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On the Shape of Meissner Solutions to a Limiting Form of Ginzburg–Landau Systems

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Abstract

In this paper we study a semilinear system involving the curl operator, which is a limiting form of the Ginzburg–Landau model for superconductors in $\mathbb{R}^3$ for a large value of the Ginzburg–Landau parameter. We consider the locations of the maximum points of the magnitude of solutions, which are associated with the nucleation of instability of the Meissner state for superconductors when the applied magnetic field is increased in the transition between the Meissner state and the vortex state. For small penetration depth, we prove that the location is not only determined by the tangential component of the applied magnetic field, but also by the normal curvatures of the boundary in some directions. This improves the result obtained by Bates and Pan in Commun. Math. Phys. **276**, 571–610 (2007). We also show that the solutions decay exponentially in the normal direction away from the boundary if the penetration depth is small.

1. Introduction

Consider the following semilinear elliptic system:

$$\begin{cases} -\lambda^2 \operatorname{curl}^2 \mathbf{Q} = (1-|\mathbf{Q}|^2)\mathbf{Q} & \text{in } \Omega, \\ \lambda\, (\operatorname{curl} \mathbf{Q})_T = \mathcal{H}_T^e & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^3$, $\mathcal{H}_T^e$ is the tangential component of a given vector field $\mathcal{H}^e$ on $\partial\Omega$, and λ is a positive parameter. Our aim in this paper is to investigate the locations of the maximum points of $|\mathbf{Q}|$ for small λ.

System (1.1) was first obtained by Chapman in [6] as a limiting form of the Ginzburg–Landau model for type II superconductors in three dimensions for a large value of the Ginzburg–Landau parameter. The parameter λ is the penetration depth of the magnetic field, generally $\lambda \ll 1$. Vector fields

$$\mathbf{Q} = (Q_1, Q_2, Q_3) \text{ and } \operatorname{curl} \mathbf{Q} = (\partial_2 Q_3 - \partial_3 Q_2, \partial_3 Q_1 - \partial_1 Q_3, \partial_1 Q_2 - \partial_2 Q_1)$$

are the gauge-invariant potential and the induced magnetic field, respectively. The domain Ω represents the shape of a superconductor placed in an applied magnetic field $\mathcal{H}^e$. We refer to [9] for the physical background of the Ginzburg–Landau model and to [4,6,12,16] for the derivation of system (1.1).

The above problem arises in the mathematical theory of superconductivity in the transition between the Meissner state and the vortex state, in which H_{sh} (known as the superheating field) is the critical field. If the applied field is below H_{sh}, there are no vortices and the magnetic field will be expelled from the interior of the sample except in thin boundary layers. In this state $|\Psi| > 0$ in $\bar{\Omega}$, where Ψ is the order parameter to describe the density of superconducting electron pairs. The solution corresponding to the Meissner state is called the Meissner solution. If the applied field is above H_{sh}, the magnetic field will penetrate the sample in the form of quantised flux tubes, each circled by a vortex of superconducting current. In this state $|\Psi|$ vanishes at some points which correspond to vortices. See [5–7,11].

There are other two critical fields H_{C_1} and H_S (where $H_{C_1} < H_S < H_{sh}$) in the Meissner state. The value of H_{C_1} is known as the lower (first) critical field to differentiate between the superconducting state and the mixed state. If the applied field is below H_{C_1}, then the Meissner solution is globally stable with respect to the Ginzburg–Landau free energy [18]. If the applied field is above H_{C_1} but below H_S, the Meissner solution is locally stable (see [11]). For the finite Ginzburg–Landau parameter the local stability is characterized in [5] by

$$\inf_{\bar{\Omega}} \left\{ |\Psi|^2 - |\mathbf{Q}|^2 \right\} > \frac{1}{3},$$

where Ψ is the order parameter and $\mathbf{Q}$ is the gauge-invariant potential as above. While for the infinite Ginzburg–Landau parameter, it is characterized by

$$|\mathbf{Q}| < \frac{1}{\sqrt{3}},$$

see [4,7,17]. If the applied field is above H_S but below H_{sh}, the Meissner solution is unstable. It was conjectured by Chapman in [6,7] that this instability will lead to the generation of vortices, and also the point at which the solution first becomes unstable corresponds to the position of the first nucleation of superconducting vortices. It is therefore important to know at which the Meissner state will begin to lose stability. For system (1.1), we need to find the position at which $|\mathbf{Q}| = 1/\sqrt{3}$ is first attained when the applied field increases.

We mention here the pioneering works in this direction by mathematicians. Berestycki et al. in [4] showed that, for a bounded domain in $\mathbb{R}^2$ (corresponds to a cylinder of superconducting material), the instability will occur first on the boundary, while for the setting of $\mathbb{R}^3$ it was shown by Monneau in [12]. Using formal analysis, Chapman in [7] further showed that, as $\lambda \to 0$, the solution $\mathbf{Q}$ first becomes unstable at the point of largest negative curvature of the boundary in two-dimensional domain. This was rigorously proved later by Pan and Kwek in [17] using the exponential decay estimate and the boundary layer analysis.

To locate the points at which $|\mathbf{Q}| = 1/\sqrt{3}$ is attained, one way is first to find the maximum points of $|\mathbf{Q}|$ when the maximal value is below and close to $1/\sqrt{3}$. Then the maximum points will tend to the position as we expected by letting $\|\mathbf{Q}\|_{L^\infty}$ go to $1/\sqrt{3}$. For bounded domains in $\mathbb{R}^3$, under the assumption that

$$\|\mathbf{Q}\|_{L^\infty(\Omega)} < \frac{1}{\sqrt{3}} \tag{1.2}$$

(which holds if $\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)} < \sqrt{5/18}$ and λ is small, see [3, Theorem 1(iii)]), Bates and Pan in [3] proved that the points where the maximum of $|\mathbf{Q}|$ attained must approach the points in $\partial\Omega(\mathcal{H}_T^e)$, as $\lambda \to 0$, where

$$\partial\Omega(\mathcal{H}_T^e) = \left\{x \in \partial\Omega \ : \ |\mathcal{H}_T^e| = \|\mathcal{H}_T^e\|_{C^0(\partial\Omega)}\right\}. \tag{1.3}$$

Therefore, in the special case of a homogeneous applied field, Bates and Pan's result showed that the gauge-invariant potential is maximal around the points on the boundary where the applied field is tangential to the surface.

Note that, for a general applied magnetic field, the set $\partial\Omega(\mathcal{H}_T^e)$, depending on the strength of the magnetic field, may still be large. We expect to obtain a more precise description for the locations of the maximum points of $|\mathbf{Q}|$ for small λ. One question is whether the geometry of superconductors influences the location as in the case of $\mathbb{R}^2$? This is the motivation of the present paper.

In this paper we will show that the location is also influenced by the normal curvatures of the boundary in some directions. Before giving the main result, we sketch our proof: let $\mathbf{H}_\lambda = \lambda \operatorname{curl} \mathbf{Q}$. Then we can reduce the system (1.1) to a quasilinear system (see [6]):

$$\begin{aligned} -\lambda^2 \operatorname{curl}\left[F\left(\lambda^2|\operatorname{curl}\mathbf{H}|^2\right)\operatorname{curl}\mathbf{H}\right] &= \mathbf{H} \qquad \text{in } \Omega,\\ \mathbf{H}_T &= \mathcal{H}_T^e \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where the function $F(t)$ is defined by the following implicit relation

$$v = F(t^2)t \Leftrightarrow t = (1 - v^2)v. \tag{1.5}$$

From [3], there is a boundary layer for $\mathbf{H}_\lambda$ if λ is small. Therefore, we consider this problem in a neighborhood of the boundary. The key step in our proof is to establish an approximation theorem for the solution $\mathbf{H}_\lambda$ to system (1.4) (a similar result see [13]): for any $X_0 \in \partial\Omega$, in the neighborhood of X_0 with the diameter λ, denote by $\mathcal{U}_{0,\lambda}$, we have:

$$\|\mathbf{H}_\lambda(x) - \mathscr{H}(\mathcal{F}(x)/\lambda) - \lambda\mathscr{W}(\mathcal{F}(x)/\lambda)\|_{C^1_\lambda\left(\overline{\mathcal{U}_{0,\lambda}\cap\Omega}\right)} \leqq O\left(\lambda^{3/2}\right), \tag{1.6}$$

where $y = \mathcal{F}(x)$ is a diffeomorphism straightening a boundary portion around X_0; the norm of C^1_λ is defined by

$$\|\mathbf{u}\|_{C^1_\lambda\left(\overline{\mathcal{U}_{0,\lambda}\cap\Omega}\right)} := \|\mathbf{u}\|_{C^0\left(\overline{\mathcal{U}_{0,\lambda}\cap\Omega}\right)} + \lambda\|D\mathbf{u}\|_{C^0\left(\overline{\mathcal{U}_{0,\lambda}\cap\Omega}\right)}; \tag{1.7}$$

the vector $\mathscr{H}(\cdot)$, obtained earlier by Bates and Pan in [3], depends on the strength of the magnetic field; the vector $\mathscr{W}(\cdot)$, which we will introduce in this paper, depends on the strength of the magnetic field but also on the geometric shape of superconductors. Thus, from (1.6) we can obtain the set that the maximum of $|\operatorname{curl}\mathbf{H}_\lambda|$

attained for small λ. Note that the maximum points of $|\operatorname{curl}\mathbf{H}_\lambda|$ correspond to the maximum points of $|\mathbf{Q}|$ since

$$\lambda|\operatorname{curl}\mathbf{H}_\lambda| = (1-|\mathbf{Q}|^2)|\mathbf{Q}|. \tag{1.8}$$

Therefore, the locations of the maximum points of $|\mathbf{Q}|$ can be determined.

We need to mention that the uniform convergence of (1.6) involves a C^1 estimate of the solution to a curl-type elliptic system. However, for such an elliptic system there is no comparison principle and no maximum principle. Our strategy is as follows: we first establish the global H^1 estimate by the method of matched asymptotic expansion, then deduce an H^2 estimate near boundary by the difference quotient technique. Hence the solution is C^α regularity by the Sobolev imbedding theorem. Finally the convergence (1.6) follows by solving a local oblique derivative problem. The method of our proof is different from Pan and Kwek in [17] where it was treated this problem in a bounded domain of $\mathbb{R}^2$ by applying the maximum principle for a single divergence-type elliptic equation.

Let

$$f_1 = \|\mathcal{H}_T^e\|_{C^0(\partial\Omega)}, \quad f_0 = \left[1-(1-2f_1^2)^{1/2}\right]^{1/2}.$$

We state our first result as follows.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Assume that*

$$\mathcal{H}_T^e \in C^{2,\alpha}(\partial\Omega,\mathbb{R}^3), \quad \|\mathcal{H}_T^e\|_{C^0(\partial\Omega)} < \sqrt{\frac{5}{18}}, \quad \operatorname{Div}\mathcal{H}_T^e = 0 \quad \text{on } \partial\Omega, \tag{1.9}$$

where Div *is the divergence operator on boundary $\partial\Omega$. Let y_1- and y_2-curves on $\partial\Omega$ be the lines of principle curvature with $\kappa_1(x), \kappa_2(x)$ corresponding to the respective principal curvatures at $x \in \partial\Omega$, and let $\theta(x)$ be the angle between the vector $\mathcal{H}_T^e$ and the y_1-curve at $x \in \partial\Omega$. Then the maximum points of the magnitude of the solution $\mathbf{Q}$ to system (1.1), as $\lambda \to 0$, approach the points in the set S defined by*

$$S = \left\{x \in \partial\Omega(\mathcal{H}_T^e) \ : \ \min_{\tilde{x}\in\partial\Omega(\mathcal{H}_T^e)} m(\tilde{x}) = m(x)\right\}, \tag{1.10}$$

where $\partial\Omega(\mathcal{H}_T^e)$ is defined by (1.3),

$$m(x) = \left(cos^2\theta(x)\kappa_1(x) + sin^2\theta(x)\kappa_2(x)\right)C_1$$
$$+\left(cos^2\theta(x)\kappa_2(x) + sin^2\theta(x)\kappa_1(x)\right)C_2,$$

and the constants C_1 and C_2 are defined by

$$C_1 = -f_0f_1 - 2 + \frac{2f_1}{f_0} < 0, \quad C_2 = 2f_0f_1 - 2 + \frac{2f_1}{f_0} > 0.$$

Remarks. We would like to point out:

(i) The assumption on $\mathcal{H}_T^e$ in (1.9) is to guarantee, for small λ, the existence of the solution $\mathbf{Q}$ to (1.1) satisfying (1.2). Thus the semilinear system (1.1) can be reduced to the quasilinear system (1.4) by letting $\mathbf{H} = \lambda \operatorname{curl} \mathbf{Q}$, see [3,15].

(ii) Theorem 1.1 can be viewed as an improvement of the result obtained by Bates and Pan in [3], since we give a more exact description for the locations of the maximum points of $|\mathbf{Q}|$. In particular, in the special case of a homogeneous applied field $\mathcal{H}^e = (0, 0, h)$ and of the superconductor being the shape of an ellipsoid

$$\Omega : \quad \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{a^2} < 1 \quad \text{with } a > b > 0.$$

Bates and Pan's result showed that $|\mathbf{Q}|$ is maximal around the curve

$$\left\{ x \in \partial\Omega \ : \ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad x_3 = 0 \right\},$$

while Theorem 1.1 gives that the maximum points approach the two points

$$P_+ = (0, b, 0), \quad P_- = (0, -b, 0).$$

We refer the readers to Example 4.1 for details.

(iii) For Example 4.1 mentioned above, Theorem 1.1 shows that for any solution $\mathbf{Q}_h$ to system (1.1) satisfying (1.2) the maximum points are always near P_+ and P_-. For this reason we expect that as the applied magnetic field mentioned above increases (by letting h increase), the maximum points of $|\mathbf{Q}_h|$ with the maximum value approaching $1/\sqrt{3}$ will approach P_+ and P_-, where the Meissner state first becomes unstable and the vortices may first appear (if Chapman's conjecture holds).

From the approximation formula (1.6), we can see the profile of the solution $\mathbf{H}_\lambda$ to system (1.4), and hence the solution $\mathbf{Q}$ to system (1.1), near the boundary. For the portion away from the boundary, Bates and Pan in [3, Lemma 8.1] showed that the solution $\mathbf{H}_\lambda$ can be made arbitrarily small if the penetration depth λ is sufficiently small. It is natural to ask whether we can show that the solution $\mathbf{Q}$ decays exponentially in the normal direction away from the boundary as is the case for the domain in $\mathbb{R}^2$? The following result verifies this conclusion.

Theorem 1.2. (Decay estimate). *Let* $\mathbf{Q}$ *be the solution of system* (1.1) *satisfying* (1.2). *Then there exists a positive constant* λ_0 *such that for any* $\lambda \in (0, \lambda_0)$ *and* $0 < \beta < 1$ *we have*

$$|\mathbf{Q}(x)| \leqq C e^{-\beta d(x,\partial\Omega)/\lambda}, \tag{1.11}$$

where the constant C *depends on* β, *the domain* Ω *and* $\mathcal{H}_T^e$.

The organization of this paper is as follows. The first order term $\mathscr{W}$ in (1.6) is formally derived in Section 2. Then we rigorously establish the asymptotic expansion (1.6) for the solution $\mathbf{H}_\lambda$ to system (1.4) in Section 3. In Section 4, by applying (1.6) we give the proof of Theorem 1.1 and obtain the asymptotic behavior of the maximum points of $|\mathbf{Q}|$. Finally, we prove the exponential decay estimate (Theorem 1.2) in Section 5.

Throughout the paper, the bold typeface is used to indicate vector quantities; normal typeface will be used for vector components and for scalars. The positive constants C are independent of λ and their numerical value may be different in each occasion.

2. Formal Asymptotic Solution to System (1.4)

As stated in the introduction, the function $|\operatorname{curl} \mathbf{H}_\lambda|$ obtains its maximum only on $\partial\Omega$ and the boundary layer appears in the neighborhood of the surface of the body if λ is small. To show how the geometry of the body influences the locations of the maximum points, we need to carry out the detailed analysis near the boundary.

In this section we shall apply the method of matched asymptotic expansion (for the detail see [10]) to derive the formal expansion with three orders for the solution $\mathbf{H}_\lambda$ with respect to the parameter λ. To obtain the global expansion, we need take the outer asymptotic expansion outside the boundary layer and the inner asymptotic expansion inside the boundary layer. The outer expansion for system (1.4) is of the form

$$\mathbf{U}(x,\lambda) = \sum_{k=0}^{\infty} \lambda^{2k} \mathbf{h}_{2k}(x), \quad \lambda \to 0.$$

Substituting $\mathbf{U}$ into the system (1.4) and equating the coefficients of the same powers of λ, one obtains

$$\mathbf{h}_{2k}(x) = 0, \quad \text{for all } k.$$

To show the inner expansion, it is necessary to seek the expansion for the solution $\mathbf{H}_\lambda$ at the points on the boundary with respect to the parameter λ. We need to mention that the leading order term of the inner expansion has been obtained earlier in [3, Lemma 8.2]. Our purpose in this section is to derive the first order term and the second order term.

