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Uniqueness of Positive Ground State Solutions of the Logarithmic Schrödinger Equation

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Dedicated to the memory of Bryce McLeod
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Abstract

We prove the uniqueness of positive ground state solutions of the problem $\frac{d^2u}{dr^2} + \frac{n-1}{r}\frac{du}{dr} + u\ln(|u|) = 0$, $u(r) > 0\ \forall r \geq 0$, and $(u(r), u'(r)) \to (0, 0)$ as $r \to \infty$. This equation is derived from the logarithmic Schrödinger equation $\mathrm{i}\psi_t = \Delta\psi + u\ln\left(|u|^2\right)$, and also from the classical equation $\frac{\partial u}{\partial t} = \Delta u + u\left(|u|^{p-1}\right) - u$. For each $n \geq 1$, a positive ground state solution is $u_0(r) = \exp\left(-\frac{r^2}{4} + \frac{n}{2}\right)$, $0 \leq r < \infty$. We combine $u_0(r)$ with energy estimates and associated Ricatti equation estimates to prove that, for each $n \in [1, 9]$, $u_0(r)$ is the *only* positive ground state. We also investigate the stability of $u_0(r)$. Several open problems are stated.

1. Introduction

We investigate the uniqueness of solutions of the ground state problem

$$u'' + \frac{n-1}{r}u' + u\ln(|u|) = 0, \tag{1}$$

$$u'(r) = 0, \quad \big(u(r), u'(r)\big) \to (0, 0) \quad \text{as } r \to \infty, \tag{2}$$

$$u(r) > 0\ \forall r \geq 0. \tag{3}$$

Equation (1) is derived (see the Appendix for details) from a rescaling of the dimensionless logarithmic Schrödinger equation

$$\mathrm{i}\psi_t = \Delta\psi + \psi\ln\left(|\psi|^2\right), \tag{4}$$

where ψ denotes the dimensionless wave function. The Appendix also shows how to derive (1), through a limiting process, as $p \to 1^+$, from the classical equation

$$\frac{\partial u}{\partial t} = \Delta u + u|u|^{p-1} - u, \tag{5}$$

and also from the non-linear Klein–Gordon equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + u|u|^{p-1} - u. \tag{6}$$

A positive ground state solution of (1), (2) and (3) is given by

$$u_0(r) = \exp\left(-\frac{r^2}{4} + \frac{n}{2}\right), \ 0 \le r < \infty, \ n \ge 1. \tag{7}$$

This solution plays a central role in applications of the logarithmic Schrödinger equation to quantum mechanics [4,5], quantum optics [8], transport and diffusion phenomena [18], information theory [7,30], quantum gravity [31], and the theory of Bose–Einstein condensation [1]. In these applications, a physically important property of $u_0(r)$ is that it is the only positive ground state solution. Thus, our goal is to prove

Theorem 1. (Uniqueness) *Let* $1 \le n \le 9$. *Then* $u_0(r)$ *given in* (7) *is the only solution of* (1), (2) *and* (3).

Our proof of Theorem 1 employs a new comparison method which combines $u_0(r)$ with energy estimates and associated Ricatti equation estimates. To understand why our approach is new we need to describe analytical techniques in previous studies. In 1987 McLeod and Serrin [22] investigated the existence and uniqueness of smooth solutions of the general equation

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \tag{8}$$

where $n \ge 1$, u satisfies (2), (3), and f satisfies assumptions

(A_1) $f \in C^1[0,\infty)$, $f(0) = 0$, $f'(0) < 0$;
(A_2) There is an $\alpha > 0$ such that $f(u) < 0$ for $u \in (0,\alpha)$, $f(u) > 0$, $u > \alpha$;
(A_3) $f'(\alpha) > 0$.

Problem (8), (2) and (3) arises in the study of solutions of the classical problem

$$\Delta u + f(u) = 0, \tag{9}$$

$$u(x) > 0 \ \forall x \in R^n, \ \ u(x) \to 0 \ \text{as} \ |x| \to \infty. \tag{10}$$

Under suitable differentiability conditions on f, Gidas, Ni and Nirenberg [15] proved that solutions of (9) and (10) must be radial and satisfy (1), (2) and (3). General conditions on f have led to proofs of the existence of positive ground states in other important studies [2,6,27]. In 1951 Finkelstein et. al. [16] analyzed $\Delta u + u^3 - u = 0$ in the context of spinor fields. In a 1973 classical paper, Coffman [9] proved the uniqueness of a positive ground state solution of the initial value problem

$$u'' + \frac{2}{r}u' + u^3 - u = 0, \ \ u(0) = \beta, \ u'(0) = 0. \tag{11}$$

Let $u(r, \beta)$ denote the solution of (11), and let $\beta_0 > 0$ such that $u_0(r) = u(r, \beta_0)$ satisfies (2) and (3). To prove the uniqueness of $u_0(r)$, Coffman analyzed the behavior of $w = \frac{\partial}{\partial\beta} u(r, \beta)$, which solves the equation of first variation

$$w'' + \frac{n-1}{r} w + (3u^2 + 1)w = 0, \quad w(0) = 1, \ w('0) = 0. \tag{12}$$

Coffman developed several functionals and inequalities involving w which help determine the behavior of $u(r, \beta)$ as β varies. He used this information to prove that $u(r, \beta_0)$ is the only positive ground state solution of (11). McLeod and Serrin [22,23] extended Coffman's study and investigated the general, classical equation

$$u'' + \frac{n-1}{r} u' + |u|^{p-1} u - u = 0. \tag{13}$$

McLeod and Serrin made use of functionals in terms of r, u and u', and technical comparison methods, to prove the uniqueness of a positive ground state of (13) in the following parameter regimes:

(i) $1 < p < \infty$ when $1 \le n < 2$,
(ii) $1 < p \le \frac{n}{n-2}$ when $2 < n < 4$,
(iii) $1 < p \le \frac{8}{n}$ when $4 \le n \le 8.71$.

Kwong [20] proved the uniqueness of positive ground state solutions of (13) when $n > 1$ and $1 < p < \frac{n+1}{n-1}$. He followed Coffman's approach and analyzed (13) by developing technical lemmas associated with the equation of first variation for (13), namely

$$\frac{d^2 w}{d^r} + \frac{n-1}{r} \frac{dw}{dr} + (p|u|^{p-1} - 1)w = 0, \quad w(0) = 1, \ w'(0) = 0. \tag{14}$$

The uniqueness of positive solutions in annular domains has been proved by Coffman [11] and Kwong and Zhang [21].
In 2000 Serrin and Tang [26] extended the results described above, and proved the uniqueness of positive ground state solutions of the quasilinear equation

$$\left(|u'|^{m-2} u'\right)' + \frac{n-1}{r} |u'|^{m=2} u' + f(u) = 0, \ r > 0, \quad n > m > 1, \tag{15}$$

where $n > m > 1$ and $f(u)$ satisfies assumptions:

(H1) There is a $b > 0$ such that f is continuous on $(0, \infty)$, with $f(u) \le 0$ on $(0, b]$ and $f(u) > 0$ for $u > b$;
(H2) $f \in C^1(b, \infty)$, with $g(u) = uf'(u)/f(u)$ non-increasing on (b, ∞).

