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Zero-Viscosity Limit of the Navier–Stokes Equations in the Analytic Setting

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Abstract

In this paper, we consider the zero-viscosity limit of the Navier–Stokes equations in a half space with non-slip boundary condition. Based on the vorticity formulation and the use of conormal derivative, we develop an energy method to justify the zero-viscosity limit for the analytic data.

1. Introduction

In this paper, we consider the zero-viscosity limit of the Navier–Stokes equations in a half space $\mathbb{R}^2_+$

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon + \partial_x p^\varepsilon - \varepsilon^2 \triangle u^\varepsilon = 0, \\ \partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon + \partial_y p^\varepsilon - \varepsilon^2 \triangle v^\varepsilon = 0, \\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0, \\ (u^\varepsilon, v^\varepsilon)|_{y=0} = 0. \end{cases} \tag{1.1}$$

Here $t \geqq 0$, $(x, y) \in \mathbb{R}^2_+$. Formally, letting $\varepsilon \to 0$, we get the Euler equations

$$\begin{cases} \partial_t u^e + u^e \partial_x u^e + v^e \partial_y u^e + \partial_x p^e = 0, \\ \partial_t v^e + u^e \partial_x v^e + v^e \partial_y v^e + \partial_y p^e = 0, \\ \partial_x u^e + \partial_y v^e = 0, \\ v^e|_{y=0} = 0. \end{cases} \tag{1.2}$$

In the absence of the boundary, it has been proved that the Navier–Stokes equations indeed converge to the Euler equations in various functional settings, see [1,4,14,31]. However, in the presence of the boundary, the inviscid limit problem will become very complicated due to the appearance of boundary layer.

For the Navier slip boundary condition, the boundary layer is weak. In such a case, ROUSSET and MASMOUDI [23] introduced the conormal functional space to justify the limit from the Navier–Stokes equations to the Euler equations. We refer to [12,13,24,34–36] and references therein for more relevant results.

Meanwhile, for the non-slip boundary condition, the boundary layer is strong. In 1904, Prandtl introduced the boundary layer theory in [28], where he derived the Prandtl boundary layer equation

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = \partial_y^2 u, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y\to\infty} u(t,x,y) = U(t,x). \end{cases} \tag{1.3}$$

This can be derived by a formal asymptotic expansion

$$\begin{cases} u^\varepsilon(t,x,y) = u^e(t,x,y) + u^p(t,x,\frac{y}{\varepsilon}) + O(\varepsilon), \\ v^\varepsilon(t,x,y) = v^e(t,x,y) + \varepsilon v^p(t,x,\frac{y}{\varepsilon}) + O(\varepsilon). \end{cases} \tag{1.4}$$

To justify this formal expansion, one of the key ingredeints is to deal with the well-posedness of the Prandtl equation. Because the nonlinear term $v\partial_y u$ loses one derivative in the horizontal direction in the energy estimates, the question of whether the Prandtl equation is well-posed in Sobolev space remains open. Up until now, the well-posedness of Prandtl equation was proved only under some special functional space. We refer to [21,29] for the analytic class. Another class of note is the monotonic class. Using the Crocco transformation, OLEINIK and SAMOKHIN [26] proved the local existence and uniqueness of classical solutions to (1.3) for the monotonic data. XIN and ZHANG [37] proved the global existence of weak solutions to this system for the favorable pressure. Recently, ALEXANDRE et al. [2] and MASMOUDI and WONG [25] independently proved the local well-posedness of the Prandtl equations in Sobolev space by the direct energy method. Their method might shed some light on the inviscid limit problem in Sobolev space. On the other hand, GERARD-VARET and DORMY [7] proved the ill-posedness in Sobolev space for the linearized Prandtl equation around non-monotonic shear flows. We refer to [8–10,18–20] and references therein for more relevant results.

Although we have a good understanding of the well-posed problem of the Prandtl equation, there are few results on the rigorous verification of the Prandtl boundary layer theory. The inviscid limit was only achieved for some specific cases, for example the analytic space [30] and the initial vorticity supported away from the boundary [22]. Recently, GUO and NGUYEN [11] made an important progress for the inviscid limit of the steady Navier–Stokes equations. Let us also mention some conditional convergence results [6,15–17,32,33], which were first considered by KATO [15].

The proof in [22,30] is based on the abstract Cauchy–Kowaleskaya theorem, where the representation formula of the solution was used in a crucial way. Hence, it seems difficult to extend their methods to the general domain. The goal of this paper is to develop an energy method for the inviscid limit problem in the analytic setting. Our method is applicable for the inviscid limit problem in general domain, which will be presented in a forthcoming paper.

In this paper, we consider the initial data with the form

$$u^\varepsilon(0,x,y)=u_0(x,y),\quad v^\varepsilon(0,x,y)=v_0(x,y),$$

which satisfies

$$\partial_x u_0+\partial_y v_0=0,\quad u_0(x,0)=v_0(x,0)=0.$$

We further assume that initial data falls into the analytic class with the bound

$$M\triangleq\sum_{m=0}^{\infty}\frac{a^{2m}}{(m!)^2}\|\partial_y^m e^{a|D|}(u_0,v_0)\|^2_{H^N_{x,y}}<\infty \tag{1.5}$$

for some $N\geq 30$. Here the constant $a>0$ denotes the analytic bandwidth. In the sequel, we take $a=2$ without loss of generality.

Let (u^e,v^e,p^e) be the solution of the Euler equations (1.2) with the initial data (u_0,v_0), and (u^p,v^p) be the solution of the Prandtl type equation

$$\begin{cases}\partial_t u^p-\partial_z^2u^p+(u^e(t,x,0)+u^p)\partial_x u^p+u^p\partial_x u^e(t,x,0)\\ \qquad\qquad+\Big(v^p-\displaystyle\int_0^\infty\partial_x u^p\,\mathrm{d}z+z\partial_y v^e(t,x,0)\Big)\partial_z u^p=0,\\ \partial_x u^p+\partial_z v^p=0,\\ u^p|_{z=0}=-u^e|_{y=0},\quad v^p|_{z=+\infty}=0,\\ u^p|_{t=0}=0.\end{cases} \tag{1.6}$$

Our main result is stated as follows.

Theorem 1.1. *There exist $T>0$ and $C>0$ independent of ε such that there exists a unique analytic solution $\big(u^\varepsilon,v^\varepsilon\big)$ of the Navier–Stokes equations (1.1) in $[0,T]$, which satisfies*

$$\big\|u^\varepsilon(t,x,y)-\big(u^e(t,x,y)+u^p(t,x,\frac{y}{\varepsilon})\big)\big\|_{L^2_{x,y}\cap L^\infty_{x,y}}\leq C\varepsilon,$$

$$\big\|v^\varepsilon(t,x,y)-\big(v^e(t,x,y)+\varepsilon v^p(t,x,\frac{y}{\varepsilon})\big)\big\|_{L^2_{x,y}\cap L^\infty_{x,y}}\leq C\varepsilon,$$

for any $t\in[0,T]$.

To prove the theorem, we will first construct an approximate solution $U^a=(u^a,v^a)$ of the Navier–Stokes equations by using the asymptotic matched expansion method. The key difficulty is to prove that the error U^R between the real solution and the approximate solution is uniformly bounded. Indeed, U^R satisfies [see (3.2)]

$$\partial_t U^R+\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R-\varepsilon^2\triangle U^R+\nabla p^R=\widetilde{R},$$

where $U^R=\big(u^\varepsilon(t),v^\varepsilon(t)\big)-\big(u^a(t),v^a(t)\big)=(u^R,v^R)$. In the error equation, the main trouble term is $v^R\cdot\partial_y(u^p(t,x,\frac{y}{\varepsilon}))$, which behaves like

$$\begin{aligned}v^R\cdot\partial_y\left(u^p\left(t,x,\frac{y}{\varepsilon}\right)\right)&=\frac{v^R}{y}\cdot\left(z\partial_z u^p\right)\left(t,x,\frac{y}{\varepsilon}\right)\\ &\sim-\partial_x u^R\cdot(z\partial_z u^p)\left(t,x,\frac{y}{\varepsilon}\right).\end{aligned}$$

This leads to one derivative loss in the x direction. Hence, it is natural to work in the analytic setting, which will help us recover one derivative. On the other hand, if we directly take the ∂_y derivative to the error equation, the Prandtl part (u^p, v^p) of the approximate solution will give rise to a bad factor, ε^{-1}. To avoid this singularity, a key idea motivated by [23] is to use the conormal derivative. However, the conormal derivative cannot provide a control in the normal direction for the error near the boundary. Motivated by [22], we will use the vorticity formulation of the Navier–Stokes equations to gain one derivative in the normal direction. In addition, some ideas from [27] and [5,38] were very useful for the construction of the analytic type energy norm and the estimates of nonlinear terms.

This paper is organized as follows. In Section 2, we construct the approximate solution by the asymptotic expansion method. In Section 3, we derive the vorticity formulation of the error equations. In Section 4, we introduce some analytic type spaces and product estimates in the analytic norm. In Section 5, we present the analytic type estimates of the velocity. Section 6 is devoted to the uniform estimates of the vorticity in the analytic space. In Section 7, we present the proof of main theorem. In Section 8, we prove the well-posedness of the Euler equations and Prandtl equations in the analytic spaces.

2. Construction of the Approximate Solution

In this section, we will construct the approximate solution of the Navier–Stokes equations by using the asymptotic matched expansion method.

2.1. Euler expansion

Away from the boundary, we construct the approximate solutions by the following expansion

$$
\begin{aligned}
u^\varepsilon(t,x,y) &= u_e^{(0)}(t,x,y) + \varepsilon u_e^{(1)}(t,x,y) + \cdots,\\
v^\varepsilon(t,x,y) &= v_e^{(0)}(t,x,y) + \varepsilon v_e^{(1)}(t,x,y) + \cdots,\\
p^\varepsilon(t,x,y) &= p_e^{(0)}(t,x,y) + \varepsilon p_e^{(1)}(t,x,y) + \cdots.
\end{aligned}
$$

Plugging this expansion into the Navier–Stokes equations (1.1) and then matching the zero order terms, we find that $(u_e^{(0)}, v_e^{(0)}, p_e^{(0)})$ should satisfy the Euler equations

$$
\begin{cases}
\partial_t u_e^{(0)} + u_e^{(0)}\partial_x u_e^{(0)} + v_e^{(0)}\partial_y u_e^{(0)} + \partial_x p_e^{(0)} = 0,\\
\partial_t v_e^{(0)} + u_e^{(0)}\partial_x v_e^{(0)} + v_e^{(0)}\partial_y v_e^{(0)} + \partial_y p_e^{(0)} = 0,\\
\partial_x u_e^{(0)} + \partial_y v_e^{(0)} = 0,\\
\big(u_e^{(0)}, v_e^{(0)}\big)|_{t=0} = (u_0, v_0).
\end{cases}
\tag{2.1}
$$

Here we impose the boundary condition

$$
v_e^{(0)}(t,x,0) = 0. \tag{2.2}
$$

By matching the ε-order terms, we get the linearized Euler equations for $(u_e^{(1)}, v_e^{(1)}, p_e^{(1)})$

$$\begin{cases} \partial_t u_e^{(1)} + u_e^{(1)}\partial_x u_e^{(0)} + v_e^{(1)}\partial_y u_e^{(0)} + u_e^{(0)}\partial_x u_e^{(1)} + v_e^{(0)}\partial_y u_e^{(1)} + \partial_x p_e^{(1)} = 0, \\ \partial_t v_e^{(1)} + u_e^{(1)}\partial_x v_e^{(0)} + v_e^{(1)}\partial_y v_e^{(0)} + u_e^{(0)}\partial_x v_e^{(1)} + v_e^{(0)}\partial_y v_e^{(1)} + \partial_y p_e^{(1)} = 0, \\ \partial_x u_e^{(1)} + \partial_y v_e^{(1)} = 0, \\ \big(u_e^{(1)}, v_e^{(1)}\big)|_{t=0} = 0. \end{cases} \tag{2.3}$$

Here the boundary condition on $v_e^{(1)}$ is determined by $v_p^{(1)}$ [see (2.6)]:

$$v_e^{(1)}|_{y=0} = -v_p^{(1)}|_{z=0}. \tag{2.4}$$

2.2. Prandtl boundary layer expansion

Near the boundary $y = 0$, we will make the Prandtl boundary layer expansion. For this, we introduce the scaled variable $z = \frac{y}{\varepsilon}$ and make the following formal expansion:

$$\begin{aligned} u^\varepsilon(t, x, y) &= u_P^{(0)}(t, x, y, z) + \varepsilon u_P^{(1)}(t, x, y, z) + \cdots, \\ v^\varepsilon(t, x, y) &= v_P^{(0)}(t, x, y, z) + \varepsilon v_P^{(1)}(t, x, y, z) + \cdots, \\ p^\varepsilon(t, x, y) &= p_P^{(0)}(t, x, y, z) + \varepsilon p_P^{(1)}(t, x, y, z) + \cdots. \end{aligned}$$

For every $i \in \{0, 1, 2, \cdots\}$, we write

$$\begin{aligned} u_P^{(i)}(t, x, y, z) &= u_e^{(i)}(t, x, y) + u_p^{(i)}(t, x, z), \\ v_P^{(i)}(t, x, y, z) &= v_e^{(i)}(t, x, y) + v_p^{(i)}(t, x, z), \\ p_P^{(i)}(t, x, y, z) &= p_e^{(i)}(t, x, y) + p_p^{(i)}(t, x, z). \end{aligned}$$

The matched boundary condition requires that

$$u_p^{(i)}(t, x, z) \to 0, \quad v_p^{(i)}(t, x, z) \to 0, \quad p_p^{(i)}(t, x, z) \to 0, \tag{2.5}$$

as $z \to +\infty$. Meanwhile, the boundary condition of $(u^\varepsilon, v^\varepsilon)$ on $y = 0$ requires that

$$u_e^{(i)}(t, x, 0) + u_p^{(i)}(t, x, 0) = 0, \quad v_e^{(i)}(t, x, 0) + v_p^{(i)}(t, x, 0) = 0 \\ i = 0, 1, 2, \cdots. \tag{2.6}$$

To derive the equations of $(u_p^{(i)}, u_p^{(i)}, p_p^{(i)})$, we plug the expansion into the Navier–Stokes equations (1.1), use the Taylor expansion in y, and then put the terms with the same order in ε together. Since the calculations are routine, we omit these complicated details.

First of all, we deduce from the ε^{-1}-th order terms that

$$v_p^{(0)} = 0, \quad p_p^{(0)} = 0. \tag{2.7}$$

Next, we deduce from the ε^0-th order terms that

$$\begin{cases} \partial_t u_p^{(0)} - \partial_z^2 u_p^{(0)} + u_p^{(0)}\partial_x u_e^{(0)}(t,x,0) + \big(u_e^{(0)}(t,x,0) + u_p^{(0)}\big)\partial_x u_p^{(0)} \\ \qquad\qquad + \big(v_e^{(1)}(t,x,0) + v_p^{(1)} + z\partial_y v_e^{(0)}(t,x,0)\big)\partial_z u_p^{(0)} = 0, \\ \partial_x u_p^{(0)} + \partial_z v_p^{(1)} = 0, \\ u_p^{(0)}|_{t=0} = 0, \end{cases} \tag{2.8}$$

together with the boundary conditions

$$u_p^{(0)}|_{z=0} = -u_e^{(0)}|_{y=0}, \quad \big(u_p^{(1)}, v_p^{(1)}\big)|_{z=+\infty} = 0. \tag{2.9}$$

Furthermore, $p_p^{(1)}$ can be determined from the ε^0-th order terms of the v^ε equation

$$\partial_z p_p^{(1)} = 0, \quad \text{thus} \quad p_p^{(1)} = 0. \tag{2.10}$$

Remark 2.1. Let

$$\begin{aligned} &\widetilde{u}_p^{(0)}(t,x,z) = u_e^{(0)}(t,x,0) + u_p^{(0)}(t,x,z), \\ &\widetilde{v}_p^{(1)}(t,x,z) = v_e^{(1)}(t,x,0) + v_p^{(1)}(t,x,z) + z\partial_y v_e^{(0)}(t,x,0). \end{aligned}$$

Then we find that

$$\begin{cases} \partial_t \widetilde{u}_p^{(0)} - \partial_{zz}\widetilde{u}_p^{(0)} + \widetilde{u}_p^{(0)}\partial_x \widetilde{u}_p^{(0)} + \widetilde{v}_p^{(1)}\partial_z \widetilde{u}_p^{(0)} + \partial_x p_e^{(0)}(t,x,0) = 0, \\ \partial_x \widetilde{u}_p^{(0)} + \partial_z \widetilde{v}_p^{(1)} = 0, \\ \lim\limits_{z\to+\infty} \widetilde{u}_p^{(0)}(t,x,z) = u_e^{(0)}(t,x,0), \end{cases}$$

which is just the Prandtl equations (1.3).