We first recall Bates and Pan's work on the derivation of the leading term in [3]. Let $X_0 \in \partial\Omega$ be fixed and consider the problem in a neighborhood of X_0, denote by $\mathcal{U}$. We take the grid of the curvature lines as the curvilinear coordinate system. Then introduce new variables y_1 and y_2 such that $\mathbf{r}(y_1, y_2)$ represents the portion of $\partial\Omega$ in $\mathcal{U}$ with $\mathbf{r}(0,0) = X_0$ and the y_1- and y_2-curves on $\partial\Omega$ are the lines of principle curvature. We use the following notations:

$$y=(y', y_3)=(y_1, y_2, y_3), \quad \mathbf{r}_1(y_1, y_2)=\partial_{y_1}\mathbf{r}(y_1, y_2), \ \mathbf{r}_2(y_1, y_2) = \partial_{y_2}\mathbf{r}(y_1, y_2),$$

and let $\mathbf{n}(y_1, y_2)$ denote the unit inner normal vector at $(y', 0) \in \partial\Omega$ defined by

$$\mathbf{n}(y_1, y_2) = \frac{\mathbf{r}_1(y_1, y_2) \times \mathbf{r}_2(y_1, y_2)}{|\mathbf{r}_1(y_1, y_2) \times \mathbf{r}_2(y_1, y_2)|}.$$

Since the domain is smooth, then for any $x \in \bar{\Omega} \bigcap \mathcal{U}$ we have a diffeomorphism map $\mathcal{F}$:

$$x = \mathcal{F}(y) = \mathbf{r}(y_1, y_2) + y_3\mathbf{n}(y_1, y_2). \tag{2.1}$$

Let

$$g_{ij}(y') = \mathbf{r}_i(y') \cdot \mathbf{r}_j(y'), \quad G_{ij}(y) = \partial_i \mathcal{F}(y) \cdot \partial_j \mathcal{F}(y). \tag{2.2}$$

Then we get

$$\begin{aligned} &G_{11}(y) = g_{11}(y')(1 - \kappa_1(y')y_3)^2, \quad G_{22}(y) = g_{22}(y')(1 - \kappa_2(y')y_3)^2, \\ &G_{33}(y) = 1, \end{aligned}$$

where $\kappa_1(y')$, $\kappa_2(y')$ are the principal curvatures of $\partial\Omega$ at the point $x = \mathcal{F}(y', 0) \in \partial\Omega$.

Introduce the new orthogonal coordinate framework $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ as follows:

$$\mathbf{E}_j(y) = \frac{\partial_j \mathcal{F}}{|\partial_j \mathcal{F}|}, \quad j = 1, 2, \qquad \mathbf{E}_3(y) = \frac{\partial_3 \mathcal{F}}{|\partial_3 \mathcal{F}|} = \mathbf{n}(y). \tag{2.3}$$

Under the above coordinate framework, the vector $\mathbf{H}(x)$ can be represented by:

$$\mathbf{H}(x) = \hat{H}_1(y)\mathbf{E}_1 + \hat{H}_2(y)\mathbf{E}_2 + \hat{H}_3(y)\mathbf{E}_3 =: \hat{\mathbf{H}}(y).$$

Then curl $\mathbf{H}(x)$ can be represented by (see [20, p. 205])

$$\operatorname{curl} \mathbf{H}(x) = \sum_{i=1}^{3} \mathcal{Q}_j(y)\mathbf{E}_j(y)$$

with

$$\begin{aligned} \mathcal{Q}_1(y) &= \frac{1}{\sqrt{G_{22}G_{33}}} \left[\partial_2 \left(\hat{H}_3\sqrt{G_{33}}\right) - \partial_3 \left(\hat{H}_2\sqrt{G_{22}}\right)\right], \\ \mathcal{Q}_2(y) &= \frac{1}{\sqrt{G_{11}G_{33}}} \left[\partial_3 \left(\hat{H}_1\sqrt{G_{11}}\right) - \partial_1 \left(\hat{H}_3\sqrt{G_{33}}\right)\right], \\ \mathcal{Q}_3(y) &= \frac{1}{\sqrt{G_{22}G_{11}}} \left[\partial_1 \left(\hat{H}_2\sqrt{G_{22}}\right) - \partial_2 \left(\hat{H}_1\sqrt{G_{11}}\right)\right]. \end{aligned}$$

For simplicity, denote

$$\mathscr{C}\mathrm{url}_y \, \hat{\mathbf{H}} = (\mathcal{Q}_1(y), \mathcal{Q}_2(y), \mathcal{Q}_3(y)). \tag{2.4}$$

Let $\hat{\mathbf{H}}_\lambda(y) = \mathbf{H}_\lambda(x)$ be the solution to system (1.4) and let $y = \lambda z$. Then in the neighborhood of X_0 we define the rescaled vector fields:

$$\tilde{\mathbf{H}}_\lambda(z) = \hat{\mathbf{H}}_\lambda(\lambda z) = \hat{\mathbf{H}}_\lambda(y), \quad \tilde{G}_{ii}(z) = G_{ii}(\lambda z) = G_{ii}(y) \quad \text{for } i = 1, 2, 3. \tag{2.5}$$

For simplicity, in the following, we always let $\tilde{\mathbf{H}}(z) = \tilde{\mathbf{H}}_\lambda(z)$. Under the new coordinate system $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, the system (1.4) can be rewritten by

$$-\mathscr{C}\mathrm{url}_z \left[F\left(|\mathscr{C}\mathrm{url}_z \, \tilde{\mathbf{H}}|^2\right) \mathscr{C}\mathrm{url}_z \, \tilde{\mathbf{H}}\right] = \tilde{\mathbf{H}}, \tag{2.6}$$

where the operator $\mathscr{C}\mathrm{url}_z$ is defined by, for $y = \lambda z$ in (2.4),

$$\mathscr{C}\mathrm{url}_z := \lambda\, \mathscr{C}\mathrm{url}_y\,. \tag{2.7}$$

We assume that the inner expansion in the neighborhood of X_0 is

$$\hat{\mathbf{H}}_\lambda\,(y) = \hat{\mathbf{H}}_0\,(y_1, y_2, z_3) + \lambda \hat{\mathbf{H}}_1\,(y_1, y_2, z_3) + \lambda^2 \hat{\mathbf{H}}_2\,(y_1, y_2, z_3) + O\left(\lambda^3\right). \tag{2.8}$$

It follows from [3, Lemma 8.2] that in local coordinates near X_0, the rescaled vector field $\tilde{\mathbf{H}}(z)$ converges in $C^{2+\alpha}_{loc}(\mathbb{R}^3_+, \mathbb{R}^3)$ to the solution $\mathbf{H}_0$ of system

$$-\operatorname{curl}\left[F(|\operatorname{curl}\mathbf{H}_0|^2)\operatorname{curl}\mathbf{H}_0\right] = \mathbf{H}_0 \quad \text{in } \mathbb{R}^3_+, \quad \mathbf{H}_{0_T} = \tilde{\mathbf{h}} \quad \text{on } \partial\mathbb{R}^3_+, \tag{2.9}$$

where $\tilde{\mathbf{h}}$ corresponds to $\mathcal{H}^e_T(X_0)$ in y-coordinate system. Hence, we can assume

$$\tilde{\mathbf{h}}(y_1, y_2, 0) = |\tilde{\mathbf{h}}|(cos\theta, sin\theta, 0), \tag{2.10}$$

where θ is the angle between the vector $\mathcal{H}^e_T$ and the y_1-curve at the point $X_0 \in \partial\Omega$.

Let f be the unique bounded solution satisfying $|f| < 1$ to the following equation

$$f'' = (1 - f^2)f \quad \text{for } z_3 > 0, \quad f'(0) = -|\tilde{\mathbf{h}}|. \tag{2.11}$$

Then we can write out the expression of $\mathbf{H}_0$ (see [3, Proposition 6.2]):

$$\mathbf{H}_0 = -f'(z_3)(cos\theta, sin\theta, 0). \tag{2.12}$$

Hence,

$$\mathbf{M}_0 := \operatorname{curl}\mathbf{H}_0 = \left(f''\,(z_3)\,sin\theta, -f''\,(z_3)\,cos\theta, 0\right). \tag{2.13}$$

From the definition of $F(t)$ in (1.5), it follows that

$$F\left(|\mathbf{M}_0|^2\right) = \frac{1}{1 - f^2}, \quad F'\left(|\mathbf{M}_0|^2\right) = \frac{1}{\left(1 - f^2\right)^3\left(1 - 3f^2\right)}. \tag{2.14}$$

Thus we have

$$\hat{\mathbf{H}}_0(0, 0, z_3) = \mathbf{H}_0.$$

Let $\hat{\mathcal{H}}^e_T(y_1, y_2)$ be the value of $\mathcal{H}^e_T(x)$ in y-coordinate system. We can do it by a similar process, and get

$$\hat{\mathbf{H}}_0(y_1, y_2, z_3) = -\frac{f'_{y_1,y_2}(z_3)}{|\hat{\mathcal{H}}^e_T|}\hat{\mathcal{H}}^e_T, \tag{2.15}$$

where $f'_{y_1,y_2}(z_3)$ is defined by (2.11) with $|\tilde{\mathbf{h}}| = |\hat{\mathcal{H}}^e_T|$. By simple calculations, we have

$$\left|\nabla\hat{\mathbf{H}}_0\right| \leqq Ce^{-z_3}, \quad \left|\nabla^2\hat{\mathbf{H}}_0\right| \leqq Ce^{-z_3},$$

here the constants C depend on $\mathcal{H}_T^e$. Denote

$$\left(p_1^1, p_1^2, 0\right) := \partial_{y_1}\hat{\mathbf{H}}_0\left(0, 0, z_3\right), \quad \left(p_2^1, p_2^2, 0\right) := \partial_{y_2}\hat{\mathbf{H}}_0\left(0, 0, z_3\right). \quad (2.16)$$

We begin to derive the first order term at X_0:

$$\hat{\mathbf{H}}_1(0, 0, z_3) := \mathbf{w}(z)|_{z=(0,0,z_3)} = (w_1, w_2, w_3)|_{z=(0,0,z_3)}. \quad (2.17)$$

Then $\mathbf{w}(z)$ satisfies the following system in $\mathbb{R}^3_+$:

$$\begin{aligned} -\operatorname{curl}\left[F\left(|\mathbf{M}_0|^2\right)\operatorname{curl}\mathbf{w} + 2F'\left(|\mathbf{M}_0|^2\right)\langle\mathbf{M}_0, \operatorname{curl}\mathbf{w}\rangle\mathbf{M}_0\right] - \mathbf{w} = \mathbf{b} \quad & \text{in } \mathbb{R}^3_+, \\ \mathbf{w}_T = 0 \quad & \text{on } \partial\mathbb{R}^3_+, \end{aligned} \quad (2.18)$$

where the vector $\mathbf{b} = (-b_1, b_2, b_3)$ is defined by

$$\begin{aligned} b_1 &= \partial_3\left(2F'\left(|\mathbf{M}_0|^2\right)\left(\kappa_2 H_2^0\partial_3 H_2^0 + \kappa_1 H_1^0\partial_3 H_1^0\right)\partial_3 H_1^0 + \kappa_1 F\left(|\mathbf{M}_0|^2\right)H_1^0\right) \\ &\quad + \kappa_2 F\left(|\mathbf{M}_0|^2\right)\partial_3 H_1^0, \\ b_2 &= \partial_3\left(2F'\left(|\mathbf{M}_0|^2\right)\left(\kappa_2 H_2^0\partial_3 H_2^0 + \kappa_1 H_1^0\partial_3 H_1^0\right)\partial_3 H_2^0 + \kappa_2 F\left(|\mathbf{M}_0|^2\right)H_2^0\right) \\ &\quad + \kappa_1 F\left(|\mathbf{M}_0|^2\right)\partial_3 H_2^0, \\ b_3 &= F\left(|\mathbf{M}_0|^2\right)\left(\Gamma_{21}^2\partial_3 H_1^0 + \Gamma_{11}^1\partial_3 H_2^0 + \partial_3 p_1^1 + \partial_3 p_2^2\right) \\ &\quad + 2F'\left(|\mathbf{M}_0|^2\right)\left(\partial_3 H_1^0\partial_3 H_2^0\partial_3 p_1^2\right. \\ &\quad \left. + \partial_3 H_1^0\partial_3 H_1^0\partial_3 p_1^1 + \partial_3 H_2^0\partial_3 H_2^0\partial_3 p_2^2 + \partial_3 H_2^0\partial_3 H_1^0\partial_3 p_2^1\right), \end{aligned} \quad (2.19)$$

Γ_{ij}^k denote the value of the Christoffel symbols at X_0 and

$$\kappa_1 = \kappa_1(X_0), \quad \kappa_2 = \kappa_2(X_0).$$

The detailed calculations will be shown in Appendix A.

In the following, we show the existence of the solution to system (2.18) and prove that the solution decays exponentially.

Let w_1 and w_2 depend only on the variable z_3, but not on z_1, z_2, and let

$$w_3 = -b_3.$$

Then system (2.18) can be reduced to

$$\begin{aligned} &\partial_3\Big(2F'\left(|\mathbf{M}_0|^2\right)\left(\partial_3 H_2^0\partial_3 w_2 + \partial_3 H_1^0\partial_3 w_1\right)\partial_3 H_1^0 \\ &\quad + F\left(|\mathbf{M}_0|^2\right)\partial_3 w_1\Big) - w_1 = b_1, \end{aligned}$$

$$\partial_3\Big(2F'\left(|\mathbf{M}_0|^2\right)\left(\partial_3 H_2^0\partial_3 w_2+\partial_3 H_1^0\partial_3 w_1\right)\partial_3 H_2^0$$
$$+F\left(|\mathbf{M}_0|^2\right)\partial_3 w_2\Big)-w_2=b_2, \tag{2.20}$$

with the boundary conditions $w_1=0$, $w_2=0$ on $\partial\mathbb{R}^3_+$.

Substituting (2.12)–(2.14) to system (2.20), then using $f''=(1-f^2)f$, we see that w_1, w_2 satisfy the equation

$$\partial_3\left(\frac{2f^2cos\theta}{(1-f^2)(1-3f^2)}\partial_3(sin\theta w_2+cos\theta w_1)+\frac{1}{1-f^2}\partial_3 w_1\right)-w_1-\kappa_2 f cos\theta$$
$$=-\partial_3\left(\frac{f'}{1-f^2}\right)cos\theta\kappa_1$$
$$-2\partial_3\left(\frac{f'f^2}{(1-f^2)(1-3f^2)}\right)\left(sin^2\theta\kappa_2+cos^2\theta\kappa_1\right)cos\theta, \tag{2.21}$$

and the equation

$$\partial_3\left(\frac{2f^2sin\theta}{(1-f^2)(1-3f^2)}\partial_3(sin\theta w_2+cos\theta w_1)+\frac{1}{1-f^2}\partial_3 w_2\right)-w_2+\kappa_1 f sin\theta$$
$$=-\partial_3\left(\frac{f'}{1-f^2}\right)sin\theta\kappa_2$$
$$-2\partial_3\left(\frac{f'f^2}{(1-f^2)(1-3f^2)}\right)\left(sin^2\theta\kappa_2+cos^2\theta\kappa_1\right)sin\theta. \tag{2.22}$$

Combining the above two equations and then setting $v=sin\theta w_2+cos\theta w_1$, we have

$$\partial_3\left[\frac{\partial_3 v}{1-3f^2}\right]-v=-\left(cos^2\theta\kappa_1+sin^2\theta\kappa_2\right)\partial_3\left(\frac{1}{1-3f^2}f'\right)$$
$$-\left(cos^2\theta\kappa_2+sin^2\theta\kappa_1\right)f, \tag{2.23}$$

with the boundary condition $v(0)=0$ and the natural condition $v(\infty)=0$.

Consider the equation:

$$\partial_3\left(\frac{\partial_3 u_1}{1-3f^2}\right)-u_1=-\partial_3\left(\frac{f'}{1-3f^2}\right),\quad z_3>0;\qquad u_1(0)=u_1(\infty)=0 \tag{2.24}$$

and the equation

$$\partial_3\left(\frac{\partial_3 u_2}{1-3f^2}\right)-u_2=-f,\quad z_3>0;\quad u_2(0)=u_2(\infty)=0. \tag{2.25}$$

Define the bilinear form $\mathcal{B}[\cdot,\cdot]$ on $H_0^1(\mathbb{R}^+)\times H_0^1(\mathbb{R}^+)$ by

$$\mathcal{B}[u,v]=\int_{\mathbb{R}^+}\left(\frac{\partial_3 u\partial_3 v}{1-3f^2}+uv\right)\mathrm{d}z_3.$$

It is easy to see that $\mathcal{B}$ is bounded and coercive. Then the existence and the uniqueness of the solutions to (2.24) and (2.25) in $H_0^1(\mathbb{R}^+)$ follow from Lax-Milgram lemma. From the standard elliptic estimates, the solutions obtained are actually smooth.

