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Splash Singularities for the One-Phase Muskat Problem in Stable Regimes

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Abstract

In this paper we show the existence in finite time of splash singularities for the one-phase Muskat problem.

1. Introduction

This paper establishes some scenarios where the 2D Muskat problem produces splash singularities; that is to say, we prove that a free boundary evolving by the Muskat problem collapses at a single point while the interface remains smooth. The situation is stable; we show geometries for initial data where the Rayleigh-Taylor condition holds.

The singularities we construct are "splash" singularities in which the interface self-intersects at a single point at the time of breakdown T_* as in Fig. 1. Our previous papers [6,7] showed the existence of a splash singularity for the water wave problem. The strategy there was to start with a "splash" singularity at the time T_*, then solve the water wave equation backwards in time. This yields a solution to the water wave equation in a time interval $[T_* - \varepsilon, T_*]$ that is well behaved at any time $[T_* - \varepsilon, T_*)$ but exhibits a splash at time T_*. In our present setting, we cannot use that strategy because the Muskat problem in the stable regime is parabolic and therefore cannot be solved backwards in time. The importance of this issue is made clear by the fact that water waves can form a "splat" singularity [7] whereas the Muskat solution cannot [15] (a "splat" occurs when, at the time of breakdown, the interface self-intersects along an arc). On the other hand, an analysis of the Muskat problem has in common with our previous work on water waves a conformal map to the "tilde domain", see [7].

Recall the Muskat problem, which describes the evolution of two fluids of different nature in porous media. Both fluids are assumed to be immiscible and incompressible, the most common example for applications being the dynamics of

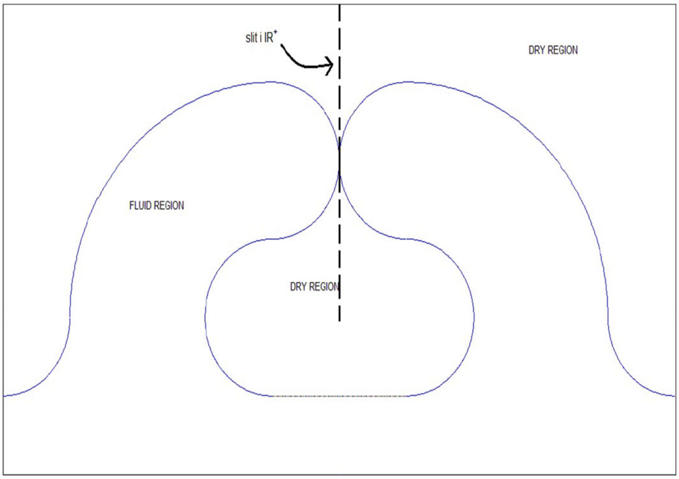

Fig. 1. Splash singularity

water and oil [3]. In two dimensions, the two fluids occupy the connected open set $D(t)$ and $\mathbb{R}^2 \smallsetminus D(t)$ respectively. The characteristics of the fluids are their constant densities and viscosities. Then the step functions $\rho(x,t)$ and $\mu(x,t)$ represent the density and viscosity respectively in the porous medium given by:

$$\rho(x,t) = \begin{cases} \rho_0, & x \in D(t), \\ \overline{\rho}_0, & x \in \mathbb{R}^2 \smallsetminus D(t), \end{cases}$$
$$\mu(x,t) = \begin{cases} \mu_0, & x \in D(t), \\ \overline{\mu}_0, & x \in \mathbb{R}^2 \smallsetminus D(t), \end{cases}$$

for $x \in \mathbb{R}^2$, $t \geqq 0$; here, $\rho_0, \overline{\rho}_0, \mu_0, \overline{\mu}_0$ constant values. The main concern is about the dynamics of the common free boundary $\partial D(t)$, which is given by using the experimental Darcy's law:

$$\mu(x,t)v(x,t) = -\nabla p(x,t) - (0, \rho(x,t)). \tag{1}$$

Here $v(x,t) = (v_1(x,t), v_2(x,t))$ is the incompressible velocity

$$\nabla \cdot v(x,t) = 0, \tag{2}$$

and $p(x,t)$ is the scalar pressure. Above, the permeability of the media and the gravity constant are set equal to one without loss of generality.

The Muskat problem is a long standing matter [26] of recognized importance, especially because of its connection with the evolution of fluids in Hele-Shaw cells. In that setting the fluids are confined inside two closely parallel flat surfaces in such a way that the dynamics is essentially two dimensional. The Hele-Shaw evolution law is given by

$$\frac{12}{b^2}\mu(x,t)v(x,t) = -\nabla p(x,t) - (0, \rho(x,t)),$$

where b is the distance between the surfaces. Therefore, it is possible to observe that for both different scenarios comparable phenomena and properties hold [29].

A main feature of the problem is the appearance of instabilities, which have been shown in different situations [27,30]. From a contour dynamics point of view, the system of equations for the free boundary is essentially ill-posed from a Hadamard point of view [12,29]. Although the Muskat problem with surface tension taken into account is well-posed [2,17], it nevertheless shows fingering [18] and exponentially growing modes [24].

On the other hand, the Muskat problem is well-posed in stable regimes without surface tension [11,12,14]. This situation is reached for the problem when the normal component of the jumps in the pressure gradients at the free interface is positive [1]. Then it is said that the Rayleigh-Taylor condition holds. In such a case, linearizing the contour equation leads to the following [12]:

$$f_t^L(\alpha, t) = -\sigma \Lambda f^L(\alpha, t), \tag{3}$$

where $(\alpha, f^L(\alpha, t))$ represents the free boundary ($\alpha \in \mathbb{R}$), σ is the Rayleigh-Taylor function and the operator Λ is the square root of the negative Laplacian. Then, the fact that $\sigma > 0$ turns the Muskat problem into a parabolic system at the linear level. This fact has been used to prove global-in-time regularity and instant analyticity for small initial data in different situations [4,9,12,18,23,29].

For the case of equal viscosities ($\mu_0 = \overline{\mu}_0$), the Rayleigh-Taylor condition holds when the more dense fluid lies below the interface and the less dense fluid lies above it [12]. In this situation, the regime is stable if the free boundary ∂D is represented by the graph of a function $(\alpha, f(\alpha, t))$. In particular, it is possible to get a decay of the L^∞ norm [13] as follows:

$$\left\| f - \frac{1}{2\pi} \int_{-\pi}^{\pi} f_0 \mathrm{d}\alpha \right\|_{L^\infty}(t) \leqq \left\| f_0 - \frac{1}{2\pi} \int_{\mathbb{T}} f_0 \mathrm{d}\alpha \right\|_{L^\infty} e^{-Ct},$$

for $f(\alpha + 2\pi, t) = f(\alpha, t)$; and with $f(\alpha, t) \in L^2(\mathbb{R})$

$$\|f\|_{L^\infty}(t) \leqq \|f_0\|_{L^\infty}(1 + Ct)^{-1}, \quad C = C(f_0) > 0.$$

It is easy to check that above formulas provide the same rate of decay as Equation (3) for f^L at the linear level. On the other hand, the L^2 norm evolution allows one to control half a derivative for f^L due to the identity

$$\|f^L\|_{L^2}^2(t) + 2\sigma \int_0^t \left\| \Lambda^{1/2} f^L \right\|_{L^2}^2 (s)\mathrm{d}s = \left\| f_0^L \right\|_{L^2}^2,$$

while at the nonlinear level the following equality

$$\|f\|_{L^2}^2(t) + \frac{\sigma}{\pi} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \ln\Big(1 + \Big(\frac{f(\alpha, s) - f(\beta, s)}{\alpha - \beta}\Big)^2\Big) \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}s = \|f_0\|_{L^2}^2,$$

does not give a chance of gaining any regularity [9].

The case of a drop on a solid substrate in porous media has been studied in [25]. This case considers the dynamics of one fluid, also known as the one-phase Muskat problem. The authors show local well-posedness of the problem with estimates independent of the contact angle.

In [8], solutions of the Muskat equation are exhibited for initial smooth stable graphs; those solutions enter an unstable regime by becoming non-graphs in finite time. The pattern is far from trivial and recently it has been shown to be richer for the inhomogeneous and confined problems (see [22] and references therein). In particular the significance of a turnover (non-graph scenario) is that the Rayleigh-Taylor condition breaks down. Furthermore, [5] there exist smooth initial data in the stable regime for the Muskat problem such that the solutions turn to the unstable regime and later the regularity breaks down. Therefore global existence is false for some large initial data in the stable regime, as the solutions develop singularities in finite time.

In this paper we show that the Muskat problem with initial data in stable regimes can develop singularities. The singularity is a splash, where for the free boundary given by

$$\partial D(t) = \{z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) : \alpha \in \mathbb{R}\}, \tag{4}$$

there exist a blow-up time $T_s > 0$ and a point $x_s \in \mathbb{R}^2$ such that $x_s = z(\alpha_1, T_s) = z(\alpha_2, T_s)$ for $\alpha_1 \neq \alpha_2$. In particular the curve is regular, and satisfies the chord-arc condition up to the time T_s:

$$|z(\alpha, t) - z(\beta, t)| \geqq C_{ca}(t)|\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}, \quad C_{ca}(t) > 0, \quad t \in [0, T_s).$$

Free boundary incompressible fluid equations can develop splash singularities. This scenario has been shown for the incompressible Euler equations in the water waves form [6,7] which considers the evolution of a free boundary given by air, with density 0, and water, with density 1 and irrotational velocity. This type of singularity can also be shown for the case with vorticity [16]. For the case of two incompressible fluids with positive densities, this scenario has been recently ruled out [20]. Similarly, for Muskat this type of singularity does not also hold in the case in which $\mu_0 = \overline{\mu}_0$ and $\rho_0 \neq \overline{\rho}_0$ [21].

In this work we show finite time splash singularities with $\overline{\rho}_0 = \overline{\mu}_0 = 0$:

$$(\rho(x, t), \mu(x, t)) = \begin{cases} (\rho_0, \mu_0) & x \in D(t), \\ (0, 0), & x \in \mathbb{R}^2 \setminus D(t), \end{cases} \tag{5}$$

dealing with one fluid dynamics with $\mathbb{R}^2 \setminus D(t)$ a dry region. In those scenarios the fluid essentially lies below the dry region: there is $M > 1$ such that $\mathbb{R} \times (-\infty, -M] \subset D(t)$. Then, the free boundary will be asymptotically flat: $z(\alpha, t) - (\alpha, 0) \to 0$ as $\alpha \to \infty$, or periodic in the x_1 direction: $z(\alpha + 2\pi, t) = z(\alpha, t) + (2\pi, 0)$. The energy of the system is finite,

$$\int_{D(t)} |v(x, t)|^2 \mathrm{d}x < \infty,$$

yielding physical relevant scenarios. In those cases we provide some geometries for the interface where the Rayleigh-Taylor condition is satisfied, getting rid of unstable situations. The main theorem of the paper is the following:

Theorem 1.1. *There exists an open set of curves $\mathcal{O} \subset H^3$, satisfying the chord-arc and Rayleigh-Taylor condition, such that for any $z_0 \in \mathcal{O}$ the solution of Muskat* (1,2,4,5) *with $z(\alpha, 0) = z_0(\alpha)$ violates the chord-arc condition at a finite time $T_s = T_s(z_0) > 0$. In addition, this holds in such a way that $z(\alpha_1, T_s) = z(\alpha_2, T_s)$ with $\alpha_1 \neq \alpha_2$.*

Remark 1.2. At the time T_s the Muskat system (1,2,4,5) breaks down.

In the rest of the paper we show the proof of above result, splitting it in several sections. In Section 2 we construct a family of curves z^l for which there is a unique self-intersection point x_s where $x_s = z^l(\alpha_1) = z^l(\alpha_2)$ with $\alpha_1 \neq \alpha_2$ and $\partial_\alpha z_1^l(\alpha_1) = \partial_\alpha z_1^l(\alpha_2) = 0$. Plugging these curves in Darcy's law, we find that the Rayleigh-Taylor condition holds. Furthermore, the velocity indicates that the self-intersection point is going to disappear going backward in time. A more general scenario can be found in Section 7. In Section 3 we show how to make sense of the problem with a self-intersecting interface, transforming the Muskat problem into a new contour dynamics equation we call P(Muskat). Up to the time of the splash we can recover Muskat from P(Muskat), but at the time of splash P(Muskat) makes sense and it is possible to go further in time. In Section 4 we prove the local existence of the P(Muskat) system. In Section 5 we show a stability result for P(Muskat). Finally, in Section 6 we show how the family of curves $z^l(\alpha)$ together with the existence and stability for P(Muskat) allow us to conclude the proof of Theorem 1.1.