Finally, we deduce from the ε-th order terms that

$$\begin{aligned} &\partial_t u_p^{(1)} - \partial_{zz}^2 u_p^{(1)} + \big(u_e^{(0)}(t,x,0) + u_p^{(0)}\big)\partial_x u_p^{(1)} \\ &\quad + \big(v_e^{(1)}(t,x,0) + v_p^{(1)} + z\partial_y v_e^{(0)}(t,x,0)\big)\partial_z u_p^{(1)} \\ &\quad + u_p^{(1)}\partial_x u_e^{(0)}(t,x,0) + (u_p^{(1)} + u_e^{(1)}(t,x,0) + z\partial_y u_e^{(0)}(t,x,0))\partial_x u_p^{(0)} \\ &\quad + (v_p^{(2)} - v_p^{(2)}(t,x,0) + z\partial_y v_e^{(1)}(t,x,0) + \frac{1}{2}z^2\partial_{yy}^2 v_e^{(0)}(t,x,0))\partial_z u_p^{(0)} \\ &= -\Big(z\partial_{xy}^2 u_e^{(0)}(t,x,0)u_p^{(0)} \\ &\quad + u_p^{(0)}\partial_x u_e^{(1)}(t,x,0) + v_p^{(1)}\partial_y u_e^{(0)}(t,x,0)\Big), \end{aligned} \tag{2.11}$$

with the initial condition $u_p^{(1)}|_{t=0} = 0$ and the boundary condition

$$u_p^{(1)}|_{z=0} = -u_e^{(1)}|_{y=0}, \quad u_p^{(1)}|_{z=+\infty} = 0. \tag{2.12}$$

Here $v_p^{(2)}$ is determined by

$$v_p^{(2)}(t,x,z) = \int_z^{+\infty} \partial_x u_p^{(1)}(t,x,z')\,\mathrm{d}z'. \tag{2.13}$$

The pressure $p_p^{(2)}$ is determined by

$$p_p^{(2)}(t,x,z) = -\int_z^{+\infty} \mathcal{P}_2^1(t,z')dz', \tag{2.14}$$

where

$$\begin{aligned}\mathcal{P}_2^1 &= \partial_{zz}^2 v_p^{(1)} - \partial_t v_p^{(1)} - u_e^{(0)}(t,x,0)\partial_x v_p^{(1)} \\ &\quad - u_p^{(0)}\big(\partial_x v_e^{(1)}(t,x,0) + \partial_x v_p^{(1)}\big) - v_p^{(1)}\partial_y v_e^{(0)}(t,x,0) \\ &\quad - \big(v_e^{(1)}(t,x,0) + v_p^{(1)}\big)\partial_z v_p^{(1)} - z\partial_{xy}^2 v_e^{(0)}(t,x,0)u_p^{(0)} - z\partial_y v_e^{(0)}(t,x,0)\partial_z v_p^{(1)}.\end{aligned}$$

Remark 2.2. These equations can be solved in the following way:

$$\left(u_e^{(0)}, v_e^{(0)}\right) \to \left(u_p^{(0)}, v_p^{(1)}\right) \to \left(u_e^{(1)}, v_e^{(1)}\right) \to \left(u_p^{(1)}, v_p^{(2)}\right).$$

See the appendix for the details.

2.3. Approximate solution

With the above asymptotic expansion, let us define the approximate solution

$$\begin{aligned}u^a &= u_e^{(0)}(t,x,y) + u_p^{(0)}\left(t,x,\frac{y}{\varepsilon}\right) + \varepsilon u_e^{(1)}(t,x,y) + \varepsilon u_p^{(1)}\left(t,x,\frac{y}{\varepsilon}\right), \\ v^a &= v_e^{(0)}(t,x,y) + \varepsilon v_e^{(1)}(t,x,y) + \varepsilon v_p^{(1)}\left(t,x,\frac{y}{\varepsilon}\right) + \varepsilon^2 v_p^{(2)}\left(t,x,\frac{y}{\varepsilon}\right), \\ p^a &= p_e^{(0)}(t,x,y) + \varepsilon p_e^{(1)}(t,x,y) + \varepsilon^2 p_p^{(2)}\left(t,x,\frac{y}{\varepsilon}\right).\end{aligned}$$

We denote

$$\begin{aligned}u^e(t,x,y) &= u_e^{(0)}(t,x,y) + \varepsilon u_e^{(1)}(t,x,y), \\ v^e(t,x,y) &= v_e^{(0)}(t,x,y) + \varepsilon v_e^{(1)}(t,x,y), \\ u^p\left(t,x,\frac{y}{\varepsilon}\right) &= u_p^{(0)}\left(t,x,\frac{y}{\varepsilon}\right) + \varepsilon u_p^{(1)}\left(t,x,\frac{y}{\varepsilon}\right), \\ v^p\left(t,x,\frac{y}{\varepsilon}\right) &= v_p^{(1)}\left(t,x,\frac{y}{\varepsilon}\right) + \varepsilon v_p^{(2)}\left(t,x,\frac{y}{\varepsilon}\right), \\ f(t,x) &= \int_0^\infty \partial_x u_p^{(1)}(t,x,z)\,\mathrm{d}z.\end{aligned}$$

Thanks to our construction, we find that the approximate solution (u^a, v^a, p^a) satisfies

$$\begin{cases}\partial_t u^a + u^a\partial_x u^a + (v^a - \varepsilon^2 f(t,x)e^{-y})\partial_y u^a - \varepsilon^2\triangle u^a + \partial_x p^a = -R_1, \\ \partial_t v^a + u^a\partial_x v^a + (v^a - \varepsilon^2 f(t,x)e^{-y})\partial_y v^a - \varepsilon^2\triangle v^a + \partial_y p^a = -R_2, \\ \partial_x u^a + \partial_y v^a = 0,\end{cases} \tag{2.15}$$

where the remainder $\widetilde{R} = (R_1, R_2)$ is given by

$$
\begin{aligned}
-R_1 = \varepsilon^2\Big(&(u_e^{(0)} + u_p^{(0)})(\partial_x u_e^{(2)} + \partial_x u_p^{(2)}) + v_e^{(1)}\partial_y u_e^{(1)} + v_p^{(1)}\partial_y u_e^{(1)} \\
&+ (v_p^{(2)} - f e^{-y})(\partial_y u_e^{(0)} + \varepsilon\partial_y u_e^{(1)}) \\
&- f e^{-y}\partial_z u_p^{(1)}\Big) + \varepsilon\Big((u_e^{(0)} - u_e^{(0)}(t,x,0))\partial_x u_p^{(1)} \\
&+ u_p^{(0)}(\partial_x u_e^{(1)} - \partial_x u_e^{(1)}(t,x,0)) \\
&+ (u_e^{(1)} - u_e^{(1)}(t,x,0))\partial_x u_p^{(0)} + u_p^{(1)}(\partial_x u_e^{(0)} \\
&- \partial_x u_e^{(0)}(t,x,0)) + (v_e^{(1)} - v_e^{(1)}(t,x,0))\partial_z u_p^{(1)} \\
&+ v_p^{(1)}(\partial_y u_e^{(0)} - \partial_y u_e^{(0)}(t,x,0)) + f(1-e^{-y})\partial_z u_p^{(0)}\Big) \\
&+ \frac{1}{\varepsilon}(v_e^{(0)} - y\partial_y v_e^{(0)}(t,x,0) - \frac{y^2}{2}\partial_y^2 v_e^{(0)}(t,x,0))\partial_z u_p^{(0)} \\
&+ (v_e^{(0)} - y\partial_y v_e^{(0)}(t,x,0))\partial_z u_p^{(1)} \\
&+ (v_e^{(1)} - v_e^{(1)}(t,x,0) - y\partial_y v_e^{(1)}(t,x,0))\partial_z u_p^{(1)} + (u_e^{(0)} \\
&- u_e^{(0)}(t,x,0) - y\partial_y u_e^{(0)}(t,x,0))\partial_x u_p^{(0)} \\
&+ u_p^{(0)}(\partial_x u_e^{(0)} - \partial_x u_e^{(0)}(t,x,0) - y\partial_{xy}^2 u_e^{(0)}(t,x,0)) \\
&- \varepsilon^2\triangle u^e - \varepsilon^2\partial_x^2 u^p + \varepsilon^2\partial_x p_p^{(2)},
\end{aligned}
$$

and

$$
\begin{aligned}
-R_2 = \varepsilon^2\Big(&\partial_t v_p^{(2)} + u_e^{(0)}\partial_x v_p^{(2)} + u_p^{(0)}\partial_x v_p^{(2)} + v_e^{(1)}\partial_y v_e^{(1)} + v_e^{(1)}\partial_z v_p^{(2)} + \varepsilon v_p^{(1)}\partial_y v_p^{(2)}\Big) \\
&+ \varepsilon^2\Big(v_p^{(1)}\partial_y v_e^{(1)} + (v_p^{(2)} - f e^{-y})\partial_y v^a\Big) \\
&+ \varepsilon\Big((u_e^{(0)} - u_e^{(0)}(t,x,0))\partial_x v_p^{(1)} + u_p^{(0)}(\partial_x v_e^{(1)} - \partial_x v_e^{(1)}(t,x,0))\Big) \\
&+ \varepsilon^2(v_e^{(1)} - v_e^{(1)}(t,x,0))\partial_y v_p^{(1)} + \varepsilon v_p^{(1)}(\partial_y v_e^{(0)} - \partial_y v_e^{(0)}(t,x,0)) \\
&+ \varepsilon u_p^{(1)}\partial_x v^a + \varepsilon^2 v_e^{(0)}\partial_y v_p^{(2)} + \varepsilon^2 u_e^{(1)}(\partial_x v_e^{(1)} + \partial_x v_p^{(1)} + \varepsilon\partial_x v_p^{(2)}) \\
&+ u_p^{(0)}(\partial_x v_e^{(0)} - y\partial_{xy}^2 v_e^{(0)}(t,x,0)) + \varepsilon(v_e^{(0)} - y\partial_y v_e^{(0)}(t,x,0))\partial_y v_p^{(1)} \\
&- \varepsilon^2\triangle v^e - \varepsilon^3\partial_x^2 v^p - \varepsilon^4\partial_y^2 v_p^{(2)}.
\end{aligned}
$$

Formally, it holds that

$$R_1 \sim \varepsilon^2, \quad R_2 \sim \varepsilon^2.$$

3. The Vorticitiy Formulation of the Error Equations

We introduce the error between the solution and the approximate solution:

$$u^R = u^\varepsilon - u^a, \quad v^R = v^\varepsilon - v^a, \quad p^R = p^\varepsilon - p^a.$$

Thanks to (2.15), it is easy to see that $U^R=(u^R,v^R)$ and p^R satisfy

$$\begin{cases}\partial_t u^R+u^R\partial_x u^\varepsilon+u^a\partial_x u^R+(v^R+\varepsilon^2 f(t,x)e^{-y})\partial_y u^\varepsilon+(v^a-\varepsilon^2 f(t,x)e^{-y})\partial_y u^R\\ \qquad\qquad -\varepsilon^2\triangle u^R+\partial_x p^R=R_1,\\ \partial_t v^R+u^R\partial_x v^\varepsilon+u^a\partial_x v^R+(v^R+\varepsilon^2 f(t,x)e^{-y})\partial_y v^\varepsilon+(v^a-\varepsilon^2 f(t,x)e^{-y})\partial_y v^R\\ \qquad\qquad -\varepsilon^2\triangle v^R+\partial_y p^R=R_2,\\ \partial_x u^R+\partial_y v^R=0,\end{cases} \tag{3.1}$$

with the boundary conditions

$$u^R|_{y=0}=0,\quad v^R|_{y=0}=-\varepsilon^2 f(t,x).$$

To simplify the notations, we denote

$$\begin{aligned}&U^a=(u^a,v^a)=(u^e,v^e)(t,x,y)+(u^p,\varepsilon v^p)(t,x,\frac{y}{\varepsilon})\triangleq U^e+U^p,\\ &\widetilde{U}^R=(u^R,v^R+\varepsilon^2 f(t,x)e^{-y}),\quad \widetilde{U}^a=(u^a,v^a-\varepsilon^2 f(t,x)e^{-y}),\\ &\quad\widetilde{R}=(R_1,R_2).\end{aligned}$$

Then the error equations (3.1) are reduced to

$$\begin{cases}\partial_t U^R+\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R-\varepsilon^2\triangle U^R+\nabla p^R=\widetilde{R},\\ \mathrm{div}U^R=0,\\ U^R|_{y=0}=(0,-\varepsilon^2 f),\\ U^R|_{t=0}=0.\end{cases} \tag{3.2}$$

Now we introduce the vorticity defined by

$$\omega^R=\mathrm{curl}\,U^R=\partial_y u^R-\partial_x v^R,\quad \omega^a=\mathrm{curl}\,U^a=\partial_y u^a-\partial_x v^a,$$

which satisfies

$$\partial_t\omega^R-\varepsilon^2\triangle\omega^R+\widetilde{U}^R\cdot\nabla(\omega^R+\omega^a)+\widetilde{U}^a\cdot\nabla\omega^R=R^\omega, \tag{3.3}$$

where

$$R^\omega=\partial_y R_1-\partial_x R_2+\varepsilon^2 f e^{-y}\partial_y u^a+\varepsilon^2 e^{-y}\partial_x f\partial_y v^a.$$

To derive the boundary condition of ω^R, we follow the derivation in [22].

Lemma 3.1. *The vorticity ω^R satisfies the following boundary condition*

$$\begin{aligned}\varepsilon^2(\partial_y+|D_x|)\omega^R\big|_{y=0}=&-\Big(\partial_y(-\Delta_D)^{-1}\,\mathrm{curl}(\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R-\widetilde{R})\Big)\Big|_{y=0}\\ &-\varepsilon^2\int_0^\infty |D_x|\partial_t u_p^{(1)}\,dz.\end{aligned}$$

Proof. We introduce a new function $\Phi(t,x,y)$ defined by

$$\triangle\Phi = 0, \quad \Phi(t,x,0) = -\varepsilon^2\int_0^\infty u_p^{(1)}\,\mathrm{d}z.$$

It is easy to see that

$$\partial_y\Phi|_{y=0} = \varepsilon^2\int_0^\infty |D_x|u_p^{(1)}\,\mathrm{d}z.$$

Let $\nabla^\perp\Phi = (\partial_y\Phi, -\partial_x\Phi)$ and recall the boundary condition of u^R

$$u^R|_{y=0} = 0, \quad v^R = -\varepsilon^2\int_0^\infty \partial_x u_p^{(1)}\,\mathrm{d}z.$$

We find that

$$\operatorname{div}(U^R + \nabla^\perp\Phi) = 0, \quad \operatorname{curl}(U^R + \nabla^\perp\Phi) = \omega^R$$

with the boundary condition

$$(v^R - \partial_x\Phi)|_{y=0} = 0.$$

Then the Biot-Savart law ensures that

$$\partial_y(-\Delta_D)^{-1}\omega_t^R|_{y=0} = \partial_t(u^R + \partial_y\Phi)|_{y=0} = \varepsilon^2\int_0^\infty |D_x|\partial_t u_p^{(1)}\,\mathrm{d}z,$$

which along with (3.3) gives

$$\begin{aligned}
&\Big(\partial_y(-\Delta_D)^{-1}\operatorname{curl}(\widetilde{U}^R\cdot\nabla(U^R+U^a) + \widetilde{U^a}\cdot\nabla U^R - \widetilde{R})\Big)\Big|_{y=0} \\
&\quad+\varepsilon^2\int_0^\infty |D_x|\partial_t u_p^{(1)}\,\mathrm{d}z \\
&= \varepsilon^2\Big(\partial_y(-\Delta_D)^{-1}\triangle\omega^R\Big)|_{y=0}.
\end{aligned} \tag{3.4}$$

We denote by $\mathcal{H}(\omega|_{y=0})$ the harmonic extension of $\omega|_{y=0}$. It holds that

$$\partial_y\mathcal{H}(\omega|_{y=0})|_{y=0} = -|D_x|\omega|_{y=0}.$$

Thus, we have

$$\begin{aligned}
(\partial_y(-\Delta_D)^{-1}\triangle\omega^R)|_{y=0} &= \Big(\partial_y(-\Delta_D)^{-1}(\triangle(\omega^R - \mathcal{H}(\omega|_{y=0})))\Big)\,\Big|_{y=0} \\
&= -\,(\partial_y + |D|)\omega^R|_{y=0},
\end{aligned}$$

which together with (3.4) implies the lemma. □

4. The Analytic Functional Spaces

4.1. Conormal Sobolev space

As in [23,24], we introduce conormal operator $Z = \varphi(y)\partial_y$, where φ is a smooth function defined by

$$\varphi(y) = \begin{cases} \delta y \quad \text{for} \quad y \leqq 1, \\ \dfrac{\delta y}{1+y} \quad \text{for} \quad y \geqq 2, \end{cases}$$

where δ is a small constant determined later. In the sequel, we denote

$$Z^k = \varphi(y)^k \partial_y^k, \quad \widetilde{Z}^k = (\delta z)^k \partial_z^k.$$

The conormal Sobolev space $\overline{H}^s$ for $s \in \mathbf{N}$ is defined by

$$\overline{H}^s \stackrel{\text{def}}{=} \Big\{ u \in L^2_{x,y}(\mathbf{R}^2_+) : \|u\|_{\overline{H}^s} = \sum_{k+\ell \leqq s} \|Z^k \partial_x^\ell u\|_{L^2_{x,y}} < \infty \Big\}.$$

We denote by $\langle \cdot, \cdot \rangle_{\overline{H}^s}$ the $\overline{H}^s$ inner product.

Let $\rho(t) \in [1, 2]$ be a decreasing function. For a function u compactly supported in Fourier space in x variable, we define

$$u_\rho(x, y) \stackrel{\text{def}}{=} e^{\rho(t)\langle D_x \rangle} u(x, y) = \mathcal{F}^{-1}_{\xi \to x}\big(e^{\rho(t)\langle \xi \rangle} \mathcal{F}_{x \to \xi} u(\xi, y)\big).$$

We have the following product estimates in $\overline{H}^s$.