Thus, we get the solution to $v-$equation:

$$v(z) = \left(cos^2\theta\kappa_1 + sin^2\theta\kappa_2\right) u_1(z) + \left(cos^2\theta\kappa_2 + sin^2\theta\kappa_1\right) u_2(z). \quad (2.26)$$

Plugging the expression of v back to (2.21) and (2.22), we see that w_1 satisfies the equation

$$\begin{aligned}
&\partial_3\left(\frac{1}{1-f^2}\partial_3 w_1\right) - w_1 - \kappa_2 f cos\theta + \partial_3\left(\frac{2f^2cos\theta}{(1-f^2)(1-3f^2)}\partial_3 v\right)\\
&= -\partial_3\left(\frac{f'}{1-f^2}\right)cos\theta\kappa_1 - 2\partial_3\left(\frac{f'f^2}{(1-f^2)(1-3f^2)}\right)\\
&\quad\times(sin^2\theta\kappa_2 + cos^2\theta\kappa_1)cos\theta,
\end{aligned}$$

and w_2 satisfies the equation

$$\begin{aligned}
&\partial_3\left(\frac{1}{1-f^2}\partial_3 w_2\right) - w_2 + \kappa_1 f sin\theta + \partial_3\left(\frac{2f^2sin\theta}{(1-f^2)(1-3f^2)}\partial_3 v\right)\\
&= -\partial_3\left(\frac{f'}{1-f^2}\right)sin\theta\kappa_2 - 2\partial_3\left(\frac{f'f^2}{(1-f^2)(1-3f^2)}\right)\\
&\quad\times(sin^2\theta\kappa_2 + cos^2\theta\kappa_1)sin\theta.
\end{aligned}$$

Then the existence of solutions w_1 and w_2 can be obtained by the Lax-Milgram lemma, the proof of which is the same as that of u_1.

We now show the exponential decays for the solutions to (2.24) and (2.25).

Lemma 2.1. *Let f be the solution to (2.11), and let u be the solution of the equation*

$$\partial_3\left(\frac{\partial_3 u}{1-3f^2}\right) - u = h, \quad z_3 > 0; \quad u(0) = u(\infty) = 0,$$

where the function h satisfies $|h| \leqq M_0 e^{-z_3}$ with M_0 being a constant. Then there exists a constant C depending on M_0 and f such that

$$|u(z_3)| \leqq Ce^{-z_3/2}.$$

Proof. Let w be the solution of

$$\partial_3\left(\frac{\partial_3 w}{1-3f^2}\right) - w = M_0 e^{-z_3}, \quad z_3 > 0; \quad w(0) = w(\infty) = 0. \quad (2.27)$$

It follows from the comparison principle that

$$w \leqq u \leqq -w.$$

By the maximum principle, there exists z_0 such that for any $z_3 \in (z_0, \infty)$ we have $w(z_3) < 0$ and $w'(z_3) > 0$. Rewriting the Eq. (2.27), we have

$$w'' - \frac{1}{2}w' - \frac{1}{2}w = M_0 e^{-z_3} + \left(-\frac{1}{2} - \frac{6ff'}{1-3f^2}\right) w' + \left(\frac{1}{2} - 3f^2\right) w \\ -3f^2 M_0 e^{-z_3}.$$

Since the function f decays exponentially with respect to z_3, there exists $\tilde{z}_0 > z_0$ such that for any $z_3 \in (\tilde{z}_0, \infty)$

$$\left(-\frac{1}{2} - \frac{6ff'}{1-3f^2}\right) w' + \left(\frac{1}{2} - 3f^2\right) w - 3f^2 M_0 e^{-z_3} \leqq 0.$$

Consider the equation

$$\tilde{w}'' - \frac{1}{2}\tilde{w}' - \frac{1}{2}\tilde{w} = M_0 e^{-z_3}, \quad z_3 > \tilde{z}_0; \quad \tilde{w}(\tilde{z}_0) = w(\tilde{z}_0), \quad w(\infty) = 0.$$

By the comparison principle again, we have

$$w \geqq \tilde{w} = M_0 e^{-z_3} + \left[w(\tilde{z}_0)e^{\tilde{z}_0/2} - M_0 e^{-\tilde{z}_0/2}\right] e^{-z_3/2}, \quad z_3 > \tilde{z}_0.$$

This gives that

$$|u| \leqq |w| \leqq |\tilde{w}| \leqq Ce^{-z_3/2}, \quad z_3 > \tilde{z}_0.$$

We end our proof. □

Combining with Lemma 2.1, we get the solution to system (2.18):

Theorem 2.2. *There exists a unique solution* $\mathbf{w} \in H^1(\mathbb{R}^3_+)$ *to system* (2.18) *with*

$$\mathbf{w} = (w_1, w_2, -b_3), \tag{2.28}$$

where w_1 *and* w_2 *is the solution pair to system* (2.21)–(2.22)*,* b_3 *is defined in* (2.19)*. Moreover, we have*

$$|\mathbf{w}| \leqq Ce^{-z_3/2},$$

where the constant C *depends on the domain* Ω *and* $\mathcal{H}^e_T$*.*

We are now in the position to show the second order term at X_0:

$$\hat{\mathbf{H}}_2(0,0,z_3) := \mathbf{\Psi}(z)\,|_{z=(0,0,z_3)} = (\Psi_1, \Psi_2, \Psi_3)\,|_{z=(0,0,z_3)}. \tag{2.29}$$

Then we can find that $\mathbf{\Psi}(z)$ satisfies the following system in $\mathbb{R}^3_+$:

$$\begin{aligned} -\operatorname{curl}\left[F\left(|\mathbf{M}_0|^2\right)\operatorname{curl}\mathbf{\Psi} + 2F'\left(|\mathbf{M}_0|^2\right)\langle \mathbf{M}_0, \operatorname{curl}\mathbf{\Psi}\rangle \mathbf{M}_0\right] - \mathbf{\Psi} = \mathbf{\Phi} \quad & \text{in } \mathbb{R}^3_+, \\ \mathbf{\Psi}_T = 0 \quad & \text{on } \partial\mathbb{R}^3_+, \end{aligned} \tag{2.30}$$

where the vector $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ depending only on the variable z_3 is determined by the domain Ω and the strength of the magnetic field $\mathcal{H}^e_T$. The detailed calculations will be shown in Appendix B.

By a similar proof of Theorem 2.2, we have the following corollary:

Corollary 2.3. *There exists a unique solution* $\mathbf{\Psi} = \mathbf{\Psi}(z_3) \in H^1(\mathbb{R}^3_+)$ *to system* (2.30). *Moreover, we have*

$$|\mathbf{\Psi}| \leqq Ce^{-z_3/2},$$

where the constant C *depends on the domain* Ω *and* $\mathcal{H}^e_T$.

3. Uniform Estimation for the Approximation Solution

In this section we use the first order term $\mathbf{w}$ obtained in Theorem 2.2 and the second order term $\mathbf{\Psi}$ obtained in Corollary 2.3 to construct an approximation solution $\mathbf{H}_{ap}$ of the solution $\mathbf{H}_\lambda$ to system (1.4). We shall prove that the approximation solution $\mathbf{H}_{ap}$ converges to $\mathbf{H}_\lambda$ in $H^1(\Omega)$ and in C^1 in the neighborhood of the boundary.

Let $\mathcal{N}_0$ be the neighborhood of the boundary $\partial\Omega$ such that for each point $X_0 \in \partial\Omega$ there is a $C^{2,\alpha}$ diffeomorphism and a ball $B_\varepsilon(X_0)$ that straighten the boundary in $\mathcal{N}_0 \bigcap B_\varepsilon(X_0)$. Denote

$$d_0 := \text{dist}\,(\partial\Omega, \Omega\backslash\mathcal{N}_0)\,, \quad \sigma_n := \left\{x \in \bar{\Omega} : \text{dist}(x, \partial\Omega) \leqq d_0/n\right\},$$

then define a fixed smooth function $\chi(x)$ (independent of λ) by

$$\chi(x) = \begin{cases} 1, & x \in \sigma_4; \\ smooth, & x \in \sigma_2\backslash\sigma_4; \\ 0, & x \in \Omega\backslash\sigma_2. \end{cases}$$

For each $x \in \mathcal{N}_0$, there exists a unique pair (y', y_3) with y_3 being the distance from the point x to $\partial\Omega$ and $y' \in \partial\Omega$ satisfying

$$\text{dist}(x, \partial\Omega) = \text{dist}(x, y').$$

Let $\mathscr{H}(y', z_3)$, $\mathscr{W}(y', z_3)$ and $\mathbf{\Psi}(y', z_3)$ be defined by (2.15), (2.28) and (2.30) with $y' = X_0$ respectively. Now we define the approximation solution $\mathbf{H}_{ap}(x)$ by

$$\mathbf{H}_{ap}(x) = \chi(x)(\mathscr{H} + \lambda\mathscr{W} + \lambda^2\mathbf{\Psi})\left(y', z_3\right) \tag{3.1}$$

with $z_3 = y_3/\lambda$. It is clear that the vector $\mathbf{H}_{ap}(x)$ can be defined everywhere in $\bar{\Omega}$.

We define the operator $\mathscr{L}_\lambda$ for any vector $\mathbf{A} \in C^2(\bar{\Omega})$ by

$$\mathscr{L}_\lambda[\mathbf{A}] = -\lambda^2\,\text{curl}\left[F(\lambda^2|\,\text{curl}\,\mathbf{A}|^2)\,\text{curl}\,\mathbf{A}\right] - \mathbf{A}. \tag{3.2}$$

In view of the calculations in Appendix A, if we replace $\tilde{\mathbf{H}}$ in Appendix A by $\mathscr{H} + \lambda\mathscr{W}$, then for any $\mathfrak{R}_i(\lambda^2)$ in Appendix A there exists λ^* such that for any $\lambda \in (0, \lambda^*)$ we have

$$\left|\mathfrak{R}_i(\lambda^2)\right| \leqq C\lambda^2 \quad \text{for } i = 3\ldots 23.$$

If we replace $\tilde{\mathbf{H}}$ in Appendix A and in Appendix B by $\mathscr{H} + \lambda\mathscr{W} + \lambda^2\mathbf{\Psi}$, then for any $\mathfrak{R}_i(\lambda^3)$ in Appendix B we have

$$\left|\mathfrak{R}_i(\lambda^3)\right| \leqq C\lambda^3 \quad \text{for } i = 24\dots 42,$$

where the constants C depend only on Ω and $\mathcal{H}_T^e$.

Denote

$$\mathbf{b}(x,\lambda) := \mathscr{L}_\lambda[\mathbf{H}_{ap}(x)].$$

Then, for $x \in \Omega\backslash\sigma_2$ we have $\mathbf{b}(x,\lambda) = 0$. Let $\tilde{\mathbf{b}}$ be the representation of $\mathbf{b}$ under the z-coordinate system (for $x \in \mathcal{N}_0$). Then for any $x \in \sigma_4$ we have

$$\left|\tilde{\mathbf{b}}\right| \leqq C\left(\Omega, \mathcal{H}_T^e\right)\lambda^3, \quad \left|\nabla_z\tilde{\mathbf{b}}\right| \leqq C\left(\Omega, \mathcal{H}_T^e\right)\lambda^3, \quad \left|\nabla_z^2\tilde{\mathbf{b}}\right| \leqq C\left(\Omega, \mathcal{H}_T^e\right)\lambda^3. \tag{3.3}$$

From the expressions of $\mathscr{H}$, $\mathscr{W}$ and $\mathbf{\Psi}$, we see that

$$\left|\mathscr{H} + \lambda\mathscr{W} + \lambda^2\mathbf{\Psi}\right| \leqq Ce^{-z_3/2}.$$

Therefore, for $x \in \sigma_2\backslash\sigma_4$ we also have (3.3) holding for λ small.

We introduce the remainder term $\mathbf{R}$ by

$$\mathbf{R} = \mathbf{H}_\lambda - \mathbf{H}_{ap}, \tag{3.4}$$

where $\mathbf{H}_\lambda$ is the solution of system (1.4). Then $\mathbf{R}$ satisfies

$$\begin{aligned} -\lambda^2\,\mathrm{curl}\left[a_1(x)\,\mathrm{curl}\,\mathbf{R} + 2\lambda^2\int_0^1 h_1(x,t)\langle\mathbf{M}, \mathrm{curl}\,\mathbf{R}\rangle\mathbf{M}\,\mathrm{d}t\right] - \mathbf{R} &= \mathbf{b}(x,\lambda) \quad \text{in } \Omega,\\ \mathbf{R}_T &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{3.5}$$

where the symbol $\langle\cdot,\cdot\rangle$ represents the inner product between two vectors, the functions $a_1(x)$ and $h_1(x,t)$ are defined by

$$\begin{aligned} a_1(x) &= \int_0^1 F\left(\lambda^2|\,\mathrm{curl}\left(\mathbf{H}_{ap} + t\mathbf{R}\right)|^2\right)\mathrm{d}t, \quad h_1(x,t)\\ &= F'\left(\lambda^2|\,\mathrm{curl}\left(\mathbf{H}_{ap} + t\mathbf{R}\right)|^2\right) \end{aligned} \tag{3.6}$$

and the vector $\mathbf{M}$ is defined by

$$\mathbf{M}(x,t) = \mathrm{curl}(\mathbf{H}_{ap} + t\mathbf{R}). \tag{3.7}$$

This is a curl-type linear elliptic system with the coefficients satisfying

$$a_1(x) \geqq 1, \quad h_1(x,t) \geqq 0.$$

We begin to consider how the regularity of the solution $\mathbf{R}$ depends on the parameter λ. Firstly, note that the integral in (3.5) with respect to t does not influence the regularity of the vector $\mathbf{R}$, this is because we can calculate the integral with respect to t finally in the process of the estimate. Hence, it suffices to consider the system

$$-\lambda^2 \operatorname{curl}(a(x) \operatorname{curl} \mathbf{R} + \lambda^2 h(x)\langle \mathbf{M}, \operatorname{curl} \mathbf{R}\rangle \mathbf{M}) - \mathbf{R} = \mathbf{b}(x,\lambda) \quad \text{in } \Omega, \qquad \mathbf{R}_T = 0 \quad \text{on } \partial\Omega. \tag{3.8}$$

Let

$$\mathrm{H}_T(\operatorname{curl}, \Omega) = \left\{ \mathbf{u} \in L^2(\Omega) \, : \, \operatorname{curl} \mathbf{u} \in L^2(\Omega), \quad \mathbf{u}_T = 0 \quad \text{on } \partial\Omega \right\}.$$

Now we state the following global H^1 estimate:

Lemma 3.1. *Let $\mathbf{R} \in H^1(\Omega)$ be the solution of system (3.8). Then we have*

$$\|\mathbf{R}\|_{L^2(\Omega)} + \|\lambda \nabla \mathbf{R}\|_{L^2(\Omega)} \leqq C \left(\|\mathbf{b}\|_{L^2(\Omega)} + \|\lambda \operatorname{div} \mathbf{b}\|_{L^2(\Omega)} \right), \tag{3.9}$$

where the constant C depends only on Ω, but not on λ.