2. Self-Intersecting Stable Curves With Suitable Sign of Velocity

In this section we show that there exits a family of splash curves satisfying the Rayleigh-Taylor condition, with velocities which separate the splash point running backward-in-time.

First we use Hopf's lemma (see [19] for example) to achieve the Rayleigh-Taylor condition. Taking divergence in Darcy's law (1) we have

$$\Delta p(x, t) = 0$$

for any $x \in D(t)$. In addition, the continuity of the pressure on the free boundary [11] and the fact that

$$-\nabla p(x, t) = (0, 0)$$

for any x in the interior of $\mathbb{R}^2 \setminus D(t)$ allow us to get

$$p(z(\alpha, t), t) = 0.$$

Also

$$\lim_{x_2 \to -\infty} v(x, t) = 0,$$

and therefore Darcy's law gives

$$\lim_{x_2\to-\infty} \partial_{x_1} p(x,t) = 0,$$
$$\lim_{x_2\to-\infty} \partial_{x_2} p(x,t) = -\rho_0.$$

It is possible to find that $p(x,t) - (-\rho_0 x_2 + c(t)) \to 0$ when $x_2 \to -\infty$ and to conclude that the pressure is positive in $D(t)$ by the maximum principle for harmonic functions. In this situation we can apply Hopf's lemma to obtain that

$$-\nabla p(z(\alpha,t),t)\cdot\partial_\alpha^\perp z(\alpha,t) > 0, \tag{6}$$

where $\partial_\alpha^\perp z(\alpha,t) = (-\partial_\alpha z_2(\alpha,t), \partial_\alpha z_1(\alpha,t))$ is the normal vector pointing out the domain $D(t)$. In the periodic setting, compactness provides

$$-\nabla p(z(\alpha,t),t)\cdot\partial_\alpha^\perp z(\alpha,t) \geqq k(t) > 0, \tag{7}$$

for any $\alpha \in \mathbb{R}$. In the asymptotically flat scenario, (6) again implies (7) provided we restrict α to lie in a bounded interval. On the other hand, for large $|\alpha|$, Darcy's law implies

$$\mu_0 v(z(\alpha,t),t)\cdot\partial_\alpha^\perp z(\alpha,t) = -\nabla p(z(\alpha,t),t)\cdot\partial_\alpha^\perp z(\alpha,t) - \rho_0\partial_\alpha z_1(\alpha).$$

Since the $v \to 0$ and $\partial_\alpha z_1(\alpha) \to 1$ as $|\alpha| \to \infty$, (7) holds for large α thus (7) holds in all cases.

Next we deal with curves $z^l(\alpha)$ with a unique splash point $x_s = z^l(\alpha_1) = z^l(\alpha_2)$ for $\alpha_1 \neq \alpha_2$ where

$$\partial_\alpha z_1^l(\alpha_1) = \partial_\alpha z_1^l(\alpha_2) = 0.$$

We show that this configuration provides a sign for the velocity at x_s. Taking the trace of Darcy's law to the curve and multiplying by $\partial_\alpha^\perp z^l(\alpha)$ we have that

$$\mu_0 v(z^l(\alpha))\cdot\partial_\alpha^\perp z^l(\alpha) = -\nabla p(z^l(\alpha))\cdot\partial_\alpha^\perp z^l(\alpha) - \rho_0\partial_\alpha z_1^l(\alpha).$$

Thanks to our choice of the splash curve it must be satisfied

$$v(z^l(\alpha_i))\cdot\partial_\alpha^\perp z^l(\alpha_i) = -\mu_0^{-1}\nabla p(z^l(\alpha_i))\cdot\partial_\alpha^\perp z^l(\alpha_i) \geqq c > 0, \quad i = 1,2, \tag{8}$$

where again we have used Hopf's Lemma (7). It is clear that (8) implies that the velocity separates the splash point backwards in time. In Fig. 1 we give a graphic sketch of the kind of splash singularities we are considering.

Theses curves yield the simplest splash scenario we can consider. In Section 7 we show the existence of different geometries that give rise to a splash singularity for the one-phase Muskat problem.

3. Transformation to a Non-Splash Scenario

This section is devoted to transform the system into a new contour evolution equation which we use to handle the splash singularity. We consider solutions of Muskat satisfying (1, 2, 4, 5) for regular $z(\alpha, t)$ satisfying the chord-arc condition. Taking limit as $x \to z(\alpha, t)$ from $D(t)$ we find

$$v(z(\alpha, t), t) = u(\alpha, t),$$

where

$$u(\alpha, t) = \mathrm{BR}(z, \omega)(\alpha, t) + \frac{\omega(\alpha, t)}{2} \frac{z_\alpha(\alpha, t)}{|z_\alpha(\alpha, t)|^2}.$$

BR stands for the Birkhoff-Rott integral, which is given by

$$\mathrm{BR}(\alpha, t) = \mathrm{BR}(z, \omega)(\alpha, t) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(z(\alpha, t) - z(\alpha - \beta, t))^{\perp}}{|z(\alpha, t) - z(\alpha - \beta, t)|^2} \omega(\alpha - \beta, t) \mathrm{d}\beta, \tag{9}$$

and ω is the amplitude of the vorticity concentrated on the free boundary:

$$(\partial_{x_1} v_2 - \partial_{x_2} v_1)(x, t) = \omega(\beta, t)\delta(x = z(\beta, t)).$$

By approaching the contour in Darcy's law and taking the dot product with $\partial_\alpha z(\alpha, t)$, it is easy to relate the amplitude of the vorticity and the free boundary by an elliptic implicit equation:

$$\omega(\alpha, t) = -2\mathrm{BR}(z, \omega)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - 2\frac{\rho_0}{\mu_0} \partial_\alpha z_2(\alpha, t). \tag{10}$$

We have the dynamics given by the contour equation

$$z_t(\alpha, t) = u(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t), \tag{11}$$

where c represents reparametrization freedom. See [11] for a detailed derivation of the system.

From now on we establish the transformation in the periodic setting. In the asymptotically flat case the map is different but the same properties follow using elementary complex variable arguments. In the periodic setting, we regard $z(\alpha, t)$ as a point in the cylinder $\mathbb{C}/2\pi\mathbb{Z}$ obtained by identifying points in the complex plane that differ by a multiple of 2π. We will transform the system with the conformal map:

$$P(w) = \left(\tan(w/2)\right)^{1/2}, \quad w \in (\mathbb{C}/2\pi\mathbb{Z})\backslash i\mathbb{R}^+$$

and $i\mathbb{R}^+$ intersects the curve only at the splash point and make sure that the splash point lies in $i\mathbb{R}^+$. Above, the branch of the square root is chosen in such a way that $P(z^l(\alpha))$ becomes a one-to-one closed curve. In this setting the inverse map of P is P^{-1} which is well defined and smooth from $\mathbb{C}$ to $\mathbb{C}/2\pi\mathbb{Z}$. See [7].

We then consider by this new transformation the curve $\tilde{z}(\alpha, t) = P(z(\alpha, t))$. This provides easily

$$\tilde{z}_\alpha(\alpha, t) = \nabla P(z(\alpha, t)) z_\alpha(\alpha, t),$$

and

$$\begin{aligned}\tilde{z}_t(\alpha, t) &= \nabla P(z(\alpha, t)) z_t(\alpha, t) = \nabla P(z(\alpha, t))(u(\alpha, t) + c(\alpha, t) z_\alpha(\alpha, t)) \\ &= \nabla P(z(\alpha, t)) u(\alpha, t) + c(\alpha, t)\tilde{z}_\alpha(\alpha, t).\end{aligned}$$

For the potential $\phi(x, t)$ ($\nabla\phi(x, t) = v(x, t)$) we define in the tilde domain $\tilde{\phi}(\tilde{x}, t) = \phi(x, t)$. Then

$$v(x, t) = \nabla\phi(x, t) = (\nabla\tilde{\phi})(P(x), t)\nabla P(x) = \nabla P(x)^T(\nabla\tilde{\phi})(P(x), t).$$

Taking the limit we find

$$u(\alpha, t) = \nabla P(z(\alpha, t))^T(\nabla\tilde{\phi})(P(z(\alpha, t)), t) = \nabla P(z(\alpha, t))^T\tilde{u}(\alpha, t),$$

where $\tilde{u}(\alpha, t) = \nabla\tilde{\phi}(\tilde{z}(\alpha, t), t)$. It yields

$$\tilde{z}_t(\alpha, t) = Q^2(\alpha, t)\tilde{u}(\alpha, t) + c(\alpha, t)\tilde{z}_\alpha(\alpha, t), \tag{12}$$

where Q^2 is given by

$$\nabla P(z(\alpha, t))\nabla P(z(\alpha, t))^T = Q^2(\alpha, t)I,$$

and I is the 2×2 identity matrix. In other words,

$$Q^2(\alpha, t) = \left|\frac{\mathrm{d}P}{\mathrm{d}w}(z(\alpha, t))\right|^2 = \left|\frac{\mathrm{d}P}{\mathrm{d}w}(P^{-1}(\tilde{z}(\alpha, t)))\right|^2. \tag{13}$$

Next we consider the velocity $\tilde{v}$ defined on the whole space by

$$\tilde{v}(\tilde{x}, t) = \nabla\tilde{\phi}(\tilde{x}, t) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(\tilde{x} - \tilde{z}(\alpha - \beta, t))^\perp}{|\tilde{x} - \tilde{z}(\alpha - \beta, t)|^2}\tilde{\omega}(\alpha - \beta, t)\mathrm{d}\beta,$$

where

$$(\partial_{\tilde{x}_1}\tilde{v}_2 - \partial_{\tilde{x}_2}\tilde{v}_1)(\tilde{x}, t) = \tilde{\omega}(\beta, t)\delta(\tilde{x} = \tilde{z}(\beta, t)),$$

in a distributional sense. Approaching the free boundary it is possible to obtain

$$\tilde{u} = \mathrm{BR}(\tilde{z}, \tilde{\omega}) + \frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\tilde{z}_\alpha. \tag{14}$$

In order to close the system we integrate Darcy's law to find

$$\mu_0\phi(z(\alpha, t), t) = -p(z(\alpha, t), t) - \rho_0 z_2(\alpha, t) = -\rho_0 z_2(\alpha, t),$$

due to the continuity of the pressure at the free boundary and the vacuum state. The conformal map P provides

$$\mu_0\tilde{\phi}(\tilde{z}(\alpha, t), t) = -\rho_0 P_2^{-1}(\tilde{z}(\alpha, t)), \tag{15}$$

where $P^{-1}(\tilde{z}) = (P_1^{-1}(\tilde{z}), P_2^{-1}(\tilde{z}))$. Taking one derivative and applying identity (14) allows us to find

$$\mu_0 \left(\mathrm{BR}(\tilde{z}, \tilde{\omega}) \cdot \tilde{z}_\alpha + \frac{\tilde{\omega}}{2} \right) = -\rho_0 \partial_\alpha (P_2^{-1}(\tilde{z})).$$

We rewrite the above identity as

$$\tilde{\omega}(\alpha, t) = -2\mathrm{BR}(\tilde{z}, \tilde{\omega})(\alpha, t) \cdot \tilde{z}_\alpha(\alpha, t) - 2\frac{\rho_0}{\mu_0} \partial_\alpha \left(P_2^{-1}(\tilde{z}(\alpha, t)) \right). \tag{16}$$

Identities (12) and (14) give

$$\tilde{z}_t(\alpha, t) = Q^2(\alpha, t)\mathrm{BR}(\tilde{z}, \tilde{\omega})(\alpha, t) + \tilde{c}(\alpha, t)\tilde{z}_\alpha(\alpha, t), \tag{17}$$

for $\tilde{c} = Q^2\tilde{\omega}/(2|\tilde{z}_\alpha|^2) + c$. We pick

$$\begin{aligned} \tilde{c}(\alpha, t) = \frac{\alpha + \pi}{2\pi} \int_{-\pi}^{\pi} \partial_\beta (Q^2 \mathrm{BR}(\tilde{z}, \tilde{\omega}))(\beta, t) \cdot \frac{\tilde{z}_\beta(\beta, t)}{|\tilde{z}_\beta(\beta, t)|^2} \mathrm{d}\beta \\ - \int_{-\pi}^{\alpha} \partial_\beta (Q^2 \mathrm{BR}(\tilde{z}, \tilde{\omega}))(\beta, t) \cdot \frac{\tilde{z}_\beta(\beta, t)}{|\tilde{z}_\beta(\beta, t)|^2} \mathrm{d}\beta, \end{aligned} \tag{18}$$

which provides a tangential component $|\tilde{z}_\alpha|$ depending only on the variable t. We end up with a contour equation given by (16–18).