Gazzola, Serrin and Tang [17] proved the existence of positive ground state solutions.

Remark 1. The Case m = 2. Equation (15) reduces to (8) when $m = 2$. Because of the constraint $n > m > 1$ in (15), it follows that, if $m = 2$ then the Serrin–Tang [26] theorem does not apply to (1) when $1 \le n \le 2$.

To prove Theorem 1 we determine the behavior of solutions of

$$u'' + \frac{n-1}{r}u' + u\ln(|u|) = 0, \quad u(0) = \beta > 0, \ u'(0) = 0. \tag{16}$$

Equation (16) is fundamentally different from the investigations described above in two important ways:

(I) The nonlinearity $f(u) = u\ln(|u|)$ is continuous on $(-\infty, \infty)$, with zeros at $u = 0$ and $u = \pm 1$, and satisfies (A_2)–(A_3) stated above. However,

$$\lim_{|u|\to 0} f'(u) = \lim_{|u|\to 0} \ln(|u|) + 1 = -\infty. \tag{17}$$

A key step in the McLeod–Serrin analysis (Lemma 3, p. 124 in [22]) makes use of the requirement $f'(0) = -m < 0$. However, when $f(u) = u\ln(|u|)$, property (17) shows that $f'(u)$ becomes unbounded as $u \to 0$. Thus, assumption (A_1) is not satisfied and it is challenging to prove the uniqueness of a ground state solution using the methods in [22]. The function $f(u) = u\ln(|u|)$ does satisfy assumptions (H1)–(H2) in [26]. However, Remark 1 shows that the uniqueness result in [26] does not apply to (1), (2) and (3) when $1 \le n \le 2$. Thus, the uniqueness of the positive ground state solution $u_0(r)$ has not previously been proved when $1 \le n \le 2$.

(II) The uniqueness proofs in [9,11,20,21,29] make extensive use of the equation of first variation for the function $w = \frac{\partial}{\partial\beta}u(r,\beta)$. For (16) this equation is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{n-1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} + (\ln(|u|) + 1)w = 0, \quad w(0) = 1, \ w'(0) = 0. \tag{18}$$

As Coffman [9] originally showed, a crucial step in proving the uniqueness of ground states is to determine the behavior of points $R_1 = R_1(\beta) > 0$ where $u(R_1, \beta) = 0$. The behavior of $R_1(\beta)$ is determined from the equation

$$\frac{\mathrm{d}R_1}{\mathrm{d}\beta} = -\frac{w\,(R_1)}{u'\,(R_1)}. \tag{19}$$

However, it is challenging to accurately determine the behavior of $u'\,(R_1)$ and $w\,(R_1)$ in (19) since the term $\ln(|u|)$ in (16) and (18) becomes unbounded as $|u| \to 0$.

The analytical difficulties described above have led us to develop a new approach to prove the uniqueness of the positive ground state $u_0(r)$ given in (7). Our approach is to combine $u_0(r)$ with energy based estimates and associated Ricatti equation estimates to determine the behavior of solutions of the initial value problem (16) for each $\beta > 0$. Thus, to prove Theorem 1 we only need to prove the equivalent result.

Theorem 2. (Uniqueness) *Let* $1 \le n \le 9$ *and* $\beta > 0$.

(i) *If* $\beta = e^{n/2}$ *then the solution of* (16) *is the positive ground state solution* (7).
(ii) *If* $\beta \ne e^{n/2}$ *then the solution of* (16) *is not a positive ground state.*

Remark 2. We prove uniqueness when $1 \le n \le 9$, approximately the same range where the McLeod–Serrin [22] theorem holds. Also, Theorem 2 applies to the previously unresolved parameter regime $1 \le n \le 2$ (see Remark 1), in particular to the physically important value $n = 2$.

Proof of Theorem 2. The first step is to consider the case $n = 1$ and observe that (16) has the first integral

$$\frac{(u')^2}{2} + \frac{u^2}{2}\left(\ln(u) - \frac{1}{2}\right) = E, \tag{20}$$

where E is constant. Substituting $\big(u(r), u'(r)\big) \to (0,0)$ as $r \to \infty$ into (20) gives $E = 0$, and it easily follows from $\frac{(u')^2}{2} + \frac{u^2}{2}\left(\ln(u) - \frac{1}{2}\right) = 0$ that the only positive ground state is $u_0(r) = \exp\left(-\frac{r^2}{4} + \frac{1}{2}\right)$. □

Outline of proof when n > 1. In Section 2 we prove Theorem 2 when $1 < n \le 9$. There, we determine the behavior of solutions in the distinct parameter regimes $0 < \beta < e^{n/2}$ and $\beta > e^{n/2}$.
(i) When $0 < \beta < e^{n/2}$ we show that $u(r) > 0$ on its maximal interval of existence, and that $u(r)$ cannot satisfy both the positivity condition $u(r) > 0\ \forall r \ge 0$, and the limiting condition

$$\lim_{r\to\infty}(u(r), u'(r)) = (0,0). \tag{21}$$

(ii) When $\beta > e^{n/2}$ we prove that there is a first $R_1 = R_1(\beta) > 0$ such that

$$u\left(R_1\right) = 0 \ \text{ and } \ u'\left(R_1\right) < 0. \tag{22}$$

Thus, the positivity condition $u(r) > 0\ \forall r \ge 0$ is violated and the solution cannot be a positive ground state.
Uniqueness proofs in the previous studies described above are necessarily very technical. In Section 2 our proof, which is also somewhat technical, is completed with the help of auxiliary lemmas. The role of each lemma is explained as we proceed. Section 3 contains conclusions and a statement of open problems. Section 4 is the Appendix.

Future study: sign changing solutions. The existence of multi-zero ground state solutions of (13) was proved by McLeod, Troy and Weissler [24], and Jones and Kupper [19]. Troy [28] proved the existence of multi-zero ground states for (1)–(2) when $f(u) = |u|^{p-1}u - u^q$. Troy [29] also proved the uniqueness of the ground state solution of (8)–(2) with exactly one positive zero when $f(u)$ is piecewise linear. For more general general $f(u)$, the existence and uniqueness of multi-zero ground state solutions was proved by Cortazar, Garcia-Huidboro and Yarur [12–14]. Their results prove the uniqueness of multi-zero ground states of (16) when $n = 3$ or $n = 4$. Uniqueness remains an open problem when $n \notin \{3, 4\}$. The proof of the uniqueness of multi-zero ground states of the classical cubic equation (11) is also unresolved. It is hoped that a combination of techniques in [12–14] and the energy and Ricatti based estimates developed in this paper may give new insights into proving the uniqueness of sign-changing ground state solutions of (16) and (11).