Proof. Note that the vector $\mathbf{R}$ can be viewed as a weak solution of (3.8) in the sense of

$$\int_\Omega \left\{ \lambda^2 (a(x) \operatorname{curl} \mathbf{R} + \lambda^2 h(x) \langle \mathbf{M}, \operatorname{curl} \mathbf{R} \rangle \mathbf{M}) \cdot \operatorname{curl} \mathbf{B} + \mathbf{R} \cdot \mathbf{B} \right\} \mathrm{d}x = \int_\Omega \mathbf{b} \cdot \mathbf{B} \mathrm{d}x \tag{3.10}$$

for all $\mathbf{B} \in \mathrm{H}_T(\operatorname{curl}, \Omega)$. Then taking $\mathbf{B} = \mathbf{R}$, we have

$$\int_\Omega \left(\lambda^2 a(x) |\operatorname{curl} \mathbf{R}|^2 + \lambda^4 h(x) |\mathbf{M} \cdot \operatorname{curl} \mathbf{R}|^2 + |\mathbf{R}|^2 \right) \mathrm{d}x = \int_\Omega \mathbf{b} \cdot \mathbf{R} \mathrm{d}x.$$

By the Cauchy's inequality and by noting that $a(x) \geqq 1$ and $h(x) \geqq 0$, we get

$$\|\lambda \operatorname{curl} \mathbf{R}\|_{L^2(\Omega)} + \|\mathbf{R}\|_{L^2(\Omega)} \leqq C \|\mathbf{b}\|_{L^2(\Omega)}.$$

From (3.8), naturally we have

$$\operatorname{div} \mathbf{R} = \operatorname{div} \mathbf{b}.$$

Applying the following inequality for $\mathbf{R} \in \mathrm{H}_T(\operatorname{curl}, \Omega)$:

$$\|\nabla \mathbf{R}\|_{L^2(\Omega)} \leqq C(\Omega) \left(\|\operatorname{curl} \mathbf{R}\|_{L^2(\Omega)} + \|\operatorname{div} \mathbf{R}\|_{L^2(\Omega)} \right),$$

we thus obtain the global H^1 estimate for $\mathbf{R}$:

$$\|\mathbf{R}\|_{L^2(\Omega)} + \|\lambda \nabla \mathbf{R}\|_{L^2(\Omega)} \leqq C \left(\|\mathbf{b}\|_{L^2(\Omega)} + \|\lambda \operatorname{div} \mathbf{b}\|_{L^2(\Omega)} \right).$$

This ends the proof. □

In the following, we shall establish the H^2 estimate near boundary. We use the notations introduced in Section 2. Let $X_0 \in \partial\Omega$ be fixed and consider the estimate in the neighborhood of X_0, denote by $\mathcal{U}$. We assume that $\mathcal{F}$ defined by (2.1) is a diffeomorphism from a half ball $B_R^+(0)$ with the center at the origin and the radius R onto $\mathcal{U} \bigcap \Omega$. For λ small, denote

$$\mathcal{O} := B_2^+(0) \subset B_{R/\lambda}^+(0), \quad \mathcal{T} := \partial B_2^+(0) \bigcap \partial\mathbb{R}_+^3.$$

Let $\tilde{\mathbf{R}}$, $\tilde{\mathbf{M}}$, $\tilde{\mathbf{b}}$, $\tilde{a}$, $\tilde{h}$ be the representations of $\mathbf{R}$, $\mathbf{M}$, $\mathbf{b}$, a, h under the z-coordinate system respectively. From (3.3), it follows that

$$\|\mathbf{b}\|_{L^2(\Omega)} + \|\lambda \operatorname{div} \mathbf{b}\|_{L^2(\Omega)} \leqq C\lambda^3$$

and

$$\|\tilde{\mathbf{b}}\|_{H^2(\mathcal{O})} \leqq C\lambda^3, \quad \|\tilde{\mathbf{b}}\|_{C^2(\mathcal{O})} \leqq C\lambda^3. \tag{3.11}$$

Then from Lemma 3.1, by scaling argument we have the following H^1 estimate for $\tilde{\mathbf{R}}$:

$$\|\tilde{\mathbf{R}}\|_{H^1(\mathcal{O})} \leqq C\lambda^{3/2}. \tag{3.12}$$

Rewriting system (3.8), we obtain

$$\begin{aligned} -\mathscr{C}\mathrm{url}_z \left[\tilde{a}(z)\,\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}} + \tilde{h}(z)\langle \tilde{\mathbf{M}}, \mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}}\rangle \tilde{\mathbf{M}}\right] - \tilde{\mathbf{R}} = \tilde{\mathbf{b}}(z,\lambda) \quad & \text{in } \mathcal{O}, \\ \tilde{\mathbf{R}}_T = 0 \quad & \text{on } \mathcal{T}, \end{aligned} \tag{3.13}$$

where the operator $\mathscr{C}\mathrm{url}_z$ is defined by (2.7) and the coefficients satisfy, for some positive constant Λ,

$$\tilde{a}(z) \geqq 1; \quad \tilde{h}(z) \geqq 0; \tag{3.14}$$

$$\|\tilde{a}(z)\|_{C^{1,\alpha}(\overline{\mathcal{O}})}, \|\tilde{h}(z)\|_{C^{1,\alpha}(\overline{\mathcal{O}})}, \|\tilde{\mathbf{M}}\|_{C^{1,\alpha}(\overline{\mathcal{O}})} \leqq \Lambda, \tag{3.15}$$

where the constant Λ depends on Ω, α and $\|\mathcal{H}_T^e\|_{C^{2,\alpha}(\partial\Omega)}$. See [3, Lemma 8.2].

We need to mention that one can check that the system (3.13) satisfies the ellipticity condition, the supplementary condition and the complementary boundary conditions as specified by Agmon et al. in [2], then we can apply Theorem 10.4 in [2] to obtain the $W^{2,p}$ estimate for $p > 3$, and hence the C^1 estimate for $\tilde{\mathbf{R}}$ (Theorem 3.3). This idea was pointed out by the referee.

We can also establish the H^2 estimate directly by using the difference-quotient technique [8] and the div-curl estimate. Then by solving a regular oblique derivative problem to obtain the C^1 estimate for $\tilde{\mathbf{R}}$. We here adopt the second method.

We first show the H^2 estimate near boundary for $\tilde{\mathbf{R}}$.

Lemma 3.2. *Let $\tilde{\mathbf{R}}$ be the solution of system (3.13) with the coefficients satisfying (3.14). Then we have the estimate*

$$\|\tilde{\mathbf{R}}\|_{H^2(B_{5/4}^+(0))} \leqq C\lambda^{3/2}, \tag{3.16}$$

where the constant C depends only on Ω and Λ, but not on λ.

Proof. For $\tilde{\mathbf{M}} = 0$, $\tilde{\mathbf{b}}(z, \lambda) = 0$ and if the $\mathscr{C}\mathrm{url}_z$ operator is replaced by the curl operator, the proof of H^2 estimate can be found in [3, Theorem 4.1].

We divide the estimate into two parts: the tangential and the normal derivatives of $\nabla\tilde{\mathbf{R}}$. We first consider the estimate on the tangential derivative of $\nabla\tilde{\mathbf{R}}$. Let $\sigma > 0$ be small. For any function ψ we define

$$\delta_\sigma \psi(z) = \frac{1}{\sigma}[\psi(z + \sigma \mathbf{e}_i) - \psi(z)],$$
$$\psi_{t,\sigma}(z) = \psi(z) + t[\psi(z + \sigma \mathbf{e}_i) - \psi(z)] = \psi(z) + \sigma t \delta_\sigma \psi(z), \quad (3.17)$$

where the unit vector $\mathbf{e}_i$ denotes the i-th direction of coordinate system. Here, we choose $i = 1, 2$.

Set

$$Q(z) = \tilde{a}(z)\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}} + \tilde{h}(z)\langle \tilde{\mathbf{M}}, \mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}}\rangle \tilde{\mathbf{M}}.$$

From (3.13), for all $\tilde{\mathbf{B}} \in \mathrm{H}_T(\mathcal{O})$ with support in the interior of $B_2(0)$, we obtain that

$$\int_{\mathcal{O}} \Big((Q(z + \sigma \mathbf{e}_i) - Q(z)) \cdot \mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{B}} + \sigma \delta_\sigma \tilde{\mathbf{R}} \cdot \tilde{\mathbf{B}} \Big) \sqrt{\tilde{G}} \mathrm{d}z$$
$$= \sigma \int_{\mathcal{O}} \delta_\sigma \tilde{\mathbf{b}} \cdot \tilde{\mathbf{B}} \sqrt{\tilde{G}} \mathrm{d}z, \quad (3.18)$$

where $\tilde{G} = \tilde{G}_{11}\tilde{G}_{22}\tilde{G}_{33}$ is defined by (2.5). By the simple calculations,

$$Q(z + \sigma \mathbf{e}_i) - Q(z)$$
$$= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \Big[\tilde{a}_{t,\sigma}(\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}})_{t,\sigma} + \tilde{h}_{t,\sigma}\langle \tilde{\mathbf{M}}_{t,\sigma}, (\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}})_{t,\sigma}\rangle \tilde{\mathbf{M}}_{t,\sigma} \Big] \,\mathrm{d}t$$
$$= \sigma \int_0^1 (I_1 + I_2 + I_3 + I_4)\, \mathrm{d}t$$

with

$$I_1 = \delta_\sigma \tilde{a}(\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}})_{t,\sigma}, \quad I_2 = \tilde{a}_{t,\sigma}\delta_\sigma(\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}}),$$
$$I_3 = \tilde{h}_{t,\sigma}\langle \tilde{\mathbf{M}}_{t,\sigma}, \delta_\sigma(\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}})\rangle \tilde{\mathbf{M}}_{t,\sigma},$$
$$I_4 = \tilde{h}_{t,\sigma}\langle \delta_\sigma \tilde{\mathbf{M}}, (\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}})_{t,\sigma}\rangle \tilde{\mathbf{M}}_{t,\sigma} + \delta_\sigma \tilde{h}\langle \tilde{\mathbf{M}}_{t,\sigma}, (\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}})_{t,\sigma}\rangle \tilde{\mathbf{M}}_{t,\sigma}$$
$$+ \tilde{h}_{t,\sigma}\langle \tilde{\mathbf{M}}_{t,\sigma}, (\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{R}})_{t,\sigma}\rangle \delta_\sigma \tilde{\mathbf{M}}.$$

We now estimate the first integral in (3.18) and choose $\tilde{\mathbf{B}} = \eta^2 \delta_\sigma \tilde{\mathbf{R}}$, where η is a cut-off function defined by

$$\eta(z) = \begin{cases} 0, & z \in B_2(0) \backslash B_{3/2}(0), \\ 1, & z \in B_{5/4}(0), \\ smooth & others. \end{cases} \quad (3.19)$$

Note that we have the inequality ([8, Lemma 7.23])

$$\int_{\mathcal{O}} |\delta_\sigma(\eta \tilde{\mathbf{R}})|^2 \mathrm{d}z \leqq \int_{\mathcal{O}} |D_i(\eta \tilde{\mathbf{R}})|^2 \mathrm{d}z. \quad (3.20)$$

Then, for σ small, by the Cauchy's inequality we have

$$\left|\int_{\mathcal{O}} I_1 \cdot \mathscr{C}\mathrm{url}_z\left(\eta^2\delta_\sigma\tilde{\mathbf{R}}\right)\sqrt{\tilde{G}}\mathrm{d}z\right| \leqq C(\Omega,\Lambda)\|\tilde{\mathbf{R}}\|^2_{H^1(\mathcal{O})} + \varepsilon\int_{\mathcal{O}}|\mathscr{C}\mathrm{url}_z\left(\eta\delta_\sigma\tilde{\mathbf{R}}\right)|^2\mathrm{d}z. \tag{3.21}$$

For the integral involving I_2,

$$\int_{\mathcal{O}} I_2 \cdot \mathscr{C}\mathrm{url}_z\left(\eta^2\delta_\sigma\tilde{\mathbf{R}}\right)\sqrt{\tilde{G}}\mathrm{d}z \geqq \frac{1}{2}\int_{\mathcal{O}}\left|\mathscr{C}\mathrm{url}_z\left(\eta\delta_\sigma\tilde{\mathbf{R}}\right)\right|^2\mathrm{d}z - C(\Omega,\Lambda)\|\tilde{\mathbf{R}}\|^2_{H^1(\mathcal{O})} - \varepsilon\int_{\mathcal{O}}\left|\mathscr{C}\mathrm{url}_z\left(\eta\delta_\sigma\tilde{\mathbf{R}}\right)\right|^2\mathrm{d}z. \tag{3.22}$$

For the integral involving I_3,

$$\int_{\mathcal{O}} I_3 \cdot \mathscr{C}\mathrm{url}_z(\eta^2\delta_\sigma\tilde{\mathbf{R}})\sqrt{\tilde{G}}\mathrm{d}z \geqq -C(\Omega,\Lambda)\|\tilde{\mathbf{R}}\|^2_{H^1(\mathcal{O})} - \varepsilon\int_{\mathcal{O}}|\mathscr{C}\mathrm{url}_z(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z. \tag{3.23}$$

We estimate the integral involving I_4. For any bounded vectors $\mathbf{A}$ and $\mathbf{B}$, we have

$$\left|\int_{\mathcal{O}}\langle\mathbf{A},(\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{R}})_{t,\sigma}\rangle\langle\mathbf{B},\mathscr{C}\mathrm{url}_z(\eta^2\delta_\sigma\tilde{\mathbf{R}})\rangle\sqrt{\tilde{G}}\mathrm{d}z\right| \leqq C(\Omega,\Lambda)\int_{\mathcal{O}}|\nabla_z\tilde{\mathbf{R}}|\cdot|\mathscr{C}\mathrm{url}_z(\eta^2\delta_\sigma\tilde{\mathbf{R}})|\mathrm{d}z + \varepsilon\int_{\mathcal{O}}|\mathscr{C}\mathrm{url}_z(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z.$$

This gives that

$$\left|\int_{\mathcal{O}} I_4 \cdot \mathscr{C}\mathrm{url}_z(\eta^2\delta_\sigma\tilde{\mathbf{R}})\sqrt{\tilde{G}}\mathrm{d}z\right| \leqq C(\Omega,\Lambda)\|\tilde{\mathbf{R}}\|^2_{H^1(\mathcal{O})} + \varepsilon\int_{\mathcal{O}}|\mathscr{C}\mathrm{url}_z(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z. \tag{3.24}$$

For the integral in the right side of (3.18), we have

$$\left|\int_{\mathcal{O}}\delta_\sigma\tilde{\mathbf{b}}\cdot\tilde{\mathbf{B}}\sqrt{\tilde{G}}\mathrm{d}z\right| \leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|^2_{H^1(\mathcal{O})} + \|\tilde{\mathbf{R}}\|^2_{H^1(\mathcal{O})}\right). \tag{3.25}$$

Plugging the inequalities (3.21)–(3.25) back to (3.18), we thus obtain the estimate on the $\mathscr{C}\mathrm{url}_z$ part:

$$\int_{\mathcal{O}}|\mathscr{C}\mathrm{url}_z(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z \leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|^2_{H^1(\mathcal{O})} + \|\tilde{\mathbf{R}}\|^2_{H^1(\mathcal{O})}\right). \tag{3.26}$$

Therefore, from the expression of $\mathscr{C}\mathrm{url}_z$ we have

$$\int_{\mathcal{O}}|\,\mathrm{curl}(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z \leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|^2_{H^1(\mathcal{O})} + \|\tilde{\mathbf{R}}\|^2_{H^1(\mathcal{O})}\right) + \varepsilon\int_{\mathcal{O}}|\nabla(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z. \tag{3.27}$$

We now estimate the divergence part. For any vector $\mathbf{U} = (U_1, U_2, U_3)$ we have

$$\lambda \operatorname{div}_x \mathbf{U} = \frac{1}{\sqrt{\tilde{G}}}\Big(\frac{\partial}{\partial z_1}\Big(\sqrt{\tilde{G}_{22}\tilde{G}_{33}}\tilde{U}_1\Big) + \frac{\partial}{\partial z_2}\Big(\sqrt{\tilde{G}_{11}\tilde{G}_{33}}\tilde{U}_2\Big) + \frac{\partial}{\partial z_3}\Big(\sqrt{\tilde{G}_{11}\tilde{G}_{22}}\tilde{U}_3\Big)\Big),$$

where $\operatorname{div}_x \mathbf{u}$ denotes the divergence of a vector $\mathbf{u}$ with respect to the variables x. From system (3.8) we see that

$$\operatorname{div}_x \mathbf{R} = \operatorname{div}_x \mathbf{b}.$$

Denote

$$\mathfrak{D}\mathrm{iv}_z\tilde{\mathbf{R}} := \frac{\partial}{\partial z_1}\left(\sqrt{\tilde{G}_{22}\tilde{G}_{33}}\tilde{R}_1\right) + \frac{\partial}{\partial z_2}\left(\sqrt{\tilde{G}_{11}\tilde{G}_{33}}\tilde{R}_2\right) + \frac{\partial}{\partial z_3}\left(\sqrt{\tilde{G}_{11}\tilde{G}_{22}}\tilde{R}_3\right).$$

Then it follows that

$$\begin{aligned}\int_{\mathcal{O}} |\operatorname{div}(\eta\delta_\sigma\tilde{\mathbf{R}})|^2 \mathrm{d}z &\leqq C\int_{\mathcal{O}}\left(|\eta\delta_\sigma\tilde{\mathbf{R}}|^2 + |\mathfrak{D}\mathrm{iv}_z(\delta_\sigma\tilde{\mathbf{R}})|^2 + \frac{\varepsilon}{C}|\nabla(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\right)\mathrm{d}z \\ &\leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|^2_{H^2(\mathcal{O})} + \|\tilde{\mathbf{R}}\|^2_{H^1(\mathcal{O})}\right) + \varepsilon\int_{\mathcal{O}}|\nabla(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z.\end{aligned} \tag{3.28}$$

Applying the L^p estimate of the gradient of vector fields, and then from (3.27) and (3.28), we obtain that

$$\begin{aligned}\int_{\mathcal{O}} |\nabla(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z &\leqq C(\Omega,\Lambda)\left(\int_{\mathcal{O}}|\operatorname{div}(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z + \int_{\mathcal{O}}|\operatorname{curl}(\eta\delta_\sigma\tilde{\mathbf{R}})|^2\mathrm{d}z\right) \\ &\leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|^2_{H^2(\mathcal{O})} + \|\tilde{\mathbf{R}}\|^2_{H^1(\mathcal{O})}\right).\end{aligned}$$

This yields that

$$\left\|\delta_\sigma(\nabla(\eta\tilde{\mathbf{R}}))\right\|_{L^2(\mathcal{O})} \leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|_{H^2(\mathcal{O})} + \|\tilde{\mathbf{R}}\|_{H^1(\mathcal{O})}\right).$$

It follows from [8, Lemma 7.24] that

$$\left\|\nabla_i(\nabla(\eta\tilde{\mathbf{R}}))\right\|_{L^2(\mathcal{O})} \leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|_{H^2(\mathcal{O})} + \|\tilde{\mathbf{R}}\|_{H^1(\mathcal{O})}\right).$$

Therefore, we have

$$\sum_{i=1}^{2}\left\|\nabla_i(\nabla\tilde{\mathbf{R}})\right\|_{L^2(B^+_{5/4})} \leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|_{H^2(\mathcal{O})} + \|\tilde{\mathbf{R}}\|_{H^1(\mathcal{O})}\right). \tag{3.29}$$