Finally we will find the Rayleigh-Taylor condition in terms of $\tilde{z}$. We define $\tilde{p}(\tilde{x}, t) = p(x, t)$ to obtain, with Darcy's law,

$$-\nabla \tilde{p}(\tilde{x}, t) = \mu_0 \nabla \tilde{\phi}(\tilde{x}, t) + \rho_0 \nabla P_2^{-1}(\tilde{x}).$$

Approaching the free boundary, we easily find that

$$\tilde{\sigma}(\alpha, t) = -\nabla \tilde{p}(\tilde{z}(\alpha, t), t) \cdot \tilde{z}_\alpha^\perp = \mu_0 \,\mathrm{BR}(\tilde{z}, \tilde{\omega}) \cdot \tilde{z}_\alpha^\perp + \rho_0 \nabla P_2^{-1}(\tilde{z}(\alpha, t)) \cdot \tilde{z}_\alpha^\perp. \tag{19}$$

4. Local-Existence in the Tilde Domain

This section is devoted to show local existence for $\tilde{z}$ solutions of (16–18) with $\tilde{z} \in C([0, T]; H^k)$ with $k \geqq 3$. The main difficulty lies in finding *a priori* estimates for the system. In what follows we skip the details on how to pass from the *a priori* estimates to obtain solutions of the system (the relevant arguments may be found in [11]). In order to simplify the exposition we suppress the time variable and the tilde in the equation. We define

$$q^0 = (0, 0), \quad q^1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad q^2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad q^3 = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right),$$
$$q^4 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right),$$

which are the singular points of the P^{-1} conformal map. We set $z(\alpha, t)$ to satisfy $\tilde{z}(\alpha, t) \neq q^l$ for $l = 0, ..., 4$. In order to get this we fix $\overline{D(0)}$ so that $\frac{dP}{dw}(w) \neq 0$ for any $w \in \overline{D(0)}$ without loss of generality. We will check that this property remains true for a short time. Next we define the quantity

$$E_k(z, t) = E_k(t) = \|z\|^2_{H^k}(t) + \|F(z)\|^2_{L^\infty}(t) + \frac{1}{m(Q^2\sigma)(t)} + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)}, \tag{20}$$

where

$$F(z)(\alpha, \beta) = \frac{|\beta|}{|z(\alpha) - z(\alpha - \beta)|}, \quad \alpha, \beta \in [-\pi, \pi],$$

and

$$m(Q^2\sigma)(t) = \min_{\alpha \in \mathbb{T}} Q^2(\alpha, t)\sigma(\alpha, t), \quad m(q^l)(t) = \min_{\alpha \in \mathbb{T}} |z(\alpha, t) - q^l|.$$

We state the main result.

Theorem 4.1. *Let* $z(\alpha, 0) = z_0(\alpha) \in H^k(\mathbb{T})$ *for* $k \geqq 3$, $F(z_0) \in L^\infty$, $m(Q^2\sigma)(0) > 0$ *and* $m(q^l)(0) > 0$ *for* $l = 0, ..., 4$. *Then there exists a time* $T > 0$ *so that there is a unique solution* $z(\alpha, t)$ *of* (16–18) *in* $C([0, T]; H^k)$.

We shall show a proof of the energy estimates.

Proposition 4.2. *Let* $z(\alpha, t)$ *be a solution of* (16–18). *Then, the following estimate holds:*

$$\frac{d}{dt} E_k(t) \leqq C(E_k(t))^p$$

for $k \geqq 3$. *The constants* C *and* p *depend only on* k.

Below we will show the proof for $k = 3$, the rest of the cases being analogous. As in [11], we can show that

$$\frac{d}{dt}\left(\|z\|^2_{L^2}(t) + \|F(z)\|^2_{L^\infty}(t) + \frac{1}{m(Q^2\sigma)(t)} + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)}\right) \leqq C(E_k(t))^p.$$

Next we study

$$\frac{d}{dt} \left\|\partial^3_\alpha z\right\|^2_{L^2}(t) = 2\int \partial^3_\alpha z(\alpha) \cdot \partial^3_\alpha z_t(\alpha) d\alpha.$$

We can estimate most of the terms as in [11]. We also quote [6] for dealing with the Q^2 factor. This factor does not introduce any unbounded character as

$$\|Q^2\|_{H^k} \leqq C(E_k(t))^p.$$

We will show how to deal with the unbounded and therefore singular terms. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_\alpha^3 z\right\|_{L^2}^2(t) \leqq C(E_k(t))^p + I,$$

for

$$I = \int \partial_\alpha^3 z(\alpha)\cdot Q^2(\alpha)\frac{1}{\pi}\int \frac{(z(\alpha)-z(\alpha-\beta))^\perp}{|z(\alpha)-z(\alpha-\beta)|^2}\partial_\alpha^3\omega(\alpha-\beta)\mathrm{d}\beta\mathrm{d}\alpha.$$

We get $I \leqq C(E_k(t))^p + II$, where

$$II = \int \partial_\alpha^3 z(\alpha)\cdot \frac{z_\alpha^\perp(\alpha)}{|z_\alpha(\alpha)|^2}Q^2(\alpha)H(\partial_\alpha^3\omega)(\alpha)\mathrm{d}\alpha.$$

The identity $H(\partial_\alpha) = \Lambda$ allows us to rewrite II as

$$II = \frac{1}{|z_\alpha(\alpha)|^2}\int \Lambda\left(\partial_\alpha^3 z\cdot z_\alpha^\perp Q^2\right)(\alpha)\partial_\alpha^2\omega(\alpha)\mathrm{d}\alpha.$$

Next we can use formula (16) to further split $II = III + IV$ where

$$III = \frac{-2}{|z_\alpha(\alpha)|^2}\int \Lambda\left(\partial_\alpha^3 z\cdot z_\alpha^\perp Q^2\right)(\alpha)\partial_\alpha^2(\mathrm{BR}(z,\omega)\cdot z_\alpha)(\alpha)\mathrm{d}\alpha,$$

and

$$IV = \frac{-2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int \Lambda\left(\partial_\alpha^3 z\cdot z_\alpha^\perp Q^2\right)(\alpha)\partial_\alpha^3(P_2^{-1}(z))(\alpha)\mathrm{d}\alpha.$$

The term III can be estimated as K_3 in p. 514 of [11]. An analogous approach provides

$$III \leqq C(E_k(t))^p - \frac{2}{|z_\alpha(\alpha)|^2}\int Q^2(\alpha)\mathrm{BR}(z,\omega)(\alpha)\cdot z_\alpha^\perp(\alpha)\partial_\alpha^3 z(\alpha) \cdot\Lambda\left(\partial_\alpha^3 z\right)(\alpha)\mathrm{d}\alpha. \tag{21}$$

For IV we consider the most singular terms as the rest are bounded: $IV \leqq C(E_k(t))^p + V$ where

$$V = -\frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int \Lambda\left(\partial_\alpha^3 z\cdot z_\alpha^\perp Q^2\right)(\alpha)\left(\nabla P_2^{-1}\right)(z(\alpha))\cdot\partial_\alpha^3 z(\alpha)\mathrm{d}\alpha.$$

Then we split further $V = VI + VII + VIII + IX$ by writing the components of the curve:

$$VI = \frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int \Lambda\left(\partial_\alpha^3 z_1\partial_\alpha z_2 Q^2\right)(\alpha)\partial_{\tilde{x}_1}P_2^{-1}(z(\alpha))\partial_\alpha^3 z_1(\alpha)\mathrm{d}\alpha,$$

$$VII = \frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int \Lambda\left(\partial_\alpha^3 z_1\partial_\alpha z_2 Q^2\right)(\alpha)\partial_{\tilde{x}_2}P_2^{-1}(z(\alpha))\partial_\alpha^3 z_2(\alpha)\mathrm{d}\alpha,$$

$$VIII = -\frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int \Lambda\left(\partial_\alpha^3 z_2\partial_\alpha z_1 Q^2\right)(\alpha)\partial_{\tilde{x}_1}P_2^{-1}(z(\alpha))\partial_\alpha^3 z_1(\alpha)\mathrm{d}\alpha,$$

$$IX = -\frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int \Lambda\left(\partial_\alpha^3 z_2\partial_\alpha z_1 Q^2\right)(\alpha)\partial_{\tilde{x}_2}P_2^{-1}(z(\alpha))\partial_\alpha^3 z_2(\alpha)\mathrm{d}\alpha.$$

The commutator estimate

$$\|\Lambda(gf) - g\Lambda f\|_{L^2} \leqq C\|g\|_{C^{1,\frac{1}{3}}}\|f\|_{L^2}$$

yields

$$VI \leqq C(E_k(t))^p + \frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int Q^2(\alpha)\partial_{\tilde{x}_1}P_2^{-1}(z(\alpha))\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda\left(\partial_\alpha^3 z_1\right)(\alpha)\mathrm{d}\alpha \tag{22}$$

and

$$IX \leqq C(E_k(t))^p - \frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int Q^2(\alpha)\partial_{\tilde{x}_2}P_2^{-1}(z(\alpha))\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_2(\alpha)\Lambda\left(\partial_\alpha^3 z_2\right)(\alpha)\mathrm{d}\alpha. \tag{23}$$

Similarly, for VII,

$$VII \leqq C(E_k(t))^p + \frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int Q^2(\alpha)\partial_{\tilde{x}_2}P_2^{-1}(z(\alpha))\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha)\Lambda\left(\partial_\alpha^3 z_1\right)(\alpha)\mathrm{d}\alpha.$$

The identity

$$\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha) = -\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha) + \left|\partial_\alpha^2 z(\alpha)\right|^2$$

provides

$$VII \leqq C(E_k(t))^p - \frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int Q^2(\alpha)\partial_{\tilde{x}_2}P_2^{-1}(z(\alpha))\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda\left(\partial_\alpha^3 z_1\right)(\alpha)\mathrm{d}\alpha. \tag{24}$$

Proceeding in a similar manner we can get

$$VIII \leqq C(E_k(t))^p + \frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int Q^2(\alpha)\partial_{\tilde{x}_1}P_2^{-1}(z(\alpha))\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha)\Lambda\left(\partial_\alpha^3 z_2\right)(\alpha)\mathrm{d}\alpha. \tag{25}$$

Adding the inequalities (22)–(25) it is easy to get

$$V \leqq C(E_k(t))^p - \frac{2\rho_0\mu_0^{-1}}{|z_\alpha(\alpha)|^2}\int Q^2(\alpha)\nabla P_2^{-1}(z(\alpha)) \cdot z_\alpha^\perp(\alpha)\partial_\alpha^3 z(\alpha)\cdot\Lambda\left(\partial_\alpha^3 z\right)(\alpha)\mathrm{d}\alpha.$$

The above inequality, together with (21), lets us obtain

$$III \leqq C(E_k(t))^p - \frac{2\mu_0^{-1}}{|z_\alpha(\alpha)|^2} \int Q^2(\alpha)\sigma(\alpha)\partial_\alpha^3 z(\alpha) \cdot \Lambda\left(\partial_\alpha^3 z\right)(\alpha)\mathrm{d}\alpha,$$

with σ given in (19).

Finally we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_\alpha^3 z\right\|_{L^2}^2(t) \leqq C(E_k(t))^p - \frac{2\mu_0^{-1}}{|z_\alpha(\alpha)|^2} \int Q^2(\alpha)\sigma(\alpha)\partial_\alpha^3 z(\alpha) \cdot \Lambda\left(\partial_\alpha^3 z\right)(\alpha)\mathrm{d}\alpha.$$

From the *a priori* energy estimates we have that $m(Q^2\sigma)(t) > 0$, which, together with the pointwise inequality $2f\Lambda(f) \geqq \Lambda(f^2)$ (see [10]), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_\alpha^3 z\right\|_{L^2}^2(t) \leqq C(E_k(t))^p - \frac{\mu_0^{-1}}{|z_\alpha(\alpha)|^2} \int Q^2(\alpha)\sigma(\alpha)\Lambda\left(\left|\partial_\alpha^3 z\right|^2\right)(\alpha)\mathrm{d}\alpha.$$

Integration by parts for the Λ operator gives the desired estimate.

5. Stability for the Muskat Problem

This section is devoted to demonstration of the following result:

Proposition 5.1. *Let $x(\alpha, t)$ and $y(\alpha, t)$ be two curves which satisfy the contour Equation (16–18). Then, the following estimate holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x-y\|_{H^1}(t) \leqq C\left(\sup_{[0,T]} E_3(x,t) + \sup_{[0,T]} E_3(y,t)\right)^p \|x-y\|_{H^1}(t).$$

Above, $E_3(x,t)$ and $E_3(y,t)$ are given by (20). The constants C and p are universal.

Proof. In order to simplify the exposition we suppress the time variable and we denote $f' = f(\alpha-\beta)$, $f = f(\alpha)$, $f_- = f - f'$ and $\int = \int_{\mathbb{T}}$.

