicitly known (see for instance [25]),

$$\left(V^{-1}\right)_{\alpha\beta} = (-1)^{\alpha-1} \frac{\sigma^{\beta}_{m-1-\alpha}}{\lambda_\beta \prod_{\nu\neq\beta}(\lambda_\nu - \lambda_\beta)}, \quad \alpha, \beta = 1, \ldots, m-1,$$

where σ_i^j, $i, j = 1, \ldots, m-2$ is the i^{th} elementary symmetric function in the $(m-2)$ variables $\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{m-1}$,

$$\sigma_i^j = \sum_{\substack{1\leqq\nu_1<\cdots<\nu_i\leqq m-1 \\ \nu_1,\ldots,\nu_i\neq j}} \lambda_{\nu_1}\cdots\lambda_{\nu_i}, \qquad \sigma_0^j := 1.$$

By means of the identity (Lemma 1 in [25]) we have

$$\sum_{i=0}^{m-2} \sigma_i^j = \prod_{\substack{\nu=1\\ \nu\neq j}}^{m-1} (1+\lambda_\nu),$$

which holds since the λ_ν are all positive, hence

$$\begin{aligned}\|V^{-1}\| &= \max_{1\leqq\beta\leqq m-1} \sum_{\alpha=1}^{m-1} \left|\left(V^{-1}\right)_{\alpha\beta}\right| = \max_{1\leqq\beta\leqq m-1} \frac{1}{\lambda_\beta \prod_{\nu\neq\beta}|\lambda_\nu - \lambda_\beta|} \sum_{\alpha=1}^{m-1} \sigma^\beta_{m-1-\alpha} \\ &= \max_{1\leqq\beta\leqq m-1} \frac{1}{\lambda_\beta} \prod_{\substack{\nu=1\\ \nu\neq\beta}}^{m-1} \frac{1+\lambda_\nu}{|\lambda_\nu - \lambda_\beta|}.\end{aligned}$$

Going back to inequality (81), we have so far proved that

$$\frac{h^k}{k!}\left|w^{(k)}(x)\right| \leqq (m-1)\|V^{-1}\| \left(2\|w\|_{L^\infty([0,1])} + \frac{h^m}{m!}\|w^{(m)}\|_{L^\infty([0,1])}\right),$$

which yields

$$\left|w^{(k)}(x)\right| \leqq (m-1)\|V^{-1}\| \left(\frac{2k!}{h^k}\|w\|_{L^\infty([0,1])} + h^{m-k}\frac{k!}{m!}\|w^{(m)}\|_{L^\infty([0,1])}\right)$$
$$\leqq (m-1)\|V^{-1}\| \left(\frac{2m!}{h^k}\|w\|_{L^\infty([0,1])} + h^{m-k}\|w^{(m)}\|_{L^\infty([0,1])}\right). \tag{82}$$

For $x \in [\frac{1}{2}, 1]$ the same calculations with h replaced by $-h$ prove inequality (82) in this case as well, so

$$\|w^{(k)}\|_{L^\infty([0,1])} \leqq (m-1)\|V^{-1}\| \left(\frac{2m!}{h^k}\|w\|_{L^\infty([0,1])} + h^{m-k}\|w^{(m)}\|_{L^\infty([0,1])}\right) \tag{83}$$

for all $h \in [0, \frac{1}{2}]$. Taking an arbitrary $u \in [0, 1]$, inequality (83) implies, with $h = \frac{u}{2} \in [0, \frac{1}{2}]$,

$$\|w^{(k)}\|_{L^\infty([0,1])} \leqq 2^m m!(m-1)\|V^{-1}\| \left(\frac{1}{u^k}\|w\|_{L^\infty([0,1])} + u^{m-k}\|w^{(m)}\|_{L^\infty([0,1])}\right),$$

which is the claimed inequality with

$$C_m = 2^m m!(m-1)\|V^{-1}\| = 2^m m!(m-1) \max_{1\leqq\beta\leqq m-1} \frac{1}{\lambda_\beta} \prod_{\substack{\nu=1\\ \nu\neq\beta}}^{m-1} \frac{1+\lambda_\nu}{|\lambda_\nu - \lambda_\beta|}. \tag{84}$$

□

Remark C.2. The constant C_m in equality (84) is far from optimal, but can be made small by minimising in the choice of the points $0 < \lambda_1 < \cdots < \lambda_{m-1} \leqq 1$, suggesting that the optimal constant might be obtained by methods from approximation theory.

