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The Initial Boundary Value Problem for the Boltzmann Equation with Soft Potential

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Abstract

Boundary effects are central to the dynamics of the dilute particles governed by the Boltzmann equation. In this paper, we study both the diffuse reflection and the specular reflection boundary value problems for the Boltzmann equation with a soft potential, in which the collision kernel is ruled by the inverse power law. For the diffuse reflection boundary condition, based on an L^2 argument and its interplay with intricate L^∞ analysis for the linearized Boltzmann equation, we first establish the global existence and then obtain the exponential decay in L^∞ space for the nonlinear Boltzmann equation in general classes of bounded domain. It turns out that the zero lower bound of the collision frequency and the singularity of the collision kernel lead to some new difficulties for achieving the a priori L^∞ estimates and time decay rates of the solution. In the course of the proof, we capture some new properties of the probability integrals along the stochastic cycles and improve the $L^2 - L^\infty$ theory to give a more direct approach to overcome those difficulties. As to the specular reflection condition, our key contribution is to develop a new time-velocity weighted L^∞ theory so that we could deal with the greater difficulties stemming from the complicated velocity relations among the specular cycles and the zero lower bound of the collision frequency. From this new point, we are also able to prove that the solutions of the linearized Boltzmann equation tend to equilibrium exponentially in L^∞ space with the aid of the L^2 theory and a bootstrap argument. These methods, in the latter case, can be applied to the Boltzmann equation with soft potential for all other types of boundary condition.

Contents

1. Introduction

1.1. The Problem and Background

Boundary effects should been taken into account when we study the dynamics of rarefied gas governed by the Boltzmann equation in a bounded domain. There are several standard classes of boundary conditions for the Boltzmann equation, cf. [27, pp. 716]. In this paper, we consider the the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad (x, v) \in \Omega \times \mathbb{R}^3, \ t > 0, \tag{1.1}$$

with initial data

$$F(0, x, v) = F_0(x, v), \quad (x, v) \in \Omega \times \mathbb{R}^3, \tag{1.2}$$

and either of the following boundary conditions:

- The diffuse reflection boundary condition

$$F(t, x, v)|_{n(x)\cdot v<0} = \mu(v) \int_{n(x)\cdot v'>0} F(t, x, v')(n(x)\cdot v')\mathrm{d}v', \ x \in \partial\Omega, \ t \geq 0; \tag{1.3}$$

- The specular reflection boundary condition

$$F(t, x, v)|_{n(x)\cdot v<0} = F(t, x, R_x v), \ R_x v = v - 2(v\cdot n(x))n(x), \ x \in \partial\Omega, \ t \geq 0. \tag{1.4}$$

Here, $F(t, x, v) \geq 0$ denotes the density distribution function of the gas particles at time $t \geq 0$, position $x \in \Omega$, and velocity $v \in \mathbb{R}^3$, Ω is a bounded domain in $\mathbb{R}^3$, $n(x)$ is the outward pointing unit norm vector at boundary $x \in \partial\Omega$ and $\mu(v)$ stands for the global Maxwellian which is normalized as

$$\mu(v) = \frac{1}{2\pi} e^{-\frac{|v|^2}{2}},$$

so that

$$\int_{n(x)\cdot v>0} \mu(v)(n(x) \cdot v)\mathrm{d}v = 1. \tag{1.5}$$

Let (u, v) and (u', v') be the velocities of the particles before and after the collision, which satisfy

$$\begin{cases} v' = v + [(u-v)\cdot\omega]\omega, \quad u' = u - [(u-v)\cdot\omega]\omega, \\ |u|^2 + |v|^2 = |u'|^2 + |v'|^2. \end{cases} \tag{1.6}$$

The Boltzmann collision operator $Q(\cdot,\cdot)$ is given as the following non-symmetric form:

$$\begin{aligned} Q(F_1, F_2) &= \int_{\mathbb{R}^3\times\mathbb{S}^2} |u-v|^{\varrho} b_0(\theta)[F_1(u')F_2(v') - F_1(u)F_2(v)]\mathrm{d}u\mathrm{d}\omega \\ &= Q^{\text{gain}}(F_1, F_2) - Q^{\text{loss}}(F_1, F_2), \end{aligned}$$

where the exponent is $\varrho = 1-\frac{4}{s}$ with inverse power $1 < s < 4$ and $\cos\theta = \omega\cdot\frac{u-v}{|u-v|}$. Through the paper, we assume

$$-3 < \varrho < 0, \ 0 < b_0(\theta) \leq C\cos\theta, \tag{1.7}$$

which are so-called soft potentials with Grad's angular cutoff. Traditionally, one labels things as hard potentials for the case when $\varrho \in (0, 1]$, Maxwellian molecules for the case when $\varrho = 0$, and the soft potentials for the case when $\varrho \in (-3, 0)$.

The boundary condition (1.3) says the incoming particles are a probability average of the outgoing particles, while boundary condition (1.4) reveals that the gas particles elastically collide against the wall like billiard balls.

The boundary effects in kinetic equations are fundamental to the dynamics of gas; for instance, the phenomena of slip boundary layer, thermal creep, curvature effects, and singularity of propagation due to the boundary [40] can be understood only with knowledge of the interaction mechanism of the particles with the boundary. Owing to the importance of the boundary effects, there have been many achievements in the mathematical study of different aspects of Boltzmann boundary value problems, see [4–6,12,19,20,32,33,37,38] and references therein. In what follows, we mention some works related to the current study of this paper. Hamdache [30] constructed the global renormalization solution to the Boltzmann equation in the case of a hard potential with an isothermal Maxwell boundary condition which in fact extends the pioneering work [9] for the Cauchy problem to the initial boundary value problem. Later on, Arkeryd and Cercinagani [1] generalized the results in [30] to more extensive situations including the case when the boundaries are not isothermal and the velocity is bounded. Arkeryd and Maslova [2] then removed the restriction on the bounded velocity introduced in [1] to study the similar issue for the Boltzmann equation and the BGK model. Except for the topic concerning the existence of the weak solution to the Botlzmann equation with initial boundary value problem mentioned above, another interesting problem is to prove the existence and uniqueness of the solution, as well as their time decay toward an absolute Maxwllian, at the appearance of compatible physical boundary conditions in a general domain, cf. [22,23]. Compared with the study for the Cauchy problem in the whole space, to our best knowledge, there are much less rigorous mathematical results of uniqueness, regularity or time-decay for the Boltzmann solutions toward a Maxwellian in a bounded domain. Although it was announced in [39] that the solutions to the Boltzmann equation near a Maxwellian would tend exponentially to the same equilibrium in a smooth bounded convex domain with specular reflection boundary condition, there is no complete rigorous proof. UKAI [44] made a rough outline for proving the existence and time convergence to a global Maxwellian for the initial boundary value problem with a hard potential. GOLSE ET AL. [21] investigated the boundary layer of stationary Boltzmann equations in one spatial dimension with a specular reflection boundary condition in the case of the hard spheres model ($\varrho = 1$). LIU and YU [35,36] studied the stationary boundary layers and the propagating fluid waves of the initial boundary value problem for the Botlzmann equation in half space by means of Green's function, introduced in [34]. Based on an elementary energy method, YANG and ZHAO [48] proved the stability of the rarefaction waves for the one dimensional Boltzmann equation in half space with a specular reflection boundary condition. Under the assumption that a priori strong Sobolev estimates can be verified, DESVILLETTES and VILLANI [7,8,47] recently established an almost exponential decay rate for Boltzmann solutions with large amplitude for general collision kernels and general boundary conditions. It should be pointed out that many of the natural physical boundary conditions create singularities in general domains [31], for which the Sobolev estimates break down in the crucial elliptic estimates for the macroscopic part [25,26]. A new $L^2 - L^\infty$ theory was developed in [27] to obtain the global existence and the exponential decay rates of the solution around a global Maxwellian in the case of hard potentials for four basic types of boundary conditions: in flow, bounce back reflection, specular reflection and diffuse reflection; we refer to [3,16,17] for the latest advancement on this topic. Different $L^2 - L^\infty$ methods have also been used in [16,45]. Thanks to the work of [27], the regularity [28,29] and hydrodynamic limits [18] for the Boltzmann equation in general classes of a bounded domain were further pondered. All of these works are focused on the case of the hard potential. A natural challenge is to extend the $L^2 - L^\infty$ analysis developed in [27] to the case of the soft potential. This is the goal of the present paper. Namely, we will investigate the global existence and the large time behaviors of the initial boundary value problem of (1.1), (1.2), (1.3) or (1.4) with the condition (1.7).

1.2. Domain, Characteristics and Perturbation

Throughout this paper, Ω is a connected and bounded domain in $\mathbb{R}^3$ and defined by the open set $\{x \mid \xi(x) < 0\}$ with $\xi(x)$ being a smooth function. Let $\nabla\xi(x) \neq 0$ at boundary $\xi(x) = 0$. The outward pointing unit normal vector at every point $x \in \partial\Omega$ is given by

$$n(x) = \frac{\nabla\xi(x)}{|\nabla\xi(x)|}.$$

We say Ω is strictly convex if there exists $c_\xi > 0$, for any $\zeta = (\zeta^1, \zeta^2, \zeta^3) \in \mathbb{R}^3$, that satisfies

$$\partial_{ij}\xi(x)\zeta^i\zeta^j \geq c_\xi|\zeta|^2. \tag{1.8}$$

We say that Ω has a rotational symmetry if there are vectors x_0 and ϖ such that, for all $x \in \partial\Omega$,

$$\{(x - x_0) \times \varpi\} \cdot n(x) \equiv 0. \tag{1.9}$$

For convenience, the phase boundary in the phase space $\Omega \times \mathbb{R}^3$ is denoted by $\gamma = \partial\Omega \times \mathbb{R}^3$, and we further split it into the following three kinds:

$$\begin{aligned}
\text{outgoing boundary} : \gamma_+ &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 \ : \ n(x) \cdot v > 0\},\\
\text{incoming boundary} : \gamma_- &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 \ : \ n(x) \cdot v < 0\},\\
\text{grazing boundary} : \gamma_0 &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 \ : \ n(x) \cdot v = 0\}.
\end{aligned}$$

As is shown in [27, pp. 715], the *backward exit time* which plays a crucial role in the study of the boundary value problem of the Botlzmann equation can be well-defined via the backward characteristic trajectory. Given (t, x, v), we let $[X(s), V(s)]$ satisfy

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s} = V(s), \quad \frac{\mathrm{d}V(s)}{\mathrm{d}s} = 0, \tag{1.10}$$

with the initial data $[X(t; t, x, v), V(t; t, x, v)] = [x, v]$. Then

$$[X(s; t, x, v), V(s; t, x, v)] = [x - (t - s)v, v] = [X(s), V(s)],$$

which is called the backward characteristic trajectory for the Boltzmann equation (1.1).

For $(x, v) \in \Omega \times \mathbb{R}^3$, the *backward exit time* $t_{\mathbf{b}}(x, v) > 0$ is defined as the first moment at which the backward characteristic line $[X(s; 0, x, v), V(s; 0, x, v)]$ emerges from $\partial\Omega$:

$$t_{\mathbf{b}}(x, v) = \inf\{\, t > 0 : x - tv \notin \partial\Omega\},$$

and we also define $x_{\mathbf{b}}(x, v) = x - t_{\mathbf{b}}(x, v)v \in \partial\Omega$. Note that for any (x, v), we use $t_{\mathbf{b}}(x, v)$ whenever it is well-defined.

Set the perturbation in a standard way $F = \mu + \sqrt{\mu} f$; the initial boundary value problem (1.1), (1.2), (1.3) and (1.4) can be reformulated as

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f), \tag{1.11}$$

$$f(0, x, v) = f_0(x, v), \tag{1.12}$$

with the boundary conditions

$$f(t, x, v)|_{\gamma_-} = \sqrt{\mu} \int_{n(x)\cdot v' > 0} f(t, x, v')\sqrt{\mu(v')} n(x) \cdot v' \mathrm{d}v', \tag{1.13}$$

and

$$f(t, x, v)|_{\gamma_-} = f(t, x, R_x v), \tag{1.14}$$

respectively. The nonlinear operator $\Gamma(\cdot, \cdot)$ and linear operator L in (1.11) are defined as

$$\Gamma(f_1, f_2) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_2),$$

and

$$Lf = -\frac{1}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)\}, \tag{1.15}$$

respectively. L can be further split into $L = \nu - K$ with K a suitable integral kernel defined by (2.1) in Section 2, and the collision frequency $\nu(v) \equiv \int_{\mathbb{R}^3\times\mathbb{S}^2} b_0(\theta)|u - v|^\varrho \mu(u)\mathrm{d}u\mathrm{d}\omega$ for $-3 < \varrho < 0$, moreover there exists a constant $C_\varrho > 0$ such that

$$\frac{1}{C_\varrho}\{1 + |v|^2\}^{\varrho/2} \le \nu(v) \le C_\varrho\{1 + |v|^2\}^{\varrho/2}. \tag{1.16}$$

Under the conditions (1.13) or (1.14), it is straightforward to check that

$$\int_{\gamma_+} f(t, x, v)\sqrt{\mu(v)}|n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v = \int_{\gamma_-} f(t, x, v)\sqrt{\mu(v)}|n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v,$$

where $\mathrm{d}S_x$ is the surface element.

Hence, in terms of perturbation $f(t, x, v)$, the mass conservation

$$\int_{\Omega\times\mathbb{R}^3} f(t, x, v)\sqrt{\mu(v)}\mathrm{d}x\mathrm{d}v = 0 \tag{1.17}$$

holds true for either of the boundary conditions (1.13) or (1.14) by further assuming that, initially, (1.1) has the same mass as the Maxwellian μ.

For the specular reflection condition (1.14), in addition to the mass conservation (1.17), the energy conservation law also holds for $t \ge 0$, that is

$$\int_{\Omega\times\mathbb{R}^3} |v|^2 f(t, x, v)\sqrt{\mu(v)}\mathrm{d}x\mathrm{d}v = 0. \tag{1.18}$$

Moreover, if the domain Ω has any axis of rotation symmetry (1.9), then we further assume that the corresponding conservation of angular momentum is valid for all $t \ge 0$:

$$\int_{\Omega\times\mathbb{R}^3} \{(x - x_0)\times\varpi\}\cdot v f(t, x, v)\sqrt{\mu}\mathrm{d}x\mathrm{d}v = 0. \tag{1.19}$$

1.3. Main Results

We introduce a weight function

$$w_{q,\theta,\vartheta} = \exp\left\{\frac{q|v|^\theta}{8} + \frac{q|v|^\theta}{8(1+t)^\vartheta}\right\},\ (q,\theta)\in\mathcal{A}_{q,\theta},\ 0\le\vartheta < -\frac{\theta}{\varrho}, \tag{1.20}$$

where

$$\mathcal{A}_{q,\theta} = \{(q,\theta)|q > 0, \text{if } 0 < \theta < 2, \quad \text{and} \quad 0 < q < 1,\ \text{if } \theta = 2\}.$$

For the sake of simplicity, we denote $w_{q,\theta,0} = w_{q,\theta} = \exp\left(\frac{q|v|^\theta}{4}\right)$ throughout the paper.

We now state our main results as follows.

Theorem 1.1. *Let $-3 < \varrho < 0$ and $(q, \theta) \in \mathcal{A}_{q,\theta}$. Assume the mass conservation (1.17) holds for $f_0(x, v)$. Then there exists a small constant $\varepsilon_0 > 0$ such that if $F_0(x, v) = \mu + \sqrt{\mu} f_0(x, v) \geq 0$ and $\|w_{q,\theta} f_0\|_\infty \leq \varepsilon_0$, there exists a unique solution $F(t, x, v) = \mu + \sqrt{\mu} f(t, x, v) \geq 0$ for the Boltzmann equation (1.1) and (1.2) with the diffuse reflection boundary condition (1.3). Moreover, there is some $C > 0$ such that*

$$\sup_{0 \leq t \leq +\infty} \|w_{q,\theta} f(t)\|_\infty \leq C \|w_{q,\theta} f_0\|_\infty. \tag{1.21}$$

Furthermore, we assume Ω is strictly convex and $f_0(x, v)$ is continuous away from the set γ_0 and

$$f_0(x, v)|_{\gamma_-} = \sqrt{\mu} \int_{n(x)\cdot v' > 0} f_0(x, v') \sqrt{\mu(v')} n(x) \cdot v' \mathrm{d}v'.$$

Then, $f(t, x, v)$ is continuous in $[0, +\infty) \times \{\overline{\Omega} \times \mathbb{R}^3 \backslash \gamma_0\}$. Moreover, let $\rho_0 = \frac{\theta}{\theta - \varrho}$, then there exist $C > 0$ and $\lambda_0 > 0$ independent of t such that

$$\|f(t)\|_\infty \leq C e^{-\lambda_0 t^{\rho_0}} \|w_{q,\theta} f_0\|_\infty. \tag{1.22}$$

Theorem 1.2. *Let $0 < \vartheta < -\frac{\theta}{\varrho}$ with $-3 < \varrho < 0$ and $(q, \theta) \in \mathcal{A}_{q,\theta}$. Assume that ξ is both strictly convex (1.8) and analytic, and the mass (1.17) and energy (1.18) are conserved for f_0. In the case that Ω has any rotational symmetry (1.9), we further require the corresponding angular momentum (1.19) is conserved for f_0. Then there exists $\varepsilon_0 > 0$ such that if $F_0(x, v) = \mu + \sqrt{\mu} f_0(x, v) \geq 0$ and $\|w_{q,\theta,\vartheta} f_0\|_\infty \leq \varepsilon_0$, there exists a unique solution $F(t, x, v) = \mu + \sqrt{\mu} f(t, x, v) \geq 0$ to the Boltzmann equation (1.1) and (1.2) with the specular reflection boundary condition (1.4). Moreover, let $\rho_1 = \frac{\theta + \vartheta\varrho}{\theta - \varrho}$, then there exist $\lambda_0 > 0$ and $C > 0$ such that*

$$\|w_{q,\theta,\vartheta} f(t)\|_\infty \leq C e^{-\lambda_0 t^{\rho_1}} \|w_{q,\theta,\vartheta} f_0\|_\infty.$$

Furthermore, if $f_0(x, v)$ is continuous except on the set γ_0 and

$$f_0(x, v) = f_0(x, R_x v) \text{ on } \partial\Omega,$$

then $f(t, x, v)$ is continuous in $[0, \infty) \times \{\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0\}$.

Remark 1.1. It should be pointed out that the method developed in Theorem 1.2 can be applied to verify Theorem 1.1, and it can also be used to handle the other two kinds of boundary conditions: in flow and bounce back reflection. Moreover, one can see that the approach developed in the proof of Theorem 1.1 is more direct and constructive while the method used in the proof of Theorem 1.2 is simpler, though both of them have their merits. In addition, it is straightforward to know that the decay exponents satisfy $\rho_0 = \lim_{\vartheta \to 0^+} \rho_1$. It is quite interesting to improve ρ_1 to ρ_0, which coincides with the decay rate for the periodic boundary condition [43].

Let us now give some comments on the difficulties associated with Theorems 1.1 and 1.2. Compared with previous works such as [3,17,27,49], a remarkable feature of our problems is that the collision frequency ν has no positive lower bound, so that the Boltzmann solution could not be expected to decay exponentially in L^∞ immediately. However, the decay rate plays a key role in establishing the global existence of the Boltzmann equation in the bounded domain, see Lemma 19 in [27, pp. 761] and also [44, pp. 81]. This time decay rate is essentially applied to eliminate the possible growth created by the k-times bounce-back reflection (k is large). Our strategy for overcoming this difficulty starts with consideration of the diffuse reflection boundary condition, where one needs some careful estimates on the integrals along the stochastic cycles so as to obtain the global existence by using only the L^2 decay. One of the key points in this paper is to develop a direct and unified approach to establishing the global existence of the linearized Boltzmann equation with the diffuse reflection boundary condition. More specifically, instead of applying the time decay in L^∞ to obtain the global existence cf. [3,17,27], we first construct a local solution via an iteration method, then directly deduce the *a priori* estimate, which is uniform in time by means of a refined estimates on the integrals defined on the stochastic cycles and the L^2 time decay for linearized equation. Finally, the global solution is obtained with the aid of the standard continuity argument. Among these steps, the main one is to establish the following type of uniform estimates:

$$
\begin{aligned}
&\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}\iint \mathrm{d}v'\mathrm{d}v''\int_{\prod_{j=1}^{k-1}\mathcal{V}'_j}\frac{e^{-\nu(v')(s-t'_1)}}{\tilde{w}_{q,\theta}(v')}\\
&\quad\times\sum_{l'=1}^{k-1}\mathbf{1}_{\{t'_{l'+1}>0\}}\left\{\int_{t'_{l'+1}}^{t'_{l'}-\frac{1}{k^2(s)}}+\int_{t'_{l'}-\frac{1}{k^2(s)}}^{t'_{l'}}\right\}\mathbf{k}_w^\chi(v_l,v')\mathbf{k}_w^\chi(v'_{l'},v'')\\
&\quad\times|h^j(s_1,x'_{l'}+(s_1-t'_{l'})v'_{l'},v'')|\mathrm{d}\Sigma_{l'}^w(s_1)\mathrm{d}s_1\mathrm{d}\Sigma_l^w(s)\mathrm{d}s\\
&\le C_{q,\theta}\left(\frac{1}{T_0^{5/4}}+\frac{1}{N}\right)\sup_{0\le s\le t_1}\|h^j(s)\|_\infty+C_N\sup_{0\le s\le t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\left\|\frac{h^j(s)}{w_{q,\theta}(v)}\right\|_2\right\},
\end{aligned}
\tag{1.23}
$$

where $k(s)=k=C'_1[\alpha(s)]^{5/4}\ge C'_1T_0^{5/4}$ and $C'_1>0$ is a constant. To derive (1.23), the following key observation is used:

$$
\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}e^{\nu(v_m)(s-t_1)}\nu(v_m)\mathrm{d}s\le\int_{t_k}^{t_1}e^{\nu(v_m)(s-t_1)}\nu(v_m)\mathrm{d}s\le C,
$$

where v_m is defined to satisfy $|v_m|=\max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}$ for $\max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}\le k$.

In addition, a delicate Banach space $\mathbf{X}_\delta(t)$ is designed to capture the properties of the solution in $L^\infty\cap L^2$ space so that the global existence and the exponential decay in the L^2-norm can be simultaneously obtained. The rapid time decay $e^{-\frac{\lambda_0}{2}s^{\rho_0}}$ in

L^2-norm is adopted to control the Jacobian determinate when we convert the $L^\infty-$norm to the L^2-norm.

It is also interesting to note that estimate (1.22) is a consequence of the interpolation technique based on the L^2 energy estimate and the weighted L^∞ estimate for the global solution as well as Young's inequality

$$e^{-\nu(v)t} w^{-1}_{q/2,\theta}(v) \le e^{-\lambda_0 t^{\rho_0}}, \quad \rho_0 = \frac{\theta}{\theta - \varrho}, \tag{1.24}$$

which means that one has to trade between the exponential decay rates and the additional exponential momentum weight on the solution itself in order to obtain the rapid time decay rates. This also reveals that the additional velocity weight imposed on the initial data in (1.22) is seen as a compensation for the exponential decay rates.

As to the specular reflection boundary condition, we cannot expect to obtain an estimate similar to (1.23). There are two mathematical difficulties: one concerns the times of the bounce back reflections k and k'; in this situation both grow exponentially in time according to Velocity Lemma 2.5, hence the summation of the integral is out of control. The other is that it is impractical to compute the Jacobian determinate

$$\det\left(\frac{\partial\{x'_{k'} + (s_1 - t'_{k'})v'_{k'}\}}{\partial v'}\right),$$

which depends on t, x, v, k and k'. In this sense, the method developed in the case of the diffuse reflection boundary condition cannot be applied to the case of the specular reflection boundary condition. Precisely speaking, one cannot first obtain the global existence of (1.11), (1.12) and (1.14) in some higher weighted L^∞ space and therefore one is not able to deduce the time decay rates in the lower weighted L^∞ space. As a consequence, we are forced to resort to the same bootstrap argument as that of [27, Lemma 19, pp. 761]. As mentioned before, to apply the bootstrap argument, the key point is to obtain the rapid time decay rates without any growth. Nevertheless, it seems impossible to achieve this due to the zero lower bound of the collision frequency. To deal with this difficulty, we introduce a time-velocity weight

$$w_{q,\theta,\vartheta} = \exp\left\{\frac{q|v|^\theta}{8} + \frac{q|v|^\theta}{8(1+t)^\vartheta}\right\},$$

which has been used in [11,13–15], to handle the non-hard sphere Boltzmann equations with self-consistent forces; by using this weight, we are able to deduce a time-dependent lower bound for a revised collision frequency, say,

$$\widetilde{\nu}(v,t) = \nu + \frac{\vartheta q|v|^\theta}{8(1+t)^{\vartheta+1}} \ge C_{\varrho,q,\vartheta}(1+t)^{\frac{(1+\vartheta)\varrho}{\theta-\varrho}}.$$

Thus, the desired time decay rates will be naturally obtained. This is another key contribution of the present paper.

Due to the singularity of the collision kernel, the integral operator K raises another difficulty when we carry out L^∞ estimates for the linearized equation. Similar to the study of the Cauchy problem of the Boltzmann equation on a torus [24,41,43], we introduce a cutoff function χ to split $K = K^\chi + K^{1-\chi}$. With this decomposition, we only need to iterate K^χ twice [46] to obtain the desired estimates, since $K^{1-\chi}$ is small and can be controlled directly.

The estimates of the nonlinear operator $\Gamma(\cdot,\cdot)$, in terms of the exponential weighted norm $\|w_{q,\theta,\vartheta}\cdot\|_\infty$, are subtle. To avoid additional weight, we estimate $w_{q/2,\theta,\vartheta}(v)$ as

$$w_{q/2,\theta,\vartheta}(v) \le \frac{1}{2}(w_{q,\theta,\vartheta}(v') + w_{q,\theta,\vartheta}(u')),$$

instead of $w_{q/2,\theta,\vartheta}(v) \le w_{q/2,\theta,\vartheta}(v')w_{q/2,\theta,\vartheta}(u')$.

The organization of the paper is as follows. In Section 2, we collect some significant estimates for later use. Section 3 is devoted to the study of the Boltzmann equation with a diffuse reflection boundary condition. The global existence and exponential time decay for the Boltzmann equation with a specular reflection boundary condition are presented in Section 4.

1.4. Notations and Norms

We now list some notations and norms used in the paper:

- Throughout this paper, C denotes some generic positive (generally large) constant, and $\lambda, \lambda_1, \lambda_2$, as well as λ_0, denote some generic positive (generally small) constants, where $C, \lambda, \lambda_1, \lambda_2$ and λ_0 may take different values in different places. $D \lesssim E$ means that there is a generic constant $C > 0$ such that $D \le CE$. $D \sim E$ means $D \lesssim E$ and $E \lesssim D$.
- Letting $1 \le p \le \infty$, we denote $\|\cdot\|_p$ as the $L^p(\Omega\times\mathbb{R}^3)-$norm or the $L^p(\Omega)-$norm or $L^p(\Omega\cup\gamma)-$norm, while $|\cdot|_\infty$ is either the $L^\infty(\partial\Omega\times\mathbb{R}^3)-$norm or the $L^\infty(\partial\Omega)-$norm at the boundary. Moreover, we denote that $\|\cdot\|_\nu \equiv \|\nu^{1/2}\cdot\|_2$, and $(\cdot,\cdot)$ denotes the L^2 inner product in $\Omega\times\mathbb{R}^3$ with the L^2 norm $\|\cdot\|_2$.
- As to the phase boundary integration, we denote that $\mathrm{d}\gamma = |n(x)\cdot v|\mathrm{d}S(x)\mathrm{d}v$, where $\mathrm{d}S(x)$ is the surface element, and for $1 \le p < +\infty$, we define $|f|_p^p = \int_\gamma |f(x,v)|^p\mathrm{d}\gamma \equiv \int_\gamma |f(x,v)|^p$ and the corresponding space as $L^p(\partial\Omega\times\mathbb{R}^3;\mathrm{d}\gamma) = L^p(\partial\Omega\times\mathbb{R}^3)$. Furthermore, $|f|_{p,\pm} = |f\mathbf{1}_{\gamma_\pm}|_p$ and $|f|_{\infty,\pm} = |f\mathbf{1}_{\gamma_\pm}|_\infty$. For simplicity, we use $|f|_p^p = \int_{\partial\Omega}|f(x)|^p\mathrm{d}S(x) \equiv \int_{\partial\Omega}|f(x)|^p$. We also denote $f_\pm = f_{\gamma_\pm} = f\mathbf{1}_{\gamma_\pm}$.
- Finally, we define

$$P_\gamma f(x,v) = \sqrt{\mu(v)}\int_{n(x)\cdot v'>0} f(x,v')\sqrt{\mu(v')}(n(x)\cdot v')\mathrm{d}v', \quad x\in\partial\Omega.$$

Thanks to (1.5), $P_\gamma f$ defined on $\partial\Omega\times\mathbb{R}^3$ is an L^2_v-projection with respect to the measure $|n(x)\cdot v|$ for any boundary function f defined on γ_+. We also denote $\{I - P_\gamma\}f = f - P_\gamma f$.

2. Preliminary

In this section, we collect some basic definitions and estimates for the later proof. We start with the analysis of K, from (1.15), and state that a standard decomposition for K is the following:

$$Kf = \int_{\mathbb{R}^3\times\mathbb{S}^2} |u-v|^{\varrho} b_0(\cos\theta)\sqrt{\mu(u)}\left\{f(u')\sqrt{\mu(v')}+f(v')\sqrt{\mu(u')}\right\}\mathrm{d}u\mathrm{d}\omega$$
$$-\sqrt{\mu(v)}\int_{\mathbb{R}^3\times\mathbb{S}^2} |u-v|^{\varrho} b_0(\cos\theta)\sqrt{\mu(u)}f(u)\mathrm{d}u\mathrm{d}\omega \stackrel{\text{def}}{=} K_2 - K_1. \tag{2.1}$$

To treat the singularity in K, we introduce a smooth cutoff function $0 \le \chi \le 1$ such that

$$\chi(s) = \begin{cases} 1, & s \ge 2\epsilon, \\ 0, & s \le \epsilon. \end{cases}$$

we use χ to split $K_2 = K_2^{\chi} + K_2^{1-\chi}$, where

$$K_2^{\chi} f = \int_{\mathbb{R}^3\times\mathbb{S}^2} \chi(|u-v|)|u-v|^{\varrho} b_0(\cos\theta)\sqrt{\mu(u)}$$
$$\times\left\{f(u')\sqrt{\mu(v')}+f(v')\sqrt{\mu(u')}\right\}\mathrm{d}u\mathrm{d}\omega.$$

With this, it follows from [43, pp. 294] that

$$K_2^{\chi} f = \int_{\mathbb{R}^3} \mathbf{k}_2^{\chi}(v,u) f(u)\mathrm{d}u,$$

where

$$|\mathbf{k}_2^{\chi}(v,u)| \le C\epsilon^{\varrho-1}\frac{\exp\left(-\frac{1}{8}|u-v|^2 - \frac{1}{8}\frac{(|v|^2-|u|^2)^2}{|v-u|^2}\right)}{|v-u|},$$

or

$$|\mathbf{k}_2^{\chi}(v,u)| \le C\frac{\exp\left(-\frac{s_2}{8}|u-v|^2 - \frac{s_1}{8}\frac{(|v|^2-|u|^2)^2}{|v-u|^2}\right)}{|v-u|(1+|v|+|u|)^{1-\varrho}}, \tag{2.2}$$

for any $0 < s_1 < s_2 < 1$. As to K_1, it is obvious to see that

$$K_1 f = \int_{\mathbb{R}^3} \mathbf{k}_1(v,u) f(u)\mathrm{d}u,$$

with $\mathbf{k}_1(v,u) = \int_{\mathbb{S}^2} |u-v|^{\varrho} b_0(\cos\theta)\sqrt{\mu(u)}\sqrt{\mu(v)}\mathrm{d}\omega$. Analogously, we also denote $K^{\chi} = K_2^{\chi} - K_1^{\chi}$ and $K^{1-\chi} = K_2^{1-\chi} - K_1^{1-\chi}$.

Prior to the study of the property of the operators K and Γ, we present the following elementary inequality:

Lemma 2.1. *If $0 < p \le 1$, for any $x, y \ge 0$, it holds that*

$$(x+y)^p \le x^p + y^p. \tag{2.3}$$

If $p > 1$, for any $x, y \ge 0$, it holds that

$$(x+y)^p \le 2^{p-1}(x^p + y^p). \tag{2.4}$$

Proof. If $y = 0$, (2.3) is obviously true. If $y > 0$, (2.3) is then equivalent to

$$\left(1+\frac{x}{y}\right)^p - \left(\frac{x}{y}\right)^p - 1 \le 0.$$

It is easy to check that the function $g(t) = (1+t)^p - t^p - 1$ is monotonically decreasing for $0 < p \le 1$, and moreover that $g(0) = 0$, therefore (2.3) is also valid for $y > 0$. If $p > 1$, (2.4) directly follows from the convexity of t^p. This completes the proof of Lemma 2.1. □

We now summarize the properties of K as follows:

Lemma 2.2. *Assume $-3 < \varrho < 0$, $(q,\theta) \in \mathcal{A}(q,\theta)$ and $\vartheta \ge 0$. It holds that for $\eta > 0$:*

$$(Kf_1, w^2_{q,\theta,\vartheta} f_2) \le \left\{\eta \|w_{q,\theta,\vartheta} f_1\|_\nu + C(\eta)\|\mathbf{1}_{|v|\le C(\eta)} f_1\|\right\} \|w_{q,\theta,\vartheta} f_2\|_\nu, \tag{2.5}$$

especially,

$$(Kf_1, w^2_{q,\theta,\vartheta} f_2) \le C\|w_{q,\theta,\vartheta} f_1\|_\nu \|w_{q,\theta,\vartheta} f_2\|_\nu, \quad (Kf_1, f_2) \le C\|f_1\|_\nu \|f_2\|_\nu. \tag{2.6}$$

In addition, for any $l \ge 0$, one has

$$\langle v\rangle^l w_{q,\theta,\vartheta} K^{1-\chi}\left(\frac{|h|}{\langle v\rangle^l w_{q,\theta,\vartheta}}\right) \le C(\mu(v))^{\min\left\{1/8q, \frac{|1-q|}{8}\right\}} \epsilon^{\varrho+3}\|h\|_\infty, \tag{2.7}$$

and

$$\langle v\rangle^l w_{q,\theta,\vartheta} \int_{\mathbb{R}^3} \mathbf{k}^\chi(v,u)\left(\frac{e^{\varepsilon|v-u|^2}|h(u)|}{\langle u\rangle^l w_{q,\theta,\vartheta}(u)}\right) \mathrm{d}u \le C_{q,\theta}\langle v\rangle^{\varrho-2}\|h\|_\infty, \tag{2.8}$$

where $\varepsilon > 0$ and is sufficiently small and $\langle v\rangle = \sqrt{1+|v|^2}$.