Lemma 4.1. *Let* $s \geqq 7$ *and* $\sigma \in [0, 1]$. *It holds that*

$$\begin{aligned} \|\langle D_x \rangle^{-\sigma} (uv)_\rho\|_{\overline{H}^s} \leqq C \Bigg(&\sum_{\ell+k \leqq s} \|Z^\ell \partial_x^k \langle D_x \rangle^{-\sigma} u_\rho\|_{L^\infty_y(L^2_x)} \|v_\rho\|_{\overline{H}^{s-3}} \\ &+ \sum_{\ell+k \leqq s-3} \|Z^\ell \partial_x^k u_\rho\|_{L^\infty_y(L^2_x)} \|\langle D_x \rangle^{-\sigma} v_\rho\|_{\overline{H}^s} \Bigg), \\ \|\langle D_x \rangle^{-\sigma} (uv)_\rho\|_{\overline{H}^s} \leqq C \Bigg(&\sum_{\ell+k \leqq s-3} \|Z^\ell \partial_x^k v_\rho\|_{L^\infty_y(L^2_x)} \|\langle D_x \rangle^{-\sigma} u_\rho\|_{\overline{H}^s} \\ &+ \sum_{\ell+k \leqq s-3} \|Z^\ell \partial_x^k u_\rho\|_{L^\infty_y(L^2_x)} \|\langle D_x \rangle^{-\sigma} v_\rho\|_{\overline{H}^s} \Bigg). \end{aligned}$$

Proof. Let us first consider the case of $\rho = 0$. By the definition of $\overline{H}^s$, we have

$$\|\langle D_x\rangle^{-\sigma}(uv)\|_{\overline{H}^s} \leqq C \sum_{\ell+k\leqq s} \|Z^\ell \partial_x^k \langle D_x\rangle^{-\sigma}(uv)\|_{L^2_{x,y}}.$$

We only consider the case of $\ell = s$, the other cases can be treated in a similar way. Recall the classical product estimate in Sobolev space(see [3]):

$$\|\langle D_x\rangle^{\sigma}(uv)\|_{L^2_x} \le C\|\langle D_x\rangle^{\sigma} u\|_{L^2_x}\|v\|_{H^1_x},$$

which gives, by a dual argument, that

$$\|\langle D_x\rangle^{-\sigma}(uv)\|_{L^2_x} \le C\|\langle D_x\rangle^{-\sigma} u\|_{L^2_x}\|v\|_{H^1_x}.$$

Thus, we deduce that

$$\begin{aligned}
&\|Z^s\langle D_x\rangle^{-\sigma}(uv)\|_{L^2_{x,y}}\\
&\quad\leqq C \sup_{\substack{s_1+s_2=s,\\ s_1\geqq s_2}} \|\langle D_x\rangle^{-\sigma}(Z^{s_1}uZ^{s_2}v)\|_{L^2_{x,y}} + C \sup_{\substack{s_1+s_2=s,\\ s_1\leqq s_2}} \|\langle D_x\rangle^{-\sigma}(Z^{s_1}uZ^{s_2}v)\|_{L^2_{x,y}}\\
&\quad\leqq C \sup_{\substack{s_1+s_2=s,\\ s_2\leqq[\frac{s}{2}]-1}} \|\langle D_x\rangle^{-\sigma} Z^{s_1}u\|_{L^\infty_y(L^2_x)}\|Z^{s_2}v\|_{L^2_y(H^1_x)}\\
&\qquad + C \sup_{\substack{s_1+s_2=s,\\ s_1\leqq[\frac{s}{2}]-1}} \|\langle D_x\rangle^{-\sigma} Z^{s_2}v\|_{L^2_y(L^2_x)}\|Z^{s_1}u\|_{L^\infty_y(H^1_x)}\\
&\qquad + C\|\langle D_x\rangle^{-\sigma} Z^{s-[\frac{s}{2}]}u\|_{L^\infty_y(H^1_x)}\|Z^{[\frac{s}{2}]}v\|_{L^2_y(L^2_x)}\\
&\qquad + C\|\langle D_x\rangle^{-\sigma} Z^{s-[\frac{s}{2}]}v\|_{L^2_y(H^1_x)}\|Z^{[\frac{s}{2}]}u\|_{L^\infty_y(L^2_x)}\\
&\qquad + C\|\langle D_x\rangle^{-\sigma} Z^{s-[\frac{s}{2}]-1}u\|_{L^\infty_y(H^1_x)}\|Z^{[\frac{s}{2}]+1}v\|_{L^2_y(L^2_x)}\\
&\qquad + C\|\langle D_x\rangle^{-\sigma} Z^{s-[\frac{s}{2}]-2}u\|_{L^\infty_y(L^2_x)}\|Z^{[\frac{s}{2}]+2}v\|_{L^2_y(H^1_x)}\\
&\quad\leqq C\sum_{\ell\le s}\|Z^\ell\langle D_x\rangle^{-\sigma}u\|_{L^\infty_y(L^2_x)}\|v\|_{\overline{H}^{s-3}} + C\sum_{\ell\leqq s-3}\|Z^\ell u\|_{L^\infty_y(L^2_x)}\|\langle D_x\rangle^{-\sigma}v\|_{\overline{H}^s}.
\end{aligned}$$

In order to deal with the general case, we only need to notice the fact that

$$\mathcal{F}\big((uv)_\rho\big)(\xi) \le \mathcal{F}\big(u^+_\rho v^+_\rho\big)(\xi),$$

where we denote by u^+ the inverse Fourier transform of $|\widehat{u}|$, and the map $u \mapsto u^+$ preserves the L^2_x norm. This proves the first inequality of the lemma. The proof of the second one is similar. □

4.2. Conormal analytic norm

Let us first introduce some conormal analytic norms

$$\|u\|_{X^s}^2 \stackrel{\text{def}}{=} \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \|Z^m u_\rho\|_{\overline{H}^s}^2,$$

$$\|u\|_{X^{s,\frac12}}^2 \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \frac{\rho(t)^{2m-1} m}{(m!)^2} \|Z^m u_\rho\|_{\overline{H}^s}^2 + \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \|Z^m \langle D_x\rangle^{\frac12} u_\rho\|_{\overline{H}^s}^2,$$

$$\|u\|_{\widetilde{X}^s}^2 \stackrel{\text{def}}{=} \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \|\widetilde{Z}^m u_\rho\|_{\overline{H}^s}^2.$$

To deal with the nonlinear term like $u\partial_x v$, we need the following lemmas.

Lemma 4.2. *Let $s \geqq 7$. It holds that*

(1) $$\sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \langle Z^m(u\partial_x v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s} \leqq C\big(\|u\|_{X^s}^2 + \|\partial_y u\|_{X^s}^2\big)\|v\|_{X^{s,\frac12}}^2 + C\|w\|_{X^{s,\frac12}}^2,$$

(2) $$\sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \langle Z^m(u\partial_x v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s} \leqq C\|u\|_{X^s}^2\big(\|v\|_{X^{s,\frac12}}^2 + \|\partial_y v\|_{X^{s,\frac12}}^2\big) + C\|w\|_{X^{s,\frac12}}^2,$$

(3) $$\sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \langle Z^m(u\partial_x v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s} \leqq C\big(\|u\|_{X^{s-2}}^2 + \|\partial_y u\|_{X^{s-2}}^2\big)\|v\|_{X^{s,\frac12}}^2 + \|u\|_{X^s}^2\big(\|v\|_{X^{s-2}}^2 + \|\partial_y v\|_{X^{s-2}}^2\big) + C\|w\|_{X^{s,\frac12}}^2.$$

Proof. Using Leibniz's rule,

$$Z^m(u\partial_x v)_\rho = \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!}\big(Z^{m_1}u \cdot Z^{m_2}\partial_x v\big)_\rho,$$

we get

$$\langle Z^m(u\partial_x v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s} \leqq \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \|\langle D_x\rangle^{-\frac12}(Z^{m_1}uZ^{m_2}\partial_x v)_\rho\|_{\overline{H}^s} \|\langle D_x\rangle^{\frac12} Z^m w_\rho\|_{\overline{H}^s}.$$

It follows from Lemma 4.1 and Sobolev embedding that

$$\begin{aligned}
&\|\langle D_x\rangle^{-\frac12}(Z^{m_1}u_\rho Z^{m_2}\partial_x v_\rho)\|_{\overline{H}^s} \\
&\leqq C\big(\|\langle D_x\rangle^{-\frac12} Z^{m_1}u_\rho\|_{\overline{H}^s} + \|\partial_y\langle D_x\rangle^{-\frac12} Z^{m_1}u_\rho\|_{\overline{H}^s}\big)\|Z^{m_2}v_\rho\|_{\overline{H}^{s-2}} \\
&\quad + C\big(\|Z^{m_1}u_\rho\|_{\overline{H}^{s-3}} + \|\partial_y Z^{m_1}u_\rho\|_{\overline{H}^{s-3}}\big)\|\langle D_x\rangle^{\frac12} Z^{m_2}v_\rho\|_{\overline{H}^s}.
\end{aligned}$$

Then we get by Young's inequality that

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u\partial_x v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}\\
&\leqq C\sum_{m=0}^{\infty}\Bigg(\sum_{m_1+m_2=m}\frac{\rho(t)^m}{m_1!m_2!}\Big(\|\langle D_x\rangle^{-\frac12}Z^{m_1}u_\rho\|_{\overline{H}^s}+\|\langle D_x\rangle^{-\frac12}Z^{m_1}\partial_y u_\rho\|_{\overline{H}^s}\Big)\\
&\qquad\qquad \|Z^{m_2}v_\rho\|_{\overline{H}^{s-2}}\Bigg)^2\\
&\quad + C\sum_{m=0}^{\infty}\Bigg(\sum_{m_1+m_2=m}\frac{\rho(t)^m}{m_1!m_2!}\Big(\|Z^{m_1}u_\rho\|_{\overline{H}^{s-3}}+\|Z^{m_1}\partial_y u_\rho\|_{\overline{H}^{s-3}}\Big)\\
&\qquad\qquad \|\langle D_x\rangle^{\frac12}Z^{m_2}v_\rho\|_{\overline{H}^s}\Bigg)^2\\
&\quad + C\|w\|^2_{X^{s,\frac12}}\\
&\leqq C\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\big(\|\langle D_x\rangle^{-\frac12}Z^m u_\rho\|_{\overline{H}^s}+\|\langle D_x\rangle^{-\frac12}Z^m\partial_y u_\rho\|_{\overline{H}^s}\big)^2\\
&\qquad\qquad \Bigg(\sum_{m=0}^{\infty}\frac{\rho(t)^m}{m!}\|Z^m v_\rho\|_{\overline{H}^{s-2}}\Bigg)^2\\
&\quad + C\Bigg(\sum_{m=0}^{\infty}\frac{\rho(t)^m}{m_1!m_2!}\Big(\|Z^m u_\rho\|_{\overline{H}^{s-3}}+\|Z^m\partial_y u_\rho\|_{\overline{H}^{s-3}}\Big)\Bigg)^2\\
&\qquad \sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\|\langle D_x\rangle^{\frac12}Z^{m_2}v_\rho\|^2_{\overline{H}^s}\\
&\quad + C\|w\|^2_{X^{s,\frac12}}\\
&\leqq C\big(\|u\|^2_{X^s}+\|\partial_y u\|^2_{X^s}\big)\|v\|^2_{X^{s,\frac12}}+C\|w\|^2_{X^{s,\frac12}}.
\end{aligned}
$$

Here we used the fact that

$$
\sum_{m=0}^{\infty}\frac{\rho(t)^m}{m!}\|Z^m f_\rho\|_{\overline{H}^{s-1}}\leq\|f\|_{X^s}.
$$

This proves the first inequality. Another two inequalities can be similarly proved by using the following product estimates(by Lemma 4.1)

$$
\begin{aligned}
&\|\langle D_x\rangle^{-\frac12}(Z^{m_1}u_\rho Z^{m_2}\partial_x v_\rho)\|_{\overline{H}^s}\\
&\quad\leqq C\|\langle D_x\rangle^{-\frac12}Z^{m_1}u_\rho\|_{\overline{H}^s}\big(\|Z^{m_2}v_\rho\|_{\overline{H}^{s-2}}+\|\partial_y Z^{m_2}v_\rho\|_{\overline{H}^{s-2}}\big)\\
&\qquad + C\|Z^{m_1}u_\rho\|_{\overline{H}^{s-2}}\big(\|\langle D_x\rangle^{\frac12}Z^{m_2}v_\rho\|_{\overline{H}^s}+\|\langle D_x\rangle^{\frac12}\partial_y Z^{m_2}v_\rho\|_{\overline{H}^s}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\langle D_x\rangle^{-\frac{1}{2}}(Z^{m_1}u_\rho Z^{m_2}\partial_x v_\rho)\|_{\overline{H}^s}\\
&\quad\leqq C\big(\|Z^{m_1}u_\rho\|_{\overline{H}^{s-3}}+\|\partial_y Z^{m_1}u_\rho\|_{\overline{H}^{s-3}}\big)\|\langle D_x\rangle^{\frac{1}{2}}Z^{m_2}v_\rho\|_{\overline{H}^s}\\
&\qquad+C\big(\|\partial_y Z^{m_2}v_\rho\|_{\overline{H}^{s-3}}+\|Z^{m_2}v_\rho\|_{\overline{H}^{s-3}}\big)\|Z^{m_1}u_\rho\|_{\overline{H}^s}.
\end{aligned}
$$

□

Using the fact that for $u(x,z)=u(x,\frac{y}{\varepsilon})$,

$$
|Z^m u|\leqq(\widetilde{Z}^m u)\left(x,\frac{y}{\varepsilon}\right),
$$

we can see from the proof of Lemma 4.2 that we have

Lemma 4.3. *Let $s\geqq 7$. For $u(x,z)=u(x,\frac{y}{\varepsilon})$, we have*

$$
\begin{aligned}
\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u\partial_x v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}\leqq C\big(\|u\|^2_{\widetilde{X}^s}+\|\partial_z u\|^2_{\widetilde{X}^s}\big)\|v\|^2_{X^{s,\frac{1}{2}}}\\
+C\|w\|^2_{X^{s,\frac{1}{2}}}.
\end{aligned}
$$

For $v(x,z)=v(x,\frac{y}{\varepsilon})$, we have

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u\partial_x v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}\\
&\quad\leqq C\|u\|^2_{X^s}\big(\|v\|^2_{\widetilde{X}^{s,\frac{1}{2}}}+\|\partial_z v\|^2_{\widetilde{X}^{s,\frac{1}{2}}}\big)+C\|w\|^2_{X^{s,\frac{1}{2}}}.
\end{aligned}
$$

The proof of Lemma 4.2 also shows that

Lemma 4.4. *Let $s\geqq 4$. Then it holds that*

$$
\begin{aligned}
&\|uv\|_{X^s}\leq C\|u\|_{X^s}\big(\|v\|_{X^s}+\|\partial_y v\|_{X^s}\big),\\
&\|\langle D_x\rangle^{\frac{1}{2}}(uv)\|_{X^s}\leq C\|u\|_{X^{s,\frac{1}{2}}}\big(\|v\|_{X^s}+\|\partial_y v\|_{X^s}\big)\\
&\qquad\qquad+C\|u\|_{X^s}\big(\|v\|_{X^{s,\frac{1}{2}}}+\|\partial_y v\|_{X^{s,\frac{1}{2}}}\big),
\end{aligned}
$$

and for $\overline{v}(x,z)=\overline{v}(x,\frac{y}{\varepsilon})$,

$$
\begin{aligned}
&\|u\overline{v}\|_{X^s}\leq C\|u\|_{X^s}\big(\|\overline{v}\|_{\widetilde{X}^s}+\|\partial_z\overline{v}\|_{\widetilde{X}^s}\big),\\
&\|\langle D_x\rangle^{\frac{1}{2}}(u\overline{v})\|_{X^s}\leq C\|u\|_{X^{s,\frac{1}{2}}}\big(\|\overline{v}\|_{\widetilde{X}^{s,\frac{1}{2}}}+\|\partial_z\overline{v}\|_{\widetilde{X}^{s,\frac{1}{2}}}\big).
\end{aligned}
$$

The following lemmas will be used to deal with the nonlinear terms like $u\partial_y v$ with $u|_{y=0}=0$.

Lemma 4.5. *Let* $s \geqq 7$. *If* $u|_{y=0} = 0$, *we have*

$$(1)\quad \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \langle Z^m (u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s} \leqq C\big(\|u\|_{X^s}^2 + \|\partial_y u\|_{X^s}^2 + \|\partial_y^2 u\|_{X^s}^2\big)\|v\|_{X^s}^2 + C\big(\|u\|_{X^{s-2}}^2 + \|\partial_y u\|_{X^{s-2}}^2 + \|\partial_y^2 u\|_{X^{s-2}}^2\big)\|v\|_{X^{s,\frac12}}^2 + C\|w\|_{X^{s,\frac12}}^2,$$

$$(2)\quad \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \langle Z^m (u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s} \leqq C\big(\|\partial_y \langle D_x\rangle^{-\frac12} u\|_{X^s}^2 + \|u\|_{X^s}^2\big)\big(\|v\|_{X^{s+1}}^2 + \|\partial_y v\|_{X^{s+1}}^2\big) + C\|w\|_{X^{s,\frac12}}^2,$$

$$(3)\quad \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \langle Z^m (u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s} \leqq \delta C\big(\|\partial_y \langle D_x\rangle^{-\frac12} u\|_{X^s}^2 + \|u\|_{X^s}^2\big)\big(\|v\|_{X^{s-2}}^2 + \|\partial_y v\|_{X^{s-2}}^2\big) + \delta C\big(\|u\|_{X^{s-2}}^2 + \|\partial_y u\|_{X^{s-2}}^2 + \|\partial_y^2 u\|_{X^{s-2}}^2\big)\|v\|_{X^{s,\frac12}}^2 + C_\delta\|w\|_{X^{s,\frac12}}^2,$$

where $\delta > 0$.