To obtain the estimate for the normal derivative of $\nabla\tilde{\mathbf{R}}$ we shall apply the equations. Rewrite system (3.13) as follows:

$$\begin{cases} \left[-\tilde{a}(z) - \tilde{h}(z)\tilde{M}_2^2\right]\partial_{33}\tilde{R}_1 + \tilde{h}(z)\tilde{M}_1\tilde{M}_2\partial_{33}\tilde{R}_2 = f, \\ \tilde{h}(z)\tilde{M}_1\tilde{M}_2\partial_{33}\tilde{R}_1 + \left[-\tilde{a}(z) - \tilde{h}(z)\tilde{M}_1^2\right]\partial_{33}\tilde{R}_2 = g, \end{cases} \tag{3.30}$$

where f and g are linear combinations of $\nabla\tilde{\mathbf{R}}$, $\nabla\nabla_1\tilde{\mathbf{R}}$, $\nabla\nabla_2\tilde{\mathbf{R}}$, $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{b}}$. Solving $\partial_{33}\tilde{R}_1$ and $\partial_{33}\tilde{R}_2$ from (3.30), and then applying (3.29), we have

$$\sum_{i=1}^{2}\|\partial_{33}\tilde{R}_i\|_{L^2(B^+_{5/4})} \leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|_{H^2(\mathcal{O})} + \|\tilde{\mathbf{R}}\|_{H^1(\mathcal{O})}\right), \tag{3.31}$$

where we have used

$$\left(\tilde{a}(z) + \tilde{h}(z)\tilde{M}_2^2\right)\left(\tilde{a}(z) + \tilde{h}(z)\tilde{M}_1^2\right) - \tilde{h}^2(z)\tilde{M}_1^2\tilde{M}_2^2 \geqq 1.$$

Using $\operatorname{div}\mathbf{R} = \operatorname{div}\mathbf{b}$ to solve $\partial_{33}\tilde{R}_3$, and then applying the estimate in (3.29) and (3.31), we obtain that

$$\|\nabla^2\tilde{\mathbf{R}}\|_{L^2(B^+_{5/4})} \leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|_{H^2(\mathcal{O})} + \|\tilde{\mathbf{R}}\|_{H^1(\mathcal{O})}\right).$$

From (3.3) and (3.12), it follows that

$$\|\nabla^2\tilde{\mathbf{R}}\|_{L^2(B^+_{5/4})} \leqq C(\Omega,\Lambda)\lambda^{3/2}.$$

This lemma is completed. □

At last, we show the C^1 estimate for $\tilde{\mathbf{R}}$ on a half ball $\overline{B_1^+}$ by solving a regular oblique derivative problem.

Theorem 3.3. *Let* $\tilde{\mathbf{R}}$ *be the solution of system* (3.13) *with the coefficients satisfying* (3.14). *Then we have the estimate*

$$\|\tilde{\mathbf{R}}\|_{C^1(\overline{B_1^+})} \leqq C\lambda^{3/2}, \tag{3.32}$$

where the constant C *depends only on* Ω *and* Λ, *but not on* λ.

Proof. Applying the Sobolev imbedding theorem ([1, Lemma 5.17]), for any β with $0 < \beta < 1/2$ we have

$$\|\tilde{\mathbf{R}}\|_{C^\beta(B^+_{5/4})} \leqq C\|\tilde{\mathbf{R}}\|_{H^2\left(B^+_{5/4}\right)},$$

where the constant C depends on β. In the following, we let β ($\beta < \alpha$) be fixed and let $\Omega_1 = B^+_{5/4}$. We now consider the following div-curl problem:

$$\operatorname{curl}\tilde{\mathbf{B}} = \tilde{\mathbf{q}}, \quad \operatorname{div}\tilde{\mathbf{B}} = 0 \quad \text{in } \Omega_1, \quad \nu\cdot\tilde{\mathbf{B}} = 0 \quad \text{on } \partial\Omega_1,$$

where

$$\tilde{\mathbf{q}} = \left(\sqrt{\tilde{G}_{22}\tilde{G}_{33}} \left(\tilde{b}_1 - \tilde{R}_1 \right), \quad \sqrt{\tilde{G}_{11}\tilde{G}_{33}} \left(\tilde{b}_2 - \tilde{R}_2 \right), \quad \sqrt{\tilde{G}_{11}\tilde{G}_{22}} \left(\tilde{b}_3 - \tilde{R}_3 \right) \right).$$

Then we have the $C^{1,\beta}$ estimate

$$\|\tilde{\mathbf{B}}\|_{C^{1,\beta}(\Omega_1)} \leqq C \left(\|\tilde{\mathbf{b}}\|_{C^{\beta}(\Omega_1)} + \|\tilde{\mathbf{R}}\|_{C^{\beta}(\Omega_1)} \right)$$

and the H^1 estimate

$$\|\tilde{\mathbf{B}}\|_{H^1(\Omega_1)} \leqq C \left(\|\tilde{\mathbf{b}}\|_{L^2(\Omega_1)} + \|\tilde{\mathbf{R}}\|_{L^2(\Omega_1)} \right). \tag{3.33}$$

Denote

$$\tilde{\mathbf{P}} = \tilde{a}(z)\,\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{R}} + \tilde{h}(z)\langle \tilde{\mathbf{M}}, \mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{R}}\rangle \tilde{\mathbf{M}}.$$

Then we can write (3.13) by

$$\mathrm{curl} \left((\tilde{P}_1\sqrt{\tilde{G}_{11}}, \tilde{P}_2\sqrt{\tilde{G}_{22}}, \tilde{P}_3\sqrt{\tilde{G}_{33}}) - \tilde{\mathbf{B}} \right) = 0 \quad \text{in } \Omega_1.$$

By global *Poincaré* lemma in bounded domains, there exists $\phi \in H^2(\Omega_1)$ satisfying $\int_{\Omega_1} \phi \mathrm{d}x = 0$ such that

$$\left(\tilde{P}_1\sqrt{\tilde{G}_{11}}, \tilde{P}_2\sqrt{\tilde{G}_{22}}, \tilde{P}_3\sqrt{\tilde{G}_{33}} \right) - \tilde{\mathbf{B}} = \nabla\phi \quad \text{in } \Omega_1. \tag{3.34}$$

By Sobolev imbedding theorem we obtain that

$$\|\phi\|_{C^{\beta}(\Omega_1)} \leqq C\|\phi\|_{H^2(\Omega_1)} \leqq C\|\nabla\phi\|_{H^1(\Omega_1)}.$$

Based on the estimates on $\tilde{\mathbf{R}}$ (Theorem 3.2) and on $\tilde{\mathbf{B}}$ (see (3.33)), we thus get

$$\|\phi\|_{C^{\beta}(\Omega_1)} \leqq C \left(\|\tilde{\mathbf{R}}\|_{H^2(\Omega_1)} + \|\tilde{\mathbf{B}}\|_{H^1(\Omega_1)} \right).$$

From (3.34), we can calculate

$$\mathrm{curl} \left(\tilde{R}_1\sqrt{\tilde{G}_{11}}, \tilde{R}_2\sqrt{\tilde{G}_{22}}, \tilde{R}_3\sqrt{\tilde{G}_{33}} \right) = (J_1, J_2, J_3),$$

where

$$J_1 = \frac{\left(\tilde{a} + \tilde{h}\tilde{M}_2^2 + \tilde{h}\tilde{M}_3^2 \right) \psi_1 - \tilde{h}\tilde{M}_1\tilde{M}_2\psi_2 - \tilde{h}\tilde{M}_1\tilde{M}_3\psi_3}{\tilde{a}^2 \left(\tilde{a} + \tilde{h}|\tilde{\mathbf{M}}|^2 \right) \sqrt{\tilde{G}_{11}}},$$

$$J_2 = \frac{\left(\tilde{a} + \tilde{h}\tilde{M}_1^2 + \tilde{h}\tilde{M}_3^2 \right) \psi_2 - \tilde{h}\tilde{M}_1\tilde{M}_2\psi_1 - \tilde{h}\tilde{M}_2\tilde{M}_3\psi_3}{\tilde{a}^2 \left(\tilde{a} + \tilde{h}|\tilde{\mathbf{M}}|^2 \right) \sqrt{\tilde{G}_{22}}},$$

$$J_3 = \frac{\left(\tilde{a} + \tilde{h}\tilde{M}_1^2 + \tilde{h}\tilde{M}_2^2 \right) \psi_3 - \tilde{h}\tilde{M}_1\tilde{M}_3\psi_1 - \tilde{h}\tilde{M}_2\tilde{M}_3\psi_2}{\tilde{a}^2 \left(\tilde{a} + \tilde{h}|\tilde{\mathbf{M}}|^2 \right) \sqrt{\tilde{G}_{33}}} \tag{3.35}$$

with

$$\psi_1 = \left(\phi_{x_1} + B_1\right)\sqrt{\tilde{G}_{22}\tilde{G}_{33}}, \quad \psi_2 = \left(\phi_{x_2} + B_2\right)\sqrt{\tilde{G}_{11}\tilde{G}_{33}},$$
$$\psi_3 = \left(\phi_{x_3} + B_3\right)\sqrt{\tilde{G}_{11}\tilde{G}_{22}}.$$

Plugging these formulas into the equation

$$\operatorname{div}(J_1, J_2, J_3) = 0 \quad \text{in } \Omega_1. \tag{3.36}$$

Then we can see that the Equation (3.36) is a uniformly elliptic equation with variable coefficients with respect to the unknown function ϕ.

We now consider the boundary condition. From the boundary conditon $\tilde{\mathbf{R}}_T = 0$ on $\mathcal{T}$, we have

$$J_3 = 0 \quad \text{on } \mathcal{T}.$$

Rewrite the above boundary condition with the form

$$\sum_{i=1}^{3} \beta_i(x) \cdot D_i\phi = \varphi(x) \quad \text{on } \mathcal{T}. \tag{3.37}$$

Then we can find that the normal component $\beta_3(x)$ of the vector $(\beta_1, \beta_2, \beta_3)$ satisfies

$$|\beta_3(z)| \geqq \gamma_0 \quad \text{for } z \in \mathcal{T}$$

for some positive constant γ_0, and the function φ has the estimate

$$\|\varphi\|_{C^{1,\beta}(\partial\Omega_1 \bigcap \mathcal{T})} \leqq C\|\tilde{\mathbf{B}}\|_{C^{1,\beta}(\partial\Omega_1 \bigcap \mathcal{T})}.$$

Therefore, the equation (3.36) with the boundary condition (3.37) is a regular oblique derivative problem. Theorem 6.30 in [8] is applicable in Ω_1, and we get the local $C^{2,\beta}$ estimate for ϕ:

$$\|\phi\|_{C^{2,\beta}(\overline{B^+_{9/8}})} \leqq C(\Lambda)\left(\|\phi\|_{C^0(\Omega_1)} + \|\tilde{\mathbf{B}}\|_{C^{1,\beta}(\overline{\Omega_1})}\right).$$

From (3.35), it follows that curl $\tilde{\mathbf{R}} \in C^{1,\beta}(\overline{B^+_{9/8}})$ with the estimate

$$\|\operatorname{curl}\tilde{\mathbf{R}}\|_{C^{1,\beta}(\overline{B^+_{9/8}})} \leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|_{H^2(\Omega_1)} + \|\tilde{\mathbf{R}}\|_{H^2(\Omega_1)}\right).$$

From this estimate and noting that div $\mathbf{R} =$ div $\mathbf{b}$, we get the C^1 estimate

$$\begin{aligned}\|\tilde{\mathbf{R}}\|_{C^1\left(\overline{B^+_1}\right)} &\leqq C\left(\|\operatorname{curl}\tilde{\mathbf{R}}\|_{C^{1,\beta}\left(\overline{B^+_{9/8}}\right)} + \|\operatorname{div}\tilde{\mathbf{R}}\|_{C^1\left(\overline{B^+_{9/8}}\right)} + \|\tilde{\mathbf{R}}\|_{C^\beta\left(\overline{B^+_{9/8}}\right)}\right)\\ &\leqq C(\Omega,\Lambda)\left(\|\tilde{\mathbf{b}}\|_{H^2(\Omega_1)} + \|\tilde{\mathbf{b}}\|_{C^2(\Omega_1)} + \|\tilde{\mathbf{R}}\|_{H^2(\Omega_1)}\right).\end{aligned}$$

This gives

$$\|\tilde{\mathbf{R}}\|_{C^1(\overline{B^+_1})} \leqq C\lambda^{3/2},$$

since the inequalities (3.11) and (3.12). This theorem is proved. □

Therefore, from Theorem 3.3 the inequality (1.6) follows immediately.

4. Proof of Theorem 1.1

Before proving Theorem 1.1, we first calculate $u_1'(0)$ and $u_2'(0)$ which will be used later, where u_1 and u_2 are the solutions of (2.24) and (2.25) respectively. Set $u_3 = u_1 - u_2 + f$. Then from (2.24)and (2.25) we have the following ODE:

$$\begin{cases} \partial_3\left(\frac{\partial_3 u_3}{1-3f^2}\right) - u_3 = 0, & y_3 > 0, \\ u_3(0) = f(0), & u_3(\infty) = 0. \end{cases}$$

We can obtain the solution

$$u_3(z_3) = \frac{1}{\left(1 - f^2(0)\right)}\left(1 - f^2(z_3)\right) f(z_3).$$

Thus we have

$$u_1'(0) - u_2'(0) = \frac{-2f^2(0)}{1 - f^2(0)} f'(0).$$

We now solve $u_2'(0)$. Let

$$g(z_3) = \frac{\partial_3 u_2}{1 - 3f^2}.$$

Then from (2.25), it follows that

$$g'' - (1 - 3f^2)g = -f', \quad z_3 > 0.$$

Using $f'' = (1 - f^2)f$, we have

$$\left(f'g' - f''g\right)' = f'g'' - f'''g = -f'^2.$$

Integrating from 0 to ∞ on both sides with respect to the variable z_3, then noting that $g'(0) = -f(0)$ we can conclude that

$$g(0) = \frac{1}{f''(0)}\left(-f(0)f'(0) - \int_0^\infty f'^2 dz_3\right).$$

This gives that

$$u_2'(0) = \frac{1 - 3f^2(0)}{f''(0)}\left(-f(0)f'(0) - \int_0^\infty f'^2 dz_3\right).$$

Note that from the solution of (2.11) we have

$$\int_0^\infty f'^2 dz_3 = \frac{1}{3}\left(2 + \frac{2f'(0)}{f(0)} - f(0)f'(0)\right), \tag{4.1}$$

see the proof in Appendix C. Then we obtain that

$$u_2'(0) = \frac{1 - 3f^2(0)}{3f''(0)}\left(-2f(0)f'(0) - 2 - \frac{2f'(0)}{f(0)}\right) \tag{4.2}$$

and

$$u_1'(0) + f'(0) = \frac{1 - 3f^2(0)}{3f''(0)} \left(f(0)f'(0) - 2 - \frac{2f'(0)}{f(0)} \right). \tag{4.3}$$

By simple calculations for (4.2) and (4.3) or by the maximum principle for the equations (2.24) and (2.25), we can show that

$$u_1'(0) + f'(0) < 0, \quad u_2'(0) > 0.$$

We now begin to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\mathbf{H}_\lambda$ be the solution of system (1.4). From Proposition 3.4 in [12] obtained by Monneau, we see that the function $|\lambda \operatorname{curl} \mathbf{H}_\lambda|$ obtains its maximum only on $\partial\Omega$. Therefore, we need to take the asymptotic expansion for $\mathbf{H}_\lambda$ near the boundary. Let $\tilde{\mathbf{H}}$ be the representation of $\mathbf{H}_\lambda$ under the z-coordinate system (see Section 2). Then for any x belonging to the neighborhood of the boundary, from Theorem 3.3, under the z-coordinate system it follows that

$$\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{H}} = \mathscr{C}\mathrm{url}_z \left(\mathscr{H}(z_1, z_2, z_3) + \lambda \mathscr{W}(z_1, z_2, z_3)\right) + O\left(\lambda^{3/2}\right),$$

where $\mathscr{H}$ and $\mathscr{W}$ are defined by (2.15) and (2.28) with $(z_1, z_2, 0) = X_0$ respectively. From (6.10) in Appendix, we have

$$\begin{aligned}
&|\mathscr{C}\mathrm{url}_z \left(\mathscr{H}(z_1, z_2, z_3) + \lambda \mathscr{W}(z_1, z_2, z_3)\right)|^2 \\
&\quad = |\operatorname{curl} \mathbf{H}_0|^2 + 2\lambda \left(\operatorname{curl} \mathbf{H}_0 \cdot \operatorname{curl} \mathbf{w} - \kappa_2 \partial_3 H_2^0 H_2^0 - \kappa_1 \partial_3 H_1^0 H_1^0\right) + O\left(\lambda^2\right),
\end{aligned}$$

where $\mathbf{H}_0$ is defined by (2.12) with $(z_1, z_2, 0) = X_0$. Therefore, we have

$$\begin{aligned}
\left|\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{H}}\right|^2 &= |\operatorname{curl} \mathbf{H}_0|^2 + 2\lambda\left(\operatorname{curl} \mathbf{H}_0 \cdot \operatorname{curl} \mathbf{w} - \kappa_2 \partial_3 H_2^0 H_2^0 - \kappa_1 \partial_3 H_1^0 H_1^0\right) \\
&\quad + O\left(\lambda^{3/2}\right).
\end{aligned}$$

In particular, for any $x \in \partial\Omega$ we obtain that, by (2.13) and (2.26),

$$|\lambda \operatorname{curl} \mathbf{H}_\lambda(x)|^2 = \left|\mathscr{C}\mathrm{url}_z\, \tilde{\mathbf{H}}\right|^2 = f_x''(0)^2 - 2\lambda f_x''(0) m(x) + O(\lambda^{3/2}), \tag{4.4}$$

where the function f depending on X_0 is defined by (2.11) and the function $m(x)$ is defined by

$$\begin{aligned}
m(x) = &\left(cos^2\theta(x)\kappa_1(x) + sin^2\theta(x)\kappa_2(x)\right)\left(u_1{}'(0) + f_x'(0)\right) \\
&+ \left(cos^2\theta(x)\kappa_2(x) + sin^2\theta(x)\kappa_1(x)\right) u_2{}'(0)
\end{aligned}$$

with u_1 and u_2 being the solutions of (2.24) and (2.25) respectively.

