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On the Vortex Filament Conjecture for Euler Flows

Robert L. Jerrard & Christian Seis

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Abstract

In this paper, we study the evolution of a vortex filament in an incompressible ideal fluid. Under the assumption that the vorticity is concentrated along a smooth curve in $\mathbb{R}^3$, we prove that the curve evolves to leading order by binormal curvature flow. Our approach combines new estimates on the distance of the corresponding Hamiltonian-Poisson structures with stability estimates recently developed in Jerrard and Smets (J Eur Math Soc (JEMS) 17(6):1487–1515, 2015).

1. Introduction

In this paper, we study the evolution of an incompressible ideal fluid described by the Euler equations. We are interested in data such that the initial vorticity is concentrated in a tube of radius $\varepsilon \ll 1$ around a smooth curve in $\mathbb{R}^3$. One might then ask:

- at later times, does the vorticity continue to concentrate around some curve, and
- if so, how does the curve evolve?

The second question is not so hard if one already has a sufficiently good answer to the first. Indeed, the literature on this question, dating back to work of da Rios in 1906 [4], shows that if one somehow knows that at some time the vorticity concentrates smoothly and symmetrically in a small tube around a smooth curve, then one can compute the instantaneous velocity of the curve to leading order. These computations suggest that the curve should evolve, after a possible rescaling in time, by an equation described in Section 2 below, known by various names, including the binormal curvature flow, the vortex filament equation, and the local induction approximation.

Our results have the same character: they provide information about curve evolution, conditional upon knowing that vorticity remains concentrated around some curve. Their new feature is that we show that for these purposes, a quite weak description of the vorticity concentration suffices. Roughly speaking, we show that for suitable initial data, as long as the vorticity remains concentrated on the same scale $\varepsilon \ll 1$ around some curve of the correct arclength, where concentration is measured by a geometric variant of a particular negative Sobolev norm, then in fact the curve evolves by the binormal curvature flow. Our results also improve on earlier work in that they require very little a priori smoothness of the curve around which the vorticity concentrates, and they apply to very rough solutions of the Euler equations.

From another perspective, the relationship between the Euler equations and the binormal curvature flow can be formally understood by a correspondence between Hamiltonian-Poisson structures giving rise to the two flows. Our results may be seen as giving quantitative estimates of a sort of distance between these Hamiltonian-Poisson structures.

The belief that one can find solutions of the Euler equations for which the vorticity remains close for a significant period of time to a filament evolving by binormal curvature flow may be called the "vortex filament conjecture" for the Euler equations. We believe that our results provide more credible evidence in favor of the conjecture than any earlier arguments that we are aware of.

We conclude the introduction with a brief overview on preliminary and related works.

- *Formal asymptotics* As mentioned earlier, the first derivation of the binormal curvature flow dates back to the work of da Rios in 1906 [4]. In his doctoral thesis, the Italian mathematician formally computed the motion law of vortex filaments with the help of potential theory. At this time, da Rios' work was mostly ignored, except by his supervisor Levi-Civita, who promoted the results in a survey article [18,19] many years later. In subsequent years, the local induction approximation was rediscovered several times, see [23] and references therein, and it is by now a classical topic in fluid dynamics. Discussions that include alternative models arising from more refined formal asymptotics can be found for example in the texts of Saffman [25, Chapter 11] or of Majda and Bertozzi [20, Chapter 7].
- *Rigid motion* An explicit example for the motion of a vortex filament in an Euler fluid is the rigid motion of a perfect vortex ring, see for example [1,12]. Here, the evolution reduces to a translation with constant speed in direction normal to the plane in which the ring is embedded. The possibility of nontrivial steady vortex configurations featuring knots and links was conjectured by Kelvin [32]. Only very recently, such (infinite energy) solutions were found by Enciso and Peralta-Salas [8,9]. Explicit knotted solutions to the binormal curvature flow were studied in [16].
- *Dimension reduction* To the best of our knowledge, the only rigorous result in favor of the vortex filament conjecture for Euler flows is restricted to flows with an axial symmetry. In [2], the authors manage to show that the (axially symmetric) vorticity remains sharply concentrated in a small tube which rigidly moves at a constant speed in the direction of the symmetry axis. The analogous problem for two-dimensional fluids is much easier and is by now well-understood. In

fact, if the vorticity is initially sharply concentrated around a number of points in $\mathbb{R}^2$ (or a subdomain), the two dimensional Euler dynamics are well described by the so-called point vortex model. For details, we refer the interested reader to Chapter 4 of the monograph [21] and the references therein.

The binormal curvature flow is also conjectured to arise as a description of dynamics of vortex filaments in certain quantum fluids, as described by the Gross–Pitaevskii equation. This problem too is very largely open, although some conditional results, similar in spirit to the ones we prove here, are established in [13].

A higher-dimensional analog of the binormal curvature flow has been shown formally to describe the motion of codimension 2 vortex submanifolds in ideal fluids in dimensions $n \geqq 4$, first in the context of quantum fluids [13], and more recently for the Euler equations [17,27].

This article is organized as follows. In Section 2, we introduce the Euler equations, the binormal curvature flow, and the notation that is used in this paper. We subsequently present our rigorous results and discuss the method of this paper. In Section 3, we present a heuristic derivation of the binormal curvature flow from the Euler equations by formally passing to the limit in the corresponding Hamiltonian-Poisson structures. The remaining Section 4 contain the proofs.

2. Mathematical Setting and Results

2.1. Notation

For notational convenience, throughout the manuscript we use the same notation for length, area and volume. More precisely, $|\cdot|$ can stand for the one- or two-dimensional Hausdorff measure $\mathcal{H}^1$ or $\mathcal{H}^2$, respectively, or the Lebesgue measure $\mathcal{L}^3$ on $\mathbb{R}^3$. It should be clear from the context, which measure is actually used.

We will always write Γ to denote a closed, oriented Lipschitz curve (normally smoother) in $\mathbb{R}^3$ of length L, and $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ an arclength parametrization of Γ. Thus $|\gamma'(s)| = 1$ for all $s \in \mathbb{R}/L\mathbb{Z}$,

$$\Gamma = \{\gamma(s) : s \in \mathbb{R}/L\mathbb{Z}\} \qquad \text{and} \quad \gamma'(s) = \tau_\Gamma := \text{unit tangent to } \Gamma \text{ at } \gamma(s).$$

Depending on the context we may freely change between the notations $\partial_s\gamma$ and γ' for the derivative of γ with respect to the arc-length parameter.

For $s, t \in \mathbb{R}/L\mathbb{Z}$, we always understand $|s-t|$ to mean distance modulo $L\mathbb{Z}$ between s and t; that is, $|s-t| = \min_{k\in\mathbb{Z}} |s-t-Lk|$.

We remark that throughout most of the paper we will normalize by setting $L = 1$.

We will write

$$A \lesssim B$$

to mean that there exists some constant C, independent of ε (as long as $0 < \varepsilon < \frac{L}{2}$), such that $A \leqq CB$. Similarly, $A = \mathcal{O}(B)$ if $|A| \lesssim B$. Except where explicitly noted otherwise, the implicit constants are absolute in the sense that they are independent of all parameters. The implicit constants appearing in all such estimates may change from line to line.

We will write μ_Γ to denote the vector-valued measure corresponding to integration over Γ, defined by

$$\int \phi \cdot \mathrm{d}\mu_\Gamma = \int_\Gamma \phi \cdot \tau_\Gamma \, \mathrm{d}\mathcal{H}^1 = \int_{\mathbb{R}/L\mathbb{Z}} \phi(\gamma(s)) \cdot \gamma'(s) \, \mathrm{d}s \quad \text{for } \phi \in C_c(\mathbb{R}^3; \mathbb{R}^3).$$

If μ is an absolutely continuous measure with density ω, that is, $\mathrm{d}\mu = \omega \mathrm{d}x$, we will occasionally identify ω with μ.

Derivatives of measures are defined in the sense of distributions.

Given γ and Γ as above, for $s \in \mathbb{R}/L\mathbb{Z}$, we define the security radius of Γ at $\gamma(s)$ by

$$r_\gamma(s) = r(s) := \sup \left\{ r \in \left(0, \frac{L}{2}\right] \, : \, \begin{array}{l} |\gamma(s+h) - \gamma(s)| \geqq \frac{r}{2} \ \text{ for all } |h| \geqq r, \ \text{ and} \\ |\gamma'(s+h) - \gamma'(s)| \leqq \frac{|h|}{r} \ \text{ for all } |h| \leqq r. \end{array} \right\}. \tag{1}$$

We set $r(s) = 0$ if γ is not differentiable at s. We will see in Lemma 3 below that within the tube of variable radius $r_\gamma/4$ around Γ, there is a well-defined orthogonal projection onto Γ.

We will also write (omitting subscripts when no confusion can result)

$$\kappa^*_\gamma(s) = \kappa^*(s) := 1/r(s). \tag{2}$$

It is easy to check that

$$\kappa^*(s) \geqq \kappa(s) = |\gamma''(s)| \quad \text{for all } s, \tag{3}$$

provided that the letter is defined. Note that all arclength parametrizations γ are translates of one another, so that quantities such as norms of κ^* depend only on the geometry of Γ, not on the parametrization. In particular, we will be interested in Lipschitz curves Γ for which

$$\|\kappa^*_\Gamma\|_{L^{1,\infty}} = \|\kappa^*\|_{L^{1,\infty}} := \sup_{\sigma > 0} \sigma \left| \left\{ s \in \mathbb{R}/L\mathbb{Z} : |\kappa^*_\gamma(s)| \geqq \sigma \right\} \right| < \infty. \tag{4}$$

This is a weak regularity condition that allows corners (but not cusps) and a finite number of self-intersections.

We will also need the following. Let ω be a vector-valued Radon measure on $\mathbb{R}^3$. We define the homogeneous flat norm of ω as

$$\|\omega\|_F := \sup \left\{ \int \xi \cdot \mathrm{d}\omega : \ \xi \in C^1_c(\mathbb{R}^3; \mathbb{R}^3) \text{ with } \|\nabla \times \xi\|_{L^\infty} \leq 1 \right\}. \tag{5}$$

If $\omega : \mathbb{R}^3 \to \mathbb{R}^3$ is a locally integrable vector field, then we write $\|\omega\|_F = \|\mathcal{L}^3 \llcorner \omega\|_F$, where $\mathcal{L}^3$ is the Lebesgue measure on $\mathbb{R}^3$.

2.2. Euler equations

We consider the time-rescaled Euler equations

$$k_\varepsilon^{-1}\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = 0, \tag{6}$$

$$\nabla \cdot u^\varepsilon = 0, \tag{7}$$

where, as usual, $u^\varepsilon : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes the fluid velocity and $p^\varepsilon : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is the pressure. Moreover, k_ε is a (dimensionless) scaling factor that has to be specified later. The initial configuration is a divergence-free vector field u_0^ε on $\mathbb{R}^3$, that is

$$u^\varepsilon(0,\,\cdot\,) = u_0^\varepsilon.$$

To be more specific, we are interested in weak solutions of the Euler equation.

Definition 1. We call $u^\varepsilon \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3;\mathbb{R}^3))$ a weak solution of (6), (7) if $\nabla \cdot u^\varepsilon = 0$ in the sense of distributions, and

$$\int_0^\infty \int k_\varepsilon^{-1}\partial_t\phi \cdot u^\varepsilon + \nabla\phi : (u^\varepsilon \otimes u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int \phi(0,\,\cdot\,)\cdot u_0^\varepsilon\,\mathrm{d}x = 0 \tag{8}$$

for every test function $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3;\mathbb{R}^3)$ such that $\nabla\cdot\phi = 0$. A weak solution is said to be conservative if

$$t \mapsto \int |u^\varepsilon(t,x)|^2 \mathrm{d}x \qquad \text{is constant.}$$

Here and in the following, we understand undetermined integrals as integrals over the whole space.

It is well-known that in (8) the pressure can be reintroduced as a Lagrange multiplier for the divergence free condition on the test functions, and is then uniquely determined up to a function that depends on time t only. The weak solutions u^ε, however, are not unique, see for example [3,6,26,28,29]. In particular, it is shown in [7] that energy conservation fails as criterion for uniqueness. On the positive side existence of weak solutions for any initial datum in L^2 was established in [33]. These solutions are non-conservative (in fact, the energy is even discontinuous). In [31], the authors construct a dense subset of L^2 for which conservative solutions exist.

From the definition of weak solutions in (8), we immediately infer the following identity.

Lemma 1. *Let $u^\varepsilon \in L^2_{loc}(\mathbb{R}\times\mathbb{R}^3;\mathbb{R}^3)$ be a weak solution to the Euler equation, and let $\omega^\varepsilon = \nabla \times u^\varepsilon$ the vorticity (measure). Then for every $\phi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \phi \cdot d\omega^\varepsilon = k_\varepsilon \int \nabla(\nabla\times\phi) : u^\varepsilon \otimes u^\varepsilon\,\mathrm{d}x \qquad \textit{distributionally in } (0,\infty). \tag{9}$$

Proof. Fix $\phi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ and $f \in C_c^\infty(0,\infty)$. Since u^ε is a weak solution,

$$-\int_0^\infty f'(t)\left(\int (\nabla\times\phi)\cdot u^\varepsilon\,\mathrm{d}x\right)\mathrm{d}t = k_\varepsilon \int_0^\infty f(t)\left(\int \nabla(\nabla\times\phi) : (u^\varepsilon\otimes u^\varepsilon)\,\mathrm{d}x\right)\mathrm{d}t,$$

and this is exactly (9). □

2.3. Binormal curvature flow

A family of smooth curves $\{\Gamma(t)\}_{t\in[0,T]} \in \mathbb{R}^3$ is said to evolve by binormal curvature flow (BCF) if

$$\partial_t \gamma = \gamma' \times \gamma'', \tag{10}$$

where for each $t \in [0, T]$, $\gamma(t, \cdot) : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ is an arc-length parametrization of $\Gamma(t)$, that is, $\Gamma(t) = \{\gamma(t, s) : s \in \mathbb{R}/L\mathbb{Z}\}$ and $|\gamma'(t, s)|^2 = 1$ for all $s \in \mathbb{R}/L\mathbb{Z}$. Here, L is the length of the curve, and we assume the curve to be closed, so that $\mathbb{R}/L\mathbb{Z}$ is the interval of periodicity of $\gamma(t, \cdot)$. In case where $\Gamma(t)$ is a Frenet curve, then we can equivalently write (10) as

$$\partial_t \gamma = \kappa b,$$

where κ is the curvature and b is the binormal vector along the curve. That arc-length parametrizations are indeed compatible with binormal curvature flows can be seen by computing

$$\frac{\partial}{\partial t}|\gamma'|^2 = 2\gamma' \cdot \partial_t \gamma' \overset{(10)}{=} 0.$$

The short-time existence of smooth solutions follows from classical arguments.

There is a striking similarity between Lemma 1 for Euler solutions and the following formula for binormal curvature flows.

Lemma 2. *Let $\{\Gamma(t)\}_{t\in[0,T]}$ be a family of smooth curves evolving by binormal curvature flow. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \phi \cdot \tau \,\mathrm{d}\mathcal{H}^1 = \int_\Gamma \nabla(\nabla \times \phi) : (I - \tau \otimes \tau)\,\mathrm{d}\mathcal{H}^1, \tag{11}$$

for all $\phi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$. Here $\tau = \tau_\Gamma$ denotes the tangent along Γ.