We consider two solutions of the system $x(\alpha, t)$ and $y(\alpha, t)$ in $C([0, T]; H^3(\mathbb{T}))$ with γ and ζ its vorticity amplitudes given by (16). We will also denote by Q_x^2, Q_y^2, BR_x, BR_y and c_x, c_y the factors Q^2, Birhoff-Rott integrals and parametrization constants associated with x and y, respectively [see (13), (9) and (18)]. During the time of existence $T > 0$ one finds $\sup_{[0,T]} E_3(x,t)$ and $\sup_{[0,T]} E_3(y,t)$ bounded so that we will write

$$C\left(\sup_{[0,T]} E_3(x,t) + \sup_{[0,T]} E_3(y,t)\right)^p \leqq C,$$

by abuse of notation.

For the function $z(\alpha,t) = x(\alpha,t) - y(\alpha,t)$, one finds

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_{L^2}^2 = \int z \cdot z_t \mathrm{d}\alpha = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int z \cdot \left(Q_x^2 - Q_y^2\right) \mathrm{BR}_x \mathrm{d}\alpha,$$
$$I_2 = \int z \cdot Q_y^2 (\mathrm{BR}_x - \mathrm{BR}_y) \mathrm{d}\alpha,$$
$$I_3 = \int z \cdot (c_x - c_y) x_\alpha \mathrm{d}\alpha,$$
$$I_4 = \int z \cdot c_y z_\alpha \mathrm{d}\alpha.$$

Then for I_1 we find that

$$I_1 \leqq \|z\|_{L^\infty} \|Q_x^2 - Q_y^2\|_{L^2} \|\mathrm{BR}_x\|_{L^2} \leqq C\|z\|_{H^1}^2.$$

In I_2 we split further, as follows:

$$I_{2,1} = \frac{1}{2\pi} \int z \cdot Q_y^2 \int \frac{z_-^\perp}{|x_-|^2} \gamma' \mathrm{d}\beta \mathrm{d}\alpha,$$
$$I_{2,2} = \frac{1}{2\pi} \int z \cdot Q_y^2 \int y_-^\perp \left(\frac{1}{|x_-|^2} - \frac{1}{|y_-|^2} \right) \gamma' \mathrm{d}\beta \mathrm{d}\alpha,$$
$$I_{2,3} = \int z \cdot Q_y^2 \mathrm{BR}(y, \omega) \mathrm{d}\alpha,$$

where $\omega = \gamma - \zeta$. In $I_{2,1}$, for the integral in β, we find a kernel of degree -2 applied to z_-; thus

$$I_{2,1} \leqq C\|z\|_{H^1}^2.$$

Since

$$I_{2,2} = \frac{-1}{2\pi} \int z \cdot Q_y^2 \int y_-^\perp \frac{(x_- + y_-) \cdot z_-}{|x_-|^2 |y_-|^2} \gamma' \mathrm{d}\beta \mathrm{d}\alpha,$$

we again recognize a kernel of degree -2 applied to z_- above, so that

$$I_{2,2} \leqq C\|z\|_{H^1}^2.$$

For $I_{2,3}$ it is easy to check that BR has a kernel of degree -1 and therefore

$$I_{2,3} \leqq C\|z\|_{L^2} \|\omega\|_{L^2}.$$

In order to deal with $\|\omega\|_{L^2}$ we write

$$\omega + 2\mathrm{BR}(x, \omega) \cdot x_\alpha = 2\mathrm{BR}(y, \zeta) \cdot y_\alpha - 2\mathrm{BR}(x, \zeta) \cdot x_\alpha + 2\frac{\rho_0}{\mu_0} \left(\nabla P_2^{-1}(y) \cdot y_\alpha - \nabla P_2^{-1}(x) \cdot x_\alpha \right).$$

Bounds for the operator $(I + 2\mathrm{BR}(x, \cdot) \cdot x_\alpha)^{-1}$ (see [11]) allow us to get

$$\|\omega\|_{L^2} \leqq C \Big\| 2\mathrm{BR}(y, \zeta) \cdot y_\alpha - 2\mathrm{BR}(x, \zeta) \cdot x_\alpha + \frac{\rho_0}{\mu_0} \left(\nabla P_2^{-1}(y) \cdot y_\alpha - \nabla P_2^{-1}(x) \cdot x_\alpha \right) \Big\|_{L^2}.$$

We proceed as before to obtain

$$\Big\| 2\mathrm{BR}(y, \zeta) \cdot y_\alpha - 2\mathrm{BR}(x, \zeta) \cdot x_\alpha + \frac{\rho_0}{\mu_0} \left(\nabla P_2^{-1}(y) \cdot y_\alpha - \nabla P_2^{-1}(x) \cdot x_\alpha \right) \Big\|_{L^2} \leqq C \|z\|_{H^1},$$

giving

$$I_{2,3} \leqq C \|z\|_{H^1}^2,$$

as desired. Next we move to I_3. We split further to deal with $c_x - c_y$ by writing $c_x - c_y = G_1 + G_2$ where

$$G_1 = \frac{\alpha+\pi}{2\pi} \int \left[\partial_\beta \left(Q_x^2 \mathrm{BR}_x \right)(\beta) \cdot \frac{x_\beta(\beta)}{|x_\beta(\beta)|^2} - \partial_\beta \left(Q_y^2 \mathrm{BR}_y \right)(\beta) \cdot \frac{y_\beta(\beta)}{|y_\beta(\beta)|^2} \right] \mathrm{d}\beta,$$

and

$$G_2 = -\int_{-\pi}^{\alpha} \left[\partial_\beta \left(Q_x^2 \mathrm{BR}_x \right)(\beta) \cdot \frac{x_\beta(\beta)}{|x_\beta(\beta)|^2} - \partial_\beta \left(Q_y^2 \mathrm{BR}_y \right)(\beta) \cdot \frac{y_\beta(\beta)}{|y_\beta(\beta)|^2} \right] \mathrm{d}\beta.$$

Then we decompose further, to find $|G_1| \leqq |G_{1,1}| + |G_{1,2}| + |G_{1,3}| + |G_{1,4}| + |G_{1,5}|$ where

$$\begin{aligned}
G_{1,1} &= \int \partial_\alpha \left(\left(Q_x^2 - Q_y^2 \right) \mathrm{BR}_x \right) \cdot \frac{x_\alpha}{|x_\alpha|^2} \mathrm{d}\alpha, \\
G_{1,2} &= \int \partial_\alpha \left(Q_y^2 \right) (\mathrm{BR}_x - \mathrm{BR}_y) \cdot \frac{x_\alpha}{|x_\alpha|^2} \mathrm{d}\alpha, \\
G_{1,3} &= \int Q_y^2 \partial_\alpha (\mathrm{BR}_x - \mathrm{BR}_y) \cdot \frac{x_\alpha}{|x_\alpha|^2} \mathrm{d}\alpha, \\
G_{1,4} &= \int \partial_\alpha \left(Q_y^2 \mathrm{BR}_y \right) \cdot \frac{z_\alpha}{|x_\alpha|^2} \mathrm{d}\alpha, \\
G_{1,5} &= \int \partial_\beta \left(Q_y^2 \mathrm{BR}_y \right) \cdot y_\alpha \left(\frac{1}{|x_\alpha|^2} - \frac{1}{|y_\alpha|^2} \right) \mathrm{d}\alpha.
\end{aligned}$$

Above, we use α variables instead of β for the sake of simplicity. We can proceed as before to get

$$|G_{1,1}| + |G_{1,2}| + |G_{1,4}| + |G_{1,5}| \leqq C \|z\|_{H^1}.$$

For the most delicate term we have to split further: $G_{1,3} = G_{1,3,1} + G_{1,3,2} + G_{1,3,3} + G_{1,3,4} + G_{1,3,5} + G_{1,3,6}$, where

$$G_{1,3,1} = \frac{1}{2\pi}\int Q_y^2 \int \frac{x_-^\perp}{|x_-|^2}\cdot\frac{x_\alpha}{|x_\alpha|^2}\omega'_\alpha \mathrm{d}\beta\mathrm{d}\alpha,$$
$$G_{1,3,2} = \frac{1}{2\pi}\int Q_y^2 \int \left[\frac{x_-^\perp}{|x_-|^2} - \frac{y_-^\perp}{|y_-|^2}\right]\cdot\frac{x_\alpha}{|x_\alpha|^2}\zeta'_\alpha \mathrm{d}\beta\mathrm{d}\alpha,$$
$$G_{1,3,3} = \frac{1}{2\pi}\int Q_y^2 \int \frac{\partial_\alpha z_-^\perp}{|x_-|^2}\cdot\frac{x_\alpha}{|x_\alpha|^2}\gamma' \mathrm{d}\beta\mathrm{d}\alpha,$$
$$G_{1,3,4} = \frac{1}{2\pi}\int Q_y^2 \int \partial_\alpha y_-^\perp\left[\frac{\gamma'}{|x_-|^2} - \frac{\zeta'}{|y_-|^2}\right]\cdot\frac{x_\alpha}{|x_\alpha|^2} \mathrm{d}\beta\mathrm{d}\alpha,$$
$$G_{1,3,5} = -\frac{1}{\pi}\int Q_y^2 \int \frac{x_-^\perp}{|x_-|^4}\cdot\frac{x_\alpha}{|x_\alpha|^2}x_-\cdot\partial_\alpha z_-\zeta' \mathrm{d}\beta\mathrm{d}\alpha,$$

and

$$G_{1,3,6} = -\frac{1}{\pi}\int Q_y^2 \int \left[\frac{x_-^\perp}{|x_-|^4}x_-\cdot\partial_\alpha y_-\gamma' - \frac{y_-^\perp}{|y_-|^4}y_-\cdot\partial_\alpha y_-\zeta'\right]\cdot\frac{x_\alpha}{|x_\alpha|^2}\mathrm{d}\beta\mathrm{d}\alpha.$$

We estimate first the less singular terms, which can be controlled as before:

$$|G_{1,3,2}| + |G_{1,3,4}| + |G_{1,3,6}| \leqq C\|z\|_{H^1}.$$

One can rewrite $G_{1,3,1}$ as follows:

$$G_{1,3,1} = \frac{1}{2\pi}\int Q_y^2 \int \frac{x_-^\perp - x_\alpha^\perp\beta}{|x_-|^2}\cdot\frac{x_\alpha}{|x_\alpha|^2}\omega'_\alpha \mathrm{d}\beta\mathrm{d}\alpha,$$

to find a kernel of degree 0 applied to ω_α. This yields

$$|G_{1,3,1}| \leqq C\|\omega\|_{L^2} \leqq C\|z\|_{H^1}.$$

Similarly,

$$G_{1,3,5} = -\frac{1}{\pi}\int Q_y^2 \int \frac{x_-^\perp - x_\alpha^\perp\beta}{|x_-|^4}\cdot\frac{x_\alpha}{|x_\alpha|^2}x_-\cdot\partial_\alpha z_-\zeta' \mathrm{d}\beta\mathrm{d}\alpha,$$

and a kernel of order -1 applied to $\partial_\alpha z_-$ yields

$$|G_{1,3,5}| \leqq C\|z\|_{H^1}.$$

It remains to deal with $G_{1,3,3}$, where we simply integrate by parts to obtain

$$G_{1,3,3} = -\frac{1}{2\pi}\int\int z_-^\perp\cdot\partial_\alpha\left(\frac{1}{|x_-|^2}Q_y^2\frac{x_\alpha}{|x_\alpha|^2}\gamma'\right)\mathrm{d}\beta\mathrm{d}\alpha.$$

We find, as before,

$$|G_{1,3,3}| \leqq C\|z\|_{H^1}.$$

Since we are done with G_1 it remains to deal with G_2. The same decompositions used to control G_1 will also be used to control G_2. The only term in G_2 that cannot be controlled by the ideas used for G_1 is the term analogous to $G_{1,3,3}$ namely:

$$G_{2,3,3} = \frac{1}{2\pi}\int_{-\pi}^{\alpha} Q_y^2(\beta)\int \frac{(\partial_\beta z(\beta)-\partial_\beta z(\beta-\xi))^{\perp}}{|x(\beta)-x(\beta-\xi)|^2}\cdot\frac{x_\beta(\beta)}{|x_\beta(\beta)|^2}\gamma(\beta-\xi)\mathrm{d}\xi\mathrm{d}\beta.$$