Proof. We infer from Leibniz's rule and Lemma 4.1 that

$$\begin{aligned}
&\langle Z^m (u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}\\
&\leqq \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \Big\|\Big(\frac{Z^{m_1}u}{\varphi} Z^{m_2+1} v\Big)_\rho\Big\|_{\overline{H}^s} \|Z^m w_\rho\|_{\overline{H}^s}\\
&\leqq C \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \sum_{k+\ell\leqq s} \Big\|Z^\ell \partial_x^k \frac{Z^{m_1}u_\rho}{\varphi}\Big\|_{L_y^\infty(L_x^2)} \|Z^{m_2} v_\rho\|_{\overline{H}^{s-3}} \|Z^m w_\rho\|_{\overline{H}^s}\\
&\quad + C \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \sum_{k+\ell\leqq s-4} \Big\|Z^\ell \partial_x^k \frac{Z^{m_1}u_\rho}{\varphi}\Big\|_{L_y^\infty(L_x^2)} \|Z^{m_2+1} v_\rho\|_{\overline{H}^s}\\
&\quad \|Z^m w_\rho\|_{\overline{H}^s}.
\end{aligned}$$

Thanks to $Z^{m_1}u_\rho|_{y=0} = 0$ and Sobolev embedding, we have

$$\begin{aligned}
\Big\|Z^\ell \partial_x^k \frac{Z^{m_1}u_\rho}{\varphi}\Big\|_{L_y^\infty(L_x^2)} &\leqq C\big(\|\partial_y Z^\ell \partial_x^k Z^{m_1} u_\rho\|_{L_y^\infty(L_x^2)} + \|Z^\ell \partial_x^k Z^{m_1} u_\rho\|_{L_y^\infty(L_x^2)}\big)\\
&\leqq C\big(\|Z^{m_1}u_\rho\|_{\overline{H}^{k+\ell}} + \|\partial_y Z^{m_1}u_\rho\|_{\overline{H}^{k+\ell}} + \|\partial_y^2 Z^{m_1}u_\rho\|_{\overline{H}^{k+\ell}}\big).
\end{aligned}$$

This gives

$$\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}$$
$$\leqq C\sum_{m=1}^{\infty}\Big(\sum_{m_1+m_2=m}\frac{\rho(t)^m}{\sqrt{m}m_1!m_2!}\big(\|Z^{m_1}u_\rho\|_{\overline{H}^s}+\|\partial_y Z^{m_1}u_\rho\|_{\overline{H}^s}$$
$$+\|\partial_y^2 Z^{m_1}u_\rho\|_{\overline{H}^s}\big)\|Z^{m_2}v_\rho\|_{\overline{H}^{s-3}}\Big)^2$$
$$+C\sum_{m=1}^{\infty}\Big(\sum_{m_1+m_2=m}\frac{\rho(t)^m}{\sqrt{m}m_1!m_2!}\big(\|Z^{m_1}u_\rho\|_{\overline{H}^{s-4}}$$
$$+\|\partial_y Z^{m_1}u_\rho\|_{\overline{H}^{s-4}}+\|\partial_y^2 Z^{m_1}u_\rho\|_{\overline{H}^{s-4}}\big)$$
$$\times\|Z^{m_2+1}v_\rho\|_{\overline{H}^s}\Big)^2+C\|w\|^2_{X^{s,\frac12}}.$$

Using the fact that

$$\|\partial_y Z^m f_\rho\|_{\overline{H}^s}\leq C\big(\|Z^m\partial_y f_\rho\|_{\overline{H}^s}+m\|Z^{m-1}\partial_y f_\rho\|_{\overline{H}^s}\big),$$

we get by Young's inequality that

$$\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}\leqq C\big(\|u\|^2_{X^s}+\|\partial_y u\|^2_{X^s}+\|\partial_y^2 u\|^2_{X^s}\big)\|v\|^2_{X^s}$$
$$+C\big(\|u\|^2_{X^{s-2}}+\|\partial_y u\|^2_{X^{s-2}}+\|\partial_y^2 u\|^2_{X^{s-2}}\big)\|v\|^2_{X^{s,\frac12}}+C\|w\|^2_{X^{s,\frac12}}.$$

The first inequality is proved. To show the second inequality, we apply Lemma 4.1 again to obtain

$$\langle Z^m(u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}$$
$$\leqq\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\|\langle D_x\rangle^{-\frac12}\Big(\frac{Z^{m_1}u}{\varphi}Z^{m_2+1}v\Big)_\rho\|_{\overline{H}^s}\|\langle D_x\rangle^{\frac12}Z^m w_\rho\|_{\overline{H}^s}$$
$$\leqq C\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\sum_{k+\ell\leqq s}\|Z^\ell\partial_x^k Z^{m_2+1}v_\rho\|_{L_y^\infty(L_x^2)}\|\frac{Z^{m_1}u_\rho}{\varphi}\|_{\overline{H}^{s-4}}\|\langle D_x\rangle^{\frac12}$$
$$Z^m w_\rho\|_{\overline{H}^s}$$
$$+C\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\sum_{k+\ell\leqq s-4}\|Z^\ell\partial_x^k Z^{m_2+1}v_\rho\|_{L_y^\infty(L_x^2)}\|\langle D_x\rangle^{-\frac12}\frac{Z^{m_1}u_\rho}{\varphi}$$
$$\|_{\overline{H}^s}\|\langle D_x\rangle^{\frac12}Z^m w_\rho\|_{\overline{H}^s},$$

and by Hardy's inequality,

$$\|\frac{Z^{m_1}u_\rho}{\varphi}\|_{\overline{H}^k}\leqq C\big(\|Z^{m_1}u_\rho\|_{\overline{H}^k}+\|\partial_y Z^{m_1}u_\rho\|_{\overline{H}^k}\big).$$

Then we deduce that

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}\\
&\leqq C\sum_{m=0}^{\infty}\Big(\sum_{m_1+m_2=m}\frac{\rho(t)^m}{m_1!m_2!}\big(\|\partial_y\langle D_x\rangle^{-\frac12}Z^{m_1}u_\rho\|_{\overline{H}^s}+\|Z^{m_1}u_\rho\|_{\overline{H}^s}\big)\\
&\quad\times\big(\|Z^{m_2+1}v_\rho\|_{\overline{H}^{s-4}}+\|\partial_y Z^{m_2+1}v_\rho\|_{\overline{H}^{s-4}}\big)\Big)^2\\
&\quad+C\sum_{m=0}^{\infty}\Big(\sum_{m_1+m_2=m}\frac{\rho(t)^m}{m_1!m_2!}\big(\|\partial_y Z^{m_1}u_\rho\|_{\overline{H}^{s-4}}+\|Z^{m_1}u_\rho\|_{\overline{H}^{s-4}}\big)\\
&\quad\times\big(\|Z^{m_2+1}v_\rho\|_{\overline{H}^s}+\|\partial_y Z^{m_2+1}v_\rho\|_{\overline{H}^s}\big)\Big)^2+C\|w\|^2_{X^{s,\frac12}}\\
&\leqq C\big(\|\partial_y\langle D_x\rangle^{-\frac12}u\|^2_{X^s}+\|u\|^2_{X^s}\big)\big(\|v\|^2_{X^{s+1}}+\|\partial_y v\|^2_{X^{s+1}}\big)+C\|w\|^2_{X^{s,\frac12}}.
\end{aligned}
$$

This proves the second inequality. By Lemma 4.1 and Hardy's inequality, we have

$$
\begin{aligned}
&\langle Z^m(u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}\\
&=\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\left\langle\left(\frac{Z^{m_1}u}{\varphi}\,Z^{m_2+1}v\right)_\rho, Z^m w_\rho\right\rangle_{\overline{H}^s}\\
&\leqq\sum_{m_1\geqq m_2}\frac{m!}{m_1!m_2!}\left\langle\left(\frac{Z^{m_1}u}{\varphi}Z^{m_2+1}v\right)_\rho, Z^m w_\rho\right\rangle_{\overline{H}^s}\\
&\quad+\sum_{m_1\leqq m_2}\frac{m!}{m_1!m_2!}\left\langle\left(\frac{Z^{m_1}u}{\varphi}Z^{m_2+1}v\right)_\rho, Z^m w_\rho\right\rangle_{\overline{H}^s}\\
&\leqq\sum_{m_1\geqq m_2}\frac{m!}{m_1!m_2!}\|\langle D_x\rangle^{-\frac12}\left(\frac{Z^{m_1}u}{\varphi}Z^{m_2+1}v\right)_\rho\|_{\overline{H}^s}\|\langle D_x\rangle^{\frac12}Z^m w_\rho\|_{\overline{H}^s}\\
&\quad+\sum_{m_1\leqq m_2}\frac{m!}{m_1!m_2!}\|\frac{Z^{m_1}u_\rho}{\varphi}\|_{\overline{H}^{s-4}}\|Z^{m_2+1}v_\rho\|_{\overline{H}^s}\|Z^m w_\rho\|_{\overline{H}^s}\\
&\quad+\sum_{m_1\leqq m_2}\frac{m!}{m_1!m_2!}\|\langle D_x\rangle^{-\frac12}\frac{Z^{m_1}u_\rho}{\varphi}\|_{\overline{H}^s}\|Z^{m_2+1}v_\rho\|_{\overline{H}^{s-4}}\|\langle D_x\rangle^{\frac12}Z^m w_\rho\|_{\overline{H}^s}\\
&\leqq\delta C\Big(\sum_{m_1\geqq m_2}\frac{m!}{m_1!m_2!}\big(\|\partial_y\langle D_x\rangle^{-\frac12}Z^{m_1}u_\rho\|_{\overline{H}^s}\\
&\quad+\|Z^{m_1}u_\rho\|_{\overline{H}^s}\big)\left(\|Z^{m_2+1}v_\rho\|_{\overline{H}^{s-4}}+\|\partial_y Z^{m_2+1}v_\rho\|_{\overline{H}^{s-4}}\right)\Big)^2\\
&\quad+\delta C\Big(\sum_{m_1\geqq m_2}\frac{m!}{m_1!m_2!}\big(\|Z^{m_1}u_\rho\|_{\overline{H}^{s-4}}+\|\partial_y Z^{m_1}u_\rho\|_{\overline{H}^{s-4}}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\partial_y^2 Z^{m_1} u_\rho\|_{\overline{H}^{s-4}}\big)\|Z^{m_2+1} v_\rho\|_{\overline{H}^s}\Big)^2\\
&+\delta C\Big(\sum_{m_1\leqq m_2}\frac{m!}{\sqrt{m}m_1!m_2!}\big(\|Z^{m_1}u_\rho\|_{\overline{H}^{s-4}}\\
&+\|\partial_y Z^{m_1} u_\rho\|_{\overline{H}^{s-4}}+\|\partial_y^2 Z^{m_1} u_\rho\|_{\overline{H}^{s-4}}\big)\|Z^{m_2+1} v_\rho\|_{\overline{H}^s}\Big)^2\\
&+\delta C\Big(\sum_{m_1\leqq m_2}\frac{m!}{\sqrt{m}m_1!m_2!}\Big(\|\langle D_x\rangle^{-\frac12}\partial_y Z^{m_1}u_\rho\|_{\overline{H}^s}+\|\langle D_x\rangle^{-\frac12}Z^{m_1}u_\rho\|_{\overline{H}^s}\Big)\\
&\quad\cdot\Big(\|Z^{m_2+1}v_\rho\|_{\overline{H}^{s-4}}+\|\partial_y Z^{m_2+1}v_\rho\|_{\overline{H}^{s-4}}\Big)\Big)^2\\
&+C\|\langle D_x\rangle^{\frac12}Z^m w_\rho\|_{\overline{H}^s}^2+Cm\|Z^m w_\rho\|_{\overline{H}^s}^2,
\end{aligned}
$$

which implies the third inequality. □

An argument similar to that of Lemma 4.5 yields

Lemma 4.6. *Let $s\geqq 7$ and $u|_{y=0}=0$. For $u(x,z)=\varepsilon\widetilde{u}(x,\frac{y}{\varepsilon})$, we have*

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}\\
&\quad\leqq C\big(\|\widetilde{u}\|_{\widetilde{X}^s}^2+\|\partial_z\widetilde{u}\|_{\widetilde{X}^s}^2+\|\partial_z^2\widetilde{u}\|_{\widetilde{X}^s}^2\big)\|v\|_{X^{s,\frac12}}^2+C\|w\|_{X^{s,\frac12}}.
\end{aligned}
$$

For $v(x,z)=v(x,\frac{y}{\varepsilon})$, we have

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u\partial_y v)_\rho, Z^m w_\rho\rangle_{\overline{H}^s}\\
&\leqq C\big(\|\partial_y\langle D_x\rangle^{-\frac12}u\|_{X^s}^2+\|u\|_{X^s}^2\big)\big(\|v\|_{\widetilde{X}^{s+1}}^2+\|\partial_z v\|_{\widetilde{X}^{s+1}}^2\big)+C\|w\|_{X^{s,\frac12}}^2.
\end{aligned}
$$

5. Conormal Analytic Type Estimates of Velocity

In the sequel, we denote by C a constant independent of ε, and by C_1 a constant independent of ε,δ, which may depend on M defined in (1.5). In the sequel, we take

$$\rho(t)=2-\lambda t$$

for some $\lambda>0$ determined later. We always assume $\rho(t)\in[1,2]$.

For the moment, let us assume the following uniform estimates for the approximate solution and the remainder $\widetilde{R}$ defined in Section 2. The proof will be presented in appendix.

Lemma 5.1. *There exists $T_a > 0$ such that for any $t \in [0, T_a]$, there holds*

$$\sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \|Z^m (u_e^{(i)}, v_e^{(i)})_\rho\|_{H^{12}_{x,y}}^2 \leq C,$$

$$\sum_{s=0}^{2}\sum_{k=0}^{12}\sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \|e^{z^2} \widetilde{Z}^{m+k} \partial_z^s (u_p^{(i)}, v_p^{(i+1)})_\rho\|_{L^2_z(H_x^{12-k})}^2 \leq C,$$

for $i = 0, 1$.

Lemma 5.2. *There exists $T_a > 0$ such that for any $t \in [0, T_a]$, there holds*

$$\|\widetilde{R}\|_{X^{10}} \leqq C\varepsilon^2, \quad \|\nabla \widetilde{R}\|_{X^{10}} \leq C\varepsilon.$$

The main result of this section is the conormal analytic type estimate of the velocity.

Proposition 5.3. *There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ so that for any $\delta \in (0, \delta_0)$ and $\varepsilon \in (0, \varepsilon_0)$, it holds that*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|U^R\|_{X^8}^2 + \|\langle D_x\rangle^9 U_\rho^R\|_{L^2}^2\right) + \lambda\left(\|U^R\|_{X^{8,\frac12}}^2 + \|\langle D_x\rangle^{9.5} U_\rho^R\|_{L^2}^2\right)\\
&\quad + \frac{1}{4}\varepsilon^2\left(\|\nabla U^R\|_{X^8}^2 + \|\nabla \langle D_x\rangle^9 U_\rho^R\|_{L^2}^2\right)\\
&\leqq C\left(1 + \|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2\right)\left(\|U^R\|_{X^{8,\frac12}}^2 + \|\langle D_x\rangle^{9.5} U_\rho^R\|_{L^2}^2 + \varepsilon^4\right)\\
&\quad + C_1\delta\varepsilon^4\|\nabla\omega^R\|_{X^8}^2 + C\|\omega^R\|_{X^{8,\frac12}}^2\|U^R\|_{X^8}^2.
\end{aligned}$$

Proof. The proposition is a direct consequence of the following Lemmas 5.4–5.8. □

5.1. Conormal analytic energy estimates

Taking div on both sides of (3.2), we obtain

$$\begin{cases}
\triangle p^R = -\mathrm{div}(\widetilde{U}^R \cdot \nabla(U^R + U^a) + \widetilde{U}^a \cdot \nabla U^R) + \mathrm{div}\widetilde{R},\\
\partial_y p^R|_{y=0} = \varepsilon^2 \partial_t f - \varepsilon^4 \partial_x^2 f + R_2|_{y=0} - \varepsilon^2 \partial_x\partial_y u^R|_{y=0}.
\end{cases} \tag{5.1}$$

We first decompose the pressure p^R as follows:

$$p^R = p^{R,NS} + p^{R,E},$$

where

$$\begin{cases}
\triangle p^{R,NS} = 0,\\
\partial_y p^{R,NS}|_{y=0} = -\varepsilon^2 \partial_x \partial_y u^R|_{y=0},
\end{cases}$$

and

$$\begin{cases}
\triangle p^{R,E} = -\mathrm{div}(\widetilde{U}^R \cdot \nabla(U^R + U^a) + \widetilde{U}^a \cdot \nabla U^R) + \mathrm{div}\widetilde{R},\\
\partial_y p^{R,E}|_{y=0} = \varepsilon^2 \partial_t f - \varepsilon^4 \partial_x^2 f + R_2|_{y=0}.
\end{cases}$$

