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Uniform Regularity and Vanishing Dissipation Limit for the Full Compressible Navier–Stokes System in Three Dimensional Bounded Domain

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Abstract

In the present paper, we study the uniform regularity and vanishing dissipation limit for the full compressible Navier–Stokes system whose viscosity and heat conductivity are allowed to vanish at different orders. The problem is studied in a three dimensional bounded domain with Navier-slip type boundary conditions. It is shown that there exists a unique strong solution to the full compressible Navier–Stokes system with the boundary conditions in a finite time interval which is independent of the viscosity and heat conductivity. The solution is uniformly bounded in $W^{1,\infty}$ and is a conormal Sobolev space. Based on such uniform estimates, we prove the convergence of the solutions of the full compressible Navier–Stokes to the corresponding solutions of the full compressible Euler system in $L^\infty(0, T; L^2)$, $L^\infty(0, T; H^1)$ and $L^\infty([0, T] \times \Omega)$ with a rate of convergence.

1. Introduction and Main Results

The motion of a compressible viscous, heat conductive, ideal polytropic fluid is governed by the following full compressible Navier–Stokes equations (FCNS):

$$\begin{cases} \rho^\varepsilon_t + \text{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ (\rho^\varepsilon u^\varepsilon)_t + \text{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla p^\varepsilon = \mu\varepsilon\Delta u^\varepsilon + (\mu+\lambda)\varepsilon\nabla\text{div}u^\varepsilon, \quad x \in \Omega,\ t > 0 \\ (\rho^\varepsilon E^\varepsilon)_t + \text{div}(\rho^\varepsilon u^\varepsilon E^\varepsilon + p^\varepsilon u^\varepsilon) = \kappa(\varepsilon)\Delta\theta^\varepsilon + \text{div}(\boldsymbol{\tau}^\varepsilon u^\varepsilon), \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^3$. Here ρ^ε, u^ε and E^ε represent density, velocity and total energy, respectively. The pressure function p^ε and total energy E^ε are given by

$$p^\varepsilon = R\rho^\varepsilon\theta^\varepsilon, \quad E^\varepsilon = c_v\left(\theta^\varepsilon + \frac{1}{2}|u^\varepsilon|^2\right),$$

where θ^ε is temperature and c_v is a positive constant. For simplicity of presentation, we normalize c_v to be 1. The tensor $\boldsymbol{\tau}^\varepsilon$ is represented as

$$\boldsymbol{\tau}^\varepsilon = \lambda\varepsilon \mathrm{div} u^\varepsilon I + 2\mu\varepsilon S u^\varepsilon, \quad \text{with } S u^\varepsilon = \frac{1}{2}(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T).$$

Here μ, λ are given constants satisfying the physical restriction

$$\mu > 0, \quad 2\mu + 3\lambda > 0, \tag{1.2}$$

and the parameter $\varepsilon > 0$ is the inverse of the Reynolds number. Also, $\kappa(\varepsilon) > 0$ is the heat conductivity which is assumed to depend on ε.

We impose the full compressible Navier–Stokes equations with the following Navier-slip type boundary conditions:

$$u^\varepsilon \cdot n = 0, \quad ((S u^\varepsilon) n)_\tau = (A u^\varepsilon)_\tau, \quad \text{and} \quad n \cdot \nabla\theta^\varepsilon = \nu\theta^\varepsilon, \quad \text{on } \partial\Omega. \tag{1.3}$$

where n is the outward unit normal to $\partial\Omega$, u_τ represents the tangential part of u, A is a smooth symmetric matrix and ν is a given constant. For smooth solutions, it is noticed that

$$(2S(v)n - (\nabla \times v) \times n)_\tau = -(2S(n)v)_\tau,$$

see [25] for details. The boundary condition (1.3) can be rewritten in the form of the vorticity as

$$u^\varepsilon \cdot n = 0, \quad n \times \omega^\varepsilon = [B u^\varepsilon]_\tau, \quad \text{and} \quad n \cdot \nabla\theta^\varepsilon = \nu\theta^\varepsilon, \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\omega^\varepsilon = \nabla \times u^\varepsilon$ is the vorticity and $B = 2(A - S(n))$ is a symmetric matrix. Actually, it turns out that the form (1.4) will be more convenient than (1.3) in the energy estimates.