We cannot integrate by parts here as in $G_{1,3,3}$. We further decompose $G_{2,3,3} = G_{2,3,3}^1 + G_{2,3,3}^2 + G_{2,3,3}^3$ where

$$\begin{aligned} G_{2,3,3}^1 &= \frac{1}{2\pi}\int_{-\pi}^{\alpha} Q_y^2(\beta)\frac{x_\beta(\beta)}{|x_\beta(\beta)|^2}\cdot\int(\partial_\beta z(\beta)-\partial_\beta z(\beta-\xi))^{\perp} \\ &\quad\times\left[\frac{\gamma(\beta-\xi)}{|x(\beta)-x(\beta-\xi)|^2}-\frac{\gamma(\beta)}{|x_\beta(\beta)|^2 4\sin^2(\beta/2)}\right]\mathrm{d}\xi\mathrm{d}\beta, \\ G_{2,3,3}^2 &= \frac{1}{2}\int_{-\pi}^{\alpha} Q_y^2(\beta)\gamma(\beta)\frac{x_\beta(\beta)}{|x_\beta(\beta)|^4}\cdot\Lambda(\partial_\beta z^{\perp})(\beta)\mathrm{d}\beta \\ &\quad-\frac{\alpha+\pi}{4\pi}\int Q_y^2(\beta)\gamma(\beta)\frac{x_\beta(\beta)}{|x_\beta(\beta)|^4}\cdot\Lambda(\partial_\beta z^{\perp})(\beta)\mathrm{d}\beta, \end{aligned}$$

and

$$G_{2,3,3}^3 = \frac{\alpha+\pi}{2}\int Q_y^2(\beta)\gamma(\beta)\frac{x_\beta(\beta)}{|x_\beta(\beta)|^4}\cdot\Lambda(\partial_\beta z^{\perp})(\beta)\mathrm{d}\beta.$$

The fact that the kernel in ξ has degree -1 allows us to get

$$\left|G_{2,3,3}^1\right| \leqq C\|z\|_{H^1}.$$

Integrating by parts (Λ is a self-adjoint operator) it is easy to obtain

$$\left|G_{2,3,3}^3\right| \leqq C\|z\|_{H^1}.$$

All the bounds above for $c_x - c_y$ allow us to get

$$I_3 \leqq C\|z\|_{H^1}^2 + \int z\cdot x_\alpha G_{2,3,3}^2\mathrm{d}\alpha.$$

Above we integrate by parts to find

$$\int z\cdot x_\alpha G_{2,3,3}^2\mathrm{d}\alpha = I_{3,1} + I_{3,2}$$

where

$$I_{3,1} = \frac{1}{2}\int\left(\int_{-\pi}^{\alpha} z(\beta)\cdot x_\beta(\beta)\mathrm{d}\beta\right)Q_y^2\gamma\frac{x_\alpha}{|x_\alpha|^4}\cdot\Lambda(\partial_\alpha z^{\perp})\mathrm{d}\alpha.$$

and

$$I_{3,2} = -\frac{1}{4\pi}\int\int_{-\pi}^{\alpha} z(\beta)\cdot x_\beta(\beta)\mathrm{d}\beta\mathrm{d}\alpha\int Q_y^2(\beta)\gamma(\beta)\frac{x_\beta(\beta)}{|x_\beta(\beta)|^4}\cdot\Lambda(\partial_\beta z^{\perp})(\beta)\mathrm{d}\beta.$$

As before, using that Λ is self-adjoint, it is easy to get

$$I_{3,2} \leqq C\|z\|_{H^1}^2.$$

Similarly,

$$I_{3,1} = \frac{1}{2}\int \Lambda\left(a\, Q_y^2 \gamma \frac{x_\alpha}{|x_\alpha|^4}\right)\cdot \partial_\alpha z^\perp \mathrm{d}\alpha, \quad \text{where} \quad a(\alpha) = \int_{-\pi}^{\alpha} z(\beta)\cdot x_\beta(\beta)\mathrm{d}\beta.$$

The fact that $\Lambda = H(\partial_\alpha)$ allows us to find

$$I_{3,2} \leqq C\|z\|_{H^1}^2,$$

and thus

$$I_3 \leqq C\|z\|_{H^1}^2.$$

At this point it is easy to get

$$I_4 \leqq C\|z\|_{H^1}^2.$$

If one combines the above inequalities the following is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_{L^2}^2 \leqq C\|z\|_{H^1}^2.$$

The next step is to analyze

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z_\alpha\|_{L^2}^2 = 2\int \partial_\alpha z\cdot\partial_\alpha z_t \mathrm{d}\gamma = I_5 + I_6 + I_7 + I_8 + I_9,$$

where

$$I_5 = 2\int z_\alpha \cdot \partial_\alpha\big(Q_x^2\big)(\mathrm{BR}_x - \mathrm{BR}_y)\mathrm{d}\alpha,$$
$$I_6 = 2\int z_\alpha \cdot Q_x^2 \partial_\alpha(\mathrm{BR}_x - \mathrm{BR}_y)\mathrm{d}\alpha$$
$$I_7 = 2\int z_\alpha \cdot \partial_\alpha\big(\big(Q_x^2 - Q_y^2\big)\mathrm{BR}_y\big)\mathrm{d}\alpha,$$
$$I_8 = 2\int z_\alpha \cdot \partial_\alpha\big((c_x - c_y)x_\alpha\big)\mathrm{d}\alpha,$$
$$I_9 = 2\int z_\alpha \cdot \partial_\alpha\big(c_y z_\alpha\big)\mathrm{d}\alpha.$$

It is easy to get

$$I_5 \leqq C\|z\|_{H^1}^2.$$

For I_6 we consider $I_6 = I_{6,1} + I_{6,2} + I_{6,3} + I_{6,4} + I_{6,5} + I_{6,6}$ where

$$I_{6,1} = \frac{1}{\pi}\int z_\alpha \cdot Q_x^2 \int \frac{x_-^\perp}{|x_-|^2}\omega'_\alpha \mathrm{d}\beta\mathrm{d}\alpha,$$

$$I_{6,2} = \frac{1}{\pi}\int z_\alpha \cdot Q_x^2 \int \left[\frac{x_-^\perp}{|x_-|^2} - \frac{y_-^\perp}{|y_-|^2}\right]\zeta'_\alpha \mathrm{d}\beta\mathrm{d}\alpha,$$

$$I_{6,3} = \frac{1}{\pi}\int z_\alpha \cdot Q_x^2 \int \frac{\partial_\alpha z_-^\perp}{|x_-|^2}\gamma' \mathrm{d}\beta\mathrm{d}\alpha,$$

$$I_{6,4} = \frac{1}{\pi}\int z_\alpha \cdot Q_x^2 \int \partial_\alpha y_-^\perp \left[\frac{\gamma'}{|x_-|^2} - \frac{\zeta'}{|y_-|^2}\right] \mathrm{d}\beta\mathrm{d}\alpha,$$

$$I_{6,5} = -\frac{2}{\pi}\int z_\alpha \cdot Q_x^2 \int \frac{x_-^\perp}{|x_-|^4}x_- \cdot \partial_\alpha z_- \zeta' \mathrm{d}\beta\mathrm{d}\alpha,$$

and

$$I_{6,6} = -\frac{2}{\pi}\int z_\alpha \cdot Q_x^2 \int \left[\frac{x_-^\perp}{|x_-|^4}x_- \cdot \partial_\alpha y_- \gamma' - \frac{y_-^\perp}{|y_-|^4}y_- \cdot \partial_\alpha y_- \zeta'\right] \mathrm{d}\beta\mathrm{d}\alpha.$$

It is easy to get

$$I_{6,2} + I_{6,4} + I_{6,6} \leqq C\|z\|_{H^1}^2.$$

We split further: $I_{6,3} = I_{6,3,1} + I_{6,3,2}$, to find

$$I_{6,3,1} = \frac{1}{2\pi}\int z_\alpha \cdot \int \frac{\partial_\alpha z_-^\perp}{|x_-|^2}\left[Q_x^2\gamma' - (Q_x^2)'\gamma\right] \mathrm{d}\beta\mathrm{d}\alpha,$$

and

$$I_{6,3,2} = \frac{1}{2\pi}\int z_\alpha \cdot \int \frac{\partial_\alpha z_-^\perp}{|x_-|^2}\left[Q_x^2\gamma' + (Q_x^2)'\gamma\right] \mathrm{d}\beta\mathrm{d}\alpha.$$

We find, as before,

$$I_{6,3,1} \leqq C\|z\|_{H^1}^2.$$

Changing variables one could obtain

$$I_{6,3,2} = \frac{1}{4\pi}\int\int \partial_\alpha z_- \cdot \frac{\partial_\alpha z_-^\perp}{|x_-|^2}\left[Q_x^2\gamma' + (Q_x^2)'\gamma\right] \mathrm{d}\beta\mathrm{d}\alpha = 0.$$

We are done with $I_{6,3}$. The term $I_{6,5}$ is decomposed as the sum of

$$I_{6,5,1} = -\frac{2}{\pi}\int z_\alpha \cdot Q_x^2 \int \left[\frac{\zeta' x_-^\perp}{|x_-|^4}x_- - \frac{\zeta x_\alpha^\perp}{|x_\alpha|^4 4\sin^2(\beta/2)}x_\alpha\right] \cdot \partial_\alpha z_- \mathrm{d}\beta\mathrm{d}\alpha,$$

$$I_{6,5,2} = -2\int z_\alpha \cdot Q_x^2 \zeta \frac{x_\alpha^\perp}{|x_\alpha|^4}x_\alpha \cdot \Lambda(z_\alpha)\mathrm{d}\alpha$$

and we have

$$I_{6,5,1} \leqq C\|z\|_{H^1}^2.$$

We use the fact that $\Lambda = H(\partial_\alpha)$, and the identity

$$x_\alpha \cdot \partial_\alpha^2 z = -x_\alpha \cdot \partial_\alpha^2 y = -z_\alpha \cdot \partial_\alpha^2 y,$$

to rewrite

$$\begin{aligned} I_{6,5,2} = 2 \int z_\alpha \cdot Q_x^2 \zeta \frac{x_\alpha^\perp}{|x_\alpha|^4} [\Lambda(x_\alpha \cdot z_\alpha) - x_\alpha \cdot \Lambda(z_\alpha)] \\ - 2 \int z_\alpha \cdot Q_x^2 \zeta \frac{x_\alpha^\perp}{|x_\alpha|^4} H\left(\partial_\alpha^2 x \cdot z_\alpha\right) \mathrm{d}\alpha \\ + 2 \int z_\alpha \cdot Q_x^2 \zeta \frac{x_\alpha^\perp}{|x_\alpha|^4} H\left(z_\alpha \cdot \partial_\alpha^2 y\right) \mathrm{d}\alpha. \end{aligned}$$

Since the commutator estimate for Λ gives

$$I_{6,5,2} \leqq C\|z\|_{H^1}^2,$$

we are done with $I_{6,5}$. It remains to deal with $I_{6,1}$, where we have to find the Rayleigh-Taylor condition. We decompose further $I_{6,1}$ as the sum of

$$\begin{aligned} I_{6,1,1} &= \frac{1}{\pi} \int z_\alpha \cdot Q_x^2 \int \left[\frac{x_-^\perp}{|x_-|^2} - \frac{x_\alpha^\perp}{|x_\alpha|^2 2\tan(\beta/2)} \right] \omega'_\alpha \mathrm{d}\beta \mathrm{d}\alpha \quad \text{and} \\ I_{6,1,2} &= \int Q_x^2 z_\alpha \cdot \frac{x_\alpha^\perp}{|x_\alpha|^2} H(\omega_\alpha) \mathrm{d}\alpha. \end{aligned}$$

As before,

$$I_{6,1,1} \leqq C\|z\|_{H^1}^2.$$

Next we will decompose ω_α, pointing out first the bounded terms, and dealing later with the unbounded. We take $\omega_\alpha = G_3 + G_4 + G_5 + G_6$ where

$$\begin{aligned} G_3 &= -2\mathrm{BR}_x \cdot \partial_\alpha^2 x + 2\mathrm{BR}_y \cdot \partial_\alpha^2 y, \\ G_4 &= -2\partial_\alpha \mathrm{BR}_x \cdot x_\alpha + 2\partial_\alpha \mathrm{BR}_y \cdot y_\alpha \\ G_5 &= -2\frac{\rho_0}{\mu_0} \left(\partial_\alpha \left(\nabla P_2^{-1}(x) \right) \cdot x_\alpha - \partial_\alpha \left(\nabla P_2^{-1}(y) \right) \cdot y_\alpha \right), \\ G_6 &= -2\frac{\rho_0}{\mu_0} \left(\nabla P_2^{-1}(x) \cdot \partial_\alpha^2 x - \nabla P_2^{-1}(y) \cdot \partial_\alpha^2 y \right). \end{aligned}$$