Proof. We only detail the proof for (2.7) and (2.8), since the strategy to prove (2.5) is basically the same as for Lemma 2 of [43, pp. 296], and (2.6) directly follows from (2.5). Notice that $K^{1-\chi} = K_2^{1-\chi} - K_1^{1-\chi}$. We first consider the estimates for $K_1^{1-\chi}$. Recall

$$w_{q,\theta,\vartheta} = \exp\left\{\frac{q|v|^\theta}{8} + \frac{q|v|^\theta}{8(1+t)^\vartheta}\right\} = \exp\left\{\frac{q}{8}(1+(1+t)^{-\vartheta})|v|^\theta\right\}, \quad (q,\theta)\in\mathcal{A}_{q,\theta}.$$

Let $\widetilde{q} = \frac{q}{2}(1+(1+t)^{-\vartheta})$, then $q/2 < \widetilde{q} \le q$. Direct calculation yields

$$\langle v\rangle^l w_{q,\theta,\vartheta}\sqrt{\mu(v)} \le C_{q,\theta}(\mu(v))^{\min\left\{1/8q, \frac{|1-q|}{8}\right\}},$$

then it is easy to obtain

$$
\begin{aligned}
&\langle v\rangle^l w_{q,\theta,\vartheta} K_1^{1-\chi}\left(\frac{|h|}{\langle v\rangle^l w_{q,\theta,\vartheta}}\right)\\
&\quad=\langle v\rangle^l w_{q,\theta,\vartheta}\sqrt{\mu(v)}\int_{\mathbb{R}^3\times\mathbb{S}^2}(1-\chi(|u-v|))|u-v|^{\varrho}b_0(\cos\theta)\sqrt{\mu(u)}\\
&\quad\quad\times\left(\frac{|h(u)|}{\langle u\rangle^l w_{q,\theta,\vartheta}(u)}\right)\mathrm{d}\omega\mathrm{d}u\\
&\quad\le C_{q,\theta,\vartheta}(\mu(v))^{\min\left\{1/8q,\frac{|1-q|}{8}\right\}}\int_{|v-u|\le 2\epsilon}|u-v|^{\varrho}\mathrm{d}u\|h\|_\infty\\
&\quad\le C(\mu(v))^{\min\left\{1/8q,\frac{|1-q|}{8}\right\}}\epsilon^{\varrho+3}\|h\|_\infty.
\end{aligned}
$$

For the contribution of $\mathbf{k}_1^{\chi}$ in (2.8), it follows that

$$
\begin{aligned}
&\langle v\rangle^l w_{q,\theta,\vartheta}\int_{\mathbb{R}^3}\mathbf{k}_1^{\chi}(v,u)\left(\frac{e^{\varepsilon|v-u|^2}|h(u)|}{\langle u\rangle^l w_{q,\theta,\vartheta}(u)}\right)\mathrm{d}u\\
&\quad=\langle v\rangle^l w_{q,\theta,\vartheta}\sqrt{\mu(v)}\int_{\mathbb{R}^3\times\mathbb{S}^2}\chi(|u-v|)|u-v|^{\varrho}b_0(\cos\theta)\sqrt{\mu(u)}\\
&\quad\quad\times\left(\frac{e^{\varepsilon|v-u|^2}|h(u)|}{\langle u\rangle^l w_{q,\theta,\vartheta}(u)}\right)\mathrm{d}\omega\mathrm{d}u\\
&\quad\le C_{q,\theta}\int_{\mathbb{R}^3}|u-v|^{\varrho}(\mu(v)\mu(u))^{\min\left\{1/8q,\frac{|1-q|}{8}\right\}}e^{\varepsilon|v-u|^2}\mathrm{d}u\|h\|_\infty\\
&\quad\le C\langle v\rangle^{\varrho}(\mu(v))^{\min\left\{1/16q,\frac{|1-q|}{16}\right\}}\|h\|_\infty,
\end{aligned}
$$

where the last inequality is due to $\int_{\mathbb{R}^3}|u-v|^{\varrho}(\mu(u))^{\min\{1/16q,\frac{|1-q|}{16}\}}\mathrm{d}u\le C\langle v\rangle^{\varrho}$.

We now turn to derive the contributions of $K_2^{1-\chi}$ in (2.7). In light of (2.1), on the one hand, we have

$$
\begin{aligned}
&\langle v\rangle^l w_{q,\theta,\vartheta} K_2^{1-\chi}\left(\frac{|h|}{\langle v\rangle^l w_{q,\theta,\vartheta}}\right)\\
&\quad=\langle v\rangle^l w_{q,\theta,\vartheta}\int_{\mathbb{R}^3\times\mathbb{S}^2}(1-\chi)(|u-v|)|u-v|^{\varrho}b_0(\cos\theta)\sqrt{\mu(u)}\\
&\quad\quad\times\left\{\frac{|h(u')|}{\langle u'\rangle^l w_{q,\theta,\vartheta}(u')}\sqrt{\mu(v')}+\frac{|h(v')|}{\langle v'\rangle^l w_{q,\theta,\vartheta}(v')}\sqrt{\mu(u')}\right\}\mathrm{d}\omega\mathrm{d}u.
\end{aligned}
\tag{2.9}
$$

On the other hand, (1.6) and $|v-u|\le 2\epsilon$ imply

$$
\begin{cases}
|v'|=|v+[(u-v)\cdot\omega]\omega|\ge|v|-|v-u|\ge|v|-2\epsilon,\\
|u'|=|v+u-v-[(u-v)\cdot\omega]\omega|\ge|v|-2|v-u|\ge|v|-4\epsilon.
\end{cases}
\tag{2.10}
$$

Using $\langle v\rangle^l w_{q,\theta,\vartheta}\sqrt{\mu(v)} \le C_{q,\theta}(\mu(v))^{\min\left\{1/8q,\frac{|1-q|}{8}\right\}}$ again, we get, from (2.9) and (2.10), that

$$\langle v\rangle^l w_{q,\theta,\vartheta} K_2^{1-\chi}\left(\frac{|h|}{\langle v\rangle^l w_{q,\theta,\vartheta}}\right) \le C_{q,\theta}(\mu(v))^{\min\left\{1/8q,\frac{|1-q|}{8}\right\}} \int_{|v-u|\le 2\epsilon} |u-v|^{\varrho}\mathrm{d}u\|h\|_\infty \le C(\mu(v))^{\min\left\{1/8q,\frac{|1-q|}{8}\right\}}\epsilon^{\varrho+3}\|h\|_\infty.$$

It remains now to deduce the contribution of $\mathbf{k}_2^\chi$ in (2.8). Recall (2.2) and take $s_0 = \min\{s_1, s_2\}$ to obtain

$$\begin{aligned}&\langle v\rangle^l w_{q,\theta,\vartheta}\int_{\mathbb{R}^3}\mathbf{k}_2^\chi(v,u)\left(\frac{e^{\varepsilon|v-u|^2}|h(u)|}{\langle u\rangle^l w_{q,\theta,\vartheta}(u)}\right)\mathrm{d}u\\ &\le C\|h\|_\infty\langle v\rangle^{\varrho-1}\langle v\rangle^l w_{q,\theta,\vartheta}\int_{\mathbb{R}^3}\frac{\exp\left(-\frac{s_0}{8}|u-v|^2-\frac{s_0}{8}\frac{(|v|^2-|u|^2)^2}{|v-u|^2}\right)}{|v-u|}\\ &\times\left(\frac{e^{\varepsilon|v-u|^2}}{\langle u\rangle^l w_{q,\theta,\vartheta}(u)}\right)\mathrm{d}u \overset{\text{def}}{=} \mathcal{K}_0.\end{aligned}$$

Next, from (1.20), we notice that for some $C_l > 0$ and $\theta = 2$,

$$\left|\frac{w_{q,\theta,\vartheta}(v)}{w_{q,\theta,\vartheta}(u)}\right| \le C_l[1+|v-u|^2]^l e^{-\widetilde{q}\{|u|^2-|v|^2\}}.$$

Let $v-u=\eta$ and $u=v-\eta$ in the integral of $\mathcal{K}_0$. We then compute the total exponent in $\mathbf{k}_2^\chi(v,u)\frac{w_{q,2,\vartheta}(v)}{w_{q,2,\vartheta}(u)}$ as:

$$\begin{aligned}&-\frac{s_0}{8}|\eta|^2-\frac{s_0}{8}\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2}-\frac{\widetilde{q}}{4}\{|v-\eta|^2-|v|^2\}\\ &=-\frac{s_0}{4}|\eta|^2+\frac{s_0}{2}v\cdot\eta-\frac{s_0}{2}\frac{|v\cdot\eta|^2}{|\eta|^2}-\frac{\widetilde{q}}{4}\{|\eta|^2-2v\cdot\eta\}\\ &=-\frac{1}{4}(\widetilde{q}+s_0)|\eta|^2+\frac{1}{2}(s_0+\widetilde{q})v\cdot\eta-\frac{s_0}{2}\frac{\{v\cdot\eta\}^2}{|\eta|^2}.\end{aligned}$$

Let $\widetilde{q} \le q < s_0$, and the discriminant of the above quadratic form of $|\eta|$ and $\frac{v\cdot\eta}{|\eta|}$ is

$$\Delta = \frac{1}{4}(s_0+\widetilde{q})^2-(\widetilde{q}+s_0)\frac{s_0}{2} = \frac{1}{4}(\widetilde{q}^2-s_0^2) < 0.$$

Notice that $q/2 < \widetilde{q} \le q$, and we thus have, for $\varepsilon > 0$ sufficiently small and $q < s_0$, that there is $C_q > 0$ independent of ϑ such that the following perturbed quadratic form is still negative definite:

$$
\begin{aligned}
-\frac{s_0-8\varepsilon}{8}|\eta|^2 - \frac{s_0-8\varepsilon}{8}\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2} - \frac{\widetilde{q}}{4}\{|\eta|^2-2v\cdot\eta\}& \\
\leq -C_q\left\{|\eta|^2+\frac{|v\cdot\eta|^2}{|\eta|^2}\right\} &= -C_q\left\{\frac{|\eta|^2}{2}+\left(\frac{|\eta|^2}{2}+\frac{|v\cdot\eta|^2}{|\eta|^2}\right)\right\} \\
\leq -C_q\left\{\frac{|\eta|^2}{2}+|v\cdot\eta|\right\}.&
\end{aligned}
\tag{2.11}
$$

If $0<\theta<2$, Lemma 2.1 yields

$$
|v|^\theta - |u|^\theta \leq C_\theta|\eta|^\theta.
$$

Therefore, one also has

$$
\begin{aligned}
-\frac{s_0-8\varepsilon}{8}|\eta|^2 - \frac{s_0}{8}\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2} + \frac{\widetilde{q}C_\theta}{4}\eta^\theta& \\
\leq -\frac{s_0-9\varepsilon}{8}|\eta|^2 - \frac{s_0-9\varepsilon}{8}\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2} + C_{q,\theta}& \\
\leq -C_{s_0}\left\{|\eta|^2+\frac{|v\cdot\eta|^2}{|\eta|^2}\right\}+C_{q,\theta} \leq -C_{s_0}\left\{\frac{|\eta|^2}{2}+|v\cdot\eta|\right\}+C_{q,\theta}.&
\end{aligned}
\tag{2.12}
$$

Plugging (2.11) or (2.12) into $\mathcal{K}_0$, we obtain

$$
\mathcal{K}_0 \leq C_{q,\theta}\langle v\rangle^{\varrho-1}\|h\|_\infty\int_{\mathbb{R}^3}\frac{\langle\eta\rangle^l}{|\eta|}\exp\left\{-C_q\left\{\frac{|\eta|^2}{2}+|v\cdot\eta|\right\}\right\}\mathrm{d}\eta.
$$

Next, we make another change of variable $\eta_\parallel = (\eta\cdot\frac{v}{|v|})\frac{v}{|v|}$ and $\eta_\perp = \eta-\eta_\parallel$ so that $v\cdot\eta = |v||\eta_\parallel|$, which leads us to

$$
\begin{aligned}
\mathcal{K}_0 &\leq C_{q,\theta}\langle v\rangle^{\varrho-1}\|h\|_\infty\int_{\mathbb{R}^2}\frac{1}{|\eta_\perp|}\exp\left\{-\frac{C_q}{4}|\eta_\perp|^2\right\}\int_{\mathbb{R}}\exp\left\{-\frac{C_q}{4}|v||\eta_\parallel|\right\}\mathrm{d}\eta_\parallel\mathrm{d}\eta_\perp \\
&\leq C_{q,\theta}\langle v\rangle^{\varrho-2}\|h\|_\infty.
\end{aligned}
$$

This finishes the proof of Lemma 2.2. □

The following lemma is concerned with the estimates on the nonlinear operator Γ.

Lemma 2.3. *It holds that*

$$
\|\nu^{-1}w_{q,\theta,\vartheta}\Gamma(f_1,f_2)\|_\infty \leq C\|w_{q,\theta,\vartheta}f_1\|_\infty\|w_{q,\theta,\vartheta}f_2\|_\infty, \tag{2.13}
$$

$$
\begin{aligned}
\|w_{q/2,\theta,\vartheta}\Gamma(f_1,f_2)\|_\infty \leq C\big\{&\|f_1\|_\infty\|w_{q,\theta,\vartheta}f_2\|_\infty \\
&+\|w_{q,\theta,\vartheta}f_1\|_\infty\|f_2\|_\infty\big\},
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
\|\nu^{-1/2}w_{q/2,\theta,\vartheta}\Gamma(f_1,f_2)\|_2^2 \leq\ & C\|w_{q,\theta,\vartheta}f_1\|_\infty^2\|w_{q/2,\theta,\vartheta}f_2\|_\nu^2 \\
&+C\|w_{q,\theta,\vartheta}f_2\|_\infty^2\|w_{q/2,\theta,\vartheta}f_1\|_\nu^2,
\end{aligned}
\tag{2.15}
$$

and

$$\|\nu^{-1/2}\Gamma(f_1, f_2)\|_2^2 \le C\|w_{q/2,\theta,\vartheta} f_1\|_\infty^2\|f_2\|_\nu^2 + C\|w_{q/2,\theta,\vartheta} f_2\|_\infty^2\|f_1\|_\nu^2. \quad (2.16)$$

Proof. The proof of (2.13) is the same as that of Lemma 5 in [27, pp. 730], for brevity we omit the details. To prove (2.14), we rewrite

$$\begin{aligned}\Gamma(f_1, f_2) &= \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_2)\\ &= \int_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^\varrho b_0(\cos\theta)\sqrt{\mu(u)} f_1(u')f_2(v')\mathrm{d}u\mathrm{d}\omega\\ &\quad - \int_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^\varrho b_0(\cos\theta)\sqrt{\mu(u)} f_1(u)f_2(v)\mathrm{d}u\mathrm{d}\omega\\ &= \Gamma^{\text{gain}}(f_1, f_2) - \Gamma^{\text{loss}}(f_1, f_2). \end{aligned} \quad (2.17)$$

For the loss term, a simple calculation directly gives

$$\|w_{q/2,\theta,\vartheta}\Gamma^{\text{loss}}(f_1, f_2)\|_\infty \le C\|w_{q/2,\theta,\vartheta} f_2\nu\|_\infty\|f_1\|_\infty \le C\|w_{q/2,\theta,\vartheta} f_2\|_\infty\|f_1\|_\infty.$$

Next, since $|v|^2 \le |v'|^2 + |u'|^2$, by virtue of (2.3), one has

$$w_{q/2,\theta,\vartheta}(v) \le w_{q/2,\theta,\vartheta}(u')w_{q/2,\theta,\vartheta}(v') \le \frac{1}{2}(w_{q,\theta,\vartheta}(u') + w_{q,\theta,\vartheta}(v')).$$

With this, we present the corresponding computation for the gain term as follows:

$$\begin{aligned}&|w_{q/2,\theta,\vartheta}\Gamma^{\text{gain}}(f_1, f_2)|\\ &\le \frac{1}{2}(w_{q,\theta,\vartheta}(u') + w_{q,\theta,\vartheta}(v'))\int_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^\varrho b_0(\cos\theta)\sqrt{\mu(u)}\\ &\quad |f_1(u')f_2(v')|\mathrm{d}u\mathrm{d}\omega\\ &\le C\left\{\|f_1\|_\infty\|w_{q,\theta,\vartheta} f_2\|_\infty + \|w_{q,\theta,\vartheta} f_1\|_\infty\|f_2\|_\infty\right\}.\end{aligned}$$

This ends the proof for (2.14). In what follows, we only prove (2.15), since (2.16) can be obtained in a similar fashion. Recalling (2.17), for the loss term, one has

$$\begin{aligned}&\|\nu^{-1/2}w_{q/2,\theta,\vartheta}\Gamma^{\text{loss}}(f_1, f_2)\|_2^2\\ &= \int_{\mathbb{R}^3\times\Omega} \nu^{-1}(v)w_{q/2,\theta,\vartheta}^2\\ &\quad \left\{\int_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^\varrho b_0(\cos\theta)\sqrt{\mu(u)} f_1(u)f_2(v)\mathrm{d}u\mathrm{d}\omega\right\}^2 \mathrm{d}v\mathrm{d}x\\ &\le \|f_1\|_\infty^2 \int_{\mathbb{R}^3\times\Omega} \nu^{-1}(v)w_{q/2,\theta,\vartheta}^2|f_2(v)|^2\\ &\quad \left\{\int_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^\varrho b_0(\cos\theta)\sqrt{\mu(u)}\mathrm{d}\omega\right\}^2 \mathrm{d}v\mathrm{d}x\\ &\le C\|f_1\|_\infty^2\|w_{q/2,\theta,\vartheta} f_2\|_\nu^2.\end{aligned}$$

As for the gain term, let us denote

$$\begin{aligned}\mathcal{I}_0 &= \|\nu^{-1/2} w_{q/2,\theta,\vartheta} \Gamma^{\text{gain}}(f_1, f_2)\|_2^2 \\ &= \int_{\mathbb{R}^3\times\Omega} \nu^{-1}(v) w_{q/2,\theta,\vartheta}^2 \\ &\quad \left\{ \int_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^{\varrho} b_0(\cos\theta)\sqrt{\mu(u)} f_1(u') f_2(v') \mathrm{d}u\omega \right\}^2 \mathrm{d}v\mathrm{d}x.\end{aligned}$$

The calculation for $\mathcal{I}_0$ is a little more delicate; we divide it into the following three cases:

Case 1, $|u| \geq |v|/2$. In this case, $\mu^{1/2}(u) \leq \mu^{1/4}(u)\mu^{1/16}(v)$. By Hölder's inequality and a change of variable $(u, v) \to (u', v')$, we have

$$\begin{aligned}\mathcal{I}_0 &\leq C \int_{\mathbb{R}^3\times\Omega} \nu^{-1}(v) w_{q/2,\theta,\vartheta}^2(v) \int_{\mathbb{R}^3} |v-u|^{\varrho} \sqrt{\mu(u)} f_1^2(u') f_2^2(v') \mathrm{d}u \\ &\quad \times \int_{\mathbb{R}^3} |v-u|^{\varrho} \sqrt{\mu(u)} \mathrm{d}u\mathrm{d}v\mathrm{d}x \\ &\leq C \int_{\mathbb{R}^3\times\mathbb{R}^3\times\Omega} w_{q/2,\theta,\vartheta}^2(u) w_{q/2,\theta,\vartheta}^2(v) |v-u|^{\varrho} \mu^{1/16}(u) \\ &\quad \times \mu^{1/16}(v) f_1^2(u) f_2^2(v) \mathrm{d}u\mathrm{d}v\mathrm{d}x \\ &\leq C \|w_{q/2,\theta,\vartheta} f_1\|_\infty^2 \|w_{q/2,\theta,\vartheta} f_2\|_\nu^2,\end{aligned}$$

where we also used the fact that $\max\{|v|, |u|\} \leq |u'| + |v'|$.

Case 2, $|u| \leq |v|/2$ *and* $|v| \leq 1$. In this situation, $|u-v| \geq |v| - |u| \geq |v|/2$ and $|u| \leq 1/2$, moreover, $|u'| + |v'| \leq 2(|u| + |v|) \leq 3|v| \leq 3$, consequently, when $(u, v) \in \{(u, v) | |u| \geq |v|/2, |v| \leq 1\}$, we have, by Hölder's inequality and a change of variable $(u, v) \to (u', v')$, that

$$\begin{aligned}\mathcal{I}_0 &\leq C \int_{\{|v|\leq 1\}\times\Omega} |v|^{\varrho} \int_{\{|u|\leq 1/2\}} |v-u|^{\varrho} \mu(u) f_1^2(u') f_2^2(v') \mathrm{d}u\mathrm{d}v\mathrm{d}x \\ &\leq C \int_{\{|v|\leq 1, |u|\leq 1/2\}\times\Omega} \min\{|u'|^{\varrho}, |v'|^{\varrho}\} f_1^2(u') f_2^2(v') \mathrm{d}u\mathrm{d}v\mathrm{d}x \\ &\leq C \int_{\{|v|\leq 3, |u|\leq 3\}\times\Omega} \min\{|u|^{\varrho}, |v|^{\varrho}\} f_1^2(u) f_2^2(v) \mathrm{d}u\mathrm{d}v\mathrm{d}x \leq C \|f_1\|_\infty^2 \|f_2\|_\nu^2.\end{aligned}$$

Case 3, $|u| \leq |v|/2$ *and* $|v| \geq 1$. One has $\max\{|u'|, |v'|\} \leq 5|v|/2$ on this occasion, hence $\nu(v) \lesssim \nu(v') + \nu(u')$, moreover, it follows that $|u-v| \geq |v| - |u| \geq |v|/2 \geq 1/2$. Notice that $w_{q/2,\theta,\vartheta}^2(v) \leq w_{q/2,\theta,\vartheta}^2(u') w_{q/2,\theta,\vartheta}^2(v')$. Apply Hölder's inequality and a change of variable $(u, v) \to (u', v')$ again to obtain

$$\begin{aligned}\mathcal{I}_0 &\leq \int_{\mathbb{R}^3\times\mathbb{R}^3\times\Omega} \nu^{-1}(v) w_{q/2,\theta,\vartheta}^2 (1+|v|)^{2\varrho} f_1^2(u') f_2^2(v') \mathrm{d}u\mathrm{d}v\mathrm{d}x \\ &\leq C \int_{\mathbb{R}^3\times\mathbb{R}^3\times\Omega} w_{q/2,\theta,\vartheta}^2(u) w_{q/2,\theta,\vartheta}^2(v) (\nu(v) + \nu(u)) f_1^2(u) f_2^2(v) \mathrm{d}u\mathrm{d}v\mathrm{d}x \\ &\leq C \|w_{q,\theta,\vartheta} f_1\|_\infty^2 \|w_{q/2,\theta,\vartheta} f_2\|_\nu^2 + C \|w_{q,\theta,\vartheta} f_2\|_\infty^2 \|w_{q/2,\theta,\vartheta} f_1\|_\nu^2,\end{aligned}$$

where the fact that $\int_{\mathbb{R}^3} w_{-q/2,\theta,\vartheta} \mathrm{d}v < +\infty$ was used.

Combining all of the estimates above, we see that (2.15) holds true, and this ends the proof of Lemma 2.3.

Next, we address the following Ukai's trace theorem whose proof can be found in Lemma 2.1 of [17, pp. 187]: □

Lemma 2.4. *Let* $\varepsilon > 0$*, and define the near-grazing set of* γ_+ *or* γ_- *as*

$$\gamma_\pm^\varepsilon \equiv \left\{(x,v) \in \gamma_\pm : |n(x)\cdot v| \le \varepsilon \text{ or } |v| \ge \frac{1}{\varepsilon} \quad \text{or} \quad |v| \le \varepsilon\right\}.$$

There exists a constant $C_{\varepsilon,\Omega} > 0$ *that depends only on* ε *and* Ω *such that*

$$\begin{aligned}&\int_s^t |f\mathbf{1}_{\gamma_+\setminus\gamma_+^\varepsilon}(\tau)|_1 \mathrm{d}\tau\\ &\quad \le C_{\varepsilon,\Omega}\left\{||f(s)||_1 + \int_s^t \Big[\|f(\tau)\|_1 + \|\{\partial_t + v\cdot\nabla_x\}f(\tau)\|_1\Big]\mathrm{d}\tau\right\}\end{aligned}$$

for any $0 \le s \le t$. □

The following lemma quoted from [27, pp. 723] is concerned with property of the kinetic distance function:

Lemma 2.5. *Let* Ω *be strictly convex defined in* (1.8)*. Define the functional along the trajectories* $\frac{\mathrm{d}X(s)}{\mathrm{d}s} = V(s)$, $\frac{\mathrm{d}V(s)}{\mathrm{d}s} = 0$ *in* (1.10) *as:*

$$\alpha(s) \equiv \xi^2(X(s)) + [V(s)\cdot\nabla\xi(X(s))]^2 - 2\{V(s)\cdot\nabla^2\xi(X(s))\cdot V(s)\}\xi(X(s)). \tag{2.18}$$

Let $X(s) \in \bar{\Omega}$ *for* $t_1 \le s \le t_2$*. Then there exists constant* $C_\xi > 0$ *such that*

$$\begin{aligned}e^{C_\xi(|V(t_1)|+1)t_1}\alpha(t_1) &\le e^{C_\xi(|V(t_1)|+1)t_2}\alpha(t_2),\\ e^{-C_\xi(|V(t_1)|+1)t_1}\alpha(t_1) &\ge e^{-C_\xi(|V(t_1)|+1)t_2}\alpha(t_2).\end{aligned}$$

Finally, we state the following significant lemma which gives a lower bound of the backward exit time $t_{\mathbf{b}}(x,v)$:

Lemma 2.6. [27, pp. 724] *Let* $x_i \in \partial\Omega$*, for* $i = 1,2$*, and let* (t_1,x_1,v) *and* (t_2,x_2,v) *be connected with the trajectory* $\frac{\mathrm{d}X(s)}{\mathrm{d}s} = V(s)$, $\frac{\mathrm{d}V(s)}{\mathrm{d}s} = 0$ *which lies inside* $\bar{\Omega}$*. Then there exists a constant* $C_\xi > 0$ *such that*

$$|t_1 - t_2| \ge \frac{|n(x_1)\cdot v|}{C_\xi|v|^2}. \tag{2.19}$$

3. Diffuse Reflection Boundary Value Problem

3.1. L^2 Existence and Decay for the Linearized Equation

As mentioned in Section 1, we mainly employ the $L^2 \cap L^\infty$ argument to solve the initial boundary value problem of (1.11), (1.12) and (1.13). To obtain the time decay rates in L^∞ space, an L^2- time decay theory must first be established, cf. [27]. However, one cannot directly obtain the time decay of (1.11), (1.12) and (1.13) by an L^2- energy method, since the positive operator L is degenerated in the sense that the inner product (Lf, f) has no positive lower bound in the large velocity domain. To overcome this difficulty, we first construct the global existence in some weighted L^2 space, then tend to deduce the time decay rates in a lower order weighed energy space via an interpolation technique. We remark that the main idea used here is similar to treating the Cauchy problem of the Boltzmann equation with soft potential [42,43]. It should be also pointed out that it is necessary to derive the L^2 time decay rates even only considering the global existence of the initial boundary value problem of (1.11), (1.12) and (1.13) in the case of soft potential.

Notice that the null space of the linear operator L is generated by $\{1, v, \frac{1}{2}(|v^2|-3)\}\sqrt{\mu}$, so we define

$$\mathbf{P}f = \left\{a + b \cdot v + \frac{1}{2}(|v^2| - 3)c\right\}\sqrt{\mu},\ (t, x, v) \in [0, +\infty) \times \Omega \times \mathbb{R}^3,$$

which is called the macroscopic part of f. The microscopic part of f is further denoted by $\{\mathbf{I} - \mathbf{P}\}f = f - \mathbf{P}f$. It is well-known that there exists $\delta_0 > 0$ such that

$$(Lf, f) \geq \delta_0 \|\{\mathbf{I} - \mathbf{P}\}f\|_\nu^2.$$

We consider the following initial boundary value problem of the linearized Boltzmann equation with soft potential:

$$\partial_t f + v \cdot \nabla_x f + Lf = g, \quad f(0) = f_0, \quad \text{in } (0, +\infty) \times \Omega \times \mathbb{R}^3, \tag{3.1}$$

with

$$f_- = P_\gamma f, \ \text{ on } \mathbb{R}_+ \times \gamma_-, \tag{3.2}$$

and g is given.

In what follows in this subsection we will prove the following:

Proposition 3.1. *Let $-3 < \varrho < 0$ and $(q, \theta) \in \mathcal{A}_{q,\theta}$. Assume that for all $t > 0$,*

$$\int_{\Omega\times\mathbb{R}^3} g(t, x, v)\sqrt{\mu}\mathrm{d}v\mathrm{d}x = 0, \quad \mathbf{P}g = 0. \tag{3.3}$$

There exists $\varepsilon_0 >$ such that if

$$\|w_{q/2,\theta} f_0\|_2^2 + |f_0|_{2,+}^2 + \int_0^t \left\|\nu^{-1/2} w_{q/2,\theta}\, g(s)\right\|_2^2 \mathrm{d}s \leq \varepsilon_0^2,$$

then there exists a unique solution to the problems (3.1) *and* (3.2) *such that for all* $t \geq 0$*,*

$$\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\sqrt{\mu}\mathrm{d}x\mathrm{d}v = 0, \tag{3.4}$$

$$\sup_{0\leq s\leq t} \|f(s)\|_2^2 + \int_0^t \|f(s)\|_\nu^2 \mathrm{d}s \leq C\|f_0\|_2^2 + C\int_0^t \left\|\nu^{-1/2}g(s)\right\|_2^2 \mathrm{d}s,$$

and

$$\begin{aligned}\sup_{0\leq s\leq t} \|w_{q/2,\theta}f(s)\|_2^2 &+ \int_0^t \|w_{q/2,\theta}f(s)\|_\nu^2 \mathrm{d}s \\ &\leq C\|w_{q/2,\theta}f_0\|_2^2 + C\int_0^t \left\|\nu^{-1/2}w_{q/2,\theta}g(s)\right\|_2^2 \mathrm{d}s.\end{aligned} \tag{3.5}$$

Moreover, let $\rho_0 = \frac{\theta}{\theta-\varrho}$*. There exists* $\lambda_1 > 0$ *that depends on* q *and* ρ_0 *such that*

$$\begin{aligned}&\|f(t)\|_2^2 + e^{-\lambda_1 t^{\rho_0}}\int_0^t e^{\lambda_1 s^{\rho_0}}\|f(s)\|_2^2 \mathrm{d}s \\ &\lesssim e^{-\lambda_1 t^{\rho_0}}\left\{\|w_{q/2,\theta}f_0\|_2^2 + \int_0^t e^{\lambda_1 s^{\rho_0}}\|\nu^{-1/2}g(s)\|_2^2 \mathrm{d}s\right. \\ &\left.+\int_0^t \|\nu^{-1/2}w_{q/2,\theta}g(s)\|_2^2 \mathrm{d}s\right\}.\end{aligned} \tag{3.6}$$

In order to construct the global existence of (3.1) and (3.2), we first deduce the global solvability of equation (3.1) with an approximation boundary condition and then we show that such an approximate solution sequence converges in L^2 for any $t \geq 0$. Once the global existence is obtained, the time-decay estimate (3.6) follows from an L^2 energy estimate and an interpolation technique. Along this line, Proposition 3.1 is an easy consequence of the following two lemmas (the first of which is concerned with *a priori* estimates for the macroscopic part of the solution of (3.1) and (3.2)):

Lemma 3.1. *Assume that* g *satisfies* (3.3) *and* f *satisfies* (3.1), (3.2) *and* (3.4)*. Then there exists a function* $G(t)$ *such that, for all* $t \geq 0$*,* $G(t) \lesssim \|f(t)\|_2^2$ *and*

$$\|\mathbf{P}f\|_\nu^2 \lesssim \frac{\mathrm{d}}{\mathrm{d}t}G(t) + \|g\|_2^2 + \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 + |\{1-P_\gamma\}f|_{2,+}^2. \tag{3.7}$$

Proof. The proof of Lemma 3.1 is similar to that of Lemma 6.1 in [17, pp. 221], and we omit the details for brevity. □

Lemma 3.2. *Assume that* g *satisfies* (3.3)*. There is a constant* $\varepsilon_0 > 0$ *such that for any* $t > 0$*, if*

$$\|f_0\|_2^2 + |f_0|_{2,+}^2 + \int_0^t \left\|\nu^{-1/2}g(s)\right\|_2^2 \mathrm{d}s \leq \varepsilon_0^2,$$

then (3.1) *and* (3.2) *admit a strong solution* $f(t,x,v)$ *in* $[0,+\infty)\times\Omega\times\mathbb{R}^3$ *satisfying*

$$\|f(t)\|_2^2+\int_0^t\|f(s)\|_\nu^2\mathrm{d}s+\int_0^t|(I-P_\gamma)f(s)|_{2,+}^2\mathrm{d}s$$
$$\leq C\int_0^t\|\nu^{-1/2}g(s)\|_2^2\mathrm{d}s+C\|f_0\|_2^2. \tag{3.8}$$

Proof. We establish a solution of (3.1) and (3.2) via the following approximate boundary value problem:

$$\partial_t f+v\cdot\nabla_x f+Lf=g,\quad f(0,x,v)=f_0, \tag{3.9}$$

with

$$f_-=\left(1-\frac{1}{j}\right)P_\gamma f,\ \ j=2,3,\ldots. \tag{3.10}$$

The proof is then divided into two steps.

Step 1. Global existence of (3.9) *and* (3.10). We start with constructing the local existence of (3.9) and (3.10) through the following sequence of iterating approximate solutions:

$$\partial_t f^{\ell+1}+v\cdot\nabla_x f^{\ell+1}+\nu f^{\ell+1}-Kf^\ell=g,\quad f^{\ell+1}(0)=f_0,\quad \ell\geq 0, \tag{3.11}$$

with

$$f_-^{\ell+1}=\left(1-\frac{1}{j}\right)P_\gamma f^\ell,\quad j=2,3,\ldots, \tag{3.12}$$

and $f^0\equiv f_0$. Let us now define

$$M(f)(t)=\|f(t)\|_2^2+\int_0^t|f(s)|_{2,+}^2\mathrm{d}s.$$

We *claim* that there exists a small $T_*>0$ such that if $\sum_{0\leq t\leq T*}M(f^\ell)(t)\leq M_1$ for $M_1>0$ then $\sum_{0\leq t\leq T*}M(f^{\ell+1})(t)\leq M_1$. Take an inner product of (3.11) with $f^{\ell+1}$ and use Green's identity as well as Lemma 2.2 to deduce

$$\|f^{\ell+1}(t)\|_2^2+(1-\varepsilon)\int_0^t\|f^{\ell+1}(s)\|_\nu^2+\int_0^t|f^{\ell+1}(s)|_{2,+}^2\mathrm{d}s$$
$$\leq\left(1-\frac{1}{j}\right)^2\int_0^t|P_\gamma f^\ell|_{2,-}^2\mathrm{d}s+C_\varepsilon\int_0^t\|f^\ell(s)\|_\nu\mathrm{d}s$$
$$+\int_0^t\|\nu^{-1/2}g(s)\|_2^2\mathrm{d}s+\|f_0\|_2^2. \tag{3.13}$$

Since

$$\|f_0\|_2^2+|f_0|_{2,+}^2+\int_0^t\|\nu^{-1/2}g(s)\|_2^2\mathrm{d}s<\varepsilon_0,\quad |P_\gamma f^\ell|_{2,-}^2\leq|f^\ell|_{2,+}^2$$

and

$$\nu(v) \sim (1+|v|^2)^{\varrho/2}, \quad -3<\varrho<0,$$

we see that

$$M(f^{\ell+1})(t) \le \max\{1, C_\varepsilon\} t M_1 + \varepsilon_0.$$

Taking $T_* > 0$ suitably small and letting $\varepsilon_0 < M_1$, one obtains $\sum_{0\le t\le T*} M(f^{\ell+1})(t) \le M_1$. This completes the proof of the *claim*.

