Arch. Rational Mech. Anal. 223 (2017) 817–843
Digital Object Identifier (DOI) 10.1007/s00205-016-1046-3

Global Axisymmetric Solutions of Three Dimensional Inhomogeneous Incompressible Navier–Stokes System with Nonzero Swirl

Hui Chen, Daoyuan Fang & Ting Zhang

Communicated by F. Lin

Abstract

In this paper, we investigate the global well-posedness for the three dimensional inhomogeneous incompressible Navier–Stokes system with axisymmetric initial data. We obtain the global existence and uniqueness of the axisymmetric solution provided that

$$\left\|\frac{a_0}{r}\right\|_\infty \text{ and } \|u_0^\theta\|_3 \text{ are sufficiently small.}$$

Furthermore, if $\mathbf{u}_0 \in L^1$ and $ru_0^\theta \in L^1 \cap L^2$, we have the decay estimate

$$\|u^\theta(t)\|_2^2 + \langle t\rangle\|\nabla(u^\theta \mathbf{e}_\theta)(t)\|_2^2 + t\langle t\rangle(\|u_t^\theta(t)\|_2^2 + \|\Delta(u^\theta \mathbf{e}_\theta)(t)\|_2^2) \leqq C\langle t\rangle^{-\frac{5}{2}}, \quad \forall\, t > 0.$$

1. Introduction

In this paper, we consider the initial value problem of three dimensional inhomogeneous incompressible Navier–Stokes equations with the axisymmetric initial data:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \Delta \mathbf{u} + \nabla \Pi = \mathbf{0}, \\ \operatorname{div} \mathbf{u} = 0, \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u_0}). \end{cases} \tag{1.1}$$

where ρ, $\mathbf{u} = (u^1, u^2, u^3)$ and Π stand for the density, the velocity of the fluid and the pressure, respectively.

The global weak solution to the above system was constructed by Simon [17] (see also Lions [13]). However, the problem of uniqueness has not been solved. The regularity of such a weak solution in three dimensions has becomes one of the open problems in the mathematical fluid mechanics.

In the case of the smooth initial data without vacuum, LADYZENSKAJA and SOLONNIKOV [10] addressed the question of the unique solvability of the initial-boundary value problem for the system (1.1) in the bounded domain, and DANCHIN [5,7] established the well-posedness of system (1.1) in the whole space $\mathbb{R}^d$. While there has been some recent progress along these line, we recall that, unless the initial data have some special structures, it is still not known if system (1.1) has a unique global smooth solution with the large smooth initial data, even for the classical Navier–Stokes system, which corresponds to $\rho = 1$ in (1.1).

The global well-posedness result for the classical axisymmetric Navier–Stokes system was first proved under a no swirl assumption, independently, by UKHOVSKII and YUDOVICH [19], LADYZHENSKAYA [9]; see, also [12] for a refined proof. In a the previous work [4], we established the global well-posedness result for the classical axisymmetric Navier–Stokes system provided that the initial swirl component u_0^θ is sufficient small, that is,

$$\|u_0^\theta\|_3 \leqq \frac{1}{C}\exp\left\{-C\|\mathbf{u_0}\|_2^2\left(\|\omega_0^\theta\|_2^2 + \left(\|\frac{\omega_0^\theta}{r}\|_2 + \|\partial_3\frac{u_0^\theta}{r}\|_2\right)^{\frac{4}{3}}\|\mathbf{u_0}\|_2^2\right)\right\}, \tag{1.2}$$

where the right hand side of the above inequality is scaling invariant. In [4], we also proved the global regularity of the solution under the assumption that

$$r^d u^\theta \in L_T^p(L_x^q),\ \text{where}\ \frac{2}{p}+\frac{3}{q}\leqq 1-d,\ \frac{3}{1-d}<q \leqq \infty,\ \frac{2}{1-d}\leqq p\leqq\infty,\ 0\leqq d<1.$$

Specifically, we proved that if there exist $\alpha > 0$ and $C > 0$ such that

$$r|u^\theta| \leqq Cr^\alpha,\ \textit{almost everywhere}\ (t,x)\in(0,T)\times\mathbb{R}^3, \tag{1.3}$$

then $\mathbf{u}$ is regular. Recently, LEI and ZHANG [11] and WEI [20] gave some technical improvements, especially the blow-up criteria of the solutions. They improved the regularity criteria (1.3) to

$$\sup_{0\leqq t<T} r|u^\theta| \leqq C|\ln r|^{-\beta}, \quad r\leqq\delta_0, \tag{1.4}$$

with $\beta = 2$ in [11] and $\beta = \frac{3}{2}$ in [20]. These works imply that the three dimensional classical axisymmetric Navier–Stokes system is in fact "critical".

For the inhomogeneous Navier–Stokes equations (1.1), ABIDI and ZHANG [2] obtained the global smooth axisymmetric solution without swirl when $\|\frac{a_0}{r}\|_\infty$ is sufficiently small, where $a_0 = \frac{1}{\rho_0} - 1$. Inspired by [2,4], we assume that the solution of (1.1) is axisymmetric, that is,

$$\rho(t,x) = \rho(t,r,x_3),\ \Pi(t,x) = \Pi(t,r,x_3),$$
$$\mathbf{u}(t,x) = u^r(t,r,x_3)\mathbf{e}_r + u^\theta(t,r,x_3)\mathbf{e}_\theta + u^3(t,r,x_3)\mathbf{e}_3,$$

where

$$\mathbf{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad \mathbf{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad \mathbf{e}_3 = (0,0,1), \quad r = \sqrt{x_1^2 + x_2^2}.$$

Then, from (1.1), we have

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \\ \rho \partial_t u^r + \rho \mathbf{u} \cdot \nabla u^r - (\Delta - \frac{1}{r^2}) u^r - \rho \frac{(u^\theta)^2}{r} + \partial_r \Pi = 0, \\ \rho \partial_t u^\theta + \rho \mathbf{u} \cdot \nabla u^\theta - (\Delta - \frac{1}{r^2}) u^\theta + \rho \frac{u^\theta u^r}{r} = 0, \\ \rho \partial_t u^3 + \rho \mathbf{u} \cdot \nabla u^3 - \Delta u^3 + \partial_3 \Pi = 0, \\ \partial_r u^r + \frac{1}{r} u^r + \partial_3 u^3 = 0, \\ (u^r, u^\theta, u^3)|_{t=0} = (u_0^r, u_0^\theta, u_0^3). \end{cases} \tag{1.5}$$

We can also compute the vorticity $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$ as follows:

$$\boldsymbol{\omega} = \omega^r \mathbf{e}_r + \omega^\theta \mathbf{e}_\theta + \omega^3 \mathbf{e}_3, \tag{1.6}$$

with

$$\omega^r = -\partial_3 u^\theta, \ \omega^\theta = \partial_3 u^r - \partial_r u_3, \ \omega^3 = \partial_r u^\theta + \frac{u^\theta}{r}. \tag{1.7}$$

Furthermore, we can deduce the equations of vorticity

$$\begin{cases} \partial_t \omega^r + \mathbf{u} \cdot \nabla \omega^r + \partial_3 \left(\frac{1}{\rho}\left(\Delta - \frac{1}{r^2}\right) u^\theta\right) - (\omega^r \partial_r + \omega^3 \partial_3) u^r = 0, \\ \partial_t \omega^\theta + \mathbf{u} \cdot \nabla \omega^\theta - \partial_3 \left(\frac{1}{\rho}\left(\left(\Delta - \frac{1}{r^2}\right) u^r - \partial_r \Pi\right)\right) + \partial_r \left(\frac{1}{\rho}\left(\Delta u^3 - \partial_3 \Pi\right)\right) \\ \quad - \frac{2u^\theta \partial_3 u^\theta}{r} - \frac{u^r \omega^\theta}{r} = 0, \\ \partial_t \omega^3 + \mathbf{u} \cdot \nabla \omega^3 - \left(\partial_r + \frac{1}{r}\right)\left(\frac{1}{\rho}\left(\Delta - \frac{1}{r^2}\right) u^\theta\right) - (\omega^r \partial_r + \omega^3 \partial_3) u^3 = 0, \\ (\omega^r, \omega^\theta, \omega^3)|_{t=0} = (\omega_0^r, \omega_0^\theta, \omega_0^3). \end{cases} \tag{1.8}$$

We state our main theorem, where we set $(\Phi, \Gamma) = (\frac{\omega^r}{r}, \frac{\omega^\theta}{r})$, $\sigma(t) = \min\{t, 1\}$, $\langle t \rangle = \sqrt{1+t^2}$:

Theorem 1.1. *Assume* $(\rho_0, \mathbf{u}_0)$ *is axisymmetric,* $a_0 = \frac{1}{\rho_0} - 1 \in L^2 \cap L^\infty$, $\frac{a_0}{r} \in L^\infty$, $\mathbf{u}_0 \in H^1$ *and* $\Gamma_0, \Phi_0 \in L^2$, $0 < m \leqq \rho_0 \leqq M$ *with some positive constants* m *and* M*. Then there exists a positive time* T_* *so that the system* (1.1) *has a unique solution* $(\rho, \mathbf{u})$ *on* $[0, T_*)$*, satisfying that for any* $T < T_*$,

$$\begin{aligned} &\rho \in L^\infty(0,T;\mathbb{R}^3), \ \mathbf{u} \in \mathcal{C}([0,T]; H^1(\mathbb{R}^3)), \ \text{and } \nabla \mathbf{u} \in L^2(0,T;H^1(\mathbb{R}^3)), \\ &\sup_{t\in[0,T]} \left(\sigma(t)(\|\mathbf{u}_t(t)\|_2^2 + \|\mathbf{u}(t)\|_{\dot{H}^2}^2 + \|\nabla \Pi(t)\|_2^2) + \int_0^t \sigma(\tau)\|\nabla \mathbf{u}_t(\tau)\|_2^2 \,\mathrm{d}\tau\right) < \infty. \end{aligned} \tag{1.9}$$

In addition, there exists a positive constant $C = C(m, M)$*, such that if*

$$\|u_0^\theta\|_3 + \left\|\frac{a_0}{r}\right\|_\infty \|\mathbf{u}_0\|_2^2 \leqq \eta_1, \ \left\|\frac{a_0}{r}\right\|_\infty^2 (\|(u_0^\theta)^2\|_2^2 + \|\nabla \mathbf{u}_0\|_2^2) \leqq \eta_1 (\|\Gamma_0\|_2^2 + \|\Phi_0\|_2^2), \tag{1.10}$$

where

$$\eta_1 = \frac{1}{2C} \exp\left(-C\|\mathbf{u}_0\|_2^3 (\|\Gamma_0\|_2 + \|\Phi_0\|_2)\right), \tag{1.11}$$

then the solution $(\rho, \mathbf{u})$ *is global, that is* $T_* = \infty$. *Furthermore, assuming that* $\mathbf{u}_0 \in L^1$ *and* $ru_0^\theta \in L^1 \cap L^2$, *we have*

$$\|\mathbf{b}(t)\|_{L^2}^2 + \langle t\rangle\|\nabla \mathbf{b}(t)\|_{L^2}^2 + t\langle t\rangle\|(\mathbf{b}_t, \Delta\mathbf{b})(t)\|_{L^2}^2 \leqq C\langle t\rangle^{-\frac{3}{2}}, \tag{1.12}$$

$$\|ru^\theta(t)\|_2^2 \leqq C\langle t\rangle^{-\frac{3}{2}},$$
$$\|u^\theta(t)\|_2^2 + \langle t\rangle\|\nabla(u^\theta\mathbf{e}_\theta)(t)\|_2^2 + t\langle t\rangle(\|u_t^\theta(t)\|_2^2 + \|\Delta(u^\theta\mathbf{e}_\theta)(t)\|_2^2) \leqq C\langle t\rangle^{-\frac{5}{2}}. \tag{1.13}$$

Remark 1.1. From the above theorem, we can obtain the global existence of the smooth axisymmetric solution for the three dimensional inhomogeneous incompressible Navier–Stokes system when

$$\left\|\frac{a_0}{r}\right\|_\infty \text{ and } \|u_0^\theta\|_3 \text{ are sufficiently small.}$$

It is well-known that the system (1.1) possesses a structure of scaling invariance: if (ρ, u, Π) is a solution of the system (1.1) on a time interval $[0, T]$ with initial data (ρ_0, u_0), then $(\rho_\lambda, u_\lambda, \Pi_\lambda)$, defined by

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \quad \Pi_\lambda(t,x) = \lambda^2\Pi(\lambda^2 t, \lambda x), \quad \rho_\lambda(t,x) = \rho(\lambda^2 t, \lambda x),$$

is also a solution of the system (1.1) on the time interval $[0, \lambda^{-2}T]$ with the initial data $(\rho_0(\lambda x), \lambda u_0(\lambda x))$. We attempt to obtain the global well-posedness result mostly under some scaling invariant conditions. Fortunately, the inequalities (1.10) are indeed scaling invariant. If we choose $\rho_0 = 1$ in Theorem 1.1, we can obtain the global well-posedness result for the three dimensional classical axisymmetric Navier–Stokes system when

$$\|u_0^\theta\|_3 \leqq \frac{1}{2C}\exp\left(-C\|\mathbf{u}_0\|_2^3(\|\Gamma_0\|_2 + \|\Phi_0\|_2)\right). \tag{1.14}$$

The above small condition is better than (1.2). If we choose $u_0^\theta = 0$ in Theorem 1.1, we can obtain the global well-posedness for the three dimensional inhomogeneous axisymmetric Navier–Stokes system without swirl when

$$\left\|\frac{a_0}{r}\right\|_\infty^2 (\|\nabla\mathbf{u}_0\|_2^2 + \|\mathbf{u}_0\|_2^4\|\Gamma_0\|_2^2) \leqq \frac{1}{2C}\exp\left(-C\|\mathbf{u}_0\|_2^3\|\Gamma_0\|_2\right)\|\Gamma_0\|_2^2. \tag{1.15}$$

This small condition is much clearer than the one in [2].