We split further $G_3 = G_{3,1} + G_{3,2}$

$$G_{3,1} = -2\mathrm{BR}_x \cdot \partial_\alpha^2 z, \quad G_{3,2} = 2(\mathrm{BR}_y - \mathrm{BR}_x) \cdot \partial_\alpha^2 y$$

to obtain, as before,

$$\|G_{3,2}\|_{L^2} \leqq C\|z\|_{H^1}.$$

The term $G_{3,1}$ is one of the unbounded characters. We continue by taking $G_4 = G_{4,1} + G_{4,2} + G_{4,3} + G_{4,4} + G_{4,5} + G_{4,6}$ where

$$\begin{aligned}
G_{4,1} &= -\frac{1}{\pi}\int \frac{\partial_\alpha z_-^\perp}{|x_-|^2}\gamma' \mathrm{d}\beta \cdot x_\alpha,\\
G_{4,2} &= -\frac{1}{\pi}\int \partial_\alpha y_-^\perp \cdot \left[\frac{\gamma'}{|x_-|^2}x_\alpha - \frac{\zeta'}{|y_-|^2}y_\alpha\right]\mathrm{d}\beta,\\
G_{4,3} &= \frac{2}{\pi}\int \frac{x_-^\perp}{|x_-|^4}\cdot x_\alpha\, x_- \cdot \partial_\alpha z_- \gamma' \mathrm{d}\beta,\\
G_{4,4} &= \frac{2}{\pi}\int \left[\frac{x_-^\perp}{|x_-|^4}\cdot x_\alpha\, x_- \gamma' - \frac{y_-^\perp}{|y_-|^4}\cdot y_\alpha\, y_- \zeta'\right]\cdot \partial_\alpha y_- \mathrm{d}\beta,\\
G_{4,5} &= -\frac{1}{\pi}\int \frac{x_-^\perp}{|x_-|^2}\omega'_\alpha \mathrm{d}\beta \cdot x_\alpha,\\
G_{4,6} &= -\frac{1}{\pi}\int \left[\frac{x_-^\perp}{|x_-|^2}\cdot x_\alpha - \frac{y_-^\perp}{|y_-|^2}\cdot y_\alpha\right]\zeta'_\alpha \mathrm{d}\beta.
\end{aligned}$$

Next, $G_{4,1}$ joins the unbounded terms and

$$\|G_{4,2}\|_{L^2} + \|G_{4,4}\|_{L^2} + \|G_{4,6}\|_{L^2} \leqq C\|z\|_{H^1}.$$

It is possible to obtain a kernel of degree -1 applied to $\partial_\alpha z_-$ in $G_{4,3}$ as follows:

$$G_{4,3} = \frac{2}{\pi}\int \frac{x_-^\perp - x_\alpha^\perp \beta}{|x_-|^4}\cdot x_\alpha\, x_- \cdot \partial_\alpha z_- \gamma' \mathrm{d}\beta.$$

Therefore

$$\|G_{4,3}\|_{L^2} \leqq C\|z\|_{H^1}.$$

Since

$$G_{4,5} = -\frac{1}{\pi}\int \frac{x_-^\perp - x_\alpha^\perp \beta}{|x_-|^2}\omega'_\alpha \mathrm{d}\beta \cdot x_\alpha,$$

we obtain, in an analogous way,

$$\|G_{4,5}\|_{L^2} \leqq C\|\omega\|_{L^2} \leqq C\|z\|_{H^1}.$$

For G_5 it is easy to get

$$\|G_5\|_{L^2} \leqq C\|z\|_{H^1},$$

but the term G_6 has to be decomposed as follows:

$$\begin{aligned}
G_{6,1} &= -2\frac{\rho_0}{\mu_0}\nabla P_2^{-1}(x)\cdot \partial_\alpha^2 z,\\
G_{6,2} &= -2\frac{\rho_0}{\mu_0}\left(\nabla P_2^{-1}(x) - \nabla P_2^{-1}(y)\right)\cdot \partial_\alpha^2 y.
\end{aligned}$$

$G_{6,1}$ remains unbounded, and

$$\|G_{6,2}\|_{L^2} \leqq C\|z\|_{H^1}$$

easily. Thanks to all this decomposition we find

$$I_{6,1,2} \leqq C\|z\|^2_{H^1} + I^1_{6,1,2} + I^2_{6,1,2} + I^3_{6,1,2},$$

where

$$I^1_{6,1,2} = \int Q^2_x z_\alpha \cdot \frac{x^\perp_\alpha}{|x_\alpha|^2} H(G_{3,1})\mathrm{d}\alpha, \quad I^2_{6,1,2} = \int Q^2_x z_\alpha \cdot \frac{x^\perp_\alpha}{|x_\alpha|^2} H(G_{4,1})\mathrm{d}\alpha,$$

and

$$I^3_{6,1,2} = \int Q^2_x z_\alpha \cdot \frac{x^\perp_\alpha}{|x_\alpha|^2} H(G_{6,1})\mathrm{d}\alpha.$$

In $I^1_{6,1,2}$ and $I^3_{6,1,2}$ the Rayleigh-Taylor condition will show up. For $I^2_{6,1,2}$ we consider the splitting

$$I^{2,1}_{6,1,2} = \int H\left(Q^2_x z_\alpha \cdot \frac{x^\perp_\alpha}{|x_\alpha|^2}\right) \frac{1}{\pi} \int \partial_\alpha z^\perp_- [\frac{\gamma'}{|x_-|^2} - \frac{\gamma}{|x_\alpha|^2 4\sin^2(\beta/2)}]\mathrm{d}\beta \cdot x_\alpha \mathrm{d}\alpha,$$

$$I^{2,2}_{6,1,2} = \int H\left(Q^2_x z_\alpha \cdot \frac{x^\perp_\alpha}{|x_\alpha|^2}\right) \frac{\gamma}{|x_\alpha|^2} \Lambda(\partial_\alpha z^\perp) \cdot x_\alpha \mathrm{d}\alpha,$$

using that H is skew-adjoint. The first term satisfies

$$I^{2,1}_{6,1,2} \leqq C\|z\|^2_{H^1}.$$

For the second term we use the commutator estimate to show that

$$I^{2,2}_{6,1,2} \leqq C\|z\|^2_{H^1} + \int H\left(Q^2_x z_\alpha \cdot \frac{x^\perp_\alpha}{|x_\alpha|^2}\right) \Lambda\left(\frac{\gamma}{|x_\alpha|^2} \partial_\alpha z^\perp \cdot x_\alpha\right) \mathrm{d}\alpha.$$

The fact that $H^2 = -I$ yields

$$I^{2,2}_{6,1,2} \leqq C\|z\|^2_{H^1} + \int Q^2_x z_\alpha \cdot \frac{x^\perp_\alpha}{|x_\alpha|^2} \partial_\alpha \left(\frac{\gamma}{|x_\alpha|^2} \partial_\alpha z^\perp \cdot x_\alpha\right) \mathrm{d}\alpha.$$

In the integral above we expand the derivative, to find out that it is possible to integrate by parts in $\partial_\alpha(z_\alpha \cdot x^\perp_\alpha)$. This yields

$$I^{2,2}_{6,1,2} \leqq C\|z\|^2_{H^1}, \quad \text{and} \quad \text{therefore} \quad I^2_{6,1,2} \leqq C\|z\|^2_{H^1}.$$

Next we consider

$$I^1_{6,1,2} = -\int Q^2_x z_\alpha \cdot \frac{x^\perp_\alpha}{|x_\alpha|^2} H\left(2\mathrm{BR}_x \cdot \partial^2_\alpha z\right) \mathrm{d}\alpha,$$

for which we use the commutator for the Hilbert transform

$$\|H(g\partial_\alpha f) - gH(\partial_\alpha f)\|_{L^2} \leqq C\|g\|_{C^{1,\frac{1}{3}}} \|f\|_{L^2},$$

to find

$$I^1_{6,1,2} \leqq -\frac{2}{|x_\alpha|^2}\int Q_x^2 z_\alpha \cdot x_\alpha^\perp \mathrm{BR}_x \cdot H\left(\partial_\alpha^2 z\right) \mathrm{d}\alpha.$$

Next we split the above integral by components:

$$I^{1,1}_{6,1,2} = \frac{2}{|x_\alpha|^2}\int Q_x^2 \partial_\alpha z_1 \partial_\alpha x_2 \mathrm{BR}_{x1} \cdot H\left(\partial_\alpha^2 z_1\right) \mathrm{d}\alpha, \tag{26}$$

$$I^{1,2}_{6,1,2} = \frac{2}{|x_\alpha|^2}\int Q_x^2 \partial_\alpha z_1 \partial_\alpha x_2 \mathrm{BR}_{x2} \cdot H\left(\partial_\alpha^2 z_2\right) \mathrm{d}\alpha,$$

$$I^{1,3}_{6,1,2} = -\frac{2}{|x_\alpha|^2}\int Q_x^2 \partial_\alpha z_2 \partial_\alpha x_1 \mathrm{BR}_{x1} \cdot H\left(\partial_\alpha^2 z_1\right) \mathrm{d}\alpha,$$

$$I^{1,4}_{6,1,2} = -\frac{2}{|x_\alpha|^2}\int Q_x^2 \partial_\alpha z_2 \partial_\alpha x_1 \mathrm{BR}_{x2} \cdot H\left(\partial_\alpha^2 z_2\right) \mathrm{d}\alpha. \tag{27}$$

The commutator estimate for the Hilbert transform allows us to obtain

$$I^{1,2}_{6,1,2} \leqq C\|z\|^2_{H^1} + \frac{2}{|x_\alpha|^2}\int Q_x^2 \partial_\alpha z_1 \mathrm{BR}_{x2} \cdot H\left(\partial_\alpha x_2 \partial_\alpha^2 z_2\right) \mathrm{d}\alpha.$$

Together with the identity

$$\partial_\alpha x_2 \partial_\alpha^2 z_2 = -\partial_\alpha x_1 \partial_\alpha^2 z_1 - \partial_\alpha z \cdot \partial_\alpha^2 y,$$

this provides

$$I^{1,2}_{6,1,2} \leqq C\|z\|^2_{H^1} - \frac{2}{|x_\alpha|^2}\int Q_x^2 \partial_\alpha z_1 \mathrm{BR}_{x2} \cdot H\left(\partial_\alpha x_1 \partial_\alpha^2 z_1\right) \mathrm{d}\alpha.$$

The commutator estimate yields

$$I^{1,2}_{6,1,2} \leqq C\|z\|^2_{H^1} - \frac{2}{|x_\alpha|^2}\int Q_x^2 \partial_\alpha z_1 \partial_\alpha x_1 \mathrm{BR}_{x2} \cdot H\left(\partial_\alpha^2 z_1\right) \mathrm{d}\alpha. \tag{28}$$

In a similar manner we find that

$$I^{1,3}_{6,1,2} \leqq C\|z\|^2_{H^1} + \frac{2}{|x_\alpha|^2}\int Q_x^2 \partial_\alpha z_2 \partial_\alpha x_2 \mathrm{BR}_{x1} \cdot H\left(\partial_\alpha^2 z_2\right) \mathrm{d}\alpha. \tag{29}$$

Adding (26)–(29) we find that

$$I^1_{6,1,2} \leqq C\|z\|^2_{H^1} - \frac{2}{|x_\alpha|^2}\int Q_x^2 \mathrm{BR}_x \cdot x_\alpha^\perp z_\alpha \Lambda(z_\alpha) \mathrm{d}\alpha. \tag{30}$$

Next,

$$I^3_{6,1,2} = -2\frac{\rho_0}{\mu_0}\int Q_x^2 z_\alpha \cdot \frac{x_\alpha^\perp}{|x_\alpha|^2} H\left(\nabla P_2^{-1}(x) \cdot \partial_\alpha^2 z\right) \mathrm{d}\alpha,$$

and a decomposition in components as before provides

$$I^3_{6,1,2} \leqq C\|z\|^2_{H^1} - 2\frac{\rho_0}{\mu_0 |x_\alpha|^2}\int Q_x^2 \nabla P_2^{-1}(x) \cdot x_\alpha^\perp z_\alpha \cdot \Lambda(z_\alpha) \mathrm{d}\alpha. \tag{31}$$

Adding (30) and (31), we find that

$$I_{6,1,2} \leqq C\|z\|_{H^1}^2 + I_{6,1,2}^1 + I_{6,1,2}^3 \leqq C\|z\|_{H^1}^2 - 2\frac{1}{\mu_0|x_\alpha|^2}\int Q_x^2\sigma_x z_\alpha \cdot \Lambda(z_\alpha)\mathrm{d}\alpha.$$

The Rayleigh-Taylor condition ($\sigma \geqq 0$) for the curve x gives

$$I_{6,1,2} \leqq C\|z\|_{H^1}^2, \quad I_{6,1} \leqq C\|z\|_{H^1}^2, \quad \text{and finally} \quad I_6 \leqq C\|z\|_{H^1}^2.$$

We easily find that

$$I_7 \leqq C\|z\|_{H^1}^2.$$

For I_8 we consider

$$I_8 = 2\int z_\alpha \cdot (c_x - c_y)\partial_\alpha^2 x\mathrm{d}\alpha + 2\int z_\alpha \cdot x_\alpha\partial_\alpha(c_x - c_y)\mathrm{d}\alpha,$$

and integrate by parts to find

$$I_8 = -2\int \partial_\alpha^2 z \cdot x_\alpha(c_x - c_y)\mathrm{d}\alpha.$$

The fact that

$$I_8 = 2\int \partial_\alpha^2 y \cdot z_\alpha(c_x - c_y)\mathrm{d}\alpha$$

allows us to deal with I_8 as for I_3 to get

$$I_8 \leqq C\|z\|_{H^1}^2.$$

Finally, integration by parts provides

$$I_9 = \int |z_\alpha|^2\partial_\alpha c_y\mathrm{d}\alpha \leqq C\|z\|_{H^1}^2.$$

This completes the proof of Proposition 5.1. □

6. Proof of Theorem 1.1

Let $\mathbb{R}/2\pi\mathbb{Z} \ni \alpha \to z^0(\alpha) \in \mathbb{C}/2\pi\mathbb{Z}$ be the curve depicted in Fig. 1. For the short time $t \in [0, t_0]$, we solve the P(Muskat) equation for $\tilde{z}(\alpha, t)$, with initial data $\tilde{z}(\alpha, 0) = P(z^0(\alpha))$ (see Section 3 for P and P^{-1}). We then set $z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) = P^{-1}(\tilde{z}(\alpha, t)) \in \mathbb{C}/2\pi\mathbb{Z}$ for $(\alpha, t) \in (\mathbb{R}/2\pi\mathbb{Z}) \times [0, t_0]$.