Next we get from the difference of the equation (3.11) for $\ell+1$ and ℓ that

$$\partial_t[f^{\ell+1}-f^\ell] + v\cdot\nabla_x[f^{\ell+1}-f^\ell] + \nu[f^{\ell+1}-f^\ell] = K[f^\ell - f^{\ell-1}],\ \ell\ge 1,$$

with $[f^{\ell+1}-f^\ell](0) \equiv 0$ and $f_-^{\ell+1} - f_-^\ell = (1-\frac{1}{j})P_\gamma[f^\ell - f^{\ell-1}]$. Performing similar calculations as to those for obtaining (3.13), one has

$$\begin{aligned}\|f^{\ell+1}(t)-f^\ell(t)\|_2 &+ \int_0^t \|f^{\ell+1}(s)-f^\ell(s)\|_\nu^2 + \int_0^t |f^{\ell+1}(s)-f^\ell(s)|_{2,+}^2 \mathrm{d}s\\ &\le \left(1-\frac{1}{j}\right)^2 \int_0^t |f^\ell - f^{\ell-1}|_{2,+}^2 + C_\varepsilon \int_0^t \|f^\ell(s)-f^{\ell-1}(s)\|_\nu^2 \mathrm{d}s,\end{aligned}$$

from which we obtain

$$\begin{aligned}\sup_{0\le t\le T_*} &\|f^{\ell+1}(t)-f^\ell(t)\|_2 + \int_0^{T_*} |f^{\ell+1}(s)-f^\ell(s)|_{2,+}^2 \mathrm{d}s\\ &\le \max\left\{\left(1-\frac{1}{j}\right)^2, T_*C_\varepsilon\right\}\\ &\quad\times\left\{\sup_{0\le t\le T_*}\|f^\ell(t)-f^{\ell-1}(t)\|_2^2 + \int_0^{T_*}|f^\ell - f^{\ell-1}|_{2,+}^2\right\}\end{aligned}$$

for $\ell \ge 1$. Thus, if $T_*C_\varepsilon < 1$, we also show that $f^\ell(t)$ is a Cauchy sequence in L^2 for $t\in[0,T_*]$ and $j\ge 2$. That is, $f^\ell \to f^j$ and f^j is a strong solution of

$$\partial_t f + v\cdot\nabla_x f + Lf = g, \quad f(0)=f_0, \quad f_- = \left(1-\frac{1}{j}\right)P_\gamma f. \tag{3.14}$$

Furthermore, for any given $j\ge 2$, assume that f^j is a strong solution of (3.11) and (3.12). By using Green's identity and $\mathbf{P}g = 0$, one then obtains the following *a priori* estimate:

$$\begin{aligned}\|f^j(t)\|_2^2 &+ \lambda\int_0^t \|(\mathbf{I}-\mathbf{P})f^j(s)\|_\nu^2 \mathrm{d}s + \int_0^t |(1-P_\gamma)f^j(s)|_{2,+}^2\mathrm{d}s\\ &+\left(\frac{2}{j}-\frac{1}{j^2}\right)\int_0^t |P_\gamma f^j(s)|_{2,+}^2 \mathrm{d}s \le \int_0^t \|\nu^{-1/2}g(s)\|_2^2\mathrm{d}s + \|f_0\|_2^2.\end{aligned} \tag{3.15}$$

Then the global existence of (3.9) and (3.10) follow from the standard continuation argument.

Step 2. For any $t > 0$, $\{f^j\}_{j=2}^{+\infty}$ *is convergent in* L^2. Notice that f^j enjoys the bound (3.15), and by taking a weak limit, we obtain a weak solution f to (3.1) and (3.2). Taking the difference, we further have that

$$\partial_t[f^j-f]+v\cdot\nabla_x[f^j-f]+L[f^j-f]=0, \quad [f^j-f]_- = P_\gamma[f^j-f]+\frac{1}{j}P_\gamma f^j, \tag{3.16}$$

with $[f^j-f](0)=0$. Utilizing standard L^2 energy estimates as for deriving (3.15), we obtain, for $\eta > 0$,

$$\begin{aligned}
&\|f^j(t)-f(t)\|_2^2+\int_0^t \|\{\mathbf{I}-\mathbf{P}\}[f^j(s)-f(s)]\|_\nu^2 \mathrm{d}s \\
&\quad +\int_0^t |\{I-P_\gamma\}[f^j(s)-f(s)]|_{2,+}^2 \mathrm{d}s \\
&\quad \lesssim \eta\int_0^t |P_\gamma[f^j(s)-f(s)]|_{2,+}^2 \mathrm{d}s+\frac{C_\eta}{j^2}\int_0^t |P_\gamma f^j|_{2,+}^2 \mathrm{d}s.
\end{aligned} \tag{3.17}$$

Since $\left(\frac{2}{j}-\frac{1}{j^2}\right)\int_0^t |P_\gamma f^j(s)|_{2,+}^2 \mathrm{d}s$ is bounded by (3.15), one can see that

$$\frac{C_\eta}{j^2}\int_0^t |P_\gamma f^j|_{2,+}^2 \mathrm{d}s \to 0, \quad \text{as} \quad j\to\infty.$$

To handle the small term $\eta\int_0^t |P_\gamma[f^j(s)-f(s)]|_{2,+}^2 \mathrm{d}s$, we resort to Ukai's trace theorem. We recall the boundary norm

$$\begin{aligned}
&\int_0^t |P_\gamma[f^j-f](s)|_{2,\pm}^2 \\
&\quad =\int_0^t \int_{\gamma_\pm}\left[\int_{\{u:n\cdot u>0\}}[f^j-f](s,x,u)\sqrt{\mu}\{n\cdot u\}\mathrm{d}u\right]^2\mu(v)\mathrm{d}\gamma\mathrm{d}s.
\end{aligned}$$

Now we split the domain of inner integration as follows:

$$\begin{aligned}
\{u\in\mathbb{R}^3: n(x)\cdot u>0\}=&\{u\in\mathbb{R}^3: 0<n(x)\cdot u<\varepsilon \quad \text{or} \quad |u|>1/\varepsilon \quad \text{or} \quad |u|<\varepsilon\} \\
&\cup\{u\in\mathbb{R}^3: \varepsilon\le n(x)\cdot u \quad \text{and} \quad |u|\le 1/\varepsilon \text{ and } |u|\ge\varepsilon\}.
\end{aligned}$$

The first set's contribution (the grazing part) of $\int_0^t |P_\gamma f^j(s)|_{2,\pm}^2 \mathrm{d}s$ is bounded by the Hölder inequality:

$$\begin{aligned}
&C\left(\int_{\substack{0<n\cdot u<\varepsilon\\ \text{or}|u|>1/\varepsilon\\ \text{or}|u|<\varepsilon}}\mu(u)\{n\cdot u\}\mathrm{d}u\right)\int_0^t\int_{\partial\Omega}\int_{\{u:n\cdot u>0\}}|[f^j-f](s)|^2\{n\cdot u\}\mathrm{d}S_x\mathrm{d}u\mathrm{d}s \\
&\quad\lesssim \varepsilon\int_0^t\int_{\gamma_+}|[f^j-f](s)|^2\mathrm{d}\gamma\mathrm{d}s.
\end{aligned} \tag{3.18}$$

For the second term, we use Lemma 2.4 and (3.16) to bound the second set's contribution (the non-grazing part) of $\int_0^t |P_\gamma[f^j - f](s)|_{2,\pm}^2 \mathrm{d}s$ by

$$\begin{aligned}
&C\int_0^t |[f^j - f](s)\mathbf{1}_{\gamma_+\setminus\gamma_+^\varepsilon}|_2^2 \mathrm{d}s \\
&\lesssim C\int_0^t \|[f^j - f](s)\|_2^2 \mathrm{d}s + C\int_0^t \|\partial_t[f^j - f]^2 + v\cdot\nabla_x[f^j - f]^2\|_1 \mathrm{d}s \\
&\lesssim C\int_0^t \|[f^j - f](s)\|_2^2 \mathrm{d}s + C\int_0^t |(L[f^j - f], [f^j - f])| \mathrm{d}s \\
&\lesssim C\int_0^t \|[f^j - f](s)\|_2^2 \mathrm{d}s + C\int_0^t \|\{\mathbf{I}-\mathbf{P}\}[f^j - f](s)\|_\nu^2 \mathrm{d}s.
\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we have, on the one hand,

$$\begin{aligned}
&\int_0^t |P_\gamma[f^j - f](s)|_{2,\pm}^2 \mathrm{d}s \\
&\quad\le \varepsilon\int_0^t\int_{\gamma_+} |[f^j - f](s)|^2 \mathrm{d}\gamma\mathrm{d}s \\
&\qquad + C_\varepsilon\left\{\int_0^t \left[\|[f^j - f](s)\|_2^2 + \|(\mathbf{I}-\mathbf{P})[f^j - f](s)\|_\nu^2\right]\mathrm{d}s\right\}.
\end{aligned} \tag{3.20}$$

On the other hand, we get, by integrating (3.17) from 0 to t,

$$\begin{aligned}
&\int_0^t ||\mathbf{P}[f^j - f](s)||_2^2 \mathrm{d}s \\
&\quad\lesssim \eta C_t\int_0^t |P_\gamma[f^j(s) - f(s)]|_{2,+}^2 \mathrm{d}s + \frac{C_t C_\eta}{j^2}\int_0^t |P_\gamma f^j|_{2,+}^2 \mathrm{d}s.
\end{aligned} \tag{3.21}$$

Letting $\varepsilon > 0$ and $\eta > 0$ be suitably small and taking an appropriate linear combination of (3.17), (3.20) and (3.21), we improve (3.17) as follows:

$$\begin{aligned}
&\|f^j(t) - f(t)\|_2^2 + \int_0^t \|f^j(s) - f(s)\|_\nu^2 \mathrm{d}s + \int_0^t |f^j(s) - f(s)|_2^2 \mathrm{d}s \\
&\lesssim \frac{C_t}{j^2}\int_0^t |P_\gamma f^j|^2 \mathrm{d}s \to 0,
\end{aligned}$$

which implies $f^j \to f$ strongly in L^2 for any given $t \ge 0$. Moreover, we can also show that such a solution is unique by L^2 energy estimates similar to these used above. As a consequence, we construct $f(t,x,v)$ as an L^2 strong solution to (3.1) and (3.2) for any $t \ge 0$. Finally, by taking the inner product of (3.1) with f over $\Omega\times\mathbb{R}^3$ and applying Green's identity again, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_2^2 + \lambda\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 + |(I - P_\gamma)f|_{2,+}^2 \le \|\nu^{-1/2}g\|^2. \tag{3.22}$$

Letting $0 < \kappa_1 \ll 1$, taking the summation of (3.22) and $\kappa_1 \times$ (3.7), we obtain

$$\frac{d}{dt}\left\{\|f\|_2^2 - \kappa_1 G(t)\right\} + \lambda\|f\|_\nu^2 + \lambda|(I-P_\gamma)f|_{2,+}^2 \leq \|\nu^{-1/2}g\|^2. \tag{3.23}$$

Then (3.8) follows from (3.23). This completes the proof of Lemma 3.2. □

With Lemmas 3.2 and 3.1 in hand, we now turn to complete

The proof of Proposition 3.1. Let $h = w_{q/2,\theta} f$, then having (3.1) and (3.2) is equivalent to

$$\begin{aligned}\partial_t h + v\cdot\nabla h + \nu h - w_{q/2,\theta} K\left(\frac{h}{w_{q/2,\theta}}\right) &= w_{q/2,\theta}\, g,\ h(0,x,v)\\ &= h_0(x,v) = w_{q/2,\theta} f_0(x,v),\end{aligned} \tag{3.24}$$

and

$$h_- = w_{q/2,\theta}\sqrt{\mu}\int_{\mathcal{V}(x)} h(t,x,v')\frac{1}{w_{q/2,\theta}(v')\sqrt{\mu}(v')}d\sigma \stackrel{\text{def}}{=} P_\gamma^w h, \tag{3.25}$$

where

$$\mathcal{V}(x) = \{v' \in \mathbb{R}^3, v'\cdot n(x) > 0\},\quad d\sigma = \mu(v')|n(x)\cdot v'|dv'.$$

Proceeding similarly to obtain the global existence of (3.1) and (3.2), one can show that (3.24) and (3.25) possess a unique solution $h(t,x,v)$. We now turn to prove (3.5) and (3.6). Taking the inner product of (3.24) with h over $\Omega\times\mathbb{R}^3$ and applying Lemma 2.2, one has

$$\frac{d}{dt}\|h\|_2^2 + |\{I-P_\gamma^w\}h|_{2,+}^2 + \|h\|_\nu^2 \leq \eta\|h\|_\nu^2 + C_\eta\|f\|_\nu^2 + C\|\nu^{-1/2}w_{q/2,\theta}g\|_2^2, \tag{3.26}$$

where we have also used the fact that $|P_\gamma^w h|_{2,+}^2 = |P_\gamma^w h|_{2,-}^2$. Integrating (3.26) with respect to the time variable over $[0,t]$ and combining it with (3.8), we obtain

$$\begin{aligned}&\|h(t)\|_2^2 + \int_0^t |\{I-P_\gamma^w\}h|_{2,+}^2 dt + \int_0^t \|h\|_\nu^2 ds + \|f(t)\|_2^2\\ &\quad + \int_0^t \|f(s)\|_\nu^2 ds + \int_0^t |(I-P_\gamma)f(s)|_{2,+}^2 ds\\ &\leq C\|w_{q/2,\theta} f_0\|_2^2 + C\int_0^t \|\nu^{-1/2}w_{q/2,\theta}g(s)\|_2^2 ds,\end{aligned}$$

which implies (3.5). It remains now to prove the time decay (3.6). Take constants $\lambda_1 > 0$ and $0 < \rho_0 < 1$, whose specific values will be determined later on, multiply $e^{\lambda_1 t^{\rho_0}}$ to (3.23) and integrate the resulting inequality with respect to the time variable over $[0,t]$ to obtain

$$\begin{aligned}&e^{\lambda_1 t^{\rho}}\|f\|_2^2 + \int_0^t e^{\lambda_1 s^{\rho}}\|f\|_\nu^2 ds + \int_0^t e^{\lambda_1 s^{\rho_0}}|(I-P_\gamma)f|_{2,+}^2 ds\\ &\leq C\|f_0\|_2^2 + C\lambda_1\rho_0\int_0^t s^{\rho-1}e^{\lambda_1 s^{\rho_0}}\|f\|_2^2 ds + C\int_0^t e^{\lambda_1 s^{\rho_0}}\|\nu^{-1/2}g\|_2^2 ds.\end{aligned} \tag{3.27}$$

To take care of the delicate term $s^{\rho_0-1}e^{\lambda_1 s^{\rho_0}}\|f\|_2^2$, we decompose the v integration domain into two parts:

$$E:\{v|s^{\rho-1}\le\kappa_0(1+|v|^2)^{\varrho/2}\},\quad E^c:\{v|s^{\rho_0-1}\ge\kappa_0(1+|v|^2)^{\varrho/2}\},$$

where $\kappa_0>0$ and is small enough. On E, it is straightforward to see that

$$s^{\rho_0-1}e^{\lambda_1 s^{\rho_0}}\|f\mathbf{1}_E\|_2^2\le C_\varrho\kappa_0 e^{\lambda_1 s^{\rho_0}}\|f\|_\nu^2, \tag{3.28}$$

where

$$\mathbf{1}_E=\begin{cases}1, & v\in E,\\ 0, & v\notin E,\end{cases}$$

and C_ϱ is determined by (1.16). While on E^c, notice that $0<\rho_0<1$, one obtains

$$2\lambda_1 s^{\rho_0}\le 2\lambda_1\kappa_0^{\frac{\rho_0}{\rho_0-1}}(1+|v|^2)^{\frac{\varrho\rho_0}{2(\rho_0-1)}}.$$

With this, we further have, by letting $\lambda_1=\frac{q}{8}\kappa_0^{\frac{\rho_0}{1-\rho_0}}$ and $\rho_0=\frac{\theta}{\theta-\varrho}$,

$$\begin{aligned}\int_0^t s^{\rho_0-1}e^{\lambda_1 s^{\rho_0}}\|f\mathbf{1}_{E^c}\|_2^2\mathrm{d}s&=\int_0^t s^{\rho_0-1}e^{-\lambda_1 s^{\rho_0}}e^{2\lambda_1 s^{\rho_0}}\|f\mathbf{1}_{E^c}\|_2^2\mathrm{d}s\\ &\le C_\theta\int_0^t s^{\rho_0-1}e^{-\lambda_1 s^{\rho_0}}\|w_{q/2,\theta}f\|_2^2\mathrm{d}s\\ &\le C\|w_{q/2,\theta}f_0\|_2^2+C\int_0^t\|w_{q/2,\theta}\nu^{-1/2}g(s)\|_2^2\mathrm{d}s.\end{aligned} \tag{3.29}$$

Here we have used (3.5) to derive the last inequality. Plugging (3.28) and (3.29) into (3.27) and dividing the resulting inequality by $e^{\lambda_1 t^{\rho_0}}$, we then show that (3.6) holds true. This concludes the proof of Proposition 3.1. □

3.2. L^∞ Existence for the Linearized Equation

In this subsection, we still consider the following initial boundary value problem:

$$\partial_t f+v\cdot\nabla_x f+Lf=g,\ \ f(0)=f_0,\quad \text{in}\quad (0,+\infty)\times\Omega\times\mathbb{R}^3, \tag{3.30}$$

with

$$f_-=P_\gamma f,\quad \text{on}\quad \mathbb{R}_+\times\gamma_-, \tag{3.31}$$

where g is given. Our purpose is to establish the global existence for (3.30) and (3.31) in a weighted L^∞ space. A key point is that we develop some new iterated integral estimates so that one can construct the L^∞ existence without using the time-decay of the solution in the $L^\infty-$norm. We stress that it is very difficult to obtain the global existence and the time-decay of the solution in L^∞ space at the same time due to the fact that the collision frequency ν has zero lower bound. The main result of this subsection is the following:

Proposition 3.2. *Let $(q, \theta) \in \mathcal{A}_{q,\theta}$, and assume that (3.3) holds true. Then the initial boundary value problem (3.1) and (3.2) admits a unique solution satisfying*

$$\|w_{q,\theta} f\|_\infty + |w_{q,\theta} f|_\infty \lesssim \|w_{q,\theta} f_0\|_\infty + \sup_{0 \le s \le t} \|\nu^{-1} w_{q,\theta} g(s)\|_\infty + \sqrt{\int_0^t e^{\lambda_1 s^{\rho_0}} \|\nu^{-1/2} g(s)\|_2^2 \mathrm{d}s} + \sqrt{\int_0^t \|\nu^{-1/2} w_{q/2,\theta} g(s)\|_2^2 \mathrm{d}s}. \tag{3.32}$$

Our proof for Proposition 3.2 relies heavily upon the estimates for the iterated integral defined on stochastic cycles. The stochastic cycles are defined as follows:

Definition 3.1 (*Stochastic Cycles*). Fixing any point (t, x, v) with $(x, v) \notin \gamma_0$, let $(t_0, x_0, v_0) = (t, x, v)$. For v_{k+1}such that $v_{k+1} \cdot n(x_{k+1}) > 0$, define the $(k+1)$-component of the back-time cycle as

$$(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_{\mathbf{b}}(x_k, v_k), x_{\mathbf{b}}(x_k, v_k), v_{k+1}). \tag{3.33}$$

Set

$$X_{\mathbf{cl}}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)\{x_k + (s - t_k)v_k\},$$
$$V_{\mathbf{cl}}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s) v_k.$$

Define $\mathcal{V}_{k+1} = \{v \in \mathbb{R}^3 \mid v \cdot n(x_{k+1}) > 0\}$, and let the iterated integral for $k \ge 2$ be defined as

$$\int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \Pi_{j=1}^{k-1} \mathrm{d}\sigma_j \equiv \int_{\mathcal{V}_1} \cdots \left\{ \int_{\mathcal{V}_{k-1}} \mathrm{d}\sigma_{k-1} \right\} \mathrm{d}\sigma_1,$$

where $\mathrm{d}\sigma_j = \mu(v)(n(x_j) \cdot v)\mathrm{d}v$ is a probability measure.

Lemma 3.3. *Let $T_0 > 0$ and be large enough, denote $\alpha(t) = \max\{t, T_0\}$; then there exist constants $C_1, C_2 > 0$ independent of $\alpha(t)$, such that for $k = C_1[\alpha(t)]^{5/4}$, and $(t, x, v) \in [0, \infty) \times \overline{\Omega} \times \mathbb{R}^3$,*

$$\int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k(t,x,v,v_1,v_2,\ldots,v_{k-1})>0\}} \Pi_{j=1}^{k-1} \mathrm{d}\sigma_j \le \left\{ \frac{1}{2} \right\}^{C_2[\alpha(t)]^{5/4}}. \tag{3.34}$$

We also have, for $(q, \theta) \in \mathcal{A}_{q,\theta}$, that there exist constants $C_3,\ C_4 > 0$ independent of k such that

$$\int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \le 0 < t_l\}} \int_0^{t_l} \mathrm{d}\Sigma_l^w(s) \mathrm{d}s \le C_3, \tag{3.35}$$

and

$$\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\mathrm{d}\Sigma_l^w(s)\mathrm{d}s\le C_4, \tag{3.36}$$

where

$$\mathrm{d}\Sigma_l^w(s)=\left\{\Pi_{j=l+1}^{k-1}\mathrm{d}\sigma_j\right\}\times\{e^{\nu(v_l)(s-t_l)}\tilde{w}_{q,\theta}(v_l)\mathrm{d}\sigma_l\}\times\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}, \tag{3.37}$$

and $\widetilde{w}_{q,\theta}=\frac{1}{w_{q,\theta}\sqrt{\mu}}$.

Proof. If $0<t\le T_0$, then $\alpha(t)=T_0$. The proof of (3.34) has already been given by Lemma 23 in [27, pp. 781]. For the case that $T_0<t<+\infty$, setting $T_0=t$ in Lemma 23 of [27, pp. 781] and performing the same computations as its proof, one sees that (3.34) is also true for $\alpha(t)=t$. In what follows, we mainly prove (3.36); the proof for (3.35) will only be briefly sketched. For any $k>0$, we first split the the left hand side of (3.36) as

$$\begin{aligned}
&\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\tilde{w}_{q,\theta}(v_l)\nu^{-1}(v_l)\{\Pi_{j=l+1}^{k-1}\mathrm{d}\sigma_j\}\{e^{\nu(v_l)(s-t_l)}\nu(v_l)\mathrm{d}\sigma_l\}\\
&\qquad\times\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\mathrm{d}s\\
&\quad=\int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}\le k}}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\tilde{w}_{q,\theta}(v_l)\nu^{-1}(v_l)\{\Pi_{j=l+1}^{k-1}\mathrm{d}\sigma_j\}\\
&\qquad\times\{e^{\nu(v_l)(s-t_l)}\nu(v_l)\mathrm{d}\sigma_l\}\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\mathrm{d}s\\
&\qquad+\int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}>k}}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\tilde{w}_{q,\theta}(v_l)\nu^{-1}(v_l)\{\Pi_{j=l+1}^{k-1}\mathrm{d}\sigma_j\}\\
&\qquad\times\{e^{\nu(v_l)(s-t_l)}\nu(v_l)\mathrm{d}\sigma_l\}\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\mathrm{d}s=\mathcal{K}_1+\mathcal{K}_2.
\end{aligned} \tag{3.38}$$

For $\mathcal{K}_1$, denote $\max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}=|v_m|$, one has

$$\begin{aligned}
\mathcal{K}_1&\le C_{q,\theta}\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}e^{\nu(v_m)(s-t_1)}\nu(v_m)\mathrm{d}s\,\tilde{w}_{q,\theta}(v_m)\nu^{-1}\\
&\qquad(v_m)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j\\
&\le C_{q,\theta}\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\int_{t_k}^{t_1}e^{\nu(v_m)(s-t_1)}\nu(v_m)\mathrm{d}s\,\tilde{w}_{q,\theta}(v_m)\nu^{-1}(v_m)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j\\
&\le\frac{C_{q,\theta}}{\sqrt{2\pi}}\int_{n(x_m)\cdot v_m>0}(n(x_m)\cdot v_m)e^{-\frac14|v_m|^2-\frac{q}{4}|v_m|^\theta}\nu^{-1}(v_m)\mathrm{d}v_m\\
&\le\frac{C_{q,\theta}}{\sqrt{2\pi}}\int_{u_{m1}>0}u_{m1}e^{-\frac14|u_m|^2}\mathrm{d}u_m\le C_{q,\theta}.
\end{aligned}$$

Here we have used the key observation

$$\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l} e^{\nu(v_m)(s-t_1)}\nu(v_m)\mathrm{d}s \le \int_{t_k}^{t_1} e^{\nu(v_m)(s-t_1)}\nu(v_m)\mathrm{d}s \le 2.$$

As to $\mathcal{K}_2$, without loss of generality, we may assume that $|v_i| > k$ for some $i \in \{1, 2, \dots, k-1\}$, then

$$\begin{aligned}
\mathcal{K}_2 &\le C\sum_{l=1}^{k-1}\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j} \tilde{w}_{q,\theta}(v_l)\nu^{-1}(v_l)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \\
&\le C\int_{\Pi_{j=1}^{i-1}\mathcal{V}_j}\Pi_{j=1}^{i-1}\mathrm{d}\sigma_j\int_{\substack{n(x_i)\cdot v_i>0\\ |v_i|>k}}(n(x_i)\cdot v_i)e^{-\frac{1}{4}|v_i|^2-\frac{q}{4}|v_i|^\theta}\nu^{-1}(v_i)\mathrm{d}v_i \\
&\quad +C\sum_{l=1}^{i-1}\int_{\Pi_{j=1}^{l-1}\mathcal{V}_j}\Pi_{j=1}^{l-1}\mathrm{d}\sigma_j\int_{n(x_l)\cdot v_l>0}(n(x_l)\cdot v_l)e^{-\frac{1}{4}|v_l|^2-\frac{q}{4}|v_l|^\theta}\nu^{-1}(v_l)\mathrm{d}v_l \\
&\quad \times\int_{\Pi_{j=l+1}^{i-1}\mathcal{V}_j}\Pi_{j=l+1}^{i-1}\mathrm{d}\sigma_j\int_{\substack{n(x_i)\cdot v_i>0\\ |v_i|>k}}e^{-\frac{|v_i|^2}{2}}(n(x_i)\cdot v_i)\mathrm{d}v_i \\
&\quad +C\sum_{l=i+1}^{k-1}\int_{\Pi_{j=1}^{i-1}\mathcal{V}_j}\Pi_{j=1}^{i-1}\mathrm{d}\sigma_j\int_{\substack{n(x_i)\cdot v_i>0\\ |v_i|>k}}e^{-\frac{|v_i|^2}{2}}(n(x_i)\cdot v_i)\mathrm{d}v_i \\
&\quad \times\int_{\Pi_{j=i+1}^{l-1}\mathcal{V}_j}\Pi_{j=i+1}^{l-1}\mathrm{d}\sigma_j\int_{n(x_l)\cdot v_l>0}(n(x_l)\cdot v_l)e^{-\frac{1}{4}|v_l|^2-\frac{q}{4}|v_l|^\theta}\nu^{-1}(v_l)\mathrm{d}v_l \\
&\le C_{q,\theta}(k-1)e^{-\frac{k^2}{8}} \le C_{q,\theta}.
\end{aligned}$$

Substituting the above estimates for $\mathcal{K}_1$ and $\mathcal{K}_2$ into (3.38), we see that (3.36) is true.

The proof for (3.35) is very similar to that of (3.36), the only difference being the following:

$$\begin{aligned}
&\int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\dots,|v_{k-1}|\}\le k}}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}\tilde{w}_{q,\theta}(v_l)\nu^{-1}(v_l)\{\Pi_{j=l+1}^{k-1}\mathrm{d}\sigma_j\} \\
&\qquad\qquad \times\{e^{\nu(v_l)(s-t_l)}\nu(v_l)\mathrm{d}\sigma_l\}\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\mathrm{d}s \\
&\qquad\le \int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\tilde{w}_{q,\theta}(v_m)\nu^{-1}(v_m)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \\
&\qquad\le \int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\tilde{w}_{q,\theta}(v_m)\nu^{-1}(v_m)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \le C_{q,\theta};
\end{aligned}$$

here the second inequality follows due to $\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}} = \mathbf{1}_{\{t_{l+1}\le 0\}}$. This finishes the proof of Lemma 3.3. □

Remark 3.1. Since $\alpha(t) \le T_0$, the upper bound on the right hand side of (3.34) can be relaxed to $\left\{\frac{1}{2}\right\}^{C_2 T_0^{5/4}}$.

Prior to proving Proposition 3.2, we first show the following global solvability of (3.1) and (3.2) in the L^∞ space without weight:

Lemma 3.4. *There exists* $\varepsilon_0 > 0$ *such that if* $\mathbf{P}g = 0$ *and*

$$\|f_0\|_{L^\infty(\Omega\cup\gamma_+)} + \|w_{q/2,\theta} f_0\|_2 + \sup_{0\le s\le t} \|\nu^{-1} g(s)\|_\infty + \sqrt{\int_0^t e^{\lambda_1 s^{\rho_0}} \|\nu^{-1/2} g(s)\|_2^2 \mathrm{d}s} + \sqrt{\int_0^t \|\nu^{-1/2} w_{q/2,\theta} g(s)\|_2} \le \varepsilon_0,$$

then (3.1) *and* (3.2) *admit a unique solution* $f(t,x,v)$ *for which it holds that*

$$\sup_{0\le s\le t} \|f\|_\infty + |f|_\infty \lesssim \|f_0\|_{L^\infty(\Omega\cup\gamma_+)} + \|w_{q/2,\theta} f_0\|_2 + \sup_{0\le s\le t} \|\nu^{-1} g(s)\|_\infty + \sqrt{\int_0^t e^{\lambda_1 s^{\rho_0}} \|\nu^{-1/2} g(s)\|_2^2 \mathrm{d}s} + \sqrt{\int_0^t \|\nu^{-1/2} w_{q/2,\theta} g(s)\|_2^2 \mathrm{d}s}. \tag{3.39}$$

Proof. As in the proof of Lemma 3.2, we use the approximate form

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + Lf = g, \quad f(0,x,v) = f_0, \\ f_- = \left(1 - \frac{1}{j}\right) P_\gamma f, \ j = 2, 3, \dots \end{cases} \tag{3.40}$$

to construct the global existence of (3.1) and (3.2), while the global solution (denoted by f^j) of (3.40) is further established by the following iteration scheme:

$$\begin{cases} \partial_t f^{\ell+1} + v\cdot\nabla_x f^{\ell+1} + \nu f^{\ell+1} - K f^\ell = g, \quad f^{\ell+1}(0) = f_0, \ \ \ell \ge 0, \ f^0 \equiv f_0, \\ f_-^{\ell+1} = \left(1 - \frac{1}{j}\right) P_\gamma f^\ell, \ j = 2, 3, \dots. \end{cases}$$

To do this, performing a similar calculation as for deriving (199) in Lemma 24 of [27, pp. 783], we find

$$\begin{aligned} |f^{\ell+1}(t,x,v)| \le{}& \underbrace{\left\{\mathbf{1}_{t_1\le 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)} |K f^\ell(s, x-(t-s)v, v)| \mathrm{d}s}_{I_1} \\ & \underbrace{+ \left\{\mathbf{1}_{t_1\le 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)} |g(s, x-(t-s)v, v)| \mathrm{d}s}_{I_3} \\ & + \mathbf{1}_{t_1\le 0} e^{-\nu(v)t} |f^{\ell+1}(0, x-tv, v)| \\ & + \mathbf{1}_{t_1>0}\left(1 - \frac{1}{j}\right) e^{-\nu(v)(t-t_1)} \int_{\mathcal{V}_1} |f^\ell| \mathrm{d}\sigma_1, \end{aligned}$$

where the last line follows from the boundary condition. A direct calculation leads us to

$$\begin{aligned}\|f^{\ell+1}\|_{L^\infty(\Omega\cup\gamma_+)} &\leq tC\|f^\ell\|_\infty+\left(1-\frac{1}{j}\right)|f^\ell|_{\infty,+}+\|f_0\|_\infty+\sup_{0\leq s\leq t}\|\nu^{-1}g(s)\|_\infty\\ &\leq tC\|f^\ell\|_\infty+\left(1-\frac{1}{j}\right)|f^\ell|_{\infty,+}+\varepsilon_0.\end{aligned}$$