2. Uniqueness

In this section we keep $1 < n \leq 9$ fixed and prove that the solution of initial value problem (16) is not a positive ground state when $\beta > 0$ and $\beta \neq e^{n/2}$. We consider two separate cases: $0 < \beta < e^{n/2}$ and $\beta > e^{n/2}$.

Case I. $0 < \beta < e^{n/2}$**.** Let u denote the solution of (16). The energy functional

$$Q = \frac{(u')^2}{2} + \frac{u^2}{2}\left(\ln(u) - \frac{1}{2}\right) \tag{23}$$

satisfies

$$Q' = -\frac{n-1}{r}\left(u'\right)^2, \quad Q'(0) = 0 \text{ and } Q(0) = \frac{\beta^2}{2}\left(\ln(\beta) - \frac{1}{2}\right). \tag{24}$$

The following result gives conditions that a positive ground state solution must satisfy.

Lemma 1. *Let* $\beta \in (0, e^{n/2})$*. A positive ground state solution of* (16) *satisfies*

$$u(0) = \beta \geq e^{1/2} \text{ and } Q(0) = \beta^2\left(\ln(\beta) - \frac{1}{2}\right) \geq 0, \tag{25}$$

$$u'(r) < 0, \ Q(r) > 0 \text{ and } Q'(r) < 0 \ \forall r > 0, \text{ and } \lim_{r\to\infty} Q(r) = 0. \tag{26}$$

Proof. A positive ground state solution satisfies conditions (2)–(3). Thus,

$$Q'(r) \leq 0 \ \forall r > 0 \quad \text{and} \quad \lim_{r\to\infty} Q(r) = 0. \tag{27}$$

These properties imply that

$$Q(r) \geq 0 \ \forall r > 0. \tag{28}$$

We conclude from (16), (23) and (28) that $Q(0) = \beta^2\left(\ln(\beta) - \frac{1}{2}\right) \geq 0$, hence $\beta \geq e^{1/2}$. Next, it follows from (16) that $u''(0) = -\beta\ln(\beta) < 0$ since $\beta \geq e^{1/2}$. This implies that $u(r) > 0$ and $u'(r) < 0$ on an interval $(0, \epsilon)$. If there is a first $\bar{r} > 0$ such that $u(\bar{r}) > 0$ and $u'(\bar{r}) = 0$ then $u''(\bar{r}) = -u(\bar{r})\ln(u(\bar{r})) \geq 0$. Thus, $\ln(u(\bar{r})) \leq 0$ and $Q(\bar{r}) = u(\bar{r})^2\left(\ln(u(\bar{r})) - \frac{1}{2}\right) < 0$, contradicting (28). We conclude that $u'(r) < 0 \ \forall r > 0$, hence $Q'(r) = -\frac{(n-1)u'(r)^2}{r} < 0 \ \forall r > 0$. This property and (27) imply that $Q(r) > 0 \ \forall r \geq 0$. This completes the proof. We use the results of Lemma 1 to prove □

Theorem 3. *Let* $1 < n \leq 9$ *and* $\beta \in (0, e^{n/2})$*. Then the solution of* (16) *is not a positive ground state solution.*

Proof. We assume that there is a $\beta \in (0, e^{n/2})$ such that the solution of (16) is a positive ground state, and obtain a contradiction. By Lemma 1, the solution must satisfy conditions (25)–(26). Thus, if we prove that one of these conditions does not hold, then we have a contradiction of the assumption that the solution is a positive ground state. First, when $0 < \beta < e^{1/2}$, it is easily verified that

$$Q(0) = \beta^2 \left(\ln(\beta) - \frac{1}{2} \right) < 0,$$

hence (25) does not hold. Next, when $e^{1/2} \leq \beta < e^{n/2}$, we claim that (26) does not hold. To prove this claim we make use of the Ricatti function $\rho = \frac{u'}{u}$, which satisfies

$$\rho' + \rho^2 + \frac{(n-1)}{r}\rho + \ln(u) = 0, \quad \rho(0) = 0 \quad \text{and} \quad \rho'(0) = -\frac{\ln(\beta)}{n} < 0. \quad (29)$$

Below we show that $\rho(r)$ decreases until it reaches a negative minimum, then increases until $\rho(\hat{r}) \in \left(-\frac{1}{\sqrt{2}}, 0\right)$ and $0 < u(\hat{r}) < 1$ at some $\hat{r} > 0$, hence

$$Q(\hat{r}) = \frac{u^2(\hat{r})}{2} \left(\left(\frac{u'(\hat{r})}{u(\hat{r})} \right)^2 + \ln\left(u(\hat{r})\right) - \frac{1}{2} \right) < 0. \quad (30)$$

Thus, (26) does not hold and Lemma 1 implies that $u(r)$ is not a ground state.
A differentiation of (29) gives

$$\rho'' + \frac{(n-1)}{r}\rho' - \frac{(n-1)}{r^2}\rho + 2\rho\rho' + \rho = 0, \quad \rho''(0) = 0. \quad (31)$$

We also use the function $\rho_0 = \frac{u_0'}{u_0}$, where $u_0(r) = \exp\left(-\frac{r^2}{4} + \frac{n}{2}\right)$ is the positive ground state given in (7). Then ρ_0 satsifies

$$\rho_0(r) = -\frac{r}{2} \quad \text{and} \quad \rho_0'(r) = -\frac{1}{2} \quad \forall r \geq 0. \quad (32)$$

Important properties are

$$\rho'(0) = -\frac{\ln(\beta)}{n} < 0 \; \forall \beta \in \left[e^{1/2}, e^{n/2}\right) \quad (33)$$

and

$$\rho'(0) - \rho_0'(0) = -\frac{\ln(\beta)}{n} + \frac{1}{2} > 0 \quad \forall \beta \in \left[e^{1/2}, e^{n/2}\right). \quad (34)$$

Next, let $(0, r_{\max})$ denote the largest interval where $-\infty < \rho(r) < 0$. That is,

$$r_{\max} = \sup\left\{\hat{r} > 0 \mid \text{if } 0 < r < \hat{r} \text{ then } -\infty < \rho(r) < 0\right\}. \quad (35)$$

To prove that condition (26) does not hold we need two auxiliary results (Lemma 2 and Lemma 3 below) which determine the behavior of ρ_0', ρ' and ρ''.
Lemma 2 gives practical lower bounds for $\rho''(r)$ and $\rho'(r) - \rho_0'(r)$ over the largest interval $(0, \bar{r}) \subset (0, r_{\max})$ where $\rho_0'(r) < \rho'(r) < 0$. Thus, define

$$\bar{r} = \sup\left\{\hat{r} \in (0, r_{\max}) \mid \text{if } 0 < r < \hat{r} \text{ then } \rho_0'(r) < \rho'(r) < 0\right\}. \quad (36)$$

It follows from (33)–(34) and continuity that $\bar{r} > 0 \;\; \forall \beta \in \left[e^{1/2}, e^{r/2}\right)$.