Remark 1.2. In [2] (Section 3), Abidi and Zhang obtained the following decay estimates,

$$\|\mathbf{u}(t)\|_{L^2}^2 + \langle t\rangle\|\nabla\mathbf{u}(t)\|_{L^2}^2 + t\langle t\rangle\|(\mathbf{u}_t, \Delta\mathbf{u})(t)\|_{L^2}^2 \leqq C\langle t\rangle^{-\frac{3}{2}}. \tag{1.16}$$

The decay estimates (1.16) also hold for the non-axisymmetric case. One cannot obtain any special behavior for the axisymmetric case from (1.16). In Theorem 1.1, we obtain that the swirl component u^θ will share better decay estimates than (u^r, u^3). One can easily show that these decay estimates are optimal under the conditions $\rho_0 \equiv 1$, $\mathbf{u}_0 \in L^1 \cap H^2$ and $ru_0^\theta \in L^1 \cap L^2$.

Thanks to the blow up criteria (for example, see [8]), to prove the global well-posedness result, we only need to prove that $\|\nabla \mathbf{u}\|_{L_T^{\infty,2}}$ is bounded for all $T > 0$. For the axisymmetric solution of (1.1) without swirl, for example, the authors used the homogeneous case [9, 12, 19] or the inhomogeneous case with $\|\frac{a_0}{r}\|_\infty$ sufficiently small [2] to prove

$$\|\Gamma(t)\|_2^2 \leqq constant, \quad \forall t \in (0, \infty),$$

then

$$\|\nabla \mathbf{u}\|_2 \approx \|w^\theta\|_2 \leqq constant, \quad \forall t \in (0, \infty).$$

However, when the solutions have nonzero swirls, the estimate of $\|\Gamma(t)\|_2$ depends on many more complicated terms. For the homogeneous case, we found in [4] that the system of the pair (Φ, Γ) has some good structures, and easily showed that

$$\|\Gamma(t)\|_2^2 + \|\Phi(t)\|_2^2 \leqq constant, \quad \forall t \in (0, \infty).$$

In such a sense, we consider the following system: for the pair (Φ, Γ):

$$\begin{cases} \partial_t \Phi + \mathbf{u} \cdot \nabla \Phi + \frac{1}{r}\partial_3 \left(\frac{1}{\rho}\left(\Delta - \frac{1}{r^2}\right) u^\theta\right) - (\omega^r \partial_r + \omega^3 \partial_3)\frac{u^r}{r} = 0, \\ \partial_t \Gamma + \mathbf{u} \cdot \nabla \Gamma - \frac{1}{r}\partial_3 \left(\frac{1}{\rho}((\Delta - \frac{1}{r^2})u^r - \partial_r \Pi)\right) + \frac{1}{r}\partial_r \left(\frac{1}{\rho}(\Delta u^3 - \partial_3 \Pi)\right) \\ \quad +2\frac{u^\theta}{r}\Phi = 0. \end{cases} \tag{1.17}$$

If we assume $a = \frac{1}{\rho} - 1, a|_{r=0} = 0$, we also have the following important new identity:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Phi\|_2^2 + \|\Gamma\|_2^2) + \|\nabla\Phi\|_2^2 + \|\nabla\Gamma\|_2^2 \\ &= \int_{\mathbb{R}^3} \Big[(\omega^r \partial_r + \omega^3 \partial_3)\frac{u^r}{r}\, \Phi - 2\Gamma\Phi + \frac{a}{r}(\partial_r \omega^3 \\ &\qquad -\partial_3 \omega^r)\partial_3 \Phi - \frac{a}{r}(\partial_3 \omega^\theta - \partial_r \Pi)\partial_3 \Gamma - \frac{a}{r}(\partial_r \omega^\theta + \Gamma + \partial_3 \Pi)\partial_r \Gamma \Big]\, \mathrm{d}x. \end{aligned}$$

This is the key to obtaining some *a priori* estimates for the solutions of the inhomogeneous axisymmetric Navier–Stokes system (1.5). However, this identity contains many more complicated terms compared with [4]. Fortunately, this can be controlled by estimates (2.16) and (2.23). Then we can reach the goal by the continuous method under the small assumptions (1.10). We may need to point out that there are two technical steps in our proofs:

(1) using $\|\Gamma(t)\|_2$ and the energy method to estimate $\|\boldsymbol{\omega}\|_2$ (see Lemmas 2.5 and 2.6);
(2) using the energy method to estimate $\|\Gamma(t)\|_2 + \|\Phi(t)\|_2$ (see Lemma 2.7).

Furthermore, since there is no pressure term in the system of (w^r, w^3) (1.8), one can use a similar argument as that in the homogeneous case [21], using $\|\mathbf{b}\|_{L^\infty}$ to estimate $\|w^r\|_2 + \|w^3\|_2$. However, in our case, we have to give a new estimate for $\|w^r\|_2 + \|w^3\|_2$ in Lemma 2.5.

Notations. We denote $\tilde{\nabla} = (\partial_r, \partial_3)$, $\tilde{\mathbf{u}} = (u^r, u^3)$, $\mathbf{b} = u^r\mathbf{e}_r + u^3\mathbf{e}_3$. If $f(x)$ is axisymmetric, that is $f(x) = f(r, x_3)$, we have

$$\mathbf{u} \cdot \nabla f = \mathbf{b} \cdot \nabla f = (u^r\partial_r + u^3\partial_3)f.$$

We introduce the Banach spaces $L_T^{p,q}$, equipped with the norms

$$\|f\|_{L_T^{p,q}} = \begin{cases} \left(\int_0^T \|f(t)\|_q^p \, \mathrm{d}t\right)^{\frac{1}{p}}, & \text{if } \ 1 \leqq p < \infty, \\ \operatorname{ess\,sup}_{t\in(0,T)} \|f(t)\|_q, & \text{if } \ p = \infty, \end{cases}$$

where

$$\|f(t)\|_q = \begin{cases} \left(\int_{\mathbb{R}^3} |f(t,x)|^q \, \mathrm{d}x\right)^{\frac{1}{q}}, & \text{if } \ 1 \leqq q < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^3} |f(t,x)|, & \text{if } \ q = \infty. \end{cases}$$

2. Preliminaries

From the Lemmas 2.2–2.4 in [4], we present the following proposition of the axisymmetric velocity, which is frequently used in the axisymmetric system.

Proposition 2.1. *Assume that* $(\rho, \mathbf{u})$ *is the smooth axisymmetric solution of* (1.1) *on* $[0, T]$ *with the initial data* $\mathbf{u_0}$*, and* curl $\mathbf{u} = \boldsymbol{\omega}$*, then we obtain:*

(i) $\mathbf{u} = u^\theta\mathbf{e}_\theta + \nabla \times (\psi\mathbf{e}_\theta) = -\partial_3\psi\boldsymbol{e_r} + u^\theta\mathbf{e}_\theta + \frac{\partial_r(r\psi)}{r}\boldsymbol{e_3}$*, with*

$$u^\theta(t, r, x_3), \quad \psi(t, r, x_3), \quad \omega^\theta(t, r, x_3) \in C^1(0, T; C^\infty(\overline{\mathbb{R}^+} \times \mathbb{R})),$$

and $u^\theta(t, 0, x_3) = \psi(t, 0, x_3) = \omega^\theta(t, 0, x_3) = 0$*;*

(ii) *There exists a positive constant* $C = C(q)$*, such that for all* $t \in [0, T]$ *and* $1 < q < \infty$*,*

$$\|\tilde{\nabla}u^r\|_q + \|\tilde{\nabla}u^3\|_q + \left\|\frac{u^r}{r}\right\|_q \leqq C\|\omega^\theta\|_q, \tag{2.1}$$

$$\|\tilde{\nabla}u^\theta\|_q + \left\|\frac{u^\theta}{r}\right\|_q \leqq C\|\nabla\mathbf{u}\|_q;$$

(iii)

$$\frac{u^r}{r} = \Delta^{-1}\partial_3(\Gamma) - 2\frac{\partial_r}{r}\Delta^{-2}\partial_3(\Gamma).$$

There exists a positive constant $C = C(q)$*, such that for all* $1 < q < \infty$*,*

$$\left\|\tilde{\nabla}\frac{u^r}{r}\right\|_q \leqq C(q)\,\|\Gamma\|_q, \tag{2.2}$$

$$\left\|\tilde{\nabla}\tilde{\nabla}\frac{u^r}{r}\right\|_q \leqq C(q)\,\|\partial_3(\Gamma)\|_q,$$

and

$$\|\frac{u^r}{r}\|_\infty \leqq C\ \|\Gamma\|_2^{\frac{1}{2}}\|\nabla\Gamma\|_2^{\frac{1}{2}}; \tag{2.3}$$

(iv) (*Sobolev–Hardy inequality*) *If* $0 \leqq s < 2$, $q_* \in [2, 2(3-s)]$, *then there exists a positive constant* $C_{q_*,s}$, *such that for all* $f \in C_0^\infty(\mathbb{R}^3)$,

$$\left\|\frac{f}{r^{\frac{s}{q_*}}}\right\|_{q_*} \leqq C_{q_*,s}\|f\|_2^{\frac{3-s}{q_*}-\frac{1}{2}}\|\nabla f\|_2^{\frac{3}{2}-\frac{3-s}{q_*}}.$$

We can extend the properties in [2] to the axisymmetric velocity with nonzero swirls, and have following identities.

Lemma 2.1. *Under the conditions in Proposition* 2.1, *we have*

$$\left(\Delta - \frac{1}{r^2}\right)u^r = \partial_3\omega^\theta, \tag{2.4}$$

$$\left(\Delta - \frac{1}{r^2}\right)u^\theta = \partial_r\omega^3 - \partial_3\omega^r, \tag{2.5}$$

$$\Delta u^3 = -\partial_r\omega^\theta - \Gamma, \qquad \nabla\cdot\boldsymbol{\omega} = 0. \tag{2.6}$$

Proof. The above identities can be deduced directly from (1.7) and the divergence-free property of $\mathbf{u}$. For instance,

$$\begin{aligned}\left(\Delta - \frac{1}{r^2}x\right)u^\theta &= \left(\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^\theta\\ &= \partial_r\left(\omega^3 - \frac{u^\theta}{r}\right) - \partial_3\omega^r + \left(\frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^\theta\\ &= \partial_r\omega^3 - \partial_3\omega^r.\end{aligned}$$

Using an argument similar to that in [2], we have

$$\begin{aligned}\left(\Delta - \frac{1}{r^2}\right)u^r &= \left(\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^r\\ &= \partial_r\left(-\frac{u^r}{r} - \partial_3u^3\right) + \partial_3(\omega^\theta + \partial_ru^3) + \left(\frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^r\\ &= \partial_3\omega^\theta,\end{aligned}$$

and

$$\Delta u^3 = -\partial_r\omega^\theta - \Gamma.$$

□

Then, we can give the following remark which is essential in the proof of Theorem 1.1.

Remark 2.1. Set $\mathbf{B} = \omega^r\mathbf{e}_r + \omega^3\mathbf{e}_3$, and

$$\nabla\cdot\mathbf{B} = 0,\quad \nabla\times\mathbf{B} = (\partial_3\omega^r - \partial_r\omega^3)\mathbf{e}_\theta.$$

Thus

$$\|\nabla\omega^r, \nabla\omega^3\|_p + \|\Phi\|_p \leqq C\|\nabla\mathbf{B}\|_p \leqq C\|\partial_3\omega^r - \partial_r\omega^3\|_p,\quad 1 < p < \infty. \tag{2.7}$$

2.1. A priori estimates

Now we shall present some *a priori* estimates in this section.

One can easily obtain the following lemma and omit the detail, see [13].

Lemma 2.2. *Under the conditions in Proposition 2.1, we obtain that for all $t \in [0, T]$,*

$$0 < m \leqq \rho \leqq M, \tag{2.8}$$

and the energy inequality

$$\frac{1}{2}\|\sqrt{\rho}\mathbf{u}\|_2^2 + \int_0^t \|\nabla \mathbf{u}\|_2^2 \leqq C\|\mathbf{u_0}\|_2^2. \tag{2.9}$$

For the convenience of the proof, we estimate the swirl component and the convection term below. The proofs of these two lemmas will be given in the Appendix.