We know that $\tilde{z} \in C([0, t_0], H^3(\mathbb{R}/2\pi\mathbb{Z}))$, therefore we also have $\partial_t\tilde{z} \in C([0, t_0], H^2(\mathbb{R}/2\pi\mathbb{Z}))$ and $\partial_t^2\tilde{z} \in C([0, t_0], H^1(\mathbb{R}/2\pi\mathbb{Z}))$ as we see from the P(Muskat) equation. In particular, $\tilde{z} \in C^2((\mathbb{R}/2\pi\mathbb{Z}) \times [0, t_0])$, and therefore $z \in C^2((\mathbb{R}/2\pi\mathbb{Z}) \times [0, t_0])$.

We have $z(\alpha, 0) = z^0(\alpha)$ for all α. We will show that the curve $\alpha \to z(\alpha, t)$ self-intersects transversely for small positive t.

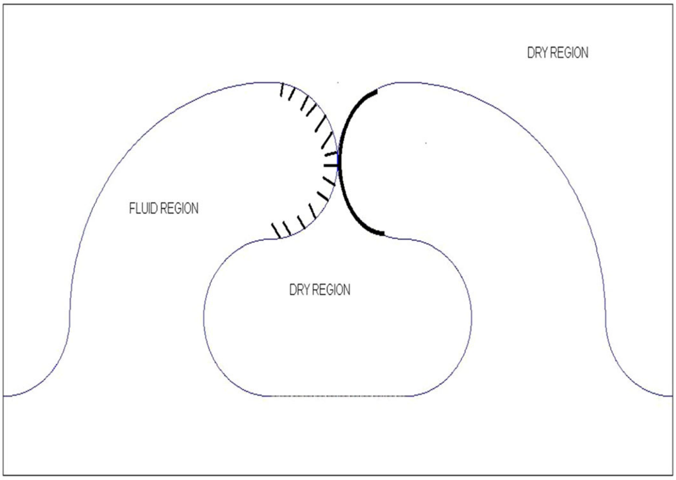

Fig. 2. The hatched and solid arcs

To see this, we examine the hatched arc $z^0(I^-)$ and the solid arc $z^0(I^+)$ depicted in Fig. 2, where I^- and I^+ are disjoint intervals in $\mathbb{R}/2\pi\mathbb{Z}$. The curve $z^0(\cdot)$ self-intersects at the point $z^0(\alpha^+) = z^0(\alpha^-)$, where $\alpha^+ \in I^+$ and $\alpha^- \in I^-$.

We may suppose that the hatched and solid arcs both have nonzero curvature at that point. The above remarks, and the discussion of the Rayleigh Taylor condition in Section 7, imply:

$$\begin{aligned}\partial_\alpha z_1(\alpha_+, 0) &= \partial_\alpha z_1(\alpha_-, 0) = 0\\ \partial_\alpha^2 z_1(\alpha_+, 0) &> 0 > \partial_\alpha^2 z_1(\alpha_-, 0)\\ \partial_\alpha z_2(\alpha_+, 0) \neq 0, &\quad \partial_\alpha z_2(\alpha_-, 0) \neq 0\\ \partial_t z_1(\alpha_-, 0) &> 0 > \partial_t z_1(\alpha_+, 0).\end{aligned}$$

Intuitively, the curve $\alpha \to z(\alpha, t)$ self-intersects transversely because the hatched arc in Fig. 2 moves to the right as time increases, while the solid arc moves to the left. This intuition is justified by the following elementary lemma and its collorary:

Lemma 6.1. *Let* $[-c_0, c_0] \times [0, c_0] \ni (\alpha, t) \to z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) \in \mathbb{R}^2$ *be* C^2*-smooth, with* $z(0, 0) = 0$, $\partial_\alpha z_1(0, 0) = 0$, $\partial_\alpha z_2(0, 0) \neq 0$. *Given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $t \in [0, \delta]$ *a part of the curve* $[-c_0, c_0] \ni \alpha \to z(\alpha, t)$ *admits a* C^2 *reparametrization as a graph* $[-\delta, \delta] \ni x_2 \to (F(x_2, t), x_2) \in \mathbb{R}^2$, *where*

$$\left| F(x_2, t) - \left[(\partial_t z_1(0, 0))t + \frac{1}{2} \frac{\partial_\alpha^2 z_1(0, 0)}{(\partial_\alpha z_2(0, 0))^2} x_2^2 \right] \right| \leqq \varepsilon \left[x_2^2 + t \right]$$

and

$$\left| \partial_{x_2} F(x_2, t) - \frac{\partial_\alpha^2 z_1(0, 0)}{(\partial_\alpha z_2(0, 0))^2} x_2 \right| \leqq \varepsilon |x_2| + Ct$$

for $(x_2, t) \in [-\delta, \delta] \times [0, \delta]$. *Here, C is independent of* (x_2, t).

Proof. Using the implicit function theorem, we produce a C^2 function $\alpha = A(x_2, t)$ that solves the equation $z_2(\alpha, t) = x_2$ for x_2, t given. We set $F(x_2, t) = z_1(A(x_2, t), t)$, and derive the desired estimates for F and $\partial_{x_2} F$ by Taylor-expanding $z(\alpha, t)$ and $A(x_2, t)$ about $(0, 0)$ and substituting $\alpha = A(x_2, t)$. Details are left to the reader. □

Corollary 6.2. *Let* $(\alpha, t) \to z^+(\alpha, t) = (z_1^+(\alpha, t), z_2^+(\alpha, t))$ *and* $(\alpha, t) \to z^-(\alpha, t) = (z_1^-(\alpha, t), z_2^-(\alpha, t))$ *satisfy the hypothesis of the above Lemma. Suppose* $\partial_t z_1^+(0, 0) < 0 < \partial_t z_1^-(0, 0)$ *and* $\partial_\alpha^2 z_1^+(0, 0) > 0 > \partial_\alpha^2 z_1^-(0, 0)$. *Then for small positive t, the curves* $\alpha \to z^+(\alpha, t)$ *and* $\alpha \to z^-(\alpha, t)$ *intersect transversely.*

Proof. Let K be a large enough positive number. We apply the lemma with $\varepsilon = K^{-10}$. Then for $t \in [0, \delta]$, our two curves contain graphs $[-\delta, \delta] \ni x_2 \to (F^+(x_2, t), x_2)$ and $[-\delta, \delta] \ni x_2 \to (F^-(x_2, t), x_2)$, where

$$\left| F^\pm(x_2, t) - \left[v^\pm t + \frac{1}{2} a^\pm x_2^2 \right] \right| \leqq K^{-10} \left[x_2^2 + t \right]$$

and

$$|\partial_{x_2} F^\pm(x_2, t) - a^\pm x_2| \leqq K^{-10} |x_2| + Ct, \tag{32}$$

and $v^+ < 0 < v^-$, $a^+ > 0 > a^-$.

It follows that $F^+(K^{-1} t^{\frac{1}{2}}, t) < F^-(K^{-1} t^{\frac{1}{2}}, t)$ and $F^+(K t^{\frac{1}{2}}, t) > F^-(K t^{\frac{1}{2}}, t)$ for small t; hence, $F^+(\tilde{x}_2, t) = F^-(\tilde{x}_2, t)$ for some $\tilde{x}_2 \in [K^{-1} t^{\frac{1}{2}}, K t^{\frac{1}{2}}]$. Thus our curves intersect at the point $(F^+(\tilde{x}_2, t), \tilde{x}_2)$. That intersection is transverse because $\partial_{x_2} F^+(\tilde{x}_2, t) > 0 > \partial_{x_2} F^-(\tilde{x}_2, t)$, thanks to (32). □

We now know that the curve $\alpha \to z(\alpha, t)$ self-intersects transversely, as promised.

We can now easily finish the proof of Theorem 1.1.

Proof. We fix a time $t_1 \in [0, t_0]$ at which the curve $\alpha \to z(\alpha, t)$ self-intersects transversely. Let $z_*^0(\cdot)$ be a simple closed curve in $(\mathbb{C}/2\pi\mathbb{Z}) \setminus i\mathbb{R}^+$ such that $||z_*^0 - z^0||_{H^3(\mathbb{R}/2\pi\mathbb{Z})} < \eta$, with η to be picked below. We solve the P(Muskat) equation with initial data $P(z_*^0(\cdot))$, obtaining a solution $\tilde{z}_*(\alpha, t)$ for which $||\tilde{z}_*(\cdot, t) - \tilde{z}(\cdot, t)||_{H^3(\mathbb{R}/2\pi\mathbb{Z})} \leqq C\eta$ for $t \in [0, t_0]$, thanks to Section 5. Here, C is independent of z_*^0 and η. Setting $z_*(\alpha, t) = P^{-1}(\tilde{z}_*(\alpha, t))$ for $(\alpha, t) \in (\mathbb{R}/2\pi\mathbb{Z}) \times [0, t_0]$, we find that:

- $z_*(\alpha, 0) = z_*^0(\alpha)$ for all α;
- $z_*(\alpha, t)$ solves the Muskat equation on $(\mathbb{R}/2\pi\mathbb{Z}) \times [0, \tau)$ provided $\alpha \to z_*(\alpha, t)$ is a simple closed curve for each $t \in [0, \tau)$;
  and
- $||z_*(\cdot, t) - z(\cdot, t)||_{H^3(\mathbb{R}/2\pi\mathbb{Z})} \leqq C\eta$ for $t \in [0, t_0]$.

For small enough t, the curve $\alpha \to z_*(\alpha, t)$ does not self-intersect, because $\alpha \to z_*(\alpha, 0) = z_*^0(\alpha)$ is a simple closed curve. On the other hand, because $\alpha \to z(\alpha, t_1)$ self-intersects transversely and $||z_*(\cdot, t) - z(\cdot, t)||_{H^3(\mathbb{R}/2\pi\mathbb{Z})} \leqq C\eta$, it follows that $\alpha \to z_*(\alpha, t)$ self-intersects transversely if we pick η small enough.

Now let $\tau = \inf\{t \in [0, t_0] : \alpha \to z_*(\alpha, t) \text{ self-intersects}\}$. Then $\tau \in (0, t_1)$, $\alpha \to z(\alpha, t)$ self intersects, and $z_*(\alpha, t)$ $(\alpha \in \mathbb{R}/2\pi\mathbb{Z}, t \in [0, \tau))$ is a Muskat solution with initial data $z_*(\alpha, 0) = z_*^0(\alpha)$. Here, $z_*^0(\cdot)$ is any simple closed curve that avoids the slit $i\mathbb{R}^+$ and lies close enough to $z^0(\cdot)$ in H^3. The set of all such z_*^0 is a nonempty open subset of H^3.