Lemma 2. *Let $e^{1/2} \leq \beta < e^{n/2}$. Then*

$$\rho''(r) > 0 \text{ and } \rho'(r) - \rho_0'(r) \geq -\frac{\ln(\beta)}{n} + \frac{1}{2} > 0 \;\forall r \in (0, \bar{r}]. \tag{37}$$

Proof. It follows from (31) and (32) that

$$\left(r^2\rho''\right)' + \left(\frac{n-1}{r} + 2\rho\right) r^2\rho'' = -2\left(\rho' - \rho_0'\right)\left(2r\rho + r^2\rho'\right). \tag{38}$$

We conclude from (38) that

$$\left(r^{n+1}e^{2\int_0^r \rho(x)\mathrm{d}x}\rho''(r)\right)' = -2\, r^{n-1}e^{2\int_0^r \rho(x)\mathrm{d}x}\left(\rho' - \rho_0'\right)\left(2r\rho + r^2\rho'\right). \tag{39}$$

The definitions of $\bar{r}$ and $r_{\max}$ imply that the right side of (39) is positive for all $r \in (0, \bar{r})$. Also, recall from (31) that $\rho''(0) = 0$. Thus, an integration of (39) gives

$$\rho''(r) > 0 \;\forall r \in (0, \bar{r}]. \tag{40}$$

It follows from (32) and (40) that $\frac{d}{dr}\left(\rho'(r) - \rho_0'(r)\right) = \rho''(r) > 0 \;\forall r \in (0, \bar{r}]$. An integration gives

$$\rho'(r) - \rho_0'(r) \geq -\frac{\ln(\beta)}{n} + \frac{1}{2} > 0 \;\forall r \in (0, \bar{r}]. \tag{41}$$

This completes the proof of Lemma 2. □

Our second auxiliary result (Lemma 3) shows that ρ' has at most one zero on $(0, r_{\max})$, and that the second inequality in (37) extends to the entire interval $(0, r_{\max})$.

Lemma 3. *Let $e^{1/2} \leq \beta < e^{n/2}$. Suppose that $\rho'(\bar{r}) = 0$ at some $\bar{r} \in (0, r_{\max})$. Then*

$$\rho'(r) < 0 \,\forall r \in (0, \bar{r}),\;\; \rho'(\bar{r}) = 0, \;\text{ and }\; \rho'(r) > 0 \,\forall r \in (\bar{r}, r_{\max}), \tag{42}$$

and

$$\rho'(r) - \rho_0'(r) \geq -\frac{\ln(\beta)}{n} + \frac{1}{2} > 0 \;\forall r \in [\bar{r}, r_{\max}). \tag{43}$$

Proof. Write equation (31) as

$$\rho'' + \left(\frac{n-1}{r} + 2\rho\right)\rho' = \left(\frac{n-1}{r^2} - 1\right)\rho. \tag{44}$$

Let $\bar{r} \in (0, r_{\max})$ denote the first positive zero of ρ'. Then $\rho''(\bar{r}) \geq 0$, and (44) gives

$$\rho''(\bar{r}) = \left(\frac{n-1}{\bar{r}^2} - 1\right)\rho(\bar{r}) \geq 0. \tag{45}$$

Lemma 2 and the definitions of $\bar{r}$ and $r_{\max}$ imply that $\rho''(\bar{r}) > 0$. Also, $\rho(\bar{r}) < 0$ since $0 < \bar{r} < r_{\max}$. Combining these properties with (45), we conclude that

$\bar{r} > \sqrt{n-1}$, and that $\rho'(r) > 0$ on an interval $(\bar{r}, \bar{r}+\epsilon)$. If there is a next zero of ρ' at some $\tilde{r} \in (\bar{r}, r_{\max})$ then

$$\rho''(\tilde{r}) \le 0. \tag{46}$$

However, $\rho(\tilde{r}) < 0$ and $\tilde{r} > \bar{r} > \sqrt{n-1}$, and therefore (44) gives $\rho''(\tilde{r}) > 0$, contradicting (46). This proves property (42). Finally, (42) shows that $\rho'(r) \ge 0 \ \forall r \in [\bar{r}, r_{\max})$. Thus, we conclude that

$$\rho'(r) - \rho_0'(r) \ge 0 + \frac{1}{2} \ge -\frac{\ln(\beta)}{n} + \frac{1}{2} > 0 \ \ \forall r \in [\bar{r}, r_{\max}). \tag{47}$$

This completes the proof of the Lemma 3. □

We now complete the proof of Theorem 3. The assumption that u is a positive ground state implies that $u(r) > 0$ and $u'(r) < 0$ for all $r > 0$, and $u(r) \to 0$ as $r \to \infty$. Thus,

$$r_{\max} = \infty \ \text{ and } \ \rho(r) = \frac{u'(r)}{u(r)} < 0 \ \ \forall r > 0, \tag{48}$$

and there is an $r^* > 0$ such that

$$0 < u(r) \le 1 \ \text{ and } \ \ln(u(r)) \le 0 \ \ \forall r \ge r^*. \tag{49}$$

Also, it follows from (23) and Lemma 1 that

$$Q(r) = \frac{u^2(r)}{2}\left(\left(\frac{u'(r)}{u(r)}\right)^2 + \ln(u(r)) - \frac{1}{2}\right) > 0 \ \ \forall r \ge 0. \tag{50}$$

Combining (48), (49) and (50), we conclude that $\rho^2(r) > \frac{1}{2} \ \forall r \ge r^*$. Thus,

$$\rho(r) < -\frac{1}{\sqrt{2}} \ \ \forall r \ge r^*. \tag{51}$$

Our goal is to prove that $\rho(r) > -\frac{1}{\sqrt{2}}$ when $r \gg r^*$, contradicting (51). It follows from (31) and identities $\rho_0'(r) = -\frac{1}{2}$ and $r\rho'' = \left(r\rho' - \rho\right)'$ that

$$\left(r\rho' - \rho\right)' + \frac{n-1}{r}\left(r\rho' - \rho\right) = -2r\rho\left(\rho' - \rho_0'\right) \ \ \forall r > 0. \tag{52}$$