Lemma 2.3. *Under the conditions in Proposition 2.1, we get that for all $t \in [0, T]$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\rho}(u^\theta)^2\right\|_2^2 + \left\|\nabla(u^\theta)^2\right\|_2^2 + \left\|\frac{(u^\theta)^2}{r}\right\|_2^2 \leqq C\|\mathbf{u}\|_2^{\frac{4}{3}}\|\Gamma\|_2^{\frac{4}{3}}\|\nabla\mathbf{u}\|_2^{\frac{10}{3}}. \tag{2.10}$$

Lemma 2.4. *Under the conditions in Proposition 2.1, we obtain that for all $t \in [0, T]$,*

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\tilde{\mathbf{u}}\|_2^2 &= \|\mathbf{u}\cdot\nabla u^r\|_2^2 + \|\mathbf{u}\cdot\nabla u^3\|_2^2 \\ &\leqq C_\delta\|\mathbf{u}\|_2^{\frac{4}{3}}\|\Gamma\|_2^{\frac{4}{3}}\|\nabla\mathbf{u}\|_2^{\frac{10}{3}} + \delta(\|\nabla\omega^\theta\|_2^2 + \|\Gamma\|_2^2),\end{aligned} \tag{2.11}$$

where δ is sufficient small.

We now evaluate the terms $\|\omega^r\|_2$ and $\|\omega^3\|_2$ by the system (1.8).

Lemma 2.5. *Under the conditions in Proposition 2.1, we have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega^r\|_2^2 + \|\omega^3\|_2^2) + \|\tilde{\nabla}\omega^r\|_2^2 + \|\tilde{\nabla}\omega^3\|_2^2 + \|\Phi\|_2^2 \\ &\leqq C_\delta\|\mathbf{u}\|_2^{\frac{4}{3}}\|\Gamma\|_2^{\frac{4}{3}}\|\nabla\mathbf{u}\|_2^{\frac{10}{3}} + \delta\|\nabla\boldsymbol{\omega}\|_2^2,\end{aligned} \tag{2.12}$$

for all $t \in [0, T]$, where δ is sufficiently small.

Proof. Multiplying the equations $(1.8)_1$ and $(1.8)_3$ by ω^r and ω^3, respectively, and using integration by parts and Lemma 2.1, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega^r\|_2^2 + \|\omega^3\|_2^2) + \int_{\mathbb{R}^3}\frac{1}{\rho}(\partial_3\omega^r - \partial_r\omega^3)^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3}(\omega^r\partial_r + \omega^3\partial_3)u^r\omega^r + (\omega^r\partial_r + \omega^3\partial_3)u^3\omega^3\,\mathrm{d}x \\ &= J_1 + J_2 + J_3 + J_4.\end{aligned} \tag{2.13}$$

Take $v_1 = (u^r, u^\theta)$, $v_2 = (\omega^r, \omega^\theta, \omega^3)$ and r_0 as in (5.3). By similar calculus to that in the proof of Lemma 2.4, we have

$$\begin{aligned}
\|v_1 v_2|_{r>r_0}\|_2^2 &\leqq \left(\int_{\mathbb{R}}\int_{r_0}^{\infty} |r^{\frac{1}{2}} v_1|^4 \mathrm{d}r\mathrm{d}x_3\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}}\int_{r_0}^{\infty} |v_2|^4 \mathrm{d}r\mathrm{d}x_3\right)^{\frac{1}{2}} \\
&\leqq C\left(\int_{\mathbb{R}}\int_{r_0}^{\infty} |r^{\frac{1}{2}} v_1|^2 \mathrm{d}r\mathrm{d}x_3\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}}\int_{r_0}^{\infty} |\tilde{\nabla}\left(r^{\frac{1}{2}} v_1\right)|^2 \mathrm{d}r\mathrm{d}x_3\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{R}}\int_{r_0}^{\infty} |v_2|^2 \mathrm{d}r\mathrm{d}x_3\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}}\int_{r_0}^{\infty} |\tilde{\nabla} v_2|^2 \mathrm{d}r\mathrm{d}x_3\right)^{\frac{1}{2}} \\
&\leqq \frac{C}{r_0}\|\mathbf{u}\|_2 \|\nabla\mathbf{u}\|_2^2 \|\nabla\boldsymbol{\omega}\|_2 \\
&\leqq C\|\mathbf{u}\|_2^{\frac{2}{3}} \|\Gamma\|_2^{\frac{2}{3}} \|\nabla\mathbf{u}\|_2^{\frac{5}{3}} \|\nabla\boldsymbol{\omega}\|_2, \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned}
\|u^r v_2|_{r\leqq r_0}\|_2^2 &\leqq r_0^2 \|\frac{u^r}{r} v_2\|_2^2 \\
&\leqq C r_0^2 \|\nabla \frac{u^r}{r}\|_2^2 \|v_2\|_3^2 \\
&\leqq C r_0^2 \|\Gamma\|_2^2 \|\nabla\mathbf{u}\|_2 \|\nabla\boldsymbol{\omega}\|_2. \\
&\leqq C\|\mathbf{u}\|_2^{\frac{2}{3}} \|\Gamma\|_2^{\frac{2}{3}} \|\nabla\mathbf{u}\|_2^{\frac{5}{3}} \|\nabla\boldsymbol{\omega}\|_2. \qquad (2.15)
\end{aligned}$$

Using integration by parts, the Cauchy–Schwarz inequality, (2.1), (2.2), (2.14), (2.15) and the fact (1.7), we obtain

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}^3} \partial_r u^r \omega^r \omega^r \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} -2u^r \omega^r \partial_r \omega^r - u^r \omega^r \Phi \,\mathrm{d}x \\
&\leqq C\|u^r \omega^r\|_2 \|\nabla\boldsymbol{\omega}\|_2 \\
&\leqq C(\|u^r \omega^r|_{r\leqq r_0}\|_2 + \|u^r \omega^r|_{r>r_0}\|_2)\|\nabla\boldsymbol{\omega}\|_2 \\
&\leqq C\|\mathbf{u}\|_2^{\frac{1}{3}} \|\Gamma\|_2^{\frac{1}{3}} \|\nabla\mathbf{u}\|_2^{\frac{5}{6}} \|\nabla\boldsymbol{\omega}\|_2^{\frac{3}{2}},
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \int_{\mathbb{R}^3} -u^r \partial_3(\omega^r \omega^3) \,\mathrm{d}x \\
&\leqq C\|u^r v_2\|_2 \|\nabla\boldsymbol{\omega}\|_2 \\
&\leqq C(\|u^r v_2|_{r\leqq r_0}\|_2 + \|u^r v_2|_{r>r_0}\|_2)\|\nabla\boldsymbol{\omega}\|_2 \\
&\leqq C\|\mathbf{u}\|_2^{\frac{1}{3}} \|\Gamma\|_2^{\frac{1}{3}} \|\nabla\mathbf{u}\|_2^{\frac{5}{6}} \|\nabla\boldsymbol{\omega}\|_2^{\frac{3}{2}}.
\end{aligned}$$

Similarly, using $\partial_r u^3 = \partial_3 u^r - \omega^\theta$, we have

$$\begin{aligned}
J_3 &= J_2 - \int_{\mathbb{R}^3} \omega^\theta \omega^r \omega^3 \, \mathrm{d}x \\
&= J_2 - \int_{\mathbb{R}} \int_{r \leqq r_0} r\Gamma \omega^r \omega^3 \, r \mathrm{d}r \mathrm{d}x_3 + \int_{\mathbb{R}} \int_{r > r_0} \partial_3 u^\theta \omega^\theta \omega^3 r \mathrm{d}r \mathrm{d}x_3 \\
&\leqq |J_2| + r_0 \int_{r \leqq r_0} |\Gamma \omega^r \omega^3| \, \mathrm{d}x + + \int_{\mathbb{R}} \int_{r > r_0} |u^\theta| |\partial_3(\omega^\theta \omega^3)| \, \mathrm{d}x \\
&\leqq |J_2| + r_0 \|\Gamma\|_2 \|\nabla \mathbf{u}\|_2^{\frac{1}{2}} \|\nabla \boldsymbol{\omega}\|_2^{\frac{3}{2}} + \|u^\theta v_2|_{r > r_0}\|_2 \|\nabla \boldsymbol{\omega}\|_2 \\
&\leqq |J_2| + C \|\mathbf{u}\|_2^{\frac{1}{3}} \|\Gamma\|_2^{\frac{1}{3}} \|\nabla \mathbf{u}\|_2^{\frac{5}{6}} \|\nabla \boldsymbol{\omega}\|_2^{\frac{3}{2}} \\
&\leqq C \|\mathbf{u}\|_2^{\frac{1}{3}} \|\Gamma\|_2^{\frac{1}{3}} \|\nabla \mathbf{u}\|_2^{\frac{5}{6}} \|\nabla \boldsymbol{\omega}\|_2^{\frac{3}{2}},
\end{aligned}$$

and

$$\begin{aligned}
J_4 &= \int_{\mathbb{R}^3} -\frac{\partial_r (r u^r)}{r} (\omega^3)^2 \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} 2u^r \omega^3 \partial_r \omega^3 \, \mathrm{d}x \\
&\leqq C \|\mathbf{u}\|_2^{\frac{1}{3}} \|\Gamma\|_2^{\frac{1}{3}} \|\nabla \mathbf{u}\|_2^{\frac{5}{6}} \|\nabla \boldsymbol{\omega}\|_2^{\frac{3}{2}}.
\end{aligned}$$

Combining the above inequalities, we get

$$\begin{aligned}
J_1 + J_2 + J_3 + J_4 &\leqq C \|\mathbf{u}\|_2^{\frac{1}{3}} \|\Gamma\|_2^{\frac{1}{3}} \|\nabla \mathbf{u}\|_2^{\frac{5}{6}} \|\nabla \boldsymbol{\omega}\|_2^{\frac{3}{2}} \\
&\leqq C_\delta \|\mathbf{u}\|_2^{\frac{4}{3}} \|\Gamma\|_2^{\frac{4}{3}} \|\nabla \mathbf{u}\|_2^{\frac{10}{3}} + \delta \|\nabla \boldsymbol{\omega}\|_2^2.
\end{aligned}$$

Recall that the density has lower bound $\rho \geq m > 0$ and (2.7), so we obtain (2.12). □

We present an essential estimate of $\|\nabla \mathbf{u}\|_{L_T^{\infty,2}}$ as follows.

Lemma 2.6. *Under the conditions in Proposition 2.1, we obtain that for all $t \in [0, T]$,*

$$\begin{aligned}
&\left\|(u^\theta)^2\right\|_{L_t^{\infty,2}}^2 + \|\boldsymbol{\omega}\|_{L_t^{\infty,2}}^2 + \left\|\nabla (u^\theta)^2\right\|_{L_t^{2,2}}^2 \\
&\quad + \left\|\frac{(u^\theta)^2}{r}\right\|_{L_t^{2,2}}^2 + \|u_t^r\|_{L_t^{2,2}}^2 + \|u_t^3\|_{L_t^{2,2}}^2 + \|\nabla \Pi\|_{L_t^{2,2}}^2 + \|\nabla \boldsymbol{\omega}\|_{L_t^{2,2}}^2 \\
&\quad \leqq C(\|(u_0^\theta)^2\|_2^2 + \|\nabla \mathbf{u}_0\|_2^2 + \|\mathbf{u}_0\|_2^4 \|\Gamma\|_{L_t^{\infty,2}}^2) \exp(C \|\mathbf{u}_0\|_2^3 \|\Gamma\|_{L_t^{\infty,2}}). \qquad (2.16)
\end{aligned}$$

Proof. • The $\dot{H}^1$ estimates of u^r, u^3.

Multiplying the equations $(1.5)_2$ and $(1.5)_4$ by $\partial_t u^r$ and $\partial_t u^3$, respectively, using integration by parts and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u^r\|_2^2 + \|\nabla u^3\|_2^2 + \left\|\frac{u^r}{r}\right\|_2^2\right) + \|\sqrt{\rho}u_t^r\|_2^2 + \|\sqrt{\rho}u_t^3\|_2^2 \\
&= -\left(\int_{\mathbb{R}^3} \rho\mathbf{u}\cdot\nabla u^r u_t^r - \rho\frac{(u^\theta)^2}{r}u_t^r + \rho\mathbf{u}\cdot\nabla u^3 u_t^3\,\mathrm{d}x\right) \\
&\leqq C\left(\|\mathbf{u}\cdot\nabla u^r\|_2^2 + \|\mathbf{u}\cdot\nabla u^3\|_2^2 + \left\|\frac{(u^\theta)^2}{r}\right\|_2^2\right) + \frac{1}{2}(\|\sqrt{\rho}u_t^r\|_2^2 + \|\sqrt{\rho}u_t^3\|_2^2).
\end{aligned} \tag{2.17}$$

• The estimates of Π and $\nabla\omega^\theta$ by the Stokes equation.