In Section 3 below, we will see that the identities (9) and (11) give rise to the Hamiltonian-Poisson structures of the Euler equation and the binormal curvature flow, respectively. In a certain sense, in Theorems 1 and 2 below, we will estimate the distance of these structures.

We briefly recall the proof of Lemma 2 from [15]; see also [13] for a more general result.

Proof. Let γ be an arc-length parametrization of Γ satisfying (10). We write ϕ and γ in components and compute (summing implicitly)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}/L\mathbb{Z}} \phi_i(\gamma)\,\partial_s\gamma_i\,\mathrm{d}s = \int_{\mathbb{R}/L\mathbb{Z}} \partial_j\phi_i(\gamma)\partial_t\gamma_j\partial_s\gamma_i + \phi_i(\gamma)\partial_s\partial_t\gamma_i\,\mathrm{d}s.$$

Integrating by parts in the second term on the right-hand side and rearranging, one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}/L\mathbb{Z}} \phi_i(\gamma)\,\partial_s\gamma_i\,\mathrm{d}s = \int_{\mathbb{R}/L\mathbb{Z}} (\nabla \times \phi)(\gamma) \cdot (\partial_t\gamma \times \partial_s\gamma)\,\mathrm{d}s.$$

Since $\partial_s\gamma \cdot \partial_{ss}\gamma = 0$, the equation (10) implies that $\partial_t\gamma \times \partial_s\gamma = -\partial_s\gamma \times \partial_t\gamma = \partial_{ss}\gamma$. One arrives at (11) by substituting this into the right-hand side, integrating by parts again, and using the fact that $\nabla(\nabla \times \phi)$ is trace-free, so that $\nabla(\nabla \times \phi) : I = 0$. □

In [15], Smets and the first author develop a notion of weak solutions of the binormal curvature flow, in the spirit of geometric measure theory, based on the identity (11), and allowing for phenomena, such as changes of topology, seen in vortex filaments in real fluids. In the present work, (10) is a suitable notion as we deal with smooth flows only. Still, the results from [15] enter into our analysis through stability estimates in the spirit of Theorem 3 in [15].

2.4. Main results

In this section we state our two main results. First, we give some conditions under which it can be shown that Euler vortex filaments evolve to leading order by binormal curvature flow. The first condition is that the vorticity concentrates on an ε-scale around a smooth curve of fixed length. Concentration is measured in terms of the flat norm, introduced in (5) above. More precisely, we introduce the concentration distance

$$D(u^\varepsilon, \Lambda) := \|\nabla \times u^\varepsilon - \mu_\Lambda\|_F,$$

and will be interested in velocity vector fields such that

$$D(u^\varepsilon, \Lambda) \leqq \varepsilon L \tag{12}$$

for some curve Λ of length L, satisfying the weak regularity condition (4).[1] We will show that (12) implies a lower bound on the kinetic energy:

$$\int \frac{1}{2}|u^\varepsilon|^2 \,\mathrm{d}x \geqq \frac{L\log(L/\varepsilon)}{4\pi} - \mathcal{O}(1). \tag{13}$$

This is contained in Theorem 1 below, though with a very indirect proof. We next fix the time rescaling factor in the Euler equations as

$$k_\varepsilon = \frac{4\pi}{|\log(\varepsilon/L)|}.$$

It is a classical fact that this is the "right" scaling to obtain binormal curvature flow in the limit $\varepsilon \to 0$. This fact follows formally from the energy scaling (13), and it is confirmed by our main results.

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[1] Notice that by imposing (12), we make an implicit assumption on dimensions. Indeed, because the concentration distance formally scales like $(\text{length})^4$ divided by time, assumption (12) enforces a rescaling of time, namely time $\sim$ $(\text{length})^2$, so that velocity fields have dimensions of $(\text{length})^{-1}$ and vorticity fields have dimensions of $(\text{length})^{-2}$. This scaling in particular entails that both the excess (defined in (14)) and the circulation $\int u^\varepsilon \cdot \mu_\Gamma$ (that will not enter our analysis) are dimensionless quantities. See also Remark 1 below.

We also introduce the excess of the kinetic energy relative to the curve Λ:

$$\mathrm{Exc}_\varepsilon(u^\varepsilon,\Lambda) := \frac{k_\varepsilon}{L}\int \frac{1}{2}|u^\varepsilon|^2\,\mathrm{d}x - 1, \quad L = |\Lambda|. \tag{14}$$

As a consequence of the time scaling imposed in (12), the excess is a dimensionless quantity. It is preserved by the evolution if u^ε is a conservative solution and Λ has constant length (for example, as a solution to the binormal curvature flow). The excess measures the extent to which the lower bound in (13) is saturated. We will be interested in velocity fields for which

$$\mathrm{Exc}_\varepsilon(u^\varepsilon,\Lambda) \leqq C k_\varepsilon. \tag{15}$$

Together, conditions (12) and (15) imply that the kinetic energy is essentially induced by vorticity.

Our first main result estimates the difference between the right-hand sides of identities (9) and (11) for Euler and the binormal curvature flow. This can be understood as an estimate of the extent to which the (distributional) instantaneous velocity of a vortex filament in a solution of the Euler equations deviates from the binormal curvature.

Theorem 1. *Let $\Gamma \subset \mathbb{R}^3$ be an oriented closed Lipschitz curve of length L and let $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ be an arclength parametrization of Γ such that*

$$\|\kappa^*\|_{L^{1,\infty}} < \infty.$$

For $\varepsilon \in (0, \frac{L}{2})$, let $u^\varepsilon \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ be divergence-free vector fields such that u^ε and Γ satisfy $D(u^\varepsilon,\Gamma) \leqq \varepsilon L$. Then

$$0 \leqq \mathrm{Exc}_\varepsilon(u_\varepsilon,\Gamma) + \mathcal{O}\left(\|\kappa^*\|^2_{L^{1,\infty}} k_\varepsilon\right),$$

and there exists an absolute constant $C < \infty$ such that

$$\begin{aligned}\frac{1}{L}&\left| k_\varepsilon \int \phi : u^\varepsilon \otimes u^\varepsilon\,\mathrm{d}x - \int_\Gamma \phi : (I - \tau_\Gamma \otimes \tau_\Gamma)\,\mathrm{d}\mathcal{H}^1\right| \\ &\leqq C\|\phi\|_{L^\infty}\,\mathrm{Exc}_\varepsilon(u^\varepsilon,\Gamma) + \mathcal{O}\left(k_\varepsilon \|\kappa^*\|^2_{L^{1,\infty}}\|\phi\|_{W^{1,\infty}_L}\right)\end{aligned} \tag{16}$$

for all $\phi \in W^{1,\infty}(\mathbb{R}^3; M^{3\times 3})$, where $M^{3\times3}$ is the space of 3×3 matrices, $\|\phi\|_{W^{1,\infty}_L} := \|\phi\|_{L^\infty} + L\|\nabla\phi\|_{L^\infty}$.

In view of Lemmas 1 and 2, the conclusion (16) implies that if $u^\varepsilon(t,x)$ is a solution of the Euler equation, $\{\Gamma(t)\}$ is a binormal curvature flow of length L, and if $D(u^\varepsilon(t_0),\Gamma(t_0)) \leqq \varepsilon L$ at some time t_0, then

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}t}\int \phi\cdot\nabla\times u^\varepsilon\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \phi\cdot\tau\,\mathrm{d}\mathcal{H}^1\right| \\ &\quad\leqq C\|\nabla(\nabla\times\phi)\|_{L^\infty}\,\mathrm{Exc}_\varepsilon(u^\varepsilon,\Gamma) + \mathcal{O}\left(k_\varepsilon\|\kappa^*\|^2_{L^{1,\infty}}\|\nabla(\nabla\times\phi)\|_{W^{1,\infty}_L}\right)\end{aligned}$$

at time t_0. This shows that (at time t_0) the vorticity in u^ε is close in a distributional sense to a binormal curvature flow if the excess is small.

The same estimate (16) may also be understood as describing the distance of the Hamiltonian-Poisson structures associated with the Euler equation, cf. (9), and the binormal curvature flow, cf. (11). We discuss this in more detail in Section 3, where we also review the Hamiltonian-Poisson structures of both evolutions.

Like all prior work on this subject, Theorem 1 addresses only the question of estimating the instantaneous velocity of a vortex filament known to concentrate at a fixed time t_0 around a curve—it does not say anything about when such a concentration condition is preserved by the dynamics.[2] However, in a number of other ways, it improves on known results:

- As far as we know, all previous studies of the dynamics of vortex filaments consist of asymptotic computations that describe only highly idealized vortex filaments, such as the "prototype velocity field" associated to an ε-regularization of a C^2 filament, introduced in Section 2.5 below.
  By contrast, Theorem 1 applies to a much larger and more physically reasonable class of velocity fields—those with vorticity concentrated in the weak sense (12) about some curve Γ, and with small excess.
- Earlier results that we are aware of do not obtain any very useful control over error terms, whereas Theorem 1 quantifies errors well enough to conclude in Theorem 2 below that, as long as the vorticity remains concentrated around some curve, one can control the closeness of the vorticity to a binormal curvature flow over a macroscopic time interval.
  One reason this is possible is that the distributional estimates that we obtain, relating vortex filament velocity and the binormal curvature flow, seem to be more useful than the pointwise estimates found in earlier work.
- Theorem 1 shows that, at least at a fixed time, the binormal curvature flow approximates the velocity of vortex filaments (in a distributional sense), even when the vorticity is concentrated around a curve of low regularity, measured by the geometric quantity $\|\kappa^*\|_{L^{1,\infty}}$.
  For example, a recent paper [5] of de la Hoz and Vega studies the binormal curvature flow for initial data given by a regular planar polygon. Owing to the weak regularity conditions imposed on the curve Γ, Theorem 1 implies that if one considers the Euler equation for initial data whose vorticity is concentrated in the sense of (12) around a polygon, and with small excess, then the distributional initial velocity of the vorticity is close to the distributional binormal curvature of the polygon—a sum of delta functions at its vertices.

In short, to the best of our knowledge, this is the first time that the dynamics of vortex filaments in Euler flows have been approached in a quantitative way.

Our second main result shows, as discussed above, that a distributional estimate such as (16) is sufficient to ensure that a vortex filament remains close to a binormal

[2] Here and in what follows, we are omitting papers [1,2,12] that assume rotational and often additional symmetries.

curvature flow over a macroscopic time interval (that is bounded below, independent of ε, as $\varepsilon \to 0$).

Theorem 2. *Let $\{\Gamma(t)\}_{t\in[0,T]}$ be a family of smooth curves evolving by binormal curvature flow (10) with $|\Gamma(t)| = L$ and with arc-length parametrizations $\gamma : [0,T]\times\mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ satisfying*

$$\sup_{0\leqq t\leqq T} \|\partial_s^4\gamma\|_{L^\infty} < \infty, \qquad \inf_{0\leqq t\leqq T, s\in\mathbb{R}/L\mathbb{Z}} r_{\gamma(t,\cdot)}(s) > 0. \tag{17}$$

For $\varepsilon \in (0, \frac{L}{2})$, let u^ε be a conservative weak solution to the Euler equation (6), (7), for initial data satisfying

$$D(u^\varepsilon(0), \Gamma(0)) \leqq \varepsilon L, \qquad \mathrm{Exc}_\varepsilon(u^\varepsilon(0), \Gamma(0)) \leqq C k_\varepsilon. \tag{18}$$

If, for every $t \in [0,T]$, there is a Lipschitz curve $\Lambda(t)$ with arc-length parametrization $\lambda(t,\cdot) : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ and such that $u^\varepsilon(t)$ and $\Lambda(t)$ satisfy (12), and if in addition

$$|\Lambda(t)| = L \text{ for all } t, \qquad \sup_{t\in[0,T]} \|\kappa^*_{\Lambda(t)}(t,\cdot)\|_{L^{1,\infty}} < \infty, \tag{19}$$

then there exists a function $\bar\sigma : [0,T] \to \mathbb{R}/L\mathbb{Z}$ and a constant $C' < \infty$, depending only on the bounds in (17), (18), (19), and in particular independent of ε and L, such that for any $t \in [0,T]$,

$$\max_{s\in\mathbb{R}/L\mathbb{Z}} L^{-1} |\lambda(t,s) - \gamma(t, s+\bar\sigma(t))| \leqq C' k_\varepsilon^{1/2}, \tag{20}$$

$$\left(L^{-1}\int_{\mathbb{R}/L\mathbb{Z}} |\partial_s\lambda(t,s) - \partial_s\gamma(t,s+\bar\sigma(t))|^2\, ds\right)^{1/2} \leqq C' k_\varepsilon^{1/2}. \tag{21}$$

Our statement contains two estimates. First, (20) shows that in the regime of small ε, the curves Λ and Γ are close one to the other uniformly in t and s. The distance of both curves is of order $|\log\varepsilon|^{-1/2}$ and thus controlled by the initial datum via (18). The somewhat weaker L^2 bound on the difference of the tangents at Λ and Γ is displayed in (21). The latter ensures the closeness of both curves in a very geometric sense: The curves are locally very nearly parallel. Because no assumptions are imposed on the arc-length parametrizations λ, the spatial translations $\bar\sigma(t)$ are necessary in both (20) and (21).

As remarked above, the existence of smooth solutions of the binormal curvature flow (10) is standard, so hypotheses (17) and (18) are assumptions about the initial data. The content of the theorem is that to prove the vortex filament conjecture for such initial data, it suffices to find a solution u^ε for which vorticity remains concentrated around some curve, in the sense of (12) and (19). As far as we know, this is the first result to describe any conditions under which vortex filaments can be related to the binormal curvature flow for a macroscopic length of time. In addition, Theorem 2, like Theorem 1 above, is quantitative in a way that one does not find in the previous literature.

The task of constructing solutions as considered in Theorem 2, should one hope to do so, is in principle made easier by the facts that u^ε need only belong to L^2, that vorticity concentration is required only in the weak sense (12), and that only the weak regularity condition (19) is a priori required for the curves $\Lambda(t)$. For example, it is conceivable that one could construct weak solutions satisfying (12), (19) using techniques inspired by convex integration, as developed in references such as [3,6, 7,31,33]. On the other hand, the theorem shows that the regime described by (12), (19) is quite "rigid", and so may well be inaccessible to these techniques, which exploit "flexible" aspects of the Euler equations.

Remark 1. For $\alpha > 0$, the statements of the theorems are invariant with respect to the rescaling

$$x \mapsto \tilde{x} = \alpha x, \qquad t \mapsto \tilde{t} = \alpha^2 t, \qquad \varepsilon \mapsto \tilde{\varepsilon} = \alpha \varepsilon, \qquad u(\cdot) \mapsto \tilde{u}(\cdot) = \frac{1}{\alpha} u\left(\frac{\cdot}{\alpha^2}, \frac{\cdot}{\alpha}\right).$$

Due to this scaling invariance, it suffices to prove the Theorems 1 and 2 for $|\Gamma| = 1$.