We are interested in the existence of strong solutions of (1.1) with uniform bounds on an interval of time independent of the viscosity and heat conductivity, and the vanishing dissipation limit to the full compressible Euler flow as ε and $\kappa(\varepsilon)$ vanish, i.e,

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla p = 0, \\ [\rho(\theta + \frac{1}{2}|u|^2)]_t + \mathrm{div}[\rho u(\theta + \frac{1}{2}|u|^2) + p u] = 0, \end{cases} \quad \text{as } \varepsilon, \kappa \to 0+, \tag{1.5}$$

with slip boundary condition

$$u \cdot n|_{\partial\Omega} = 0. \tag{1.6}$$

There is a lot of literature on the inviscid limit for *incompressible* Navier–Stokes equations. The inviscid limit of the Cauchy problem has been studied by many authors, see for instance [5,6,11,13]. However, in the presence of a physical boundary, the problems become challenging due to the appearance of boundary layers. As illustrated by Prandtl's theory, the inviscid limit of the incompressible Navier–Stokes with a no-slip boundary condition to the incompressible Euler flows with a slip boundary condition (1.6) is a very difficult problem. SAMMARTINO and CAFLISCH [17,18] proved the convergence of the incompressible Navier–Stokes flows to the Euler flows away from the boundary and to the prandtl flows near the boundary at the inviscid limit for the analytic initial data. Recently, Maekawa [12] proved a such limit when the initial vorticity is located away from the boundary in the two dimensional half plane. On the other hand, for the incompressible Navier–Stokes system with a Navier-slip boundary condition (1.3) (without the heat flux part), lots of important progress has been made on the problem. The uniform H^3 bound and the inviscid limit to the Euler flow was proved by Xiao and Xin [23] for flat boundaries, and this was generalized to $W^{k,p}$ in [2,3] by Veiga–Crispo soon later. However, one cannot obtain such results for general curved boundaries since boundary layers may appear due to non-trivial curvature, as pointed out by Iftimie and Sueur [10], where the inviscid limit was also obtained in $L^\infty(0, T, ; L^2)$ by a careful construction of boundary layer expansions. In order to investigate precisely the asymptotic structure and get the convergence in stronger norms such as $L^\infty(0, T; H^s)(s > 0)$, stronger estimates are needed. Recently, Masmoudi and Rousset [14] established a conormal uniform estimate for three dimensional domains with the Navier-slip boundary condition, which implies the uniform boundedness of the Lipschitz-norm for the velocity field. This allows one to obtain the inviscid limit in the L^∞-norm by a compactness argument. Based on the uniform estimates in [14], better convergence with rates have been obtained in [8,24]. In particular, Xiao and Xin [24] proved the convergence in $L^\infty(0, T; H^1)$ with a rate.

For the *isentropic compressible* Navier–Stokes equations, Xin and Yanagisawa [26] studied the vanishing viscosity limit of the linearized problem with the no-slip boundary condition in a two dimensional half plane. For the Navier-slip boundary condition case, Wang and Williams [21] constructed a boundary layer solution of the compressible Navier–Stokes equations in a two dimensional half plane. The layers constructed in [21] are of width $O(\sqrt{\varepsilon})$, as the Prandtl boundary layer, but the amplitude layers are of $O(\sqrt{\varepsilon})$, which is similar to the one [10] for the incompressible case. It is also shown [21] that the boundary layers for the density are weaker than the one for the velocity, so, in general, it is impossible to obtain the H^3 or $W^{2,p}(p > 3)$ estimates for the compressible Navier–Stokes system with the Navier-slip boundary condition. Recently, Paddick [16] obtained an uniform conormal Sobolev estimate for the isentropic compressible Navier–Stokes system in the three dimensional half-space. Wang et al. [22] also obtained uniform regularity for isentropic compressible Navier–Stokes equations with Navier-slip a boundary conditions in a three dimensional domain with curvature; the inviscid limit was also obtained with a rate of convergence in $L^\infty([0, T] \times \Omega)$ and $L^\infty([0, T]; H^1)$. The fact that the boundary layer for density is weaker than the one for velocity fields was also shown in [22].