Combining (37), (43), (51) and (52) gives

$$\left(r\rho' - \rho\right)' + \frac{n-1}{r}\left(r\rho' - \rho\right) \ge \mathrm{A}r \ \ \forall r \ge r^*, \tag{53}$$

where $\mathrm{A} = \sqrt{2}\left(-\frac{\ln(\beta)}{n} + \frac{1}{2}\right) > 0$ since $\beta \in (e^{1/2}, e^{n/2})$. Integrating (53) gives

$$r^{n-1}\left(r\rho' - \rho\right) \ge \frac{\mathrm{A}r^{n+1}}{n+1} + \mathrm{B} \ \ \forall r \ge r^*, \tag{54}$$

where $\mathrm{B} = -\frac{\mathrm{A}(r^*)^{n+1}}{n+1} + (r^*)^{n-1}\left(r^*\rho'(r^*) - \rho(r^*)\right)$. Next, divide (54) by r^{n+1} and get

$$\left(\frac{\rho}{r}\right)' \geq \frac{\mathrm{A}}{n+1} + \frac{\mathrm{B}}{r^{n+1}} \ \ \forall r \geq r^*. \tag{55}$$

An integration of (55) gives

$$\frac{\rho(r)}{r} \geq \frac{\rho(r^*)}{r^*} + \frac{\mathrm{A}}{n+1}\left(r - r^*\right) + \frac{\mathrm{B}}{n}\left(\frac{1}{(r^*)^n} - \frac{1}{r^n}\right) \ \ \forall r \geq r^*. \tag{56}$$

Because $\mathrm{A} > 0$, the right side of (56) is positive when $r \gg 1$. Thus, $\rho(r) > -\frac{1}{\sqrt{2}}$ when $r \gg 1$, contradicting (51). This completes the proof of Theorem 3.
Case (II) $\beta > e^{n/2}$. In this regime we prove that the solution of (16) is not a ground state solution by showing that there is an $r_1 > 0$ such that

$$u(r) > 0 \ \text{ and } \ u'(r) < 0 \ \ \forall r \in (0, r_1), \ \ u(r_1) = 0 \ \text{ and } \ u'(r_1) < 0. \tag{57}$$

To prove (57) we make use of the transformation

$$f(r) = u(r) \exp\left(\frac{r^2}{4} - \frac{n}{2}\right). \tag{58}$$

Define $\alpha = \beta e^{-n/2}$. Then $\beta > e^{n/2} \iff \alpha > 1$, and f satisfies

$$f'' + \left(\frac{n-1}{r} - r\right) f' + f \ln(f) = 0, \ \ f(0) = \alpha > 1, \ \ f'(0) = 0. \tag{59}$$

It follows from (59) that

$$f''(0) = -\frac{\alpha \ln(\alpha)}{n} < 0 \ \ \forall \alpha > 1. \tag{60}$$

We need to determine the behavior of $f(r)$ for each $\alpha \geq 1$. When $\alpha = 1$, uniqueness of the constant solution $f = 1$ implies that $f(r) = 1 \ \forall r \geq 0$. The corresponding solution of (16) is ground state solution (7) since

$$u(r) = f(r) \exp\left(-\frac{r^2}{4} + \frac{n}{2}\right) = \exp\left(-\frac{r^2}{4} + \frac{n}{2}\right) \ \ \forall r \geq 0. \tag{61}$$

Goal. When $\alpha > 1$ we show that $f(r)$ reaches $f = 0$ at a finite r value. Define

$$r_1 = \sup\{\hat{r} > 0 \mid \text{if } 0 < r < \hat{r} \text{ then } f(r) > 0\}. \tag{62}$$

Theorem 4. *Let $\alpha > 1$ and $1 < n \leq 9$. The solution of* (59) *satisfies $r_1 < \infty$,*

$$f(r) > 0 \text{ and } f'(r) < 0 \ \forall r \in (0, r_1), \ \ f(r_1) = 0 \ \text{ and } \ f'(r_1) < 0. \tag{63}$$

Implications of Theorem 4. Let $\beta > e^{n/2}$ so that $\alpha = \beta e^{-n/2} > 1$, and let f denote the solution of (59). It follows from (58) and Theorem 4 that the corresponding solution of (16) satisfies $u(0) = \beta > e^{n/2}$ and $u'(0) = 0$,

$$u(r) = f(r)\exp\left(-\frac{r^2}{4} + \frac{n}{2}\right) > 0 \ \ \forall r \in (0, r_1), \tag{64}$$

$$u'(r) = \left(f'(r) - \frac{r}{2}f(r)\right)\exp\left(-\frac{r^2}{4} + \frac{n}{2}\right) < 0 \ \ \forall r \in (0, r_1), \tag{65}$$

$$u(r_1) = f(r_1)e^{\left(-\frac{r_1^2}{4}+\frac{n}{2}\right)} = 0 \ \text{ and } \ u'(r_1) = f'(r_1)e^{\left(-\frac{r^2}{4}+\frac{n}{2}\right)} < 0. \tag{66}$$

Thus, property (57) is verified and the proof of Theorem 2 is complete □

Proof of Theorem 4. The first step is to show that $f(r)$ decreases until $f(r_0) = 1$ at a finite $r_0 > 0$. Thus, define

$$r_0 = \sup\{\hat{r} \in (0, r_1) \mid \text{if } 0 < r < \hat{r} \text{ then } f(r) > 1\}. \tag{67}$$

Lemma 4. *Let* $\alpha > 1$. *Then* $r_0 < \infty$,

$$f(r) > 1 \text{ and } f'(r) < 0\,\forall r \in (0, r_0), \quad f(r_0) = 1 \text{ and } f'(r_0) < 0. \tag{68}$$

Proof. It follows from (59) and definition (67) that

$$\left(r^{n-1}e^{\left(-\frac{r^2}{2}\right)}f'(r)\right)' = -r^{n-1}e^{\left(-\frac{r^2}{2}\right)}f\ln(f) < 0\,\forall r \in (0, r_0). \tag{69}$$

Combining (59) with(69) gives $f(r) > 1$ and $f'(r) < 0 \ \forall r \in (0, r_0)$. This proves the first part of (68). It remains to show that r_0 is finite, and that

$$f(r_0) = 1 \text{ and } f'(r_0) < 0. \tag{70}$$

Suppose, however, that $r_0 = \infty$ for some $\alpha > 1$. Then (59) implies that $f'(r) < 0$, $f(r) > 1$ and $f''(r) = \left(r - \frac{n-1}{r}\right)f' - f\ln(f) < 0\ \forall r > \sqrt{n-1}$. These properties imply that $f(r) = 1$ at a finite $r > \sqrt{n-1}$, contradicting the supposition $r_0 = \infty$. Thus, $r_0 < \infty$, $f(r_0) = 1$ and $f'(r_0) \le 0$. The uniqueness of the constant solution $f \equiv 1$ implies that $f'(r_0) < 0$. This proves (70). □

Remark. It follows from Lemma 4 that $f(r) > 0$ and $f'(r) < 0$ on an interval $(r_0, r_0{+}\epsilon)$. This fact and the definitions of r_0 and r_1 imply that $0 < r_0 < r_1\ \forall \alpha > 1$.