In particular, we remark that if we write κ^* and $\tilde{\kappa}^*$ for the functions defined in (2), associated to parametrizations of Γ and $\tilde{\Gamma}$ respectively, then it is straightforward to check that $\|\kappa^*\|_{L^{1,\infty}} = \|\tilde{\kappa}^*\|_{L^{1,\infty}}$. We believe that this invariance makes $\|\kappa^*\|_{L^{1,\infty}}$ a natural quantity to consider.

2.5. Method of this paper

In this subsection, we explain the ideas of the proofs of our main results, Theorems 1 and 2 above. In view of Remark 1, it is enough to consider the case $L = 1$ in the following.

The main estimate (16) of Theorem 1 is derived by estimating the (Euler) velocity u^ε against a prototype velocity field v^ε, which itself satisfies the assertions of the theorem. To be more specific, given the curve Γ from the hypothesis, we construct v^ε by

$$v^\varepsilon = \nabla \times (-\Delta)^{-1}(\rho^\varepsilon * \mu_\Gamma).$$

Here, $\{\rho^\varepsilon\}_{\varepsilon \downarrow 0}$ denotes a sequence of radially symmetric standard mollifiers in $\mathbb{R}^3$, supported in a ball of radius ε and such that $\rho^\varepsilon(x) = \varepsilon^{-3}\rho^1(\varepsilon^{-1}x)$. The convolution of ρ^ε with a measure μ is defined as

$$\rho^\varepsilon * \mu(x) = \int_{\mathbb{R}^3} \rho^\varepsilon(x-y)\, \mathrm{d}\mu(y).$$

Moreover, the nonlocal differential operator $\nabla \times (-\Delta)^{-1}$ associates to a vorticity field ω a vector potential v via the Biot–Savart law

$$v(x) = \nabla \times (-\Delta)^{-1}\omega(x) = -\frac{1}{4\pi} \int \frac{x-y}{|x-y|^3} \times \omega(y)\, \mathrm{d}y.$$

That is, v is the unique divergence free vector field satisfying $\nabla \times v = \omega$. Notice that the Biot–Savart kernel is obtained as the curl of the Newtonian potential. We will sometimes write $(\nabla\times)^{-1} = \nabla \times (-\Delta)^{-1}$. The vector field v^ε can thus be written as

$$v^\varepsilon(x) = \frac{1}{4\pi}\int\int_0^1 \rho^\varepsilon(x-y)\frac{\gamma(s)-y}{|\gamma(s)-y|^3}\times\gamma'(s)\,\mathrm{d}s\,\mathrm{d}y$$

where, as above, $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is an arclength parametrization of a smooth non-self-intersecting curve Γ.

The main properties of v^ε that we will use later on are summarized in the following proposition.

Proposition 1. *The following estimates hold:*

$$\|\nabla\times v^\varepsilon - \mu_\Gamma\|_F \leqq \varepsilon, \tag{22}$$

$$\|v^\varepsilon\|_{L^q} \lesssim \varepsilon^{\frac{2}{q}-1}\|\kappa^*\|_{L^{1,\infty}} \quad \textit{for all } q\in(2,\infty], \tag{23}$$

$$\|v^\varepsilon\|_{L^2}^2 = \frac{1}{2\pi}|\log\varepsilon| + \mathcal{O}\left(\|\kappa^*\|_{L^{1,\infty}}^2\right), \tag{24}$$

and

$$\frac{4\pi}{|\log\varepsilon|}\int\phi : v^\varepsilon\otimes v^\varepsilon\,\mathrm{d}x = \int_\Gamma \phi : (I-\tau\otimes\tau)\,\mathrm{d}\mathcal{H}^1 + \mathcal{O}\left(\frac{\|\kappa^*\|_{L^{1,\infty}}^2\|\phi\|_{W^{1,\infty}}}{|\log\varepsilon|}\right), \tag{25}$$

for all $\phi\in W^{1,\infty}(\mathbb{R}^3; M^{3\times3})$*, where* $M^{3\times3}$ *denotes the space of* 3×3 *matrices.*

Hence, in view of (22), (24) and (25), the prototype velocity field v^ε satisfies, among others, the concentration condition and the two statements of Theorem 1. Estimates (23) and (24) will be used to bound error terms. In order to verify Theorem 1, we thus need to control how much u^ε deviates from v^ε. This is the content of Section 4.3 below. The concentration condition (22) will be established in Section 4.2 below. The remaining statements of Proposition 1 are proved in Section 4.5.

Let us now discuss our approach to Theorem 2. The hypothesis and the Hamiltonian structure of the Euler equations guarantee that the excess $\mathrm{Exc}_\varepsilon(u^\varepsilon,\Lambda)$ is of the order of the error term, that is $\mathcal{O}(|\log\varepsilon|^{-1})$. In view of the main estimate (16) of Theorem 1 and the weak formulation of Lemma 2 for the binormal curvature flow, the question of how close $\{\Lambda(t)\}_{t\in[0,T]}$ is to the binormal curvature flow $\{\Gamma(t)\}_{t\in[0,T]}$ amounts thus to the study of stability of the latter. Stability properties of binormal curvature flows have been recently investigated by Smets and the first author. In [15], it is shown that for a certain decaying C^2 continuation X of τ_Γ it holds that

$$|\partial_t X\cdot\xi - \nabla(\nabla\times X) : \xi\otimes\xi| \leq K(1 - X\cdot\xi), \tag{26}$$

for some $K = K(\Gamma)$ and all unit vector fields ξ. This remarkable property is reminiscent of (11). The *binormal curvature flow defect* of the flow $\{\Lambda(t)\}_{t\in[0,T]}$ is thus controlled by the distance of the tangent fields τ_Λ and τ_Γ. We use this stability estimate to deduce Theorem 2 from Theorem 1.

3. The Hamiltonian–Poisson Structure

The goal of this section is to give an abstract geometric interpretation of our first main result, Theorem 1. More precisely, we will show that our main estimate (16) can be formally seen as an estimate on the distance of the Poisson brackets that constitute the Euler equation and the binormal curvature flow, respectively. The present section has no impact on the rest of the paper and can be skipped by the impatient reader.

We will briefly review Marsden and Weinstein's interpretation of the Euler equations and the binormal curvature flow as Hamiltonian systems on Poisson manifolds [22], see also [17,27] for more recent and more general discussions. As opposed to the original paper, where this interpretation is worked out in the language of Lie algebras, our approach is based on vector calculus which allows us to identify the involved ingredients directly with various quantities that appear in the statements of our theorems.

We recall that we need three ingredients to constitute a Hamiltonian-Poisson system:

- a differentiable manifold $\mathcal{M}$;
- a Poisson bracket on $\mathcal{M}$, that is, a bilinear map that takes functions on $\mathcal{M}$ to functions on $\mathcal{M}$, is skew-symmetric, that is, $\{F, G\} + \{G, F\} = 0$, satisfies the Jacobi identity, that is, $\{E, \{F, G\}\} + \{F, \{G, E\}\} + \{G, \{E, F\}\} = 0$, and obeys the Leibniz rule, that is, $\{EF, G\} = \{E, G\}F + \{F, G\}E$;
- and a function $H : \mathcal{M} \to \mathbb{R}$.

We call a dynamical system $p(t)$ in $\mathcal{M}$ a Hamiltonian-Poisson system (or a Hamiltonian system with Poisson structure) if it satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} F(p) = \{F, H\}(p) \tag{27}$$

for any $F : \mathcal{M} \to \mathbb{R}$. The function H is referred to as the Hamiltonian.

We first explain the formal Hamiltonian-Poisson structure of the Euler equations. As we are interested in the situation where the vorticity is concentrated along curves, it is convenient to consider vorticity fields as main objects. We accordingly take the manifold to be the phase space of vorticity fields

$$\mathcal{M} = \{\omega : \exists u \text{ s.t. } \nabla \cdot u = 0 \text{ and } \omega = \nabla \times u\}.$$

The Hamiltonian is the kinetic energy

$$H(\omega) = \frac{1}{2} \int |(\nabla\times)^{-1} \omega|^2 \, \mathrm{d}x.$$

The Poisson bracket $\{F, H\}$ is induced by a symplectic form, that is, a skew-symmetric bilinear form, for the gradients of F and H,

$$\{F, H\}(\omega) = \int (\operatorname{grad} F_\omega \times \operatorname{grad} H_\omega) \cdot \omega \, \mathrm{d}x.$$

Recall that on a Riemannian manifold $(\mathcal{M}, g)$ the gradient is the tangent vector related to the differential via the Riemannian metric: $\mathrm{d}\, F_\omega(v) = g_\omega(\operatorname{grad} F_\omega, v)$ for any tangent vector $v \in \mathcal{T}_\omega\mathcal{M}$. In the case of the Euler equation, we think of tangent vectors as velocity fields that transport the vortex filaments. We will thus identify

$$\mathcal{T}_\omega\mathcal{M} \cong \{v :\ \nabla \cdot v = 0\}.$$

In the Euler case, the metric tensor is the L^2 inner product

$$g_\omega(v_1, v_1) = \int v_1 \cdot v_2 \,\mathrm{d}x.$$

The gradient of the Hamiltonian can thus be read directly from the differential $\mathrm{d}\, H_\omega(v) = \int (\nabla\times)^{-1}\omega \cdot v \,\mathrm{d}x$: It holds $\operatorname{grad} H_\omega = (\nabla\times)^{-1}\omega =: u$.

We now calculate the dynamics defined through (27). We let F be an arbitrary function on $\mathcal{M}$. One the one hand, by the definition of the differential, $\frac{\mathrm{d}}{\mathrm{d}t} F(\omega) = \mathrm{d}\, F_\omega(\partial_t u)$. On the other hand, by the definition of the Poisson bracket and elementary computations, $\{F, H\}(\omega) = -\int \operatorname{grad} F_\omega \cdot (\omega \times u)\,\mathrm{d}x$. Since F was arbitrary,

$$\mathrm{d}\, F_\omega\, (\partial_t u + \omega \times u) = 0 \quad \text{for all} \quad F : \mathcal{M} \to \mathbb{R}.$$

In particular, $\partial_t u + \omega \times u \in (\mathcal{T}_\omega\mathcal{M})^\perp$, and thus $\partial_t u + \omega \times u = \nabla q$ for some scalar q on $\mathbb{R}^3$. Notice that the latter can be rewritten as

$$\partial_t u + u \cdot \nabla u + \nabla p = 0$$

for some function p, because $\omega \times u = u \cdot \nabla u - \nabla(\frac{1}{2}|u|^2)$. This formal computation describes the Hamiltonian-Poisson structure of the Euler equation.

Let thus now describe the Hamiltonian-Poisson structure of the binormal curvature flow. The manifold is that of perfect vortex filaments,

$$\mathcal{M} = \Big\{\text{oriented, closed curves } \Gamma \text{ in } \mathbb{R}^3\Big\},$$

and the Hamiltonian is the length of the curve

$$H(\Gamma) = |\Gamma|.$$

The Poisson bracket is essentially the same as for the Euler equation,

$$\{F, H\}(\Gamma) = \int_\Gamma (\operatorname{grad} F_\Gamma \times \operatorname{grad} H_\Gamma) \cdot \tau_\Gamma \,\mathrm{d}\mathcal{H}^1,$$

where, as before, the gradients are induced by an L^2 inner product

$$g_\Gamma(v_1, v_2) = \int_\Gamma v_1 \cdot v_2 \,\mathrm{d}\mathcal{H}^1.$$

Infinitesimal variations of curves can be represented by vector fields. We thus identify

$$\mathcal{T}_\Gamma\mathcal{M} \cong \{\text{vector fields } v \text{ on } \Gamma\}.$$

We now compute the Hamiltonian-Poisson system generated by these ingredients. If γ is an arc-length parametrization of Γ, then $\mathrm{d}\,H_\gamma(v) = -\int_0^L \gamma''(s)\cdot v(\gamma(s))\,\mathrm{d}s$ and thus $\operatorname{grad} H_\gamma = -\gamma''$. It thus follows that

$$\{F, H\}(\Gamma) = -\int_0^L \left(\operatorname{grad} F_\gamma \times \gamma''\right)\cdot\gamma'\,\mathrm{d}s = \int_0^L \operatorname{grad} F_\gamma \cdot \left(\gamma' \times \gamma''\right)\,\mathrm{d}s.$$

Hence, using similar arguments to those above, the Hamilton-Poisson system (27) becomes

$$\mathrm{d}\,F_\Gamma\left(\partial_t \gamma - \gamma' \times \gamma''\right) = 0.$$

Since F was arbitrary, this shows that the curve evolves by binormal curvature flow.

In the following, we will make a connection between Marsden and Weinstein's Hamiltonian interpretations of the vortex (filament) dynamics and Theorem 1. We will show that the left-hand side of (16) is nothing but the distance of the Poisson brackets generating, respectively, the Euler equation and the binormal curvature flow. With that, we give Theorem 1 an abstract geometric meaning.

Starting with the Euler term, we let F be a linear function induced by some velocity field v, i.e, $F(\omega) = \int v\cdot(\nabla\times)^{-1}\omega\,\mathrm{d}x$, so that $\operatorname{grad} F = v$ and $\{F, H\} = -\int v\cdot(\omega\times u)\,\mathrm{d}x = \int \nabla v : u\otimes u\,\mathrm{d}x$. Because v is itself divergence-free, we can furthermore write $v = \nabla\times\phi$ and $F(\omega) = \int \phi\cdot\omega\,\mathrm{d}x$ and get

$$\{F, H\} = \int \nabla(\nabla\times\phi) : u\otimes u\,\mathrm{d}x. \tag{28}$$

Likewise, in the BCF case, where ω reduces to the tangent field on Γ, we consider $F(\phi) = \int_\Gamma \phi\cdot\tau_\Gamma\,\mathrm{d}\mathcal{H}^1$. Then $\operatorname{grad} F_\Gamma = \tau_\Gamma\times(\nabla\times\phi)$ and thus

$$\{F, H\} = -\int \nabla(\nabla\times\phi) : \tau_\Gamma\otimes\tau_\Gamma\,d\mathcal{H}^1 = \int \nabla(\nabla\times\phi) : (I - \tau_\Gamma\otimes\tau_\Gamma)\,d\mathcal{H}^1. \tag{29}$$

In view of (28) and (29), our first main result, Theorem 1, provides a quantitative version of the statement that if the vorticity concentrates in an ε neighborhood of a curve, then the Euler Poisson bracket is close to the BCF Poisson bracket.

4. Proofs

The remainder of the paper is devoted to the rigorous justification of Theorems 2 and 1. We start with some preliminaries on geometric properties of the class of curves that we consider, and we also derive some properties of the flat norm.