For the *full compressible* Navier–Stokes equations, the extent of study is quite limited. Under the assumption that the viscosity and heat conductivity converge to zero at the same order, Ding and Jiang [7] studied the zero viscosity and heat conductivity limit for the linearized compressible Navier–Stokes–Fourier equations with a no-slip boundary condition in the half plane. However, there is no uniform regularity and vanishing dissipation limit results for the *full compressible* Navier–Stokes equations (1.1) with Navier-slip type boundary conditions (1.3) in a bounded domain. The aim of this paper is to investigate the uniform regularity for the solutions of the full compressible Navier–Stokes system (1.1) even if the viscosity and heat conductivity converge to zero at different orders. Compared to the isentropic case [16,22], it is difficult to obtain the Lipschitz estimates for the solutions of (1.1) due to the appearance of temperature and the strongly coupled system of $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$. On the other hand, the amplitude and width of boundary layers for velocity and temperature may be not at the same order if the viscous and heat conductivity vanish at a different order. In that case the interaction of the two different amplitude boundary layers, may throw up difficulties in the analysis in terms of obtaining uniform regularity, especially, in terms of the Lipschitz estimates. To overcome these difficulties, some new ideas and observations are needed. It is also very important to study the vanishing dissipation limit. In addition, and very importantly we shall investigate how the rate of convergence is influenced by the thermal boundary layers.

Before stating our main results, we first explain the notations and conventions used throughout this paper. Similar to [14,22], one assumes that the bounded domain $\Omega \subset \mathbb{R}^3$ has a covering such that

$$\Omega \subset \Omega_0 \cup_{k=1}^n \Omega_k, \tag{1.7}$$

where $\overline{\Omega}_0 \subset \Omega$, and in each Ω_k there exists a function ψ_k such that

$$\begin{aligned}\Omega \cap \Omega_k &= \{x = (x_1, x_2, x_3) \mid x_3 > \psi_k(x_1, x_2)\} \cap \Omega_k,\\ \partial\Omega \cap \Omega_k &= \{x_3 = \psi_k(x_1, x_2)\} \cap \Omega_k.\end{aligned}$$

Here, Ω is said to be $\mathcal{C}^m$ if the functions ψ_k are a $\mathcal{C}^m$-function. To define the Sobolev conormal spaces, one considers $(Z_k)_{1\leqq k\leqq N}$ to be a finite set of generators of vector fields that are tangential to $\partial\Omega$, and sets

$$H_{co}^m = \{f \in L^2(\Omega) | Z^I f \in L^2(\Omega), \quad \text{for } |I| \leqq m\},$$

where $I = (k_1, \ldots, k_m)$. The following notations will be used:

$$\|u\|_m^2 = \|u\|_{H_{co}^m}^2 = \sum_{j=1}^3 \sum_{|I|\leqq m} \|Z^I u_j\|_{L^2}^2,$$

$$\|u\|_{m,\infty}^2 = \sum_{|I|\leqq m} \|Z^I u\|_{L^\infty}^2 \quad \text{and} \quad \|\nabla Z^m u\|^2 = \sum_{|I|=m} \|\nabla Z^I u\|_{L^2}^2.$$

Note that by using the covering of Ω, one can always assume that each vector field (ρ, u, θ) is supported in one of the Ω_i, moreover, in Ω_0, the norm $\|\cdot\|_m$ yields a control of the standard H^m norm, whereas if $\Omega_i \cap \partial\Omega \neq \emptyset$, there is no control of the normal derivatives.