With this, one can show that there exists $T^{**}>0$ ($CT^{**}<1$) such that if $\sup_{0\leq t\leq T^{**}}\|f^\ell\|_{L^\infty(\Omega\cup\gamma_+)}\leq 2\varepsilon_0$, then

$$\sup_{0\leq t\leq T^{**}}\|f^{\ell+1}\|_{L^\infty(\Omega\cup\gamma_+)}\leq 2\varepsilon_0,$$

thus $\{\|f^\ell\|_{L^\infty(\Omega\cup\gamma_+)}\}$ is uniformly bounded with respect to ℓ in a short time interval $[0, T^{**}]$. In fact, we can further prove that $\{f^\ell\}$ is also a Cauchy sequence in $L^\infty(\Omega\cup\gamma_+)$, provided that $CT^{**}<1$, thus we obtain a local solution f^j for (3.14). To construct the global existence, it suffices to obtain the following *a priori estimates*:

$$\begin{aligned}&\sup_{0\leq s\leq t}\{\|f^j\|_\infty+|f^j|_{\infty,+}\}\\ &\quad\lesssim\|f_0\|_{L^\infty(\Omega\cup\gamma_+)}+\sup_{0\leq s\leq t}\|\nu^{-1}g(s)\|_\infty+\|w_{q/2,\theta}f_0\|_2\\ &\qquad+\sqrt{\int_0^t e^{\lambda_1 s^{\rho_0}}\|\nu^{-1/2}g(s)\|_2^2\mathrm{d}s}+\sqrt{\int_0^t\|\nu^{-1/2}w_{q/2,\theta}g(s)\|_2^2\mathrm{d}s},\end{aligned}\tag{3.41}$$

for all $j\geq 2$. In fact, (3.41) follows from a tedious calculation for the following inequality:

$$\begin{aligned}|f^j(t,x,v)|\leq&\left\{\mathbf{1}_{t_1\leq 0}\int_0^t+\mathbf{1}_{t_1>0}\int_{t_1}^t\right\}e^{-\nu(v)(t-s)}|K^{1-\chi}f^j(s,x-(t-s)v,v)|\mathrm{d}s\\ &+\left\{\mathbf{1}_{t_1\leq 0}\int_0^t+\mathbf{1}_{t_1>0}\int_{t_1}^t\right\}e^{-\nu(v)(t-s)}|K^{\chi}f^j(s,x-(t-s)v,v)|\mathrm{d}s\\ &+\left\{\mathbf{1}_{t_1\leq 0}\int_0^t+\mathbf{1}_{t_1>0}\int_{t_1}^t\right\}e^{-\nu(v)(t-s)}|g(s,x-(t-s)v,v)|\mathrm{d}s\\ &+\sum_{n=1}^5 I_n,\end{aligned}\tag{3.42}$$

with

$$\begin{aligned}I_1=&\,\mathbf{1}_{t_1\leq 0}e^{-\nu(v)t}|f(0,x-tv,v)|\\ &+e^{-\nu(v)(t-t_1)}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\leq 0<t_l\}}|f(0,x_l-t_lv_l,v_l)|\mathrm{d}\Sigma_l(0),\end{aligned}$$

$$I_2 = e^{-\nu(v)(t-t_1)}\sqrt{\mu}\bigg\{ \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \sum_{l=1}^{k-1} \int_0^{t_l}$$
$$\times |[K^{1-\chi} f^j](s, x_l-(t_l-s)v_l, v_l)| \mathrm{d}\Sigma_l(s)\mathrm{d}s$$
$$+ \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{0<t_{l+1}\}} \int_{t_{l+1}}^{t_l} |[K^{1-\chi} f^j](s, x_l-(t_l-s)v_l, v_l)| \mathrm{d}\Sigma_l(s)\mathrm{d}s\bigg\},$$
$$I_3 = e^{-\nu(v)(t-t_1)}\sqrt{\mu}\bigg\{ \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \sum_{l=1}^{k-1} \int_0^{t_l}$$
$$\times |[K^{\chi} f^j](s, x_l-(t_l-s)v_l, v_l)| \mathrm{d}\Sigma_l(s)\mathrm{d}s$$
$$+ \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{0<t_{l+1}\}} \int_{t_{l+1}}^{t_l} |[K^{\chi} f^j](s, x_l-(t_l-s)v_l, v_l)| \mathrm{d}\Sigma_l(s)\mathrm{d}s\bigg\},$$
$$I_4 = e^{-\nu(v)(t-t_1)}\sqrt{\mu}\bigg\{ \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}}$$
$$\int_0^{t_l} |g(s, x_l-(t_l-s)v_l, v_l)| \mathrm{d}\Sigma_l(s)\mathrm{d}s$$
$$+ \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{0<t_{l+1}\}} \int_{t_{l+1}}^{t_l} |g(s, x_l-(t_l-s)v_l, v_l)| \mathrm{d}\Sigma_l(s)\mathrm{d}s\bigg\},$$
$$I_5 = e^{-\nu(v)(t-t_1)}\sqrt{\mu} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{0<t_k\}} |f^j(t_k, x_k, v_{k-1})| \mathrm{d}\Sigma_{k-1}(t_k), \quad k \ge 2,$$

and

$$\mathrm{d}\Sigma_l(s) = \{\Pi_{j=l+1}^{k-1}\mathrm{d}\sigma_j\} \times \{e^{\nu(v_l)(s-t_l)}\mu^{-1/2}(v_l)\mathrm{d}\sigma_l\} \times \Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}. \tag{3.43}$$

We point out that (3.42) is deduced from (3.14) by means of a similar argument as for obtaining (199) in [27, pp. 783]. The estimates for the corresponding terms on the right hand side of (3.42) are very similar to that of $\mathcal{I}_n$ $(1 \le n \le 8)$ in (3.54). To avoid needless repetition, we are not going to detail the computations here. When (3.41) is derived, the global existence of (3.11) and (3.12) follow from a standard continuation argument. Notice that (3.41) is unform in j, and that $\{f^j\}_{j=1}^{\infty}$ possesses (up to a subsequence) a weak$-*$ limit f which satisfies (3.1) and (3.2) in the weak sense. Again, by taking a difference, one has

$$\begin{cases} \partial_t[f^j-f] + v\cdot\nabla_x[f^j-f] + L[f^j-f] = 0, & [f^j-f](0) = 0, \\ [f^j-f]_- = P_\gamma[f^j-f] + \frac{1}{j}P_\gamma f^j, \end{cases}$$

from which we have by an argument similar to that for obtaining (3.42)

$$
\begin{aligned}
&|[f^j - f](t,x,v)| \\
&\le \left\{\mathbf{1}_{t_1\le 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|K^{1-\chi}[f^j - f](s, x-(t-s)v, v)|\mathrm{d}s \\
&\quad + \left\{\mathbf{1}_{t_1\le 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|K^{\chi}[f^j - f](s, x-(t-s)v, v)|\mathrm{d}s \\
&\quad + \sum_{n=6}^{9} I_n,
\end{aligned}
\tag{3.44}
$$

with

$$
\begin{aligned}
I_6 &= \frac{1}{j}\mathbf{1}_{t_1>0}e^{-\nu(v)(t-t_1)}|(P_\gamma f^j)(t_1, x_1, v)| \\
&\quad + \frac{1}{j}e^{-\nu(v)(t-t_1)}\sqrt{\mu}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}>0\}}|(P_\gamma f^j)(t_{l+1}, x_{l+1}, v_l)|\mathrm{d}\Sigma_l(t_{l+1}), \\
I_7 &= e^{-\nu(v)(t-t_1)}\sqrt{\mu}\Bigg\{\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \\
&\quad \times \int_0^{t_l}|[K^{1-\chi}[f^j - f]](s, x_l-(t_l-s)v_l, v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s \\
&\quad + \int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{0<t_{l+1}\}}\int_{t_{l+1}}^{t_l}|[K^{1-\chi}[f^j - f]] \\
&\quad (s, x_l-(t_l-s)v_l, v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s\Bigg\}, \\
I_8 &= e^{-\nu(v)(t-t_1)}\sqrt{\mu}\Bigg\{\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \\
&\quad \times \int_0^{t_l}|[K^{\chi}[f^j - f]](s, x_l-(t_l-s)v_l, v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s \\
&\quad + \int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{0<t_{l+1}\}}\int_{t_{l+1}}^{t_l}|[K^{\chi}[f^j - f]] \\
&\quad (s, x_l-(t_l-s)v_l, v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s\Bigg\}, \\
I_9 &= e^{-\nu(v)(t-t_1)}\sqrt{\mu}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{0<t_k\}}|[f^j - f](t_k, x_k, v_{k-1})|\mathrm{d}\Sigma_{k-1}(t_k), \quad k\ge 2.
\end{aligned}
$$

Comparing (3.44) with (3.42), one obtains

$$
\sup_{0\le s\le t}\{\|[f^j - f]\|_\infty(s) + |[f^j - f]|_{\infty,+}(s)\} \lesssim C\sup_{0\le s\le t}|I_6|. \tag{3.45}
$$

On the other hand, from Lemma 3.3, it follows that

$$|I_6| \leq \frac{C}{j}|P_\gamma f^j|_{\infty,-} \leq \frac{C}{j}|f^j|_{\infty,+}. \tag{3.46}$$

Then, (3.45) and (3.46) lead us to

$$\sup_{0\leq s\leq t}\{\|[f^j - f](s)_\infty + |[f^j - f](s)|_{\infty,+}\} \lesssim \frac{C}{j}\sup_{0\leq s\leq t}|f^j|_{\infty,+},$$

from which, along with the bound (3.41), we see that f^j converges to f strongly in L^∞ and that f satisfies (3.39), and this completes the proof of Lemma 3.4. □

With Lemma 3.4 in hand, we are now ready to tackle

The proof of Proposition 3.2. Similar to the analysis in Section 3.1, denote

$$h^\ell = w_{q,\theta}f^\ell,\quad \ell \geq 0,\quad \text{and}\quad K_w(\cdot) = w_{q,\theta}K\left(\frac{\cdot}{w_{q,\theta}}\right),$$

where f^ℓ is determined by (3.11) and (3.12). The solution $h^j(t,x,v) = w_{q,\theta}f^j$ of the problem

$$\partial_t h + v\cdot\nabla_x h + \nu h - K_w h = w_{q,\theta}g,\quad h(0,x,v) = h_0(x,v) = w_{q,\theta}f_0(x,v), \tag{3.47}$$

and

$$h_- = \frac{1-\frac{1}{j}}{\widetilde{w}_{q,\theta}}\int_{\mathcal{V}(x)} h(t,x,v')\widetilde{w}_{q,\theta}(v')\mathrm{d}\sigma \tag{3.48}$$

will be constructed with the help of an *abstract iteration* scheme defined in the following way:

$$\begin{cases} \partial_t h^{\ell+1} + v\cdot\nabla_x h^{\ell+1} + \nu h^{\ell+1} - K_w h^\ell = w_{q,\theta}g, \\ h^{\ell+1}(0,x,v) = h_0^{\ell+1}(x,v) = w_{q,\theta}f_0(x,v), \ell \geq 0, \end{cases} \tag{3.49}$$

with $h^0 = h_0 = w_{q,\theta}f_0(x,v)$ and

$$h_-^{\ell+1} = \frac{1-\frac{1}{j}}{\widetilde{w}_{q,\theta}}\int_{\mathcal{V}(x)} h^\ell(t,x,v')\widetilde{w}_{q,\theta}(v')\mathrm{d}\sigma. \tag{3.50}$$

From (3.49) and (3.50), it is straightforward to check that

$$\begin{aligned} |h^{\ell+1}(t,x,v)| \leq{}& \left\{\mathbf{1}_{t_1\leq 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|K_w h^\ell(s,x-(t-s)v,v)|\mathrm{d}s \\ &+ \left\{\mathbf{1}_{t_1\leq 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|w_{q,\theta}g(s,x-(t-s)v,v)|\mathrm{d}s \\ &+ \mathbf{1}_{t_1\leq 0}e^{-\nu(v)t}|h^{\ell+1}(0,x-tv,v)| \\ &+ \mathbf{1}_{t_1>0}\left(1-\frac{1}{j}\right)\frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}_{q,\theta}(v)}\int_{\mathcal{V}_1}|h^\ell(t_1,x_1,v_1)|\tilde{w}_{q,\theta}(v_1)\mathrm{d}\sigma_1. \end{aligned} \tag{3.51}$$

Since $\frac{1}{\tilde{w}_{q,\theta}(v)} \leq C_{q,\theta}$, and $\int_{n\cdot v>0}\sqrt{\mu}(n\cdot v)\mathrm{d}v < \infty$, we get from (3.51) and Lemma 2.2 that

$$\|h^{\ell+1}(t)\|_\infty \leq Ct\|h^\ell(t,x,v)\|_\infty + C\|h_0\|_\infty + C\sup_{0\leq s\leq t}\|\nu^{-1}w_{q,\theta}g(s)\|_\infty \\ + C_{q,\theta}\max_\ell \sup_{0\leq s\leq t}|f^\ell|_{\infty,+}. \tag{3.52}$$

Recalling Lemma 3.4, we have shown that $f^\ell \to f^j$ and f^j bears the bound (3.41), therefore $\max_\ell \sup_{0\leq s\leq t}|f^\ell|_{\infty,+} < \infty$. From this and (3.52), for any given $j \geq 2$, the existence of a local solution h^j to (3.47) and (3.48) is guaranteed by an argument similar to the proof of Lemma 3.4.

To obtain the global existence of (3.47) and (3.48), a central part of the deduction is the following *a priori* estimate:

$$\sup_{0\leq s\leq t}\|h^j(s)\|_\infty \leq C\|h_0\|_\infty + C\sup_{0\leq s\leq t}\|\nu^{-1}w_{q,\theta}g(s)\|_\infty + C\|w_{q/2,\theta}f_0\|_2 \\ + C\sqrt{\int_0^t e^{\lambda_1 s^{\rho_0}}\|\nu^{-1/2}g(s)\|_2^2\mathrm{d}s} \\ + C\sqrt{\int_0^t \|\nu^{-1/2}w_{q/2,\theta}g(s)\|_2^2\mathrm{d}s}. \tag{3.53}$$

Once again using (3.47) and (3.48), we proceed as for deducing (199) in [27, pp. 783] to obtain

$$|h^j(t,x,v)| \leq \underbrace{\left\{\mathbf{1}_{t_1\leq 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|K_w^{1-\chi}h^j(s,x-(t-s)v,v)|\mathrm{d}s}_{\mathcal{I}_1} \\ + \underbrace{\left\{\mathbf{1}_{t_1\leq 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|K_w^{\chi}h^j(s,x-(t-s)v,v)|\mathrm{d}s}_{\mathcal{I}_2} \\ + \underbrace{\left\{\mathbf{1}_{t_1\leq 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|w_{q,\theta}g(s,x-(t-s)v,v)|\mathrm{d}s}_{\mathcal{I}_3} \\ + \sum_{n=4}^{8}\mathcal{I}_n, \tag{3.54}$$

with

$$
\begin{aligned}
\mathcal{I}_4 &= \mathbf{1}_{t_1\le 0} e^{-\nu(v)t}|h(0,x-tv,v)| \\
&\quad + \frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}_{q,\theta}(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}} |h(0,x_l-t_lv_l,v_l)|\mathrm{d}\Sigma_l^w(0), \\
\mathcal{I}_5 &= \frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}_{q,\theta}(v)} \Bigg\{ \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \int_0^{t_l} \\
&\quad \times |[K_w^{1-\chi}h^j](s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l^w(s)\mathrm{d}s \\
&\quad + \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{0<t_{l+1}\}} \int_{t_{l+1}}^{t_l} |[K_w^{1-\chi}h^j](s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l^w(s)\mathrm{d}s \Bigg\}, \\
\mathcal{I}_6 &= \frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}_{q,\theta}(v)} \Bigg\{ \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \int_0^{t_l} \\
&\quad \times |[K_w^{\chi}h^j](s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l^w(s)\mathrm{d}s \\
&\quad + \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{0<t_{l+1}\}} \int_{t_{l+1}}^{t_l} |[K_w^{\chi}h^j](s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l^w(s)\mathrm{d}s \Bigg\}, \\
\mathcal{I}_7 &= \frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}_{q,\theta}(v)} \Bigg\{ \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \int_0^{t_l} \\
&\quad \times |w_{q,\theta}g(s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l^w(s)\mathrm{d}s \\
&\quad + \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{0<t_{l+1}\}} \int_{t_{l+1}}^{t_l} |w_{q,\theta}g(s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l^w(s)\mathrm{d}s \Bigg\}, \\
\mathcal{I}_8 &= \frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}_{q,\theta}(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{0<t_k\}} |h^j(t_k,x_k,v_{k-1})|\mathrm{d}\Sigma_{k-1}^w(t_k), \quad k\ge 2,
\end{aligned}
$$

where $\mathrm{d}\Sigma_l^w(s)$ is given by (3.37). The main difference between this proof and that of Lemma 3.4 is that we now have an additional velocity weight $w_{q,\theta}$. We now turn to compute $\mathcal{I}_n$ $(1\le n\le 8)$ in (3.54) term by term. *Estimates for* $\mathcal{I}_1$ *and* $\mathcal{I}_5$. Notice that

$$
\int_0^t e^{-\nu(v)(t-s)}\nu(v)\mathrm{d}s < +\infty. \tag{3.55}
$$

From Lemma 2.2, it follows that

$$
\mathcal{I}_1 \le C\epsilon^{\varrho+3} \sup_{0\le s\le t} \|h^j(s)\|_\infty.
$$

Since $\tilde{w}_{q,\theta}^{-1}(v) \leq C_{q,\theta}$, Lemma 2.2 and (3.35) imply that the first term in $\mathcal{I}_5$ can be bounded by

$$C_{q,\theta}\epsilon^{\varrho+3}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\leq 0<t_l\}}\int_0^{t_l}\left\|h^j(s)\right\|_\infty \mathrm{d}\Sigma_l(s)\mathrm{d}s$$
$$\leq C_{q,\theta}\epsilon^{\varrho+3}\sup_{0\leq s\leq t}\left\|h^j(s)\right\|_\infty.$$

As for the second term in $\mathcal{I}_5$, by Lemma 2.2 and (3.36), we get the upper bound

$$C_{q,\theta}\epsilon^{\varrho+3}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{0<t_{l+1}\}}\int_{t_{l+1}}^{t_l}\left\|h^j(s)\right\|_\infty \mathrm{d}\Sigma_l(s)\mathrm{d}s$$
$$\leq C_{q,\theta}\epsilon^{\varrho+3}\sup_{0\leq s\leq t}\left\|h^j(s)\right\|_\infty.$$

Estimates for $\mathcal{I}_3$ *and* $\mathcal{I}_7$. From (3.55), it is straightforward to check that

$$\mathcal{I}_3 \leq C\sup_{0\leq s\leq t}\left\|\nu^{-1}(v)w_{q,\theta}g(s)\right\|_\infty.$$

In view of (3.35), one sees that the first term in $\mathcal{I}_7$ can be dominated by

$$C_{q,\theta}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\leq 0<t_l\}}\int_0^{t_l}\left\|w_{q,\theta}g(s)\right\|_\infty \mathrm{d}\Sigma_l(s)\mathrm{d}s$$
$$\leq C_{q,\theta}\sup_{0\leq s\leq t}\left\|w_{q,\theta}g(s)\right\|_\infty. \tag{3.56}$$

Thanks to (3.55) and (3.35), we bound the second term in $\mathcal{I}_7$ by

$$C_{q,\theta}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{0<t_{l+1}\}}\int_{t_{l+1}}^{t_l}\left\|w_{q,\theta}g(s)\right\|_\infty \mathrm{d}\Sigma_l(s)\mathrm{d}s \leq C_{q,\theta}\sup_{0\leq s\leq t}\left\|w_{q,\theta}g(s)\right\|_\infty.$$

Estimates for $\mathcal{I}_4$. In a manner similar to that for obtaining (3.56), we have

$$\mathcal{I}_4 \leq \|h(0)\|_\infty + C_{q,\theta}\|h(0)\|_\infty\int\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\leq 0<t_l\}}\mathrm{d}\Sigma_l(0) \leq C_{q,\theta}\|h(0)\|_\infty.$$

Estimates for $\mathcal{I}_8$. Since

$$\int_{\mathcal{V}_{k-1}}\tilde{w}_{q,\theta}(v_{k-1})\mathrm{d}\sigma_{k-1}$$
$$\leq \frac{1}{\sqrt{2\pi}}\int_{n(x_{k-1})\cdot v_{k-1}>0}(n(x_{k-1})\cdot v_{k-1})e^{-\frac{1}{4}|v_{k-1}|^2-\frac{q}{4}|v_{k-1}|^\theta}\mathrm{d}v_{k-1} \leq C_{q,\theta},$$

by applying (3.34) in Lemma 3.3, we have

$$
\begin{aligned}
\mathcal{I}_8 &\leq C_{q,\theta} \int_{\prod_{j=1}^{k-2} \mathcal{V}_j} \mathbf{1}_{\{0<t_{k-1}\}} \Pi_{j=1}^{k-2} \mathrm{d}\sigma_j \sup_{0\leq s\leq t} \|h^j(s)\|_\infty \\
&\leq C_{q,\theta} \left\{\frac{1}{2}\right\}^{C_2 T_0^{5/4}} \sup_{0\leq s\leq t} \|h(s)\|_\infty .
\end{aligned}
$$

We cannot obtain the desired estimates for $\mathcal{I}_2$ and $\mathcal{I}_6$ for the time being, and they will be treated by using iteration (3.54) for h^j again. To illustrate this more clearly, we first combine the above estimates for $\mathcal{I}_1, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_7$ and $\mathcal{I}_8$ to conclude that

$$
\begin{aligned}
|h^j(t,x,v)| \leq & \left\{\mathbf{1}_{t_1\leq 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)} |K_w^\chi h^j(s, x-(t-s)v, v)| \mathrm{d}s \\
& + \frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}_{q,\theta}(v)} \times \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \left\{ \int_0^{t_l} \mathbf{1}_{\{t_{l+1}\leq 0<t_l\}} |K_w^\chi h^j(s, X_{\mathbf{cl}}(s), v_l)| \right. \\
& \left. + \int_{t_{l+1}}^{t_l} \mathbf{1}_{\{0<t_{l+1}\}} |K_w^\chi h^j(s, X_{\mathbf{cl}}(s), v_l)| \right\} \mathrm{d}\Sigma_l(s)\mathrm{d}s + A_1(t) \\
= & \ \mathcal{I}_2 + \mathcal{I}_6 + A_1(t), \qquad (3.57)
\end{aligned}
$$

where $A_1(t)$ denotes

$$
\begin{aligned}
A_1(t) = & \ C_{q,\theta} \sup_{0\leq s\leq t} \left\| \nu^{-1} w_{q,\theta} g(s) \right\|_\infty + C_{q,\theta}\|h(0)\|_\infty \\
& + C_{q,\theta} \left(\frac{1}{2}\right)^{C_2 T_0^{5/4}} \sup_{0\leq s\leq t} \|h^j(s)\|_\infty + C_{q,\theta}\epsilon^{3+\varrho} \sup_{0\leq s\leq t} \left\| h^j(s) \right\|_\infty .
\end{aligned}
$$

Recall the back-time cycles: $X_{\mathbf{cl}}(s) = \sum_l \mathbf{1}_{[t_{l+1},t_l)}(s)\{x_l - (t_l - s)v_l\}$. Let $(t_0', x_0', v_0') = (s, X_{\mathbf{cl}}(s), v')$, for $v_{l'+1}' \in \mathcal{V}_{l'+1}' = \{v_{l'+1}' \cdot n(x_{l'+1}') > 0\}$; we define a new back-time cycle as

$$
(t_{l'+1}', x_{l'+1}', v_{l'+1}') = (t_{l'}' - t_{\mathbf{b}}(x_{l'}', v_{l'}'), x_{\mathbf{b}}(x_{l'}', v_{l'}'), v_{l'+1}').
$$

We now iterate (3.57) to get the representation for $K_w^\chi h^j(s, X_{\mathbf{cl}}(s), v_l)$ as

$$
\begin{aligned}
& K_w^\chi h^j(s, X_{\mathbf{cl}}(s), v_l) \\
& \leq \int_{\mathbf{R}^3} \mathbf{k}_w^\chi(v_l, v') |h^j(s, X_{\mathbf{cl}}(s), v')| \mathrm{d}v' \\
& \leq \iint \left\{ \mathbf{1}_{t_1'\leq 0} \int_0^s + \mathbf{1}_{t_1'>0}\int_{t_1'}^s \right\} e^{-\nu(v')(s-s_1)} \mathbf{k}_w^\chi(v_l, v')\mathbf{k}_w^\chi(v', v'') \\
& \quad \times |h^j(s_1, X_{\mathbf{cl}}(s) - (s-s_1)v', v'')| \mathrm{d}s_1 \mathrm{d}v' \mathrm{d}v''
\end{aligned}
$$

$$
\begin{aligned}
&+\iint \mathrm{d}v'\mathrm{d}v'' \int_{\prod_{j=1}^{k-1}\mathcal{V}_j'} \frac{e^{-\nu(v')(s-t_1')}}{\tilde{w}_{q,\theta}(v')}\\
&\times \sum_{l'=1}^{k-1}\int_0^{t_{l'}'} \mathrm{d}s_1 \mathbf{1}_{\{t_{l'+1}'\le 0<t_{l'}'\}}\mathbf{k}_w^{\chi}(v_l,v')\\
&\times \mathbf{k}_w^{\chi}(v_{l'}',v'')|h^j(s_1,x_{l'}'+(s_1-t_{l'}')v_{l'}',v'')|\mathrm{d}\Sigma_{l'}^w(s_1)\\
&+\iint \mathrm{d}v'\mathrm{d}v'' \int_{\prod_{j=1}^{k-1}\mathcal{V}_j'} \frac{e^{-\nu(v')(s-t_1')}}{\tilde{w}_{q,\theta}(v')}\\
&\times \sum_{l'=1}^{k-1}\int_{t_{l'+1}'}^{t_{l'}'} \mathrm{d}s_1 \mathbf{1}_{\{t_{l'+1}'>0\}}\mathbf{k}_w^{\chi}(v_l,v')\\
&\times \mathbf{k}_w^{\chi}(v_{l'}',v'')|h^j(s_1,x_{l'}'+(s_1-t_{l'}')v_{l'}',v'')|\mathrm{d}\Sigma_{l'}^w(s_1)\\
&+\int_{\mathbb{R}^3}\mathbf{k}_w^{\chi}(v_l,v')\mathrm{d}v' A_1(s)=\sum_{n=1}^4 J_n,
\end{aligned}
\tag{3.58}
$$

where $\mathbf{k}_w^{\chi}(\cdot)=w_{q,\theta}\mathbf{k}^{\chi}(\frac{\cdot}{w_{q,\theta}})$ and J_n $(1\le n\le 4)$ denote the corresponding four terms on the right hand side of the last inequality.

In what follows, we only give an explicit computation for the delicate term $\mathcal{I}_6$; the appropriate estimates for $\mathcal{I}_2$ are similar and much easier and will be omitted for the sake of brevity.

Estimates for $\mathcal{I}_6$. Substituting (3.58) into $\mathcal{I}_6$, one has

$$
\mathcal{I}_6 \le C_{q,\theta}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\left\{\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}+\mathbf{1}_{\{0<t_{l+1}\}}\int_{t_{l+1}}^{t_l}\right\}\sum_{n=1}^{4} J_n\,\mathrm{d}\Sigma_l^w(s)\mathrm{d}s.
\tag{3.59}
$$

Continuing, we first consider the simpler terms involving $A_1(s)$ in $\mathcal{I}_6$, that is, the terms containing J_4. Since $\int \mathbf{k}_w^{\chi}(v_l,v')\mathrm{d}v'<\infty$, the summation of all contributions from A_1 lead to the bound

$$
\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\left\{\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}A_1(s)+\mathbf{1}_{\{0<t_l\}}\int_{t_{l+1}}^{t_l}A_1(s)\right\}\mathrm{d}\Sigma_l^w\mathrm{d}s\le CA_1(t),
$$

according to Lemma 3.3.

Next, we only compute the terms containing J_2 and J_3, because the estimates for the terms involving J_1 are similar and easier. Let us first show that there exists a constant $N>0$ such that

$$
\begin{aligned}
&\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_0^{t_l}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\,J_3\,\mathrm{d}\Sigma_l^w(s)\mathrm{d}s\\
&=\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}\iint \mathrm{d}v'\mathrm{d}v''
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{\prod_{j=1}^{k-1}\mathcal{V}_j'} \frac{e^{-\nu(v')(s-t_1')}}{\tilde{w}_{q,\theta}(v')} \sum_{l'=1}^{k-1} \int_{t'_{l'+1}}^{t'_{l'}} \mathbf{1}_{\{t'_{l'+1}>0\}} \\
&\times \mathbf{k}_w^{\chi}(v_l, v')\mathbf{k}_w^{\chi}(v'_{l'}, v'')|h^j(s_1, x'_{l'} + (s_1 - t'_{l'})v'_{l'}, v'')|\mathrm{d}\Sigma_{l'}^w(s_1)\mathrm{d}s_1\mathrm{d}\Sigma_l^w(s)\mathrm{d}s \\
\leq{}& C_{q,\theta}\left(\frac{1}{T_0^{5/4}} + \frac{1}{N}\right) \sup_{0\leq s\leq t} \|h^j(s)\|_\infty + C_N \sup_{0\leq s\leq t_1} \left\{ e^{\frac{\lambda_0}{2}s^{\rho_0}} \left\| \frac{h^j(s)}{w_{q,\theta}(v)} \right\|_2 \right\}.
\end{aligned} \tag{3.60}
$$

To prove (3.60), we decompose the velocity-time integration into several regions and treat them independently. Recalling that $\{(t'_{l'}, x'_{l'}, v'_{l'})\}_{l'=1}^k$ start from $(s, X_{\mathbf{cl}}, v')$, in order to avoid confusion, let us denote

$$
k(s) = k = C_1'[\alpha(s)]^{5/4}. \tag{3.61}
$$

For any $1 \leq l' \leq k-1$, we consider the following splitting:

$$
\int_{t_{l'+1}}^{t'_{l'}} = \int_{t_{l'+1}}^{t'_{l'}-\frac{1}{k^2(s)}} + \int_{t'_{l'}-\frac{1}{k^2(s)}}^{t'_{l'}},
$$

and treat the second integral first, specifically, we intend to obtain

$$
\begin{aligned}
&\int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\leq 0<t_l\}} \int_0^{t_l} \iint \mathrm{d}v'\mathrm{d}v'' \int_{\prod_{j=1}^{k-1}\mathcal{V}_j'} \frac{e^{-\nu(v')(s-t_1')}}{\tilde{w}_{q,\theta}(v')} \sum_{l'=1}^{k-1} \int_{t'_{l'}-\frac{1}{k^2(s)}}^{t'_{l'}} \\
&\qquad \times \mathbf{1}_{\{t'_{l'+1}>0\}}\mathbf{k}_w^{\chi}(v_l, v')\mathbf{k}_w^{\chi}(v'_{l'}, v'')|h^j(s_1, x'_{l'} + (s_1 - t'_{l'})v'_{l'}, v'')| \\
&\qquad \times \mathrm{d}\Sigma_{l'}^w(s_1)\mathrm{d}s_1\mathrm{d}\Sigma_l^w(s)\mathrm{d}s \\
&\quad \leq \frac{C_{q,\theta}}{T_0^{5/4}} \sup_{0\leq s\leq t} \|h^j(s)\|_{L^\infty}.
\end{aligned} \tag{3.62}
$$

Indeed, since $t'_{l'} - s_1 \leq 1/k^2(s)$, and $k(s) \geq C_1' T_0^{5/4}$, it follows from Lemma 2.2 and (3.36) that the right hand side of (3.62) is bounded by

$$
\begin{aligned}
&C_{q,\theta} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\leq 0<t_l\}} \int_0^{t_l} \mathrm{d}\Sigma_l^w(s) \frac{1}{k^2(s)} \\
&\qquad \times \sup_{0\leq s_1\leq s} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j'} \sum_{l'=1}^{k-1} \mathbf{1}_{\{t'_{l'+1}>0\}}\mathrm{d}\Sigma_{l'}^w(s_1)\mathrm{d}s \sup_{0\leq s_1\leq t_1} \|h^j(s_1)\|_{L^\infty} \\
&\quad \leq C_{q,\theta} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\leq 0<t_l\}} \int_0^{t_l} \mathrm{d}\Sigma_l^w(s) \frac{1}{k(s)}\mathrm{d}s \sup_{0\leq s_1\leq t_1} \|h^j(s_1)\|_{L^\infty} \\
&\quad \leq \frac{C_{q,\theta}}{T_0^{5/4}} \sup_{0\leq s\leq t_1} \|h^j(s)\|_\infty,
\end{aligned}
$$

where we also used the following significant estimate:

$$\sup_{0\le s_1\le s}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j'}\sum_{l'=1}^{k-1}\mathbf{1}_{\{t_{l'+1}'>0\}}\mathrm{d}\Sigma_{l'}^{w}(s_1)\le C_{q,\theta}k(s).$$

As for the first integral,

$$\begin{aligned}&\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}\iint \mathrm{d}v'\mathrm{d}v''\int_{\prod_{j=1}^{k-1}\mathcal{V}_j'}\frac{e^{-\nu(v')(s-t_1')}}{\tilde{w}_{q,\theta}(v')}\sum_{l'=1}^{k-1}\mathbf{1}_{\{t_{l'+1}'>0\}}\\&\quad\times\int_{t_{l'+1}'}^{t_{l'}'-\frac{1}{k^2(s)}}\mathbf{k}_w^{\chi}(v_l,v')\mathbf{k}_w^{\chi}(v_{l'}',v'')|h^j(s_1,x_{l'}'+(s_1-t_{l'}')v_{l'}',v'')|\\&\quad\times \mathrm{d}\Sigma_{l'}^{w}(s_1)\mathrm{d}s_1\mathrm{d}\Sigma_l^{w}(s)\mathrm{d}s,\end{aligned}\tag{3.63}$$

we divide our computations into the following three cases:

Case 1. $|v_l|\ge N$ or $|v_{l'}'|\ge N$ with N suitably large. From Lemma 2.2, it follows that

$$\begin{aligned}\int \mathbf{k}_w^{\chi}(v_l,v')\mathrm{d}v'&\le \frac{C_\epsilon}{(1+|v_l|)^{-\varrho}}\le\frac{C_\epsilon}{N},\\ \nu^{-1}(v_{l'}')\int \mathbf{k}_w^{\chi}(v_{l'}',v'')\mathrm{d}v''&\le\frac{C_\epsilon}{(1+|v_{l'}'|)^{-\varrho}}\le\frac{C_\epsilon}{N}.\end{aligned}$$

Using this, for $|v_l|\ge N$ or $|v_{l'}'|\ge N$, we know thanks to (3.35) and (3.36) that

$$\begin{aligned}(3.63)\le{}&\frac{C_\epsilon C_{q,\theta}}{N}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}\mathrm{d}\Sigma_l^{w}(s)\\&\times\int_{\prod_{j=1}^{k-1}\mathcal{V}_j'}\sum_{l'=1}^{k-1}\mathbf{1}_{\{t_{l'+1}'>0\}}\int_{t_{l'+1}'}^{t_{l'}'}\mathrm{d}\Sigma_{l'}^{w}(s_1)\mathrm{d}s_1\mathrm{d}s\sup_{0\le s\le t_1}\|h^j(s)\|_\infty\\ \le{}&\frac{C_{\epsilon,q,\theta}}{N}\sup_{0\le s\le t}\|h^j(s)\|_\infty.\end{aligned}\tag{3.64}$$

Case 2. $|v_l|\le N$ and $|v'|\ge 2N$, or $|v_{l'}'|\le N$ and $|v''|\ge 2N$. Notice that we have either $|v_l-v'|\ge N$ or $|v_{l'}'-v''|\ge N$, and that either of the following holds:

$$\begin{aligned}&\mathbf{k}_w^{\chi}(v_l,v')\le Ce^{-\frac{\varepsilon N^2}{16}}\mathbf{k}_w^{\chi}(v_l,v')e^{\frac{\varepsilon|v_l-v'|^2}{16}},\\ \text{or }&\mathbf{k}_w^{\chi}(v_{l'}',v'')\le Ce^{-\frac{\varepsilon N^2}{16}}\mathbf{k}_w^{\chi}(v_{l'}',v'')e^{\frac{\varepsilon|v_{l'}'-v''|^2}{16}}.\end{aligned}$$

By virtue of Lemma 2.2, one sees that both

$$\int \mathbf{k}_w^{\chi}(v_l,v')e^{\frac{\varepsilon|v_l-v'|^2}{16}}\quad\text{and}\quad\int \mathbf{k}_w^{\chi}(v_{l'}',v'')e^{\frac{\varepsilon|v_{l'}'-v''|^2}{16}}$$

are still bounded. In this situation, we have, by an argument similar to that for obtaining (3.64), that

$$(3.63) \le C_{q,\theta} e^{-\frac{\varepsilon N^2}{16}} \sup_{0\le s\le t} \|h^j(s)\|_\infty. \tag{3.65}$$

To obtain the final bound for (3.63), we are now in a position to handle the last case:

Case 3. $|v_l| \le N$, $|v'| \le 2N$, $|v'_{l'}| \le N$ and $|v''| \le 2N$. Recall that there is a lower bound $t'_{l'} - s_1 \ge 1/k^2$, so that one can convert the bound in $L^\infty-$norm to the one in the L^2-norm, which has been well-established in Section 3.1. To do this, for any large $N > 0$, we choose a number $m(N)$ to define

$$\mathbf{k}^\chi_{m,w}(p, v') \equiv \mathbf{1}_{|p-v'|\ge \frac{1}{m}, |v'|\le m} \mathbf{k}^\chi_w(p, v'), \tag{3.66}$$

such that $\sup_p \int_{\mathbf{R}^3} |\mathbf{k}^\chi_m(p, v') - \mathbf{k}^\chi_w(p, v')| \mathrm{d}v' \le \frac{1}{N}$. We split

$$\begin{aligned}
\mathbf{k}^\chi_w(v_l, v')\mathbf{k}^\chi_w(v'_{l'}, v'') = {} & \{\mathbf{k}^\chi_w(v_l, v') - \mathbf{k}^\chi_{m,w}(v_l, v')\}\mathbf{k}^\chi_w(v'_{l'}, v'') \\
& + \{\mathbf{k}^\chi_w(v'_{l'}, v'') - \mathbf{k}^\chi_{m,w}(v'_{l'}, v'')\}\mathbf{k}^\chi_{m,w}(v_l, v') \\
& + \mathbf{k}^\chi_{m,w}(v_l, v')\mathbf{k}^\chi_{m,w}(v'_{l'}, v''),
\end{aligned}$$

and from Lemma 3.3, the first two differences lead to a small contribution in (3.63)

$$\frac{C_{q,\theta}}{N} \sup_{0\le s\le t} \|h^j(s)\|_\infty. \tag{3.67}$$

For the remaining main contribution of $\mathbf{k}^\chi_{m,w}(v_l, v')\mathbf{k}^\chi_{m,w}(v'_{l'}, v'')$, by a change of variable, $y = x'_{l'} + (s_1 - t'_{l'})v_{l'}$. Noticing that $x'_{l'}$ is independent of v'_l, we see that $\left|\frac{\mathrm{d}y}{\mathrm{d}v'_l}\right| \ge (k(s))^{-6}$. Consequently, as in Case 4 in the proof of Theorem 6 in [27, pp. 754], we obtain

$$\begin{aligned}
(3.63) \le {} & \frac{C_{q,\theta}}{N} \sup_{0\le s\le t} \|h^j(s)\|_\infty \\
& + C_N \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\le 0 < t_l\}} \int_0^{t_l} \mathrm{d}\Sigma^w_l(s)(k(s))^6 \\
& \times \int_{\prod_{j=1}^{l'-1} \mathcal{V}'_j \prod_{j=l'+1}^{k-1} \mathcal{V}'_j} \sum_{l'=1}^{k-1} \mathbf{1}_{\{t'_{l'+1} > 0\}} \int_{t'_{l'+1}}^{t'_{l'}} \int_{\Omega\times\{|v''|\le 2N\}} \left|\frac{h^j(s_1)}{w_{q,\theta}(v)}\right| \mathrm{d}y \mathrm{d}v'' \\
& \times e^{-\lambda_0 (t'_{l'} - s_1)^{\rho_0}} \{\Pi_{j'=l'+1}^{k-1} \mathrm{d}\sigma_{j'}\} \times \Pi_{j'=1}^{l'-1} \{e^{\nu(v'_{j'})(t'_{j'+1} - t'_{j'})} \mathrm{d}\sigma_{j'}\} \mathrm{d}s_1 \mathrm{d}s.
\end{aligned} \tag{3.68}$$

In light of Lemma 3.5 in Section 3.4, we see that (3.68) can be further dominated by

$$
\begin{aligned}
(3.63) \leq\ & \frac{C_{q,\theta}}{N}\sup_{0\leq s\leq t}\|h^j(s)\|_\infty \\
& + C_N\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\leq 0<t_l\}}\int_0^{t_l}\mathrm{d}\Sigma_l^w(s)(k(s))^7 e^{-\frac{\lambda_0}{2}s^{\rho_0}}\mathrm{d}s \\
& \times\sup_{0\leq s\leq t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\left\|\frac{h^j(s)}{w_{q,\theta}(v)}\right\|_2\right\} \\
\leq\ & \frac{C_{q,\theta}}{N}\sup_{0\leq s\leq t}\|h^j(s)\|_\infty + C_N\sup_{0\leq s\leq t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\left\|\frac{h^j(s)}{w_{q,\theta}(v)}\right\|_2\right\}. \qquad (3.69)
\end{aligned}
$$

Putting the estimates (3.62), (3.64), (3.65), (3.67) and (3.69) together, one sees that (3.60) is valid. Furthermore, by an argument similar to that for proving (3.60), we can also show that the remaining terms in (3.59) and $\mathcal{I}_2$ share the same bound as (3.60), and we thus arrive at

$$
\begin{aligned}
\mathcal{I}_2,\ \mathcal{I}_6 \leq\ & C_{q,\theta}\left(\frac{1}{T_0^{5/4}}+\frac{1}{N}\right)\sup_{0\leq s\leq t}\|h^j(s)\|_\infty \\
& + C_N\sup_{0\leq s\leq t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\left\|\frac{h^j(s)}{w_{q,\theta}(v)}\right\|_2\right\} + CA_1(t).
\end{aligned}
$$

Now choose $T_0, N > 0$ large, and plug the estimates for $\mathcal{I}_2$, $\mathcal{I}_6$ and $A_1(t)$ into (3.57) to obtain

$$
\begin{aligned}
\sup_{0\leq s\leq t}\|h^j(s)\|_\infty \leq\ & C\|h_0\|_\infty + C\sup_{0\leq s\leq t}\|\nu^{-1}w_{q,\theta}g(s)\|_\infty \\
& + C\sup_{0\leq s\leq t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\left\|f^j(s)\right\|_2\right\}.
\end{aligned} \qquad (3.70)
$$

On the other hand, from (3.6) in Proposition (3.1), one has, by taking $\lambda_0 \leq \lambda_1$, that

$$
\begin{aligned}
& \sup_{0\leq s\leq t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\left\|f^j(s)\right\|_2\right\} \\
& \leq C\|w_{q/2,\theta}f_0\|_2 + C\sqrt{\int_0^t e^{\lambda_1 s^{\rho_0}}\|\nu^{-1/2}g(s)\|_2^2\mathrm{d}s} \qquad (3.71) \\
& + C\sqrt{\int_0^t\|\nu^{-1/2}w_{q/2,\theta}g(s)\|_2^2\mathrm{d}s}.
\end{aligned}
$$

We then have that (3.53) follows from (3.70), (3.71) and (3.15). This allows us to construct a global solution $h^j(t,x,v)$ to (3.47) and (3.48). Since (3.53) is uniform in j, one can further show that $\{h^j\}_{j=2}^\infty$ converges to h strongly in L^∞ via an argument similar to that used in the end of the proof for Lemma 3.4. Finally, by (3.53), we also have

$$\sup_{0\le s\le t}\{\|h(s)\|_\infty+|h(s)|_\infty\}\lesssim\|w_{q,\theta}f_0\|_\infty+\sup_{0\le s\le t}\|\nu^{-1}w_{q,\theta}g(s)\|_\infty$$
$$+\|w_{q/2,\theta}f_0\|_2+\sqrt{\int_0^t e^{\lambda_1 s^{\rho_0}}\|\nu^{-1/2}g(s)\|_2^2\mathrm{d}s}$$
$$+\sqrt{\int_0^t\|\nu^{-1/2}w_{q/2,\theta}g(s)\|_2^2\mathrm{d}s}. \tag{3.72}$$

Then, (3.72) and the inequality $\|w_{q/2,\theta}f_0\|_2\lesssim\|w_{q,\theta}f_0\|_\infty$ imply (3.32). This completes the proof of Proposition 3.2. □

3.3. Nonlinear Existence

Our aim in this subsection is to prove

The global existence of (1.11)**,** (1.12) **and** (1.13)**.** Recalling the initial boundary value problem for the linearized equation (3.1) and (3.2), we design the following iteration sequence:

$$\partial_t f^{\ell+1}+v\cdot\nabla_x f^{\ell+1}+Lf^{\ell+1}=\Gamma(f^\ell,f^\ell),\ \ f^{\ell+1}(0,x,v)=f_0(x,v), \tag{3.73}$$

with $f_-^{\ell+1}=P_\gamma f^\ell$ and $f^0=f_0(x,v)$. Clearly, $\mathbf{P}\{\Gamma(f^\ell,f^\ell)\}=0$.

Note that the iteration scheme (3.73) does not provide us with the positivity of the solution of the original equation (1.1), however it coincides with the linearized equation (3.1) so that Propositions 3.1 and 3.2 can be used directly. Our strategy to proving the global existence (1.11), (1.12) and (1.13) can be outlined as follows: we first show that the sequence $\{f^\ell\}_{\ell=0}^\infty$ determined by (3.73) is well-defined in a suitable Banach space via Propositions 3.1 and 3.2, then we prove that such a sequence is in fact a Cauchy sequence and that the limit is a desired global solution. Let us now define the following energy functional:

$$\mathcal{E}(f)(t)=\|w_{q,\theta}f\|_\infty^2+|w_{q,\theta}f|_{\infty,+}^2+e^{\lambda_1 t^{\rho_0}}\|f\|_2^2+\|w_{q/2,\theta}f\|_2^2,$$

and dissipation rate

$$\mathcal{D}(f)(t)=\|w_{q/2,\theta}f\|_\nu^2+e^{\lambda_1 t^{\rho_0}}\|f\|_\nu^2.$$

For later use, we also define a Banach space

$$\mathbf{X}_\delta=\left\{f\mid\sup_{0\le s\le t}\mathcal{E}(f)(s)+\int_0^t\mathcal{D}(f)(s)\mathrm{d}s<\delta,\ \ \delta>0\right\},$$

associated with the norm

$$\mathbf{X}_\delta(f)(t)=\sup_{0\le s\le t}\mathcal{E}(f)(s)+\int_0^t\mathcal{D}(f)(s)\mathrm{d}s.$$

We now show that $f^{\ell+1} \in \mathbf{X}_\delta$ if $f^\ell \in \mathbf{X}_\delta$. For this, on the one hand, we know from (3.32), (3.5) and (3.6), with $f = f^{\ell+1}$ and $g = \Gamma(f^\ell, f^\ell)$, that (3.73) admits a unique solution $f^{\ell+1}$ satisfying

$$\begin{aligned}
&\sup_{0\le s\le t} \mathcal{E}(f^{\ell+1})(s) + \int_0^t \mathcal{D}(f^{\ell+1})(s)\mathrm{d}s \\
&\quad \le C\mathcal{E}(f)(0) + C \sup_{0\le s\le t} \|\nu^{-1} w_{q,\theta}\Gamma(f^\ell, f^\ell)(s)\|_\infty^2 \\
&\qquad + C\int_0^t \left\|\nu^{-1/2} w_{q/2,\theta}\Gamma(f^\ell, f^\ell)(s)\right\|_2^2 \mathrm{d}s \\
&\qquad + C\int_0^t e^{\lambda_1 s^{\rho_0}} \|\nu^{-1/2}\Gamma(f^\ell, f^\ell)(s)\|_2^2 \mathrm{d}s,
\end{aligned} \tag{3.74}$$

provided that the right hand side is finite. On the another hand, thanks to Lemma 2.3, it follows that

$$\begin{aligned}
&\int_0^t \left\|\nu^{-1/2} w_{q/2,\theta}\Gamma(f^\ell, f^\ell)\right\|_2^2 \mathrm{d}s \\
&\quad \le C \sup_{0\le s\le t} \|w_{q,\theta} f(s)\|_\infty^2 \int_0^t \|w_{q/2,\theta} f(s)\|_\nu^2 \mathrm{d}s \\
&\quad \le C \sup_{0\le s\le t} \mathcal{E}(f^\ell)(s) \int_0^t \mathcal{D}(f^\ell)(s)\mathrm{d}s,
\end{aligned} \tag{3.75}$$

$$\begin{aligned}
&\int_0^t e^{\lambda_1 s^{\rho_0}} \|\nu^{-1/2}\Gamma(f^\ell, f^\ell)(s)\|_2^2 \mathrm{d}s \\
&\quad \le C \sup_{0\le s\le t} \|w_{q/2,\theta} f(s)\|_\infty^2 \int_0^t e^{\lambda_1 s^{\rho_0}} \|f(s)\|_\nu^2 \mathrm{d}s \\
&\quad \le C \sup_{0\le s\le t} \mathcal{E}(f^\ell)(s) \int_0^t \mathcal{D}(f^\ell)(s)\mathrm{d}s,
\end{aligned} \tag{3.76}$$

and

$$\sup_{0\le s\le t} \left\|\nu^{-1} w_{q,\theta}\Gamma(f^\ell, f^\ell)\right\|_\infty \le C \sup_{0\le s\le t} \mathcal{E}(f^\ell)(s). \tag{3.77}$$

As a consequence, one has from (3.74), (3.75), (3.76) and (3.77) that

$$\mathbf{X}_\delta(f^{\ell+1})(t) \le C\mathcal{E}(f)(0) + C\mathbf{X}_\delta^2(f^\ell)(t), \tag{3.78}$$

which further yields that $\mathbf{X}_\delta(f^{\ell+1})(t) < \delta$, supposing $f^\ell \in \mathbf{X}_\delta$ with δ and $\mathcal{E}(f)(0)$ small enough.

We now prove the strong convergence of the iteration sequence $\{f^\ell\}_{\ell=0}^\infty$ constructed above. To do this, by taking the difference of the equations that $f^{\ell+1}$ and f^ℓ satisfy, we deduce that

$$\begin{cases}
\partial_t[f^{\ell+1} - f^\ell] + v\cdot\nabla_x[f^{\ell+1} - f^\ell] + L[f^{\ell+1} - f^\ell] \\
\qquad = \Gamma(f^\ell - f^{\ell-1}, f^\ell) + \Gamma(f^{\ell-1}, f^\ell - f^{\ell-1}), \\
[f^{\ell+1} - f^\ell]_- = P_\gamma[f^{\ell+1} - f^\ell],
\end{cases}$$

with $f^{\ell+1} - f^{\ell} = 0$ initially. Repeating the same argument as that for obtaining (3.78), we obtain

$$\mathbf{X}_\delta(f^{\ell+1} - f^{\ell})(t) \le C\left\{\mathbf{X}_\delta(f^{\ell}) + \mathbf{X}_\delta(f^{\ell-1})\right\}\mathbf{X}_\delta(f^{\ell} - f^{\ell-1})(t).$$

This implies that $\{f^{\ell}\}_{\ell=0}^{\infty}$ is a Cauchy sequence in $\mathbf{X}_\delta$ for δ suitably small. Moreover, take f as the limit of the sequence $\{f^{\ell}\}_{\ell=0}^{\infty}$ in $\mathbf{X}_\delta$, then f satisfies

$$\sup_{0\le s\le t} \mathcal{E}(f)(s) + \int_0^t \mathcal{D}(f)(s)\mathrm{d}s \le C\mathcal{E}(f)(0) \le C\|w_{q,\theta} f_0\|_\infty^2. \tag{3.79}$$

Since we have L^∞ convergence at each step, as [27, pp. 788], we deduce that $w_{q,\theta} f$ is continuous away from γ_0 when Ω is strictly convex. The uniqueness is standard. We now turn to prove the positivity of $\mu + \sqrt{\mu} f$. As mentioned at the beginning of this subsection, we need to design a different iterative sequence. We use the following one:

$$\begin{cases} \{\partial_t + v\cdot\nabla_x\}\, F^{\ell+1} + F^{\ell+1}(v)\nu(F^{\ell}) \\ \quad = \int_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^{\varrho} b_0(\theta) F^{\ell}(u')F^{\ell}(v')\,\mathrm{d}u\mathrm{d}\omega = \Gamma^{\text{gain}}(F^{\ell}, F^{\ell}), \\ F_-^{\ell+1} = \mu \int_{n(x)\cdot v>0} F^{\ell}(v) n(x)\cdot v\mathrm{d}v, \\ F^{\ell+1}(0,x,v) = F_0(x,v), \end{cases}$$

starting with $F^0(t,x,v) = F_0(x,v)$, where $\nu(F^{\ell}) = \int_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^{\varrho} b_0(\theta) F^{\ell}(u)\mathrm{d}u\mathrm{d}\omega$. By a procedure similar to the proof of Theorem 4 in [27, pp. 806–807], one can easily verify that such an iteration preserves the non-negativity. We now need to prove that F^{ℓ} is convergent in order to conclude the non-negativity of the limit $F(t) \ge 0$. Noticing that $F^{\ell+1} = \mu + \mu^{1/2} f^{\ell+1}$, equivalently, we need to solve $f^{\ell+1}$ such that

$$\begin{aligned} &\{\partial_t + v\cdot\nabla_x + \nu\}\, f^{\ell+1} - Kf^{\ell} = \Gamma^{\text{gain}}(f^{\ell}, f^{\ell}) - f^{\ell+1}(v)\nu(\sqrt{\mu} f^{\ell}), \\ &f_-^{\ell+1} = P_\gamma f^{\ell}, \quad f^{\ell+1}(0,x,v) = f_0(x,v). \end{aligned} \tag{3.80}$$

In fact, since $|\nu(\sqrt{\mu} f^{\ell})| \le C\varepsilon_0 \nu$ for $\|w_{q,\theta} f^{\ell}\|_\infty \le \varepsilon_0$, one can rewrite (3.80) as

$$\begin{aligned} &\{\partial_t + v\cdot\nabla_x + \overline{\nu}\}\, f^{\ell+1} = Kf^{\ell} + \Gamma^{\text{gain}}(f^{\ell}, f^{\ell}), \\ &f_-^{\ell+1} = P_\gamma f^{\ell}, \quad f^{\ell+1}(0,x,v) = f_0(x,v), \end{aligned}$$

with $\overline{\nu} = \nu + \nu(\sqrt{\mu} f^{\ell})$. As with the proof of Lemma 3.2, it follows from a routine procedure to show that $\{h^{\ell+1} = w_{q,\theta} f^{\ell+1}\}_{\ell=0}^{\infty}$ is indeed convergent in L^∞ local in time $[0, T_*]$. This ends the proof of the first part of Theorem 1.1. We leave the second part to the next subsection. □

3.4. Nonlinear L^∞ Exponential Decay

In this subsection, we are going to deduce the L^∞ exponential time decay rates for the initial boundary value problems (1.11), (1.12) and (1.13) based on the global existence constructed in Section 3.3. For this, let us first present the following refined estimates for integrals on the stochastic cycles given by Definition 3.1:

Lemma 3.5. *Denote $\|\cdot\|_{\mathbf{Y}} = \|\cdot\|_2$ or $\|\cdot\|_\infty$. Assume $(q,\theta)\in\mathcal{A}_{q,\theta}$. There exists constant $\lambda_0>0$ such that for $\rho_0=\frac{\theta}{\theta-\varrho}$,*

$$\begin{aligned}\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_0^{t_l}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\|f(s,\cdot,v_l)\|_{\mathbf{Y}}\mathrm{d}\Sigma_l(s)\mathrm{d}s&\\ \le Ce^{-\frac{\lambda_0}{2}t_1^{\rho_0}}\sup_{0\le s\le t_1}e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_{\mathbf{Y}},&\end{aligned}\tag{3.81}$$

and

$$\begin{aligned}\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{t_{l+1}>0\}}\|f(s,\cdot,v_l)\|_{\mathbf{Y}}\mathrm{d}\Sigma_l(s)\mathrm{d}s&\\ \le Ce^{-\frac{\lambda_0}{2}t_1^{\rho_0}}\sup_{0\le s\le t_1}e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_{\mathbf{Y}},&\end{aligned}\tag{3.82}$$

where $C>0$ and is independent of k.

Moreover, for any $\epsilon_0>0$, it holds that

$$\begin{aligned}&\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\int_{t_l-\epsilon_0}^{t_l}\|f(s,\cdot,v_l)\|_{\mathbf{Y}}\mathrm{d}\Sigma_l(s)\mathrm{d}s\\ &\le C\epsilon_0e^{-\frac{\lambda_0}{2}t_1^{\rho_0}}\sup_{0\le s\le t_1}e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s,\cdot,v_l)\|_{\mathbf{Y}},\end{aligned}\tag{3.83}$$

$$\begin{aligned}&\int_{\prod_{j=1}^{l-1}\mathcal{V}_j\prod_{j=l+1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\|f(s,\cdot,v_l)\|_{\mathbf{Y}}e^{-\lambda_0(t_l-s_1)^{\rho_0}}\{\Pi_{j=l+1}^{k-1}\mathrm{d}\sigma_j\}\\ &\quad\times\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\mathrm{d}s_1\\ &\le Ce^{-\frac{\lambda_0}{2}t_1^{\rho_0}}\sup_{0\le s\le t_1}e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s,\cdot,v_l)\|_{\mathbf{Y}},\end{aligned}\tag{3.84}$$

and

$$\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{0<t_k\}}|f(t_k,\cdot,v_{k-1})|\mathrm{d}\Sigma_{k-1}(t_k)\le C\epsilon_0e^{-\frac{\lambda_0}{2}t_1^{\rho_0}}\sup_{0\le s\le t_1}e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty.\tag{3.85}$$

Proof. We first prove (3.82). Recall the decomposition (3.38), and also rewrite

$$\begin{aligned}
&\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{t_{l+1}>0\}}\|f(s)\|_{\mathbf{Y}}e^{\nu(v_l)(s-t_l)}\mu^{-1/2}(v_l)\mathrm{d}\sigma_l\mathrm{d}s\\
&\qquad\times\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\\
&\quad=\int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}\le k}}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{t_{l+1}>0\}}\|f(s)\|_{\mathbf{Y}}\\
&\qquad\quad\times\mu^{-1/2}(v_l)e^{\nu(v_l)(s-t_l)}\mathrm{d}\sigma_l\mathrm{d}s\,\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\\
&\qquad+\int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}>k}}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{t_{l+1}>0\}}\|f(s)\|_{\mathbf{Y}}\\
&\qquad\quad\times\mu^{-1/2}(v_l)e^{\nu(v_l)(s-t_l)}\mathrm{d}\sigma_l\mathrm{d}s\,\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\\
&\quad\overset{\text{def}}{=}\mathcal{K}_3+\mathcal{K}_4.
\end{aligned}$$

To estimate $\mathcal{K}_3$, as in the proof for (3.36), we denote $\max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}=|v_m|$ again, then it follows that

$$\begin{aligned}
\mathcal{K}_3\le&\int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}\le k}}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{t_{l+1}>0\}}\|f(s)\|_{\mathbf{Y}}\\
&\times e^{\nu(v_m)(s-t_1)}\mu^{-1/2}(v_m)\mathrm{d}\sigma_l\mathrm{d}s\,\Pi_{j=1}^{l-1}\mathrm{d}\sigma_j.
\end{aligned}$$

Meanwhile, by Young's inequality, we find

$$e^{-\nu(v)t}w_{q/2,\theta}^{-1}(v)\le e^{-\lambda_0t^{\rho_0}},\ \ \rho_0=\frac{\theta}{\theta-\varrho},\tag{3.86}$$

where λ_0 is given by

$$0<\lambda_0\le(C_\varrho\rho_0)^{-\rho_0}\left(\frac{q}{8(1-\rho_0)}\right)^{1-\rho_0}>0.$$

Using (3.86), we obtain, for $\rho_0=\frac{\theta}{\theta-\varrho}$,

$$\begin{aligned}
\mathcal{K}_3\le&\int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}\le k}}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{t_{l+1}>0\}}e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_{\mathbf{Y}}e^{-\lambda_0(t_1-s)^{\rho_0}}e^{-\frac{\lambda_0}{2}s^{\rho_0}}\\
&\times w_{q/2,\theta}(v_m)\mu^{-1/2}(v_m)\mathrm{d}\sigma_l\mathrm{d}s\,\Pi_{j=1}^{l-1}\mathrm{d}\sigma_j\\
\le&\sqrt{2\pi}e^{-\frac{\lambda_0}{2}t_1^{\rho_0}}\int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\ldots,|v_{k-1}|\}\le k}}e^{\frac{q}{8}|v_m|^\theta}e^{\frac{|v_m|^2}{4}}\\
&\times\left\{\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}e^{-\frac{\lambda_0}{2}(t_1-s)^{\rho_0}}\mathrm{d}s\right\}\sup_{0\le s\le t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_{\mathbf{Y}}\right\}\Pi_{j=1}^{l-1}\mathrm{d}\sigma_j
\end{aligned}$$

$$
\begin{aligned}
&\le e^{-\frac{\lambda_0}{2}t_1^{\rho_0}} \sup_{0\le s\le t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_{\mathbf{Y}}\right\} \\
&\quad\times \frac{1}{\sqrt{2\pi}}\int_{n(x_m)\cdot v_m>0}(n(x_m)\cdot v_m)e^{-\frac{1}{4}|v_m|^2+\frac{q}{8}|v_m|^{\theta}}\mathrm{d}v_m \\
&\le Ce^{-\frac{\lambda_0}{2}t_1^{\rho_0}} \sup_{0\le s\le t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_{\mathbf{Y}}\right\}.
\end{aligned}
\tag{3.87}
$$

Here Lemma 2.1 is also used to guarantee $e^{-\frac{\lambda_0}{2}(t_1-s)^{\rho_0}}e^{-\frac{\lambda_0}{2}s^{\rho_0}} \le e^{-\frac{\lambda_0}{2}t_1^{\rho_0}}$ for $0<\rho_0<1$.

As to $\mathcal{K}_4$, assume with no loss of generality that $|v_i|\ge k$; following the calculations for $\mathcal{K}_2$ in the proof of Lemma 3.3, one has

$$
\begin{aligned}
\mathcal{K}_4 &\le \int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}e^{-\lambda_0(t_l-s)^{\rho}}\mathrm{d}s\mathbf{1}_{\{t_{l+1}>0\}}\sup_{t_{l+1}\le s\le t_l}\|f(s)\|_{\mathbf{Y}} \\
&\quad\times w_{q/2,\theta}(v_l)\mu^{-1/2}(v_l)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \\
&\le C\sum_{l=1}^{k-1}\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}w_{q/2,\theta}(v_l)\mu^{-1/2}(v_l)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j\sup_{0\le s\le t_1}\|f(s)\|_{\mathbf{Y}} \\
&\le C\int_{\Pi_{j=1}^{i-1}\mathcal{V}_j}\Pi_{j=1}^{i-1}\mathrm{d}\sigma_j\int_{\substack{n(x_i)\cdot v_i>0\\ |v_i|>k}}(n(x_i)\cdot v_i)e^{-\frac{1}{4}|v_i|^2+\frac{q}{8}|v_i|^{\theta}}\mathrm{d}v_i\sup_{0\le s\le t_1}\|f(s)\|_{\mathbf{Y}} \\
&\quad+C\sum_{l=1}^{i-1}\int_{\Pi_{j=1}^{l-1}\mathcal{V}_j}\Pi_{j=1}^{l-1}\mathrm{d}\sigma_j\int_{n(x_l)\cdot v_l>0}(n(x_l)\cdot v_l)e^{-\frac{1}{4}|v_l|^2+\frac{q}{8}|v_l|^{\theta}}\mathrm{d}v_l \\
&\quad\times\int_{\Pi_{j=l+1}^{i-1}\mathcal{V}_j}\Pi_{j=l+1}^{i-1}\mathrm{d}\sigma_j\int_{\substack{n(x_i)\cdot v_i>0\\ |v_i|>k}}e^{-\frac{|v_i|^2}{2}}(n(x_i)\cdot v_i)\mathrm{d}v_i\sup_{0\le s\le t_1}\|f(s)\|_{\mathbf{Y}} \\
&\quad+C\sum_{l=i+1}^{k-1}\int_{\Pi_{j=1}^{i-1}\mathcal{V}_j}\Pi_{j=1}^{i-1}\mathrm{d}\sigma_j\int_{\substack{n(x_i)\cdot v_i>0\\ |v_i|>k}}e^{-\frac{|v_i|^2}{2}}(n(x_i)\cdot v_i)\mathrm{d}v_i \\
&\quad\times\int_{\Pi_{j=i+1}^{l-1}\mathcal{V}_j}\Pi_{j=i+1}^{l-1}\mathrm{d}\sigma_j\int_{n(x_l)\cdot v_l>0}n(x_l)\cdot v_l \\
&\quad e^{-\frac{1}{4}|v_l|^2+\frac{q}{8}|v_l|^{\theta}}\mathrm{d}v_l\sup_{0\le s\le t_1}\|f(s)\|_{\mathbf{Y}} \\
&\le C_{q,\theta}(k-1)e^{-\frac{k^2}{8}}\sup_{0\le s\le t_1}\|f(s)\|_{\mathbf{Y}}\le C_{q,\theta}e^{-\frac{k^2}{16}}\sup_{0\le s\le t_1}\|f(s)\|_{\mathbf{Y}}.
\end{aligned}
\tag{3.88}
$$

Notice that $k=C_1[\alpha(t)]^{5/4}$; (3.82) then follows from (3.87) and (3.88). Just like the proof for Lemma 3.3, (3.81) can be handled in a similar way as to (3.82), and the proofs for (3.83) and (3.84), being similar and easier, we omit for the sake of brevity. It remains now to prove (3.85). To do that, we have, using a decomposition as in (3.38) again,

$$\begin{aligned}
&\int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{0<t_k\}}|f(t_k,\cdot,v_{k-1})|\mu^{-1/2}(v_{k-1})\mathrm{d}\Sigma_{k-1}(t_k)\\
&\quad= \int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\dots,|v_{k-1}|\}\le k}} \mathbf{1}_{\{0<t_k\}}|f(t_k,\cdot,v_{k-1})|\mu^{-1/2}(v_{k-1})\\
&\qquad\times \Pi_{j=1}^{k-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\\
&\qquad+ \int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j\\ \max\{|v_1|,|v_2|,\dots,|v_{k-1}|\}> k}} \mathbf{1}_{\{0<t_k\}}|f(t_k,\cdot,v_{k-1})|\mu^{-1/2}(v_{k-1})\\
&\qquad\times \Pi_{j=1}^{k-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}\mathrm{d}\sigma_j\}\\
&\quad\overset{\text{def}}{=}\mathcal{K}_5+\mathcal{K}_6.
\end{aligned}$$

To compute $\mathcal{K}_5$, let us denote $\max\{|v_1|,|v_2|,\dots,|v_{k-1}|\} = |v_m|$ again. We first prove that there exists a constant $C > 0$ independent of t such that for all $1 \le m \le k-1$ and small $\epsilon_0 > 0$,

$$\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{0<t_k\}} w_{q/2,\theta}(v_m)\mu^{-1/2}(v_m)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \le C\epsilon_0. \tag{3.89}$$

For this, we define non-grazing sets for $1 \le j \le k-1$ as $\mathcal{V}_j^{\mathfrak{z}} = \{v_j \in \mathcal{V}_j : v_j\cdot n(x_j) \ge \mathfrak{z}\} \cap \{v_j \in \mathcal{V}_j : |v_j| \le \frac{1}{\mathfrak{z}}\}$ with $\mathfrak{z} > 0$ and sufficiently small. Notice that $(q,\theta) \in \mathcal{A}_{q,\theta}$; we obtain, by a direct calculation,

$$\begin{aligned}
&\int_{\mathcal{V}_j\setminus\mathcal{V}_j^{\mathfrak{z}}} w_{q/2,\theta}(v_j)\mu^{-1/2}(v_j)\mathrm{d}\sigma_j\\
&\qquad\le \int_{0<v_j\cdot n(x_j)\le\mathfrak{z}} w_{q/2,\theta}(v_j)\mu^{-1/2}(v_j)\mathrm{d}\sigma_j\\
&\qquad\quad+ \int_{|v_j|\ge\frac{1}{\mathfrak{z}}} w_{q/2,\theta}(v_j)\mu^{-1/2}(v_j)\mathrm{d}\sigma_j\\
&\qquad\le C_{q,\theta}\int_{0<v_j\cdot n(x_j)\le\mathfrak{z}} \mu^{1/4}(v_j)v_j\cdot n(x_j)\mathrm{d}v_j\\
&\qquad\quad+ C_{q,\theta}\int_{|v_j|\ge\frac{1}{\mathfrak{z}}} \mu^{1/4}(v_j)v_j\cdot n(x_j)\mathrm{d}v_j \le C\mathfrak{z},
\end{aligned} \tag{3.90}$$

and

$$\int_{\mathcal{V}_j} w_{q/2,\theta}(v_j)\mu^{-1/2}(v_j)\mathrm{d}\sigma_j \le C, \tag{3.91}$$

where C is independent of j. On the other hand, if $v_j \in \mathcal{V}_j^{\mathfrak{z}}$, we know from the definition of the diffusive back-time cycle (3.33) that $x_j - x_{j+1} = (t_j - t_{j+1})v_j$. Since $|v_j| \le \frac{1}{\mathfrak{z}}$, and $v_j\cdot n(x_j) \ge \mathfrak{z}$, thanks to Lemma 2.6, it follows that $(t_j -$

$t_{j+1}) \geq \frac{\mathfrak{z}^3}{C_\xi}$. Hence, when $t_k(t, x, v, v_1, v_2, \ldots, v_{k-1}) > 0$, there can be, at most, $\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right] + 1$ number of $v_j \in \mathcal{V}_j^{\mathfrak{z}}$ for $1 \leq j \leq k-1$. We therefore compute

$$
\begin{aligned}
&\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{0<t_k\}} w_{q/2,\theta}(v_m)\mu^{-1/2}(v_m)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \\
&\quad \leq \sum_{l=1}^{\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]+1} \int_{\mathcal{V}_1^\dagger} \Pi_{j=1}^{k-1} w_{q/2,\theta}(v_m)\mu^{-1/2}(v_m)\mathrm{d}\sigma_j \\
&\quad\quad + \sum_{l=1}^{\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]+1} \int_{\mathcal{V}_2^\dagger} \Pi_{j=1}^{k-1} w_{q/2,\theta}(v_m)\mu^{-1/2}(v_m)\mathrm{d}\sigma_j \\
&\quad \leq \sum_{l=1}^{\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]+1} \binom{k-1}{l} \left|\sup_j \int_{\mathcal{V}_j^{\mathfrak{z}}} \mathrm{d}\sigma_j\right|^{l-1} \int_{\mathcal{V}_m^{\mathfrak{z}}} w_{q,\theta}(v_m)\mu^{-1/2}(v_m)\mathrm{d}\sigma_m \\
&\quad\quad \times \left\{\sup_j \int_{\mathcal{V}_j \setminus \mathcal{V}_j^{\mathfrak{z}}} \mathrm{d}\sigma_j\right\}^{k-l-1} \\
&\quad\quad + \sum_{l=1}^{\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]+1} \binom{k-1}{l} \left|\sup_j \int_{\mathcal{V}_j^{\mathfrak{z}}} \mathrm{d}\sigma_j\right|^{l} \int_{\mathcal{V}_m / \mathcal{V}_m^{\mathfrak{z}}} w_{q,\theta}(v_m)\mu^{-1/2}(v_m)\mathrm{d}\sigma_m \\
&\quad\quad \times \left\{\sup_j \int_{\mathcal{V}_j \setminus \mathcal{V}_j^{\mathfrak{z}}} \mathrm{d}\sigma_j\right\}^{k-l-2},
\end{aligned}
$$

where $\mathcal{V}_1^\dagger$ is the set where there are exactly l of $v_{j_i} \in \mathcal{V}_{j_i}^{\mathfrak{z}}$, including $v_m \in \mathcal{V}_m^{\mathfrak{z}}$, and $k-1-l$ of $v_{j_i} \notin \mathcal{V}_{j_i}^{\mathfrak{z}}$, while $\mathcal{V}_2^\dagger$ is the set where there are exactly l of $v_{j_i} \in \mathcal{V}_{j_i}^{\mathfrak{z}}$, and $k-1-l$ of $v_{j_i} \notin \mathcal{V}_{j_i}^{\mathfrak{z}}$ and also $v_m \notin \mathcal{V}_m^{\mathfrak{z}}$. Since $\mathrm{d}\sigma$ is a probability measure, $\int_{\mathcal{V}_j^{\mathfrak{z}}} \mathrm{d}\sigma_j \leq 1$, and

$$
\left\{\int_{\mathcal{V}_j \setminus \mathcal{V}_j^{\mathfrak{z}}} \mathrm{d}\sigma_j\right\}^{k-l-1} \leq \left\{\int_{\mathcal{V}_j \setminus \mathcal{V}_j^{\mathfrak{z}}} \mathrm{d}\sigma_j\right\}^{k-2-\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]} \leq \{C\mathfrak{z}\}^{k-2-\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]}.
$$