Lemma 5.4. *There exists* $\delta_0 > 0$ *so that for any* $\delta \in (0, \delta_0)$, *it holds that*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U^R\|_{X^8}^2 + \lambda\|U^R\|_{X^{8,\frac{1}{2}}}^2 + \frac{1}{2}\varepsilon^2\|\nabla U^R\|_{X^8}^2\\ &\leqq C\Big(1 + \|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2\Big)\Big(\|U^R\|_{X^{8,\frac{1}{2}}}^2 + \varepsilon^4\Big)\\ &\quad + C_1\delta\|\nabla p^{R,NS}\|_{X^8}^2 + C_1\delta\|\langle D_x\rangle^{\frac{1}{2}}\nabla p^{R,E}\|_{X^7}^2.\end{aligned}$$

Proof. We first take $e^{\rho(t)\langle D_x\rangle}$ on both sides of (3.2) to obtain

$$\partial_t U_\rho^R + \lambda\langle D_x\rangle U_\rho^R + \big(\widetilde{U}^R\cdot\nabla(U^R + U^a) + \widetilde{U}^a\cdot\nabla U^R\big)_\rho - \varepsilon^2\triangle U_\rho^R + \nabla p_\rho^R = \widetilde{R}_\rho. \tag{5.2}$$

Taking $\frac{\rho(t)^m}{m!}Z^m$ on the both sides of (5.2), and then taking $\overline{H}^8$ inner product with $\frac{\rho(t)^m}{m!}Z^m U_\rho^R$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U^R\|_{X^8}^2 + \lambda\|U^R\|_{X^{8,\frac{1}{2}}}^2 - \varepsilon^2\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\Big\langle Z^m\triangle U_\rho^R, Z^m U_\rho^R\Big\rangle_{\overline{H}^8}\\ &\leqq \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\Big\langle Z^m\big(\widetilde{U}^R\cdot\nabla(U^R + U^a) + \widetilde{U}^a\cdot\nabla U^R + \nabla p^R - \widetilde{R}\big)_\rho, Z^m U_\rho^R\Big\rangle_{\overline{H}^8}\Big|\\ &\leqq \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(\widetilde{U}^R\cdot\nabla(U^R + U^a))_\rho, Z^m U_\rho^R\rangle_{\overline{H}^8}\Big|\\ &\quad + \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(\widetilde{U}^a\cdot\nabla U^R)_\rho, Z^m U_\rho^R\rangle_{\overline{H}^8}\Big|\\ &\quad + \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(\nabla p^R)_\rho, Z^m U_\rho^R\rangle_{\overline{H}^8}\Big| + \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m\widetilde{R}_\rho, Z^m U_\rho^R\rangle_{\overline{H}^8}\Big|\\ &\triangleq I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Step 1. Estimate of I_1. We rewrite I_1 as

$$\widetilde{U}^R\cdot\nabla(U^R + U^a) = u^R\partial_x(U^R + U^a) + (v^R + \varepsilon^2 f e^{-y})\partial_y(U^R + U^a).$$

Then we infer that

$$\begin{aligned}I_1 \leqq &\Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u^R\partial_x U^R)_\rho, Z^m U_\rho^R\rangle_{\overline{H}^8}\Big|\\ &+ \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u^R\partial_x U^a)_\rho, Z^m U_\rho^R\rangle_{\overline{H}^8}\Big|\\ &+ \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m((v^R + \varepsilon^2 f e^{-y})\partial_y u^R)_\rho, Z^m u_\rho^R\rangle_{\overline{H}^8}\Big|\\ &+ \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m((v^R + \varepsilon^2 f e^{-y})\partial_y v^R)_\rho, Z^m v_\rho^R\rangle_{\overline{H}^8}\Big|\end{aligned}$$

$$+\Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m((v^R+\varepsilon^2 fe^{-y})\partial_y U^a)_\rho, Z^m U^R_\rho\rangle_{\overline{H}^8}\Big|$$
$$\triangleq I_{11}+I_{12}+I_{13}+I_{14}+I_{15}.$$

Estimate of I_{11}. By Lemma 4.2 (3), we get

$$\begin{aligned} I_{11} \leqq& C\big(1+\|U^R\|^2_{X^7}+\|\partial_y U^R\|^2_{X^7}\big)\|U^R\|^2_{X^{8,\frac12}}\\ \leqq& C\big(1+\|U^R\|^2_{X^8}+\|\omega^R\|^2_{X^7}\big)\|U^R\|^2_{X^{8,\frac12}}. \end{aligned}$$

Here we used the fact that

$$\|\partial_y U^R\|_{X^7}\leq \|\omega^R\|_{X^8}+\|U^R\|^2_{X^8}. \tag{5.3}$$

Estimate of I_{12}. By Lemmas 4.2(2), 4.3 and 5.1, we get

$$\begin{aligned} I_{12} \leqq& C\|U^R\|^2_{X^8}\big(\|U^e\|^2_{X^{8,\frac12}}+\|\partial_y U^e\|^2_{X^{8,\frac12}}+\|U^p\|^2_{\widetilde{X}^{8,\frac12}}+\|\partial_z U^p\|^2_{\widetilde{X}^{8,\frac12}}\big)\\ &+C\|U^R\|^2_{X^{8,\frac12}}\\ \leqq& C\|U^R\|^2_{X^{8,\frac12}}. \end{aligned}$$

Estimate of I_{13}. Thanks to $v^R+\varepsilon^2 fe^{-y}|_{y=0}=0$, we get by Lemma 4.5(3) that

$$\begin{aligned} I_{13} \leqq& C\big(\|\partial_y\langle D_x\rangle^{-\frac12}(v^R+\varepsilon^2 fe^{-y})\|^2_{X^8}+\|v^R+\varepsilon^2 fe^{-y}\|^2_{X^8}\big)\\ &\big(\|U^R\|^2_{X^7}+\|\partial_y U^R\|^2_{X^7}\big)\\ &+C\big(\|v^R+\varepsilon^2 fe^{-y}\|^2_{X^6}+\|\partial_y(v^R+\varepsilon^2 fe^{-y})\|^2_{X^6}\\ &+\|\partial_y^2(v^R+\varepsilon^2 fe^{-y})\|^2_{X^6}\big)\|U^R\|^2_{X^{8,\frac12}}\\ &+C\|U^R\|^2_{X^{8,\frac12}}\\ \leqq& C\big(1+\|U^R\|^2_{X^8}+\|\omega^R\|^2_{X^8}\big)\big(\|U^R\|^2_{X^{8,\frac12}}+\varepsilon^4\big). \end{aligned}$$

Here we used the facts that

$$\|\partial_y\langle D_x\rangle^{-\frac12}v^R\|_{X^8}\leq \|u^R\|_{X^{8,\frac12}} \text{ and } \|\partial_y^2 v^R\|^2_{X^6}\leq \|U^R\|_{X^8}+\|\omega^R\|^2_{X^8}. \tag{5.4}$$

Estimate of I_{14}. Using

$$(v^R+\varepsilon^2 fe^{-y})\partial_y v^R=-(v^R+\varepsilon^2 fe^{-y})\partial_x u^R,$$

we get by Lemma 4.2(3) that

$$I_{14}\leqq C\big(1+\|U^R\|^2_{X^8}+\|\omega^R\|^2_{X^8}\big)\|U^R\|^2_{X^{8,\frac12}}.$$

Estimate of I_{15}. Thanks to $v^R+\varepsilon fe^{-y}|_{y=0}=0$, we get by Lemmas 4.5(2), 4.6 and 5.1 that

$$I_{15}\leqq C\big(\|U^R\|^2_{X^{8,\frac12}}+\varepsilon^4\big).$$

Here we also used (5.4).

Putting the estimates of $I_{11}, \ldots, I_{15}$ together, we conclude that

$$I_1 \leqq C\big(1 + \|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2\big)\big(\|U^R\|_{X^{8,\frac12}}^2 + \varepsilon^4\big).$$

Step 2. Estimate of I_2 and I_4. Recall that $U^p = (u^p, \varepsilon v^p)(t, x, \frac{y}{\varepsilon})$. It follows from Lemmas 4.2(1), 4.3, 4.5(1), 4.6 and 5.1 that

$$I_2 \leqq C\|U^R\|_{X^{8,\frac12}}^2.$$

We infer from Lemma 5.2 that

$$I_4 \leqq C\varepsilon^4 + \|U^R\|_{X^8}^2.$$

Step 3. Estimate of I_3. Using the formula

$$[Z^k Z^m, \partial_y] = -m\varphi' Z^k Z^{m-1}\partial_y - k\varphi' Z^{k-1}\partial_y Z^m, \tag{5.5}$$

we get by integration by parts that

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m \nabla p_\rho^R, Z^m U_\rho^R\rangle_{\overline{H}^8} = -\sum_{m=1}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m p_\rho^R, Z^m \mathrm{div} U_\rho^R\rangle_{\overline{H}^8}\\
&\quad + \sum_{k+\ell\le 8}\sum_{m=1}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2}\langle [Z^k Z^m, \partial_y]\partial_x^\ell p_\rho^R, Z^k Z^m \partial_x^\ell v_\rho^R\rangle\\
&\quad + \sum_{k+\ell\le 8}\sum_{m=1}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2}\langle Z^k Z^m \partial_x^\ell p_\rho^R, [Z^k Z^m, \partial_y]\partial_x^\ell v_\rho^R\rangle + \langle \nabla p_\rho^R, U_\rho^R\rangle_{\overline{H}^8}\\
&\leqq C_1\delta \sum_{k+\ell\le 8}\sum_{m=1}^{\infty} \frac{\rho(t)^{2m} m}{(m!)^2}\big|\langle Z^k Z^m \partial_x^\ell p_\rho^R, Z^k Z^{m-1}\partial_x^\ell \partial_y v_\rho^R\rangle\big|\\
&\quad + C\|\nabla P^R\|_{X^7}\|U^R\|_{X^8} + \int_{y=0} \partial_x^8 p_\rho^R \cdot \partial_x^8 v^R dx\\
&\leqq C_1\delta\|\nabla p^{R,NS}\|_{X^8}^2 + C_1\delta\|\langle D_x\rangle^{\frac12}\nabla p^{R,E}\|_{X^7}^2\\
&\quad + C\|U^R\|_{X^{8,\frac12}}^2 + C\|U^R\|_{X^8}^2 + C\varepsilon^4.
\end{aligned}$$

Here we used $\partial_y v^R = -\partial_x u^R$ and $v^R|_{y=0} = -\varepsilon^2 f$. This proves that

$$I_3 \leqq C_1\delta\|\nabla p^{R,NS}\|_{X^8}^2 + C_1\delta\|\langle D_x\rangle^{\frac12}\nabla p^{R,E}\|_{X^7}^2 + C\|U^R\|_{X^{8,\frac12}}^2 + C\varepsilon^4.$$

Step 4. Estimate of the dissipation term. Using $u^R|_{y=0} = 0$ and $\mathrm{div} U^R = 0$, we get by integration by parts that

$$
\begin{aligned}
&-\varepsilon^2\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m\triangle U_\rho^R, Z^m U_\rho^R\rangle_{\overline{H}^8}\\
&\quad=-\varepsilon^2\sum_{k+\ell\le 8}\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle \operatorname{div}(Z^kZ^m\partial_x^\ell\nabla U_\rho^R), Z^kZ^m\partial_x^\ell U_\rho^R\rangle\\
&\qquad+\varepsilon^2\sum_{k+\ell\le 8}\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle [\operatorname{div}, Z^kZ^m]\partial_x^\ell\nabla U_\rho^R, Z^kZ^m\partial_x^\ell U_\rho^R\rangle\\
&\quad=\varepsilon^2\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m\nabla U_\rho^R, Z^m\nabla U_\rho^R\rangle_{\overline{H}^8}\\
&\qquad+\varepsilon^2\sum_{k+\ell\le 8}\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle [\operatorname{div}, Z^kZ^m]\partial_x^\ell\nabla U_\rho^R, Z^kZ^m\partial_x^\ell U_\rho^R\rangle\\
&\qquad+\varepsilon^2\sum_{k+\ell\le 8}\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^kZ^m\partial_x^\ell\nabla U_\rho^R, [\nabla, Z^kZ^m]\partial_x^\ell U_\rho^R\rangle\\
&\quad\geqq \varepsilon^2\|\nabla U^R\|_{X^8}^2-C_1\delta\varepsilon^2\|\nabla U^R\|_{X^8}^2\ge\frac{1}{2}\varepsilon^2\|\nabla U^R\|_{X^8}^2,
\end{aligned}
$$

if we take δ small so that $C_1\delta\le\frac{1}{2}$.

Now, the lemma follows, from Steps 1–4. □

Next, we give the higher order tangential derivative estimates of the velocity.

Lemma 5.5. *There exists $\delta_0>0$ and $\varepsilon_0>0$ so that for any $\delta\in(0,\delta_0)$ and $\varepsilon\in(0,\varepsilon_0)$, it holds that*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle D_x\rangle^9U_\rho^R\|_{L^2}^2+\lambda\|\langle D_x\rangle^{9.5}U_\rho^R\|_{L^2}^2+\frac{1}{4}\varepsilon^2\|\nabla\langle D_x\rangle^9U_\rho^R\|_{L^2}^2\\
&\quad\leqq C\big(1+\|U^R\|_{X^8}^2+\|\omega^R\|_{X^8}^2\big)\big(\|U^R\|_{X^{8,\frac{1}{2}}}^2+\|\langle D_x\rangle^{9.5}U_\rho^R\|_{L^2}^2+\varepsilon^4\big)\\
&\qquad+C_1\delta\|\nabla p^R\|_{X^7}^2.
\end{aligned}
$$

Proof. Taking $\langle D_x\rangle^9$ on the both sides of (5.2), and then taking L^2 inner product with $\langle D_x\rangle^9U_\rho^R$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle D_x\rangle^9U_\rho^R\|_{L^2}^2+\lambda\|\langle D_x\rangle^9\langle D_x\rangle^{\frac{1}{2}}U_\rho^R\|_{L^2}^2-\varepsilon^2\langle\langle D_x\rangle^9\triangle U_\rho^R,\langle D_x\rangle^9U_\rho^R\rangle\\
&\quad\leqq\Big|\langle\langle D_x\rangle^9\big(\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R+\nabla p^R-\widetilde{R}\big)_\rho,\langle D_x\rangle^9U_\rho^R\rangle\Big|\\
&\quad\leqq\Big|\langle\langle D_x\rangle^9(\widetilde{U}^R\cdot\nabla(U^R+U^a))_\rho,\langle D_x\rangle^9U_\rho^R\rangle\Big|\\
&\qquad+\Big|\langle\langle D_x\rangle^9(\widetilde{U}^a\cdot\nabla U^R)_\rho,\langle D_x\rangle^9U_\rho^R\rangle\Big|\\
&\qquad+\Big|\langle\langle D_x\rangle^9(\nabla p^R)_\rho,\langle D_x\rangle^9U_\rho^R\rangle\Big|+\Big|\langle\langle D_x\rangle^9\widetilde{R}_\rho,\langle D_x\rangle^9U_\rho^R\rangle\Big|\\
&\quad\triangleq\widetilde{I}_1+\widetilde{I}_2+\widetilde{I}_3+\widetilde{I}_4.
\end{aligned}
$$

Following the proofs of I_1, I_2, I_4 in Lemma 5.4 , we have

$$\widetilde{I}_1+\widetilde{I}_2+\widetilde{I}_4 \leqq C\big(1+\|U^R\|_{X^8}^2+\|\omega^R\|_{X^8}^2\big)\big(\|U^R\|_{X^{8,\frac{1}{2}}}^2+\|\langle D_x\rangle^{9.5}U_\rho^R\|_{L^2}^2+\varepsilon^4\big),$$

Because of $\mathrm{div}U^R=0$ and $U^R|_{y=0}=(0,-\varepsilon^2 f)$, we have

$$\begin{aligned}\widetilde{I}_3 &\leqq \|\nabla p^R\|_{X^7}\|\langle D_x\rangle^9 U_\rho^R\|_{L^2}+\Big|\int_{y=0}|D_x|^8\langle D_x\rangle p_\rho^R\cdot\langle D_x\rangle^9 v_\rho^R dx\Big|\\ &\leqq C_1\delta\|\nabla p^R\|_{X^7}^2+C\|\langle D_x\rangle^9 U_\rho^R\|_{L^2}^2+C\varepsilon^4.\end{aligned}$$

By $u^R|_{y=0}=\partial_y v^R|_{y=0}=0$, we have

$$-\langle\langle D_x\rangle^9\triangle U_\rho^R,\langle D_x\rangle^9 U_\rho^R\rangle_{L^2}=\|\nabla\langle D_x\rangle^9 U_\rho^R\|_{L^2}^2.$$

This shows the lemma. □

5.2. Conormal analytic estimates of the pressure

To close the estimates of U^R, it remains to estimate the pressure in conormal analytic space.