Denote by C_k a positive constant independent of $\varepsilon, \kappa \in (0, 1]$ which depends only on the $\mathcal{C}^k$-norm of the functions ψ_j, $j = 1\ldots, n$. Since $\partial\Omega$ is given locally by $x_3 = \psi(x_1, x_2)$ (we omit the subscript j for notational convenience), it is convenient to use the coordinates:

$$\Psi\colon (y, z) \longmapsto (y, \psi(y) + z) = x.$$

A local basis is thus given by the vector fields (e_{y^1}, e_{y^2}, e_z) where $e_{y^1} = (1, 0, \partial_1\psi)^t$, $e_{y^2} = (0, 1, \partial_2\psi)^t$ and $e_z = (0, 0, -1)^t$. On the boundary, e_{y^1} and e_{y^2} are tangential to $\partial\Omega$, and in general, e_z is not a normal vector field. By using this parametrization, one can take as suitable vector fields compactly supported in Ω_j in the definition of the $\|\cdot\|_m$ norms:

$$Z_i = \partial_{y^i} = \partial_i + \partial_i\psi\partial_z, \quad i = 1, 2, \quad Z_3 = \varphi(z)\partial_z, \tag{1.8}$$

where $\varphi(z) = \frac{z}{1+z}$ is smooth, supported in $\mathbb{R}_+$ with the property $\varphi(0) = 0, \varphi'(0) > 0, \varphi(z) > 0$ for $z > 0$. It is easy to check that

$$Z_k Z_j = Z_j Z_k, \quad j,\ k = 1, 2, 3,$$

and

$$\partial_z Z_i = Z_i \partial_z, \quad i = 1, 2, \quad \text{and} \quad \partial_z Z_3 \neq Z_3 \partial_z.$$

In this paper, we shall still denote by $\partial_j,\ \ j = 1, 2, 3$ or ∇ the derivatives in the physical space. The coordinates of a vector field u in the basis (e_{y^1}, e_{y^2}, e_z) will be denoted by u^i, thus

$$u = u^1 e_{y^1} + u^2 e_{y^2} + u^3 e_z. \tag{1.9}$$

We shall denote by u_j the coordinates in the standard basis of $\mathbb{R}^3$, i.e, $u = u_1 e_1 + u_2 e_2 + u_3 e_3$. Denote by n the unit outward normal in the physical space which is given locally by

$$n(x) \equiv n(\Psi(y, z)) = \frac{1}{\sqrt{1 + |\nabla\psi(y)|^2}} \begin{pmatrix} \partial_1\psi(y) \\ \partial_2\psi(y) \\ -1 \end{pmatrix} \doteq \frac{-N(y)}{\sqrt{1 + |\nabla\psi(y)|^2}}, \tag{1.10}$$

and by Π, the orthogonal projection

$$\Pi u \equiv \Pi(\Psi(y, z))u = u - [u \cdot n(\Psi(y, z))]n(\Psi(y, z)), \tag{1.11}$$

which gives the orthogonal projection onto the tangent space of the boundary. Note that n and Π are defined in the whole Ω_k and do not depend on z.

For later use and notational convenience, set

$$\mathcal{Z}^\alpha = \partial_t^{\alpha_0} Z^{\alpha_1} = \partial_t^{\alpha_0} Z_1^{\alpha_{11}} Z_2^{\alpha_{12}} Z_3^{\alpha_{13}}, \tag{1.12}$$

where $\alpha, \alpha_0, \alpha_1$ are the differential multi-indices with $\alpha \doteq (\alpha_0, \alpha_1),\ \alpha_1 = (a_{11}, \alpha_{12}, \alpha_{13})$, and we also use the notations

$$\|f(t)\|_{\mathcal{H}^m}^2 = \sum_{|\alpha| \leqq m} \|\mathcal{Z}^\alpha f(t)\|_{L_x^2}^2, \quad \|f(t)\|_{\mathcal{H}^{k,\infty}} = \sum_{|\alpha| \leqq k} \|\mathcal{Z}^\alpha f(t)\|_{L_x^\infty} \tag{1.13}$$

for the smooth space-time function $f(x, t)$.