4.1. Curves for which $\|\kappa^*\|_{L^{1,\infty}} < \infty$.

Assume that Γ is a closed Lipschitz curve such that

$$|\Gamma| = 1, \qquad \text{and} \qquad \|\kappa_\Gamma^*\|_{L^{1,\infty}} < \infty. \tag{30}$$

In particular, this implies that $r_\gamma(s) > 0$ almost everywhere for any arclength parametrization γ. We will use the notation

$$\mathcal{T} := \left\{ \gamma(s) + v : s \in \mathbb{R}/\mathbb{Z}, v \in \mathbb{R}^3, v \cdot \gamma'(s) = 0, |v| < \frac{1}{4} r(s) \right\}.$$

Thus $\mathcal{T}$ is a tube of varying thickness around Γ. In general, $\mathcal{T}$ need not be open, but it is straightforward to check that it is measurable. We record some of its properties:

Lemma 3. *For every $x \in \mathcal{T}$, there exists a unique closest point $P(x)$ in Γ, characterized by* $\operatorname{dist}(x, \Gamma) = |x - P(x)|$, *and we may define $\zeta : \mathcal{T} \to \mathbb{R}/\mathbb{Z}$ by requiring that*

$$P(x) = \gamma(\zeta(x)) \quad \textit{for all} \;\; x \in \mathcal{T}.$$

If γ is C^2 then ζ is C^1 in $\mathcal{T}$, with

$$\nabla \zeta(x) = \frac{\gamma'(\zeta(x))}{1 - (x - \gamma(\zeta(x))) \cdot \gamma''(\zeta(x))} \quad \textit{for all} \;\; x \in \mathcal{T}. \tag{31}$$

More generally, if we only assume (30)*, then ζ is locally Lipschitz in $\mathcal{T}$, and*

$$\left| \frac{1}{|\nabla \zeta(x)|} - 1 \right| \leqq \frac{\operatorname{dist}(x, \Gamma)}{r(\zeta(x))} \quad \textit{for almost everywhere} \;\; x \in \mathcal{T}. \tag{32}$$

A similar function ζ, but for smoother Γ and defined on a tube of uniform diameter, can be found in [15, Prop. 4].

Proof. Fix $x \in \mathcal{T}$; then there exist $s \in \mathbb{R}/\mathbb{Z}$ and $v \in \mathbb{R}^3$ such that

$$x = \gamma(s) + v, \qquad v \cdot \gamma'(s) = 0, \qquad |v| < \frac{1}{4} r(s). \tag{33}$$

We may assume, by changing variables and reparametrizing γ, that $s = 0$, $\gamma(0) = 0 \in \mathbb{R}^3$ and $\gamma'(0) = (0, 0, 1)$, and we will write

$$\gamma = (\gamma_\perp, \gamma_\parallel) \in \mathbb{R}^2 \times \mathbb{R}, \qquad x = v = (v_\perp, 0), \qquad r_0 = r(0).$$

We will show that $P(x)$ is well-defined by proving that $\gamma(0)$ is the unique closest point in Γ to x, or in other words that

$$\text{if } 0 < |h| \leqq 1/2, \;\; \text{then } |x - \gamma(h)| > |v| = |v_\perp|. \tag{34}$$

If $|h| \geqq r_0$, then the definition of $r(0)$ implies that $|\gamma(h)| = |\gamma(h) - \gamma(0)| \geqq \frac{1}{2} r_0$, so by (33),

$$|\gamma(h) - x| \geqq |\gamma(h)| - |v| \geqq \frac{1}{4} r_0 > |v|.$$

We thus assume $|h| < r_0$. By the definition of $r(\cdot)$,

$$\gamma_\parallel'(h) = \gamma'(h) \cdot \gamma'(0) = 1 - \frac{1}{2} |\gamma'(h) - \gamma'(0)|^2 \geqq 1 - \frac{h^2}{2 r_0^2} \tag{35}$$

for almost everywhere $|h| \leqq r_0$. It follows that

$$|\gamma'_\perp(h)| = (1 - (\gamma'_\parallel)^2)^{1/2} \leq \frac{|h|}{r_0}. \tag{36}$$

For $0 < h < r_0$, we integrate these inequalities to obtain

$$h \geqq \gamma_\parallel(h) \geqq h\left(1 - \frac{h^2}{6r_0^2}\right), \qquad |\gamma_\perp(h)| \leqq \frac{h^2}{2r_0}. \tag{37}$$

Thus

$$|\gamma(h) - x|^2 \geq \begin{cases} h^2\left(1 - \frac{h^2}{6r_0^2}\right)^2 + \left(|v_\perp| - \frac{h^2}{2r_0}\right)^2 & \text{if } \frac{h^2}{2r_0} \leqq |v_\perp|, \\ h^2\left(1 - \frac{h^2}{6r_0^2}\right)^2 & \text{if not.} \end{cases}$$

It follows from this by elementary calculations, using the fact that $|v_\perp| < \frac{1}{4}r_0$, that $|\gamma(h) - x|^2 > |v_\perp|^2$. The same estimate holds for $-r_0 < h < 0$, by essentially the same argument, completing the proof of (34).

The uniqueness of the closest point implies that $P(\cdot)$ and $\zeta(\cdot)$ are continuous in $\mathcal{T}$. Indeed, if $x_k \to x$ in $\mathcal{T}$ and $P(x_k) \to y$, then it is clear that $y \in \Gamma$ and that $\operatorname{dist}(x, y) = \lim_k \operatorname{dist}(x_k, \Gamma) = \operatorname{dist}(x, \Gamma)$, and hence that $y = P(x)$.

To verify that ζ is Lipschitz, consider $y \in \mathcal{T}$ near x, and let $h := \zeta(y)$. If y is sufficiently close to x, then h satisfies

$$|h| < r_0, \qquad y = \gamma(h) + w, \qquad w \cdot \gamma'(h) = 0.$$

We would like to estimate $|h| = |\zeta(x) - \zeta(y)|$ in terms of $|x - y|$. To do this, note that $|y - x| \geqq |\gamma'(h) \cdot (y - x)| = |\gamma'(h) \cdot (\gamma(h) - x)|$. By writing

$$\gamma' \cdot (\gamma - x) = (\gamma'_\perp, \gamma'_\parallel) \cdot (\gamma_\perp - v_\perp, \gamma_\parallel)$$

and using the estimates for various components of γ and γ' in (35), (36) and (37), we find that

$$|h|\left(1 - \frac{|v_\perp|}{r_0} - \frac{7}{6}\frac{h^2}{r_0^2}\right) \leq |x - y| \tag{38}$$

as long as $|h| < r_0$. Since $|v_\perp| < \frac{1}{4}r_0$, it follows that $|x - y| \geqq \frac{1}{2}|h|$ as long as $|h| < \frac{1}{4}r_0$. Because x was arbitrary, this shows that for $x, y \in \mathcal{T}$,

$$|x - y| \geqq \frac{1}{2}|\zeta(x) - \zeta(y)| \text{ as long as } |\zeta(x) - \zeta(y)| \leqq \frac{1}{4}r(\zeta(x)).$$

It follows that ζ is locally Lipschitz in $\mathcal{T}$ and hence almost everywhere differentiable in $\mathcal{T}$.

Next, at every point of differentiability, it follows from (38) that

$$\frac{1}{|\nabla\zeta(x)|} = \liminf_{y \to x} \frac{|x - y|}{|\zeta(x) - \zeta(y)|} \geqq 1 - \frac{|v_\perp|}{r_0} = 1 - \frac{\operatorname{dist}(x, \Gamma)}{r(\zeta(x))}.$$

On the other hand, if ζ is differentiable at x and $\mathcal{T}$ has density 1 at x in the sense that $\lim_{r\searrow 0}\frac{|B_r(x)\cap\mathcal{T}|}{|B_r(x)|}=1$, then there exists a sequence $y_k\to x$ such that, writing $h_k:=\zeta(y_k)$ and still assuming $\zeta(x)=0$,

$$1=\lim_{k\to\infty}\frac{\gamma'(0)\cdot(y_k-x)}{|y_k-x|}=\lim_{k\to\infty}\frac{\gamma'(h_k)\cdot(y_k-x)}{|y_k-x|}$$
$$=\lim_{k\to\infty}\frac{\gamma'(h_k)\cdot(\gamma(h_k)-x)}{|y_k-x|}.$$

Thus, again decomposing γ and γ' and using (35), (36) and (37), we check that

$$\begin{aligned}1&\le\lim_{k\to\infty}\frac{\gamma_\parallel'(h_k)\gamma_\parallel(h_k)+|\gamma_\perp'(h_k)|\,(|\gamma_\perp(h_k)|+|v_\perp|)}{|y_k-x|}\\&\le\lim_{k\to\infty}\frac{h_k}{|y_k-x|}\left(1+\frac{|v_\perp|}{r_0}\right)\end{aligned}$$

which implies that $|\nabla\zeta(x)|^{-1}\le 1+\frac{|v_\perp|}{r_0}=1+\frac{\operatorname{dist}(x,\Gamma)}{r(\zeta(x))}$. Thus we have proved (32).

Finally, if γ is C^2, then it follows from what we have said above (which shows that $\zeta(x)=s$ when (33) holds) that

$$\varphi(x,\zeta(x))=0\qquad\text{for }\varphi(x,t)=(x-\gamma(t))\cdot\gamma'(t).$$

Because $|\gamma''(s)|\leqq\frac{1}{r(s)}$, the definition of $\mathcal{T}$ implies that $\partial_t\varphi(x,t)\leqq-3/4$ everywhere in $\mathcal{T}$. Thus the implicit function theorem implies that ζ is C^1 and that

$$\gamma_i'(\zeta(x))-\big[1-(x-\gamma(\zeta(x)))\cdot\gamma''(\zeta(x))\big]\,\partial_i\zeta(x)=0.$$

We obtain (31) by rearranging. □

The following Lemma is an important reason for the relevance of the weak L^1 norm of κ^*.

Lemma 4. *There exists an absolute constant C such that for every $x\in\mathbb{R}^3$ and every $r>0$,*

$$|\{s\in\mathbb{R}/\mathbb{Z}:|\gamma(s)-x|<r\}|\leqq Cr\|\kappa^*\|_{L^{1,\infty}}.\tag{39}$$

In fact the proof will show that we may take $C=8$.

Proof. Fix $x\in\mathbb{R}^3$ and $r>0$, and let $A:=\{s\in\mathbb{R}/\mathbb{Z}:|\gamma(s)-x|<r\}$. We consider 2 cases.

Case 1: $|A|\leqq 8r$.

It is clear from the definition of $r(\cdot)$ that $\frac{1}{\kappa^*(s)}=r(s)\leqq 1$ for every s, and hence that $|\{s\in\mathbb{R}/\mathbb{Z}:\kappa^*(s)\geqq 1\}|\geqq 1$, which implies that $\|\kappa^*\|_{L^{1,\infty}}\geqq 1$. Thus (39) holds.

Case 2: $|A| > 8r$.

For any $s \in A$, there must then exist $t \in A$ such that $4r < |s-t| < 1/2$. The definition of A and the triangle inequality imply that $|\gamma(s) - \gamma(t)| < 2r$. Then

$$2r > \left|\gamma'(s) \cdot (\gamma(t) - \gamma(s))\right| = \left|\gamma'(s) \cdot \gamma'(\tau)\right| \; |t-s|$$

for some τ between s and t. It follows that $\gamma'(s) \cdot \gamma'(\tau) < \frac{1}{2}$, and hence that $|\gamma'(s) - \gamma'(\tau)|^2 = 2 - 2\gamma'(s) \cdot \gamma'(\tau) > 1$.

These facts imply that $r(s) \leqq 4r$. To check this, we must show that for every $\rho > 4r$, one of two inequalities appearing in the definition (1) of $r(\cdot)$ is violated. The first of these inequalities is

$$|\gamma(s+h) - \gamma(s)| \geqq \frac{\rho}{2} \quad \text{ for all } \; |h| \geq \rho,$$

which is violated by $s + h = t$, if $4r < \rho < |t-s|$. The other inequality is

$$|\gamma'(s+h) - \gamma'(s)| \leqq \frac{|h|}{\rho} \quad \text{ for all } \; |h| \leqq \rho,$$

and is violated by $s + h = \tau$, if $|\tau - s| < \rho$. Thus at least one of these must fail when $\rho > 4r$, so $r(s) \leqq 4r$ as claimed.

Since s was arbitrary, we conclude $A \subset \{s : \kappa^*(s) \geqq \frac{1}{4r}\}$, and hence that

$$|A| \leqq 4r\|\kappa^*\|_{L^{1,\infty}}.$$

This proves Lemma 4. □

4.2. Some properties of the flat norm

Lemma 5. *Let ω be a divergence-free vector-valued measure on $\mathbb{R}^3$ such that $\|\omega\|_F < \infty$. Then*

$$\|\omega\|_F = \inf\{|\varphi|(\mathbb{R}^3) \; : \; \varphi \in \mathcal{M}(\mathbb{R}^3; \mathbb{R}^3), \; \nabla \times \varphi = \omega\} \tag{40}$$

and the infimum is attained.

In the statement of the lemma, $\mathcal{M}(\mathbb{R}^3; \mathbb{R}^3)$ denotes the space of vector-valued measures on $\mathbb{R}^3$, and $|\varphi|$ denotes the total variation of the measure φ, defined by

$$|\varphi|(\mathbb{R}^3) = \sup\left\{\int f \cdot \mathrm{d}\varphi : \; f \in C_c(\mathbb{R}^3; \mathbb{R}^3), \; \|f\|_{L^\infty} \leqq 1\right\}.$$

Clearly, if φ is absolutely continuous and integrable, then $|\varphi|(\mathbb{R}^3) = \|\varphi\|_{L^1}$.

The lemma is a variant of [11, 4.1.12], rewritten in the language of vector calculus. We provide a proof for the reader's convenience.