Recall that

$$f_x''(0) = \left[1 - \left(1 - 2|\tilde{\mathbf{h}}|^2\right)^{1/2}\right]\left(1 - 2|\tilde{\mathbf{h}}|^2\right) > 0,$$

where $\tilde{\mathbf{h}}$ is defined in (2.10). Then from (4.4) we see that, as $\lambda \to 0$, the position where the maximum of $|\lambda \operatorname{curl} \mathbf{H}_\lambda|$ is attained must approach the points in $\partial\Omega(\mathcal{H}_T^e)$ defined by (1.3). This result has been proved earlier by Bates and Pan in [3].

For precisely, we need to check the second term in the right side of (4.4). It is obvious to see that if $x \in \partial\Omega(\mathcal{H}_T^e)$ then the minimum points of $m(x)$ correspond to the maximum points of $|\lambda \operatorname{curl} \mathbf{H}_\lambda|$ for small λ. Define

$$S = \left\{ x \in \partial\Omega\left(\mathcal{H}_T^e\right) \ : \ \min_{\tilde{x} \in \partial\Omega(\mathcal{H}_T^e)} m(\tilde{x}) = m(x) \right\}. \tag{4.5}$$

From the formula (1.8), we see that the maximum points of $|\mathbf{Q}|$ correspond to the maximum points of $|\lambda \operatorname{curl} \mathbf{H}_\lambda|$. Therefore, the maximum points of $|\mathbf{Q}|$ must be around the points in the set S for small λ. This theorem is proved. □

In the special case of Theorem 1.1, we consider a type-II superconductor sample in the shape of an ellipsoid.

Example 4.1. Let $\mathcal{H}^e = (0, 0, h)$ with the constant h satisfying $|h| < \sqrt{5/18}$. Suppose that the domain is an ellipsoid:

$$\Omega : \quad \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{a^2} < 1, \quad a > b > 0.$$

Then the maximum points of $|\mathbf{Q}|$ must approach, as $\lambda \to 0$, the points in the set

$$S_b = \{(0, b, 0), (0, -b, 0)\}.$$

To see this, from [3], it follows that

$$\partial\Omega(\mathcal{H}_T^e) = \left\{ x \in \partial\Omega \ : \ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, x_3 = 0 \right\}.$$

Let $\partial\Omega(\mathcal{H}_T^e)$ be the principal direction corresponding to $\kappa_1(x)$. Using the polar-coordinate transformation: $x_1 = a\cos\alpha$, $x_2 = b\sin\alpha$, for any point $x \in \partial\Omega(\mathcal{H}_T^e)$ we have

$$\kappa_1(x(\alpha)) = \frac{ab}{(a^2\sin^2\alpha + b^2\cos^2\alpha)^{\frac{3}{2}}}, \quad \kappa_2(x(\alpha)) = \frac{ab}{a^2(a^2\sin^2\alpha + b^2\cos^2\alpha)^{\frac{1}{2}}}.$$

Thus, we have

$$m(x(\alpha)) = \kappa_2(x(\alpha))\left(u_1{}'(0) + f_h{}'(0)\right) + \kappa_1(x(\alpha))u_2{}'(0).$$

We can calculate that the function $m(x(\alpha))$ is monotone decreasing with respect to α in $[0, \pi/2]$ for any h satisfying $|h| < \sqrt{5/18}$. Therefore, from Theorem 1.1 we obtain the set S_b.

5. Proof of Theorem 1.2

In this section we prove that the solution $\mathbf{H}_\lambda$ to system (1.4) in the normal direction away from the boundary decays exponentially for small λ. But before that, we first show that the solution away from the boundary is uniformly bounded in the norm of C^3_λ (see (5.2)) and can be arbitrarily small.

Denote

$$\Omega_n := \{x \in \Omega : \operatorname{dist}(x, \Omega) > n\lambda\}. \tag{5.1}$$

Define the norm of C^k_λ on $\overline{\Omega_n}$ (for some n) by

$$\|\mathbf{u}\|_{C^k_\lambda(\overline{\Omega_n})} := \|\mathbf{u}\|_{C^0(\overline{\Omega_n})} + \sum_{i=1}^{k} \lambda^i \|D^i \mathbf{u}\|_{C^0(\overline{\Omega_n})}, \quad k = 1 \ldots 3. \tag{5.2}$$

We begin to establish the $C^3_\lambda\left(\overline{\Omega_2}\right)$ estimate.

Lemma 5.1. *Let* $\mathbf{H}$ *be the solution of system* (1.4)*. Then we have*

$$\|\mathbf{H}\|_{C^3_\lambda(\overline{\Omega_2})} \leqq C\left(\|\mathcal{H}^e_T\|_{C^0(\partial\Omega)}\right),$$

where the constant C *depends on* Ω *and* $\mathcal{H}^e_T$*, but not on* λ*.*

Proof. Let $\check{\mathbf{H}}(t) = \mathbf{H}(\lambda t)$, and let

$$\Omega_\lambda = \{t \; : \; t = x/\lambda, \quad x \in \Omega\}. \tag{5.3}$$

Denote

$$\omega_n := \left\{t \in \Omega_\lambda : \operatorname{dist}(t, \partial\Omega_\lambda) \geqq n\right\}. \tag{5.4}$$

The proof is straightforward, we here give the outline.

Step 1 Interior H^1 estimate. This follows from Lemma 8.1 in [3] that for any unit ball $B_1 \in \omega_1$ we have

$$\left\|\check{\mathbf{H}}\right\|_{H^1(B_1)} \leqq C, \tag{5.5}$$

where the constant C depends only on Ω and $\mathcal{H}^e_T$, but not on λ.

Step 2 Interior H^2 estimate. The proof is similar to that of Theorem 4.1 in [3] and of Lemma A.4 in [19]. We can get that

$$\left\|\check{\mathbf{H}}\right\|_{H^2(B_{1/2})} \leqq C\left\|\check{\mathbf{H}}\right\|_{H^1(B_1)}.$$

Step 3 From step 2 and step 3 in the proof of Theorem 2.1 in [19], it follows that

$$\|\check{\mathbf{H}}\|_{C^3(B_{1/32})} \leqq C(\|\check{\mathbf{H}}\|_{H^2(B_{1/2})}).$$

For any ball $B_{1/32} \in \omega_2$ the above inequality always holds, and hence

$$\left\|\check{\mathbf{H}}\right\|_{C^3(\omega_2)} \leqq C,$$

where the constant C depends on Ω and $\mathcal{H}^e_T$. From the inequalities in step 1-step 3, we obtain this lemma by the scaling argument. □

Let $\mathbf{R}$ be defined by (3.4). Then by Lemma 5.1 the coefficients in the system (3.8) satisfy, for some positive constant Λ,

$$\begin{gathered} a(x) \geqq 1; \quad h(x) \geqq 0; \\ \|a\|_{C^2_\lambda(\overline{\Omega_2})}, \quad \|h\|_{C^2_\lambda(\overline{\Omega_2})}, \quad \|\mathbf{M}\|_{C^2_\lambda(\overline{\Omega_2})} \leqq \Lambda. \end{gathered} \tag{5.6}$$

From (3.3), we have

$$\|\mathbf{b}\|_{C^2_\lambda(\overline{\Omega_2})} \leqq C\lambda^3. \tag{5.7}$$

We now show that the solution of system (3.8) can be arbitrarily small in the norm of C^2_λ if λ is sufficiently small.

Lemma 5.2. *Let* $\mathbf{R}$ *be the solution of system* (3.8). *Then there exists a constant* C *depending on* Ω *and* Λ, *but not on* λ *such that*

$$\|\mathbf{R}\|_{C^2_\lambda(\overline{\Omega_4})} \leqq C\lambda^{3/2}, \tag{5.8}$$

where Ω_4 *is defined in* (5.4).

Proof. The proof is similar to that of Lemma 5.1. Let $\check{\mathbf{R}}(t) = \mathbf{R}(\lambda t)$. From Lemma 3.1, by the scaling argument we have the H^1 estimate for $\check{\mathbf{R}}$:

$$\|\check{\mathbf{R}}\|_{H^1(\Omega_\lambda)} \leqq C\lambda^{3/2},$$

where the constant C depends on Ω and $\mathcal{H}^e_T$. Then by the difference-quotient technique, for any unit ball $B_1 \in \omega_3$ we get the H^2 estimate

$$\|\check{\mathbf{R}}\|_{H^2(B_1)} \leqq C\lambda^{3/2},$$

where ω_3 is defined by (5.4). Similar to the proof of Theorem 3.3 where treated the boundary estimate (or by the Schauder estimate for elliptic systems), one can get the interior estimate

$$\|\check{\mathbf{R}}\|_{C^2(\overline{\omega_4})} \leqq C\lambda^{3/2}.$$

This completes the proof by the scaling argument. □

At last, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. From (3.4) and Lemma 5.2, there exist constants $\alpha \in (\beta, 1)$, N and λ_1 such that for any $x \in \Omega_N$ and $\lambda \in (0, \lambda_1)$ we have

$$\alpha^2 < |F\left(|\lambda \operatorname{curl} \mathbf{H}|^2\right)(x)|^{-1},$$

where Ω_N is defined by (5.1).

Taking the inner product of (1.4) by $\eta^2\mathbf{H}$ for any $\eta \in \mathbf{H}^1_0(\Omega)$, we obtain

$$\begin{aligned} &\int_\Omega \left(\lambda^2 F(|\lambda \operatorname{curl} \mathbf{H}|^2)|\operatorname{curl}(\eta\mathbf{H})|^2 + |\eta\mathbf{H}|^2\right) \mathrm{d}x \\ &= \lambda^2 \int_\Omega F(|\lambda \operatorname{curl} \mathbf{H}|^2)|\nabla\eta \times \mathbf{H}|^2 \mathrm{d}x. \end{aligned}$$

Then one sets

$$\eta(x) = \zeta(x) e^{\alpha \frac{d(x,\partial\Omega)}{\lambda}},$$

where $\zeta \in C_0^\infty(\Omega, [0, 1])$ is a cutoff function satisfying

$\zeta(x) = 1$ if $\mathrm{d}(x, \partial\Omega) > (N+1)\lambda$; $\zeta(x) = 0$ if $d(x, \partial\Omega) < N\lambda$; and $|\nabla\zeta| \leqq 2/\lambda$.

Combining with (5.5), we thus get

$$\int_\Omega e^{2\alpha \frac{d(x,\partial\Omega)}{\lambda}} |\mathbf{H}|^2 \mathrm{d}x \leqq C, \tag{5.9}$$

where the constant C depends on Ω and $\mathcal{H}_T^e$, but not on λ.

Let $\mathbf{A} = e^{\alpha \frac{d(x,\partial\Omega)}{\lambda}} \mathbf{H}$. Then $\mathbf{A}$ satisfies

$$\begin{aligned} -\lambda^2 \operatorname{curl}[b(x) \operatorname{curl} \mathbf{A}] + \lambda \operatorname{curl}[b(x)\mathbf{c}(x) \times \mathbf{A}] \\ + b(x)\mathbf{c}(x) \times [\lambda \operatorname{curl} \mathbf{A} - \mathbf{c}(x) \times \mathbf{A}] = \mathbf{A} \quad \text{in } \Omega \end{aligned}$$

with the compatibility condition

$$\lambda \operatorname{div} \mathbf{A} = g(x) \quad \text{in } \Omega$$

and the boundary condition

$$\mathbf{A}_T = \mathcal{H}_T^e \quad \text{on } \partial\Omega,$$

where

$$b(x) = F(|\lambda \operatorname{curl} \mathbf{H}|^2), \quad \mathbf{c}(x) = \alpha\nabla d(x, \partial\Omega), \quad g(x) = \mathbf{c}(x) \cdot \mathbf{A}.$$

This is an elliptic boundary value problem in the sense of Agmon-Douglis-Nirenberg [2]. Let $\check{\mathbf{A}}(t) = \mathbf{A}(\lambda t)$. Then from [2, Theorem 10.5] we have

$$\|\check{\mathbf{A}}\|_{H^2(\omega_5)} \leqq C\|\check{\mathbf{A}}\|_{L^2(\omega_0)} \leqq C\lambda^{-3/2},$$

where ω_n is defined by (5.4). Therefore, $\check{\mathbf{A}} \in C^\gamma(\omega_5)$ with $0 < \gamma < 1/2$. Then by the Schauder estimate ([2, Theorem 10.7]), we obtain that

$$\|\check{\mathbf{A}}\|_{C^2(\omega_6)} \leqq C\|\check{\mathbf{A}}\|_{C^\gamma(\omega_5)}.$$

Hence,

$$\|\mathbf{A}\|_{C^2_\lambda(\overline{\Omega_6})} \leqq C\lambda^{-3/2}.$$

This gives that

$$\lambda|\operatorname{curl} \mathbf{H}(x)| \leqq C\lambda^{-3/2} e^{-\alpha \frac{d(x,\partial\Omega)}{\lambda}} \qquad \text{for } x \in \Omega_6.$$

Therefore, there exists $\lambda_0 \leqq \lambda_1$ such that for any $\lambda \in (0, \lambda_0)$ we have

$$\lambda|\operatorname{curl} \mathbf{H}(x)| \leqq Ce^{-\beta \frac{d(x,\partial\Omega)}{\lambda}} \quad \text{for } x \in \bar{\Omega}.$$

This completes the proof since the equality (1.8). □

Remark 5.3. We need to mention that the proof of Theorem 1.2 based on the Agmon estimates is completely given by the referee, and the decay rate is optimal. In Appendix D, we will give another proof by applying the comparison principle, but we only obtain the decay rate $\beta < 1/\sqrt{2}$.

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Appendix A. Derivation of System (2.18)

In this section we shall derive the system (2.18). The analysis is based on the knowledge of differential geometry and the asymptotic expansions. For the local coordinate expansions we refer to Section 3 in [14].

Let $X_0 \in \partial\Omega$ be fixed. We now consider the problem in the neighborhood $\mathcal{U}$ of X_0. Note that the system (1.4) is invariant to the rotation of coordinate. Thus we may assume that the unit inward normal vector of $\partial\Omega$ at X_0 is $\mathbf{k} = (0, 0, 1)$, one of the principle direction at the point X_0 is $\mathbf{e}_1 = \mathbf{i} = (1, 0, 0)$, the other principle direction is $\mathbf{e}_2 = \mathbf{j} = (0, 1, 0)$. Let $\kappa_1(X_0)$ and $\kappa_2(X_0)$ be the principle curvatures of $\partial\Omega$ at X_0 corresponding to the principle direction $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively.