Firstly, we consider the uniform regularity of the solutions of the full compressible Navier–Stokes system (1.1) with the Navier-slip type boundary conditions (1.3). Since the viscous and thermal boundary layers may appear in the presence of physical boundaries, one needs to design a suitable functional space.

Here, motivated by [14,22], the functional space $X_m^\varepsilon(T)$ for functions $(\rho, u, \theta) = (\rho, u, \theta)(x, t)$ is defined as

$$X_m^\varepsilon(T) = \left\{ (\rho, u, \theta) \in L^\infty([0, T], L^2); \quad \operatorname*{esssup}_{0 \leqq t \leqq T} \|(\rho, u, \theta)(t)\|_{X_m^\varepsilon} < +\infty \right\}, \tag{1.14}$$

where the norm $\|(\cdot, \cdot, \cdot)\|_{X_m^\varepsilon}$ is given by

$$\begin{aligned} &\|(\rho, u, \theta)(t)\|_{X_m^\varepsilon}^2 \\ &\quad = \|(\rho, u, \theta)(t)\|_{\mathcal{H}^m}^2 + \|\nabla u(t)\|_{\mathcal{H}^{m-1}}^2 + \sum_{k=0}^{m-2} \|\partial_t^k \nabla(\rho, \theta)(t)\|_{m-1-k}^2 \\ &\qquad + \varepsilon \|\nabla \partial_t^{m-1} \rho(t)\|^2 + \varepsilon \|\nabla \partial_t^{m-2} \mathrm{div} u(t)\|^2 + \kappa(\varepsilon) \|\Delta \partial_t^{m-2} \theta(t)\|^2 \\ &\qquad + \|\nabla(\rho, u, \theta)(t)\|_{\mathcal{H}^{1,\infty}}^2 + \varepsilon \|\nabla^2 u(t)\|_{L^\infty}^2 + \varepsilon \|\nabla(\rho\theta)(t)\|_{\mathcal{H}^{2,\infty}}^2. \end{aligned} \tag{1.15}$$

We remark that the term $\varepsilon \|\nabla(\rho\theta)(t)\|_{\mathcal{H}^{2,\infty}}^2$, included in (1.15), is important for obtaining the Lipschitz estimates even though such a term is slightly strange. We will explain the reason for including such term after Theorem 1.1 below.

In the present paper, we supplement the full compressible Navier–Stokes equations (1.1) with the initial data

$$(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)(x, 0) = (\rho_0^\varepsilon, u_0^\varepsilon, \theta^\varepsilon)(x), \tag{1.16}$$

such that

$$\sup_{0<\varepsilon \leqq 1} \|(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)\|_{X_m^\varepsilon}^2 \leqq \tilde{C}_0, \quad 0 < \hat{C}_0^{-1} \leqq \rho_0^\varepsilon, \quad \theta_0^\varepsilon \leqq \hat{C}_0 < \infty, \tag{1.17}$$

where $\hat{C}_0 > 0$, $\tilde{C}_0 > 0$ are positive constants independent of $\varepsilon \in (0, 1]$ and the time derivatives of initial data in (1.17) are defined through the full compressible Navier–Stokes system (1.1). Thus, the initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)$ is assumed to have a higher space regularity and compatibilities. Notice that the *a priori* estimates in Theorem 3.1 below are obtained in the case that the approximate solution is sufficiently smooth up to the boundary, therefore, in order to obtain a self-contained result, one needs to assume that the approximate initial data satisfies the boundary compatibility conditions, that is (1.3) (or equivalent to (1.4)). For the initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)$ satisfying (1.17), it is not clear if there exists an approximate sequence $(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, , \theta_0^{\varepsilon,\delta})$ (δ being a regularization parameter), which satisfies the boundary compatibilities and $\|(\rho_0^{\varepsilon,\delta} - \rho_0^\varepsilon, u_0^{\varepsilon,\delta} - u_0^\varepsilon, \theta_0^{\varepsilon,\delta} - \theta_0^\varepsilon)\|_{X_m^\varepsilon} \to 0$ as $\delta \to 0$. Therefore, we set