By Lemma 2.1, we can deduce the Stokes system

$$\begin{cases}
-\partial_3\omega^\theta + \partial_r\Pi = -\rho\partial_t u^r - \rho\mathbf{u}\cdot\nabla u^r + \rho\frac{(u^\theta)^2}{r}, \\
\partial_r\omega^\theta + \Gamma + \partial_3\Pi = -\rho\partial_t u^3 - \rho\mathbf{u}\cdot\nabla u^3.
\end{cases}$$

Multiplying the above equations by $\partial_r\Pi$ and $\partial_3\Pi$ respectively, using integration by parts, the Cauchy–Schwarz inequality and the fact that $\omega^\theta|_{r=0} = 0$, we obtain

$$\begin{aligned}
\|\nabla\Pi\|_2^2 &= -\left(\int_{\mathbb{R}^3} \rho\mathbf{u}\cdot\nabla u^r\partial_r\Pi - \rho\frac{(u^\theta)^2}{r}\partial_r\Pi + \rho\mathbf{u}\cdot\nabla u^3\partial_3\Pi\,\mathrm{d}x\right) \\
&\quad -\left(\int_{\mathbb{R}^3} \rho\partial_t u^r\partial_r\Pi + \rho\partial_t u^3\partial_3\Pi\,\mathrm{d}x\right) \\
&\leqq C\left(\|\mathbf{u}\cdot\nabla u^r\|_2^2 + \|\mathbf{u}\cdot\nabla u^3\|_2^2 + \left\|\frac{(u^\theta)^2}{r}\right\|_2^2\right) \\
&\quad + C(\|\sqrt{\rho}u_t^r\|_2^2 + \|\sqrt{\rho}u_t^3\|_2^2) + \frac{1}{2}\|\nabla\Pi\|_2^2.
\end{aligned} \tag{2.18}$$

Along the same lines, multiplying the above system by $-\partial_3\omega^\theta$ and $\partial_r\omega^\theta + \Gamma$ respectively, we have

$$\begin{aligned}
\|\nabla\omega^\theta\|_2^2 + \|\Gamma\|_2^2 &= \int_{\mathbb{R}^3}\left(\rho\partial_t u^r + \rho\mathbf{u}\cdot\nabla u^r - \rho\frac{(u^\theta)^2}{r}\right)\partial_3\omega^\theta\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3}(\rho\partial_t u^3 + \rho\mathbf{u}\cdot\nabla u^3)(\partial_r\omega^\theta + \Gamma)\,\mathrm{d}x \\
&\leqq C\left(\|\mathbf{u}\cdot\nabla u^r\|_2^2 + \|\mathbf{u}\cdot\nabla u^3\|_2^2 + \left\|\frac{(u^\theta)^2}{r}\right\|_2^2\right) \\
&\quad + C(\|\sqrt{\rho}u_t^r\|_2^2 + \|\sqrt{\rho}u_t^3\|_2^2) \\
&\quad + \frac{1}{2}(\|\nabla\omega^\theta\|_2^2 + \|\Gamma\|_2^2).
\end{aligned} \tag{2.19}$$

Combining (2.18) and (2.19), we get

$$\begin{aligned}
&\|\nabla \Pi\|_2^2 + \|\nabla\omega^\theta\|_2^2 + \|\Gamma\|_2^2 \\
&\leqq C\left(\|\mathbf{u}\cdot\nabla u^r\|_2^2 + \|\mathbf{u}\cdot\nabla u^3\|_2^2 + \left\|\frac{(u^\theta)^2}{r}\right\|_2^2\right) \\
&\quad + C\left(\|\sqrt{\rho}u_t^r\|_2^2 + \|\sqrt{\rho}u_t^3\|_2^2\right).
\end{aligned} \tag{2.20}$$

Combining the above estimates, (2.10), (2.11), (2.17) and (2.20), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\sqrt{\rho}(u^\theta)^2\right\|_2^2 + \|\nabla u^r\|_2^2 + \|\nabla u^3\|_2^2 + \left\|\frac{u^r}{r}\right\|_2^2\right) + \left\|\nabla(u^\theta)^2\right\|_2^2 + \left\|\frac{(u^\theta)^2}{r}\right\|_2^2 \\
&+(\|\sqrt{\rho}u_t^r\|_2^2 + \|\sqrt{\rho}u_t^3\|_2^2 + \|\nabla\Pi\|_2^2 + \|\nabla\omega^\theta\|_2^2 + \|\Gamma\|_2^2) \\
&\leqq C(\|\mathbf{u}\cdot\nabla u^r\|_2^2 + \|\mathbf{u}\cdot\nabla u^3\|_2^2 + \|\mathbf{u}\|_2^{\frac{4}{3}}\|\Gamma\|_2^{\frac{4}{3}}\|\nabla\mathbf{u}\|_2^{\frac{10}{3}}) \\
&\leqq C_\delta\|\mathbf{u}\|_2^{\frac{4}{3}}\|\Gamma\|_2^{\frac{4}{3}}\|\nabla\mathbf{u}\|_2^{\frac{10}{3}} + \delta(\|\nabla\omega^\theta\|_2^2 + \|\Gamma\|_2^2).
\end{aligned} \tag{2.21}$$

The inequalities (2.12) and (2.21) imply that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\sqrt{\rho}(u^\theta)^2\right\|_2^2 + \|\nabla u^r\|_2^2 + \|\nabla u^3\|_2^2 + \left\|\frac{u^r}{r}\right\|_2^2 + \|\omega^r\|_2^2 + \|\omega^3\|_2^2\right) \\
&\quad + \left\|\nabla(u^\theta)^2\right\|_2^2 + \left\|\frac{(u^\theta)^2}{r}\right\|_2^2 \\
&\quad + \left(\|\sqrt{\rho}u_t^r\|_2^2 + \|\sqrt{\rho}u_t^3\|_2^2 + \|\nabla\Pi\|_2^2 + \|\nabla\omega^\theta\|_2^2 + \|\Gamma\|_2^2\right. \\
&\quad \left.+\|\nabla\omega^r\|_2^2 + \|\nabla\omega^3\|_2^2 + \|\Phi\|_2^2\right) \\
&\leqq C\|\mathbf{u}\|_2^{\frac{4}{3}}\|\Gamma\|_2^{\frac{4}{3}}\|\nabla\mathbf{u}\|_2^{\frac{10}{3}} \\
&\leqq C\left(\|\mathbf{u}\|_2^2\|\Gamma\|_2^2\|\nabla\mathbf{u}\|_2^2\right)^{\frac{1}{3}}(\|\mathbf{u}\|_2\|\Gamma\|_2\|\nabla\mathbf{u}\|_2^4)^{\frac{2}{3}} \\
&\leqq C\|\mathbf{u}\|_2^2\|\Gamma\|_2^2\|\nabla\mathbf{u}\|_2^2 + C\|\mathbf{u}\|_2\|\Gamma\|_2\|\nabla\mathbf{u}\|_2^4.
\end{aligned}$$

Applying Gronwall's inequality and Lemma 2.2, we have (2.16). □

Using the ideas in [4], we consider the L^2 estimate of the pair (Φ, Γ) as follows.

Lemma 2.7. *Under the conditions in Proposition 2.1, assume $a = 1/\rho - 1$ and $a|_{r=0} = 0$, we obtain that for all $t \in [0, T]$,*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Phi\|_2^2 + \|\Gamma\|_2^2) + \|\nabla\Phi\|_2^2 + \|\nabla\Gamma\|_2^2 \\
&\leqq C\left\|\frac{a}{r}\right\|_\infty(\|\nabla\Pi\|_2 + \|\nabla\omega^\theta\|_2 \\
&\quad + \|\Gamma\|_2 + \|\partial_r\omega^3 - \partial_3\omega^r\|_2)(\|\nabla\Gamma\|_2 + \|\nabla\Phi\|_2) \\
&\quad + C_0\|u^\theta\|_3\|\nabla\Gamma\|_2\|\nabla\Phi\|_2.
\end{aligned} \tag{2.22}$$

Proof. Multiplying the equations (1.17) by (Φ, Γ) respectively, we have

$$
\begin{aligned}
0 = & \int_{\mathbb{R}^3} \left(\partial_t \Phi + \mathbf{u} \cdot \nabla \Phi + \frac{1}{r} \partial_3 \left(\frac{1}{\rho} \left(\Delta - \frac{1}{r^2} \right) u^\theta \right) \right. \\
& \left. - (\omega^r \partial_r + \omega^3 \partial_3) \frac{u^r}{r} \right) \cdot \Phi \, \mathrm{d}x \\
& + \int_{\mathbb{R}^3} \left(\partial_t \Gamma + \mathbf{u} \cdot \nabla \Gamma - \frac{1}{r} \partial_3 \left(\frac{1}{\rho} \left(\left(\Delta - \frac{1}{r^2} \right) u^r - \partial_r \Pi \right) \right) \right. \\
& \left. + \frac{1}{r} \partial_r \left(\frac{1}{\rho} \left(\Delta u^3 - \partial_3 \Pi \right) \right) + 2 \frac{u^\theta}{r} \Phi \right) \cdot \Gamma \, \mathrm{d}x \\
:= & \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\|\Phi\|_2^2 + \|\Gamma\|_2^2) + I_1 + I_2 + I_3.
\end{aligned}
$$

Then, using integration by parts, (1.7) and Lemma 2.1, we obtain

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^3} \frac{1}{r} \partial_3 \left((1+a) \left(\Delta - \frac{1}{r^2} \right) u^\theta \right) \cdot \Phi \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{1}{r} \partial_3 \left(\left(\Delta - \frac{1}{r^2} \right) u^\theta \right) \cdot \Phi - \frac{a}{r} \left(\left(\Delta - \frac{1}{r^2} \right) u^\theta \right) \cdot \partial_3 \Phi \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} - \left(\Delta + \frac{2}{r} \partial_r \right) \Phi \cdot \Phi - \frac{a}{r} (\partial_r \omega^3 - \partial_3 \omega^r) \cdot \partial_3 \Phi \, \mathrm{d}x \\
&= \|\nabla \Phi\|_2^2 - 2\pi \int_{\mathbb{R}} \int_0^\infty \partial_r (\Phi)^2 \, \mathrm{d}r \mathrm{d}x_3 - \int_{\mathbb{R}^3} \frac{a}{r} (\partial_r \omega^3 - \partial_3 \omega^r) \cdot \partial_3 \Phi \, \mathrm{d}x \\
&= \|\nabla \Phi\|_2^2 + 2\pi \int_{\mathbb{R}} \Phi^2|_{r=0} \, \mathrm{d}r \mathrm{d}x_3 - \int_{\mathbb{R}^3} \frac{a}{r} (\partial_r \omega^3 - \partial_3 \omega^r) \cdot \partial_3 \Phi \, \mathrm{d}x \\
&\geq \|\nabla \Phi\|_2^2 - \int_{\mathbb{R}^3} \frac{a}{r} (\partial_r \omega^3 - \partial_3 \omega^r) \cdot \partial_3 \Phi \, \mathrm{d}x.
\end{aligned}
$$

Similarly, since we have the assumption $a|_{r=0} = 0$, we get

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^3} \left(-\frac{1}{r} \partial_3 \left((1+a) \left(\left(\Delta - \frac{1}{r^2} \right) u^r - \partial_r \Pi \right) \right) \right. \\
& \quad \left. + \frac{1}{r} \partial_r \left((1+a) \left(\Delta u^3 - \partial_3 \Pi \right) \right) \right) \cdot \Gamma \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \left(-\frac{1}{r} \partial_3 ((1+a)(\partial_3 \omega^\theta - \partial_r \Pi)) \right. \\
& \quad \left. - \frac{1}{r} \partial_r ((1+a)(\partial_r \omega^\theta + \Gamma + \partial_3 \Pi)) \right) \cdot \Gamma \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} -\frac{1}{r} \left(\Delta - \frac{1}{r^2} \right) \omega^\theta \cdot \Gamma + \frac{a}{r} (\partial_3 \omega^\theta - \partial_r \Pi) \cdot \partial_3 \Gamma \\
& \quad + \frac{a}{r} (\partial_r \omega^\theta + \Gamma + \partial_3 \Pi) \partial_r \Gamma \, \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} -\left(\Delta + \frac{2}{r}\partial_r\right)\Gamma \cdot \Gamma + \frac{a}{r}(\partial_3\omega^\theta - \partial_r\Pi)\cdot\partial_3\Gamma \\
&\quad + \frac{a}{r}(\partial_r\omega^\theta + \Gamma + \partial_3\Pi)\partial_r\Gamma \,\mathrm{d}x \\
&= \|\nabla\Gamma\|_2^2 - 2\pi\int_{\mathbb{R}}\int_0^\infty \partial_r(\Gamma)^2\,\mathrm{d}r\mathrm{d}x_3 + \int_{\mathbb{R}^3}\frac{a}{r}(\partial_3\omega^\theta - \partial_r\Pi)\cdot\partial_3\Gamma \\
&\quad + \frac{a}{r}(\partial_r\omega^\theta + \Gamma + \partial_3\Pi)\partial_r\Gamma \,\mathrm{d}x \\
&= \|\nabla\Gamma\|_2^2 + 2\pi\int_{\mathbb{R}}\Gamma^2|_{r=0}\,\mathrm{d}x_3 + \int_{\mathbb{R}^3}\frac{a}{r}(\partial_3\omega^\theta - \partial_r\Pi)\cdot\partial_3\Gamma \\
&\quad + \frac{a}{r}(\partial_r\omega^\theta + \Gamma + \partial_3\Pi)\partial_r\Gamma \,\mathrm{d}x \\
&\geq \|\nabla\Gamma\|_2^2 + \int_{\mathbb{R}^3}\frac{a}{r}(\partial_3\omega^\theta - \partial_r\Pi)\cdot\partial_3\Gamma + \frac{a}{r}(\partial_r\omega^\theta + \Gamma + \partial_3\Pi)\partial_r\Gamma\,\mathrm{d}x.
\end{aligned}
$$