Indeed, by a more refined argument making use of numerical differentiation formulas, the minimisers of the associated multiplicative Kolmogorov–Landau inequality, i.e., extremisers of

$$M_k(\sigma) := \sup\{\|w^{(k)}\|_{L^\infty([0,1])} : w \in W^{m,\infty}([0,1]), \|w\|_{L^\infty([0,1])} \leqq 1, \|w^m\|_{L^\infty([0,1])} \leqq \sigma\}$$

are explicitly known (at least for a wide range of parameters $m \in \mathbb{N}$ and $\sigma \geqq 0$). The optimal Kolmogorov–Landau constants in these cases are given by the end-point values of certain Chebyshev type perfect splines. We refer to the papers by Pinkus [43] and Karlin [29], as well as the recent article by A. Shadrin [44] and references therein.

D Proof of Lemma 1.1

Proof. Let $f \in L^1_2(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$. Then

$$|H(f)| = \int_{\mathbb{R}^d} f \log_+ f \, \mathrm{d}v + \int_{\mathbb{R}^d} f \log_- f \, \mathrm{d}v.$$

The positive part is bounded by $\int f \log(1+f)\,\mathrm{d}v = \|f\|_{L\log L}$. The negative part can be controlled by

$$\int_{\mathbb{R}^d} f \log_- f \, \mathrm{d}v = \int_{\{f \leqq 1\}} f \log \frac{1}{f} \, \mathrm{d}v \leqq C_\delta \int_{\{f \leqq 1\}} f^{1-\delta} \, \mathrm{d}v$$
$$\leqq C_\delta \left(\int_{\mathbb{R}^d} (1+|v|^2)^{-\frac{1-\delta}{\delta}} \, \mathrm{d}v \right)^{\delta} \|f\|_{L^1_2}^{1-\delta},$$

which is finite for $0 < \delta < \frac{2}{d+2}$, having used that for any $\delta > 0$ there exists a constant C_δ such that $\log t \leqq C_\delta t^\delta$ for all $t \geqq 1$.
Conversely, let $f \in L^1_2(\mathbb{R}^d)$ with finite entropy $H(f)$. Then

$$\int_{\mathbb{R}^d} f \log(1+f)\,\mathrm{d}v = \int_{\{f \le 1\}} f \log(1+f)\,\mathrm{d}v + \int_{\{f>1\}} f \log(1+f)\,\mathrm{d}v.$$

On the set where $f \le 1$, we replace f by 1 and where $f > 1$, we bound $1+f$ by $2f$ leading to

$$\int_{\mathbb{R}^d} f \log(1+f)\,\mathrm{d}v \leqq \log 2 \int_{\mathbb{R}^d} f \,\mathrm{d}v + \int_{\mathbb{R}^d} f \log f \,\mathrm{d}v + \int_{\mathbb{R}^d} f \log_- f \,\mathrm{d}v.$$

As above, we conclude

$$\int_{\mathbb{R}^d} f \log(1+f)\,\mathrm{d}v \leqq \log 2 \|f\|_{L^1(\mathbb{R}^d)} + H(f) + C_{\delta,d} \|f\|_{L^1_2(\mathbb{R}^d)}^{1-\delta}, \tag{85}$$

with a finite constant $C_{\delta,d}$ for $0 < \delta < \frac{2}{d+2}$. □

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