$$X^{\varepsilon,m}_{NS,ap} = \{(\rho, u, \theta) \in H^{3m}(\Omega) | \partial_t^k \rho, \partial_t^k \theta,\ \partial_t^k u, k = 1, \dots, m \text{ are defined through the system (1.1) and } \partial_t^k u, \partial_t^k \theta, k = 0, \dots, m-1 \text{ satisfy the boundary compatibility conditions}\}, \tag{1.18}$$

and

$$X^{\varepsilon,m}_{NS} = \text{The closure of } X^{\varepsilon,m}_{NS,ap} \text{ in the norm } \|(\cdot,\cdot,\cdot)\|_{X^{\varepsilon}_m}. \tag{1.19}$$

If the heat conductivity $\kappa(\varepsilon)$ decays too fast as $\varepsilon \to 0+$, then the possible interaction between the viscous boundary layers and thermal boundary layers is strong and it is hard to get the uniform regularity. Thus, in order to control the possible interaction between the viscous boundary layers and the thermal boundary layers, throughout this paper, we assume that the heat conductivity is a continuous function of ε and satisfies

$$\varepsilon^4 \leqq C\kappa(\varepsilon) < \infty, \quad \text{for } \varepsilon \in (0, 1], \tag{1.20}$$

where $C > 0$ is some positive constant. Then our uniform regularity result is as follows:

Theorem 1.1. (Uniform regularity) *Let m be an integer satisfying $m \geqq 6$, $\kappa(\varepsilon)$ satisfy (1.20), Ω be a $\mathcal{C}^{m+2}$ domain and $A \in \mathcal{C}^{m+1}(\partial\Omega)$. Consider the initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon) \in X^{\varepsilon,m}_{NS}$ given in (1.16) and satisfying (1.17). Then there exists a time $T_0 > 0$ and $\tilde{C}_1 > 0$ independent of $\varepsilon \in (0, 1]$ such that there exists a unique solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ of (1.1), (1.3) and (1.16) on $[0, T_0]$ that satisfies the estimates:*

$$(2\hat{C}_0)^{-1} \leqq \rho^\varepsilon(t), \theta^\varepsilon(t) \leqq 2\hat{C}_0 \quad \forall t \in [0, T_0], \tag{1.21}$$

and

$$\begin{aligned}
&\sup_{0 \leqq t \leqq T_0} \|(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)(t)\|^2_{X^\varepsilon_m} + \int_0^{T_0} \|\nabla \partial_t^{m-1}(\rho^\varepsilon, \theta^\varepsilon)(t)\|^2 + \|\nabla(\rho^\varepsilon \theta^\varepsilon)(t)\|^2_{\mathcal{H}^{2,\infty}} \mathrm{d}t \\
&+ \varepsilon \int_0^{T_0} \|\nabla u^\varepsilon(t)\|^2_{\mathcal{H}^m} \mathrm{d}t + \kappa(\varepsilon) \int_0^{T_0} \|\nabla \theta^\varepsilon(t)\|^2_{\mathcal{H}^m} \mathrm{d}t \\
&+ \sum_{k=0}^{m-2} \int_0^{T_0} \varepsilon \|\nabla^2 \partial_t^k u^\varepsilon(t)\|^2_{m-1-k} + \kappa(\varepsilon) \|\Delta \partial_t^k \theta^\varepsilon(t)\|^2_{m-1-k} \mathrm{d}t \\
&+ \kappa(\varepsilon)^2 \int_0^{T_0} \|\Delta \partial_t^{m-1} \theta^\varepsilon(t)\|^2 + \|\nabla \mathcal{Z}^{m-2} \Delta \theta^\varepsilon(t)\|^2 \mathrm{d}t \\
&+ \varepsilon^2 \int_0^{T_0} \|\nabla^2 \partial_t^{m-1} u^\varepsilon(t)\|^2 \mathrm{d}t \leqq \tilde{C}_1 < \infty,
\end{aligned} \tag{1.22}$$

where $\tilde{C}_1$ depends only on $\hat{C}_0$, $\tilde{C}_0$ and C_{m+2}.