With this, from (3.90), (3.91) and $\binom{k-1}{l} \leq \{k-1\}^l \leq \{k-1\}^{\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]+1}$, we deduce that

$$
\begin{aligned}
&\int \mathbf{1}_{\{t_k>0\}} w_{q/2,\theta}(v_m)\mu^{-1/2}(v_m))\Pi_{l=1}^{k-1}\mathrm{d}\sigma_l \\
&\quad \leq C\left(\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]+1\right)(k-1)^{\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]+1} (C\mathfrak{z})^{k-2-\left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right]}.
\end{aligned}
$$

For $\epsilon_0 > 0$, (3.89) follows for $C_{\mathfrak{z}} < 1$, and $k >> \left[\frac{C_\xi \alpha(t)}{\mathfrak{z}^3}\right] + 2$. We now go back to $\mathcal{K}_5$, and from (3.89) and (3.86), it follows that

$$\begin{aligned}
\mathcal{K}_5 &\le \int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j \\ \max\{|v_1|,|v_2|,\dots,|v_{k-1}|\}\le k}} \mathbf{1}_{\{0<t_k\}}|f(t_k,\cdot,v_{k-1})|e^{-\nu(v_m)(t_1-t_k)}\mu^{-1/2}(v_m)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \\
&\le \int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j \\ \max\{|v_1|,|v_2|,\dots,|v_{k-1}|\}\le k}} \mathbf{1}_{\{0<t_k\}}e^{-\lambda_0 t_k^{\rho_0}}e^{-\lambda_0(t_1-t_k)^{\rho_0}}w_{q/2,\theta}(v_m)\mu^{-1/2}(v_m) \\
&\quad \times \Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \sup_{0\le t_k\le t_1}\left\{e^{\frac{\lambda_0}{2}t_k^{\rho_0}}\|f(t_k)\|_\infty\right\} \\
&\le C\epsilon_0 e^{-\frac{\lambda_0}{2}t_1^{\rho_0}} \sup_{0\le s\le t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\right\}.
\end{aligned}$$

As for $\mathcal{K}_6$, assume, with no loss of generality, that $|v_i| \ge k$, and apply (3.86) to obtain

$$\begin{aligned}
\mathcal{K}_6 &\le \int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j \\ \max\{|v_1|,|v_2|,\dots,|v_{k-1}|\}\ge k}} \mathbf{1}_{\{0<t_k\}}|f(t_k,\cdot,v_{k-1})|\{\Pi_{j=l}^{k-1}e^{-\nu(v_l)(t_l-t_{l+1})}\} \\
&\quad \times \mu^{-1/2}(v_k)\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \\
&\le \int_{\substack{\Pi_{j=1}^{k-1}\mathcal{V}_j \\ \max\{|v_1|,|v_2|,\dots,|v_{k-1}|\}\ge k}} \mathbf{1}_{\{0<t_k\}}e^{-\lambda_0 t_k^{\rho_0}}e^{-\lambda_0(t_1-t_k)^{\rho_0}}\left\{\Pi_{l=1}^{k-1}w_{q/2,\theta}(v_l)\right\}\mu^{-1/2}(v_k) \\
&\quad \times \Pi_{j=1}^{k-1}\mathrm{d}\sigma_j \sup_{0\le t_k\le t_1}\left\{e^{\frac{\lambda_0}{2}t_k^{\rho_0}}\|f(t_k)\|_\infty\right\} \\
&\le C e^{-\frac{\lambda_0}{2}t_1^{\rho_0}} \sup_{0\le s\le t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\right\}\left(\int_{\mathcal{V}_l} w_{q/2,\theta}(v_l)\mu^{-1/2}(v_l)\mathrm{d}\sigma_l\right)^{k-2} \\
&\quad \times \int_{\mathcal{V}_i} w_{q/2,\theta}(v_i)\mu^{-1/2}(v_i)\mathrm{d}\sigma_i \\
&\le C e^{-\frac{\lambda_0}{2}t_1^{\rho_0}} \sup_{0\le s\le t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\right\} C_{q,\theta}^{k-1}e^{-k^2/16}.
\end{aligned} \tag{3.92}$$

Choosing k suitably large so that $C_{q,\theta}^{k-1}e^{-k^2/16} < \epsilon_0$, one sees that (3.92) also enjoys the bound (3.85). This completes the proof of Lemma 3.5. □

We now turn to prove exponential decay using Lemma 3.5 and the uniform bound (1.21). The main difficulty with proving rapid decay (1.22) is created by the fact that the collision frequency has no positive lower bound in the case of the soft potential. However, as is shown in (3.86), one can trade between exponential decay rates and the additional exponential momentum weight on the initial data and the solution itself.

The proof of (1.22). Recall that $f(t,x,v)$ satisfies

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \nu f = Kf + \Gamma(f,f), & f(0,x,v) = f_0 \\ f_- = P_\gamma f. \end{cases}$$

With this, by a same kind of computation as for obtaining (3.42), one has

$$
\begin{aligned}
|f(t,x,v)| \le{}& \underbrace{\left\{\mathbf{1}_{t_1\le 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|K^{1-\chi} f(s,x-(t-s)v,v)|\mathrm{d}s}_{\mathcal{J}_1}\\
&+\underbrace{\left\{\mathbf{1}_{t_1\le 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|K^{\chi} f(s,x-(t-s)v,v)|\mathrm{d}s}_{\mathcal{J}_2}\\
&+\underbrace{\left\{\mathbf{1}_{t_1\le 0}\int_0^t + \mathbf{1}_{t_1>0}\int_{t_1}^t\right\} e^{-\nu(v)(t-s)}|g_f(s,x-(t-s)v,v)|\mathrm{d}s}_{\mathcal{J}_3}\\
&+\sum_{n=4}^{8}\mathcal{J}_n,
\end{aligned}
\tag{3.93}
$$

with

$$
\begin{aligned}
\mathcal{J}_4 ={}& \mathbf{1}_{t_1\le 0} e^{-\nu(v)t}|f(0,x-tv,v)|\\
&+e^{-\nu(v)(t-t_1)}\sqrt{\mu}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}|f(0,x_l-t_lv_l,v_l)|\mathrm{d}\Sigma_l(0),\\
\mathcal{J}_5 ={}& e^{-\nu(v)(t-t_1)}\sqrt{\mu}\Bigg\{\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_0^{t_l}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\\
&\times|[K^{1-\chi}f](s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s\\
&+\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{0<t_{l+1}\}}|[K^{1-\chi}f](s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s\Bigg\},\\
\mathcal{J}_6 ={}& e^{-\nu(v)(t-t_1)}\sqrt{\mu}\Bigg\{\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_0^{t_l}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\\
&\times|[K^{\chi}f](s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s\\
&+\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{0<t_{l+1}\}}|[K^{\chi}f](s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s\Bigg\},\\
\mathcal{J}_7 ={}& e^{-\nu(v)(t-t_1)}\sqrt{\mu}\Bigg\{\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_0^{t_l}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\\
&\times|g_f(s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s\\
&+\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{0<t_{l+1}\}}|g_f(s,x_l-(t_l-s)v_l,v_l)|\mathrm{d}\Sigma_l(s)\mathrm{d}s\Bigg\},\\
\mathcal{J}_8 ={}& e^{-\nu(v)(t-t_1)}\sqrt{\mu}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{0<t_k\}}|f(t_k,x_k,v_{k-1})|\mathrm{d}\Sigma_{k-1}(t_k),\quad k\ge 2,
\end{aligned}
$$

where $g_f = \Gamma(f, f)$ and $\Sigma_l(s)$ $(l = 1, 2, \dots,)$ is given by (3.43). We now turn to compute $\mathcal{J}_n$ $(n = 1, 2, \dots, 8)$, term by term. As the way to deal with (3.54), let us first compute $\mathcal{J}_1, \mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_7$ and $\mathcal{J}_8$; the estimates for the delicate terms $\mathcal{J}_2$ and $\mathcal{J}_6$ will be postponed to a later step when estimations such as (3.57) are derived.

Estimates on $\mathcal{J}_1$ *and* $\mathcal{J}_5$. It follows from Lemma 2.2 and (3.86) that

$$\begin{aligned}\mathcal{J}_1 \le C \sup_{0\le s\le t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\right\}\int_0^t e^{-\frac{\lambda_0}{2}(t-s)^{\rho_0}}e^{-\frac{\lambda_0}{2}(t-s)^{\rho_0}}\\ \times e^{-\frac{\lambda_0}{2}s^{\rho_0}}\mathrm{d}s\, w_{q/2,\theta}\int_{\mathbb{R}^3}K^{1-\chi}\mathrm{d}v\\ \le C\epsilon^{\varrho+3}e^{-\frac{\lambda_0}{2}t^{\rho_0}}\sup_{0\le s\le t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\right\}.\end{aligned}$$

Likewise, Lemmas 2.2 and 3.5 and inequality (3.86) imply

$$\begin{aligned}\mathcal{J}_5 &\le C\epsilon^{\varrho+3}e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}}e^{-\frac{\lambda_0}{2}t_1^{\rho_0}}w_{q/2,\theta}(v)\sqrt{\mu}(v)\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\right\}\\ &\le C\epsilon^{\varrho+3}e^{-\frac{\lambda_0}{2}t^{\rho_0}}\sup_{0\le s\le t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\right\}.\end{aligned}$$

Estimates on $\mathcal{J}_3$ *and* $\mathcal{J}_7$. We have, using (3.86), that

$$\begin{aligned}\mathcal{J}_3 &\le C\sup_{0\le s\le t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|w_{q/2,\theta}g_f(s)\|_\infty\right\}\int_0^t e^{-\frac{\lambda_0}{2}(t-s)^{\rho_0}}e^{-\frac{\lambda_0}{2}(t-s)^{\rho_0}}e^{-\frac{\lambda_0}{2}s^{\rho_0}}\mathrm{d}s\\ &\le Ce^{-\frac{\lambda_0}{2}t^{\rho_0}}\sup_{0\le s\le t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|w_{q/2,\theta}g_f(s)\|_\infty\right\}.\end{aligned}$$

Similarly, applying Lemma 3.5 and inequality (3.86) again leads to

$$\begin{aligned}\mathcal{J}_7 &\le C\epsilon^{\varrho+3}e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}}e^{-\frac{\lambda_0}{2}t_1^{\rho_0}}w_{q/2,\theta}\sqrt{\mu}\sup_{0\le s\le t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|w_{q/2,\theta}g_f(s)\|_\infty\right\}\\ &\le Ce^{-\frac{\lambda_0}{2}t^{\rho_0}}\sup_{0\le s\le t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|w_{q/2,\theta}g_f(s)\|_\infty\right\}.\end{aligned}$$

Estimates on $\mathcal{J}_4$. For the first term in $\mathcal{J}_4$, one directly has from (3.86) that

$$\mathbf{1}_{t_1\le 0}e^{-\nu(v)t}|f(0, x-tv, v)| \le e^{-\frac{\lambda_0}{2}t^{\rho_0}}\|w_{q/2,\theta}f_0\|_\infty.$$

As to the second term, applying calculations similar to the proof of Lemma 3.5, we obtain

$$\begin{aligned}&e^{-\nu(v)(t-t_1)}\sqrt{\mu}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}|f(0, x_l-t_lv_l, v_l)|\mathrm{d}\Sigma_l(0)\\ &\qquad\le Ce^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}}e^{-\frac{\lambda_0}{2}t_1^{\rho_0}}w_{q/2,\theta}\sqrt{\mu}\|f_0\|_\infty \le Ce^{-\frac{\lambda_0}{2}t^{\rho_0}}\|f_0\|_\infty.\end{aligned}$$

Gathering the above two kinds of estimates, we have

$$\mathcal{J}_4 \le Ce^{-\frac{\lambda_0}{2}t^{\rho_0}}\|w_{q/2,\theta}f_0\|_\infty.$$

Estimates on $\mathcal{J}_8$. (3.85) in Lemma 3.5 directly yields

$$\mathcal{J}_8 \le C\epsilon_0 e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}} w_{q/2,\theta}(v)\sqrt{\mu(v)} e^{-\frac{\lambda_0}{2}t_1^{\rho_0}} \sup_{0\le s\le t_1}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\right\}$$
$$\le C\epsilon_0 e^{-\frac{\lambda_0}{2}t^{\rho_0}} \sup_{0\le s\le t}\left\{e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\right\}.$$

Substituting all of the above estimates into (3.93), we arrive at

$$|f(t,x,v)| \le \mathcal{J}_2 + \mathcal{J}_6 + A_2(t), \tag{3.94}$$

with

$$A_2(t) = Ce^{-\frac{\lambda_0}{2}t^{\rho_0}}\left\{\|w_{q/2,\theta} f_0\|_\infty + (\epsilon_0 + \epsilon^{\varrho+3}) \sup_{0\le s\le t} e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty \right.$$
$$\left. + \sup_{0\le s\le t} e^{\frac{\lambda_0}{2}s^{\rho_0}}\|w_{q/2,\theta} g_f(s)\|_\infty\right\}.$$

Next, plug (3.94) into $K^\chi f$ and perform a calculation similar to (3.58) to obtain

$$\begin{aligned}
&K^\chi f(s, X_{\mathbf{cl}}(s), v_l)\\
&\le \int_{\mathbf{R}^3} \mathbf{k}^\chi(v_l, v')|f(s, X_{\mathbf{cl}}(s), v')|\mathrm{d}v'\\
&\le \iint \left\{\mathbf{1}_{t_1'\le 0}\int_0^s + \mathbf{1}_{t_1'>0}\int_{t_1'}^s\right\} e^{-\nu(v')(s-s_1)}\mathbf{k}^\chi(v_l, v')\mathbf{k}^\chi(v', v'')\\
&\quad \times|f(s_1, X_{\mathbf{cl}}(s) - (s-s_1)v', v'')|\mathrm{d}s_1\mathrm{d}v'\mathrm{d}v''\\
&\quad + \iint \mathrm{d}v'\mathrm{d}v'' \int_{\prod_{j=1}^{k-1}\mathcal{V}_j'} e^{-\nu(v')(s-t_1')}\sqrt{\mu}(v')\\
&\quad \times \sum_{l'=1}^{k-1}\int_0^{t_{l'}'} \mathrm{d}s_1 \mathbf{1}_{\{t_{l'+1}'\le 0< t_{l'}'\}}\mathbf{k}^\chi(v_l, v')\mathbf{k}^\chi(v_{l'}', v'')\\
&\quad \times|f(s_1, x_{l'}' + (s_1 - t_{l'}')v_{l'}', v'')|\mathrm{d}\Sigma_{l'}(s_1)\\
&\quad + \iint \mathrm{d}v'\mathrm{d}v'' \int_{\prod_{j=1}^{k-1}\mathcal{V}_j'} e^{-\nu(v')(s-t_1')}\sqrt{\mu}(v')\\
&\quad \times \sum_{l'=1}^{k-1}\int_{t_{l'+1}'}^{t_{l'}'} \mathrm{d}s_1 \mathbf{1}_{\{t_{l'+1}'>0\}}\mathbf{k}^\chi(v_l, v')\mathbf{k}^\chi(v_{l'}', v'')\\
&\quad \times|f(s_1, x_{l'}' + (s_1 - t_{l'}')v_{l'}', v'')|\mathrm{d}\Sigma_{l'}(s_1)\\
&\quad + \int_{\mathbb{R}^3}\mathbf{k}^\chi(v_l, v')\mathrm{d}v' A_2(s) \overset{\text{def}}{=} \sum_{n=1}^4 \mathcal{L}_n.
\end{aligned} \tag{3.95}$$

We now estimate $\mathcal{J}_6$ with the aid of (3.95). Substituting (3.95) into $\mathcal{J}_6$ and applying (3.86) leads us to

$$\mathcal{J}_6 \le C_{q,\theta} e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \left\{ \int_0^{t_l} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}} + \int_{t_{l+1}}^{t_l} \mathbf{1}_{\{0<t_{l+1}\}} \right\}$$
$$\times \sum_{n=1}^{4} \mathcal{L}_n \mathrm{d}\Sigma_l(s)\mathrm{d}s = \sum_{n=1}^{4} \mathcal{J}_{6,n}, \tag{3.96}$$

where $\mathcal{J}_{6,n}$ $(1 \le n \le 4)$ denote four terms on the right hand side of (3.96) containing $\mathcal{L}_n$ $(1 \le n \le 4)$, respectively. We now estimate $\mathcal{J}_{6,n}$ $(1 \le n \le 4)$ term by term. We first consider the simple term $\mathcal{J}_{6,4}$, since $\int_{\mathbb{R}^3} \mathbf{k}^{\chi}(v_l, v')\mathrm{d}v' < \infty$. In light of Lemma 3.5, it is straightforward to check that

$$\begin{aligned}
\mathcal{J}_{6,4} &\le C_{q,\theta} e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}} e^{-\frac{\lambda_0}{2}t_1^{\rho_0}} \sup_{0\le s\le t} \left\{ e^{\frac{\lambda_0}{2}s^{\rho_0}} A_2(s) \right\} \\
&\le C_{q,\theta} e^{-\frac{\lambda_0}{2}t^{\rho_0}} \Big\{ \|w_{q/2,\theta} f_0\|_\infty + (\epsilon_0 + \epsilon^{\varrho+3}) \sup_{0\le s\le t} e^{\frac{\lambda_0}{2}s^{\rho_0}} \|f(s)\|_\infty \\
&\quad + \sup_{0\le s\le t} e^{\frac{\lambda_0}{2}s^{\rho_0}} \|w_{q/2,\theta} g_f(s)\|_\infty \Big\}.
\end{aligned}$$

For $\mathcal{J}_{6,2}$, we first show that there exists a sufficiently large $N > 0$ such that

$$\begin{aligned}
\mathcal{J}_{6,2}^1 &= C_{q,\theta} e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \int_0^{t_l} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \iint \mathrm{d}v'\mathrm{d}v'' \\
&\quad \times \int_{\prod_{j=1}^{k-1}\mathcal{V}_j'} e^{-\nu(v')(s-t_1')} \sqrt{\mu}(v') \sum_{l'=1}^{k-1} \int_0^{t_{l'}'} \mathrm{d}s_1 \mathbf{1}_{\{t_{l'+1}'\le 0<t_{l'}'\}} \\
&\quad \times \mathbf{k}^{\chi}(v_l, v')\mathbf{k}^{\chi}(v_{l'}', v'')|f(s_1, x_{l'}' + (s_1 - t_{l'}')v_{l'}', v'')|\mathrm{d}\Sigma_{l'}(s_1)\Sigma_l(s)\mathrm{d}s \\
&\le C_{q,\theta} \left(T_0^{5/4} + \frac{1}{N} \right) e^{-\frac{\lambda_0}{2}t^{\rho_0}} \sup_{0\le s\le t} e^{\frac{\lambda_0}{2}s^{\rho_0}} \|f(s)\|_\infty \\
&\quad + C_{q,\theta} e^{-\frac{\lambda_0}{2}t^{\rho_0}} \sup_{0\le s\le t} e^{\frac{\lambda_0}{2}s^{\rho_0}} \|f(s)\|_2.
\end{aligned} \tag{3.97}$$

As the proof for (3.60), our computation for $\mathcal{J}_{6,2}^1$ is divided into the following several cases:

Case I: $s_1 > t_{l'}' - \frac{1}{k^2(s)}$, $k(s)$ is given by (3.61). From Lemma 2.2, we see that

$$\iint \mathbf{k}^{\chi}(v_l, v')\mathbf{k}^{\chi}(v_{l'}', v'') < \infty.$$

Then, (3.86) implies that

$$e^{-\nu(v')(s-t_1')} \sqrt{\mu}(v') \le C_{q,\theta} e^{-\lambda_0(s-t_1')^{\rho_0}},$$

and we get from (3.83) in Lemma 3.5 that

$$\int_{\prod_{j=1}^{k-1}\mathcal{V}_j'}\sum_{l'=1}^{k-1}\int_{t_{l'}'-\frac{1}{k^2(s)}}^{t_{l'}'}\mathrm{d}s_1\mathbf{1}_{\{t_{l'+1}'\le 0<t_{l'}'\}}|f(s_1,x_{l'}'+(s_1-t_{l'}')v_{l'}',v'')|\mathrm{d}\Sigma_{l'}(s_1)$$
$$\le\frac{C}{k(s)}e^{-\frac{\lambda_0}{2}(t_1')^{\rho_0}}\sup_{0\le s_1\le t_1'}e^{\frac{\lambda_0}{2}s_1^{\rho_0}}\|f(s_1)\|_\infty.$$

Substituting the above estimates into $\mathcal{J}_{6,2}^1$ and applying (3.81), one has

$$\begin{aligned}\mathcal{J}_{6,2}^1\le&\frac{C_{q,\theta}}{T_0^{5/4}}e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_0^{t_l}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}e^{-\lambda_0(s-t_1')^{\rho_0}}e^{-\lambda_0(t_1')^{\rho_0}}\\&\times\sup_{0\le s_1\le t_1'}e^{\lambda_0 s_1^{\rho_0}}\|f(s_1)\|_\infty\Sigma_l(s)\mathrm{d}s\\\le&\frac{C_{q,\theta}}{T_0^{5/4}}e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_0^{t_l}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\\&\times\left\{e^{-\frac{\lambda_0}{2}s^{\rho_0}}\sup_{0\le s_1\le s}e^{\frac{\lambda_0}{2}s_1^{\rho_0}}\|f(s_1)\|_\infty\right\}\Sigma_l(s)\mathrm{d}s\\\le&\frac{C_{q,\theta}}{T_0^{5/4}}e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}}e^{-\frac{\lambda_0}{2}t_1^{\rho_0}}\sup_{0\le s\le t_1}e^{\frac{\lambda_0}{2}s^{\rho_0}}\left\{e^{-\frac{\lambda_0}{2}s^{\rho_0}}\sup_{0\le s_1\le s}e^{\frac{\lambda_0}{2}s_1^{\rho_0}}\|f(s_1)\|_\infty\right\}\\\le&\frac{C_{q,\theta}}{T_0^{5/4}}e^{-\frac{\lambda_0}{2}t^{\rho_0}}\sup_{0\le s\le t}e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty.\end{aligned}$$

Case II: $s_1\le t_{l'}'-\frac{1}{k^2(s)}$, by an argument similar to Case 1 and Case 2 in the proof of (3.60), one can show that if $|v_l|\ge N$ or $|v_{l'}'|\ge N$ or $|v_l|\le N$ and $|v'|\ge 2N$, or $|v_{l'}'|\le N$ and $|v''|\ge 2N$ with N large enough, $\mathcal{J}_{6,2}^1$ bears the bound

$$\frac{C_{q,\theta}}{N}e^{-\frac{\lambda_0}{2}t^{\rho_0}}\sup_{0\le s\le t}e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty.$$

Therefore, we need only to treat the cases $|v_l|\le N$, $|v'|\le 2N$, $|v_{l'}'|\le N$ and $|v''|\le 2N$. As with Case 3 in the proof of (3.60), in this situation, one may also use the similar approximation (3.66) to obtain

$$\begin{aligned}\mathcal{J}_{6,2}^1\le&\frac{C_{q,\theta}}{N}e^{-\frac{\lambda_0}{2}t^{\rho_0}}\sup_{0\le s\le t}e^{\frac{\lambda_0}{2}s^{\rho_0}}\|f(s)\|_\infty\\&+C_{q,\theta}e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\sum_{l=1}^{k-1}\int_0^{t_l}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\iint\mathrm{d}v'\mathrm{d}v''\int_{\prod_{j=1}^{k-1}\mathcal{V}_j'}\\&\times e^{-\lambda_0(s-t_1')^{\rho_0}}\sum_{l'=1}^{k-1}\int_0^{t_{l'}'}\mathrm{d}s_1\mathbf{1}_{\{t_{l'+1}'\le 0<t_{l'}'\}}|f(s_1)|\mathrm{d}\Sigma_{l'}(s_1)\Sigma_l(s)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C_{q,\theta}}{N} e^{-\frac{\lambda_0}{2}t^{\rho_0}} \sup_{0\leq s\leq t} e^{\frac{\lambda_0}{2}s^{\rho_0}} \|f(s)\|_\infty \\
&\quad + C_{q,\theta} e^{-\frac{\lambda_0}{2}(t-t_1)^{\rho_0}} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \int_0^{t_l} \mathbf{1}_{\{t_{l+1}\leq 0<t_l\}} \\
&\quad \times \left\{ e^{-\frac{\lambda_1}{2}s^{\rho_0}} (k(s))^7 \sup_{0\leq s_1\leq s} e^{\frac{\lambda_1}{2}s_1^{\rho_0}} \|f(s_1)\|_2 \right\} \Sigma_l(s) \mathrm{d}s \\
&\leq \frac{C_{q,\theta}}{N} e^{-\frac{\lambda_0}{2}t^{\rho_0}} \sup_{0\leq s\leq t} e^{\frac{\lambda_0}{2}s^{\rho_0}} \|f(s)\|_\infty + C_{q,\theta} e^{-\frac{\lambda_0}{2}t^{\rho_0}} \sup_{0\leq s\leq t} e^{\frac{\lambda_1}{2}s^{\rho_0}} \|f(s)\|_2.
\end{aligned}
$$

Here λ_0 is chosen to be smaller than λ_1 so that $e^{-\frac{\lambda_1}{2}s^{\rho_0}}(k(s))^7 \leq Ce^{-\frac{\lambda_0}{2}s^{\rho_0}}$.

Gathering the above estimates for $\mathcal{J}_{6,2}^1$, we see that (3.97) is true. Once (3.97) is obtained, the other terms in $\mathcal{J}_6$ and $\mathcal{J}_2$ can be treated in a similar fashion and after tedious calculations it turns out that they share the same bound as (3.97). Namely, we obtain

$$
\begin{aligned}
\mathcal{J}_2,\ \mathcal{J}_6 \leq C_{q,\theta}\left(\frac{1}{T_0^{5/4}}+\frac{1}{N}\right) e^{-\frac{\lambda_0}{2}t^{\rho_0}} \sup_{0\leq s\leq t} e^{\frac{\lambda_0}{2}s^{\rho_0}} \|f(s)\|_\infty \\
+ C_{q,\theta} e^{-\frac{\lambda_0}{2}t^{\rho_0}} \sup_{0\leq s\leq t} e^{\frac{\lambda_1}{2}s^{\rho_0}} \|f(s)\|_2.
\end{aligned}
\tag{3.98}
$$

Now, substituting (3.98) into (3.94) and choosing $\epsilon, \epsilon_0 > 0$ suitably small and $N, T_0 > 0$ sufficiently large, we have

$$
\begin{aligned}
e^{\frac{\lambda_0}{2}t^{\rho_0}} \|f(t)\|_\infty \leq C\|w_{q/2,\theta} f_0\|_\infty + C \sup_{0\leq s\leq t} e^{\frac{\lambda_0}{2}s^{\rho_0}} \|w_{q/2,\theta} g_f(s)\|_\infty \\
+ C \sup_{0\leq s\leq t} e^{\frac{\lambda_1}{2}s^{\rho_0}} \|f(s)\|_2.
\end{aligned}
\tag{3.99}
$$

Next, from (2.14) and (3.79), it follows that

$$
\begin{aligned}
\|w_{q/2,\theta} g_f(s)\|_\infty = \|w_{q/2,\theta}\Gamma(f,f)(s)\|_\infty &\leq C\|w_{q,\theta} f(s)\|_\infty \|f(s)\|_\infty \\
&\leq C\varepsilon_0 \|f(s)\|_\infty.
\end{aligned}
\tag{3.100}
$$

To control the last term in (3.99), we appeal to deduce the exponential decay of f in L^2. Notice that $f(t,x,v)$, as a global solution to (1.11), (1.12) and (1.13), satisfies (3.79). We know, thanks to (3.6) in Proposition 3.1, that $f(t,x,v)$ also satisfies

$$
\begin{aligned}
\|f(t)\|_2 \lesssim e^{-\frac{\lambda_1}{2}t^{\rho_0}} \Bigg\{ \|w_{q/2,\theta} f_0\|_2 + \sqrt{\int_0^t e^{\lambda_1 s^{\rho_0}} \|\nu^{-1/2}\Gamma(f,f)(s)\|_2^2 \mathrm{d}s} \\
+ \sqrt{\int_0^t \|\nu^{-1/2} w_{q/2,\theta}\Gamma(f,f)(s)\|_2^2 \mathrm{d}s} \Bigg\}.
\end{aligned}
\tag{3.101}
$$

On the other hand, from Lemma 2.3 and the bound (3.79), it follows that

$$\begin{aligned}\int_0^t e^{\lambda_1 s^{\rho_0}} \|\nu^{-1/2}\Gamma(f,f)(s)\|_2^2 \mathrm{d}s &\leq C \int_0^t e^{\lambda_1 s^{\rho_0}} \|w_{q/2,\theta} f(s)\|_\infty^2 \|f(s)\|_\nu^2 \mathrm{d}s \\ &\leq C \sup_{0\leq s\leq t} \|w_{q/2,\theta} f(s)\|_\infty^2 \int_0^t e^{\lambda_1 s^{\rho_0}} \|f(s)\|_\nu^2 \mathrm{d}s \\ &\leq C\varepsilon_0^2 \sup_{0\leq s\leq t} \|w_{q/2,\theta} f(s)\|_\infty^2,\end{aligned} \tag{3.102}$$

and similarly

$$\begin{aligned}\int_0^t \|\nu^{-1/2} w_{q/2,\theta}\Gamma(f,f)(s)\|_2^2 \mathrm{d}s &\leq C \int_0^t \|w_{q,\theta} f(s)\|_\infty^2 \|w_{q/2,\theta} f(s)\|_\nu^2 \mathrm{d}s \\ &\leq C\varepsilon_0^2 \int_0^t \|w_{q/2,\theta} f(s)\|_\nu^2 \mathrm{d}s \leq C\|w_{q,\theta} f_0\|_\infty^2.\end{aligned} \tag{3.103}$$

Consequently, (3.101), (3.102) and (3.103) give rise to

$$e^{\frac{\lambda_1}{2} t^{\rho_0}} \|f(t)\|_2 \leq C\|w_{q,\theta} f_0\|_\infty + C\varepsilon_0 \sup_{0\leq s\leq t} \|f(s)\|_\infty. \tag{3.104}$$

Now, plugging (3.104) and (3.100) into (3.99) leads us to

$$e^{\frac{\lambda_0}{2} t^{\rho_0}} \|f(t)\|_\infty \leq C\|w_{q,\theta} f_0\|_\infty.$$

This completes the proof of the second part of Theorem 1.1. Therefore we conclude the proof of Theorem 1.1. □

4. Specular Reflection Boundary Value Problem

4.1. L^2 Theory for the Linearized Equation

Let us look at the boundary value problem for the linearized homogeneous equation

$$\partial_t f + v\cdot\nabla_x f + Lf = 0, \quad f(0) = f_0, \quad \text{in} \quad (0,\infty)\times\Omega\times\mathbb{R}^3, \tag{4.1}$$

$$f(t,x,v)|_{\gamma_-} = f(t,x,R_x v), \quad \text{on} \quad [0,\infty)\times\gamma_-. \tag{4.2}$$

We first show that the macroscopic part of the solution of (4.1) and (4.2) can be dominated by the microscopic part on the time interval [0, 1].

Proposition 4.1. *Let* $f(t,x,v) \in L^\infty([0,1], L^2(\Omega\times\mathbb{R}^3))$ *be a solution to* (4.1) *and* (4.2)*, and* $f_\gamma \in L^2([0,1], L^2(\partial\Omega\times\mathbb{R}^3))$*, then there exists* $\delta_0 > 0$ *such that*

$$\int_0^1 (Lf, f)\mathrm{d}s \geq \delta_0 \int_0^1 \|f(s)\|_\nu^2 \mathrm{d}s. \tag{4.3}$$

Proof. The proof is based on contradiction and is divided into four steps.

Step 1. Proof of contradiction. If Proposition 4.1 is false, then no δ_0 exists as in Proposition 4.1. Hence, for any $n \geq 1$, there exists a sequence of non-zero $f_n \in L^\infty([0,1], L^2(\Omega \times \mathbb{R}^3))$ relevant to the linearized Boltzmann equation (4.1) such that

$$0 \leq \int_0^1 (Lf_n, f_n)\mathrm{d}s \leq \frac{1}{n} \int_0^1 \|f_n(s)\|_\nu^2 \mathrm{d}s, \tag{4.4}$$

since f_n satisfies

$$\partial_t f_n + v \cdot \nabla_x f_n + Lf_n = 0, \quad \text{in } (0,1] \times \Omega \times \mathbb{R}^3,$$

and

$$f_n(t,x,v)|_{\gamma_-} = f_n(t,x,R_x v), \quad \text{on } [0,1] \times \gamma_-.$$

With this, and by an argument similar to that for obtaining Lemma 8 in [24, pp. 340], one has

$$\sup_{0\leq t\leq 1} \|\nu^{1/2} f_n(t)\|_2^2 \leq C\|\nu^{1/2} f_n(0)\|_2^2, \; \int_0^1 \|f_n(s)\|_\nu^2 \mathrm{d}s \geq C\|\nu^{1/2} f_n(0)\|_2^2. \tag{4.5}$$

Assume that $f_n(0)$ is not identical to zero and set

$$Z_n = \frac{f_n(t,x,v)}{\sqrt{\int_0^1 \|f_n(s)\|_\nu^2 \mathrm{d}s}},$$

then

$$\int_0^1 \|Z_n(s)\|_\nu^2 \mathrm{d}s = 1, \tag{4.6}$$

and (4.4) is equivalent to

$$0 \leq \int_0^1 (LZ_n, Z_n)\mathrm{d}s \leq \frac{1}{n}. \tag{4.7}$$

Then (4.6) and (4.7) imply that there exists $Z(t,x,v)$ such that

$$Z_n \to Z \text{ weakly in } \int_0^1 \|\cdot\|_\nu^2 \mathrm{d}s,$$

and

$$\int_0^1 (LZ_n, Z_n)\mathrm{d}s = \int_0^1 (L(\mathbf{I}-\mathbf{P})Z_n, (\mathbf{I}-\mathbf{P})Z_n)\mathrm{d}s \to 0. \tag{4.8}$$

Notice that it is straightforward to verify

$$\mathbf{P}Z_n \to \mathbf{P}Z, \; (\mathbf{I}-\mathbf{P})Z_n \to (\mathbf{I}-\mathbf{P})Z, \; \text{ weakly in } \int_0^1 ||\cdot||_\nu^2 \mathrm{d}s.$$

It follows from (4.8) that $(\mathbf{I}-\mathbf{P})Z = 0$, therefore,

$$Z(t,x,v) = \{a(t,x) + v \cdot b(t,x) + |v|^2 c(t,x)\}\sqrt{\mu}.$$

Moreover, we have from $\partial_t f_n + v \cdot \nabla_x f_n + Lf_n = 0$ that

$$\partial_t Z_n + v \cdot \nabla_x Z_n + LZ_n = 0, \tag{4.9}$$

which yields

$$\partial_t Z + v \cdot \nabla_x Z = 0. \tag{4.10}$$

In what follows, we will show, on the one hand, that $Z = 0$ from (4.10) and the inherited boundary condition (4.2). On the other hand, Z_n will be proven to converge strongly to Z in $\int_0^1 \|\cdot\|_\nu^2 \mathrm{d}s$, and $\int_0^1 \|Z\|_\nu^2 \mathrm{d}s \neq 0$. This leads to a contradiction.