Proof. We first observe that for any $\xi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with $\|\nabla\times\xi\|_{C^0} \le 1$ and any vector-valued Radon measure φ such that $\nabla\times\varphi = \omega$,

$$\int \xi \cdot \mathrm{d}\omega = \int \nabla\times\xi\cdot \mathrm{d}\varphi \le |\varphi|(\mathbb{R}^3).$$

It follows that

$$\|\omega\|_F \leqq \inf\{|\varphi|(\mathbb{R}^3)\ :\ \varphi\in\mathcal{M}(\mathbb{R}^3;\mathbb{R}^3),\ \nabla\times\varphi = \omega\}.$$

The other inequality is a nice application of the Hahn–Banach theorem. We denote by X the vector space $C_c(\mathbb{R}^3;\mathbb{R}^3)$ equipped with the maximum norm, and define the linear subspace

$$Y := \left\{ f\in X :\ \exists\xi\in C_c^1(\mathbb{R}^3;\mathbb{R}^3) \text{ with } f = \nabla\times\xi\right\},$$

also equipped with the maximum norm. We furthermore define, implicitly,

$$L(\nabla\times\xi) := \int \xi\cdot d\,\omega$$

for vector fields $\nabla\times\xi\in Y$. This linear operator is well-defined thanks to the divergence-free condition for ω. Indeed, if ξ_1 and ξ_2 are such that $\nabla\times\xi_1 = \nabla\times\xi_2$, then $\xi_1-\xi_2 = \nabla\rho$ for some C_c^2 function ρ, and thus

$$\int \xi_1\cdot \mathrm{d}\omega - \int \xi_2\cdot d\omega = \int \nabla\rho\cdot d\omega = 0,$$

because $\nabla\cdot\omega = 0$. Moreover, $L : Y\to\mathbb{R}$ is bounded with operator-norm

$$\|L\|_{Y\to\mathbb{R}} = \sup\{L(f) :\ f\in Y \text{ with } \|f\|_{L^\infty}\le 1\} = \|\omega\|_F.$$

By the Hahn–Banach theorem (see, for example, [24, Theorem 3.3]), there exists a linear function $\bar{L} : X\to\mathbb{R}$ such that $\bar{L}$ agrees with L on Y, and whose norm is not larger than that of L:

$$\|\bar{L}\|_{X\to\mathbb{R}} = \|L\|_{Y\to\mathbb{R}} = \|\omega\|_F.$$

By duality (see, for example, [10, Chapter 1.8]), there exists a vector-valued Radon measure φ on $\mathbb{R}^3$ such that

$$\bar{L}(f) = \int f\cdot \mathrm{d}\varphi \quad \text{for all } f\in C_c(\mathbb{R}^3;\mathbb{R}^3).$$

It follows that

$$\int \xi\cdot \mathrm{d}\omega = L(\nabla\times\xi) = \int \nabla\times\xi\cdot \mathrm{d}\varphi$$

for all $\xi\in C_c^1(\mathbb{R}^3;\mathbb{R}^3)$, that is, $\omega = \nabla\times\varphi$. Furthermore,

$$\begin{aligned}\|\omega\|_F &= \|\bar{L}\|_{X\to\mathbb{R}}\\ &= \sup\left\{\int f\cdot \mathrm{d}\varphi :\ f\in C_c(\mathbb{R}^3:\mathbb{R}^3),\ \|f\|_{L^\infty}\leqq 1\right\} = |\varphi|(\mathbb{R}^3).\end{aligned}$$

This completes the proof of (40), and also shows that the infimum is attained. □

Lemma 6. *Assume that* $w \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ *and that* $\nabla \cdot w = 0$. *If* $\|\nabla \times w\|_F < \infty$, *then*

$$\|w\|_{L^{1,\infty}} \lesssim \|\nabla \times w\|_F.$$

Proof. By Lemma 5, there exists a vector-valued measure φ on $\mathbb{R}^3$ such that $\nabla \times \varphi = \nabla \times w$ and $|\varphi|(\mathbb{R}^3) = \|\nabla \times w\|_F$. The linear relation between φ and w is retained by convolution, and thus, denoting by φ^ε and w^ε the convolutions of φ and w, respectively, with a smooth symmetric mollification kernel ρ^ε, on the Fourier level it holds

$$\widehat{w^\varepsilon}(\xi) = m(\xi)\widehat{\varphi^\varepsilon}(\xi) \quad \text{where } m(\xi) = \mathbb{1} - \frac{\xi \otimes \xi}{|\xi|^2},$$

as can be derived from the elementary formula $\nabla \times \nabla \times A = -\Delta A + \nabla(\nabla \cdot A)$. Because m is homogeneous of degree zero, bounded, and smooth away from the origin, we infer from Hörmander's multiplier theorem [30, Theorem 0.2.6] that

$$\|w^\varepsilon\|_{L^{1,\infty}} \lesssim \|\varphi^\varepsilon\|_{L^1}. \tag{41}$$

Next, we want to pass to the limit $\varepsilon \to 0$ in this estimate. It is clear that $w^\varepsilon \to w$ in L^1_{loc}, so that for every positive R and σ,

$$\frac{\sigma}{2}|\{x \in B(R) : |w^\varepsilon(x)| \leqq \sigma/2 < \sigma \leqq |w(x)|\}| \leqq \int_{B(R)} |w - w^\varepsilon| \mathrm{d}x \to 0$$

as $\varepsilon \to 0$. It follows that

$$\sigma|\{x \in B(R) : |w(x)| \geqq \sigma\}| \leqq 2\sigma \left|\left\{x \in B(R) : |w^\varepsilon(x)| \geq \frac{1}{2}\sigma\right\}\right|$$

if ε is sufficiently small. Then (41) then implies that

$$\sigma|\{x \in \mathbb{R}^3 : |w(x)| \geqq \sigma\}| = \lim_{R\to\infty} \sigma|\{x \in B(R) : |w(x)| \geqq \sigma\}| \lesssim \|\varphi^\varepsilon\|_{L^1}.$$

Taking the supremum over σ yields $\|w\|_{L^{1,\infty}} \lesssim |\varphi|(\mathbb{R}^3)$. To conclude, it remains to combine this estimate with the statement of Lemma 5. □

Lemma 7. *Let* $\mu \in \mathcal{M}(\mathbb{R}^3; \mathbb{R}^3)$ *be compactly supported and divergence-free. If* ρ^ε *is a nonnegative function such that supp*$(\rho^\varepsilon) \subset B_\varepsilon(0)$ *and* $\int \rho^\varepsilon \, \mathrm{d}x = 1$, *then*

$$\|\rho^\varepsilon * \mu - \mu\|_F \leqq \varepsilon|\mu|(\mathbb{R}^3).$$

Proof. For any $z \in \mathbb{R}^3$, let us write $\sigma_z\mu$ to denote the measure defined by

$$\int \phi \cdot \mathrm{d}(\sigma_z\mu) := \int \phi(\cdot + z) \cdot \mathrm{d}\mu.$$

We also define a vector-valued measure R_z by

$$\int \phi \cdot \mathrm{d}R_z = \int_0^1 \int (\phi(\cdot + sz) \times z) \cdot \mathrm{d}\mu \, \mathrm{d}s.$$

It is a standard fact that

$$\nabla \times R_z = \sigma_z \mu - \mu. \tag{42}$$

We recall the proof: for any $\phi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$,

$$\int \phi \cdot \mathrm{d}(\nabla \times R_z) = \int \nabla \times \phi \cdot \mathrm{d}R_z = \int_0^1 \int ((\nabla \times \phi)(\cdot + sz) \times z) \cdot \mathrm{d}\mu \, \mathrm{d}s.$$

Straightforward computations show that

$$((\nabla \times \phi)(\cdot + sz) \times z) \cdot \mathrm{d}\mu = \frac{\partial}{\partial s}\phi(\cdot + sz) \cdot \mathrm{d}\mu - \sum_j z_j \nabla \phi_j(\cdot + sz) \cdot \mathrm{d}\mu.$$

Clearly,

$$\int_0^1 \int z_j \nabla \phi_j(\cdot + sz) \cdot \mathrm{d}\mu \, \mathrm{d}s = 0,$$

since the inner integral vanishes for every s because μ is divergence-free. Thus we conclude from the fundamental theorem of calculus that

$$\int \phi \cdot \mathrm{d}\,(\nabla \times R_z) = \int (\phi(\cdot + z) - \phi) \cdot \mathrm{d}\mu,$$

which proves (42). It follows that

$$\rho^\varepsilon * \mu - \mu = \int \rho^\varepsilon(z)(\sigma_z \mu - \mu)\, \mathrm{d}z = \nabla \times \int \rho^\varepsilon(z) R_z \, \mathrm{d}z$$

in the sense of distributions. Hence by Lemma 5,

$$\begin{aligned} \|\rho^\varepsilon * \mu - \mu\|_F \; &\leqq \; \left| \int \rho_\varepsilon(z) R_z \, \mathrm{d}z \right| (\mathbb{R}^3) \; \leqq \; \int \rho_\varepsilon(z) \, |R_z| \, (\mathbb{R}^3) \, \mathrm{d}z \\ &\leqq \; \sup_{|z| \leqq \varepsilon} |R_z| \, (\mathbb{R}^3). \end{aligned}$$

However, it is easy to check from the definition that $|R_z|\,(\mathbb{R}^3) \leqq |z| \; |\mu|(\mathbb{R}^3)$ for every z, so the conclusion follows. □

4.3. Proof of Theorem 1

Proof of Theorem 1. We will write Exc_ε for $\mathrm{Exc}_\varepsilon(u^\varepsilon, \Gamma)$ in the following. The fact that $0 \leqq \mathrm{Exc}_\varepsilon + \mathcal{O}\left(\|\kappa^*\|^2_{L^{1,\infty}} |\log \varepsilon|^{-1}\right)$ follows immediately from the main estimate (16).

We prove (16) componentwise. For this purpose, we fix $i, j \in \{1, 2, 3\}$ and write

$$\int \phi u_i^\varepsilon u_j^\varepsilon \, \mathrm{d}x = \int \phi v_i^\varepsilon v_j^\varepsilon \, \mathrm{d}x + \int \phi \left(u_i^\varepsilon - v_i^\varepsilon\right) \left(u_j^\varepsilon - v_j^\varepsilon\right) \mathrm{d}x + \int \phi \left(u_i^\varepsilon - v_i^\varepsilon\right) v_j^\varepsilon \, \mathrm{d}x + \int \phi \left(u_j^\varepsilon - v_j^\varepsilon\right) v_i^\varepsilon \, \mathrm{d}x.$$

Thanks to (25) in Proposition 1 (whose proof appears in Section 4.5 below), conclusion (16) follows from the estimates

$$\int \phi \left(u_i^\varepsilon - v_i^\varepsilon\right) \left(u_j^\varepsilon - v_j^\varepsilon\right) \mathrm{d}x = \frac{|\log \varepsilon|}{2\pi} \|\phi\|_{L^\infty} \operatorname{Exc}_\varepsilon + \mathcal{O}(\|\phi\|_{L^\infty} \|\kappa^*\|^2_{L^{1,\infty}}), \tag{43}$$

$$\int \phi \left(u_i^\varepsilon - v_i^\varepsilon\right) v_j^\varepsilon \, \mathrm{d}x \leq C |\log \varepsilon| \|\phi\|_{L^\infty} \operatorname{Exc}_\varepsilon + \mathcal{O}(\|\phi\|_{L^\infty} \|\kappa^*\|^2_{L^{1,\infty}}), \tag{44}$$

for all $i, j \in \{1, 2, 3\}$ and all $\phi \in W^{1,\infty}(\mathbb{R}^3)$.

To prove (43) and (44), we need some preparation. First, note from assumption $D(u^\varepsilon, \Gamma) \leq \varepsilon$ and estimate (22) in Proposition 1 that

$$\|\nabla \times u^\varepsilon - \nabla \times v^\varepsilon\|_F \leqq 2\varepsilon. \tag{45}$$

Next, we write v^ε in the form $v^\varepsilon = \nabla \times \Phi^\varepsilon$, for $\Phi^\varepsilon = \mathcal{G} * \rho^\varepsilon * \mu_\Gamma$, where $\mathcal{G}$ denotes the Newtonian potential $\mathcal{G}(z) = \frac{1}{4\pi |z|}$. We recall that $\Phi^\varepsilon(x) = (\mathcal{G} * \mu_\Gamma)(x)$ for all x such that $\operatorname{dist}(x, \Gamma) \geqq \varepsilon$ by the mean value property of harmonic functions. Notice also that $\int \mathrm{d}\mu_\Gamma = 0$, because Γ is a closed curve. As a consequence,

$$|\Phi^\varepsilon(x)| = \left| \int (\mathcal{G}(x - z) - \mathcal{G}(x)) \, \mathrm{d}\mu_\Gamma(z) \right| \lesssim \int \frac{|z|}{|x|^2} \, d|\mu_\Gamma| \lesssim \frac{1}{|x|^2},$$

whenever $|x|$ is sufficiently large. Now let $\chi \in C_c^\infty(B_2(0))$ be a function such that $\chi = 1$ in $B_1(0)$, and let $\chi_\lambda(x) = \chi(x/\lambda)$. Then using the above decay of Φ^ε, one easily checks that

$$\begin{aligned}\int v^\varepsilon \cdot \left(v^\varepsilon - u^\varepsilon\right) \mathrm{d}x &= \lim_{\lambda \to \infty} \int (\chi_\lambda \Phi^\varepsilon) \cdot \nabla \times (v^\varepsilon - u^\varepsilon) \, \mathrm{d}x \\ &\leq \liminf_{\lambda \to \infty} \|\nabla \times (\chi_\lambda \Phi^\varepsilon)\|_{L^\infty} \|\nabla \times v^\varepsilon - \nabla \times u^\varepsilon\|_F.\end{aligned}$$

Also, after again using the above decay of Φ^ε, we find

$$\limsup_{\lambda \to \infty} \|\nabla \times (\chi_\lambda \Phi^\varepsilon)\|_{L^\infty} = \|v^\varepsilon\|_{L^\infty}.$$

Thanks to (23) (with $q = \infty$) in Proposition 1 and (45) we conclude that

$$\left| \int v^\varepsilon \cdot \left(v^\varepsilon - u^\varepsilon\right) \mathrm{d}x \right| \lesssim \|\kappa^*\|_{L^{1,\infty}}.$$

It thus follows by (24) in Proposition 1 and the definition (14) of $\operatorname{Exc}_\varepsilon$ that

$$\begin{aligned}\int |u^\varepsilon - v^\varepsilon|^2 \,\mathrm{d}x &= \int |u^\varepsilon|^2 \,\mathrm{d}x - \int |v^\varepsilon|^2 \,\mathrm{d}x + 2\int v^\varepsilon \cdot (v^\varepsilon - u^\varepsilon)\,\mathrm{d}x \\ &= \frac{|\log \varepsilon|}{2\pi} \operatorname{Exc}_\varepsilon + \mathcal{O}\left(\|\kappa^*\|_{L^{1,\infty}}^2\right). \end{aligned} \tag{46}$$

(Recall that $\kappa^* \geqq 1$.) From this estimate, we readily deduce (43).

We turn to the proof of (44). We let $1 < p < 2 < q < \infty$ be arbitrarily fixed such that $1 = \frac{1}{p} + \frac{1}{q}$. Then by Hölder's inequality,

$$\left| \int \phi \left(u_i^\varepsilon - v_i^\varepsilon\right) v_j^\varepsilon \,\mathrm{d}x \right| \leq \|\phi\|_{L^\infty} \|u^\varepsilon - v^\varepsilon\|_{L^p} \|v^\varepsilon\|_{L^q}.$$

We invoke the interpolation inequality $\|f\|_{L^p} \lesssim \|f\|_{L^{1,\infty}}^{\frac{2-p}{p}} \|f\|_{L^2}^{\frac{2p-2}{p}}$ (the short proof of which can be found in the appendix). We also have

$$\|u^\varepsilon - v^\varepsilon\|_{L^{1,\infty}} \lesssim \|\nabla \times u^\varepsilon - \nabla \times v^\varepsilon\|_F \lesssim \varepsilon,$$

by Lemma 6 and (45). Combining this with (23) in Proposition 1, we find that

$$\left| \int \phi \left(u_i^\varepsilon - v_i^\varepsilon\right) v_j^\varepsilon \,\mathrm{d}x \right| \lesssim \|\phi\|_{L^\infty} \|u^\varepsilon - v^\varepsilon\|_{L^2}^{\frac{2p-2}{p}} \|\kappa^*\|_{L^{1,\infty}}.$$

Now we choose $p = 4/3$ and apply Young's inequality $ab \leqq a^4/4 + 3b^{4/3}/4$ together with (46) and the fact that $\kappa^* \geqq 1$ to deduce (44). This proves Theorem 1. $\square$

4.4. Proof of Theorem 2

Proof of Theorem 2. Throughout this proof, implicit constants hidden in symbols such as $\lesssim$ or $\mathcal{O}(\cdots)$ may depend on quantities appearing in assumptions (17), (18), and (19), but are independent of ε and of properties of Λ, Γ and u^ε not appearing in the assumptions.