Using a calculus similar to [4], (2.2) and Sobolev–Hardy inequality in Proposition 2.1, we have

$$
\begin{aligned}
|I_3| &= \left|\int_{\mathbb{R}^3} -(\omega^r\partial_r + \omega^3\partial_3)\frac{u^r}{r}\cdot\Phi + 2\frac{u^\theta}{r}\Phi\cdot\Gamma\,\mathrm{d}x\right| \\
&= \left|2\pi\int_{\mathbb{R}}\int_0^\infty\left(\partial_3 u^\theta\partial_r\frac{u^r}{r}\Phi - \frac{\partial_r(ru^\theta)}{r}\partial_3\frac{u^r}{r}\Phi\right) + 2\frac{u^\theta}{r}\Phi\cdot\Gamma\; r\mathrm{d}r\mathrm{d}x_3\right| \\
&= \left|\int_{\mathbb{R}^3} -u^\theta\left(\partial_3\partial_r\frac{u^r}{r}\Phi + \partial_r\frac{u^r}{r}\partial_3\Phi\right)\mathrm{d}x\right. \\
&\quad \left. + \int_{\mathbb{R}^3} u^\theta\left(\partial_r\partial_3\frac{u^r}{r}\Phi + \partial_3\frac{u^r}{r}\partial_r\Phi\right) + 2\frac{u^\theta}{r}\Phi\cdot\Gamma\,\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}^3} u^\theta\left(-\partial_r\frac{u^r}{r}\partial_3\Phi + \partial_3\frac{u^r}{r}\partial_r\Phi\right) + 2\frac{u^\theta}{r}\Phi\cdot\Gamma\,\mathrm{d}x\right| \\
&\leqq \|u^\theta\|_3\left(\left\|\partial_r\frac{u^r}{r}\right\|_6\|\partial_3\Phi\|_2 + \left\|\partial_3\frac{u^r}{r}\right\|_6\|\partial_r\Phi\|_2\right) + 2\|u^\theta\|_3\left\|\frac{\Phi}{r^{\frac{1}{2}}}\right\|_3\left\|\frac{\Gamma}{r^{\frac{1}{2}}}\right\|_3 \\
&\leqq C\|u^\theta\|_3\|\nabla\Gamma\|_2\|\nabla\Phi\|_2.
\end{aligned}
$$

Combining the above estimates, we get (2.22). □

Lemma 2.8. *Under the conditions in Lemma 2.7, we obtain that for all $t \in [0, T]$,*

$$
\left\|\frac{a}{r}\right\|_{L_t^{\infty,\infty}} \leqq \left\|\frac{a_0}{r}\right\|_\infty \exp\left(t^{\frac{3}{4}}\|\Gamma\|_{L_t^{\infty,2}}^{\frac{1}{2}}\|\nabla\Gamma\|_{L_t^{2,2}}^{\frac{1}{2}}\right). \tag{2.23}
$$

Proof. It follows from the transport equation of (1.5) that

$$
\partial_t a + \mathbf{u}\cdot\nabla a = 0, \tag{2.24}
$$

and

$$
\partial_t\frac{a}{r} + \mathbf{u}\cdot\nabla\frac{a}{r} + \frac{u^r}{r}\,\frac{a}{r} = 0, \tag{2.25}
$$

which yields (2.23), by applying (2.3). □

3. Proof of the Well-Posedness Part of Theorem 1.1

In this section, we are going to complete the proof of the well-posedness part of Theorem 1.1. It is well known that if the initial data $(\rho_0, \mathbf{u_0})$ satisfy

$$0 < m \leqq \rho_0 \leqq M, \qquad \mathbf{u_0} \in H^1,$$

then the system (1.1) has a unique local solution $(\rho, \mathbf{u})$ on $[0, T_*)$ satisfying (1.9) (see [18] for instance).

We mollify the initial data $(\rho_0, \mathbf{u_0})$ as following. Let $J^\varepsilon = \varepsilon^{-3} J(\frac{r}{\varepsilon}, \frac{x_3}{\varepsilon})$ be mollifiers, with

$$0 \leqq J \leqq 1, \qquad \mathrm{supp} J \subset \{0 \leqq r \leqq 2, -1 \leqq x_3 \leqq 1\},$$
$$J \equiv 1, \quad \text{if } x \in \left\{0 \leqq r \leqq \frac{1}{2}, -\frac{1}{2} \leqq x_3 \leqq \frac{1}{2}\right\}, \qquad \int J \,\mathrm{d}x = 1,$$

and

$$\rho_0^\varepsilon = J^\varepsilon * \rho_0 - (J^\varepsilon * (\rho_0 - 1))(0, x_3), \quad \mathbf{u}_0^\varepsilon = J^\varepsilon * \mathbf{u}_0. \tag{3.1}$$

Obviously, $\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon$ are still axisymmetric. we claim that the system (1.1) has a unique global smooth axisymmetric solution $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$ with the initial data $(\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon)$, provided that (1.10) is satisfied. Then the conclusion of the global existence of the solution in Theorem 1.1 follows from uniform estimates (2.8), (3.5), and a standard compactness argument.

There are some properties of the initial data $(\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon)$. For the convenience of the reader, we give the proof of the following lemma in the Appendix.

Lemma 3.1. *If ε is sufficient small, and ρ_0 satisfies $0 < m \leqq \rho_0 \leqq M$, then*

$$\rho_0^\varepsilon = 1, \ \textit{if}\ r = 0,$$
$$0 < \frac{m}{2} \leqq \rho_0^\varepsilon \leqq \frac{M}{2},$$
$$|\rho_0^\varepsilon - 1| \leqq C \left\| \frac{\rho_0 - 1}{r} \right\|_\infty r.$$

It is easy to show that $a_0, \mathbf{u_0} \in H^\infty$. From the local well-posedness result in [6] (Corollary 0.8) and [3], it ensures that the following system derived from (1.1) admits a unique axisymmetric solution $(a^\varepsilon, \mathbf{u}^\varepsilon, \nabla \Pi^\varepsilon)$ in $[0, T_*^\varepsilon)$,

$$\begin{cases} \partial_t a + \mathrm{div}(a\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - (1 + a)\Delta \mathbf{u} + (1 + a)\nabla \Pi = \mathbf{0}, \\ \mathrm{div}\mathbf{u} = 0, \\ (a, \mathbf{u})|_{t=0} = (a_0, \mathbf{u_0}) \end{cases} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3.$$

For any $T^\varepsilon < T_*^\varepsilon$, the solution satisfies

$$a^\varepsilon \in \mathcal{C}([0, T^\varepsilon]; H^s), \ \mathbf{u}^\varepsilon \in \mathcal{C}([0, T^\varepsilon]; H^s) \cap \tilde{L}^1(0, T^\varepsilon; H^{s+2}),$$
$$\nabla \Pi^\varepsilon \in \tilde{L}^1(0, T^\varepsilon; H^s), \quad s > \frac{5}{2}.$$

Then, using the method of proof by contradiction, we will show that the maximal existence time $T_*^\varepsilon = \infty$ follows, provided that (1.10) is satisfied.

Without loss of generality, we denote $\rho = \rho^\varepsilon$, $\mathbf{u} = \mathbf{u}^\varepsilon$, $\Pi = \Pi^\varepsilon$, and so on, and we assume $T_* < \infty$.

Lemma 3.2. *We claim that* $a|_{r=0} = 0$.

Proof. We can define the unique trajectory $\boldsymbol{\chi}(t, x)$ of $\mathbf{u}(t, x)$ by

$$\partial_t \boldsymbol{\chi}(t, x) = \mathbf{u}(t, \boldsymbol{\chi}(t, x)), \quad \boldsymbol{\chi}(0, x) = x.$$

Since $u^r|_{r=0} = u^\theta|_{r=0} = 0$, we have that $\widetilde{\boldsymbol{\chi}}(t, (0, 0, x_3)) = (0, 0, \chi^3(t, x_3))$ is the trajectory from the initial point $(0, 0, x_3)$, satisfying

$$\partial_t \chi^3(t, x_3) = u^3(t, (0, 0, \chi^3(t, x_3))), \quad \chi^3(0, x_3) = x_3.$$

Therefore, by (2.24), we have

$$a(t, \widetilde{\boldsymbol{\chi}}(t, (0, 0, x_3))) = a_0(0, 0, x_3) = 0.$$

□

Lemma 3.3. *There exists a positive constant* C_1*, such that if* $T_* > N \triangleq C_1\|\mathbf{u_0}\|_2^4$*, then*

$$\|\nabla \mathbf{u}(t)\|_2^2 + \int_N^t \|\nabla^2 \mathbf{u}(\tau)\|_2^2 + \|\nabla \Pi(\tau)\|_2^2 \, \mathrm{d}\tau \leqq C \frac{1}{\|\mathbf{u}_0\|_2^2}, \quad \forall t \in [N, T_*). \quad (3.2)$$

Proof. By (2.9), there exists a positive constant K_1, such that

$$\sup_{t \in [0, T^*)} \|\mathbf{u}(t)\|_2^2 + \int_0^{T^*} \|\nabla \mathbf{u}\|_2^2 \leqq K_1 \|\mathbf{u_0}\|_2^2.$$

Then, there exists a time $t_0 \in (0, N)$, such that

$$\|\nabla \mathbf{u}(t_0)\|_2^2 \leqq K_1 \frac{\|\mathbf{u_0}\|_2^2}{N}.$$

Thus

$$\|\mathbf{u}(t_0)\|_2^2 \|\nabla \mathbf{u}(t_0)\|_2^2 \leqq \frac{K_1^2}{C_1}.$$

From an argument similar to that in the proof of *a priori* estimate revealed in Lemma 2.2 in [2], one can easily obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \mathbf{u}\|_2^2 + \|\sqrt{\rho} \mathbf{u}_t\|_2^2 + \|\nabla^2 \mathbf{u}\|_2^2 + \|\nabla \Pi\|_2^2 \leqq K_2 \|\mathbf{u}\|_2 \|\nabla \mathbf{u}\|_2 \|\nabla^2 \mathbf{u}\|_2^2, \quad t \in [t_0, T_*).$$

We can pick that $C_1 > 4K_1^2 K_2^2$ is sufficiently large. By using the continuous method, it is evident that

$$\|\nabla \mathbf{u}(t)\|_2^2 + \int_{t_0}^t \|\nabla^2 \mathbf{u}(\tau)\|_2^2 + \|\nabla \Pi(\tau)\|_2^2 \, \mathrm{d}\tau \leqq \|\nabla \mathbf{u}(t_0)\|_2^2 \leqq \frac{K_1}{C_1 \|\mathbf{u}_0\|_2^2},$$
$$t \in [t_0, T_*).$$

□

Now, we can deduce the contradiction by the continuity method.

Let $C_2 = 4C_0$, where C_0 is a positive constant in (2.22). We assume that there exists a maximal time $T_0 \leqq \min\{T_*, N\}$, such that for all $t \in [0, T_0)$,

$$\begin{cases} \|\Gamma\|^2_{L^{\infty,2}_t} + \|\Phi\|^2_{L^{\infty,2}_t} + \|\nabla \Gamma\|^2_{L^{2,2}_t} \leqq 2(\|\Gamma_0\|_2^2 + \|\Phi_0\|_2^2), \\ \|u^\theta\|_{L^{\infty,3}_t} \leqq \frac{1}{C_2}. \end{cases} \tag{3.3}$$

Then, from (2.22) and Hölder's inequality, we obtain that for all $t \in [0, T_0)$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\Phi\|_2^2 + \|\Gamma\|_2^2) + \|\tilde{\nabla}\Phi\|_2^2 + \|\tilde{\nabla}\Gamma\|_2^2 \\ &\leqq \frac{1}{2}(\|\tilde{\nabla}\Phi\|_2^2 + \|\tilde{\nabla}\Gamma\|_2^2) + C\|\frac{a}{r}\|^2_\infty(\|\nabla \Pi\|_2 + \|\nabla\omega^\theta\|_2 + \|\Gamma\|_2 \\ &\quad + \|\partial_r\omega^3 - \partial_3\omega^r\|_2)^2. \end{aligned}$$

Using (1.10), (2.16), (2.23) and $t < T_0 \leqq N = C\|\mathbf{u}_0\|_2^4$, we have

$$\begin{aligned} &\|\Phi\|^2_{L^{\infty,2}_t} + \|\Gamma\|^2_{L^{\infty,2}_t} + \|\tilde{\nabla}\Phi\|^2_{L^{2,2}_t} + \|\tilde{\nabla}\Gamma\|^2_{L^{2,2}_t} \\ &\leqq \|\Gamma_0\|_2^2 + \|\Phi_0\|_2^2 + C\left\|\frac{a_0}{r}\right\|^2_\infty \exp(CN^{\frac{3}{4}}(\|\Gamma_0\|_2 + \|\Phi_0\|_2))(\|(u_0^\theta)^2\|_2^2 \\ &\quad + \|\nabla \mathbf{u}_0\|_2^2 + \|\mathbf{u}_0\|_2^4(\|\Gamma_0\|_2^2 + \|\Phi_0\|_2^2)) \exp(C\|\mathbf{u}_0\|_2^3(\|\Gamma_0\|_2 + \|\Phi_0\|_2)) \\ &\leqq \|\Gamma_0\|_2^2 + \|\Phi_0\|_2^2 + C\left\|\frac{a_0}{r}\right\|^2_\infty (\|(u_0^\theta)^2\|_2^2 + \|\nabla \mathbf{u}_0\|_2^2 \\ &\quad + \|\mathbf{u}_0\|_2^4(\|\Gamma_0\|_2^2 + \|\Phi_0\|_2^2)) \exp(C\|\mathbf{u}_0\|_2^3(\|\Gamma_0\|_2 + \|\Phi_0\|_2)) \\ &\leqq \frac{3}{2}(\|\Gamma_0\|_2^2 + \|\Phi_0\|_2^2). \end{aligned}$$

Multiplying the equation $(1.5)_3$ by $(u^\theta)^2$, and using integration by parts, we have

$$\begin{aligned} &\frac{1}{3}\left\|\sqrt{\rho}(u^\theta)^{\frac{3}{2}}\right\|_2^2 + \frac{8}{9}\|\nabla(|u^\theta|^{\frac{3}{2}})\|_2^2 + \|r^{-2}(u^\theta)^3\|_1 \\ &\quad = -\int_{\mathbb{R}^3} \rho\frac{u^r}{r}\,|u^\theta|^3 \mathrm{d}x \\ &\quad \leqq \left\|\frac{u^r}{r}\right\|_\infty \left\|\sqrt{\rho}(u^\theta)^{\frac{3}{2}}\right\|_2^2. \end{aligned}$$

Then, by (2.3) and (1.11), we obtain

$$\begin{aligned} \|u^\theta\|_{L^{\infty,3}_t} &\leqq C\|u_0^\theta\|_3 \exp\left(C\left\|\frac{u^r}{r}\right\|_{L^{1,\infty}_t}\right) \\ &\leqq C\|u_0^\theta\|_3 \exp(CN^{\frac{3}{4}}(\|\Gamma_0\|_2 + \|\Phi_0\|_2)) \\ &\leqq \frac{1}{2C_2}. \end{aligned}$$

Applying the continuity method, we have the conclusion that $T_0 = \min\{T_*, N\}$, and (3.3) holds for any $t \in [0, T_0)$.