The next step in the proof of Theorem 4 is to develop a criterion which guarantees that $r_1 < \infty$, $f(r_1) = 0$ and $f'(r_1) < 0$. We do this in

Lemma 5. *Let* $n > 1$ *and* $\alpha > 1$. *If* $r_0 \ge \sqrt{n-1}$ *then* $r_1 < \infty$,

$$0 < f(r) \le 1, \ f'(r) < 0 \ \ \forall r \in [r_0, r_1), \ f(r_1) = 0 \ \text{ and } \ f'(r_1) < 0. \tag{71}$$

Proof. It follows from (59) and (70) that

$$f'(r_0) < 0 \text{ and } f''(r_0) \le 0. \tag{72}$$

A differentiation of (59) gives

$$f''' + \left(\frac{n-1}{r} - r\right) f'' + \left(\ln(f) - \frac{n-1}{r^2}\right) f' = 0. \tag{73}$$

We conclude from (70), (72) and (73) that

$$\left(r^{n-1} \exp\left(-\frac{r^2}{2}\right) f''(r)\right)' = r^{n-1} \exp\left(-\frac{r^2}{2}\right)\left(\frac{n-1}{r^2} - \ln(f)\right) f'(r) < 0 \tag{74}$$

for all r in an interval $(r_0, r_0 + \epsilon)$. It follows from (72) and (74) that $0 < f(r) < 1$, $f'(r) < 0$ and $f''(r) < 0\ \forall r \in (r_0, r_1)$. In turn this implies that $f'(r) < f'(r_0) < 0\ \forall r \in (r_0, r_1)$, and we conclude that $r_1 < \infty$, $f(r_1) = 0$ and $f'(r_1) \le f'(r_0) < 0$. This completes the proof of Lemma 5. □

We now use Lemmas 4 and 5 to determine the behavior of f and f' for each $\alpha > 1$.

Lemma 6. *Let* $1 < \alpha \le e$. *Then* $\sqrt{n-1} < r_0 < r_1 < \infty$,

$$f'(r) < 0\ \forall r \in (0, r_1] \text{ and } f(r_1) = 0. \tag{75}$$

Proof. First, Lemma 4 and the fact that $f'(0) = 0$ imply that

$$f'(r) < 0 \text{ and } 1 \le f(r) \le e\ \forall r \in (0, r_0). \tag{76}$$

Next, we show that $r_0 > \sqrt{n-1}\ \forall \alpha \in (1, e]$. Suppose, for contradiction, that there is an $\alpha \in (1, e]$ for which $0 < r_0 \le \sqrt{n-1}$. This and property (76) imply that

$$\frac{n-1}{r^2} - \ln(f(r)) \ge \frac{n-1}{r^2} - 1 > 0\ \forall r \in (0, r_0). \tag{77}$$

Recall from (60) that $f''(0) < 0$. Then Lemma 4, (74) and (77) imply that

$$f''(r) < 0\ \forall r \in (0, r_0]. \tag{78}$$

Since we assume that $0 < r_0 \le \sqrt{n-1}$, we conclude from (76) and (59) that $f''(r_0) \ge 0$, contradicting (78). Therefore, it must be the case that $r_0 > \sqrt{n-1}$. Thus, since $r_0 > \sqrt{n-1}$, Lemma 5 implies that $r_1 < \infty$, $f(r_1) = 0$ and $f'(r_1) < 0$. This completes the proof of Lemma 6. □

To complete the proof of Theorem 4 we need to show that property (63) holds when $\alpha > e$. First, if $r_0 \ge \sqrt{n-1}$, then Lemma 5 guarantees that (63) holds. It remains to prove that (63) also holds when $0 < r_0 < \sqrt{n-1}$. This is done in

Lemma 7. *Let* $1 < n \le 9$ *and* $\alpha > e$. *If* $0 < r_0 < \sqrt{n-1}$ *then* $r_0 < r_1 < \infty$,

$$f(r) > 0 \text{ and } f'(r) < 0\ \ \forall r \in [r_0, r_1),\ \ f(r_1) = 0 \text{ and } f'(r_1) < 0. \tag{79}$$

Proof. It follows from (59), the assumption $0 < r_0 < \sqrt{n-1}$, and Lemma 4 that

$$f(r_0) = 1,\ \ f'(r_0) < 0 \text{ and } f''(r_0) = \left(r_0 - \frac{n-1}{r_0}\right) f'(r_0) > 0. \tag{80}$$

We claim that (80) implies that there is an $r_A \in (0, r_0)$ such that

$$f(r_A) = \exp\left(\frac{n-1}{r_A^2}\right) \text{ and } f'(r_A) \leq -\frac{2(n-1)}{r_A^3} \exp\left(\frac{n-1}{r_A^2}\right). \tag{81}$$

Suppose, however, that there is an $\alpha > e$ such that

$$1 < f(r) < \exp\left(\frac{n-1}{r^2}\right)\ \forall r \in (0, r_0). \tag{82}$$

Then $\ln(f(r)) < \frac{n-1}{r^2}\ \forall r \in (0, r_0)$, and it follows from (59), (60) and (73) that

$$\left(r^{n-1} e^{\left(-\frac{r^2}{2}\right)} f''(r)\right)' = r^{n-1} e^{\left(-\frac{r^2}{2}\right)} \left(\frac{n-1}{r^2} - \ln(f)\right) f'(r) < 0 \tag{83}$$

when $r \in (0, r_0)$. An integration gives $f''(r) < 0\ \forall r \in (0, r_0]$, hence $f''(r_0) < 0$, contradicting (80). We conclude that there is an $r_A \in (0, r_0)$ such that $f(r_A) = \exp\left(\frac{n-1}{r_A^2}\right)$. This property, and the fact that $f(r_0) = 1 < \exp\left(\frac{n-1}{r_0^2}\right)$, imply that we can choose r_A such that

$$f(r_A) = \exp\left(\frac{n-1}{r_A^2}\right) \text{ and } 1 < f(r) < \exp\left(\frac{n-1}{r^2}\right)\ \forall r \in (r_A, r_0). \tag{84}$$

It follows from (84) that $f'(r_A) \leq -\frac{2(n-1)}{r_A^3} \exp\left(\frac{n-1}{r_A^2}\right)$, and (81) is proved. □