Let us define

$$\begin{aligned} R_\gamma &:= \frac{1}{4} \inf \left\{ r_{\gamma(t,\cdot)}(s) \ : \ 0 \leqq t \leqq T, s \in \mathbb{R}/\mathbb{Z} \right\}, \\ N(\Gamma, t) &:= \left\{ x \in \mathbb{R}^3 : \operatorname{dist}(x, \Gamma(t)) < R_\gamma \right\}. \end{aligned}$$

We have assumed that $R_\gamma > 0$, see (17). For every $t \in [0, T]$, according Lemma 3, there is a (well-defined) map $\zeta_t : N(\Gamma, t) \to \mathbb{R}/\mathbb{Z}$ characterized by

$$|x - \gamma(t, \zeta_t(x))| = \operatorname{dist}(x, \Gamma(t)).$$

Recall from (2), (3) that $r_{\gamma(t)}(s) \leqq |\partial_{ss}\gamma(t,s)|^{-1}$ for all t, s. Thus the definitions entail that

$$|x - \gamma(t, \zeta_t(x))| |\partial_{ss}\gamma(t, \zeta_t(x))| \leq 1/4 \qquad \text{in } N(\Gamma, t)$$

so that $\|\nabla\zeta_t\|_{W^{2,\infty}} \lesssim 1$ thanks to assumption (17) and (31) in Lemma 3. Similar estimates hold for the temporal derivatives of ζ_t and $\nabla\zeta_t$. Indeed, differentiating the defining condition $(x - \gamma(t, \zeta_t(x))\cdot\partial_s\gamma(t, \zeta_t(x)) = 0$ with respect to t and recalling that γ is a solution to the binormal curvature flow, we compute the identity

$$\partial_t\zeta_t(x) = \frac{(x-\gamma(t,\zeta_t(x)))\cdot(\partial_s\gamma(t,\zeta_t(x))\times\partial_s^3\gamma(t,\zeta_t(x)))}{1-(x-\gamma(t,\zeta_t(x)))\cdot\partial_s^2\gamma(t,\zeta_t(x))}.$$

In view of (17), it is thus not difficult to infer $\|\partial_t\zeta_t\|_{W^{1,\infty}} \lesssim 1$.

Following [15], we now define

$$f(r^2) := \begin{cases} \left(1-(\frac{4r^2}{R_\gamma^2})\right)^3 & \text{if } r \le \frac{1}{2}R_\gamma \\ 0 & \text{if not,} \end{cases}$$

and for $x \in \mathbb{R}^3$ and $0 \leqq t \leqq T$, we define

$$X_{\gamma(t)}(x) := \begin{cases} f\left(\mathrm{dist}^2(x,\Gamma(t))\right)\,\partial_s\gamma(t,\zeta_t(x)) & \text{if } x\in N(\Gamma,t) \\ 0 & \text{if not.}\end{cases}$$

We remark that as a result of (17), the above bounds on ζ_t, and because γ is solution of the binormal curvature flow, we find that

$$\sup_{t\in[0,T]}\|X_{\gamma(t)}(\cdot)\|_{W^{3,\infty}} \lesssim 1, \qquad \sup_{t\in[0,T]}\|\nabla\times\partial_t X_{\gamma(t)}\|_{L^\infty} \lesssim 1\,. \tag{47}$$

In addition, the fact that γ is a binormal curvature flow endows X_γ with certain remarkable properties (see (26) above), established in [15], which will be recalled below.

We define

$$E_\gamma(\Lambda,t) := 1 - \int X_{\gamma(t)}\cdot \mathrm{d}\mu_{\Lambda(t)} = \int_{\Lambda(t)} (1 - X_{\gamma(t)}\cdot\tau_{\Lambda(t)})\,\mathrm{d}\mathcal{H}^1$$

and

$$E_\gamma(\mu^\varepsilon,t) := 1 - \int X_{\gamma(t)}\cdot\mathrm{d}\mu^\varepsilon(t), \quad \mu^\varepsilon := \nabla\times u^\varepsilon(t,\cdot).$$

Then for every t, by assumption (12) and (47),

$$\left|E_\gamma(\mu^\varepsilon,t) - E_\gamma(\Lambda,t)\right| = \left|\int X_{\gamma(t)}\cdot\mathrm{d}(\mu^\varepsilon(t)-\mu_{\Lambda(t)})\right| \lesssim \varepsilon. \tag{48}$$

Also, it follows from assumptions (12) and (18) that $\|\mu_{\Gamma(0)} - \mu_{\Lambda(0)}\|_F \leqq 2\varepsilon$, so via (47)

$$|E_\gamma(\Lambda,0)| = \left|\int X_{\gamma(0)}\cdot\mathrm{d}(\mu_{\Gamma(0)}-\mu_{\Lambda(0)})\right| \lesssim \varepsilon. \tag{49}$$

Moreover, (suppressing for readability the dependence on t of various quantities) it follows from (9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\gamma(\mu^\varepsilon, t) = -\int \partial_t X_\gamma \cdot \mathrm{d}\mu^\varepsilon \; - \frac{4\pi}{|\log\varepsilon|}\int \nabla(\nabla\times X_\gamma) : u^\varepsilon \otimes u^\varepsilon \,\mathrm{d}x.$$

From hypothesis (12) and (47),

$$\int \partial_t X_\gamma \cdot \mathrm{d}\mu^\varepsilon = \int \partial_t X_\gamma \cdot \mathrm{d}\mu_\Lambda + \mathcal{O}(\varepsilon).$$

Because $\mathrm{Exc}_\varepsilon(u^\varepsilon(t), \Lambda(t)) = \mathrm{Exc}_\varepsilon(u^\varepsilon(0), \Gamma(0))$, Theorem 1, assumptions (19), (18) and (47) imply that

$$\begin{aligned}\frac{4\pi}{|\log\varepsilon|}\int \nabla(\nabla\times X_\gamma) : u^\varepsilon \otimes u^\varepsilon \,\mathrm{d}x &= \int_\Lambda \nabla(\nabla\times X_\gamma) : (I - \tau_\Lambda\otimes\tau_\Lambda)\mathrm{d}\mathcal{H}^1\\ &\quad + \mathcal{O}\left(|\log\varepsilon|^{-1}\right).\end{aligned}$$

Note also that for every vector field ϕ,

$$\nabla(\nabla\times\phi) : I = \partial_i(\varepsilon_{jkl}\partial_k\phi^l)\delta_{ij} = \varepsilon_{jkl}\partial_j\partial_k\phi^l = 0.$$

Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\gamma(\mu^\varepsilon, t) &= -\int_\Lambda \left[\partial_t X_\gamma\cdot\tau_\Lambda - \nabla(\nabla\times X_\gamma) : \tau_\Lambda\otimes\tau_\Lambda\right]\,\mathrm{d}\mathcal{H}^1\\ &\quad + \mathcal{O}\left(|\log\varepsilon|^{-1}\right).\end{aligned} \tag{50}$$

However, it is proved in [15, Prop. 4] that for any unit vector ξ,

$$\left|\partial_t X_\gamma\cdot\xi - \nabla(\nabla\times X_\gamma) : \xi\otimes\xi\right| \leqq K(1 - X_\gamma\cdot\xi)$$

where K depends only on R_γ and $\sup_{0\leqq t\leq T}\|\partial_s^3\gamma(t,\cdot)\|_{L^\infty}$. This is the remarkable property mentioned above, reflecting the fact that γ is a binormal curvature flow. As a result,

$$\left|\int_\Lambda \partial_t X_\gamma\cdot\tau_\Lambda - \nabla(\nabla\times X_\gamma) : \tau_\Lambda\otimes\tau_\Lambda \,\mathrm{d}\mathcal{H}^1\right| \leqq K E_\gamma(\Lambda, t).$$

Combining this with (48), (49) and (50), we conclude that

$$\begin{aligned}E_\gamma(\Lambda, t) &\leqq \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tilde t}E_\gamma(\mu^\varepsilon, \tilde t)\,\mathrm{d}\tilde t \; + E_\gamma(\mu^\varepsilon, 0) \; + \; \mathcal{O}(\varepsilon)\\ &\leqq K\int_0^t E_\gamma(\Lambda, \tilde t)\,\mathrm{d}\tilde t + \mathcal{O}\left(|\log\varepsilon|^{-1}\right)\end{aligned}$$

for $0 < t \leqq T$. It then follows from Grönwall's inequality that

$$E_\gamma(\Lambda, t) \lesssim \frac{e^{Kt}}{|\log\varepsilon|} = \mathcal{O}\left(|\log\varepsilon|^{-1}\right) \quad \text{for } 0 \leqq t \leq T. \tag{51}$$

Finally, we show that $E_\gamma(\Lambda, t)$ controls a certain distance between $\Gamma(t)$ and $\Lambda(t)$. We will suppress the variable t, as it is not relevant here. Let $\lambda : [0, T] \times \mathbb{R}/Z \to \mathbb{R}^3$ be an arc-length parametrization of Λ having the same orientation as γ. For $s \in \mathbb{R}/\mathbb{Z}$, let $\delta_\Gamma(s) := \mathrm{dist}(\lambda(s), \Gamma)$. We will show that for all small enough ε and for every $t \in [0, T]$,

$$\sup_s \inf_\sigma |\lambda(s) - \gamma(\sigma)|^2 = \sup_s \delta_\Gamma^2(s) \lesssim E_\gamma(\Lambda) \lesssim |\log \varepsilon|^{-1} \tag{52}$$

for all sufficiently small ε. The proof of (52) is essentially contained in [14, Lemmas 4–5], but we recall the argument for the convenience of the reader.

First, it follows from the definition of f that if $x = \lambda(s)$ for any $s \in \mathbb{R}/\mathbb{Z}$, then

$$1 - X_\gamma(x) \cdot \tau_\Lambda(x) \geqq 1 - |X_\gamma(x)| = 1 - f(\delta_\Gamma^2(s)) \gtrsim \min\left\{1, \delta_\Gamma^2(s)\right\}.$$

Thus thanks to (51)

$$\int_{\mathbb{R}/\mathbb{Z}} \min\{1, \delta_\Gamma^2(s)\}\, \mathrm{d}s \lesssim E_\gamma(\Lambda) \lesssim |\log \varepsilon|^{-1}. \tag{53}$$

We now consider $s \in \mathbb{R}/\mathbb{Z}$ such that $\delta_\Gamma(s) < R_\gamma$, and hence ζ is well-defined near $x = \lambda(s)$. For such s, we will write $\sigma(s) = \zeta(\lambda(s))$, so that $\gamma(\sigma(s)) = P(\lambda(s))$. Note that if $\gamma'(\sigma(s)) \cdot \lambda'(s) \leqq 0$, then $1 - f(\delta_\Gamma^2)\gamma'(\sigma(s)) \cdot \lambda'(s) \geqq 1$, and if not, then

$$1 - f(\delta_\Gamma^2)\gamma'(\sigma(s)) \cdot \lambda'(s) \geqq 1 - \gamma'(\sigma(s)) \cdot \lambda'(s) = \frac{1}{2}|\gamma'(\sigma(s)) - \lambda'(s)|^2.$$

Either way, it follows that

$$1 - X_\gamma(x) \cdot \tau_\Lambda(x) \geqq \frac{1}{4}|\gamma'(\sigma(s)) - \lambda'(s)|^2. \tag{54}$$

Next, recalling that $(x - \gamma(\zeta(x)) \cdot \gamma'(\zeta(x)) = 0$, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\delta_\Gamma^2(s) &= \big(\lambda(s) - \gamma(\sigma(s))\big) \cdot \big(\lambda'(s) - \gamma'(\sigma(s))\sigma'(s)\big) \\ &= \big(\lambda(s) - \gamma(\sigma(s))\big) \cdot \big(\lambda'(s) - \gamma'(\sigma(s))\big). \end{aligned}$$

We now use the Cauchy–Schwarz inequality and (54) and obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\delta_\Gamma(s)\right| \lesssim \big(1 - X_\gamma(\lambda(s)) \cdot \lambda'(s)\big)^{1/2} \tag{55}$$

as long as $\delta_\Gamma(s) < R_\gamma$. If $I \subset \mathbb{R}/\mathbb{Z}$ is any interval on which $\delta_\Gamma(s) < R_\gamma$, we can integrate (55) over I and use Jensen's inequality to find that

$$\begin{aligned} (\sup_I \delta_\Gamma) - (\inf_I \delta_\Gamma) &\lesssim |I|^{1/2} \left(\int_{s \in I} \big(1 - X_\gamma(\lambda(s)) \cdot \lambda'(s)\big)\, \mathrm{d}s \right)^{1/2} \\ &\lesssim E_\gamma(\Lambda)^{1/2} \lesssim |\log \varepsilon|^{-1/2}. \end{aligned}$$

It easily follows from this and (53) that in fact $\delta_\Gamma(s) < R_\gamma$ for all s, when ε is small enough, and hence (52) holds.

Moreover, we integrate over (54) and use the definition of $E_\gamma(\Lambda, t)$ and get

$$\int_{\mathbb{R}/\mathbb{Z}} |\gamma'(\sigma(s)) - \lambda'(s)|^2 \, ds \lesssim |\log \varepsilon|^{-1}. \tag{56}$$

The two statements of the theorem, estimates (20) and (21), now follow from (52) and (56) via

$$\sup_{s\in\mathbb{R}/\mathbb{Z}} |\sigma(s) - (s + \bar\sigma)| \lesssim |\log \varepsilon|^{-1} \tag{57}$$

with $\bar\sigma = \bar\sigma(t) = \sigma(0, t)$, because $\|\gamma'\|_{L^\infty} + \|\gamma''\|_{L^\infty} < \infty$. It thus remains to prove (57). For almost everywhere $s \in \mathbb{R}/\mathbb{Z}$ we find from (31) that

$$\sigma'(s) = \nabla\zeta(\lambda(s)) \cdot \lambda'(s) = \frac{\gamma'(\sigma(s)) \cdot \lambda'(s)}{1 - (\lambda(s) - \gamma(\sigma(s)) \cdot \gamma''(\sigma(s))}.$$

Recalling that $|\gamma''(s)| \leqq r_\gamma(s) \leqq R_\gamma$, we use (52) to deduce

$$|1 - \sigma'(s)| \leqq 1 - \gamma'(\sigma(s)) \cdot \lambda'(s) + \mathcal{O}(|\log \varepsilon|^{-1}).$$

Thus $\|1 - \sigma'\|_{L^1} \lesssim E_\gamma(\Lambda) \lesssim |\log \varepsilon|^{-1}$. Using the continuous embedding of $W^{1,1}$ into L^∞ we find (57). This completes the proof of Theorem 2. □

4.5. *Proof of Proposition 1*

In this subsection, we provide the proof of Proposition 1. Notice that the statement in (22) was established in Lemma 7 in Section 4.2. The remaining estimates (23) will be proved in Lemma 9 and estimates (24) and (25) will be proved in Lemma 12.