Moreover, combining (2.16) and (3.3), we obtain that for any $t \in [0, T_0)$,

$$\|\nabla \mathbf{u}\|^2_{L^{\infty,2}_t} + \|\nabla^2 \mathbf{u}\|^2_{L^{2,2}_t} + \|\nabla \Pi\|^2_{L^{2,2}_t} \leqq C\mathcal{G}, \tag{3.4}$$

where

$$\mathcal{G} = C(\|(u_0^\theta)^2\|_2^2 + \|\nabla \mathbf{u}_0\|_2^2 + \|\mathbf{u}_0\|_2^4(\|\Gamma_0\|_2^2 + \|\Phi_0\|_2^2)) \\ \exp(C\|\mathbf{u}_0\|_2^3(\|\Gamma_0\|_2^2 + \|\Phi_0\|_2^2)).$$

Recall Lemma 3.3 and the conversation law (2.9), we have that for all $t < T_*$,

$$\|\mathbf{u}\|^2_{L^\infty((0,t);H^1)} + \int_0^t \|\nabla \mathbf{u}(\tau)\|^2_{H^1} + \|\nabla \Pi(\tau)\|_2^2 \, \mathrm{d}\tau \leqq C\mathcal{G} + C\frac{1}{\|\mathbf{u}_0\|_2^2}. \tag{3.5}$$

Thanks to the calculus in [8] and the blow up criteria (See Proposition 0.6 in [6], for instance), we deduce the contradiction with the fact that T_* is the blow up time of the solution. Thus, we obtain that $T_* = \infty$, and finish the proof of the well-posedness part of Theorem 1.1. □

4. Proof of the Decay Estimates Part of Theorem 1.1

When $\mathbf{u}_0 \in L^1(\mathbb{R}^3)$, from the proof in [2] (Section 3), we can obtain the decay estimates (1.16) and omit the details.

Proof of the decay estimate (1.13).

• The decay estimate of $\|ru^\theta\|_2^2$.

From (1.5), we have

$$\rho\partial_t(ru^\theta) + \rho\mathbf{u}\cdot\nabla(ru^\theta) - \left(\Delta - \frac{2}{r}\partial_r\right)(ru^\theta) = 0. \tag{4.1}$$

Multiplying the equation by $|ru^\theta|^{p-2}ru^\theta$, $1 < p < \infty$, and using integration by parts, we get

$$\|ru^\theta(t)\|_p \leqq \|ru_0^\theta\|_p.$$

Then, we can easily obtain that

$$\|ru^\theta(t)\|_1 \leqq \|ru_0^\theta\|_1. \tag{4.2}$$

Moreover, if $\|ru_0^\theta\|_{L^1\cap L^2} \leqq C$, from (4.1), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}ru^\theta\|_2^2 + \|\nabla(ru^\theta)\|_2^2 = 0. \tag{4.3}$$

By the Sobolev embedding theorem and (4.2), we obtain

$$\begin{aligned} \|ru^\theta\|_2 &\leqq C\|ru^\theta\|_1^{\frac{2}{5}}\|\nabla(ru^\theta)\|_2^{\frac{3}{5}} \\ &\leqq C\|ru_0^\theta\|_1^{\frac{2}{5}}\|\nabla(ru^\theta)\|_2^{\frac{3}{5}} \\ &\leqq C\|\nabla(ru^\theta)\|_2^{\frac{3}{5}}. \end{aligned} \tag{4.4}$$

From (4.3)–(4.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}ru^\theta\|_2^2 \leqq -C(\|ru^\theta\|_2^2)^{\frac{5}{3}} \leqq -C(\|\sqrt{\rho}ru^\theta\|_2^2)^{\frac{5}{3}},$$

and

$$\|ru^\theta\|_2^2 \leqq C \leqq \|\sqrt{\rho}ru^\theta\|_2^2 \leqq C\langle t\rangle^{-\frac{3}{2}}. \tag{4.5}$$

• The decay estimate of $\|u^\theta(t)\|_2^2$.
Multiplying the equation $(1.5)_3$ by u^θ, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u^\theta\|_2^2 + \|\nabla u^\theta\|_2^2 + \left\|\frac{u^\theta}{r}\right\|_2^2 &= -\int_{\mathbb{R}^3}\rho\frac{u^r}{r}(u^\theta)^2\,\mathrm{d}x\\ &\leqq C\left\|\frac{u^r}{r}\right\|_2^4\|u^\theta\|_2^2 + \frac{1}{2}\|\nabla u^\theta\|_2^2. \end{aligned}\tag{4.6}$$

Applying the decay estimates (1.16), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u^\theta\|_2^2 + \left\|\frac{u^\theta}{r}\right\|_2^2 \leqq C\left\|\frac{u^r}{r}\right\|_2^4\|u^\theta\|_2^2 \leqq C\langle t\rangle^{-\frac{13}{2}}.$$

Set $S(t) = \{x|r \leqq M^{-\frac{1}{2}}g(t)^{-1}\}$, $g(t) = \sqrt{\gamma}\ \ (1+t)^{-\frac{1}{2}}$, $\gamma > \frac{5}{2}$. From (4.5), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u^\theta(t)\|_2^2 + g(t)^2\|\sqrt{\rho}u^\theta(t)\|_2^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u^\theta\|_2^2 + g(t)^2\\ &\left(\int_{S(t)}\rho|u^\theta|^2\,\mathrm{d}x + \int_{S(t)^c}\rho|u^\theta|^2\,\mathrm{d}x\right)\\ &\leqq \frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u^\theta\|_2^2 + \int_{S(t)}|\frac{u^\theta}{r}|^2\,\mathrm{d}x + g(t)^2\int_{S(t)^c}\rho|u^\theta|^2\,\mathrm{d}x\\ &\leqq C\langle t\rangle^{-\frac{13}{2}} + Mg(t)^2\int_{S(t)^c}\frac{1}{r^2}|ru^\theta|^2\,\mathrm{d}x\\ &\leqq C\langle t\rangle^{-\frac{13}{2}} + M^2g(t)^4\|ru^\theta\|_2^2\\ &\leqq C\langle t\rangle^{-\frac{7}{2}},\end{aligned}$$

and

$$e^{\int_0^t g(\tau)^2 d\tau}\|\sqrt{\rho}u^\theta(t)\|_2^2 \leqq \|\sqrt{\rho_0}u_0^\theta\|_2^2 + C\int_0^t e^{\int_0^\tau g(s)^2 ds}\langle\tau\rangle^{-\frac{7}{2}}\,d\tau.$$

Since $e^{\int_0^t g(\tau)^2 d\tau} \approx \langle t\rangle^\gamma$, $\gamma > \frac{5}{2}$, we have

$$\langle t\rangle^\gamma\|\sqrt{\rho}u^\theta(t)\|_2^2 \leqq C\|\sqrt{\rho_0}u_0^\theta\|_2^2 + C\langle t\rangle^{\gamma-\frac{5}{2}},$$

and

$$\|u^\theta(t)\|_2^2 \leqq C\|\sqrt{\rho}u^\theta(t)\|_2^2 \leqq C\langle t\rangle^{-\frac{5}{2}}. \tag{4.7}$$

• The decay estimate of $\|\nabla(u^\theta\mathbf{e}_\theta)\|_2^2$.

We notice that

$$\|\nabla(u^\theta \mathbf{e}_\theta)\|_2^2 = \|\nabla u^\theta\|_2^2 + \|\frac{u^\theta}{r}\|_2^2 = \|\omega^r\|_2^2 + \|\omega^3\|_2^2, \tag{4.8}$$

$$\Delta(u^\theta \mathbf{e}_\theta) = \left(\Delta - \frac{1}{r^2}\right) u^\theta \mathbf{e}_\theta. \tag{4.9}$$

Applying (1.7) and (2.7), we obtain, directly from $(1.5)_3$, that

$$\begin{aligned}
\left\|\left(\Delta - \frac{1}{r^2}\right) u^\theta\right\|_2 &\leqq \|\rho u_t^\theta\|_2 + \left\|\rho\left(\mathbf{u}\cdot\nabla + \frac{u^r}{r}\right) u^\theta\right\|_2 \\
&\leqq C\|\sqrt{\rho} u_t^\theta\|_2 + C\|u^r\omega^3\|_2 + C\|u^3\omega^r\|_2 \\
&\leqq C\|\sqrt{\rho} u_t^\theta\|_2 + C\|\nabla\mathbf{u}\|_2^{\frac{3}{2}}\left(\|\nabla\omega^r\|_2 + \|\nabla\omega^3\|_2\right)^{\frac{1}{2}} \\
&\leqq C\|\sqrt{\rho} u_t^\theta\|_2 + C\|\nabla\mathbf{u}\|_2^{\frac{3}{2}}\left\|\left(\Delta - \frac{1}{r^2}\right) u^\theta\right\|_2^{\frac{1}{2}} \\
&\leqq C\|\sqrt{\rho} u_t^\theta\|_2 + C\|\nabla\mathbf{u}\|_2^3 + \frac{1}{2}\left\|\left(\Delta - \frac{1}{r^2}\right) u^\theta\right\|_2.
\end{aligned} \tag{4.10}$$

Set $s = \frac{t}{2}$. From (1.16) and (4.6), applying Gronwall's inequality, we have

$$\begin{aligned}
&\|\sqrt{\rho} u^\theta(t)\|_2^2 + \int_s^t \|\nabla u^\theta(\tau)\|_2^2 + \left\|\frac{u^\theta(\tau)}{r}\right\|_2^2 \mathrm{d}\tau \\
&\leqq C\|\sqrt{\rho} u^\theta(s)\|_2^2 \exp\left(C\int_s^t \|\nabla\mathbf{u}(\tau)\|_2^4 \,\mathrm{d}\tau\right) \\
&\leqq C\|u^\theta(s)\|_2^2 \\
&\leqq C\langle t\rangle^{-\frac{5}{2}}.
\end{aligned} \tag{4.11}$$

Multiplying the equation $(1.5)_3$ by u_t^θ, and using integration by parts, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u^\theta\|_2^2 + \|\frac{u^\theta}{r}\|_2^2) + \|\sqrt{\rho} u_t^\theta\|_2^2 \\
&= -\int_{\mathbb{R}^3} \rho\left(\mathbf{u}\cdot\nabla u^\theta + \frac{u^r}{r} u^\theta\right) u_t^\theta \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \rho(u^r\omega^3 - u^3\omega^r) u_t^\theta \,\mathrm{d}x \\
&\leqq C_\delta\|\nabla\mathbf{u}\|_2^4(\|\omega^r\|_2^2 + \|\omega^3\|_2^2) + \delta(\|\nabla\omega^r\|_2^2 \\
&\quad + \|\nabla\omega^3\|_2^2 + \|\sqrt{\rho} u_t^\theta\|_2^2).
\end{aligned} \tag{4.12}$$

From (2.13), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega^r\|_2^2 + \|\omega^3\|_2^2) + \int_{\mathbb{R}^3} \frac{1}{\rho}(\partial_3\omega^r - \partial_r\omega^3)^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} (\omega^r\partial_r + \omega^3\partial_3)u^r\omega^r + (\omega^r\partial_r + \omega^3\partial_3)u^3\omega^3 \,\mathrm{d}x \\
&\leqq C_\delta\|\nabla\mathbf{u}\|_2^4(\|\omega^r\|_2^2 + \|\omega^3\|_2^2) + \delta(\|\nabla\omega^r\|_2^2 + \|\nabla\omega^3\|_2^2).
\end{aligned} \tag{4.13}$$

Let $f_1(t) = \|\omega^r(t)\|_2^2 + \|\omega^3(t)\|_2^2$. From (4.11), it satisfies that

$$\int_s^t f_1(\tau)\mathrm{d}\tau \leqq C\langle t\rangle^{-\frac{5}{2}}. \tag{4.14}$$