The proof of Theorem 1.1. is complete. □

7. A Remark on the Family of Splash Singularities

The scenario in Section 2 is the simplest one to obtain a splash singularity. However the one-phase Muskat problem can develop this kind of point-wise collapse for more geometries. In order to check that we proceed as follows. Let $z(\alpha)$ be a splash curve with $\alpha_1 \neq \alpha_2$ such that $z(\alpha_1) = z(\alpha_2)$ and $|\partial_\alpha z(\alpha)| > 0$ for every α. To consider a different situation from that introduced in Section 2, we also assume that $\partial_\alpha z_1(\alpha_1) \neq 0$. We make the following distinction between α_1 and α_2: there exists a neighborhood U_{α_1} of α_1 and a neighborhood U_{α_2} of α_2 such that, if $z_1(\beta_1) = z_1(\beta_2)$ for $\beta_1 \in U_{\alpha_1}$ and $\beta_2 \in U_{\alpha_2}$, then $z_2(\beta_1) \leqq z_2(\beta_2)$. Roughly speaking, we just mean that $z(\alpha_2)$ is the upper splash point and $z(\alpha_1)$ is the lower splash point.

Let us analyze the normal velocity at α_2 and α_1. For α_2 we have

$$\mu_0 u(\alpha_2) \cdot n(\alpha_2) = -\nabla p(z(\alpha_2)) \cdot n(\alpha_2) - \rho_0 \frac{\partial_\alpha z_1(\alpha_2)}{|\partial_\alpha z(\alpha_2)|},$$

where $n(\alpha) = \partial_\alpha^\perp z(\alpha)/|\partial_\alpha z(\alpha)|$ and $u(\alpha) = v(z(\alpha, t), t)$. As in Section 2, Hopf's lemma yields

$$-\nabla p(z(\alpha_2)) \cdot n(\alpha_2) > 0,$$

and we consider

$$\frac{\partial_\alpha z_1(\alpha_2)}{|\partial_\alpha z(\alpha_2)|} < 0.$$

On the other hand, we have

$$\mu_0 u(\alpha_1) \cdot n(\alpha_1) = -\nabla p(z(\alpha_1)) \cdot n(\alpha_1) - \rho_0 \frac{\partial_\alpha z_1(\alpha_1)}{|\partial_\alpha z(\alpha_1)|},$$

with

$$-\nabla p(z(\alpha_1)) \cdot n(\alpha_1) > 0,$$

and

$$\frac{\partial_\alpha z_1(\alpha_1)}{|\partial_\alpha z(\alpha_1)|} > 0.$$

Then the sign of $\partial_\alpha z(\alpha_1)$ is bad for our purpose. However we can notice that

$$\frac{\partial_\alpha z_1(\alpha_1)}{|\partial_\alpha z(\alpha_1)|} = -\frac{\partial_\alpha z_1(\alpha_2)}{|\partial_\alpha z(\alpha_2)|},$$

and therefore

$$u(\alpha_2)\cdot n(\alpha_2) > -u(\alpha_1)\cdot n(\alpha_1).$$

The last inequality is enough to show that the velocity separates the splash points backward in time. Unfortunately this is not enough to ensure that we can produce a splash singularity by using the previous analysis. It is possible to find $u(\alpha_1)\cdot n(\alpha_1)$ negative. Then the solution would cross the branch of P backwards in time. This is a mere technical problem that we can solve as follows:

Let's define a velocity

$$\underline{v}(x_1,x_2,t) = v\left(x_1, x_2-\frac{\rho_0}{\mu_0}t, t\right)+\left(0,\frac{\rho_0}{\mu_0}\right),$$

a density

$$\underline{\rho}(x_1,x_2,t) = \rho\left(x_1, x_2-\frac{\rho_0}{\mu_0}t, t\right),$$

and a viscosity

$$\underline{\mu}(x_1,x_2,t) = \mu\left(x_1, x_2-\frac{\rho_0}{\mu_0}t, t\right).$$

Then we have:

$$\begin{aligned}
\partial_t\underline{\rho}(x_1,x_2,t) &= (\partial_t\rho)\left(x_1, x_2-\frac{\rho_0}{\mu_0}t, t\right)-\frac{\rho_0}{\mu_0}(\partial_{x_2}\rho)\left(x_1, x_2-\frac{\rho_0}{\mu_0}t, t\right)\\
&= -v\left(x_1, x_2-\frac{\rho_0}{\mu_0}, t\right)\cdot(\nabla\rho)\left(x_1, x_2-\frac{\rho_0}{\mu_0}t, t\right)\\
&\quad -\frac{\rho_0}{\mu_0}(\partial_{x_2}\rho)\left(x_1, x_2-\frac{\rho_0}{\mu_0}t, t\right)\\
&= -v\left(x_1, x_2-\frac{\rho_0}{\mu_0}, t\right)\cdot\nabla\underline{\rho}(x_1,x_2,t)-\frac{\rho_0}{\mu_0}\partial_{x_2}\underline{\rho}(x_1,x_2,t)\\
&= -\left(v\left(x_1, x_2-\frac{\rho_0}{\mu_0}t, t\right)+\left(0,\frac{\rho_0}{\mu_0}\right)\right)\cdot\nabla\underline{\rho}(x_1,x_2,t).
\end{aligned}$$

Thus we have that $\underline{\rho}$ satisfies

$$\partial_t\underline{\rho}+\underline{v}\cdot\nabla\underline{\rho}=0,$$

and in a similar manner it is easy to get

$$\partial_t\underline{\mu}+\underline{v}\cdot\nabla\underline{\mu}=0.$$

On the other hand,

$$\underline{\mu}\,\underline{v}=-\nabla\underline{p},$$

where we consider

$$\underline{p}(x_1, x_2, t) = p\left(x_1, x_2 - \frac{\rho_0}{\mu_0}t, t\right).$$

Then, by using $\underline{v}$, $\underline{\rho}$, $\underline{\mu}$ and $\underline{p}$, we can write our Muskat problem as the system

$$\begin{aligned} \partial_t \underline{\rho} + \underline{v} \cdot \nabla \underline{\rho} &= 0, \\ \partial_t \underline{\mu} + \underline{v} \cdot \nabla \underline{\mu} &= 0, \\ \underline{\mu}\, \underline{v} &= -\nabla \underline{p}, \\ \nabla \cdot \underline{v} &= 0, \end{aligned}$$

with the boundary condition

$$\lim_{x_2 \to -\infty} \left(\underline{v}(x_1, x_2, t) - \left(0, \frac{\rho_0}{\mu_0}\right)\right) = 0.$$

In this new system we find the following: if $z(\alpha)$ is a splash curve such that $z(\alpha_1) = z(\alpha_2)$, then

$$\begin{aligned} \mu_0 \underline{v}(z(\alpha_1)) \cdot n(\alpha_1) &= -\nabla \underline{p}(z(\alpha_1)) \cdot n(\alpha_1), \\ \mu_0 \underline{v}(z(\alpha_2)) \cdot n(\alpha_2) &= -\nabla \underline{p}(z(\alpha_2)) \cdot n(\alpha_2), \end{aligned}$$

and again we can invoke Hopf's lemma to obtain that

$$-\nabla \underline{p}(z(\alpha_1)) \cdot n(\alpha_1) > 0, \quad -\nabla \underline{p}(z(\alpha_2)) \cdot n(\alpha_2) > 0.$$

Then, the velocity separates the splash point and $\underline{u}(\alpha_1) \cdot n(\alpha_1) n(\alpha_1)$ points in the opposite direction to $\underline{u}(\alpha_2) \cdot n(\alpha_2) n(\alpha_2)$. Therefore we can carry out the same analysis that we did for the simpler case of Section 2.

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References

1. Ambrose, D.M.: Well-posedness of two-phase Hele-Shaw flow without surface tension. *European J. Appl. Math.* **15**(5), 597–607 (2004)
2. Ambrose, D.M.: The zero surface tension limit of two-dimensional interfacial Darcy flow. *J. Math. Fluid Mech.* **16**(1):105–143 (2014)
3. Bear, J.: *Dynamics of fluids in porous media*. Dover Publications, 1988
4. Beck, T., Sosoe, P., Wong, P.: Duchon-Robert solutions for the Rayleigh-Taylor and Muskat problems. *Journal of Differential Equations* **256**(1), 206–222 (2014)
5. Castro, A., Córdoba, D., Fefferman, C., Gancedo, F.: Breakdown of smoothness for the Muskat problem. *Arch. Ration. Mech. Anal.*, **208**(3), 805–909 (2013)

6. Castro, A., Córdoba, D., Fefferman, C., Gancedo, F., Gómez-Serrano, J.: Splash singularity for water waves. *Proc. Natl. Acad. Sci.*, **109**(3), 733–738 (2012)
7. Castro, A., Córdoba, D., Fefferman, C., Gancedo, F., Gómez-Serrano, J.: Finite time singularities for the free boundary incompressible Euler equations. *Ann. Math.* **178**(3), 1061–1134 (2013)
8. Castro, A., Córdoba, D., Fefferman, C., Gancedo, F., López-Fernández, M.: Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves. *Ann. Math.* **175**(2), 909–948 (2012)
9. Constantin, P., Córdoba, D., Gancedo, F., Strain, R.M.: On the global existence for the Muskat problem. *J. Eur. Math. Soc.* **15**, 201–227 (2013)
10. Córdoba, A., Córdoba, D.: A pointwise estimate for fractionary derivatives with applications to P.D.E. *Proc. Natl. Acad. Sci.*, **100**(26), 15316–15317 (2003)
11. Córdoba, A., Córdoba, D., Gancedo, F.: Interface evolution: the Hele-Shaw and Muskat problems. *Ann. Math.*, **173**(1), 477–542 (2011)
12. Córdoba, D., Gancedo, F.: Contour dynamics of incompressible 3-D fluids in a porous medium with different densities. *Comm. Math. Phys.* **273**(2), 445–471 (2007)
13. Córdoba, D., Gancedo, F.: A maximum principle for the Muskat problem for fluids with different densities. *Comm. Math. Phys.*, **286**(2), 681–696 (2009)
14. Córdoba, D., Granero, R., Orive, R.: The confined Muskat problem: differences with the deep water regime. *Comm. Math. Sci.* **12**(3), 423–455 (2014)
15. Córdoba, D., Pernas-Castaño, T.: Non-splat singularity for the one-phase Muskat problem. *Trans. Am. Math. Soc.* (2014). arXiv:1409.2483
16. Coutand, D., Shkoller, S.: On the finite-time splash and splat singularities for the 3-D free-surface Euler equations. *Comm. Math. Phys.* 325(1), 143–183 (2014)
17. Escher, J., Simonett, G.: Classical solutions for Hele-Shaw models with surface tension. *Adv. Differ. Equations*, **2**, 619–642 (1997)
18. Escher, J., Matioc, B.V.: On the parabolicity of the Muskat problem: Well-posedness, fingering, and stability results. *Z. Anal. Anwend.* **30**(2), 193–218 (2011)
19. Evans, L.C.: *Partial Differential Equations. Graduate Studies in Mathematics, 19*. American Mathematical Society, Providence, 1998
20. Fefferman, C., Ionescu, A., Lie, V.: On the absence of splash singularities in the case of two-fluid interfaces. *Duke Math. J.* **165**(3), 417–462 (2016)
21. Gancedo, F., Strain, R.M.: Absence of splash singularities for SQG sharp fronts and the Muskat problem. *Proc. Natl. Acad. Sci.* **111**(2), 635–639 (2014)
22. Gómez-Serrano, J., Granero, R.: On turning waves for the inhomogeneous Muskat problem: a computer-assisted proof. *Nonlinearity* **27**(6), 1471–1498 (2014)
23. Granero, R.: Global existence for the confined Muskat problem. *SIAM J. Math. Anal.* **46**(2), 1651–1680 (2014)
24. Guo, Y., Hallstrom, C., Spirn, D.: Dynamics near unstable, interfacial fluids. *Comm. Math. Phys.* **270**(3), 635–689 (2007)
25. Knüpfer, H., Masmoudi, N.: Darcy's flow with prescribed contact angle–Well-posedness and lubrication approximation. *Arch. Ration. Mech. Anal.* **218**(2), 589–646 (2015)
26. Muskat, M.: Two Fluid systems in porous media. The encroachment of water into an oil sand. *Physics* **5**, 250–264 (1934)
27. Otto, F.: Evolution of microstructure in unstable porous media flow: a relaxational approach. *Comm. Pure Appl. Maths* **52**, 873–915 (1999)
28. Saffman, P.G., Taylor, G.: The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc. R. Soc. London, Ser. A* **245**, 312–329 (1958)
29. Siegel, M., Caflisch, R., Howison, S.: Global Existence, Singular Solutions, and Ill-Posedness for the Muskat Problem. *Comm. Pure Appl. Math.* **57**, 1374–1411 (2004)
30. Székelyhidi, L. Jr.: Relaxation of the incompressible porous media equation. *Ann. Sci. Éc. Norm. Supér.* **45**(3), 491–509 (2012)

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