In our computations we will occasionally encounter error terms of the form $\|g(\kappa^*)\|_{L^1(\mathbb{R}/\mathbb{Z})}$, where for example $g(t) = |\log t|^p$ for some $p \geqq 1$. These can always be absorbed into the $\|\kappa^*\|_{L^{1,\infty}}$ term, since (recalling that $\kappa^* \geqq 1$ everywhere) we have

$$\begin{aligned}\int g(\kappa^*(s)) ds &= \int_1^\infty g'(\alpha) |\{s \in \mathbb{R}/\mathbb{Z} : \kappa^*(s) \geqq \alpha\}| \, d\alpha \\ &\leqq \|\kappa^*\|_{L^{1,\infty}} \int_1^\infty \frac{g'(\alpha)}{\alpha} \, d\alpha \lesssim \|\kappa^*\|_{L^{1,\infty}}\end{aligned} \tag{58}$$

by the virtue of the coarea formula [10, Ch. 3.4].

We now start to establish pointwise estimates of v^ε. We begin with rather crude estimates that are valid everywhere; these will be sufficient for L^q estimates of v^ε, for $q > 2$. For $q = 2$, we will later prove sharper estimates in the tube $\mathcal{T}$.

Lemma 8. *For every* $x \in \mathbb{R}^3$,

$$|v^\varepsilon(x)| \lesssim \min\left\{ \frac{1}{\operatorname{dist}(x, \Gamma)^2}, \frac{\|\kappa^*\|_{L^{1,\infty}}}{\operatorname{dist}(x, \Gamma)}, \frac{1}{\varepsilon}\|\kappa^*\|_{L^{1,\infty}} \right\}.$$

Proof. Notice first that the v^ε can be written as

$$v^\varepsilon(x) = \int_\Gamma \mathcal{K}^\varepsilon(x-z) \times \tau_\Gamma(z)\,\mathrm{d}\mathcal{H}^1,$$

where $\mathcal{K}^\varepsilon = \rho^\varepsilon * \mathcal{K}$, and $\mathcal{K}(z) = -\frac{z}{4\pi|z|^3}$ for $z \in \mathbb{R}^3\backslash\{0\}$ is the gradient of the Newtonian potential in $\mathbb{R}^3$. The mean value property for harmonic functions implies that $\mathcal{K}^\varepsilon(x) = \mathcal{K}(x)$ if $|x| > \varepsilon$. If $|x| \leqq \varepsilon$, then

$$|\left(\rho^\varepsilon * \mathcal{K}\right)(x)| \lesssim \int \frac{\rho^\varepsilon(x-y)}{|y|^2}\,\mathrm{d}y \lesssim \frac{1}{\varepsilon^3}\int_{B_{2\varepsilon}(0)} \frac{1}{|y|^2}\,\mathrm{d}y \lesssim \frac{1}{\varepsilon^2}.$$

In particular,

$$|\mathcal{K}^\varepsilon(x)| \lesssim \min\left\{|x|^{-2}, \varepsilon^{-2}\right\}.$$

It easily follows $|v^\varepsilon(x)| \lesssim \operatorname{dist}(x,\Gamma)^{-2}$ for every x.

We fix x and write $\delta := \operatorname{dist}(x,\Gamma)$. We first assume that $\delta \geqq \varepsilon$. Then by Lemma 4, because $|\mathcal{K}^\varepsilon(x)| \lesssim |x|^{-2}$,

$$\begin{aligned}|v^\varepsilon(x)| &\lesssim \sum_{j=0}^{\infty} \int_{\{s:2^j\delta\leq|\gamma(s)-x|<2^{j+1}\delta\}} |x-\gamma(s)|^{-2}\,\mathrm{d}s \\ &\lesssim \|\kappa^*\|_{L^{1,\infty}} \sum_{j=0}^{\infty}(2^j\delta)^{-1} \lesssim \frac{1}{\delta}\|\kappa^*\|_{L^{1,\infty}}.\end{aligned}$$

Hence $|v^\varepsilon(x)| \lesssim \delta^{-1}\|\kappa^*\|_{L^{1,\infty}}$ if $\delta \geqq \varepsilon$. Otherwise, if $\delta < \varepsilon$ then we similarly appeal to Lemma 4 to find that

$$\begin{aligned}|v^\varepsilon(x)| &\lesssim \int_{\{s:|\gamma(s)-x|<\varepsilon\}} \varepsilon^{-2}\,\mathrm{d}s + \sum_{j=0}^{\infty}\int_{\{s:2^j\varepsilon\leqq|\gamma(s)-x|<2^{j+1}\varepsilon\}} |x-\gamma(s)|^{-2}\,\mathrm{d}s \\ &\lesssim \frac{1}{\varepsilon}\|\kappa^*\|_{L^{1,\infty}}.\end{aligned}$$

This proves the lemma. □

We can now establish L^q estimates of v^ε for $q > 2$.

Lemma 9. *Estimates* (23) *hold.*

Proof. Inequality (23) in the case $q = \infty$ is already contained in the previous lemma. We thus focus on $q < \infty$.

We set $N_r(\Gamma) := \{x \in \mathbb{R}^3 : \operatorname{dist}(x,\Gamma) < r\}$ and note that for every $r > 0$,

$$|N_r(\Gamma)| \lesssim r^2 + r^3. \tag{59}$$

Indeed, if we let $M = \lfloor 1/r \rfloor$, then

$$N_r(\Gamma) \subset \bigcup_{k=0}^{M} B_{2r}(p_k) \quad \text{where} \quad p_k := \gamma(kr).$$

Thus $|N_r(\Gamma)| \lesssim (M+1)r^3 \lesssim (1/r+1)r^3$, proving (59).

We now write

$$H^j(\Gamma) := N_{2^j\varepsilon}(\Gamma)\setminus N_{2^{j-1}\varepsilon}(\Gamma) = \left\{x \in \mathbb{R}^3 : 2^{j-1}\varepsilon \leqq \operatorname{dist}(x,\Gamma) < 2^j\varepsilon\right\}$$

for $j \geqq 1$ and set $H^0(\Gamma) = N_\varepsilon(\Gamma)$ for notational convenience. Then clearly

$$\|v^\varepsilon\|_{L^q}^q = \sum_{j=0}^{\infty} \int_{H^j(\Gamma)} |v^\varepsilon|^q \,\mathrm{d}x.$$

We now fix J such that $2^J\varepsilon \leqq 1 \leqq 2^{J+1}\varepsilon$, and we use Lemma 8 to estimate

$$|v^\varepsilon| \lesssim \min\left\{(2^j\varepsilon)^{-2}, (2^j\varepsilon)^{-1}\|\kappa^*\|_{L^{1,\infty}}\right\} \quad \text{in } H^j(\Gamma).$$

Moreover, with the help of (59) we obtain

$$|H^j(\Gamma)| \leqq |N_{2^j\varepsilon}(\Gamma)| \lesssim \begin{cases} (2^j\varepsilon)^2 & \text{if} \quad 0 \leqq j \leqq J, \\ (2^j\varepsilon)^3 & \text{if} \quad j > J. \end{cases}$$

Thus

$$\int_{H^j(\Gamma)} |v^\varepsilon|^q \mathrm{d}x \;\lesssim \begin{cases} \varepsilon^{2-q}\|\kappa^*\|_{L^{1,\infty}}^q 2^{-j(q-2)} & \text{if} \quad 0 \leqq j \leqq J, \\ \varepsilon^{3-2q} 2^{-j(2q-3)} & \text{if} \quad j > J. \end{cases}$$

We thus obtain (23) by summing over j. □

We require sharper estimates for the L^2 norms of v^ε and associated quantities, and for these we establish a more precise description of v^ε in the tube $\mathcal{T}$, defined earlier in Section 4.1.

Lemma 10. *If $x \in \mathcal{T}$ and dist$(x,\Gamma) \geqq \varepsilon$ then*

$$\left| v^\varepsilon(x) - \frac{1}{2\pi}\frac{(\gamma(\zeta(x)) - x) \times \gamma'(\zeta(x))}{\operatorname{dist}(x,\Gamma)^2} \right| \lesssim \frac{1}{r(\zeta(x))} |\log \operatorname{dist}(x,\Gamma)| + \frac{\|\kappa^*\|_{L^{1,\infty}}}{r(\zeta(x))}.$$

Proof. Fix $x \in \mathcal{T}$ with $\operatorname{dist}(x,\Gamma) > \varepsilon$. We use the same notation as in the proof of Lemma 8, and recall that $\mathcal{K}^\varepsilon(x) = \mathcal{K}(x)$ for $|x| \geqq \varepsilon$. In particular,

$$v^\varepsilon(x) = \frac{1}{4\pi}\int_{-1/2}^{1/2} \frac{\gamma(s) - x}{|\gamma(s) - x|^3} \times \gamma'(s)\,\mathrm{d}s.$$

For notational convenience, we assume in the following discussion that $\zeta(x) = 0$ and we set $\delta := \operatorname{dist}(x,\Gamma)$ and $r_0 := r(0) = r(\zeta(x))$. Then defining $\gamma_0(s) = \gamma(0) + s\gamma'(0)$, we see that

$$\begin{aligned} 4\pi v^\varepsilon(x) = &\int_{-r_0}^{r_0} \frac{\gamma_0(s) - x}{|\gamma_0(s) - x|^3} \times \gamma_0'(s)\,\mathrm{d}s + \int_{-r_0}^{r_0} F(s)\,\mathrm{d}s \\ &+ \int_{r_0<|s|<1/2} \frac{\gamma(s) - x}{|\gamma(s) - x|^3} \times \gamma'(s)\,\mathrm{d}s, \end{aligned} \tag{60}$$

where

$$F(s) := f(\gamma(s), \gamma'(s)) - f(\gamma_0(s), \gamma_0'(s)), \qquad \text{for } f(p, \tau) := \frac{p - x}{|p - x|^3} \times \tau.$$

The last integral in (60) can be estimated by exactly the arguments in the proof of Lemma 8, leading to

$$\left| \int_{r_0 < |s| < 1/2} \frac{\gamma(s) - x}{|\gamma(s) - x|^3} \times \gamma'(s)\, \mathrm{d}s \right| \lesssim \frac{1}{r_0} \|\kappa^*\|_{L^{1,\infty}}.$$

To evaluate the first integral in (60), we observe that since $(\gamma(0) - x) \cdot \gamma'(0) = 0$,

$$\int_{-r_0}^{r_0} \frac{\gamma_0(s) - x}{|\gamma_0(s) - x|^3} \times \gamma_0'(s)\, \mathrm{d}s = (\gamma(0) - x) \times \gamma'(0) \int_{-r_0}^{r_0} \frac{1}{(\delta^2 + s^2)^{3/2}} \mathrm{d}s$$

with

$$\int_{-r_0}^{r_0} \frac{1}{(\delta^2 + s^2)^{3/2}} \mathrm{d}s = \frac{2}{\delta^2} \left(1 + \left(\frac{\delta}{r_0} \right)^2 \right)^{-1/2} = \frac{2}{\delta^2} + \mathcal{O}\left(r_0^{-2} \right)$$

for $\delta \leqq r_0$. So it only remains to estimate the second integral in (60). By the mean value theorem (of calculus), we may write the integrand as

$$F(s) = \nabla_{p,\tau} f(\gamma_\lambda(s), \gamma_\lambda'(s)) \cdot (\gamma(s) - \gamma_0(s), \gamma'(s) - \gamma_0'(s))$$

where $\gamma_\lambda(s) = \lambda\gamma(s) + (1 - \lambda)\gamma_0(s)$ for some λ between 0 and 1. Straightforward calculations then imply that

$$|F(s)| \lesssim \frac{|\gamma(s) - \gamma_0(s)|}{|\gamma_\lambda(s) - x|^3} + \frac{|\gamma'(s) - \gamma_0'(s)|}{|\gamma_\lambda(s) - x|^2}.$$

Since $\gamma_0'(s) = \gamma'(0)$ for all s, it follows from the definition of r_0 that

$$|\gamma'(s) - \gamma_0'(s)| \leqq \frac{|s|}{r_0}$$

for all $|s| \leqq r_0$, and hence that

$$|\gamma(s) - \gamma_0(s)| \leqq \frac{s^2}{2r_0}, \qquad |\gamma_\lambda(s) - x| \geqq \frac{1}{2}(\delta^2 + s^2)^{1/2}. \tag{61}$$

We may therefore complete the proof by estimating the integral as

$$\left| \int_{-r_0}^{r_0} F(s)\, \mathrm{d}s \right| \lesssim \frac{1}{r_0} \left(\log\left(\frac{r_0}{\delta} \right) + 1 \right),$$

because then

$$\left| v^\varepsilon(x) - \frac{1}{2\pi} \frac{(\gamma(\zeta(x)) - x) \times \gamma'(\zeta(x))}{\operatorname{dist}(x, \Gamma)^2} \right| \lesssim \frac{1}{r_0} \left(\log\left(\frac{r_0}{\delta} \right) + 1 \right) + \frac{\delta}{r_0^2} + \frac{\|\kappa^*\|_{L^{1,\infty}}}{r_0}.$$

Since $\delta \leqq r_0$ and $\|\kappa^*\|_{L^{1,\infty}} \geqq 1$, the statement follows. $\square$

Lemma 11. *For every* $\varepsilon \in (0, \frac{1}{2})$

$$\|v^\varepsilon\|_{L^2}^2 \leqq \frac{1}{2\pi}|\log\varepsilon| + \mathcal{O}(\|\kappa^*\|_{L^{1,\infty}}^2).$$

Proof. Let $\mathcal{G}$ denote the Newtonian potential $\mathcal{G}(z) = \frac{1}{4\pi|z|}$, and define $\Phi := \mathcal{G}*\mu_\Gamma$, so that $-\Delta\Phi = \mu_\Gamma$. Then we can write $v^\varepsilon = \rho^\varepsilon * \nabla\times\Phi = \nabla\times\rho^\varepsilon*\Phi$. It is easy to see, by arguing as in the proof of Lemma 8, that $|\Phi(x)| \lesssim |x|^{-1}$ for $|x|$ large, and together with the conclusions of Lemma 8, this gives sufficient decay to justify integrating by parts as follows:

$$\begin{aligned}\int |v^\varepsilon|^2 \mathrm{d}x &= \int \nabla\times(\rho^\varepsilon*\Phi)\cdot v^\varepsilon\,\mathrm{d}x = \int \rho^\varepsilon*\Phi\cdot\nabla\times v^\varepsilon\,\mathrm{d}x\\ &= \int \rho^\varepsilon*\Phi\cdot\rho^\varepsilon*\mu_\Gamma\,\mathrm{d}x = \int \rho^\varepsilon*\rho^\varepsilon*\Phi\cdot\mathrm{d}\mu_\Gamma.\end{aligned}$$

In the last identity, we have used the radial symmetry of ρ^ε. Setting $\eta^\varepsilon := \rho^\varepsilon*\rho^\varepsilon$, it follows that

$$\int |v^\varepsilon|^2\,\mathrm{d}x \leqq \int_{\mathbb{R}/\mathbb{Z}} \left|\eta^\varepsilon*\Phi(\gamma(s))\right|\,\mathrm{d}s = \int_{\mathbb{R}/\mathbb{Z}} \left|(\eta^\varepsilon*\mathcal{G}*\mu_\Gamma)(\gamma(s))\right|\,\mathrm{d}s. \tag{62}$$

Below we will repeatedly use the facts that

$$\eta^\varepsilon*\mathcal{G}(z) = \mathcal{G}(z) \quad \text{for every } |x| > 2\varepsilon, \quad \eta^\varepsilon*\mathcal{G} \lesssim \frac{1}{\varepsilon} \quad \text{everywhere.} \tag{63}$$

The first of these follows from the mean value property for harmonic functions, and the second is easy to verify.