Recall that $p^R=p^{R,NS}+p^{R,E}$. First of all, we introduce a new function $\widetilde{p}^{R,2}$ defined by

$$\widetilde{p}^{R,E}=p^{R,E}+e^{-y}(\varepsilon^2\partial_t f-\varepsilon^4\partial_x^2 f)\triangleq p^{R,E}+g.$$

Here, by the definition of f, we have

$$g=\partial_x e^{-y}\Big(\varepsilon^2\int_0^\infty \partial_t u_p^{(1)}(t,x,z)\,\mathrm{d}z-\varepsilon^4\partial_x f\Big)\triangleq\partial_x G.$$

Thus, we have

$$\begin{cases}\mathrm{div}(\nabla\widetilde{p}^{R,E}-\widetilde{R})=-\mathrm{div}(\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R)+\partial_x\triangle G,\\ \partial_y\widetilde{p}^{R,E}|_{y=0}=R_2|_{y=0}.\end{cases}\tag{5.6}$$

Lemma 5.6. *There exists* $\delta_0>0$ *such that for any* $\delta\in(0,\delta_0)$*, it holds that*

$$\|\langle D_x\rangle^{\frac{1}{2}}\nabla p^{R,E}\|_{X^7}^2\leqq C\big(1+\|U^R\|_{X^{8,\frac{1}{2}}}^2+\|\omega^R\|_{X^{8,\frac{1}{2}}}^2\big)\|U^R\|_{X^8}^2+C\varepsilon^4.$$

Proof. Taking $\langle D_x\rangle Z^m e^{\rho(t)\langle D_x\rangle}$ on both sides of (5.6), and then taking $\overline{H}^7$ inner product with $\frac{\rho(t)^{2m}}{(m!)^2}Z^m\widetilde{p}_\rho^{R,E}$, we obtain

$$\begin{aligned}&-\sum_{m=0}^\infty\frac{\rho(t)^{2m}}{(m!)^2}\big\langle\langle D_x\rangle Z^m\mathrm{div}(\nabla\widetilde{p}^{R,E}-\widetilde{R})_\rho,Z^m\widetilde{p}_\rho^{R,E}\big\rangle_{\overline{H}^7}\\ &=\sum_{m=0}^\infty\frac{\rho(t)^{2m}}{(m!)^2}\big\langle\langle D_x\rangle Z^m(\mathrm{div}(\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R)_\rho\\ &\quad+\partial_x\langle D_x\rangle\triangle G_\rho,Z^m\widetilde{p}_\rho^{R,E}\big\rangle_{\overline{H}^7}.\end{aligned}\tag{5.7}$$

By Lemma 5.2, the term on the left hand side of (5.7) is bounded from below

$$
\begin{aligned}
&-\sum_{k+\ell\le 7}\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\big\langle\langle D_x\rangle \mathrm{div} Z^kZ^m\partial_x^\ell(\nabla\widetilde{p}^{R,E}-\widetilde{R})_\rho,\, Z^kZ^m\partial_x^\ell\widetilde{p}^{R,E}_\rho\big\rangle\\
&\quad+\sum_{k+\ell\le 7}\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\big\langle\langle D_x\rangle [\mathrm{div},\, Z^kZ^m]\partial_x^\ell(\nabla\widetilde{p}^{R,E}-\widetilde{R})_\rho,\, Z^kZ^m\partial_x^\ell\widetilde{p}^{R,E}_\rho\big\rangle\\
&\geqq \frac{1}{2}\|\langle D_x\rangle^{\frac12}\nabla\widetilde{p}^{R,E}\|_{X^7}^2 - C\|\widetilde{R}\|^2_{X^{7,\frac12}}\\
&\quad+\sum_{k+\ell\le 7}\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\big\langle\langle D_x\rangle Z^kZ^m\partial_x^\ell(\nabla\widetilde{p}^{R,E}-\widetilde{R})_\rho,\, [\nabla,\, Z^kZ^m]\partial_x^\ell\widetilde{p}^{R,E}_\rho\big\rangle\\
&\quad+\sum_{k+\ell\le 7}\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\big\langle\langle D_x\rangle [\mathrm{div},\, Z^kZ^m]\partial_x^\ell(\nabla\widetilde{p}^{R,E}-\widetilde{R})_\rho,\, Z^kZ^m\partial_x^\ell\widetilde{p}^{R,E}_\rho\big\rangle\\
&\geqq \frac{1}{2}\|\langle D_x\rangle^{\frac12}\nabla\widetilde{p}^{R,E}\|_{X^7} - C\varepsilon^4 - C_1\delta\|\langle D_x\rangle^{\frac12}\nabla\widetilde{p}^{R,E}\|_{X^7}^2.
\end{aligned}
$$

Thanks to $\widetilde{U}^R|_{y=0}=\widetilde{U}^a|_{y=0}=0$, the terms on the right hand side of (5.7) are bounded from above as follows:

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\big\langle\langle D_x\rangle Z^m\big(\mathrm{div}(\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R)+\partial_x\triangle G\big)_\rho,\\
&\quad Z^m\widetilde{p}^{R,E}_\rho\big\rangle_{\overline{H}^7}\\
&\quad\leqq \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\big\langle\langle D_x\rangle Z^m(\widetilde{U}^R\cdot\nabla(U^R+U^a))_\rho + Z^m(\widetilde{U}^a\cdot\nabla U^R)_\rho,\\
&\quad\quad \nabla Z^m\widetilde{p}^{R,E}_\rho\big\rangle_{\overline{H}^7}\Big|\\
&\quad\quad +\Big|\sum_{k+\ell\le 7}\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\big\langle\langle D_x\rangle [Z^kZ^m,\mathrm{div}]\partial_x^\ell\big(\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R\big)_\rho,\\
&\quad Z^kZ^m\partial_x^\ell\widetilde{p}^{R,E}_\rho\big\rangle\Big|\\
&\quad\quad +C\|\triangle G\|^2_{X^{7,\frac12}}+\frac18\|\langle D_x\rangle^{\frac12}\nabla\widetilde{p}^{R,E}\|^2_{X^7}\\
&\quad\triangleq I_5+I_6+C\|\triangle G\|^2_{X^{7,\frac12}}+\frac18\|\langle D_x\rangle^{\frac12}\nabla\widetilde{p}^{R,E}\|^2_{X^7}.
\end{aligned}
$$

We infer from Lemmas 4.4 and 5.1 that

$$
\begin{aligned}
I_5 &\leqq C\|\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R\|_{X^{7,\frac12}}\|\langle D_x\rangle^{\frac12}\nabla\widetilde{p}^{R,E}\|_{X^7}\\
&\le C\Big(\|U^R\|_{X^8}\big(\|\omega^R\|_{X^{8,\frac12}}+\|U^R\|_{X^{8,\frac12}}\big)+\|U^R\|_{X^{8,\frac12}}\Big)\|\langle D_x\rangle^{\frac12}\nabla\widetilde{p}^{R,E}\|_{X^7}\\
&\le \frac18\|\langle D_x\rangle^{\frac12}\nabla\widetilde{p}^{R,E}\|^2_{X^7}+C\big(\|U^R\|^2_{X^{8,\frac12}}+\|\omega^R\|^2_{X^{8,\frac12}}+1\big)\|U^R\|^2_{X^8}.
\end{aligned}
$$

In a similar way, we have

$$I_6 \leq \frac{1}{8}\|\langle D_x\rangle^{\frac{1}{2}}\nabla\widetilde{p}^{R,E}\|_{X^7}^2 + C\left(\|U^R\|_{X^{8,\frac{1}{2}}}^2 + \|\omega^R\|_{X^{8,\frac{1}{2}}}^2 + 1\right)\|U^R\|_{X^8}^2.$$

It is easy to see that

$$\|\triangle G\|_{X^{7,\frac{1}{2}}} \leqq C\varepsilon^2.$$

Now, the lemma follows by taking δ small enough. □

For another part of the pressure, we have more decay in ε.

Lemma 5.7. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, it holds that*

$$\|\nabla p^{R,NS}\|_{X^8}^2 \leqq C_1\varepsilon^4\|\nabla\omega^R\|_{X^8}^2.$$

Proof. Again, we introduce a new function $\widetilde{p}^{R,1}$ defined by

$$\widetilde{p}^{R,NS} = p^{R,NS} + \varepsilon^2\partial_x u^R,$$

which satisfies

$$\begin{cases} \triangle\widetilde{p}^{R,NS} = \varepsilon^2\triangle\partial_x u^R, \\ \partial_y\widetilde{p}^{R,NS}|_{y=0} = 0. \end{cases} \tag{5.8}$$

Taking $Z^m e^{\rho(t)\langle D_x\rangle}$ on both sides of (5.8), and then taking $\overline{H}^8$ inner product with $\frac{\rho(t)^{2m}}{(m!)^2} Z^m\widetilde{p}_\rho^{R,NS}$, we obtain

$$-\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m \mathrm{div}\nabla\widetilde{p}_\rho^{R,NS}, Z^m\widetilde{p}_\rho^{R,NS}\rangle_{\overline{H}^8}$$
$$= \varepsilon^2\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m\partial_x\triangle u_\rho^R, Z^m\widetilde{p}_\rho^{R,NS}\rangle_{\overline{H}^8}.$$

In a manner similar to the proof of Lemma 5.6, we have

$$-\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m \mathrm{div}\nabla\widetilde{p}_\rho^{R,NS}, Z^m\widetilde{p}_\rho^{R,NS}\rangle_{\overline{H}^8} \geqq \frac{1}{2}\|\nabla\widetilde{p}^{R,NS}\|_{X^8}^2,$$
$$\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m\partial_x\triangle u_\rho^R, Z^m\widetilde{p}_\rho^{R,NS}\rangle_{\overline{H}^8} \leqq \|\triangle u^R\|_{X^8}\|\nabla\widetilde{p}^{R,NS}\|_{X^8}.$$

On the other hand, we have

$$\|\Delta u^R\|_{X^8} \leqq \|\nabla\omega^R\|_{X^8}.$$

Then the lemma follows. □

Combining Lemmas 5.6 and 5.7, we get

Lemma 5.8. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, it holds that*

$$\|\nabla p^R\|_{X^7}^2 \leqq C\left(1 + \|U^R\|_{X^{8,\frac{1}{2}}}^2 + \|\omega^R\|_{X^{8,\frac{1}{2}}}^2\right)\|U^R\|_{X^8}^2 + C_1\varepsilon^4\|\nabla\omega^R\|_{X^8}^2 + C\varepsilon^4.$$

6. Conormal Analytic Estimate of the Vorticity

To gain one order regularity of the error in the normal direction, we need to use the vorticity formulation of the error equations. Recall that the vorticity ω^R of U^R satisfies

$$\partial_t \omega^R - \varepsilon^2 \triangle \omega^R + \widetilde{U}^R \cdot \nabla(\omega^R + \omega^a) + \widetilde{U}^a \cdot \nabla \omega^R = R^\omega \tag{6.1}$$

with the boundary condition

$$\begin{aligned} &\varepsilon^2(\partial_y + |D_x|)\omega^R\big|_{y=0} \\ &\quad = -\Big(\partial_y(-\Delta_D)^{-1}\operatorname{curl}(\widetilde{U}^R \cdot \nabla(U^R + U^a) + \widetilde{U}^a \cdot \nabla U^R - \widetilde{R})\Big)\Big|_{y=0} \\ &\qquad - \varepsilon^2 \int_0^\infty |D_x|\partial_t u_p^{(1)}\,\mathrm{d}z. \end{aligned}$$

Before we give the estimates of vorticity, we have the following relationship between U^R and ω^R.

Lemma 6.1. *There exists a constant C such that*

$$\begin{aligned} &\|U^R\|_{X^9} \leqq C\big(\|\omega^R\|_{X^8} + \|U^R\|_{X^8}\big), \\ &\|U^R\|_{X^{9,\frac{1}{2}}} \leqq C\big(\|\omega^R\|_{X^{8,\frac{1}{2}}} + \|U^R\|_{X^{8,\frac{1}{2}}}\big). \end{aligned}$$

Proof. Firstly, by the definition of ω^R, we have

$$\triangle u^R = \partial_y \omega^R$$

with the boundary condition $u^R|_{y=0} = 0$.

Taking $Z^m e^{\rho(t)\langle D_x\rangle}$ on both sides of the above equation, and then taking $\overline{H}^8$ inner product with $\frac{\rho(t)^{2m}}{(m!)^2} Z^m u_\rho^R$, we obtain

$$-\sum_{m=0}^\infty \frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m \operatorname{div}\nabla u_\rho^R, Z^m u_\rho^R\rangle_{\overline{H}^8} = -\sum_{m=0}^\infty \frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m \partial_y \omega_\rho^R, Z^m u_\rho^R\rangle_{\overline{H}^8},$$

which implies

$$\|\nabla u^R\|_{X^8}^2 \leqq C\|\omega\|_{X^8}^2. \tag{6.2}$$

Therefore,

$$\|u^R\|_{X^9} \leqq C(\|\omega^R\|_{X^8} + \|U^R\|_{X^8}).$$

On the other hand,

$$\partial_x v^R = \partial_y u^R - \omega \text{ and } \partial_y v^R = -\partial_x u^R,$$

which along with (6.2) yield that

$$\|v^R\|_{X^9} \leqq C\big(\|\omega^R\|_{X^8} + \|U^R\|_{X^8}\big).$$

The second inequality can be proved in a similar way. □

Proposition 6.1. *There exists* $\delta_0 > 0$ *such that for any* $\delta \in (0, \delta_0)$, *it holds that*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^R\|_{X^8}^2 + \lambda\|\omega^R\|_{X^{8,\frac{1}{2}}}^2 + \frac{1}{2}\varepsilon^2\|\nabla w^R\|_{X^8}^2\\&\le C\big(1 + \|\omega^R\|_{X^8}^2 + \|U^R\|_{X^8}^2\big)\big(\varepsilon^2 + \|\omega^R\|_{X^{8,\frac{1}{2}}}^2 + \|U^R\|_{X^{8,\frac{1}{2}}}^2\big)\\&\quad + C\varepsilon^{-2}\big(1 + \|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2\big)\big(\|\langle D_x\rangle^9 U_\rho^R\|_{L^2}^2 + \|U^R\|_{X^8}\big)\\&\quad + C_1\delta\big(\|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2\big)\|\nabla\omega^R\|_{X^8}^2 + C_1\varepsilon^2\|\nabla\langle D_x\rangle^9 U_\rho^R\|_{L^2}^2.\end{aligned}$$

Proof. Acting $e^{\rho(t)\langle D_x\rangle}$ on both sides of (6.1), we obtain

$$\partial_t\omega_\rho^R + \lambda\langle D_x\rangle\omega_\rho^R - \varepsilon^2\triangle\omega_\rho^R + (\widetilde{U}^R\cdot\nabla(\omega^R + \omega^a) + \widetilde{U}^a\cdot\nabla\omega^R)_\rho = R_\rho^\omega. \quad (6.3)$$

Taking $\frac{\rho(t)^m}{m!}Z^m$ on both sides of (6.3), and then taking $\overline{H}^8$ inner product with $\frac{\rho(t)^m}{m!}Z^m\omega_\rho^R$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^R\|_{X^8}^2 + \lambda\|\omega^R\|_{X^{8,\frac{1}{2}}}^2 - \varepsilon^2\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m\triangle\omega_\rho^R, Z^m\omega_\rho^R\rangle_{\overline{H}^8}\\&\leqq\Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(\widetilde{U}^R\cdot\nabla(\omega^R + \omega^a))_\rho, Z^m\omega_\rho^R\rangle_{\overline{H}^8}\Big|\\&\quad + \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(\widetilde{U}^a\cdot\nabla\omega^R)_\rho, Z^m\omega_\rho^R\rangle_{\overline{H}^8}\Big|\\&\quad + \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m R_\rho^\omega, Z^m\omega_\rho^R\rangle_{\overline{H}^8}\Big|\\&\triangleq II_1 + II_2 + II_3.\end{aligned}$$

Step 1. Estimates of II_1. Notice that

$$\begin{aligned}\widetilde{U}^R\cdot\nabla(\omega^R + \omega^a) &= u^R\partial_x\omega^R + (v^R + \varepsilon^2 fe^{-y})\partial_y\omega^R + u^R\partial_x\omega^a\\&\quad + (v^R + \varepsilon^2 fe^{-y})\partial_y\omega^a.\end{aligned}$$

We have

$$\begin{aligned}II_1 &\leqq \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u^R\partial_x\omega^R)_\rho, Z^m\omega_\rho^R\rangle_{\overline{H}^8}\Big|\\&\quad + \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m(u^R\partial_x\omega^a)_\rho, Z^m\omega_\rho^R\rangle_{\overline{H}^8}\Big|\\&\quad + \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m((v^R + \varepsilon^2 fe^{-y})\partial_y\omega^R)_\rho, Z^m\omega_\rho^R\rangle_{\overline{H}^8}\Big|\\&\quad + \Big|\sum_{m=0}^{\infty}\frac{\rho(t)^{2m}}{(m!)^2}\langle Z^m((v^R + \varepsilon^2 fe^{-y})\partial_y\omega^a)_\rho, Z^m\omega_\rho^R\rangle_{\overline{H}^8}\Big|\\&\triangleq II_{11} + II_{12} + II_{13} + II_{14}.\end{aligned}$$

Estimate of II_{11}. We get by Lemma 4.2 (1) that

$$II_{11} \leqq C\big(\|u^R\|_{X^8}^2 + \|\partial_y u^R\|_{X^8}^2\big)\|\omega^R\|_{X^{8,\frac12}}^2 + C\|\omega^R\|_{X^{8,\frac12}}^2$$
$$\leqq C\big(1 + \|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2\big)\|\omega^R\|_{X^{8,\frac12}}^2.$$

Estimate of II_{12}. We only consider the worst term $\frac{u^R}{\varepsilon}(\partial_x\partial_z u^p)(t,x,\frac{y}{\varepsilon}) = u^R\partial_y(\partial_x u^p(t,x,\frac{y}{\varepsilon}))$. Thanks to $u^R|_{y=0} = 0$, we deduce from Lemmas 4.6 and 5.1 that the worst part is bounded by

$$C\big(\|\partial_y\langle D_x\rangle^{-\frac12}u^R\|_{X^8}^2 + \|u^R\|_{X^8}^2\big)\left(\|\partial_z\partial_x u^p\|_{\widetilde{X}^9}^2 + \|\partial_x u^p\|_{\widetilde{X}^9}^2\right) + C\|\omega^R\|_{X^{8,\frac12}}^2.$$