To complete the proof of Lemma 7, we make use of (81) and energy functional

$$S = \frac{(f')^2}{2} + \frac{f^2}{2}\left(\ln(f) - \frac{1}{2}\right), \tag{85}$$

which satisfies

$$S' = \left(r - \frac{n-1}{r}\right)(f')^2. \tag{86}$$

It follows from (85) and (86) that $S'(0) = 0$ and $S(0) = \frac{\alpha^2}{2}\left(\ln(\alpha) - \frac{1}{2}\right) > 0$, and that S also satisfies

$$S' + \left(\frac{2(n-1)}{r} - 2r\right) S = \left(r - \frac{n-1}{r}\right) f^2 \left(\frac{1}{2} - \ln(f)\right). \tag{87}$$

Observe that $f^2\left(\frac{1}{2} - \ln(f)\right) \leq \frac{1}{2}\ \ \forall f > 0$. Then (87) reduces to

$$S' + \left(\frac{2(n-1)}{r} - 2r\right) S \geq \frac{1}{2}\left(r - \frac{n-1}{r}\right) \tag{88}$$

when $r \in \left(0, \min\{\sqrt{n-1}, r_1\}\right)$. It follows from (88) that

$$\left(r^{2n-2}e^{-r^2}S\right)' \geq \frac{1}{2}\left(r^{2n-1} - (n-1)r^{2n-3}\right)e^{-r^2} \tag{89}$$

when $r \in \left(r_A, \min\{\sqrt{n-1}, r_1\}\right)$. Integration of both sides of (89) from r_A to r gives

$$r^{2n-2}e^{-r^2}S(r) \geq r_A^{2n-2}e^{-r_A^2}S(r_A) + \frac{1}{4}\left(r_A^{2n-2}e^{-r_A^2} - r^{2n-2}e^{-r^2}\right), \tag{90}$$

where $r \in \left(r_A, \min\{\sqrt{n-1}, r_1\}\right)$. It follows from (81), (85), and the fact that $0 < r_A^2 \leq n-1$, that a practical lower bound on $S(r_A)$ is

$$S(r_A) \geq \frac{2(n-1)^2}{r_A^6}e^{\frac{2(n-1)}{r_A^2}} + \frac{e^{\frac{2(n-1)}{r_A^2}}}{2}\left(\frac{n-1}{r_A^2} - \frac{1}{2}\right) \geq \frac{2(n-1)^2}{r_A^6}e^{\frac{2(n-1)}{r_A^2}}. \tag{91}$$

To complete the proof of Lemma 7 we also need practical lower bounds on the product $r^{2n-2}S(r)$. We consider two cases: Case A: $1 < n \leq 6$ and Case B: $6 < n \leq 9$.

Case A: $1 < n \leq 6$. Combine (90) and (91), multiply by e^{r^2}, and get

$$r^{2n-2}S(r) \geq 2(n-1)^2 r_A^{2n-8}e^{\frac{2(n-1)}{r_A^2}} - \frac{r^{2n-2}}{4} \tag{92}$$

when $r_A \leq r \leq \min\{\sqrt{n-1}, r_1\}$. The term $r_A^{2n-8}e^{\frac{2(n-1)}{r_A^2}}$ is decreasing in r_A when $1 < r_A \leq \sqrt{n-1}$ and $1 < n \leq 6$. Thus, we substitute $r_A = \sqrt{n-1}$ and the upper bound $r = \sqrt{n-1}$ into (92) and conclude that, if $1 < n \leq 6$, then

$$r^{2n-2}S(r) \geq \frac{(n-1)^{n-1}}{4}\left[\frac{8}{n-1}e^2 - 1\right] > 0 \tag{93}$$

when $r_A \leq r \leq \min\{\sqrt{n-1}, r_1\}$.

Case B: $6 < n \leq 9$. The term $r_A^{2n-8}e^{\frac{2(n-1)}{r_A^2}}$ attains its positive relative minimum at $r_A = \sqrt{\frac{2(n-1)}{n-4}} < \sqrt{n-1}$ when $n > 6$. Substitute $r_A = \sqrt{\frac{2(n-1)}{n-4}}$ and the upper bound $r = \sqrt{n-1}$ into (92) and conclude that, if $6 < n \leq 9$, then

$$r^{2n-2}S(r) \geq \frac{(n-1)^{n-1}}{4}\left(\frac{8}{n-1}\left(\frac{2e}{n-4}\right)^{n-4} - 1\right) > 0 \tag{94}$$

when $r_A \leq r \leq \min\{\sqrt{n-1}, r_1\}$. We now complete the proof of Lemma 7. First, define

$$\mathrm{K}_n = \frac{(n-1)^{n-1}}{4}\left[\frac{8e^2}{n-1} - 1\right] > 0 \text{ if } 1 < n \leq 6, \tag{95}$$

$$\mathrm{K}_n = \frac{(n-1)^{n-1}}{4}\left[\frac{8}{n-1}\left(\frac{2e}{n-4}\right)^{n-4} - 1\right] > 0 \text{ if } 6 < n \leq 9. \tag{96}$$

Next, let $n \in (1, 9]$ be fixed. It follows from (93), (94), (95), and (96) that

$$r^{2n-2} S(r) \geq K_n, \ r_A \leq r \leq \min\{\sqrt{n-1}, r_1\}. \tag{97}$$

We combine (85) with (97) and conclude that

$$\frac{\left(f'(r)\right)^2}{2} + \frac{(f(r))^2}{2}\left(\ln(f(r) - \frac{1}{2}\right) \geq \frac{K_n}{(n-1)^{n-1}}, \tag{98}$$

when $r_A \leq r \leq \min\{\sqrt{n-1}, r_1\}$. Suppose that $r_0 < r_1 \leq \sqrt{n-1}$. It follows from (98) and initial conditions $f(r_0) = 1$ and $f'(r_0) < 0$ that

$$f'(r) < -\sqrt{\frac{2K_n}{(n-1)^{n-1}}}, \ \ 0 < f(r) < 1 \ \ \text{and} \ \ \ln(f(r)) < 0 \, \forall r \in (r_0, r_1). \tag{99}$$

We conclude from (99), the assumption that $r_1 < \sqrt{n-1}$, and continuity, that

$$f(r_1) = 0 \ \text{ and } \ f'(r_1) \leq -\sqrt{\frac{2K_n}{(n-1)^{n-1}}}. \tag{100}$$

Finally, suppose that $r_1 > \sqrt{n-1}$. Again, it follows from (98) and initial conditions $f(r_0) = 1$ and $f'(r_0) < 0$ that

$$f'(r) < -\sqrt{\frac{2K_n}{(n-1)^{n-1}}} \ \text{ and } \ 0 < f(r) < 1 \ \ \forall r \in (r_0, \sqrt{n-1}). \tag{101}$$

At $r = \sqrt{n-1}$, we conclude from (85), (98), (101) and continuity that

$$S\left(\sqrt{n-1}\right) \geq \frac{K_n}{(n-1)^{n-1}}, \tag{102}$$

$$f'\left(\sqrt{n-1}\right) < -\sqrt{\frac{2K_n}{(n-1)^{n-1}}} \ \text{ and } \ 0 < f(\sqrt{n-1}) < 1. \tag{103}$$