Now consider an arbitrary point in $\mathbb{R}/\mathbb{Z}$, which we take for convenience to be $s = 0$, and let $x := \gamma(0)$. Then

$$\left|(\eta^\varepsilon*\mathcal{G}*\mu_\Gamma)(x)\right| \leq \int_{-1/2}^{1/2} (\eta^\varepsilon*\mathcal{G})(x-\gamma(s))\,\mathrm{d}s.$$

Let $r_0 := r(0)$. If $r_0 < 4\varepsilon$, then we use (63) and Lemma 4 to compute

$$\begin{aligned}&\left|(\eta^\varepsilon*\mathcal{G}*\mu_\Gamma)(x)\right|\\ &\quad\lesssim \int_{\{s:|\gamma(s)-x|\leqq 2\varepsilon\}} \frac{1}{\varepsilon}\,\mathrm{d}s + \sum_{j=1}^{J}\int_{\{s:2^j\varepsilon\leqq|\gamma(s)-x|\leqq 2^{j+1}\varepsilon\}} \frac{1}{|\gamma(s)-x|}\,\mathrm{d}s\\ &\quad\lesssim \|\kappa^*\|_{L^{1,\infty}}|\log\varepsilon|,\end{aligned} \tag{64}$$

where $J \lesssim |\log\varepsilon|$ because $|\gamma(s)-x| \leq 1$ for all s. For $r_0 \geqq 4\varepsilon$ we proceed very much as in the proof of Lemma 10, writing

$$\begin{aligned}\int_{-1/2}^{1/2}(\eta^\varepsilon*\mathcal{G})(x-\gamma(s))\,\mathrm{d}s &= \int_{-r_0}^{r_0}(\eta^\varepsilon*\mathcal{G})(s\gamma'(0))\,\mathrm{d}s \;+\; \int_{-r_0}^{r_0} F(s)\,\mathrm{d}s\\ &\quad+\int_{r_0<|s|\leqq\frac{1}{2}}(\eta^\varepsilon*\mathcal{G})(x-\gamma(s))\,\mathrm{d}s\end{aligned}$$

where

$$F(s) = (\eta^\varepsilon * \mathcal{G})(x - \gamma(s)) - (\eta^\varepsilon * \mathcal{G})(x - \gamma_0(s)), \qquad \gamma_0(s) = x + s\gamma'(0).$$

Arguing as in the proof of (64) above, it follows from (63) and Lemma 4 that

$$\int_{r_0 < |s| \leqq \frac{1}{2}} (\eta^\varepsilon * \mathcal{G})(x - \gamma(s))\, \mathrm{d}s \;\lesssim\; |\log r_0| \|\kappa^*\|_{L^{1,\infty}}.$$

Next, again appealing to (63), it is straightforward to check that

$$\int_{-r_0}^{r_0} (\eta^\varepsilon * \mathcal{G})(s\gamma'(0))\, \mathrm{d}s \;=\; \frac{|\log(r_0/\varepsilon)|}{2\pi} + \mathcal{O}(1) = \frac{|\log\varepsilon|}{2\pi} + \mathcal{O}(|\log r_0|)$$

where we have used the fact that $r(s) \leqq \frac{1}{2}$ for all s to simplify the error terms.

Notice that in view of the second estimate in (61) (applied both with $\lambda = 0$ and $\lambda = 1$), we have $F(s) = \mathcal{G}(x - \gamma(s)) - \mathcal{G}(x - \gamma_0(s))$ for $4\varepsilon \leqq |s| \leqq r_0$. Hence, there exists some $\gamma_\lambda(s)$, a convex combination of $\gamma_0(s)$ and $\gamma(s)$, such that

$$F(s) \;=\; \nabla\mathcal{G}(x - \gamma_\lambda(s)) \cdot (\gamma(s) - \gamma_0(s)) \;\lesssim\; \frac{|\gamma(s) - \gamma_0(s)|}{|x - \gamma_\lambda(s)|^2}.$$

Again using (61), we find that $F(s) \lesssim \frac{1}{r_0}$ if $4\varepsilon \leqq |s| \leqq r_0$. Since $F(s) \lesssim \frac{1}{\varepsilon}$ trivially by (63) for all s, we thus obtain

$$\int_{-r_0}^{r_0} F(s)\, \mathrm{d}s \lesssim 1.$$

Combining these, we find that if $r_0 \geqq 4\varepsilon$, then

$$|(\eta^\varepsilon * \mathcal{G} * \mu_\Gamma)(\gamma(s))| \leq \frac{|\log\varepsilon|}{2\pi} + \mathcal{O}\left(|\log r(s)| \|\kappa^*\|_{L^{1,\infty}}\right).$$

Recalling (62) and (64), we can now integrate and recall (58) to find that

$$\begin{aligned}\int |v^\varepsilon|^2\, \mathrm{d}x &\leqq \frac{|\log\varepsilon|}{2\pi} + C\int |\log\kappa^*(s)|\, \mathrm{d}s \|\kappa^*\|_{L^{1,\infty}} \\ &\quad + C\|\kappa^*\|_{L^{1,\infty}} |\log\varepsilon| \left| \left\{ s \in \mathbb{R}/\mathbb{Z} : \kappa^*(s) > \frac{1}{4\varepsilon} \right\} \right| \\ &\leqq \frac{|\log\varepsilon|}{2\pi} + C\|\kappa^*\|^2_{L^{1,\infty}} + C\varepsilon|\log\varepsilon| \|\kappa^*\|^2_{L^{1,\infty}}.\end{aligned}$$

The statement follows because $\varepsilon|\log\varepsilon| \leqq 1$. □

The following Lemma completes the proof of Proposition 1.

Lemma 12. *Estimates* (24) *and* (25) *hold.*

Proof. We first claim that it suffices to show that

$$\frac{4\pi}{|\log\varepsilon|}\int_{\mathcal{T}}\phi : v^\varepsilon\otimes v^\varepsilon\,\mathrm{d}x=\int_\Gamma\phi:(I-\tau\otimes\tau)\,\mathrm{d}\mathcal{H}^1+\mathcal{O}\left(\frac{\|\kappa^*\|^2_{L^{1,\infty}}\|\phi\|_{W^{1,\infty}}}{|\log\varepsilon|}\right). \tag{65}$$

Indeed, if this holds, then we may take $\phi=I$ in (65) to find that

$$\int_{\mathcal{T}}|v^\varepsilon|^2\,\mathrm{d}x=\frac{|\log\varepsilon|}{2\pi}+\mathcal{O}\left(\|\kappa^*\|^2_{L^{1,\infty}}\right).$$

This, together with Lemma 11, implies that

$$\int_{\mathbb{R}^3\setminus\mathcal{T}}|v^\varepsilon|^2\,\mathrm{d}x=\mathcal{O}\left(\|\kappa^*\|^2_{L^{1,\infty}}\right),$$

and from this we see that (65) implies (25). Similarly, combining the previous two estimates, we directly obtain (24).

To prove (65), we first use the coarea formula to rewrite the integral on the left-hand side as

$$\int_{\mathcal{T}}\phi:v^\varepsilon\otimes v^\varepsilon\,\mathrm{d}x=\int_{\mathbb{R}/\mathbb{Z}}\left(\int_{\zeta^{-1}(s)}\phi:v^\varepsilon\otimes v^\varepsilon\,|\nabla\zeta|^{-1}\mathrm{d}\mathcal{H}^2\right)\mathrm{d}s. \tag{66}$$

We now consider some $s\in\mathbb{R}/\mathbb{Z}$. It is convenient to choose coordinates so that $\gamma(s)=0$ and $\gamma'(s)=(0,0,1)$. We will also write $r=(x_1^2+x_2^2)^{1/2}$, and we remark that $r=\operatorname{dist}(x,\Gamma)$ in $\zeta^{-1}(s)$. In these coordinates,

$$\zeta^{-1}(s)=\left\{x:x_3=0,r<\frac{1}{4}r(s)\right\},$$

and for x in this set, according to Lemma 8

$$|v^\varepsilon(x)|\lesssim\|\kappa^*\|_{L^{1,\infty}}\min\left\{\frac{1}{\varepsilon},\frac{1}{r}\right\}.$$

Moreover, it follows from Lemma 3 and Fubini's Theorem that for almost everywhere s,

$$\left||\nabla\zeta(x)|^{-1}-1\right|\leqq\frac{r}{r(s)}=r\kappa^*(s)\quad\text{for almost everywhere}\quad x\in\zeta^{-1}(s).$$

We henceforth restrict our attention to s for which this holds. We therefore have $|v^\varepsilon|\lesssim\varepsilon^{-1}\|\kappa^*\|_{L^{1,\infty}}$ if $r\leqq\varepsilon$, and otherwise

$$|\phi(x):v^\varepsilon\otimes v^\varepsilon\,|\nabla\zeta|^{-1}-\phi(0):v^\varepsilon\otimes v^\varepsilon|\lesssim\frac{1}{r}\|\kappa^*\|^2_{L^{1,\infty}}\left(\frac{\|\phi\|_{L^\infty}}{r(s)}+\|\nabla\phi\|_{L^\infty}\right).$$

It follows that

$$\int_{\zeta^{-1}(s)} \phi : v^\varepsilon \otimes v^\varepsilon \, |\nabla\zeta|^{-1} \mathrm{d}\mathcal{H}^2 = \int_{\zeta^{-1}(s)\setminus B_\varepsilon} \phi(0) : v^\varepsilon \otimes v^\varepsilon \, \mathrm{d}\mathcal{H}^2 + \mathcal{O}\left(\|\kappa^*\|^2_{L^{1,\infty}}(\|\phi\|_{L^\infty} + r(s)\|\nabla\phi\|_{L^\infty})\right). \tag{67}$$

Next, the estimates in Lemma 10 imply that for $v_*(x) := \frac{1}{2\pi}\frac{(-x_2,x_1,0)}{r^2}$, we have

$$\left|v^\varepsilon \otimes v^\varepsilon - v_* \otimes v_*\right| \lesssim \left(\frac{|\log r|}{r(s)} + \frac{\|\kappa^*\|_{L^{1,\infty}}}{r(s)}\right)\frac{\|\kappa^*\|_{L^{1,\infty}}}{r}$$

for $\varepsilon < r < r(s)$. So we integrate to find that

$$\int_{\zeta^{-1}(s)\setminus B_\varepsilon} v^\varepsilon \otimes v^\varepsilon \, \mathrm{d}\mathcal{H}^\in = \int_{\zeta^{-1}(s)\setminus B_\varepsilon} v_* \otimes v_* \, \mathrm{d}\mathcal{H}^2 + \mathcal{O}\left(\|\kappa^*\|_{L^{1,\infty}}|\log\kappa^*(s)| + \|\kappa^*\|^2_{L^{1,\infty}}\right). \tag{68}$$

For example, one of the two error terms is estimated by

$$\frac{\|\kappa^*\|_{L^{1,\infty}}}{r(s)}\int_{\zeta^{-1}(s)\setminus B_\varepsilon}\frac{|\log r|}{r}\mathrm{d}\mathcal{H}^2 \sim \frac{\|\kappa^*\|_{L^{1,\infty}}}{r(s)}\int_\varepsilon^{r(s)}|\log r|\,\mathrm{d}r \lesssim \|\kappa^*\|_{L^{1,\infty}}|\log(r(s))| = \|\kappa^*\|_{L^{1,\infty}}|\log\kappa^*(s)|.$$

The other terms is similar. Moving on, it is easy to check that

$$\int_{\zeta^{-1}(s)\setminus B_\varepsilon} v_* \otimes v_* \, \mathrm{d}\mathcal{H}^2 = \frac{1}{4\pi}\log\left(\frac{r(s)}{\varepsilon}\right)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{69}$$

Indeed, it is clear that any term involving the 3rd component of v_* must vanish. Among the remaining terms, symmetry considerations imply that the off-diagonal terms vanish and that the diagonal terms are equal. Since their sum is

$$\int_{\zeta^{-1}(s)\setminus B_\varepsilon}|v_*|^2\mathrm{d}\mathcal{H}^2 = \frac{1}{2\pi}\int_\varepsilon^{r(s)}\frac{1}{r}\,\mathrm{d}r = \frac{1}{2\pi}\log\left(\frac{r(s)}{\varepsilon}\right),$$

the claim (69) follows.

Now by combining (67), (68) and (69), and recalling that $r(s) = \frac{1}{\kappa^*(s)} \leqq \frac{1}{2}$ for all s, we find that

$$\left|\int_{\zeta^{-1}(s)} \phi : v^\varepsilon \otimes v^\varepsilon \, |\nabla\zeta|^{-1}\mathrm{d}\mathcal{H}^2 - \frac{|\log\varepsilon|}{4\pi}\phi(\gamma(s)) : (I - \gamma'(s)\otimes\gamma'(s))\right| \lesssim \left(|\log\kappa^*(s)|\|\kappa^*\|_{L^{1,\infty}} + \|\kappa^*\|^2_{L^{1,\infty}}\right)\|\phi\|_{W^{1,\infty}}.$$

We deduce (65), and hence complete the proof of the lemma, by substituting this into (66), integrating and using (58) to simplify some of the error terms. □

Appendix

In this appendix, we provide the proof of the interpolation inequality

$$\|f\|_{L^p} \lesssim \|f\|_{L^{1,\infty}}^{\frac{2-p}{p}} \|f\|_{L^2}^{\frac{2p-2}{p}}.$$

Recall that $\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha)\,\mathrm{d}\alpha$, where $d_f(\alpha) = \left|\{x \in \mathbb{R}^N : |f(x)| > \alpha\}\right|$. Then, letting $\ell > 0$ be arbitrary, we write

$$\int |f|^p\,\mathrm{d}x = p\int_0^\ell \alpha^{p-1} d_f(\alpha)\,\mathrm{d}\alpha + p\int_\ell^\infty \alpha^{p-1} d_f(\alpha)\,\mathrm{d}\alpha.$$

Clearly

$$\int_0^\ell \alpha^{p-1} d_f(\alpha)\,\mathrm{d}\alpha \leqq \ell^{p-1}\|f\|_{L^{1,\infty}}, \quad \text{and} \quad \int_\ell^\infty \alpha^{p-1} d_f(\alpha)\,\mathrm{d}\alpha \leqq \frac{\ell^{p-2}}{2}\|f\|_{L^2}^2,$$

where we have used that $1 < p < 2$. Hence,

$$\|f\|_{L^p}^p \lesssim \ell^{p-1}\|f\|_{L^{1,\infty}} + \ell^{p-2}\|f\|_{L^2}^2.$$

Optimizing in ℓ yields the desired result.

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R. L. Jerrard
Department of Mathematics,
University of Toronto,
Toronto ON M5S 2E4, Canada

and

C. Seis
Institut für Angewandte Mathematik
Universität Bonn,
53115 Bonn, Germany
e-mail: seis@iam.uni-bonn.de

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