According to the notations defined in Section 2, we have

$$\mathbf{r}_1(y) = \left(1 + \Gamma^1_{1j} y_j\right)\mathbf{i} + \Gamma^2_{1j} y_j \mathbf{j} + \kappa_1 y_1 \mathbf{k} + O\left(y_1^2 + y_2^2\right),$$
$$\mathbf{r}_2(y) = \Gamma^1_{2j} y_j \mathbf{i} + \left(1 + \Gamma^2_{2j} y_j\right)\mathbf{j} + \kappa_2 y_2 \mathbf{k} + O\left(y_1^2 + y_2^2\right),$$

where $\kappa_1 = \kappa_1(X_0)$, $\kappa_2 = \kappa_2(X_0)$ and Γ^k_{ij} denote the value of the Christoffel symbols at X_0, the subscript j means the summation from 1 to 2. From (2.2) and (2.5), we see that

$$\tilde{G}_{11}(z) = 1 + 2\lambda\left(\Gamma^1_{1j} z_j - \kappa_1 z_3\right) + \mathfrak{R}_1\left(\lambda^2\right), \tag{6.1}$$
$$\tilde{G}_{22}(z) = 1 + 2\lambda\left(\Gamma^2_{2j} z_j - \kappa_2 z_3\right) + \mathfrak{R}_2\left(\lambda^2\right). \tag{6.2}$$

This gives that

$$\sqrt{\tilde{G}_{11}(z)} = 1 + \lambda\left(\Gamma^1_{1j} z_j - \kappa_1 z_3\right) + O\left(\lambda^2\right), \tag{6.3}$$
$$\sqrt{\tilde{G}_{22}(z)} = 1 + \lambda\left(\Gamma^2_{2j} z_j - \kappa_2 z_3\right) + O\left(\lambda^2\right). \tag{6.4}$$

We first take the formal asymptotic expansion for $\tilde{\mathbf{H}}_\lambda(z) = \hat{\mathbf{H}}_\lambda(\lambda z)$ (defined in (2.5)) with respect to the variables z_1 and z_2 at the point $(0, 0, z_3)$. By Taylor expansion for $\hat{\mathbf{H}}_0(y_1, y_2, z_3)$ (defined by (2.15)) at the point $(0, 0, z_3)$, we have

$$\hat{\mathbf{H}}_0(y_1, y_2, z_3) = \mathbf{H}_0 + y_1\partial_{y_1}\hat{\mathbf{H}}_0(0,0,z_3) + y_2\partial_{y_2}\hat{\mathbf{H}}_0(0,0,z_3) + \mathfrak{R}_0\left(|y_1^2 + y_2^2|\right). \tag{6.5}$$

Denote

$$\left(p_1^1, p_1^2, 0\right) := \partial_{y_1}\hat{\mathbf{H}}_0(0,0,z_3), \quad \left(p_2^1, p_2^2, 0\right) := \partial_{y_2}\hat{\mathbf{H}}_0(0,0,z_3),$$

and let

$$q_1 := w_1 + z_1 p_1^1 + z_2 p_2^1, \quad q_2 := w_2 + z_1 p_1^2 + z_2 p_2^2.$$

Then from the inner expansion (2.8), we have the following formal asymptotic expansion:

$$\hat{\mathbf{H}}_\lambda(\lambda z) = \left(H_1^0 + \lambda q_1, H_2^0 + \lambda q_2, \lambda w_3\right) + \mathfrak{R}_3\left(\lambda^2\right). \tag{6.6}$$

From the definition of (2.4) for $\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}$ with $y = \lambda z$, it follows that

$$\mathcal{Q}_1(\lambda z) = -\partial_3 H_2^0 + \lambda\left(\partial_2 w_3 - \partial_3 q_2 + \kappa_2 H_2^0\right) + \mathfrak{R}_4\left(\lambda^2\right), \tag{6.7}$$

$$\mathcal{Q}_2(\lambda z) = \partial_3 H_1^0 + \lambda\left(\partial_3 q_1 - \partial_1 w_3 - \kappa_1 H_1^0\right) + \mathfrak{R}_5\left(\lambda^2\right), \tag{6.8}$$

$$\mathcal{Q}_3(\lambda z) = \lambda\left(\partial_1 q_2 - \partial_2 q_1 + \Gamma_{21}^2 H_2^0 - \Gamma_{12}^1 H_1^0\right) + \mathfrak{R}_6\left(\lambda^2\right). \tag{6.9}$$

We introduce

$$\mathbf{v} = (q_1, q_2, w_3),$$

then we have

$$\mathrm{curl}\,\mathbf{H}_0 \cdot \mathrm{curl}\,\mathbf{v} = -\partial_3 H_2^0(\partial_2 w_3 - \partial_3 q_2) + \partial_3 H_1^0(\partial_3 q_1 - \partial_1 w_3).$$

From (6.3), we see that

$$|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2 - |\mathrm{curl}\,\mathbf{H}_0|^2 = 2\lambda\left(\mathrm{curl}\,\mathbf{H}_0 \cdot \mathrm{curl}\,\mathbf{v} - \kappa_2\partial_3 H_2^0 H_2^0 - \kappa_1\partial_3 H_1^0 H_1^0\right) + \mathfrak{R}_7\left(\lambda^2\right). \tag{6.10}$$

By applying the equality

$$\begin{aligned} &F\left(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\right) - F\left(|\mathrm{curl}\,\mathbf{H}_0|^2\right) \\ &= \int_0^1 F'\left(|\mathrm{curl}\,\mathbf{H}_0|^2 + t\left(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2 - |\mathrm{curl}\,\mathbf{H}_0|^2\right)\right)\mathrm{d}t\left(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2 \right.\\ &\left. -|\mathrm{curl}\,\mathbf{H}_0|^2\right), \end{aligned}$$

we thus obtain that

$$F\left(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\right) - F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right) = 2\lambda F'\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\left(\mathrm{curl}\,\mathbf{H}_0\cdot\mathrm{curl}\,\mathbf{v}\right.$$
$$\left. - \kappa_2\partial_3 H_2^0 H_2^0 - \kappa_1\partial_3 H_1^0 H_1^0\right) + \mathfrak{R}_8\left(\lambda^2\right). \tag{6.11}$$

We now take the formal asymptotic expansion for system (2.8) with respect to the parameter λ. Note that

$$F\left(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\right)\mathcal{Q}_3\left(\lambda z\right)\sqrt{\tilde{G}_{33}}$$
$$= \lambda F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\left(\partial_1 q_2 - \partial_2 q_1 + \Gamma_{21}^2 H_2^0 - \Gamma_{12}^1 H_1^0\right) + \mathfrak{R}_9\left(\lambda^2\right).$$

This gives that, for $i = 1, 2$,

$$\partial_i\left(F\left(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\right)\mathcal{Q}_3\left(\lambda z\right)\sqrt{\tilde{G}_{33}}\right) = \lambda\partial_i\left(F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\left(\partial_1 w_2 - \partial_2 w_1\right)\right)$$
$$+\mathfrak{R}_{10}^i\left(\lambda^2\right). \tag{6.12}$$

Denote

$$c_1 = 2F'\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\left(\kappa_2 H_2^0\partial_3 H_2^0 + \kappa_1 H_1^0\partial_3 H_1^0\right)\partial_3 H_1^0 + \kappa_1 F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)H_1^0,$$
$$c_2 = 2F'\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\left(\kappa_2 H_2^0\partial_3 H_2^0 + \kappa_1 H_1^0\partial_3 H_1^0\right)\partial_3 H_2^0 + \kappa_2 F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)H_2^0.$$

Then from (6.4), (6.8) and (6.11)

$$F\big(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\big)\mathcal{Q}_2\big(\lambda z\big)\sqrt{\tilde{G}_{22}} - F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\partial_3 H_1^0$$
$$= \lambda\big(2\partial_3 H_1^0 F'\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\,\mathrm{curl}\,\mathbf{H}_0\cdot\mathrm{curl}\,\mathbf{v} + F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(\partial_3 q_1 - \partial_1 w_3\big)$$
$$+ \partial_3 H_1^0 F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(\Gamma_{2j}^2 z_j - \kappa_2 z_3\big)\big) - \lambda c_1 + \mathfrak{R}_{11}\big(\lambda^2\big) \tag{6.13}$$

and from (6.3), (6.7) and (6.11)

$$F\big(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\big)\mathcal{Q}_1\big(\lambda z\big)\sqrt{\tilde{G}_{11}} - F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(-\partial_3 H_2^0\big)$$
$$= \lambda\big(2\big(-\partial_3 H_2^0\big)F'\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\,\mathrm{curl}\,\mathbf{H}_0\cdot\mathrm{curl}\,\mathbf{v} + F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(\partial_2 w_3 - \partial_3 q_2\big)$$
$$- \partial_3 H_2^0 F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(\Gamma_{1j}^1 z_j - \kappa_1 z_3\big)\big) + \lambda c_2 + \mathfrak{R}_{12}\big(\lambda^2\big). \tag{6.14}$$

These show that

$$\partial_1\big(F\big(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\big)\mathcal{Q}_2\big(\lambda z\big)\sqrt{\tilde{G}_{22}}\big)$$
$$= \lambda\partial_1\big(2\partial_3 H_1^0 F'\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\,\mathrm{curl}\,\mathbf{H}_0\cdot\mathrm{curl}\,\mathbf{v} + F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(\partial_3 q_1 - \partial_1 w_3\big)\big)$$
$$+ \lambda\Gamma_{21}^2\partial_3 H_1^0 F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big) + \mathfrak{R}_{13}\big(\lambda^2\big) \tag{6.15}$$

and

$$\begin{aligned}
&\partial_2\big(F\big(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\big)\mathcal{Q}_1\big(\lambda z\big)\sqrt{\tilde{G}_{11}}\big)\\
&= \lambda\partial_2\big(2\big(-\partial_3 H_2^0\big)F'\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\,\mathrm{curl}\,\mathbf{H}_0\cdot\mathrm{curl}\,\mathbf{v}\\
&\quad + F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(\partial_2 w_3-\partial_3 q_2\big)\big)\\
&\quad - \lambda\Gamma_{11}^1\partial_3 H_2^0 F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)+\mathfrak{R}_{14}\big(\lambda^2\big).
\end{aligned} \tag{6.16}$$

From (2.9), we have

$$\partial_3\left[F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\partial_3 H_1^0\right]=H_1^0,\quad \partial_3\left[F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\partial_3 H_2^0\right]=H_2^0.$$

By using the above equalities, then from (6.12) we have

$$\begin{aligned}
&\partial_3\left(F\left(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\right)\mathcal{Q}_2\left(\lambda z\right)\sqrt{\tilde{G}_{22}}\right)-H_1^0\\
&= \lambda\partial_3\big(2\partial_3 H_1^0 F'\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\mathrm{curl}\,\mathbf{H}_0\cdot\mathrm{curl}\,\mathbf{v}\\
&\quad + F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\left(\partial_3 q_1-\partial_1 w_3\right)\big)\\
&\quad + \lambda H_1^0\left(\Gamma_{2j}^2 z_j-\kappa_2 z_3\right)-\lambda b_1+\mathfrak{R}_{15}\left(\lambda^2\right)
\end{aligned} \tag{6.17}$$

and from (6.13) we have

$$\begin{aligned}
&\partial_3\big(F\big(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|^2\big)\mathcal{Q}_1\big(\lambda z\big)\sqrt{\tilde{G}_{11}}\big)+H_2^0\\
&= \lambda\partial_3\big(2\big(-\partial_3 H_2^0\big)F'\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\,\mathrm{curl}\,\mathbf{H}_0\cdot\mathrm{curl}\,\mathbf{v}\\
&\quad + F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(\partial_2 w_3-\partial_3 q_2\big)\big)\\
&\quad - \lambda H_2^0\big(\Gamma_{1j}^1 z_j-\kappa_1 z_3\big)+\lambda b_2+\mathfrak{R}_{16}\big(\lambda^2\big),
\end{aligned} \tag{6.18}$$

where the functions $b_1=b_1(z_3)$ and $b_2=b_2(z_3)$ are defined by (2.19).
Denote

$$(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)=\mathscr{C}\mathrm{url}_z\left[F\left(|\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}|\right)^2\mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{H}}\right].$$

We can obtain that, from (2.4), (6.12) and (6.17)

$$\begin{aligned}
\mathcal{P}_1 &= \lambda\partial_2\big(F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(\partial_1 w_2-\partial_2 w_1\big)\big)-\lambda\partial_3\big(F\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\big(\partial_3 q_1-\partial_1 w_3\big)\\
&\quad + 2\partial_3 H_1^0 F'\big(|\,\mathrm{curl}\,\mathbf{H}_0|^2\big)\,\mathrm{curl}\,\mathbf{H}_0\cdot\mathrm{curl}\,\mathbf{v}\big)-H_1^0-\lambda b_1+\mathfrak{R}_{17}\big(\lambda^2\big),
\end{aligned} \tag{6.19}$$

from (2.4), (6.12) and (6.18)

$$\begin{aligned}
\mathcal{P}_2 &= -\lambda\partial_1\left(F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\left(\partial_1 w_2-\partial_2 w_1\right)\right)\\
&\quad + \lambda\partial_3\big(F\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\left(\partial_2 w_3-\partial_3 q_2\right)\\
&\quad + 2\left(-\partial_3 H_2^0\right)F'\left(|\,\mathrm{curl}\,\mathbf{H}_0|^2\right)\mathrm{curl}\,\mathbf{H}_0\cdot\mathrm{curl}\,\mathbf{v}\big)-H_2^0+\lambda b_2+\mathfrak{R}_{18}\left(\lambda^2\right)
\end{aligned} \tag{6.20}$$

and from (2.4), (6.15) and (6.16)

$$\begin{aligned}\mathcal{P}_3 &= \lambda\partial_1\big(2\partial_3 H_1^0 F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\mathbf{v} + F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_3 q_1 - \partial_1 w_3\big)\big)\\ &\quad - \lambda\partial_2\big(2\big(-\partial_3 H_2^0\big)F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\mathbf{v}\\ &\quad + F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_2 w_3 - \partial_3 q_2\big)\big)\\ &\quad + \lambda\Gamma_{21}^2\partial_3 H_1^0 F\big(|\operatorname{curl}\mathbf{H}_0|^2\big) + \lambda\Gamma_{11}^1\partial_3 H_2^0 F\big(|\operatorname{curl}\mathbf{H}_0|^2\big) + \mathfrak{R}_{19}\big(\lambda^2\big).\end{aligned} \tag{6.21}$$

Therefore, at the point $(0, 0, z_3)$ we have

$$\begin{aligned}\mathcal{P}_1\big(0,0,z_3\big) &= \lambda\partial_2\Big(F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_1 w_2 - \partial_2 w_1\big)\Big)\\ &\quad - \lambda\partial_3\big(F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_3 w_1 - \partial_1 w_3\big)\\ &\quad + 2\partial_3 H_1^0 F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\mathbf{w}\big) - H_1^0 - \lambda b_1 + \mathfrak{R}_{20}\big(\lambda^2\big),\\ \mathcal{P}_2\big(0,0,z_3\big) &= \lambda\partial_3\big(F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_2 w_3 - \partial_3 w_2\big)\big)\\ &\quad - \lambda\partial_1\Big(F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_1 w_2 - \partial_2 w_1\big)\Big)\\ &\quad + 2\big(-\partial_3 H_2^0\big)F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\mathbf{w}\big)\\ &\quad - H_2^0 + \lambda b_2 + \mathfrak{R}_{21}\big(\lambda^2\big)\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}_3\big(0,0,z_3\big) &= \lambda\partial_1\big(2\partial_3 H_1^0 F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\mathbf{w}\\ &\quad + F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_3 w_1 - \partial_1 w_3\big)\big)\\ &\quad - \lambda\partial_2\big(2\big(-\partial_3 H_2^0\big)F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\mathbf{w}\\ &\quad + F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_2 w_3 - \partial_3 w_2\big)\big) + \lambda b_3 + \mathfrak{R}_{22}\big(\lambda^2\big).\end{aligned}$$

Plugging the expressions of $\mathcal{P}_i$ back to (2.6) at the point $z = (0, 0, z_3)$, we get

$$\begin{aligned}&-\lambda\operatorname{curl}\Big[F\left(|\mathbf{M}_0|\right)^2\operatorname{curl}\mathbf{w} + 2F'\Big(|\mathbf{M}_0|^2\Big)\langle\mathbf{M}_0, \operatorname{curl}\mathbf{w}\rangle\mathbf{M}_0\Big] - \lambda\mathbf{w}\\ &\quad = \lambda\mathbf{b} + \mathfrak{R}_{23}\Big(\lambda^2\Big).\end{aligned}$$

where $\mathbf{M}_0 = \operatorname{curl}\mathbf{H}_0$. Thus we can obtain the limiting system (2.18) for $\mathbf{w}$ we required.