Step 2. The limit function $Z(t, x, v)$.

Lemma 4.1. *There exist constants a_0, c_0, c_1, c_2, and constant vectors b_0, b_1 and ϖ such that $Z(t, x, v)$ takes the form:*

$$\left(\left\{ \frac{c_0}{2}|x|^2 - b_0 \cdot x + a_0 \right\} + \{-c_0 t x - c_1 x + \varpi \times x + b_0 t + b_1\} \cdot v \right.$$
$$\left. + \left\{ \frac{c_0 t^2}{2} + c_1 t + c_2 \right\} |v|^2 \right) \sqrt{\mu}.$$

Moreover, these constants are finite:

$$|a_0| + |c_0| + |c_1| + |c_2| + |b_0| + |b_1| + |\varpi| < +\infty.$$

Proof. See Lemma 6 in [27, pp. 736]. □

Step 3. Compactness. To show the strong convergence $\lim_{n\to\infty} \int_0^1 \|\{Z_n - Z\}(s)\|_\nu^2 \mathrm{d}s = 0,$ we resort to the Averaging Lemma.

Lemma 4.2. *Up to a subsequence, it holds that* $\lim_{k\to\infty} \int_0^1 \|\{Z_n - Z\}(s)\|_\nu^2 \mathrm{d}s = 0.$

Proof. Define

$$\Omega_{\varepsilon^4} \equiv \{x \in \Omega : \xi(x) < -\varepsilon^4\}.$$

Choose any $\eta_0 > 0$ and a smooth cutoff function $\chi_1(t, x, v)$ in $(0, 1) \times \Omega \times \mathbb{R}^3$, such that $\chi_1(t, x, v) = 1$ in $[\eta_0, 1 - \eta_0] \times \Omega \backslash \Omega_{\varepsilon^4} \times \{|v| \leq \frac{1}{\eta_0}\}$. Next, multiplying the equation (4.9) by χ_1, we obtain

$$[\partial_t + v \cdot \nabla_x]\{\chi_1 Z_n\} = \{[\partial_t + v \cdot \nabla_x]\chi_1\} Z_n - \chi_1 L Z_n.$$

Since $f_n \in L^\infty([0, 1], L^2(\Omega \times \mathbb{R}^3))$, one sees that $\chi_1 Z_n \in L^2([0, 1], L^2(\Omega \times \mathbb{R}^3))$ and $\{[\partial_t + v \cdot \nabla_x]\chi_1\} Z_n - \chi_1 L Z_n \in L^2([0, 1], L^2(\Omega \times \mathbb{R}^3))$, then we know from the Averaging Lemma cf. [9, 10], that $\int \chi_1 Z_n e(v) \mathrm{d}v$ are compact in $L^2([0, 1] \times \Omega)$ for any exponential decay function $e(v)$. On the other hand, as with (4.5), from (4.9), it follows that

$$\sup_{0\leq t\leq 1} \|\nu^{1/2} Z_n(t)\|_2^2 \leq C\|\nu^{1/2} Z_n(0)\|_2^2, \quad \int_0^1 \|Z_n(s)\|_\nu^2 \mathrm{d}s \geq C\|\nu^{1/2} Z_n(0)\|_2^2.$$

Using this, one deduces

$$\begin{aligned}\int_0^1\int_\Omega\left(\int(1-\chi_1)Z_ne(v)\mathrm{d}v\right)^2\mathrm{d}x\mathrm{d}s+\int_0^1\int_\Omega\left(\int(1-\chi_1)Ze(v)\mathrm{d}v\right)^2\mathrm{d}x\mathrm{d}s\\ \le C\int_0^1\int_{\Omega\times\mathbb{R}^3}\left\{(1-\chi_1)^2Z_n^2e(v)+(1-\chi_1)^2Z^2e(v)\right\}\mathrm{d}v\mathrm{d}x\mathrm{d}s\\ \le C\int_{0\le s\le\eta_0}\int_{\Omega\times\mathbb{R}^3}+C\int_{1-\eta_0\le s\le 1}\int_{\Omega\times\mathbb{R}^3}+C\int_0^1\int_\Omega\int_{|v|\ge\frac{1}{\eta_0}}\\ \le C\eta_0\sum_{0\le s\le 1}\int_{\Omega\times\mathbb{R}^3}(1+|v|)^\varrho(Z_n^2+Z^2)\mathrm{d}v\mathrm{d}x\le C\eta_0.\end{aligned}$$

Therefore, up to a subsequence, the macroscopic parts of Z_k satisfy $\mathbf{P}Z_k\to\mathbf{P}Z=Z$ strongly in $L^2([0,1]\times\Omega\times\mathbb{R}^3)$. Therefore, in light of $\int_0^1||(\mathbf{I}-\mathbf{P})Z_k(s)||_\nu^2\mathrm{d}s\to 0$ in (4.8), we conclude our lemma. □

Step 4. Boundary condition leads to $Z=0$. Performing the same calculations as that of Section 3.6 in [27, pp. 747], we see that $Z=0$, and this leads to a contradiction, so finishes up the proof of Proposition 4.1. □

Once the coercivity estimate (4.3) is obtained, like Proposition 3.1, one can now deduce the basic energy estimates and time decay rates as follows:

Lemma 4.3. *Assume that* $f(t,x,v)$ *satisfies* (4.1) *and* (4.2)*, then it holds that*

$$\|f(t)\|_2^2+\int_0^t\|f\|_\nu^2\le C\|f_0\|_2^2, \tag{4.11}$$

and

$$\|w_{q/4,\theta}f(t)\|_2^2+\int_0^t\|w_{q/4,\theta}f\|_\nu^2\le C\|w_{q/4,\theta}f_0\|_2^2. \tag{4.12}$$

Moreover, there exists $\lambda>0$ *such that*

$$\|f(t)\|_2^2+e^{-\lambda t^{\rho_0}}\int_0^t e^{\lambda s^{\rho_0}}\|f\|_\nu^2\le Ce^{-\lambda t^{\rho_0}}\|w_{q/4,\theta}f_0\|_2^2, \tag{4.13}$$

here ρ_0 *is given as in Proposition* 3.1.

Proof. We prove (4.13) only, the proof for (4.11) and (4.12) being similar and easier. Taking the inner product of (4.1) with $e^{\lambda t^{\rho_0}}f$ over $\Omega\times\mathbb{R}^3$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{e^{\lambda t^{\rho_0}}\|f(t)\|_2^2\right\}+2(e^{\lambda t^{\rho_0}}Lf,f)=\lambda\rho_0t^{\rho_0-1}e^{\lambda t^{\rho_0}}\|f(t)\|_2^2. \tag{4.14}$$

For any $t>0$, there exists a nonnegative integer N such that $t\in[N,N+1)$. For the time interval $[0,N]$ (we may assume without lose of generality that $N\ge 1$), it follows that

$$e^{\lambda N^{\rho_0}}\|f(N)\|_2^2+2\int_0^N(e^{\lambda s^{\rho_0}}Lf,f)\mathrm{d}s=\|f_0\|_2^2+\lambda\rho_0\int_0^N s^{\rho_0-1}e^{\lambda s^{\rho_0}}\|f(s)\|_2^2\mathrm{d}s.$$

Split the time interval into $\cup_{j=0}^{N-1}[j, j+1)$ and define $f_j(s, x, v) = f(j+s, x, v)$ for $j = 0, 1, 2, \ldots, N-1$, to deduce

$$
\begin{aligned}
&e^{\lambda N^{\rho_0}}\|f(N)\|_2^2 + 2\sum_{j=1}^{N-1}\int_0^1 (e^{\lambda(j+s)^{\rho_0}} Lf_j, f_j)\mathrm{d}s \\
&\qquad\qquad \le \|f_0\|_2^2 + \lambda\rho_0 \sum_{j=1}^{N-1}\int_0^1 (j+s)^{\rho_0-1} e^{\lambda(j+s)^{\rho_0}}\|f_j(s)\|_2^2 \mathrm{d}s,
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
&e^{\lambda N^{\rho_0}}\|f(N)\|_2^2 + 2\sum_{j=1}^{N-1}\int_0^1 (e^{\lambda j^{\rho_0}} Lf_j, f_j)\mathrm{d}s \\
&\quad \le \|f_0\|_2^2 + C\lambda\rho_0 \sum_{j=1}^{N-1}\int_0^1 j^{\rho_0-1} e^{\lambda j^{\rho_0}}\|f_j(s)\|_2^2 \mathrm{d}s
\end{aligned}
\tag{4.15}
$$

for $0 < \rho_0 < 1$.

On the other hand, we get from (4.3) that

$$
\sum_{j=1}^{N-1}\int_0^1 (e^{\lambda j^{\rho_0}} Lf_j, f_j)\mathrm{d}s \ge \delta_0 \sum_{j=1}^{N-1}\int_0^1 e^{\lambda j^{\rho_0}}\|f_j\|_\nu^2 \mathrm{d}s. \tag{4.16}
$$

Substituting (4.16) into (4.15) leads us to

$$
\begin{aligned}
&e^{\lambda N^{\rho_0}}\|f(N)\|_2^2 + \sum_{j=1}^{N-1}\int_0^1 e^{\lambda j^{\rho_0}}\|f_j\|_\nu^2 \mathrm{d}s \\
&\quad \le C\|f_0\|_2^2 + C\lambda\rho_0 \sum_{j=1}^{N-1}\int_0^1 j^{\rho_0-1} e^{\lambda j^{\rho_0}}\|f_j(s)\|_2^2 \mathrm{d}s.
\end{aligned}
\tag{4.17}
$$

To handle the integral on the right hand side of the above inequality, we decompose the velocity integration domain as

$$
E_j = \{v \mid j^{\rho_0-1} \le \kappa_0'\nu\}, \quad E_j^c = \{v \mid j^{\rho_0-1} > \kappa_0'\nu\},
$$

where $\kappa_0' > 0$ and small enough. Therefore, for $\lambda = \frac{q}{16}(\kappa_0')^{\frac{\rho_0}{1-\rho_0}}$, it follows that

$$
\begin{aligned}
&\sum_{j=1}^{N-1}\int_0^1 j^{\rho_0-1} e^{\lambda j^{\rho_0}}\|f_j(s)\|_2^2 \mathrm{d}s \\
&\quad \le \kappa_0' \sum_{j=1}^{N-1}\int_0^1 e^{\lambda j^{\rho_0}}\|f_j(s)\|_\nu^2 \mathrm{d}s
\end{aligned}
$$

$$+ C \sum_{j=1}^{N-1} \int_0^1 j^{\rho_0 - 1} e^{-\lambda j^{\rho_0}} e^{2\lambda (\kappa_0')^{\frac{\rho_0}{\rho_0-1}} \nu^{\frac{\rho_0}{\rho_0-1}}} \|\mathbf{1}_{E_j^c} f_j(s)\|_2^2 \mathrm{d}s$$
$$\le \kappa_0' \sum_{j=1}^{N-1} \int_0^1 e^{\lambda j^{\rho_0}} \|f_j(s)\|_\nu^2 \mathrm{d}s + C \sum_{0 \le s \le N} \|w_{q/4,\theta} f(s)\|_2^2 \sum_{j=1}^{N-1} j^{\rho_0-1} e^{-\lambda j^{\rho_0}}. \tag{4.18}$$

Putting (4.18) back into (4.17), and noticing that $\sum_{j=1}^{N-1} j^{\rho_0-1} e^{-\lambda j^{\rho_0}} < \infty$, we arrive at

$$e^{\lambda N^{\rho_0}} \|f(N)\|_2^2 + \sum_{j=1}^{N-1} \int_0^1 e^{\lambda j^{\rho_0}} \|f_j\|_\nu^2 \mathrm{d}s \le C\|f_0\|_2^2 + C\|w_{q/4,\theta} f_0\|_2^2,$$

where we used (4.12). Changing back to $f_j(s) = f(s+j)$ and using $e^{(j+s)^{\rho_0} - s^{\rho_0}} \le e^{j^{\rho_0}}$, one further has

$$e^{\lambda N^{\rho_0}} \|f(N)\|_2^2 + \int_0^N e^{\lambda s^{\rho_0}} \|f(s)\|_\nu^2 \mathrm{d}s \le C\|f_0\|_2^2 + C\|w_{q/4,\theta} f_0\|_2^2. \tag{4.19}$$

Now integrate (4.14) over $[N, t]$ to obtain

$$e^{\lambda t^{\rho_0}} \|f(t)\|_2^2 + \int_N^t e^{\lambda s^{\rho_0}} (Lf, f) \mathrm{d}s$$
$$\le \lambda \rho_0 \int_N^t s^{\rho_0-1} e^{\lambda s^{\rho_0}} \|f(s)\|_2^2 \mathrm{d}s + e^{\lambda N^{\rho_0}} \|f(N)\|_2^2. \tag{4.20}$$

Thanks to Lemma 2.2, one has

$$\int_N^t e^{\lambda s^{\rho_0}} (Lf, f) \mathrm{d}s \ge \delta \int_N^t e^{\lambda s^{\rho_0}} \|f(s)\|_\nu^2 \mathrm{d}s - C \int_N^t e^{\lambda s^{\rho_0}} \|\mathbf{1}_{|v| \le C} f(s)\|_\nu^2 \mathrm{d}s. \tag{4.21}$$

From (4.20) and (4.21), it follows that

$$e^{\lambda t^{\rho_0}} \|f(t)\|_2^2 + \delta \int_N^t e^{\lambda s^{\rho_0}} \|f(s)\|_\nu^2 \mathrm{d}s$$
$$\le \lambda \rho_0 \int_N^t s^{\rho_0-1} e^{\lambda s^{\rho_0}} \|f(s)\|_2^2 \mathrm{d}s + C\|f_0\|_2^2 + e^{\lambda N^{\rho_0}} \|f(N)\|_2^2,$$

where the fact that $\int_N^t e^{\lambda s^{\rho_0}} \|\mathbf{1}_{|v| \le C} f(s)\|_\nu^2 \mathrm{d}s \le C \sum_{0 \le s \le t} \|f(s)\|_2^2 \le C\|f_0\|_2^2$ was used. We then have, by performing calculations similar to those as for obtaining (4.19),

$$e^{\lambda t^{\rho_0}} \|f(t)\|_2^2 + \int_N^t e^{\lambda s^{\rho_0}} \|f(s)\|_\nu^2 \mathrm{d}s \le C\|f_0\|_2^2 + C\|w_{q/4,\theta} f_0\|_2^2 + Ce^{\lambda N^{\rho_0}} \|f(N)\|_2^2. \tag{4.22}$$

Thereby, (4.13) follows from (4.19) and (4.22). This finishes the proof of Lemma 4.3. □

4.2. L^∞ Theory for the Linearized Equation

Recall

$$w_{q,\theta,\vartheta} = \exp\left\{\frac{q|v|^\theta}{8} + \frac{q|v|^\theta}{8(1+t)^\vartheta}\right\}, \quad (q,\theta) \in \mathcal{A}_{q,\theta}, \quad 0 \le \vartheta < -\frac{\theta}{\varrho}.$$

Let $h = w_{q,\theta,\vartheta}(t,v)f(t,x,v)$. The problem, (4.1) and (4.2) are now equivalent to

$$\partial_t h + v\cdot\nabla_x h + \left(\nu + \frac{\vartheta q|v|^\theta}{8(1+t)^{\vartheta+1}}\right)h = K_{\overline{w}}h, \quad h(0) = h_0, \quad \text{in} \quad (0,\infty)\times\Omega\times\mathbb{R}^3, \tag{4.23}$$

with

$$h(t,x,v)|_{\gamma_-} = h(t,x,R_x v), \quad \text{on} \quad [0,\infty)\times\gamma_-. \tag{4.24}$$

Here $K_{\overline{w}}h = w_{q,\theta,\vartheta}K\left(\frac{h}{w_{q,\theta,\vartheta}}\right)$ as in Section 3.1.

We express solution $h(t,x,v)$ to (4.23) and (4.24) through semigroup $U(t)$ as

$$h(t,x,v) = \{U(t)h_0\}(x,v),$$

with initial boundary data given by

$$\{U(0)h_0\}(x,v) = h_0(x,v), \quad \text{and} \quad U(0)h_0(x,v)|_{\gamma_-} = h_0(x,R_x v).$$

For the sake of simplicity, we denote

$$\widetilde{\nu}(v,t) = \nu + \frac{\vartheta q|v|^\theta}{8(1+t)^{\vartheta+1}}.$$

It is obvious to see $\widetilde{\nu}^{-1} < \nu^{-1}$, which plays a significant role in the later proof.

Applying Young's inequality, one can see that there exists $C_{\varrho,q,\vartheta} > 0$ independent of v such that

$$\widetilde{\nu}(v,t) \ge C_{\varrho,q,\vartheta}(1+t)^{\frac{(1+\vartheta)\varrho}{\theta-\varrho}}, \tag{4.25}$$

and for $t > 0$, one sees that

$$C_{\varrho,q,\vartheta}(1+t)^{\frac{(1+\vartheta)\varrho}{\theta-\varrho}} \sim C_{\varrho,q,\vartheta}t^{\frac{(1+\vartheta)\varrho}{\theta-\varrho}}. \tag{4.26}$$

From (4.25) and (4.26), it follows that

$$e^{-\int_s^t \widetilde{\nu}(v,\tau)\mathrm{d}\tau} \le \exp\left(-\lambda_2\left\{t^{\frac{\theta+\vartheta\varrho}{\theta-\varrho}} - s^{\frac{\theta+\vartheta\varrho}{\theta-\varrho}}\right\}\right) \overset{\text{def}}{=} e^{\lambda_2 s^{\rho_1} - \lambda_2 t^{\rho_1}}, \quad t \ge s \ge 0. \tag{4.27}$$

Here, $\rho_1 = \frac{\theta+\vartheta\varrho}{\theta-\varrho}$ with $\theta + \vartheta\varrho > 0$, moreover $\lambda_2 > 0$ is independent of v.

Our goal in this subsection will be to prove the following:

Proposition 4.2. *Let $0 < \vartheta < -\frac{\theta}{\varrho}$ with $-3 < \varrho < 0$ and $(q,\theta) \in \mathcal{A}_{q,\theta}$. Assume that ξ is both strictly convex (1.8) and analytic, and the mass (1.17) and energy (1.18) are conserved. In the case that Ω has rotational symmetry (1.9), we also assume the conservation of corresponding angular momentum (1.19). Let $h_0 \in L^\infty$. Then there exist $\lambda_0 > 0$ and $C > 0$ such that (4.23) and (4.24) admit a unique solution $U(t)h_0$ satisfying*

$$\|U(t)h_0\|_\infty \le Ce^{-\frac{\lambda_0}{2}t^{\rho_1}} \|h_0\|_\infty, \tag{4.28}$$

where $\rho_1 = \frac{\theta+\vartheta\varrho}{\theta-\varrho}$.

The Duhamel Principle will be applied to prove Proposition 4.2 and the first step is an appropriate decomposition. Initially, we look for solutions to the linearized equation (4.23) with the almost compact operator $K_{\overline{w}}$ removed. Namely, we first consider

$$\partial_t h + v \cdot \nabla_x h + \widetilde{\nu}(v,t)h = 0, \quad h(0) = h_0, \quad \text{in} \quad (0,\infty) \times \Omega \times \mathbb{R}^3, \tag{4.29}$$

with

$$h(t,x,v)|_{\gamma_-} = h(t,x,R_x v), \quad \text{on} \quad [0,\infty) \times \gamma_-. \tag{4.30}$$

Let us denote the solution to (4.29) and (4.30) as semigroup $G(t)h_0$.

Prior to investigating the properties of the solution operators $U(t)$ and $G(t)$, we give the following definition:

Definition 4.1. Let Ω be convex (1.8). Fix any point $(t,x,v) \notin \gamma_0 \cap \gamma_-$, and define $(t_0,x_0,v_0) = (t,x,v)$, and for $k \ge 1$

$$(t_{k+1},x_{k+1},v_{k+1}) = (t_k - t_{\mathbf{b}}(t_k,x_k,v_k), x_{\mathbf{b}}(x_k,v_k), R_{x_{k+1}}v_k), \tag{4.31}$$

where $R_{x_{k+1}}v_k = v_k - 2(v_k \cdot n(x_{k+1}))n(x_{k+1})$. We define the specular back-time cycle as

$$X_{\mathbf{cl}}(s) \equiv \sum_{k=1} \mathbf{1}_{[t_{k+1},t_k)}(s)\{x_k + v_k(s-t_k)\}, \quad V_{\mathbf{cl}}(s) \equiv \sum_{k=1} \mathbf{1}_{[t_{k+1},t_k)}(s)v_k.$$

Lemma 4.4. *Let $h_0 \in L^\infty(\Omega \times \mathbb{R}^3)$. There exists a unique solution $G(t)h_0$ to*

$$\{\partial_t + v \cdot \nabla_x + \widetilde{\nu}(v,t)\}\{G(t)h_0\} = 0, \quad \{G(0)h_0\} = h_0,$$

with the specular reflection $\{G(0)h_0\}(t,x,v) = \{G(0)h_0\}(t,x,R_x v)$ for $x \in \partial\Omega$. For almost any $(x,v) \in \overline{\Omega} \times \mathbb{R}^3 \backslash \gamma_0$,

$$\begin{aligned}\{G(t)h_0\}(t,x,v) &= e^{-\int_0^t \widetilde{\nu}(v,\tau)\mathrm{d}\tau} h_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \\ &= \sum_k^\infty \mathbf{1}_{[t_{k+1},t_k)}(0)e^{-\int_0^t \widetilde{\nu}(v,\tau)\mathrm{d}\tau} h_0(x_k - t_k v_k, v_k).\end{aligned} \tag{4.32}$$

Here, we define $t_k = 0$ if $t_k < 0$.

Moreover, it holds that

$$\|G(t)h_0\|_\infty \leq \left\| e^{-\int_0^t \widetilde{\nu}(v,\tau)\mathrm{d}\tau} h_0 \right\|_\infty , \tag{4.33}$$

and there exists $\lambda_2 > 0$ *such that*

$$\|G(t)h_0\|_\infty \leq Ce^{-\lambda_2 t^{\rho_1}} \|h_0\|_\infty , \quad t \geq 0, \tag{4.34}$$

and

$$\|G(t-s)h(s)\|_\infty \leq Ce^{-\lambda_2\{t^{\rho_1}-s^{\rho_1}\}} \|h(s)\|_\infty , \quad t \geq s \geq 0. \tag{4.35}$$

Proof. The proof for (4.32) and (4.33) is the same as that of Lemma 15 in [27, pp. 757]. As such, (4.34) and (4.35) directly follow from (4.27) and (4.33), and this completes the proof of Lemma 4.4. □

The following lemma shows that the solution operator $G(t)h_0$ is indeed continuous away from the grazing set:

Lemma 4.5. [27, Lemma 21, pp. 768] *Let* ξ *be convex as in* (1.8)*. Let* h_0 *be continuous in* $\bar{\Omega}\times\mathbb{R}^3\backslash\gamma_0$ *and* $g(t,x,v)$ *be continuous in the interior of* $[0,\infty)\times\Omega\times\mathbb{R}^3$ *and* $\sup_{[0,\infty)\times\Omega\times\mathbb{R}^3}|\frac{g(t,x,v)}{\widetilde{\nu}(v,t)}| < \infty$*. Assume that on* γ_-*,* $h_0(x,v) = h_0(x, R(x)v)$*. Then the specular solution* $h(t,x,v)$ *to*

$$\partial_t h + v\cdot\nabla_x h + \widetilde{\nu}(v,t)h = g(t,x,v), \quad h(0) = h_0, \quad \textit{in} \quad (0,\infty)\times\Omega\times\mathbb{R}^3,$$

with

$$h(t,x,v)|_{\gamma_-} = h(t,x,R_x v), \quad \textit{on} \quad [0,\infty)\times\gamma_-,$$

is continuous on $[0,\infty)\times\{\bar{\Omega}\times\mathbb{R}^3\backslash\gamma_0\}$.

We now go back to (4.23) and (4.24). From the Duhamel formula, it follows that

$$\{U(t)h_0\}(x,v) = G(t)h_0(x,v) + \int_0^t \mathrm{d}s\; G(t-s)K_{\overline{w}}\{U(s)h_0\}(x,v).$$

Employing the decomposition $K_{\overline{w}} = K_{\overline{w}}^{\chi} + K_{\overline{w}}^{1-\chi}$ again, we then expand out:

$$\begin{aligned}\{U(t)h_0\}(x,v) = G(t)h_0(x,v) &+ \int_0^t \mathrm{d}s\; G(t-s)K_{\overline{w}}^{1-\chi}\{U(s)h_0\}(x,v)\\ &+ \int_0^t \mathrm{d}s\; G(t-s)K_{\overline{w}}^{\chi}\{U(s)h_0\}(x,v).\end{aligned}$$

We further iterate the Duhamel formula of the last term, as was done in [46]:

$$\{U(s)h_0\}(x,v) = G(s)h_0(x,v) + \int_0^s \mathrm{d}s_1\; G(s-s_1)K_{\overline{w}}\{U(s_1)h_0\}(x,v).$$

Substituting this into the previous expression and using $K_{\overline{w}} = K_{\overline{w}}^{\chi} + K_{\overline{w}}^{1-\chi}$ again yields a more elaborate formula:

$$\begin{aligned}\{U(t)h_0\}(x,v) = {} & G(t)h_0(x,v) + \int_0^t \mathrm{d}s\, G(t-s)K_{\overline{w}}^{1-\chi}\{U(s)h_0\}(x,v) \\ & + \int_0^t \mathrm{d}s\, G(t-s)K_{\overline{w}}^{\chi}\{G(s)h_0\}(x,v) \\ & + \int_0^t \mathrm{d}s \int_0^s \mathrm{d}s_1\, G(t-s)K_{\overline{w}}^{\chi}G(s-s_1)K_{\overline{w}}^{1-\chi}\{U(s_1)h_0\}(x,v) \\ & + \int_0^t \mathrm{d}s \int_0^s \mathrm{d}s_1\, G(t-s)K_{\overline{w}}^{\chi}G(s-s_1)K_{\overline{w}}^{\chi}\{U(s_1)h_0\}(x,v) \\ \stackrel{\text{def}}{=} {} & \sum_{l=1}^{5} H_l(t,x,v). \end{aligned} \tag{4.36}$$

For any fixed point (t, x, v) with $(x, v) \notin \gamma_0$, let the back-time specular cycle of (t, x, v) be $[x_{\mathbf{cl}}(s), v_{\mathbf{cl}}(s)]$, then the most delicate term H_5 in (4.36) can be rewritten as

$$\begin{aligned} H_5(t,x,v) = {} & \int_0^t \mathrm{d}s \int_0^s \mathrm{d}s_1 \int \mathrm{d}v' dv'' e^{-\int_s^t \widetilde{\nu}(v,\tau)\mathrm{d}\tau - \int_{s_1}^s \widetilde{\nu}(v',\tau)\mathrm{d}\tau} \\ & \times \mathbf{k}_{\overline{w}}^{\chi}(V_{\mathbf{cl}}(s), v')\mathbf{k}_{\overline{w}}^{\chi}(V_{\mathbf{cl}}'(s_1), v'')h\left(X_{\mathbf{cl}}'(s_1), v''\right), \end{aligned}$$

where $\mathbf{k}_{\overline{w}}^{\chi}(\cdot) = w_{q,\theta,\vartheta}\mathbf{k}^{\chi}(\frac{\cdot}{w_{q,\theta,\vartheta}})$ and the back-time specular cycle from $(s, X_{\mathbf{cl}}(s), v')$ is denoted by

$$X_{\mathbf{cl}}'(s_1) = X_{\mathbf{cl}}(s_1; s, X_{\mathbf{cl}}(s), v'), \qquad V_{\mathbf{cl}}'(s_1) = V_{\mathbf{cl}}(s_1; s, X_{\mathbf{cl}}(s), v'). \tag{4.37}$$

More explicitly, let t_k and $t_{k'}'$ be the corresponding times for both specular cycles, as in (4.31). For $t_{k+1} \le s < t_k$, $t_{k'+1}' \le s_1 < t_{k'}'$

$$X_{\mathbf{cl}}'(s_1) = X_{\mathbf{cl}}(s_1; s, X_{\mathbf{cl}}(s), v') \equiv x_{k'}' + (s_1 - t_{k'}')v_{k'}', \tag{4.38}$$

where $x_{k'}' = X_{\mathbf{cl}}(t_{k'}'; s, x_k + (s - t_k)v_k, v')$, $v_{k'}' = V_{\mathbf{cl}}(t_{k'}'; s, x_k + (s - t_k)v_k, v')$. Recall α in (2.18) and define, naturally,

$$\alpha(x,v) \equiv \alpha(t) = \xi^2(x) + [v \cdot \nabla\xi(x)]^2 - 2[v \cdot \nabla^2\xi(x) \cdot v]\xi(x).$$

We define the main set

$$A_\alpha = \left\{(x,v) : x \in \bar{\Omega}, \quad \frac{1}{N} \le |v| \le N, \quad \text{and} \quad \alpha(x,v) \ge \frac{1}{N}\right\}. \tag{4.39}$$

Lemma 4.6. [27, Lemma 22, pp. 775] *Fix k and k'. Define for $t_{k+1} \le s \le t_k$, $s_1 \in \mathbb{R}$ and*

$$J \equiv J_{k,k'}(t,x,v,s,s_1,v') \equiv \det\left(\frac{\partial\{x_{k'}' + (s_1 - t_{k'}')v_{k'}'\}}{\partial v'}\right).$$

For any $\varepsilon > 0$ *sufficiently small, there is* $\widetilde{\delta}(N, \varepsilon, T_0, k, k') > 0$ *and an open covering* $\cup_{i=1}^m B(t_i, x_i, v_i; r_i)$ *of* $[0, T_0] \times A_\alpha$ *and corresponding open sets* O_{t_i,x_i,v_i} *for* $[t_{k+1} + \varepsilon, t_k - \varepsilon] \times \mathbb{R} \times \mathbb{R}^3$ *with* $|O_{t_i,x_i,v_i}| < \varepsilon$*, such that*

$$|J_{k,k'}(t, x, v, s, s_1, v')| \geq \widetilde{\delta} > 0$$

for $0 \leq t \leq T_0$, $(x, v) \in A_\alpha$ *and* (s, s_1, v') *in*

$$O^c_{t_i,x_i,v_i} \cap [t_{k+1} + \varepsilon, t_k - \varepsilon] \times [0, T_0] \times \{|v'| \leq 2N\}.$$

In order to prove Proposition 4.2, we first show the following crucial estimates with the aid of Lemmas 4.5 and 4.6:

Lemma 4.7. *There exist constants* $T_0 > 0$ *and* $C_{T_0} > 0$ *such that*

$$\|U(T_0)h_0\|_\infty \leq e^{-\lambda_0 T_0^{\rho_1}} \|h_0\|_\infty + C_{T_0} \int_0^{T_0} \|f(s)\|_2 \mathrm{d}s. \tag{4.40}$$

Proof. Our proof is divided into two steps.