In all, we have

$$II_{12} \leqq C\left(\|\omega^R\|_{X^{8,\frac12}}^2 + \|U^R\|_{X^{8,\frac12}}^2\right).$$

Estimate of II_{13}. Thanks to $v^R + \varepsilon^2 f e^{-y}|_{y=0} = 0$, we get by Lemma 4.5(3) that

$$\begin{aligned}
II_{13} \leqq\ & \delta\big(\|\partial_y\langle D_x\rangle^{-\frac12}(v^R + \varepsilon^2 f e^{-y})\|_{X^8}^2 \\
& + \|(v^R + \varepsilon^2 f e^{-y})\|_{X^8}^2\big)\big(\|\omega^R\|_{X^6}^2 + \|\partial_y\omega^R\|_{X^6}^2\big) \\
& + \big(\|v^R + \varepsilon^2 f e^{-y}\|_{X^6}^2 + \|\partial_y(v^R + \varepsilon^2 f e^{-y})\|_{X^6}^2 \\
& + \|\partial_y^2(v^R + \varepsilon^2 f e^{-y})\|_{X^6}^2\big)\|\omega^R\|_{X^{8,\frac12}}^2 \\
& + C\|\omega^R\|_{X^{8,\frac12}}^2 \\
\leqq\ & \delta C_1\big(\|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2 + \varepsilon^4\big)\|\partial_y\omega^R\|_{X^8}^2 + C\big(1 + \|U^R\|_{X^8}^2 \\
& + \|\omega^R\|_{X^8}^2\big)\|\omega^R\|_{X^{8,\frac12}}^2.
\end{aligned}$$

Estimate of II_{14}. For $\partial_y\omega^a$, we only focus on $\partial_y^2 u^p$. The other terms can be treated by the same argument. By the Leibniz rule, we have

$$Z^m\left((v^R + \varepsilon^2 f e^{-y})\left(\frac{\partial_z^2 u^p}{\varepsilon^2}\right)\right) = \sum_{m_1=0}^{m}\frac{m!}{m_1!(m-m_1)!}Z^{m_1}(v^R + \varepsilon^2 f e^{-y})$$
$$\cdot Z^{m-m_1}\left(\frac{\partial_z^2 u^p}{\varepsilon^2}\right).$$

For $m_1 = 0$, we write

$$\begin{aligned}
&\left(v^R + \varepsilon^2 f e^{-y}\right)Z^m\left(\frac{\partial_z^2 u^p}{\varepsilon^2}\right)\left(t,x,\frac{y}{\varepsilon}\right) \\
&= -\frac{1}{y^2}\int_0^y\int_0^{y'}\partial_{xy}^2 u^R dy'' dy' z^2 Z^m(\partial_z^2 u^p)\left(t,x,\frac{y}{\varepsilon}\right) \\
&\quad -\frac{\varepsilon(1-e^{-y})f}{y} z Z^m(\partial_z^2 u^p)\left(t,x,\frac{y}{\varepsilon}\right).
\end{aligned}$$

For $m_1 = 1$, we write

$$
\begin{aligned}
&Z(v^R + \varepsilon^2 f e^{-y}) Z^{m-1} \left(\frac{\partial_z^2 u^p}{\varepsilon^2} \right) \Big(t, x, \frac{y}{\varepsilon}\Big) \\
&= \frac{\varphi(y)}{y^2} \partial_y (v^R + \varepsilon^2 f e^{-y}) \cdot z^2 Z^{m-1} (\partial_z^2 u^p) \Big(t, x, \frac{y}{\varepsilon}\Big) \\
&= -\frac{\varphi(y)}{y^2} \int_0^y \partial_{xy}^2 u^R dy' z^2 Z^{m-1} (\partial_z^2 u^p) \Big(t, x, \frac{y}{\varepsilon}\Big) \\
&\quad -\varepsilon \frac{\varphi}{y} e^{-y} f z Z^{m-1} (\partial_z u^p) \Big(t, x, \frac{y}{\varepsilon}\Big).
\end{aligned}
$$

For $m_1 \geqq 2$, we write

$$
\begin{aligned}
&Z^{m_1} (v^R + \varepsilon^2 f e^{-y}) Z^{m-m_1} \left(\frac{\partial_z^2 u^p}{\varepsilon^2} \right) \Big(t, x, \frac{y}{\varepsilon}\Big) \\
&= -\frac{\varphi(y)^2}{y^2} Z^{m_1-2} \partial_y^2 (v^R + \varepsilon^2 f e^{-y}) z^2 \widetilde{Z}^{m-m_1} (\partial_z^2 u^p) \Big(t, x, \frac{y}{\varepsilon}\Big) \Big| \\
&= \frac{\varphi(y)^2}{y^2} Z^{m_1-2} (\partial_{xy}^2 u^R + \varepsilon^2 f e^{-y}) z^2 Z^{m-m_1} (\partial_z^2 u^p) \Big(t, x, \frac{y}{\varepsilon}\Big).
\end{aligned}
$$

With the above preparations, we can deduce by using Lemma 4.1, Hardy's inequality and Lemma 5.1 that

$$
II_{14} \leqq C \left(\varepsilon^2 + \|\omega^R\|_{X^{8,\frac{1}{2}}}^2 + \|U^R\|_{X^{8,\frac{1}{2}}}^2 \right).
$$

Putting the estimates of $II_{11} - II_{14}$ together, we conclude that

$$
\begin{aligned}
II_1 \leqq\ & C\big(1 + \|\omega^R\|_{X^8}^2 + \|U^R\|_{X^8}^2\big)(\varepsilon^2 + \|\omega^R\|_{X^{8,\frac{1}{2}}}^2 + \|U^R\|_{X^{8,\frac{1}{2}}}^2) \\
&+ \delta C_1\big(\|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2 + \varepsilon^4\big)\|\partial_y \omega^R\|_{X^8}^2.
\end{aligned}
$$

Step 2. Estimates of II_2 and II_3. We write

$$
\widetilde{U}^a \cdot \nabla \omega^R = u^a \partial_x \omega^R + (v^a - \varepsilon^2 f e^{-y}) \partial_y \omega^R.
$$

Thanks to $v^a - \varepsilon^2 f e^{-y}|_{y=0} = 0$, we infer from Lemmas 4.2(1), 4.3, 4.5(1), 4.6 and 5.1 that

$$
II_2 \leqq \|\omega^R\|_{X^{8,\frac{1}{2}}}^2.
$$

For II_3, we get by Lemma 5.2 that

$$
II_3 \leqq C\|R^\omega\|_{X^8}^2 + C\|\omega^R\|_{X^8}^2 \leq C\varepsilon^2 + C\|\omega^R\|_{X^8}^2.
$$

Step 3. Estimates of the dissipation term. Then we get by integration by parts that

$$
\begin{aligned}
&-\varepsilon^2 \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \langle Z^m \triangle \omega_\rho^R, Z^m \omega_\rho^R \rangle_{\overline{H}^8} \\
&= \varepsilon^2 \sum_{m=0}^{\infty} \frac{\rho^{2m}}{(m!)^2} \|Z^m \nabla \omega_\rho^R\|_{\overline{H}^8}^2 - \varepsilon^2 \langle \partial_y \omega_\rho^R, \omega_\rho^R \rangle_{H_x^8(\partial \mathbf{R}_+^2)} \\
&\quad - \varepsilon^2 \sum_{k+\ell \le 8} \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \langle [Z^k Z^m, \partial_y] \partial_x^\ell \partial_y \omega_\rho^R, Z^k Z^m \partial_x^\ell \omega_\rho^R \rangle \\
&\quad + \varepsilon^2 \sum_{k+\ell \le 8} \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} \langle Z^k Z^m \partial_y \partial_x^\ell \omega_\rho^R, [\partial_y, Z^k Z^m] \partial_x^\ell \omega_\rho^R \rangle \\
&\triangleq \varepsilon^2 \|\nabla \omega^R\|_{X^8}^2 + II_4 + II_5 + II_6.
\end{aligned}
$$

Estimates for II_5 and II_6. By (5.5), the term II_5 is bounded as

$$
\begin{aligned}
&\varepsilon^2 \sum_{k+\ell \le 8} \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} m \langle \varphi' Z^k Z^{m-1} \partial_y \partial_x^\ell \partial_y \omega_\rho^R, Z^k Z^m \partial_x^\ell \omega_\rho^R \rangle \\
&\quad + \varepsilon^2 \sum_{k+\ell \le 8} \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} k \langle \varphi' Z^{k-1} Z^m \partial_y \partial_x^\ell \partial_y \omega_\rho^R, Z^k Z^m \partial_x^\ell \omega_\rho^R \rangle \\
&= \varepsilon^2 \sum_{k+\ell \le 8} \sum_{m=0}^{\infty} \frac{\rho(t)^{2m}}{(m!)^2} (m+k) \langle \varphi' Z^k Z^m \partial_x^\ell \partial_y \omega_\rho^R, Z^k Z^{m-1} \partial_y \partial_x^\ell \omega_\rho^R \rangle \\
&\quad + \text{similar lower order terms},
\end{aligned}
$$

where the main term is bounded by

$$
\begin{aligned}
&C_1 \delta \varepsilon^2 \sum_{m=1}^{\infty} \frac{\rho(t)^m}{m!} \|Z^m \partial_y \omega^R\|_{\overline{H}^8} \cdot \frac{\rho(t)^m}{(m-1)!} \|Z^{m-1} \partial_y \omega^R\|_{\overline{H}^8} \\
&\le C_1 \delta \varepsilon^2 \|\partial_y \omega^R\|_{X^8}^2,
\end{aligned}
$$

and the same bound holds for the lower order terms. Similarly, we have

$$
II_6 \leqq C_1 \delta \varepsilon^2 \|\partial_y \omega^R\|_{X^8}^2.
$$

Estimates of II_4. Using the boundary condition of the vorticity, we deduce that

$$
\begin{aligned}
II_4 =& \varepsilon^2 \sum_{k \le 8} \int_{y=0} \partial_x^k \partial_y \omega_\rho^R \cdot \partial_x^k \omega_\rho^R dx \\
=& -\sum_{k \le 8} \int_{y=0} \partial_x^k \partial_y (-\triangle_D)^{-1} (\widetilde{U}^R \cdot \nabla(\omega^R + \omega^a) + U^a \cdot \nabla \omega^R - R^\omega)_\rho \\
&\cdot \partial_x^k \omega_\rho^R dx
\end{aligned}
$$

$$
\begin{aligned}
&-\varepsilon^2\sum_{k\leqq 8}\int_{y=0}\left(\partial_x^k\int_0^\infty |D_x|\partial_t u_p^{(1)}(t,x,z')\,\mathrm{d}z'\right)_\rho\cdot\partial_x^k\omega_\rho^R dx\\
&-\varepsilon^2\sum_{k\leq 8}\int_{y=0}\partial_x^k|D_x|\omega_\rho^R\cdot\partial_x^k\omega_\rho^R dx\\
=&\sum_{k\leq 8}\int_{\mathbf{R}^2_+}\langle D_x\rangle^{-\frac12}\partial_x^k\partial_y\partial_y(-\triangle_D)^{-1}(\widetilde{U}^R\cdot\nabla(\omega^R+\omega^a)+\widetilde{U}^a\cdot\nabla\omega^R-R^\omega)_\rho\\
&\cdot\langle D_x\rangle^{\frac12}\partial_x^k\omega_\rho^R dxdy\\
&+\sum_{k\leq 8}\int_{\mathbf{R}^2_+}\partial_x^k\partial_y(-\triangle_D)^{-1}(\widetilde{U}^R\cdot\nabla(\omega^R+\omega^a)+\widetilde{U}^a\cdot\nabla\omega^R-R^\omega)_\rho\\
&\cdot\partial_y\partial_x^k\omega_\rho^R dxdy\\
&-\varepsilon^2\sum_{k\leqq 8}\int_{y=0}\left(\partial_x^k\int_0^\infty |D_x|\partial_t u_p^{(1)}(t,x,z')\,\mathrm{d}z'\right)_\rho\cdot\partial_x^k\omega_\rho^R dx\\
&-2\varepsilon^2\sum_{k\leq 8}\int_{\mathbf{R}^2_+}\partial_x^k\partial_y|D_x|\omega_\rho^R\cdot\partial_x^k\omega_\rho^R dxdy\\
\triangleq& II_{41}+II_{42}+II_{43}+II_{44}.
\end{aligned}
$$

Since $\partial_y^2(-\triangle_D)^{-1}$ is an L^2 bounded operator, we get

$$
\begin{aligned}
II_{41}\leqq&\|\langle D_x\rangle^{8-\frac12}(\widetilde{U}^R\cdot\nabla(\omega^R+\omega^a)\\
&+\widetilde{U}^a\cdot\nabla\omega^R-R^\omega)_\rho\|_{L^2_{x,y}}\cdot\|\langle D_x\rangle^{8+\frac12}\omega_\rho^R\|_{L^2_{x,y}}.
\end{aligned}
$$

Similar to estimates of II_1-II_3, we can obtain

$$
\begin{aligned}
II_{41}\leqq& C\big(1+\|\omega^R\|^2_{X^8}+\|U^R\|^2_{X^8}\big)(\varepsilon^2+\|\omega^R\|^2_{X^{8,\frac12}}+\|U^R\|^2_{X^{8,\frac12}})\\
&+C_1\delta\big(\|U^R\|^2_{X^8}+\|\omega^R\|^2_{X^8}+\varepsilon^4\big)\|\partial_y\omega^R\|^2_{X^8}.
\end{aligned}
$$

Notice that $\partial_y(-\triangle_D)^{-1}$ curl is also an L^2 bounded operator. In a similar way as I_5-I_6 in Lemma 5.6, we can deduce that

$$
\begin{aligned}
II_{42}\leqq&\|\langle D_x\rangle^8\partial_y(-\triangle_D)^{-1}(\widetilde{U}^R\cdot\nabla(\omega^R+\omega^a)+\widetilde{U}^a\cdot\nabla\omega^R-R^\omega)_\rho\|_{L^2_{x,y}}\\
&\cdot\|\langle D_x\rangle^8\nabla\omega_\rho^R\|_{L^2_{x,y}}\\
\leqq& C\|\langle D_x\rangle^8\partial_y(-\triangle_D)^{-1}\,\mathrm{curl}(\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R-\widetilde{R})_\rho\\
&\|_{L^2_{x,y}}\|\nabla\omega^R\|_{X^8}\\
\leqq& C\varepsilon^{-2}\|\langle D_x\rangle^8(\widetilde{U}^R\cdot\nabla(U^R+U^a)+\widetilde{U}^a\cdot\nabla U^R-\widetilde{R})_\rho\|^2_{L^2_{x,y}}+\delta\varepsilon^2\|\nabla\omega^R\|^2_{X^8}\\
\leq& C\varepsilon^{-2}\big(1+\|U^R\|^2_{X^8}+\|\omega^R\|^2_{X^8}\big)\\
&(\|U^R\|_{X^8}+\|\langle D_x\rangle^9U^R_\rho\|_{L^2})^2+C\varepsilon^2+\delta\varepsilon^2\|\nabla\omega^R\|^2_{X^8}.
\end{aligned}
$$

For II_{43}, we get by Sobolev embedding and Lemma 5.2 that

$$II_{43} \leqq C\varepsilon^2 + C\varepsilon^2\|\omega^R\|_{X^8}^2 + C_1\delta\varepsilon^2\|\partial_y\omega^R\|_{X^8}^2.$$

For II_{44}, we have

$$II_{44} \leqq \frac{1}{4}\varepsilon^2\|\nabla\omega^R\|_{X^8} + C_1\varepsilon^2\|\nabla\langle D_x\rangle^9 U_\rho^R\|_{L^2}^2.$$

In all, we conclude that

$$\begin{aligned} II_4 \leq & C\big(1 + \|\omega^R\|_{X^8}^2 + \|U^R\|_{X^8}^2\big)\big(\varepsilon^2 + \|\omega^R\|_{X^{8,\frac{1}{2}}}^2 + \|U^R\|_{X^{8,\frac{1}{2}}}^2\big) \\ & + \delta C_1\big(\|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2 + \varepsilon^2\big)\|\nabla\omega^R\|_{X^8}^2 + \frac{1}{4}\varepsilon^2\|\nabla\omega^R\|_{X^8} \\ & + C_1\varepsilon^2\|\nabla\langle D_x\rangle^9 U_\rho^R\|_{L^2}^2 \\ & + C\varepsilon^{-2}\big(1 + \|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2\big)\big(\|U^R\|_{X^8} + \|\langle D_x\rangle^9 U_\rho^R\|_{L^2}\big)^2. \end{aligned}$$

Thus, by taking δ small enough, we obtain

$$\begin{aligned} & -\varepsilon^2\sum_{m=0}^{\infty}\frac{\rho^{2m}}{(m!)^2}\langle Z^m\triangle\omega_\rho^R, Z^m\omega_\rho^R\rangle_{\overline{H}^8} \\ & \geqq \frac{1}{2}\varepsilon^2\|\nabla\omega^R\|_{X^8}^2 - C\big(1 + \|\omega^R\|_{X^8}^2 + \|U^R\|_{X^8}^2\big)\big(\varepsilon^2 + \|\omega^R\|_{X^{8,\frac{1}{2}}}^2 + \|U^R\|_{X^{8,\frac{1}{2}}}^2\big) \\ & \quad - C_1\varepsilon^2\|\nabla\langle D_x\rangle^9 U_\rho^R\|_{L^2}^2 - C_1\delta\big(\|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2\big)\|\nabla\omega^R\|_{X^8}^2 \\ & \quad - C\varepsilon^{-2}\big(1 + \|U^R\|_{X^8}^2 + \|\omega^R\|_{X^8}^2\big)\big(\|U^R\|_{X^8} + \|\langle D_x\rangle^9 U_\rho^R\|_{L^2}\big)^2. \end{aligned}$$

Now, the proposition follows by summing up Steps 1–3. □

7. Proof of Theorem 1.1

This section is devoted to the Proof of Theorem 1.1.