Combining (4.12) and (4.13), applying (2.7), (2.8) and (4.8), picking δ sufficiently small, we have

$$\frac{}{\mathrm{d}t} f_1(t) + \left\|\left(\Delta - \frac{1}{r^2}\right)u^\theta\right\|_2^2 + \|\sqrt{\rho}u_t^\theta\|_2^2 \leqq C\|\nabla\mathbf{u}\|_2^4 f_1(t). \tag{4.15}$$

Multiplying the above inequality by $(t-s)$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}((t-s)f_1(t)) \leqq f_1(t) + C\|\nabla\mathbf{u}\|_2^4(t-s)f_1(t).$$

Applying Gronwall's inequality, we obtain

$$(t-s)f_1(t) \leqq \int_s^t f_1(\tau)\mathrm{d}\tau \exp(C\int_s^t \|\nabla\mathbf{u}(\tau)\|_2^4\mathrm{d}\tau) \leqq C\int_s^t f_1(\tau)\mathrm{d}\tau.$$

Take $s = \frac{t}{2}$, from (4.14), we get

$$f_1(t) \leqq Ct^{-1}\langle t\rangle^{-\frac{5}{2}},$$

and

$$\|\nabla(u^\theta\mathbf{e}_\theta)(t)\|_2^2 = f_1(t) \leqq C\langle t\rangle^{-\frac{7}{2}}. \tag{4.16}$$

• The decay estimate of $\|u_t^\theta(t)\|_2^2 + \|(\Delta - \frac{1}{r^2})u^\theta(t)\|_2^2$.
Applying Gronwall's lemma to (4.15) over $[s,t]$, $s = \frac{t}{2}$, we have

$$\begin{aligned} f_1(t) &+ \int_s^t \left\|\left(\Delta - \frac{1}{r^2}\right)u^\theta\right\|_2^2 + \|\sqrt{\rho}u_t^\theta\|_2^2\mathrm{d}\tau \\ &\leqq f_1(s)\exp(C\int_s^t \|\nabla\mathbf{u}\|_2^4 d\tau) \leqq Cf_1(s) \leqq C\langle t\rangle^{-\frac{7}{2}}. \end{aligned} \tag{4.17}$$

Applying (1.16), we get

$$\int_s^t (\tau - s)\|\mathbf{u}_t\|_2^2 \leqq Ct\int_s^t \tau^{-1}\langle\tau\rangle^{-\frac{5}{2}}\mathrm{d}\tau \leqq C\langle t\rangle^{-\frac{3}{2}}. \tag{4.18}$$

Taking ∂_t to $(1.5)_3$, we obtain

$$\rho u_{tt}^\theta + \rho\mathbf{u}\cdot\nabla u_t^\theta - \left(\Delta - \frac{1}{r^2}\right)u_t^\theta = -\rho_t u_t^\theta - \partial_t(\rho\mathbf{u})\cdot\nabla u^\theta - \partial_t\left(\rho\frac{u^r}{r}u^\theta\right). \tag{4.19}$$

By L^2 inner product of the above equation with u_t^θ, using the transport equation $(1.5)_1$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u_t^\theta\|_2^2+\|\nabla u_t^\theta\|_2^2+\left\|\frac{u_t^\theta}{r}\right\|_2^2\\
&=\int_{\mathbb{R}^3}\mathrm{div}(\rho\mathbf{u})\left(u_t^\theta+\mathbf{u}\cdot\nabla u^\theta+u^r\frac{u^\theta}{r}\right)u_t^\theta\\
&-\rho\mathbf{u}_t\cdot\nabla u^\theta u_t^\theta-\rho\frac{u_t^r u^\theta+u^r u_t^\theta}{r}u_t^\theta\,\mathrm{d}x\\
&=\int_{\mathbb{R}^3}\mathrm{div}(\rho\mathbf{u})\left(u_t^\theta+u^3\partial_3 u^\theta+u^r\omega^3\right)u_t^\theta-\rho u_t^r\omega^3 u_t^\theta\\
&-\rho u_t^3\partial_3 u^\theta u_t^\theta-\rho\frac{u^r}{r}u_t^\theta u_t^\theta\,\mathrm{d}x.
\end{aligned}\tag{4.20}$$

Using integration by parts, Hölder's inequality, Sobolev's inequality, the Cauchy–Schwarz inequality, (2.8) and (4.10), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u_t^\theta\|_2^2+\|\nabla u_t^\theta\|_2^2+\left\|\frac{u_t^\theta}{r}\right\|_2^2\leqq C\|\nabla\mathbf{u}\|_2\|u_t^\theta\|_3\\
&\times\left(\|\nabla u_t^\theta\|_2+\left\|\frac{u_t^\theta}{r}\right\|_2\right)+C\|\nabla\mathbf{u}\|_2^2(\|\nabla\omega^r\|_2+\|\nabla\omega^3\|_2)\|\nabla u_t^\theta\|_2\\
&+C\|\mathbf{u}_t\|_2(\|\omega^r\|_3+\|\omega^3\|_3)\|\nabla u_t^\theta\|_2\\
&\leqq C\|\nabla\mathbf{u}\|_2^4\|\sqrt{\rho}u_t^\theta\|_2^2+C\|\nabla\mathbf{u}\|_2^{10}\\
&+C(\|\nabla\mathbf{u}\|_2^4+\|\nabla\mathbf{u}\|_2\|u_t^\theta\|_2)\|\mathbf{u}_t\|_2^2+\frac{1}{2}\left(\|\nabla u_t^\theta\|_2^2+\left\|\frac{u_t^\theta}{r}\right\|_2^2\right).
\end{aligned}\tag{4.21}$$

Multiplying the above inequality by $(t-s)$, and applying Gronwall's inequality on $[s,t]$, we get

$$\begin{aligned}
&(t-s)\|\sqrt{\rho}u_t^\theta(t)\|_2^2\\
&\leqq C\left(\int_s^t\|\sqrt{\rho}u_t^\theta(\tau)\|_2^2+(\tau-s)\|\nabla\mathbf{u}\|_2^{10}+(\tau-s)(\|\nabla\mathbf{u}\|_2^4\right.\\
&\left.+\|\nabla\mathbf{u}\|_2\|u_t^\theta\|_2)\|\mathbf{u}_t\|_2^2\mathrm{d}\tau\right)\exp\left(C\int_s^t\|\nabla\mathbf{u}\|_2^4\mathrm{d}\tau\right).
\end{aligned}\tag{4.22}$$

Take $s=\frac{t}{2}$, $t>1$. Applying (1.16), (4.17) and (4.18), we have

$$\begin{aligned}
t\|\sqrt{\rho}u_t^\theta(t)\|_2^2\leqq C\Big(&\int_s^t\|\sqrt{\rho}u_t^\theta(\tau)\|_2^2+(\tau-s)\|\nabla\mathbf{u}\|_2^{10}\\
&+(\tau-s)(\|\nabla\mathbf{u}\|_2^4+\|\nabla\mathbf{u}\|_2\|u_t^\theta\|_2)\|\mathbf{u}_t\|_2^2\mathrm{d}\tau\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leqq C\left(\langle t\rangle^{-\frac{7}{2}}+t\sup_{\tau\in[s,t]}\|\nabla\mathbf{u}(\tau)\|_2^8+\sup_{\tau\in[s,t]}(\|\nabla\mathbf{u}\|_2^4\right.\\
&\qquad\left.+\|\nabla\mathbf{u}\|_2\|u_t^\theta\|_2)\int_s^t(\tau-s)\|\mathbf{u}_t\|_2^2\mathrm{d}\tau\right)\\
&\leqq C\left(\langle t\rangle^{-\frac{7}{2}}+\langle t\rangle^{-9}+(\langle t\rangle^{-5}+\langle t\rangle^{-\frac{5}{4}}t^{-\frac{1}{2}}\langle t\rangle^{-\frac{5}{4}})\langle t\rangle^{-\frac{3}{2}}\right)\\
&\leqq C\langle t\rangle^{-\frac{7}{2}}. \qquad (4.23)
\end{aligned}
$$

Then, by (4.10), we get

$$
\|u_t^\theta(t)\|_2^2+\left\|\left(\Delta-\frac{1}{r^2}\right)u^\theta(t)\right\|_2^2\leqq Ct^{-1}\langle t\rangle^{-\frac{7}{2}},\quad\forall t>0. \qquad (4.24)
$$

Therefore the result (1.13) can be directly derived, and Theorem 1.1 is proved. □

5. Appendix

Proof of Lemma 2.3.
Multiplying the equation $(1.5)_3$ by $(u^\theta)^3$, using integration by parts, applying Hölder's inequality, Sobolev's inequality, the Cauchy–Schwarz inequality and (2.2), we have

$$
\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\rho}(u^\theta)^2\right\|_2^2+\frac{3}{4}\left\|\nabla(u^\theta)^2\right\|_2^2+\left\|\frac{(u^\theta)^2}{r}\right\|_2^2\\
&=-\int_{\mathbb{R}^3}\rho\frac{u^r}{r}(u^\theta)^2(u^\theta)^2\,\mathrm{d}x\\
&\leqq C\left\|\frac{u^r}{r}\right\|_{\frac{18}{5}}\|u^\theta\|_{\frac{18}{5}}^2\left\|(u^\theta)^2\right\|_6\\
&\leqq C\left\|\frac{u^r}{r}\right\|_2^{\frac{1}{3}}\left\|\nabla\frac{u^r}{r}\right\|_2^{\frac{2}{3}}\|u^\theta\|_2^{\frac{2}{3}}\|\nabla u^\theta\|_2^{\frac{4}{3}}\|\nabla(u^\theta)^2\|_2\\
&\leqq C\|u^\theta\|_2^{\frac{2}{3}}\|\varGamma\|_2^{\frac{2}{3}}\|\nabla\mathbf{u}\|_2^{\frac{5}{3}}\left\|\nabla(u^\theta)^2\right\|_2\\
&\leqq C\|\mathbf{u}\|_2^{\frac{4}{3}}\|\varGamma\|_2^{\frac{4}{3}}\|\nabla\mathbf{u}\|_2^{\frac{10}{3}}+\frac{1}{2}\left\|\nabla(u^\theta)^2\right\|_2^2.
\end{aligned}
$$

□

Proof of Lemma 2.4.
Let $r_0>0$. Applying the Gagliardo–Nirenberg–Sobolev inequality, the Cauchy–Schwarz inequality, (2.1), (2.2) and Lemma 2.1, we get

$$
\begin{aligned}
&\|u_3\partial_3u^3\|_2^2\leqq\|u^3\partial_ru^r\|_2^2+\left\|u^3\frac{u^r}{r}\right\|_2^2\\
&\quad=\|u^3\partial_ru^r(|_{r\leqq r_0}+|_{r>r_0})\|_2^2+\left\|u^3\frac{u^r}{r}\right\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
&\leqq \|u^3\left(r\partial_r\frac{u^r}{r}+\frac{u^r}{r}\right)|_{r\leqq r_0}\|_2^2+\|u^3\partial_r u^r|_{r>r_0}\|_2^2+\left\|u^3\frac{u^r}{r}\right\|_2^2\\
&\leqq r_0^2\left\|u^3\partial_r\frac{u^r}{r}\right\|_2^2+\|u^3\partial_r u^r|_{r>r_0}\|_2^2+2\left\|u^3\frac{u^r}{r}\right\|_2^2\\
&\leqq r_0^2\|u^3\|_\infty^2\left\|\partial_r\frac{u^r}{r}\right\|_2^2+\|u^3\partial_r u^r|_{r>r_0}\|_2^2+2\|u^3\|_9^2\left\|\frac{u^r}{r}\right\|_{\frac{18}{7}}^2\\
&\leqq Cr_0^2\|u^3\|_{\dot H^1}\|u^3\|_{\dot H^2}\left\|\partial_r\frac{u^r}{r}\right\|_2^2+\|u^3\partial_r u^r|_{r>r_0}\|_2^2\\
&\quad+C\|u^3\|_{\frac{18}{7}}\|\Delta u^3\|_2\left\|\frac{u^r}{r}\right\|_2^{\frac43}\left\|\nabla\frac{u^r}{r}\right\|_2^{\frac23}\\
&\leqq Cr_0^2\|\nabla u^3\|_2\|\Delta u^3\|_2\|\Gamma\|_2^2+\|u^3\partial_r u^r|_{r>r_0}\|_2^2+C\|u^3\|_2^{\frac23}\|\nabla u^3\|_2^{\frac13}(\|\nabla\omega^\theta\|_2\\
&\quad+\|\Gamma\|_2)\|\frac{u^r}{r}\|_2^{\frac43}\|\Gamma\|_2^{\frac23}\\
&\leqq Cr_0^2\|\nabla\mathbf{u}\|_2\|(\|\partial_r\omega^\theta\|_2+\|\Gamma\|_2)\|\Gamma\|_2^2+\|u^3\partial_r u^r|_{r>r_0}\|_2^2\\
&\quad+C\|\mathbf{u}\|_2^{\frac23}\|\Gamma\|_2^{\frac23}\|\nabla\mathbf{u}\|_2^{\frac53}(\|\nabla\omega^\theta\|_2+\|\Gamma\|_2),
\end{aligned}
$$