Step 1. Estimate of $h\mathbf{1}_{A_\alpha}$. Let us split $h = h\mathbf{1}_{A_\alpha} + h(1 - \mathbf{1}_{A_\alpha})$. We first express and estimate the main part, $h\mathbf{1}_{A_\alpha}$, through (4.36). By utilizing (4.27) and Lemmas 2.2 and 4.4, we see that

$$\begin{aligned}
|H_1(t, x, v)| &\leq Ce^{-\lambda_2 t^{\rho_1}} \|h_0\|_\infty, \\
|H_2(t, x, v)| &\leq C \int_0^t e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} \widetilde{\nu}(v) e^{-\frac{\lambda_2}{2}(t^{\rho_1} - s^{\rho_1})} e^{-\frac{\lambda_2}{2}s^{\rho_1}} e^{\frac{\lambda_2}{2}s^{\rho_1}} \\
&\quad \times [K_{\overline{w}}^{1-\chi} U(s)h(s)] \widetilde{\nu}^{-1}(v)\mathrm{d}s \\
&\leq C\epsilon^{3+\varrho} e^{-\frac{\lambda_2}{2}t^{\rho_1}} \sup_{0 \leq s \leq t} \left\| e^{\frac{\lambda_2}{2}s^{\rho_1}} U(s)h(s) \right\|_\infty \int_0^t e^{-\frac{1}{2}\int_0^s \widetilde{\nu}(v')\mathrm{d}\tau} \widetilde{\nu}(v)\mathrm{d}s \\
&\leq C\epsilon^{3+\varrho} e^{-\frac{\lambda_2}{2}t^{\rho_1}} \sup_{0 \leq s \leq t} \left\| e^{\frac{\lambda_2}{2}s^{\rho_1}} U(s)h(s) \right\|_\infty.
\end{aligned}$$

Here we have used the fact that $\int_0^t e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} \widetilde{\nu}(v)\mathrm{d}s < \infty$, as well as the significant observation $\widetilde{\nu}^{-1} \leq \nu^{-1}$. Continuing, one has

$$\begin{aligned}
|H_3(t, x, v)| &\leq C \|h_0\|_\infty \int_{\mathbb{R}^3} \int_0^t e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} \widetilde{\nu}(v) e^{-\frac{\lambda_2}{2}(t^{\rho_1} - s^{\rho_1})} e^{-\frac{\lambda_2}{2}s^{\rho_1}} \nu^{-1}(v) \\
&\quad \times \mathbf{k}_{\overline{w}}^\chi(V_{\mathbf{cl}}(s), v')\mathrm{d}s\mathrm{d}v' \\
&\leq Ce^{-\frac{\lambda_2}{2}t^{\rho_1}} \|h_0\|_\infty,
\end{aligned}$$

and

$$\begin{aligned}
|H_4(t, x, v)| &\leq C\epsilon^{3+\varrho} \sup_{0 \leq s \leq t} \left\| e^{\frac{\lambda_2}{2}s^{\rho_1}} U(s)h(s) \right\|_\infty \\
&\quad \times \int_{\mathbb{R}^3} \int_0^t \int_0^s e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} e^{-\frac{1}{2}\int_{s_1}^s \widetilde{\nu}(v')\mathrm{d}\tau} \widetilde{\nu}(v)\widetilde{\nu}(v') \\
&\quad \times e^{-\frac{\lambda_2}{2}(t^{\rho_1} - s^{\rho_1})} e^{-\frac{\lambda_2}{2}(s^{\rho_1} - s_1^{\rho_1})} e^{-\frac{\lambda_2}{2}s_1^{\rho_1}} \nu^{-1}(v) \mathbf{k}_{\overline{w}}^\chi(V_{\mathbf{cl}}(s), v')\mathrm{d}s\mathrm{d}s_1\mathrm{d}v' \\
&\leq C\epsilon^{3+\varrho} e^{-\frac{\lambda_2}{2}t^{\rho_1}} \sup_{0 \leq s \leq t} \left\| e^{\frac{\lambda_2}{2}s^{\rho_1}} U(s)h(s) \right\|_\infty.
\end{aligned}$$

For the main contribution H_5, notice that along the back-time specular cycles $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$ and $[X'_{\mathbf{cl}}(s_1), V'_{\mathbf{cl}}(s_1)]$ in (4.37), $|V_{\mathbf{cl}}(s)| \equiv |v|$ and $|V'_{\mathbf{cl}}(s_1)| \equiv |v'|$. Therefore, the integration over $|v| > N$ or $|v'| \geq 2N$ or $|v'| \leq 2N$ and $|v''| \geq 3N$ are bounded by

$$C\left\{e^{-\frac{\varepsilon N^2}{16}} + \frac{1}{N}\right\} e^{-\frac{\lambda_2}{2}t^{\rho_1}} \sup_{0\leq s\leq t}\left\|e^{\frac{\lambda_2}{2}s^{\rho_1}}h(s)\right\|_\infty .$$

As in Case 3 in Section 3.2, by using the same approximation, we only need to concentrate on the bounded set $\{|v| \leq N,\ |v'| \leq 2N$ and $|v''| \leq 3N\}$ of

$$\begin{aligned}\int_0^t\int_0^s\int_{|v'|\leq 2N, |v''|\leq 3N} & e^{-\int_s^t \widetilde{\nu}(v)\tau - \int_{s_1}^s \widetilde{\nu}(v')\mathrm{d}\tau}\left|h\left(s_1, X'_{\mathbf{cl}}(s_1), v''\right)\right| \mathrm{d}v'\mathrm{d}v''\mathrm{d}s_1\mathrm{d}s \\ &= \int_{\substack{\alpha(X_{\mathbf{cl}}(s),v')<\varepsilon\\ |v'|\leq 2N,|v''|\leq 3N}} + \int_{\substack{\alpha(X_{\mathbf{cl}}(s),v')\geq\varepsilon\\ |v'|\leq 2N,|v''|\leq 3N}} = H_{5,1} + H_{5,2}.\end{aligned}$$

In the case $\alpha(X_{\mathbf{cl}}(s), v') \leq \varepsilon$, $\xi^2(X_{\mathbf{cl}}(s)) + [v' \cdot \nabla\xi(X_{\mathbf{cl}}(s))]^2 \leq \varepsilon$, notice that $|\nabla\xi(X_{\mathbf{cl}}(s))| \geq c > 0$, hence for ε small and $X_{\mathbf{cl}}(s) \backsim \partial\Omega$, $H_{5,1}$ is dominated by

$$\begin{aligned}H_{5,1} \leq C_N &\int_0^t\int_0^s e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} e^{-\frac{1}{2}\int_{s_1}^s \widetilde{\nu}(v')\mathrm{d}\tau}\widetilde{\nu}(v)\widetilde{\nu}(v') \\ &\times e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s^{\rho_1})}e^{-\frac{\lambda_2}{2}(s^{\rho_1}-s_1^{\rho_1})}e^{-\frac{\lambda_2}{2}s_1^{\rho_1}}e^{\frac{\lambda_2}{2}s_1^{\rho_1}}\left\|\widetilde{\nu}^{-1}h(s_1)\right\|_\infty \mathrm{d}s\mathrm{d}s_1 \\ &\times \int_{\substack{\alpha(X_{\mathbf{cl}}(s),v')\leq\varepsilon\\ |v'|\leq 2N,|v''|\leq 3N}} \\ &\leq C_N e^{-\frac{\lambda_2}{2}t^{\rho_1}} \sup_{0\leq s\leq t}\left\|e^{\frac{\lambda_2}{2}s^{\rho_1}}h(s)\right\|_\infty \int_{|v'\cdot\frac{\nabla\xi(X_{\mathbf{cl}}(s))}{|\nabla\xi(X_{\mathbf{cl}}(s))|}|\leq c\varepsilon, |v'|\leq 2N, |v''|\leq 3N} \\ &\leq C_N\varepsilon e^{-\frac{\lambda_2}{2}t^{\rho_1}} \sup_{0\leq s\leq t}\left\|e^{\frac{\lambda_2}{2}s^{\rho_1}}h(s)\right\|_\infty .\end{aligned}$$

As for case $\alpha(X_{\mathbf{cl}}(s), v') \geq \varepsilon$ from (4.38), we bound $H_{5,2}$ as

$$\begin{aligned}C_N\int_0^t & e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s_1^{\rho_1})}\int_0^s\int_{\substack{\alpha(X_{\mathbf{cl}}(s),v')\geq\varepsilon\\ |v'|\leq 2N,|v''|\leq 3N}} |h\left(s_1, X'_{\mathbf{cl}}(s_1), v''\right)|\mathrm{d}v'\mathrm{d}v'' \\ &= C_N\sum_{k,k'}\int_{t_{k+1}}^{t_k}\int_{t'_{k'+1}}^{t'_{k'}}\int_{\substack{\alpha(X_{\mathbf{cl}}(s),v')\geq\varepsilon\\ |v'|\leq 2N,|v''|\leq 3N}} e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s_1^{\rho_1})} \\ &\quad\times |h\left(s_1, x'_{k'} + (s_1 - t'_{k'})v'_{k'}, v''\right)|\mathrm{d}v'\mathrm{d}v'',\end{aligned}$$

where $[t'_{k'}, x'_{k'}, v'_{k'}]$ is the back-time cycle of $(s, x_k + (s - t_k)v_k, v_k)$, for $t_{k+1} \leq s \leq t_k$.

We now study $x'_{k'} + (s_1 - t'_{k'})v'_{k'}$. By repeatedly using Velocity Lemma 2.5, we deduce for $(t, x, v) \in A_\alpha$ and $0 < t \leq T_0$ and $\alpha(X_{\mathbf{cl}}(s), v') \geq \varepsilon$ that

$$\begin{aligned}\alpha(t_l) &\sim \{v_l \cdot n_{x_l}\}^2 \geq e^{-\{C_\xi N-1\}T_0}\alpha(t) \geq C_{T_0,\xi,N} > 0;\\ \alpha(t'_{l'}) &\sim \{v'_{l'} \cdot n_{x'_{l'}}\}^2 \geq e^{-\{C_\xi N-1\}T_0}\alpha(X_{\mathbf{cl}}(s), v') \geq C_{T_0,\xi,N}\varepsilon > 0.\end{aligned}$$

Therefore, applying (2.19) in Lemma 2.6 yields $t_l - t_{l+1} \geq \frac{c_{T_0,\xi,N}}{N^2}$ and $t'_{l'} - t'_{l'+1} \geq \frac{c_{T_0,\xi,N}\varepsilon}{4N^2}$ so that

$$k \leq \frac{T_0 N^2}{c_{T_0,\xi,N}} = C_{T_0,\xi,N}, \quad k' \leq \frac{T_0 N^2}{c_{T_0,\xi,N}\varepsilon} = C_{T_0,\xi,N,\varepsilon}.$$

With this, one can further split the $s-$integral as

$$\begin{aligned} C_N \int_{t_{k+1}}^{t_k} \int_{|v'|\leq 2N, |v''|\leq 3N} \sum_{k\leq C_{T_0,N}, k'\leq C_{T_0,N,\varepsilon}} \int_{t'_{k'+1}}^{t'_{k'}} \mathbf{1}_{A_\alpha} e^{-\frac{\lambda_2}{2}(t^{\rho_1} - s_1^{\rho_1})} \\ \times \left| h\left(s_1, x'_{k'} + (s_1 - t'_{k'})v'_{k'}, v''\right) \right| \\ = \int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon} + \int_{t_k-\varepsilon}^{t_k} + \int_{t_{k+1}}^{t_{k+1}+\varepsilon}. \end{aligned}$$

Noticing that $\sum_{k'} \int_{t'_{k'+1}}^{t'_{k'}} = \int_0^s$, the last two terms make a small contribution as

$$\begin{aligned} \varepsilon C_N \sup_{0\leq s\leq t} e^{-\frac{\lambda_2}{2}(t^{\rho_1} - s^{\rho_1})} ||h(s)||_\infty \int_0^{T_0} \int_{|v'|\leq 2N, |v''|\leq 3N} \\ = \varepsilon C_{N,T_0} \sup_{0\leq s\leq t} e^{-\frac{\lambda_2}{2}(t^{\rho_1} - s^{\rho_1})} \|h(s)\|_\infty. \end{aligned}$$

For the main contribution $\int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon}$, by Lemma 4.6, on the set $O^c_{t_i,x_i,v_i} \cap [t_{k+1} + \varepsilon, t_k - \varepsilon] \times [0, T_0] \times \{|v'| \leq N\}$, we can define a change of variable

$$y \equiv x'_{k'} + (s_1 - t'_{k'})v'_{k'},$$

so that $\det(\frac{\partial y}{\partial v'}) > \delta$ on the same set. By the Implicit Function Theorem, there is a finite open covering $\cup_{j=1}^m V_j$ of $O^c_{t_i,x_i,v_i} \cap [t_{k+1} + \varepsilon, t_k - \varepsilon] \times [0, T_0] \times \{|v'| \leq N\}$, and a smooth function F_j such that $v' = F_j(t, x, v, y, s_1, s)$ in V_j. We therefore have

$$\begin{aligned} \sum_{k,k'} \int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon} \int_{|v'|\leq 2N, |v''|\leq 3N} \int_{t'_{k'+1}}^{t'_{k'}} \leq \sum_{k,k'} \int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon} \int_{|v'|\leq 2N, |v''|\leq 3N} \int_{t'_{k'+1}}^{t'_{k'}} \mathbf{1}_{O_{t_i,x_i,v_i}} \\ + \sum_{j,k,k'} \int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon} \int_{|v'|\leq 2N, |v''|\leq 3N} \int_{t'_{k'+1}}^{t'_{k'}} \mathbf{1}_{V_j}. \end{aligned}$$

Since $\sum_{k'} \int_{t'_{k'+1}}^{t'_{k'}} = \int_0^s \leq \int_0^{T_0}$ and $|O_{t_i,x_i,v_i}| < \varepsilon$, the first part is bounded by

$$C_{N,T_0} \varepsilon e^{-\frac{\lambda_2}{2}t^{\rho_1}} \sup_{0\leq s\leq t} \left\{ e^{\frac{\lambda_2}{2}s^{\rho_1}} \|h(s)\|_\infty \right\}.$$

For the second part, we can make a change of variable $v' \to y = x'_{k'} + (s_1 - t'_{k'})v'_{k'}$ on each V_j to get

$$
\begin{aligned}
& C_{\varepsilon,T_0,N} \sum_{j,k,k'} \int_{V_j} \int_{|v''|\le 3N} e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s_1^{\rho_1})} |h\left(s_1, x'_{k'} + (s_1 - t'_{k'})v'_{k'}, v''\right)| \\
&= C_{\varepsilon,T_0,N} \sum_{j} \int_{V_j} \int_{|v''|\le 3N} e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s_1^{\rho_1})} |h\left(s_1, y, v''\right)| \frac{1}{\left|\det\left\{\frac{\partial y}{\partial v'}\right\}\right|} dy \mathrm{d}v'' \mathrm{d}s \mathrm{d}s_1 \\
&\le \frac{C_{\varepsilon,T_0,N}}{\delta} \int_0^t \int_0^s e^{-\frac{\lambda_2}{2}t^{\rho_1}} \int_{|v''|\le 3N} e^{\frac{\lambda_2}{2}s_1^{\rho_1}} \left\{\int_\Omega h^2\left(s_1, y, v''\right) dy\right\}^{1/2} \mathrm{d}v'' \mathrm{d}s \mathrm{d}s_1 \\
&\le C_{\varepsilon,T_0,N} \int_0^t \|f(s)\|_2 \mathrm{d}s,
\end{aligned}
$$

where $f = \frac{h}{w_{q,\theta,\vartheta}}$. We therefore conclude, summing over k and k', and collecting terms

$$
\begin{aligned}
\left\|h(t,x,v)\mathbf{1}_{A_\alpha}\right\|_\infty &\le C e^{-\frac{\lambda_2}{2}t^{\rho_1}} \|h_0\|_\infty + C_{\varepsilon,T_0,N} \int_0^t \|f(s)\|_2 \mathrm{d}s \\
&\quad + \left\{\frac{C}{N} + C_{N,T_0}\varepsilon\right\} e^{-\frac{\lambda_2}{2}t^{\rho_1}} \sup_{0\le s\le t} e^{\frac{\lambda_2}{2}s^{\rho_1}} \|h(s)\|_\infty. \qquad (4.41)
\end{aligned}
$$

Step 2: Estimate of h. We first get, from $h(t,x,v) = G(t)h_0 + \int_0^t G(t,s)K_{\overline{w}}h(s)\mathrm{d}s$, that

$$
\begin{aligned}
\|h(t)\|_\infty &\le e^{-\lambda_2 t^{\rho_2}} \|h_0\|_\infty + \int_0^t e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} \widetilde{\nu}(v) e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s^{\rho_1})} \\
&\qquad \times \left\|\nu^{-1}(v) K_{\overline{w}}^{1-\chi} h\right\|_\infty (s)\mathrm{d}s \\
&\qquad + \int_0^t e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} \widetilde{\nu}(v) e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s^{\rho_1})} \left\|\nu^{-1}(v) K_{\overline{w}}^{\chi} h\right\|_\infty (s)\mathrm{d}s \\
&\le e^{-\lambda_2 t^{\rho_2}} \|h_0\|_\infty + C\epsilon^{3+\varrho} e^{-\frac{\lambda_2}{2}t^{\rho_2}} \sup_{0\le s\le t} e^{\frac{\lambda_2}{2}s^{\rho_1}} \|h(s)\|_\infty \\
&\qquad + \int_0^t e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} \widetilde{\nu}(v) e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s^{\rho_1})} \left\|\nu^{-1}(v) K_{\overline{w}}^{\chi} h\right\|_\infty (s)\mathrm{d}s. \qquad (4.42)
\end{aligned}
$$

Next, since $\{K_{\overline{w}}^{\chi} h\}(s,x,v) = \int \mathbf{k}_{\overline{w}}^{\chi}(v,v')h(s,x,v')\mathrm{d}v'$, we then rewrite

$$
\begin{aligned}
& \int_0^t e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} \widetilde{\nu}(v) e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s^{\rho_1})} \left\|\nu^{-1}(v) K_{\overline{w}}^{\chi} h\right\|_\infty (s)\mathrm{d}s \\
&\quad = \int_0^t e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} \widetilde{\nu}(v) e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s^{\rho_1})} \\
&\qquad \times \left\|\nu^{-1} \int \mathbf{k}_{\overline{w}}^{\chi}(v,v')h(s,x,v')\{1 - \mathbf{1}_{A_\alpha(x,v')}\}\mathrm{d}v'\right\|_\infty \mathrm{d}s \\
&\qquad + \int_0^t e^{-\frac{1}{2}\int_s^t \widetilde{\nu}(v)\mathrm{d}\tau} \widetilde{\nu}(v) e^{-\frac{\lambda_2}{2}(t^{\rho_1}-s^{\rho_1})}
\end{aligned}
$$

$$\left\| \nu^{-1} \int \mathbf{k}_{\frac{\chi}{w}}(v, v') h(s, x, v') \mathbf{1}_{A_\alpha(x,v')} \mathrm{d}v' \right\|_\infty \mathrm{d}s$$
$$\stackrel{\text{def}}{=} H_6 + H_7.$$

From the definition of A_α in (4.39), it follows that

$$H_6 \le C \left(\int_{|v'|\ge N,\text{ or } |v'|\le \frac{1}{N}} |\nu^{-1} \mathbf{k}_{\frac{\chi}{w}}(v, v')| \mathrm{d}v' + \int_{\alpha(x,v')\le \frac{1}{N}} |\nu^{-1} \mathbf{k}_{\frac{\chi}{w}}(v, v')| \right)$$
$$\times e^{-\frac{\lambda_2}{2} t^{\rho_1}} \sup_{0\le s\le t} e^{\frac{\lambda_2}{2} s^{\rho_1}} \|h(s)\|_\infty .$$

By approximation, if necessary, one sees $\int_{|v'|\ge N,\text{ or } |v'|\le \frac{1}{N}} |\nu^{-1} \mathbf{k}_{\frac{\chi}{w}}(v, v')| \mathrm{d}v' = o(1)$ as $N \to \infty$. From $\alpha(x, v') \le \frac{1}{N}$, $\xi^2(x) + [v' \cdot \nabla\xi(x)]^2 \le \frac{1}{N}$. For N large, $x \backsim \partial\Omega$ and $|\nabla\xi(x)| \ge c$ so that

$$\int_{\alpha(x,v')\le \frac{1}{N}} |\nu^{-1} \mathbf{k}_{\frac{\chi}{w}}(v, v')| \mathrm{d}v' \le \int_{|v' \cdot \frac{\nabla\xi(x)}{|\nabla\xi(x)|}| \le \frac{1}{c\sqrt{N}}} |\nu^{-1} \mathbf{k}_{\frac{\chi}{w}}(v, v')| \mathrm{d}v' = o(1),$$

as $N \to \infty$. As a consequence, it follows that

$$H_6 \le o(1) e^{-\frac{\lambda_2}{2} t^{\rho_1}} \sup_{0\le s\le t} e^{\frac{\lambda_2}{2} s^{\rho_1}} \|h(s)\|_\infty .$$

As to H_7, in view of (4.41), one has

$$H_7 \le C e^{-\frac{\lambda_2}{2} t^{\rho_1}} \|h_0\|_\infty + \left\{ \frac{C}{N} + C_{N,T_0}\varepsilon \right\} e^{-\frac{\lambda_2}{2} t^{\rho_1}} \sup_{0\le s\le t} e^{\frac{\lambda_2}{2} s^{\rho_1}} \|h(s)\|_\infty$$
$$+ C_{\varepsilon,T_0,N} \int_0^t \|f(s)\|_2 \mathrm{d}s.$$

Hence, substituting the estimates for H_6 and H_7 into (4.42), we arrive at

$$\|h(t)\|_\infty \le C e^{-\frac{\lambda_2}{2} t^{\rho_1}} \|h_0\|_\infty$$
$$+ \left\{ \frac{C}{N} + C_{N,T_0}\varepsilon + o(1) \right\} e^{-\frac{\lambda_2}{2} t^{\rho_1}} \sup_{0\le s\le t} e^{\frac{\lambda_2}{2} s^{\rho_1}} \|h(s)\|_\infty$$
$$+ C_{\varepsilon,T_0,N} \int_0^t \|f(s)\|_2 \mathrm{d}s.$$

We choose T_0 large such that $2Ce^{-\frac{\lambda_2}{2} T_0^{\rho_1}} = e^{-\lambda_0 T_0^{\rho_1}}$, for some $\lambda_0 > 0$. We then further choose N large, and then ε sufficiently small such that $C\{o(1) + \frac{1}{N} + C_{N,T_0}\varepsilon\} < \frac{1}{2}$. Therefore, one has

$$\sup_{0\le s\le t} \left\{ e^{\frac{\lambda_2}{2} s^{\rho_1}} \|h(s)\|_\infty \right\} \le 2C \|h_0\|_\infty + C_{T_0} \int_0^t \|f(s)\|_2 \mathrm{d}s.$$

Choosing $s = t = T_0$, we deduce the finite-time estimate (4.40), and the proof of Lemma 4.7 is completed. □

We are ready to present

The proof of Proposition 4.2. It suffices to only prove (4.28) for $t \geq 1$. For any $m \geq 1$, we employ the finite-time estimate (4.40) repeatedly to functions $h(lT_0+s)$ for $l = m-1, m-2, \dots 0$ to deduce

$$\begin{aligned}
\|h(mT_0)\|_\infty &\leq e^{-\lambda_0 T_0^{\rho_1}} \|h(\{m-1\}T_0)\|_\infty + C_{T_0} \int_0^{T_0} \|f(\{m-1\}T_0+s)\|_2 \mathrm{d}s \\
&= e^{-\lambda_0 T_0^{\rho_1}} \|h(\{m-1\}T_0)\|_\infty + C_{T_0} \int_{\{m-1\}T_0}^{mT_0} \|f(s)\|_2 \mathrm{d}s \\
&\leq e^{-2\lambda_0 T_0^{\rho_1}} \|h(\{m-2\}T_0)\|_\infty + e^{-\lambda_0 T_0^{\rho_1}} C_{T_0} \int_{\{m-2\}T_0}^{\{m-1\}T_0} \|f(s)\|_2 \mathrm{d}s \\
&\quad + C_{T_0} \int_{\{m-1\}T_0}^{mT_0} \|f(s)\|_2 \, \mathrm{d}s \\
&\leq e^{-m\lambda_0 T_0^{\rho_1}} \|h(0)\|_\infty + C_{T_0} \sum_{k=0}^{m-1} e^{-k\lambda_0 T_0^{\rho_1}} \int_{\{m-k-1\}T_0}^{\{m-k\}T_0} \|f(s)\|_2 \mathrm{d}s,
\end{aligned}$$

where $h(t) = U(t)h_0$.

Next, by the L^2 decay constructed in Lemma 4.3, in the interval $\{m-k-1\}T_0 \leq s \leq \{m-k\}T_0$, one has

$$\|f(s)\|_2 \leq e^{-\lambda s^{\rho_0}} \|w_{q/4,\theta} f_0\|_2 \leq e^{-\lambda(\{m-k-1\}T_0)^{\rho_0}} \|w_{q/4,\theta} f_0\|_2.$$

Noticing that $\rho_0 = \frac{\theta}{\theta-\varrho} > \frac{\theta+\vartheta\varrho}{\theta-\varrho} = \rho_1$, taking $\lambda_0 = \min\{\lambda, \lambda_0\}$ and applying $(k+1)T_0^{\rho_1} \geq ((k+1)T_0)^{\rho_1}$ for $0 < \rho_1 < 1$, we further obtain

$$\begin{aligned}
\|h(mT_0)\|_\infty &\leq e^{-m\lambda_0 T_0^{\rho_1}} \|h(0)\|_\infty + C_{T_0} \sum_{k=0}^{m-1} e^{-k\lambda_0 T_0^{\rho_1}} \int_{\{m-k-1\}T_0}^{\{m-k\}T_0} \\
&\qquad \times e^{-\lambda(\{m-k-1\}T_0)^{\rho_0}} \|w_{q/4,\theta} f_0\|_2 \mathrm{d}s \\
&\leq e^{-m\lambda_0 T_0^{\rho_1}} \|h(0)\|_\infty + C_{T_0} e^{\lambda_0 T_0^{\rho_1}} mT_0 e^{-\lambda_0 m^{\rho_1} T_0^{\rho_1}} \|w_{q/4,\theta} f_0\|_2 \\
&\leq C_{T_0,\lambda_0} e^{-\frac{\lambda_0 m^{\rho_1} T_0^{\rho_1}}{2}} \|h(0)\|_\infty,
\end{aligned}$$

where we also used the fact that

$$\|w_{q/4,\theta} f_0\|_2 = \left\| w_{q/4,\theta} w_{q,\theta,\vartheta}^{-1} h_0 \right\|_2 \leq C \|h_0\|_\infty,$$

and

$$(\{m-k-1\}T_0)^{\rho_1} + (\{k+1\}T_0)^{\rho_1} \geq (mT_0)^{\rho_1}, \quad mT_0 e^{-\lambda_0 m^{\rho_1} T_0^{\rho_1}} \leq e^{-\frac{\lambda_0 m^{\rho_1} T_0^{\rho_1}}{2}}.$$

Finally, for any t, we can find m such that $mT_0 \leq t \leq \{m+1\}T_0$, and

$$\begin{aligned}
\|h(t)\|_\infty &\leq C \|h(mT_0)\|_\infty \leq C_{T_0,\lambda_0} e^{-\frac{\lambda_0 m^{\rho_1} T_0^{\rho_1}}{2}} \|h(0)\|_\infty \\
&\leq \left\{ C_{T_0,\lambda_0} e^{\lambda_0 T_0^{\rho_1}} \right\} e^{-\frac{\lambda_0}{2} t^{\rho_1}} \|h(0)\|_\infty,
\end{aligned}$$

according to the fact $e^{-\frac{\lambda_0 m^{\rho_1} T_0^{\rho_1}}{2}} \le e^{-\frac{\lambda_0}{2} t^{\rho_1}} e^{\frac{\lambda_0 T_0^{\rho_1}}{2}}$. This ends the proof of Proposition 4.2. □

4.3. Nonlinear Existence and Time Exponential Decay

In this subsection, we make use of Proposition 4.2 to prove the global existence and time exponential decay of the nonlinear Boltzmann equation with a specular reflection boundary condition. Namely, we complete

The proof of Theorem 1.2. We start with the following iteration scheme:

$$\begin{cases} \partial_t h^{\ell+1} + v \cdot \nabla_x h^{\ell+1} + \widetilde{\nu} h^{\ell+1} - K_{\overline{w}} h^{\ell+1} = w_{q,\theta,\vartheta} \Gamma\left(\frac{h^\ell}{w_{q,\theta,\vartheta}}, \frac{h^\ell}{w_{q,\theta,\vartheta}}\right), \\ h^{\ell+1}(0,x,v) = h_0(x,v), \end{cases} \tag{4.43}$$

with $h_-^{\ell+1}(t,x,v) = h^{\ell+1}(t,x,R_x v)$ and $h^0 = h_0(x,v)$. Here $h^\ell = f^\ell w_{q,\theta,\vartheta}$. From the Duhamel principle, it follows that

$$h^{\ell+1} = U(t)h_0 + \int_0^t U(t-s) w_{q,\theta,\vartheta} \Gamma\left(\frac{h^\ell}{w_{q,\theta,\vartheta}}, \frac{h^\ell}{w_{q,\theta,\vartheta}}\right)(s) \mathrm{d}s.$$

We then get from, Proposition 4.2 and Lemma 2.3 that

$$\begin{aligned} \left\| h^{\ell+1}(t) \right\|_\infty &\le C e^{-\frac{\lambda_0}{2} t^{\rho_1}} \|h_0\|_\infty \\ &\quad + \left\| \int_0^t U(t-s) w_{q,\theta,\vartheta} \Gamma\left(\frac{h^\ell}{w_{q,\theta,\vartheta}}, \frac{h^\ell}{w_{q,\theta,\vartheta}}\right)(s) \mathrm{d}s \right\|_\infty \\ &\le C e^{-\frac{\lambda_0 t^{\rho_1}}{2}} \|h_0\|_\infty + \int_0^t e^{-\frac{\lambda_0}{2}(t-s)^{\rho_1} - \lambda_0 s^{\rho_1}} \mathrm{d}s \sup_{0 \le s \le t} \left\| e^{\frac{\lambda_0}{2} s^{\rho_1}} h^\ell(s) \right\|_\infty^2 \\ &\le C e^{-\frac{\lambda_0 t^{\rho_1}}{2}} \|h_0\|_\infty + e^{-\frac{\lambda_0}{2} t^{\rho_1}} \sup_{0 \le s \le t} \left\| e^{\frac{\lambda_0}{2} s^{\rho_1}} h^\ell(s) \right\|_\infty^2, \end{aligned} \tag{4.44}$$

where the fact that $\nu(v) < C$ was used. This implies that

$$\sup_\ell \sup_{0 \le t \le \infty} \left\{ e^{\frac{\lambda_0}{2} t^{\rho_1}} \left\| h^\ell(t) \right\|_\infty \right\} \le C \|h_0\|_\infty$$

for $\|h_0\|_\infty$ sufficiently small. Moreover, subtracting $h^{\ell+1} - h^\ell$ yields

$$\begin{aligned} &\{\partial_t + v \cdot \nabla_x + \widetilde{\nu} - K_{\overline{w}}\}\{h^{\ell+1} - h^\ell\} \\ &\quad = w_{q,\theta,\vartheta} \left\{ \Gamma\left(\frac{h^\ell}{w_{q,\theta,\vartheta}}, \frac{h^\ell}{w_{q,\theta,\vartheta}}\right) - \Gamma\left(\frac{h^{\ell-1}}{w_{q,\theta,\vartheta}}, \frac{h^{\ell-1}}{w_{q,\theta,\vartheta}}\right) \right\}, \end{aligned}$$

with $\{h^{\ell+1}-h^{\ell}\}(0,x,v)=0$ and $\{h^{\ell+1}-h^{\ell}\}(t,x,v)|_{-}=\{h^{\ell+1}-h^{\ell}\}(t,x,R_x v)$ by the decomposition

$$\begin{aligned}\Gamma\left(\frac{h^{\ell}}{w_{q,\theta,\vartheta}},\frac{h^{\ell}}{w_{q,\theta,\vartheta}}\right)&-\Gamma\left(\frac{h^{\ell-1}}{w_{q,\theta,\vartheta}},\frac{h^{\ell-1}}{w_{q,\theta,\vartheta}}\right)\\&=\Gamma\left(\frac{h^{\ell}-h^{\ell-1}}{w_{q,\theta,\vartheta}},\frac{h^{\ell}}{w_{q,\theta,\vartheta}}\right)-\Gamma\left(\frac{h^{\ell-1}}{w_{q,\theta,\vartheta}},\frac{h^{\ell-1}-h^{\ell}}{w_{q,\theta,\vartheta}}\right).\end{aligned}$$

Performing a calculation similar to (4.44), we then obtain

$$\begin{aligned}\left\|\{h^{\ell+1}-h^{\ell}\}(t)\right\|_{\infty}&\leq\left\|\int_0^t U(t-s)w_{q,\theta,\vartheta}\Gamma\left(\frac{h^{\ell}-h^{\ell-1}}{w_{q,\theta,\vartheta}},\frac{h^{\ell}}{w_{q,\theta,\vartheta}}\right)(s)\mathrm{d}s\right\|_{\infty}\\&\quad+\left\|\int_0^t U(t-s)w_{q,\theta,\vartheta}\Gamma\left(\frac{h^{\ell-1}}{w_{q,\theta,\vartheta}},\frac{h^{\ell-1}-h^{\ell}}{w_{q,\theta,\vartheta}}\right)(s)\mathrm{d}s\right\|_{\infty}\\&\leq Ce^{-\frac{\lambda_0}{2}t^{\rho_1}}\sup_{0\leq s\leq t}\left\{\left\|e^{\frac{\lambda_0}{2}s^{\rho_1}}h^{\ell}(s)\right\|_{\infty}+\left\|e^{\frac{\lambda_0}{2}s^{\rho_1}}h^{\ell-1}(s)\right\|_{\infty}\right\}\\&\quad\times\sup_{0\leq s\leq t}\left\|e^{\frac{\lambda_0}{2}s^{\rho_1}}\{h^{\ell}(s)-h^{\ell-1}(s)\}\right\|_{\infty}.\end{aligned}$$

Hence h^{ℓ} is a Cauchy sequence and the limit h is a desired unique solution satisfing

$$\sup_{0\leq t\leq\infty}\left\|e^{\frac{\lambda_0}{2}t^{\rho_1}}h(t)\right\|_{\infty}\leq C||h_0||_{\infty}.$$

In addition, if Ω is strictly convex, we *claim* that $h^{\ell+1}$ is continuous in $[0,\infty)\times\{\bar{\Omega}\times\mathbb{R}^3\backslash\gamma_0\}$ inductively. To prove this claim, for any given fixed ℓ, we can use another iteration to solve the linear problem for $h^{\ell+1}$ in (4.43) as the limit of $\ell'\to\infty$:

$$\{\partial_t+v\cdot\nabla_x+\widetilde{\nu}\}h^{\ell+1,\ell'+1}=K_{\overline{w}}h^{\ell+1,\ell'}+w_{q,\theta,\vartheta}\Gamma\left(\frac{h^{\ell}}{w_{q,\theta,\vartheta}},\frac{h^{\ell}}{w_{q,\theta,\vartheta}}\right),$$

with the initial boundary condition:

$$h_{-}^{\ell+1,\ell'+1}(t,x,v)=h^{\ell+1,\ell'+1}(t,x,R_x v),\ h^{\ell+1,\ell'+1}(0)=h_0(x,v)$$

and $h^{\ell+1,0}\equiv h_0(x,v)$. By induction over ℓ', $h^{\ell+1,\ell'}$ is continuous in $[0,\infty)\times\{\bar{\Omega}\times\mathbb{R}^3\backslash\gamma_0\}$, and by Lemma 2.2, it is standard to show that $K_{\overline{w}}h^{\ell+1,\ell'}$ is continuous in the interior of $[0,\infty)\times\Omega\times\mathbb{R}^3$. From the induction hypothesis on the continuity of h^{ℓ} in $[0,\infty)\times\{\bar{\Omega}\times\mathbb{R}^3\backslash\gamma_0\}$, it is also straightforward and routine to verify that $w_{q,\theta,\vartheta}\Gamma(\frac{h^{\ell}}{w_{q,\theta,\vartheta}},\frac{h^{\ell}}{w_{q,\theta,\vartheta}})$ is continuous in the interior of $[0,\infty)\times\Omega\times\mathbb{R}^3$. In view of Lemma 4.5, we thus deduce that $h^{\ell+1,\ell'+1}$ is continuous in $[0,\infty)\times\{\bar{\Omega}\times\mathbb{R}^3\backslash\gamma_0\}$. Furthermore, it follows that

$$\{\partial_t+v\cdot\nabla_x+\widetilde{\nu}\}\{h^{\ell+1,\ell'+1}-h^{\ell+1,\ell'}\}=K_{\overline{w}}\{h^{\ell+1,\ell'}-h^{\ell+1,\ell'-1}\}$$

with $\{h^{\ell+1,\ell'+1} - h^{\ell+1,\ell'}\}(t,x,v)|_- = \{h^{\ell+1,\ell'+1} - h^{\ell+1,\ell'}\}(t,x,R_x v)$ and $\{h^{\ell+1,\ell'+1} - h^{\ell+1,\ell'}\}(0) = 0$. With this, one deduces that

$$\sup_{0\le t\le T} \left\| h^{\ell+1,\ell'+1}(t) - h^{\ell+1,\ell'}(t) \right\|_\infty$$
$$\le C_K \int_0^T \left\| h^{\ell+1,\ell'}(s) - h^{\ell+1,\ell'-1}(s) \right\|_\infty \mathrm{d}s \le \cdots \le C\frac{\{C_K T\}^{\ell'}}{\ell'!}.$$

Therefore, $\{h^{\ell+1,\ell'}\}_{\ell'=1}^\infty$ is a Cauchy sequence in L^∞, and its limit $h^{\ell+1}$ is continuous in $[0,\infty) \times \{\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0\}$. We conclude our *claim*. Once h^ℓ is continuous, its limit h is continuous as well.

Finally, the uniqueness and positivity of F follows the same argument as the proof of Theorem 3 in [27, pp. 804]; we omit the details for brevity. This finishes the proof of Theorem 1.2. □

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