It follows from (86) and (102) that $S'(r) > 0$ and $S(r) \geq \frac{K_n}{(n-1)^{n-1}}$ when $r > \sqrt{n-1}$. Combining these properties with (85), we conclude that inequality (98) extends to

$$\frac{\left(f'(r)\right)^2}{2} + \frac{(f(r))^2}{2}\left(\ln(f(r) - \frac{1}{2}\right) \geq \frac{K_n}{(n-1)^{n-1}} \ \ \forall r \in \left[\sqrt{n-1}, r_1\right). \tag{104}$$

It easily follows from (103) and (104) that(101) holds on $\left[\sqrt{n-1}, r_1\right)$. That is,

$$f'(r) < -\sqrt{\frac{2K_n}{(n-1)^{n-1}}} \ \text{ and } \ 0 < f(r) < 1 \ \forall r \in \left[\sqrt{n-1}, r_1\right). \tag{105}$$

An integration of $f'(r) < -\sqrt{\frac{2K_n}{(n-1)^{n-1}}}$ implies that $r_1 < \infty$, $f(r_1) = 0$ and $f'(r_1) \leq -\sqrt{\frac{2K_n}{(n-1)^{n-1}}}$. This completes the proof of Lemma 7. □

3. Conclusions

In this paper we prove the uniqueness of the positive ground state $u_0(r) = \exp\left(-\frac{r^2}{4} + \frac{n}{2}\right)$, which satisfies (1), (2) and (3). Our main theoretical advance is to develop an approach to proving the uniqueness which is different from traditional methods. Our method combines $u_0(r)$ with estimates derived from associated energy functionals and Ricatti equations. It is hoped that future extensions of our techniques can be combined with methods in previous studies to resolve open problems such as the following:

Problem 1. When $1 < n < 9$ determine whether $u_0(r)$ is the only positive solution of

$$\Delta u + u\ln(|u|)) = 0, \tag{106}$$

such that

$$u\,(x_1, x_2, .., x_N) \to 0 \ \text{ as } \ |(x_1, x_2, .., x_N)| \to \infty. \tag{107}$$

Problem 2. When $n > 1$ are sign changing solutions of (1) and (2) with prescribed numbers of zeros unique? What is the physical role of these solutions for the logarithmic Schrödinger equation (4)? Do they represent higher energy states? Are they stable?

Problem 3. Real variable models. Determine the stability of the positive ground state solution $u_0(r)$ of the real variable partial differential equations

$$\frac{\partial u}{\partial t} = \Delta u + u\ln(|u|)), \tag{108}$$

and

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + u\ln(|u|)). \tag{109}$$

The Appendix shows how (108) and (109) arise from classical models through a limiting process as $p \to 1^+$. A first step in proving stability is to linearize (108) around $u_0(r)$ and set $u = u_0 + \epsilon e^{\lambda t} v$, where $\epsilon \ll 1$. To first order in ϵ, v satisfies

$$\Delta v + (\ln(|u_0(r)|)) + 1 - \lambda)\, v = 0. \tag{110}$$

The bounded, positive function $v = u_0(r)$ satisfies (110) when $\lambda = 1$. This suggests that $u_0(r)$ is linearly unstable. It remains to resolve the following:

(i) Is $u_0(r)$ also unstable as a solution of the nonlinear equation (108)?
(ii) Are there solutions of of (108) which blow up in finite time, or as $t \to \infty$?
(iii) Investigates the same issues for equation (109).

4. Appendix

Here we have three goals. In part I below we give a standard derivation of

$$u'' + \frac{n-1}{r}u' + u\ln(|u|) = 0 \tag{111}$$

from the dimensionless logarithmic Schrödinger equation

$$\mathrm{i}\psi_t = \Delta\psi + \psi\ln\left(|\psi|^2\right). \tag{112}$$

In parts II and III we apply a limiting process (as $p \to 1^+$) to derive (111) from the classical equation

$$\frac{\partial u}{\partial t} = \Delta u + u|u|^{p-1} - u, \tag{113}$$

and the non-linear Klein–Gordon equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + u|u|^{p-1} - u. \tag{114}$$

I. Recall that $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ and set the wavefunction

$$\psi = \exp\left(-\mathrm{i}\omega t + \frac{\omega}{2}\right) u(x_1, .., x_N).$$

Then (112) reduces to

$$\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} + 2u\ln(|u|) = 0. \tag{115}$$

Define $\tilde{x}_i = \sqrt{2}x_i,\ i = 1, .., N$ and transform (115) into

$$\sum_{i=1}^{N} \frac{\partial^2 u}{\partial \tilde{x}_i^2} + u\ln(|u|) = 0. \tag{116}$$

Substitute $r = \sqrt{\sum_{i=1}^{N} \tilde{x}_i^2}$ into (116) and get (111).

II. Next, we derive (111) from (113). Set $\tilde{t} = (p-1)t$ and $\tilde{x}_i = \sqrt{p-1}x_i,\ i = 1, .., N,$ and recast (113) in terms of these new coordinates. Divide the resulting equation by $p-1$ and get

$$\frac{\partial u}{\partial \tilde{t}} = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial \tilde{x}_i^2} + \frac{\left(u\left(|u|^{p-1}\right) - u\right)}{p-1}. \tag{117}$$

A formal application of L'Hôpital's rule gives $\lim_{p\to 1^+} \frac{(u(|u|^{p-1})-u)}{p-1} = u\ln(|u|)$. Combining this result with (117) gives

$$\frac{\partial u}{\partial \tilde{t}} = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial \tilde{x}_i^2} + u\ln(|u|). \tag{118}$$

Time independent, radially symmetric solutions of (118) satisfy (111).
III. Finally, consider the nonlinear Klein–Gordon equation (114). The same process described above in part II (with $\tilde{t} = (p-1)t$ replaced by $\tilde{t} = \sqrt{p-1}t$) reduces (114) to

$$\frac{\partial^2 u}{\partial \tilde{t}^2} = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial \tilde{x}_i^2} + u\ln(|u|). \tag{119}$$

Time-independent, radially symmetric solutions of (119) satisfy (111).

Problem 4. Berestycki and Lions [2] proved that (114) has positive ground state solutions when $N \geq 1$. Berestycki and Cazenave [3] (see also [25]) proved strong instability of the ground state when $n \geq 3$ and $1 < p < 1 + \frac{4}{n-2}$, and that perturbations from the ground state blow up in finite time.

(a) Determine stability properties of the positive ground state solution $u_0(\tilde{r})$ of (111) as a solution of the time dependent equation (119).
(b) For equation (119), determine whether perturbations from the ground state can blow up in finite time, or as $t \to \infty$.

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