$$
\begin{aligned}
\|u^r\partial_r\tilde{\mathbf{u}}\|_2^2&\leqq\left\|r\frac{u^r}{r}\partial_r\tilde{\mathbf{u}}(|_{r\leqq r_0}+|_{r>r_0})\right\|_2^2\\
&\leqq r_0^2\left\|\frac{u^r}{r}\partial_r\tilde{\mathbf{u}}\right\|_2^2+\|u^r\partial_r\tilde{\mathbf{u}}|_{r>r_0}\|_2^2\\
&\leqq Cr_0^2\left\|\frac{u^r}{r}\right\|_6^2\|\partial_r\tilde{\mathbf{u}}\|_3^2+\|u^r\partial_r\tilde{\mathbf{u}}|_{r>r_0}\|_2^2\\
&\leqq Cr_0^2\|\Gamma\|_2^2\|\omega^\theta\|_3^2+\|u^r\partial_r\tilde{\mathbf{u}}|_{r>r_0}\|_2^2\\
&\leqq Cr_0^2\|\nabla\mathbf{u}\|_2\|\nabla\omega^\theta\|_2\|\Gamma\|_2^2+\|u^r\partial_r\tilde{\mathbf{u}}|_{r>r_0}\|_2^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\|u^3\partial_3 u^r\|_2^2&\leqq\left\|u^3r\partial_3\frac{u^r}{r}(|_{r\leqq r_0}+|_{r>r_0})\right\|_2^2\\
&\leqq r_0^2\left\|u^3\partial_3\frac{u^r}{r}\right\|_2^2+\|u^3\partial_3u^r|_{r>r_0}\|_2^2\\
&\leqq r_0^2\|u^3\|_\infty^2\|\partial_3\frac{u^r}{r}\|_2^2+\|u^3\partial_3u^r|_{r>r_0}\|_2^2\\
&\leqq Cr_0^2\|\nabla u^3\|_2\|\Delta u^3\|_2\|\Gamma\|_2^2+\|u^3\partial_3u^r|_{r>r_0}\|_2^2\\
&\leqq Cr_0^2\|\nabla\mathbf{u}\|_2\|\Gamma\|_2^2(\|\nabla\omega^\theta\|_2+\|\Gamma\|_2)+\|u^3\partial_3u^r|_{r>r_0}\|_2^2.
\end{aligned}
$$

Applying (1.7), (2.2) and Lemma 2.1, we obtain

$$
\begin{aligned}
\|\tilde\nabla\tilde\nabla\tilde{\mathbf{u}}\|_2&\leqq C\|\Delta u^3\|_2+\|\tilde\nabla\partial_3u^r\|_2+\|\tilde\nabla\partial_r u^r\|_2\\
&\leqq C\|\Delta u^3\|_2+\|\tilde\nabla(\omega^\theta+\partial_r u^3)\|_2+\left\|\tilde\nabla\left(\partial_3u^3+\frac{u^r}{r}\right)\right\|_2
\end{aligned}
$$

$$\leqq C\|\Delta u^3\|_2 + \|\tilde{\nabla}\omega^\theta\|_2 + \left\|\tilde{\nabla}\frac{u^r}{r}\right\|_2$$
$$\leqq C\|\nabla\omega^\theta\|_2 + C\|\varGamma\|_2. \tag{5.1}$$

Thus, applying Sobolev's inequality in two dimension, (2.1), (2.2), (5.1) and Lemma 2.1, we have

$$\begin{aligned}
\|\tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{\mathbf{u}}|_{r>r_0}\|_2^2 &= \int_{\mathbb{R}}\int_{r_0}^{\infty} |\tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{\mathbf{u}}|^2\, r\mathrm{d}r\mathrm{d}x_3\\
&\leqq \left(\sup_{r>r_0, x_3\in\mathbb{R}} |\tilde{\mathbf{u}}|^2\right)\int_{\mathbb{R}}\int_{r_0}^{\infty} |\tilde{\nabla}\tilde{\mathbf{u}}|^2\, r\mathrm{d}r\mathrm{d}x_3\\
&\leqq C\left(\int_{\mathbb{R}}\int_{r_0}^{\infty}|\tilde{\mathbf{u}}|^2\mathrm{d}r\mathrm{d}x_3\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\int_{r_0}^{\infty}|\tilde{\nabla}\tilde{\nabla}\tilde{\mathbf{u}}|^2\mathrm{d}r\mathrm{d}x_3\right)^{\frac{1}{2}}\|\nabla\mathbf{u}\|_2^2\\
&\leqq \frac{C}{r_0}\|\mathbf{u}\|_2(\|\varGamma\|_2 + \|\nabla\omega^\theta\|_2)\|\nabla\mathbf{u}\|_2^2.
\end{aligned} \tag{5.2}$$

Let

$$r_0 = \left(\frac{\|\mathbf{u}\|_2\|\nabla\mathbf{u}\|_2}{\|\varGamma\|_2^2}\right)^{\frac{1}{3}}. \tag{5.3}$$

Combing the above estimates, we get

$$\begin{aligned}
&\|\tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{\mathbf{u}}\|_2^2\\
&\leqq C\|\mathbf{u}\|_2^{\frac{2}{3}}\|\varGamma\|_2^{\frac{2}{3}}\|\nabla\mathbf{u}\|_2^{\frac{5}{3}}(\|\nabla\omega^\theta\|_2 + \|\varGamma\|_2)\\
&\leqq C_\delta\|\mathbf{u}\|_2^{\frac{4}{3}}\|\varGamma\|_2^{\frac{4}{3}}\|\nabla\mathbf{u}\|_2^{\frac{10}{3}} + \delta(\|\nabla\omega^\theta\|_2^2 + \|\varGamma\|_2^2).
\end{aligned}$$

□

Proof of Lemma 3.1.

It is easy to see that $m \leqq J^\varepsilon * \rho_0 \leqq M$, and

$$|(J^\varepsilon * (\rho_0 - 1))(0, x_3)| = \int J^\varepsilon(r, z)|\rho_0(r, x_3 - z) - 1|\, r\mathrm{d}r dz$$
$$\leqq C_0\int J^\varepsilon\, r\, \mathrm{d}x \leqq C\varepsilon.$$

Then $0 < \frac{m}{2} \leqq \rho_0^\varepsilon \leqq \frac{M}{2}$ when ε is sufficient small.

Set $r = \sqrt{x_1^2 + x_2^2}$, $s = \sqrt{y_1^2 + y_2^2}$ and $C_0 = \|\frac{\rho_0 - 1}{r}\|_\infty$. When $r \leqq 5\varepsilon$, it can be deduced directly,

$$\begin{aligned}
|\rho_0^\varepsilon(x) - 1| &= |J^\varepsilon * (\rho_0 - 1) - (J^\varepsilon * (\rho_0 - 1))(0, x_3)|\\
&\leqq \left|\int (J^\varepsilon(x - y) - J^\varepsilon(s, x_3 - y_3))(\rho_0(y) - 1)\, \mathrm{d}y\right|
\end{aligned}$$

$$\leqq C_0 \int |J^\varepsilon(\sqrt{r^2+s^2-2rs\cos(\tau)}, x_3-y_3)$$
$$-J^\varepsilon(s, x_3-y_3)|s^2 \mathrm{d}s\mathrm{d}\tau\mathrm{d}y_3$$
$$\leqq C_0 \int_{s\leqq 6\varepsilon, |x_3-y_3|\leqq\varepsilon} |\sqrt{r^2+s^2-2rs\cos(\tau)}$$
$$-s||\nabla J^\varepsilon|_\infty s^2 \mathrm{d}s\mathrm{d}\tau\mathrm{d}y_3$$
$$\leqq C_0 C r \int_{s\leqq 6\varepsilon, |x_3-y_3|\leqq\varepsilon} \varepsilon^{-4} s\mathrm{d}y$$
$$\leqq C_0 C r.$$

When $r > 5\varepsilon$, we have that $s \in [\frac{r}{2}, 2r]$, if $(x-y) \in \mathrm{supp} J^\varepsilon$, and

$$|\rho_0^\varepsilon(x)-1| = |J^\varepsilon * (\rho_0-1) - (J^\varepsilon * (\rho_0-1))(0,x_3)|$$
$$\leqq C_0 \int J^\varepsilon(x-y) s dy + C_0 \int J^\varepsilon(s, x_3-y_3) s dy$$
$$\leqq 2C_0 r + C_0\varepsilon$$
$$\leqq C C_0 r.$$

□

Acknowledgments. This work is partially supported by NSF of China under Grants 11271322, 11331005 and 11271017, National Program for Special Support of Top-Notch Young Professionals. The authors would like to thank the anonymous referees for their carefully reading the manuscript and giving many important comments and suggestions which significantly improve the paper.

References

1. ABIDI H., GUI G.L., Zhang P.: On the wellposedness of three-dimensional inhomogeneous Navier–Stokes equations in the critical spaces. *Arch. Rational Mech. Anal.* **204**(1), 189–230 (2012). doi:10.1007/s00205-011-0473-4
2. ABIDI H., ZHANG P.: Global smooth axisymmetric solutions of 3-D inhomogeneous incompressible Navier–Stokes system. *Calc. Var. Partial Differ. Equ.* **54**(3), 3251–3276 (2015). doi:10.1007/s00526-015-0902-6
3. CHAE D., LEE J.: On the regularity of the axisymmetric solutions of the Navier–Stokes equations. MATH. Z. **239**(4), 645–671 (2002). doi:10.1007/s002090100317
4. CHEN H., FANG D.Y., Zhang T.: Regularity of 3D axisymmetric Navier–Stokes equations. arXiv:1505.00905
5. DANCHIN R.: Local and global well-posedness results for flows of inhomogeneous viscous fluids. *Adv. Differential Equations* **9**(3–4), 353–386 (2004).
6. DANCHIN R.: The inviscid limit for density-dependent incompressible fluids. *Ann. Fac. Sci. Toulouse Math. (6)* **15**(4), 637–688 (2006). doi:10.5802/afst.1133
7. DANCHIN R.: A Lagrangian approach for the incompressible Navier–Stokes equations with variable density. *Commun. Pure Appl. Math.* **65**(10), 1458–1480 (2012). doi:10.1002/cpa.21409
8. KIM H.: A blow-up criterion for the nonhomogeneous incompressible Navier–Stokes equations. *SIAM J. Math. Anal.* **37**(5), 1417–1434 (2006). doi:10.1137/S0036141004442197

9. Ladyženskaja O.A.: Unique global solvability of the three-dimensional Cauchy problem for the Navier–Stokes equations in the presence of axial symmetry. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **7**, 155–177 (1968).
10. Ladyženskaja O.A., Solonnikov V.A.: The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids. Boundary value problems of mathematical physics, and related questions of the theory of functions, 8. Zap. Naučn. *Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **52**, 52–109, 218–219 (1975).
11. Lei Z., Zhang Qi S.: Criticality of the axially symmetric Navier–Stokes equations. arXiv:1505.02628
12. Leonardi S., Málek J., Nečas J., Pokorný M.: On axially symmetric flows in $\mathbb{R}^3$. *Z. Anal. Anwendungen* **18**(3), 639–649 (1999). doi:10.4171/ZAA/903
13. Lions P.L.: *Mathematical topics in fluid mechanics*. Incompressible models, vol. 1. The Clarendon Press, Oxford University Press, New York (1996)
14. Liu J.G., Wang W.C.: Energy and helicity preserving schemes for hydro- and magnetohydro-dynamics flows with symmetry. *J. Comput. Phys.* **200**(1), 8–33 (2004). doi:10.1016/j.jcp.2004.03.005
15. Liu J.G., Wang W.C.: Convergence analysis of the energy and helicity preserving scheme for axisymmetric flows. *SIAM J. Numer. Anal.* **44**(6), 2456–2480 (2006). doi:10.1137/050639314
16. Liu J.G., Wang W.C.: Characterization and regularity for axisymmetric solenoidal vector fields with application to Navier–Stokes equation. *SIAM J. Math. Anal.* **41**(5), 1825–1850 (2009). doi:10.1137/080739744
17. Simon J.: Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure. *SIAM J. Math. Anal.* **21**(5), 1093–1117 (1990). doi:10.1137/0521061
18. Paicu M., Zhang P., Zhang Z.F.: Global unique solvability of inhomogeneous Navier–Stokes equations with bounded density. *Comm. Partial Differential Equations* **38**(7), 1208–1234 (2013). doi:10.1080/03605302.2013.780079
19. Ukhovskii M.R., Iudovich V.I.: Axially symmetric flows of ideal and viscous fluids filling the whole space. *J. Appl. Math. Mech.* **32**(1), 52–61 (1968). doi:10.1016/0021-8928(68)90147-0
20. Wei D.Y.: Regularity criterion to the axially symmetric Navier–Stokes equations. *J. Math. Anal. Appl.,* **435**(1), 402–413 (2016). doi:10.1016/j.jmaa.2015.09.088
21. Zhang P., Zhang T.: Global axisymmetric solutions to three-dimensional Navier–Stokes system. *Int. Math. Res. Not.* **2014**(3), 610–642 (2014)

H. Chen, D. Fang and T. Zhang
Department of Mathematics,
Zhejiang University,
Hangzhou 310027
China.

H. Chen
e-mail: poorpaul@zju.edu.cn

D. Fang
e-mail: dyf@zju.edu.cn

T. Zhang
e-mail: zhangting79@zju.edu.cn
(Corresponding author)

(*Received September 14, 2015 / Accepted September 27, 2016*)
Published online October 17, 2016 –