First of all, the local well-posedness of the Navier–Stokes equations in the analytic space can be proved by following the energy estimates in Section 4. More precisely, there exists $T_\varepsilon > 0$ depending on ε so that the solution of (1.1) has a unique solution $(u^\varepsilon, v^\varepsilon)$ in $[0, T_\varepsilon]$ that satisfies

$$(u^\varepsilon, v^\varepsilon) \in C\big([0, T_\varepsilon]; X^N\big), \quad \nabla_{x,y}(u^\varepsilon, v^\varepsilon) \in L^2(0, T_\varepsilon; X^N).$$

Moreover, let T^* be the maximal existence time of the solution; the solution can be extended after $t = T^*$ if

$$\|(u^\varepsilon, v^\varepsilon)(t)\|_{X^9} + \|\nabla(u^\varepsilon, v^\varepsilon)(t)\|_{X^8} \leq C \quad \text{for any} \quad t \in [0, T^*).$$

Let us first assume Lemmas 5.1 and 5.2 so that Propositions 5.3 and 6.1 can be applied. We introduce

$$
\begin{aligned}
E(t) &\triangleq \varepsilon^{-2}\|U^R(t)\|_{X^8}^2 + \varepsilon^{-2}\|\langle D_x\rangle^9 U_\rho^R\|_{L^2} + \|\omega^R(t)\|_{X^8}^2,\\
F(t) &\triangleq \varepsilon^{-2}\|U^R(t)\|_{X^{8,\frac12}}^2 + \varepsilon^{-2}\|\langle D_x\rangle^{9.5} U_\rho^R\|_{L^2} + \|\omega^R(t)\|_{X^{8,\frac12}}^2,\\
G(t) &\triangleq \|\nabla U^R(t)\|_{X^8}^2 + \|\nabla\langle D_x\rangle^9 U_\rho^R\|_{L^2} + \varepsilon^2\|\nabla\omega^R(t)\|_{X^8}^2.
\end{aligned}
$$

Propositions 5.3 and 6.1 ensure that

$$
\begin{aligned}
&\frac12\frac{\mathrm d}{\mathrm dt}E(t) + \lambda F(t) + \frac14 G(t)\\
&\quad \leqq C\big(1+E(t)\big)(\varepsilon^2 + F(t) + E(t)) + C_1\big(\delta E(t)\varepsilon^{-2} + \varepsilon^2 + \delta\big)G(t),
\end{aligned}
$$

where C_1 is a constant independent of δ.

We first take ε and δ small enough so that $C_1(\varepsilon^2+\delta) \le \frac{1}{16}$. Then we obtain

$$
\begin{aligned}
&\frac12\frac{\mathrm d}{\mathrm dt}E(t) + \lambda F(t) + \frac18 G(t)\\
&\quad \leqq C\big(1+E(t)\big)(\varepsilon^2 + F(t) + E(t)) + C_1\delta\varepsilon^{-2}E(t)G(t).
\end{aligned}
\tag{7.1}
$$

Next we choose $\lambda = 4C$ and $T_1 > 0$ so that

$$
\rho(t) \ge 1 \quad \text{for} \quad t\in[0,T_1], \quad 8CT_1 \le 1.
$$

Let $T = \min(T_a, T_1)$. Then it follows from (7.1) and a continuous argument that the solution of the Navier–Stokes equations exists on $[0,T]$ and the error satisfies

$$
E(t) = \varepsilon^{-2}\|U^R(t)\|_{X^8}^2 + \varepsilon^{-2}\|\langle D_x\rangle^9 U_\rho^R\|_{L^2} + \|\omega^R(t)\|_{X^8}^2 \leqq 2\varepsilon^2
$$

for any $t\in[0,T]$. In particular, it holds that

$$
\varepsilon^{-1}\sum_{k=0}^{2}\|\nabla_x^k U^R(t)\|_{L^2} + \|\omega^R(t)\|_{L^2} \leqq C\varepsilon
$$

for any $t\in[0,T]$. Then the Sobolev embedding implies Theorem 1.1.

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8. Appendix

In this appendix, we prove the well-posedness of the Euler equations and the Prandtl equation in the analytic space. The proof of well-poseness of the linearized Euler equations and Prandtl equation is similar, and is thus omitted.

8.1. Well-posedness of the Euler equations

Let us first introduce analytic norms

$$\|U\|_{Y^s}^2 \stackrel{\text{def}}{=} \sum_{m=0}^{\infty} \frac{\rho_E(t)^{2m}}{(m!)^2} \|\partial_y^m U_{\rho_E}\|_{H^s},$$

$$\|U\|_{Y^{s,\frac{1}{2}}}^2 \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \frac{\rho_E(t)^{2m-1} m}{(m!)^2} \|\partial_y^m U_{\rho_E}\|_{H^s}^2 + \sum_{m=0}^{\infty} \frac{\rho_E(t)^{2m}}{(m!)^2} \|\partial_y^m \langle D_x\rangle^{\frac{1}{2}} U_{\rho_E}\|_{H^s}^2,$$

where $\rho_E(t) = 2 - \lambda_E t \geq 1$ with λ_E defined later.

Proposition 8.1. *Let* (u_0, v_0) *satisfy* (1.5). *Then there exists* $T_E > 0$ *so that the Euler equation* (1.2) *has a unique solution* $U^e = (u^e, v^e)$ *in* $[0, T_E]$, *which satisfies*

$$\|U^e(t)\|_{Y^N}^2 \leqq C, \quad \|\partial_t^k U^e(t)\|_{Y^{N-k}} \leq C \quad k = 1, 2$$

for any $t \in [0, T_E]$.

Proof. Here we just present *a priori* estimates of the solution. We consider the vorticity equation

$$\partial_t \omega^e + U^e \cdot \nabla \omega^e = 0.$$

Acting $\partial_y^m e^{\rho_E(t)\langle D_x\rangle}$ on the both sides of the above equation and taking H^{N-1} inner product with $(\frac{\rho_E^m(t)}{m!})^2 \partial_y^m \omega_{\rho_E}^e$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^e\|_{Y^{N-1}}^2 + \lambda_E \|\omega^e\|_{Y^{N-1,\frac{1}{2}}}^2 \leqq \sum_{m=0}^{\infty} \frac{\rho_E(t)^{2m}}{(m!)^2} \langle \partial_y^m (U^e \cdot \nabla \omega^e)_{\rho_E}, \partial_y^m \omega_{\rho_E}^e \rangle_{H^{N-1}}.$$

Following the proof of Lemma 4.2, it is easy to show that

$$\sum_{m=0}^{\infty} \frac{\rho_E(t)^{2m}}{(m!)^2} \langle \partial_y^m (U^e \cdot \nabla \omega^e)_{\rho_E}, \partial_y^m \omega_{\rho_E}^e \rangle_{H^{N-1}} \leqq C \|U^e\|_{Y^{N-1}} \|\omega^e\|_{Y^{N-1,\frac{1}{2}}}^2.$$

To close the estimates, we have to recover the estimates of the velocity. Here, by the definition of vorticity, we have

$$\triangle v^e = \partial_x \omega^e, \quad v^e|_{y=0} = 0.$$

Using the energy method, it is easy to show that

$$\|\nabla_{x,y} v^e\|_{Y^{N-1}} \leqq C \|\omega^e\|_{Y^{N-1}}.$$

Notice that

$$\partial_x u^e = -\partial_y v^e, \quad \partial_y u^e = \omega^e + \partial_x v^e,$$

which implies that

$$\|\nabla_{x,y} u^e\|_{Y^{N-1}} \leqq C \|\omega^e\|_{Y^{N-1}}.$$

On the other hand, we can deduce by using the velocity equation that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U^e_{\rho_E}\|^2_{L^2_{x,y}} \le C\|U^e_{\rho_E}\|^3_{H^2}.$$

In all, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\omega^e\|^2_{Y^{N-1}} + \|U^e_{\rho_E}\|^2_{L^2_{x,y}}\big) + \lambda_E\|\omega^e\|^2_{Y^{N-1,\frac{1}{2}}}\\&\quad \le C\big(\|\omega^e\|_{Y^{N-1}} + \|U^e_{\rho_E}\|_{L^2_{x,y}}\big)\|\omega^e\|^2_{Y^{N-1,\frac{1}{2}}} + C\big(\|\omega^e\|_{Y^{N-1}} + \|U^e_{\rho_E}\|_{L^2_{x,y}}\big)^3.\end{aligned}$$

Here we take $\lambda_E = 4CM^{\frac{1}{2}}$ and $T_E > 0$ so that

$$\rho_E(t) \ge 1 \quad \text{for } t \in [0, T_e], \quad CT_E M \le \frac{1}{2}.$$

Then a continuous argument ensures that

$$\|\omega^e(t)\|_{Y^{N-1}} + \|U^e_{\rho_E}(t)\|_{L^2_{x,y}} \leqq C$$

for any $t \in [0, T_E]$. This gives the first estimate. The second estimate can be deduced by using the equation. □

8.2. Well-posedness of the Prandtl equation

We denote by H^s_c the conormal Sobolev space, whose norm is defined by

$$\|U\|_{H^s_c} \overset{\text{def}}{=} \sum_{k+\ell\leqq s}\|\widetilde{Z}^k\partial_x^\ell U\|_{L^2_{x,y}}.$$

Let us introduce some weighted analytic norms:

$$\begin{aligned}\|U\|^2_{X^s_w} &\overset{\text{def}}{=} \sum_{m=0}^{\infty}\frac{\rho_P(t)^{2m}}{(m!)^2}\|e^{\phi}\widetilde{Z}^m U_{\rho_P}\|^2_{H^s_c},\\ \|U\|^2_{X^{s,\frac{1}{2}}_w} &\overset{\text{def}}{=} \sum_{m=1}^{\infty}\frac{\rho_P(t)^{2m-1}m}{(m!)^2}\|e^{\phi}\widetilde{Z}^m U_{\rho_P}\|^2_{H^s_c}\\ &\quad + \sum_{m=0}^{\infty}\frac{\rho_P(t)^{2m}}{(m!)^2}\|e^{\phi}\langle D_x\rangle^{\frac{1}{2}}\widetilde{Z}^m U_{\rho_P}\|^2_{H^s_c}\\ &\quad + \sum_{m=0}^{\infty}\frac{\rho_P(t)^{2m}}{(m!)^2}\|ze^{\phi}\widetilde{Z}^m U_{\rho_P}\|^2_{H^s_c},\end{aligned}$$

where $\rho_P(t) = 2 - \lambda_P t \ge 1$ and $\phi(t,z) = \rho_P(t)z^2$ with λ_P defined later.

Proposition 8.2. *Let (u^e, v^e) be given by Proposition 8.1. There exists $T_P > 0$ so that the Prandtl equation (1.6) has a unique solution (u^p, v^p) in $[0, T_P]$, which satisfies*

$$\|u^p\|_{X_w^{N-3}} + \|v^p\|_{X_w^{N-4}} \leqq C,$$
$$\|\partial_z^2 u^p\|_{X_w^{N-5}} + \|\partial_z^2 v^p\|_{X_w^{N-6}} \leq C.$$

Proof. Again, we just present *a priori* estimates of the solution. We introduce a new function

$$\overline{u}^p = u^p + e^{-2\phi(t,z)} u^e(t, x, 0) \triangleq u^p + g.$$

It is easy to verify that $\overline{u}^p$ satisfies

$$\partial_t \overline{u}^p - \partial_z^2 \overline{u}^p + F^p = 0,$$

where F^p is given by

$$F^p = \overline{u}^p \partial_1 \overline{u}^p + \overline{u}^p \partial_1 u^e(t, x, 0) + (u^e(t, x, 0) - g)\partial_x(\overline{u}^p - g) - \overline{u}^p \partial_x g$$
$$- g\partial_x u^e(t, x, 0) + \Big(z\partial_y v^e(t, x, 0) - \int_0^z \partial_1 u^p \, \mathrm{d}z'\Big)\partial_z(\overline{u}^p - g) - \partial_t g + \partial_z^2 g.$$

Acting $e^\phi \widetilde{Z}^m e^{\rho_P(t)\langle D_x\rangle}$ on both sides of the equation of $\overline{u}^p$, and taking H_c^{N-3} inner product with $(\frac{\rho_P(t)^m}{m!})^2 e^\phi \widetilde{Z}^m \overline{u}^p_{\rho_P}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}^p\|^2_{X_w^{N-3}} + \lambda_P \|\overline{u}^p\|^2_{X_w^{N-3,\frac12}} - \sum_{m=0}^\infty \frac{\rho_P(t)^{2m}}{(m!)^2}$$
$$\big\langle e^\phi \widetilde{Z}^m \partial_z^2 \overline{u}^p_{\rho_P}, e^\phi \widetilde{Z}^m \overline{u}^p_{\rho_P}\big\rangle_{H_c^{N-3}}$$
$$= -\sum_{m=0}^\infty \frac{\rho_P(t)^{2m}}{(m!)^2}\big\langle e^\phi \widetilde{Z}^m F^p_{\rho_P}, e^\phi \widetilde{Z}^m \overline{u}^p_{\rho_P}\big\rangle_{H_c^{N-3}}.$$

For the dissipation term, we have

$$-\sum_{m=0}^\infty \frac{\rho_P(t)^{2m}}{(m!)^2}\big\langle e^\phi \widetilde{Z}^m \partial_z^2 \overline{u}^p_{\rho_P}, e^\phi \widetilde{Z}^m \overline{u}^p_{\rho_P}\big\rangle_{H_c^{N-3}}$$
$$= \|\partial_z \overline{u}^p\|^2_{X_w^{N-3}} - \sum_{k+\ell\leq N-3}\sum_{m=0}^\infty \frac{\rho_P(t)^{2m}}{(m!)^2}$$
$$\big\langle [\widetilde{Z}^k e^\phi \widetilde{Z}^m, \partial_z]\partial_x^\ell \partial_z \overline{u}^p_{\rho_P}, \widetilde{Z}^k \widetilde{Z}^m \partial_x^\ell \overline{u}^p_{\rho_P} e^\phi\big\rangle_{H_c^{N-3}}$$
$$- \sum_{k+\ell\leq N-3}\sum_{m=0}^\infty \frac{\rho_P(t)^{2m}}{(m!)^2}\big\langle \widetilde{Z}^k \widetilde{Z}^m \partial_x^\ell \partial_z \overline{u}^p_{\rho_P}, [\widetilde{Z}^k e^\phi \widetilde{Z}^m, \partial_z]\partial_x^\ell \overline{u}^p_{\rho_P} e^\phi\big\rangle_{H_c^{N-3}}$$
$$\geq \big(1-\delta\big)\|\partial_z \overline{u}^p\|^2_{X_w^{N-3}} - C\|\overline{u}^p\|^2_{X_w^{N-3,\frac12}}.$$

For the other term, we have

$$\sum_{m=0}^{\infty}\frac{\rho_P(t)^{2m}}{(m!)^2}\big\langle e^{\phi}\widetilde{Z}^m F^p_{\rho_P}, e^{\phi}\widetilde{Z}^m\overline{u}^p_{\rho_P}\big\rangle_{H_c^{N-3}}$$
$$\leq C\big(1+\|\overline{u}^p\|^2_{X_w^{N-3}}\big)\big(1+\|\overline{u}^p\|^2_{X_w^{N-3,\frac12}}\big)+\frac12\|\partial_z\overline{u}^p\|^2_{X_w^{N-3}}.$$

One can refer to [38] for more details.
In all, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}^p\|^2_{X_w^{N-3}}+\big(\lambda_P-C-C\|\overline{u}^p\|_{X_w^{N-3}}\big)\|\overline{u}^p\|^2_{X_w^{N-3,\frac12}}+\left(\frac12-\delta\right)\|\partial_z\overline{u}^p\|^2_{X_w^{N-3}}\leq C.$$

With this, a continuous argument ensures that there exists $T_p>0$ so that

$$\rho_P(t)\geq 1,\quad \|\overline{u}^p(t)\|^2_{X_w^{N-3}}\leq C$$

for any $t\in[0,T_p]$. This implies the first estimate. For the second estimate, one can first prove that

$$\|\partial_t\overline{u}^p(t)\|^2_{X_w^{N-5}}\leq C.$$

Then the desired estimate can be deduced by using the equation of u^p. □

8.3. Proof of Lemmas 5.1 and 5.2

First of all, Proposition 8.1 gives the existence of the solution $(u_e^{(0)}, v_e^{(0)})$ of (2.1) and (2.2) with the bound

$$\|(u_e^{(0)}, v_e^{(0)})\|_{Y^N}\leq C.$$

With $(u_e^{(0)}, v_e^{(0)})$, Proposition 8.2 gives the existence of the solution $(u_p^{(0)}, v_p^{(1)})$ of (2.8) and (2.9) with the bound

$$\|u_p^{(0)}\|_{X_w^{N-3}}+\|v_p^{(0)}\|_{X_w^{N-4}}\leq C.$$

Next, we can solve the linearized Euler equation (2.3) and (2.4) in Y^{N-5}. Finally, we solve the linearized Prandtl equation (2.11) and (2.12) in X_w^{N-8}. Then Lemma 5.1 follows easily.

While, Lemma 5.2 can be deduced by using Lemma 4.4. Here we omit the details.

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