Appendix B. Derivation of System (2.30)

We follow the notations used in Section 2 and in appendix A. To derive system (2.30), we need to take the expansions for $\mathfrak{R}_i(\lambda^2)$ in appendix A in the form of

$$\mathfrak{R}_i(\lambda^2) = \lambda^2 R + \mathfrak{R}_i(\lambda^3) \quad \text{for } i = 1, \ldots, 23.$$

Since the domain is smooth, we have the expansions of $\mathfrak{R}_1(\lambda^2)$ in (6.1) and of $\mathfrak{R}_2(\lambda^2)$ in (6.2):

$$\mathfrak{R}_1\left(\lambda^2\right) = \lambda^2 \sum_{i,j=1}^{3} a_{ij} z_i z_j + \mathfrak{R}_{24}\left(\lambda^3\right), \tag{7.1}$$

$$\mathfrak{R}_2\left(\lambda^2\right) = \lambda^2 \sum_{i,j=1}^{3} b_{ij} z_i z_j + \mathfrak{R}_{25}\left(\lambda^3\right), \tag{7.2}$$

where the coefficients a_{ij} and b_{ij} are determined by the domain Ω. Then by Taylor expansion for $\hat{\mathbf{H}}_0(y_1, y_2, z_3)$ (defined by (2.15)) at the point $(0, 0, z_3)$, we have the expansion for $\mathfrak{R}_0$ in (6.5):

$$\mathfrak{R}_0(|y_1^2 + y_2^2|) = \frac{1}{2} \sum_{i,j=1}^{2} y_i y_j \partial_{y_i y_j} \hat{\mathbf{H}}_0(0, 0, z_3) + O\left(|y_1^2 + y_2^2|^{3/2}\right)$$

and the expansion for $\hat{\mathbf{H}}_1(y_1, y_2, z_3)$ (defined by (2.17)) at the point $(0, 0, z_3)$,

$$\begin{aligned}\hat{\mathbf{H}}_1\left(y_1, y_2, z_3\right) = \mathbf{w}\left(0, 0, z_3\right) + y_1 \partial_{y_1} \hat{\mathbf{H}}_1\left(0, 0, z_3\right) \\ + y_2 \partial_{y_2} \hat{\mathbf{H}}_1\left(0, 0, z_3\right) + O\left(|y_1^2 + y_2^2|\right).\end{aligned}$$

Thus from the inner expansion (2.8) and the expansion (6.6), we have the expansion for $\mathfrak{R}_3$ in (6.6):

$$\begin{aligned}\mathfrak{R}_3(\lambda^2) = \lambda^2 \left(\boldsymbol{\Psi} + \frac{1}{2} \sum_{i,j=1}^{2} z_i z_j \partial_{y_i y_j} \hat{\mathbf{H}}_0(0, 0, z_3) + \sum_{i=1}^{2} z_i \partial_{y_i} \hat{\mathbf{H}}_1(0, 0, z_3)\right) \\ + \mathfrak{R}_{26}(\lambda^3).\end{aligned}$$

Then we have the expansions for $\mathfrak{R}_4$ in (6.7), for $\mathfrak{R}_5$ in (6.8) and for $\mathfrak{R}_6$ in (6.9):

$$\mathfrak{R}_4(\lambda^2) = \lambda^2(\partial_2 \Psi_3 - \partial_3 \Psi_2 + \rho_1) + \mathfrak{R}_{27}(\lambda^3), \tag{7.3}$$

$$\mathfrak{R}_5(\lambda^2) = \lambda^2(\partial_3 \Psi_1 - \partial_1 \Psi_3 + \rho_2) + \mathfrak{R}_{28}(\lambda^3), \tag{7.4}$$

$$\mathfrak{R}_6(\lambda^2) = \lambda^2(\partial_1 \Psi_2 - \partial_2 \Psi_1 + \rho_3) + \mathfrak{R}_{29}(\lambda^3), \tag{7.5}$$

where the functions ρ_1, ρ_2, ρ_3 are determined by $a_{ij}, b_{ij}, \kappa_i, \Gamma_{ij}^k, \hat{\mathbf{H}}_0$ and $\hat{\mathbf{H}}_1$, and hence depends on the domain Ω and the strength of the magnetic field $\mathcal{H}_T^e$.
From (7.3)–(7.5) and (6.10), we see that

$$\mathfrak{R}_7\left(\lambda^2\right) = 2\lambda^2\left(\operatorname{curl} \mathbf{H}_0 \cdot \operatorname{curl} \boldsymbol{\Psi} + \rho_4\right) + \mathfrak{R}_{30}\left(\lambda^3\right). \tag{7.6}$$

We thus obtain that

$$\mathfrak{R}_8\left(\lambda^2\right) = 2\lambda^2 F'\left(|\operatorname{curl} \mathbf{H}_0|^2\right)\left(\operatorname{curl} \mathbf{H}_0 \cdot \operatorname{curl} \boldsymbol{\Psi} + \rho_5\right) + \mathfrak{R}_{31}\left(\lambda^3\right). \tag{7.7}$$

From (6.12) and (7.7), it follows that, for $i = 1, 2$,

$$\mathfrak{R}^i_{10}\left(\lambda^2\right) = \lambda^2\left[\partial_i\left(F\left(|\operatorname{curl}\mathbf{H}_0|^2\right)(\partial_1\Psi_2 - \partial_2\Psi_1)\right) + \rho_6\right] + \mathfrak{R}^i_{32}\left(\lambda^3\right). \tag{7.8}$$

From (7.7) and (6.13), we have

$$\begin{aligned}\mathfrak{R}_{11}\big(\lambda^2\big) = \lambda^2\big(&2\partial_3 H^0_1 F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\\ &+ F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_3\Psi_1 - \partial_1\Psi_3\big) + \rho_7\big) + \mathfrak{R}_{33}\big(\lambda^3\big),\end{aligned} \tag{7.9}$$

and from (7.7) and (6.14) we have

$$\begin{aligned}\mathfrak{R}_{12}\big(\lambda^2\big) = \lambda^2\big(&2\big(-\partial_3 H^0_2\big) F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\\ &+ F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_2\Psi_3 - \partial_3\Psi_2\big) + \rho_8\big) + \mathfrak{R}_{34}\big(\lambda^3\big).\end{aligned} \tag{7.10}$$

Therefore, (7.9) and (6.15) show that

$$\begin{aligned}\mathfrak{R}_{13}\big(\lambda^2\big) = \lambda^2\big[&\partial_1\big(2\partial_3 H^0_1 F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\\ &+ F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_3\Psi_1 - \partial_1\Psi_3\big)\big) + \rho_9\big] + \mathfrak{R}_{35}\big(\lambda^3\big),\end{aligned} \tag{7.11}$$

(7.10) and (6.16) show that

$$\begin{aligned}\mathfrak{R}_{14}\big(\lambda^2\big) = \lambda^2\big[&\partial_2\big(2\big(-\partial_3 H^0_2\big) F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\\ &+ F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_2\Psi_3 - \partial_3\Psi_2\big)\big) + \rho_{10}\big] + \mathfrak{R}_{36}\big(\lambda^3\big).\end{aligned} \tag{7.12}$$

From (6.17) and (7.9) we have

$$\begin{aligned}\mathfrak{R}_{15}\big(\lambda^2\big) = \lambda^2\big[&\partial_3\big(2\partial_3 H^0_1 F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\\ &+ F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_3\Psi_1 - \partial_1\Psi_3\big)\big) + \rho_{11}\big] + \mathfrak{R}_{37}\big(\lambda^3\big).\end{aligned} \tag{7.13}$$

From (6.18) and (7.10) we have

$$\begin{aligned}\mathfrak{R}_{16}\big(\lambda^2\big) = \lambda^2\big[&\partial_3\big(2\big(-\partial_3 H^0_2\big) F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\\ &+ F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_2\Psi_3 - \partial_3\Psi_2\big)\big) + \rho_{12}\big] + \mathfrak{R}_{38}\big(\lambda^3\big).\end{aligned} \tag{7.14}$$

We can obtain that: from (6.19), (7.8) and (7.13) we have

$$\begin{aligned}\mathfrak{R}_{17}\big(\lambda^2\big) = {}&\lambda^2\partial_2\left(F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_1\Psi_2 - \partial_2\Psi_1\big)\right)\\ &-\lambda^2\partial_3\big(F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_3\Psi_1 - \partial_1\Psi_3\big)\\ &+ 2\partial_3 H^0_1 F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\big) + \lambda^2\rho_{13} + \mathfrak{R}_{39}\big(\lambda^3\big);\end{aligned}$$

from (6.20), (7.8) and (7.14) we have

$$\begin{aligned}\mathfrak{R}_{18}\big(\lambda^2\big) = {}&-\lambda^2\partial_1\left(F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_1\Psi_2 - \partial_2\Psi_1\big)\right)\\ &+\lambda^2\partial_3\big(F\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\big(\partial_2\Psi_3 - \partial_3\Psi_2\big)\\ &+ 2\big(-\partial_3 H^0_2\big) F'\big(|\operatorname{curl}\mathbf{H}_0|^2\big)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\big) + \lambda^2\rho_{14} + \mathfrak{R}_{40}\big(\lambda^3\big)\end{aligned}$$

and from (6.21), (7.11) and (7.12) we have

$$\begin{aligned}\mathfrak{R}_{19}(\lambda^2) = \lambda^2\partial_1\big(&2\partial_3 H_1^0 F'(|\operatorname{curl}\mathbf{H}_0|^2)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\\ &+F(|\operatorname{curl}\mathbf{H}_0|^2)(\partial_3\Psi_1-\partial_1\Psi_3)\big)\\ &-\lambda^2\partial_2\big(2(-\partial_3 H_2^0)F'(|\operatorname{curl}\mathbf{H}_0|^2)\operatorname{curl}\mathbf{H}_0\cdot\operatorname{curl}\boldsymbol{\Psi}\\ &+F(|\operatorname{curl}\mathbf{H}_0|^2)(\partial_2\Psi_3-\partial_3\Psi_2)\big)+\lambda^2\rho_{15}+\mathfrak{R}_{41}(\lambda^3),\end{aligned}$$

where the functions $\rho_{13}, \rho_{14}, \rho_{15}$ are determined by the domain Ω and the boundary data $\mathcal{H}_T^e$. We need to mention that for the above ρ_i we always have

$$|\rho_i(z)| \leqq Ce^{-z_3} \quad \text{for } i = 1, \ldots, 15.$$

Therefore, at the point $(0, 0, z_3)$ we have

$$\begin{aligned}&-\lambda^2\operatorname{curl}\Big[F(|\mathbf{M}_0|)^2\operatorname{curl}\boldsymbol{\Psi}+2F'\left(|\mathbf{M}_0|^2\right)\langle\mathbf{M}_0,\operatorname{curl}\boldsymbol{\Psi}\rangle\mathbf{M}_0\Big]\\ &-\lambda^2\boldsymbol{\Psi}=\lambda^2\boldsymbol{\Phi}+\mathfrak{R}_{42}\left(\lambda^3\right)\end{aligned}$$

for some $\boldsymbol{\Phi}$ depending on ρ_{13}, ρ_{14} and ρ_{15} and also satisfying

$$|\boldsymbol{\Phi}| \leqq Ce^{-z_3}.$$

Thus we can obtain the limiting system (2.30) for $\boldsymbol{\Psi}$ we required.

Appendix C. Proof of the Equality (4.1)

Proof. Consider the following equation

$$f'' = \left(1 - f^2\right) f \qquad \text{for } z_3 > 0, \quad f'(0) = -|\tilde{\mathbf{h}}|.$$

Multiplying f' and then integrating the equaiton on both sides, we obtain that

$$f'^2 = f^2 - \frac{f^4}{2}.$$

Solving the above equation, we have

$$f^2(0) = 1 - \sqrt{1 - 2f'^2(0)} \qquad \text{and} \qquad f^2(z_3) = \frac{8Ce^{2z_3}}{(Ce^{2z_3}+1)^2}$$

with the constant C satisfying

$$f^2(0) = \frac{8C}{(C+1)^2}.$$

Note that

$$\begin{aligned}\int_0^\infty f'^2 dz_3 &= \int_0^\infty \left(f^2 - \frac{f^4}{2}\right) dz_3 = \frac{1}{2}\int_0^\infty f^2 dz_3 + \frac{1}{2}\int_0^\infty ff'' dz_3\\ &= \frac{1}{2}\int_0^\infty f^2 dz_3 + \frac{1}{2}\left(-f(0)f'(0) - \int_0^\infty f'^2 dz_3\right).\end{aligned}$$

This gives that

$$\int_0^\infty f'^2 \mathrm{d}z_3 = \frac{1}{3}\left(\int_0^\infty f^2 \mathrm{d}z_3 - f(0)f'(0)\right).$$

Since

$$\int_0^\infty f^2 \mathrm{d}z_3 = \int_0^\infty \frac{8Ce^{2z_3}}{(Ce^{2z_3}+1)^2}\mathrm{d}z_3 = \frac{4}{C+1} = 2 - \sqrt{4-2f^2(0)}$$

and

$$\sqrt{4-2f^2(0)} = \sqrt{2+2(1-f^2(0))} = \sqrt{2+2\sqrt{1-2f'^2(0)}} = -\frac{2f'(0)}{f(0)},$$

we obtain that

$$\int_0^\infty f'^2 \mathrm{d}z_3 = \frac{1}{3}\left(2+\frac{2f'(0)}{f(0)} - f(0)f'(0)\right).$$

We end our proof. □

Appendix D. Proof of Theorem 1.2

Proof of Theorem 1.2. in the case of $\beta < 1/\sqrt{2}$. Fix $x_0 \in \Omega$ and let $R_0 = \mathrm{dist}(x_0, \partial\Omega)$. Denote

$$\mathbf{J} = \lambda \,\mathrm{curl}\, \mathbf{H}, \quad u = \mathbf{J}\cdot\mathbf{J}.$$

From Lemmas 5.1 and 5.2, we see that for any $\varepsilon > 0$ there exists r_0 sufficiently large (independent of λ) such that for any $x \in B_{R_0-\lambda r_0}(x_0)$ (assume λ small) we have

$$|\mathbf{J}| < \varepsilon, \quad \lambda|\nabla\mathbf{J}| < \varepsilon, \quad \lambda^2|\nabla^2\mathbf{J}| \leqq C. \tag{9.1}$$

Taking curl on both sides of the quasilinear system (1.4), and then applying the formula

$$-\mathrm{curl}\,\mathrm{curl}\,\mathbf{B} = \Delta\mathbf{B} - \nabla\,\mathrm{div}\,\mathbf{B},$$

we then get

$$\lambda^2\Delta(F(u)\mathbf{J}) - \lambda^2\nabla(F'(u)\mathbf{J}\cdot\nabla u) - \mathbf{J} = 0. \tag{9.2}$$

By the simple computations,

$$\begin{aligned}\Delta(F(u)\mathbf{J}) &= F'(u)\Delta u\mathbf{J} + F''(u)|\nabla u|^2\mathbf{J} + 2F'(u)\nabla u\cdot\nabla\mathbf{J} + F(u)\Delta\mathbf{J},\\ \nabla(F'(u)\mathbf{J}\cdot\nabla u) &= F''(u)(\mathbf{J}\cdot\nabla u)\nabla u + F'(u)\nabla(\mathbf{J}\cdot\nabla u).\end{aligned}$$

Using the fact that $\nabla u = \mathbf{J}\cdot\nabla\mathbf{J}$, and then by (9.1), for λ sufficiently small we have the estimates

(i) $\lambda^2\nabla(F'(u)\mathbf{J}\cdot\nabla u)\cdot\mathbf{J} = o(1)\mathbf{J}\cdot\mathbf{J}$,
(ii) $\lambda^2(F'(u)\Delta u\mathbf{J} + F''(u)|\nabla u|^2\mathbf{J} + 2F'(u)\nabla u\cdot\nabla\mathbf{J})\cdot\mathbf{J} = o(1)\mathbf{J}\cdot\mathbf{J}$,
(iii) $\frac{1}{2}\lambda^2(F(u)-1)\Delta u = o(1)\mathbf{J}\cdot\mathbf{J}$.

Since

$$\begin{aligned}F(u)\Delta\mathbf{J}\cdot\mathbf{J} &= \frac{1}{2}F(u)\Delta u - F(u)|\nabla\mathbf{J}|^2\\ &= \frac{1}{2}\Delta u + \frac{1}{2}(F(u)-1)\Delta u - F(u)|\nabla\mathbf{J}|^2,\end{aligned}$$

by (i)–(iii), (9.2) turns to

$$\frac{1}{2}\lambda^2\Delta u - (1+\varepsilon(x))u - \lambda^2F(u)|\nabla\mathbf{J}|^2 = 0 \quad \text{in } B_{R_0-\lambda r_0}(x_0) \tag{9.3}$$

for λ sufficiently small, where $\varepsilon(x) = o(1)$ as $\lambda\to 0$. Now we choose r_0 sufficiently large (independent of λ) such that

$$|\varepsilon(x)| \leqq \varepsilon_0 \quad \text{for } x\in B_{R_0-\lambda r_0}(x_0).$$

Let v be the radially symmetric solution

$$\lambda^2\Delta v - (2-2\varepsilon_0)v = 0 \quad \text{in } B_{R_0-\lambda r_0}(x_0), \quad v = v_0 \quad \text{on } \partial B_{R_0-\lambda r_0}(x_0)$$

with the constant v_0 satisfying

$$v_0 = \|\lambda\,\mathrm{curl}\,\mathbf{H}\|^2_{L^\infty(\Omega)}.$$

Let $w = u - v$. Then w satisfies

$$\begin{aligned}\lambda^2\Delta w - (2-2\varepsilon_0)w = 2(\varepsilon_0+\varepsilon(x))u + 2\lambda F(u)|\nabla\mathbf{J}|^2 &\geqq 0 \qquad \text{in } B_{R_0-\lambda r_0}(x_0),\\ w &\leqq 0 \qquad \text{on } \partial B_{R_0-\lambda r_0}(x_0).\end{aligned}$$

By the maximum principle, we see that $w\leqq 0$. Therefore, there exists λ_0 such that for any $\lambda\in(0,\lambda_0)$ we have, for $x\in B_{R_0-\lambda r_0}(x_0)$,

$$|\lambda\,\mathrm{curl}\,\mathbf{H}|^2 = u \leqq v \leqq C\lambda^{-1}e^{-(\sqrt{2-2\varepsilon_0})\frac{d(x)}{\lambda}},$$

where the constant C depends on Ω and $\mathcal{H}^e_T$. By (1.6), there exists a positive constant λ_0 such that for any $\lambda\in(0,\lambda_0)$ and $\beta < 1/\sqrt{2}$ we have

$$|\mathbf{Q}(x)| \leqq Ce^{-\beta d(x,\partial\Omega)/\lambda},$$

where the constant C depends on β, Ω and $\mathcal{H}^e_T$. The proof is finished. □

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