Remark 1.2. The novelty of this work is that we allow the viscous and heat conductivity to vanish at different orders. Futhermore, it is noted that there are many functions that satisfy condition (1.20). For example, it is easy to see that (1.20) holds provided that $\kappa(\varepsilon) = \varepsilon^b$, where b is constant such that $0 \leqq b \leqq 4$.

Remark 1.3. In order to obtain the Lipschitz estimates included in (1.15), one needs to use the pointwise estimates because the boundary layers prevent one from obtaining a uniform estimate in $H^3(\Omega)$ (or $W^{2,p}$, $p > 0$). Thus, one has to deal with the possible interaction of the viscous and thermal boundary layers in the pointwise estimates. Indeed, restriction (1.20) is used to control such possible interactions, see Lemmas 3.14–3.18 for details.

Remark 1.4. For the solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)(t)$ of (1.1), (1.4) and (1.16), the boundary conditions (1.3) (or equivalently (1.4)) are satisfied in the trace sense for every fixed $\varepsilon \in (0, T_0]$ and $t \in (0, T_0]$.

We now describe the main ideas of the proof of Theorem 1.1. It turns out that it suffices to establish the estimates (1.22). It is noted that there are two parts included in (1.22), that is, the conormal energy estimates part and the pointwise estimates part. Firstly, by complicated conormal energy estimates, one can obtain

$$\begin{aligned}
&\|(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)(t)\|_{\mathcal{H}^m}^2 + \sum_{k=0}^{m-2} \|\partial_t^k \nabla(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)(t)\|_{m-1-k}^2 + \|\partial_t^{m-1}\omega^\varepsilon(t)\|^2 \\
&\quad + \varepsilon\|\nabla\partial_t^{m-1}\rho^\varepsilon(t)\|^2 + \varepsilon\int_0^t \|\nabla u^\varepsilon(\tau)\|_{\mathcal{H}^m}^2 + \int_0^t \varepsilon^2\|\nabla^2\partial_t^{m-1}u^\varepsilon(\tau)\|^2 \\
&\quad + \kappa\int_0^t \|\nabla\theta^\varepsilon(\tau)\|_{\mathcal{H}^m}^2 + \sum_{k=0}^{m-2}\int_0^t \|\partial_t^k(\sqrt{\varepsilon}\nabla^2 u^\varepsilon, \sqrt{\kappa}\Delta\theta^\varepsilon)(\tau)\|_{m-1-k}^2, \qquad (1.23)
\end{aligned}$$

at the cost of

$$\int_0^t \|\nabla\mathcal{Z}^{m-1}\mathrm{div}u^\varepsilon\|^2 + \kappa^2\|\nabla\mathcal{Z}^{m-2}\Delta\theta^\varepsilon\|^2\, d\tau \quad \text{and} \quad \int_0^t \|\partial_t^{m-1}\nabla(\rho^\varepsilon, \theta^\varepsilon)\|^2\, d\tau, \qquad (1.24)$$

see Lemmas 3.2–3.6 and 3.11 below. By using the structure of mass equation and energy equation, respectively, one can bound the first part of (1.24) at the cost of $\int_0^t \|\partial_t^{m-1}\nabla(\rho^\varepsilon, \theta^\varepsilon)\|^2\, d\tau$, see Lemma 3.7 below, so, it suffices to bound the second part of (1.24). Considering $\int_0^t\int \partial_t^{m-2}(3.7)_2 \cdot \nabla\partial_t^{m-2}\mathrm{div}u^\varepsilon + \partial_t^{m-1}\nabla(3.7)_3 \cdot \frac{\nabla\partial_t^{m-1}\theta^\varepsilon}{\theta^\varepsilon}\, dx\, d\tau$, one can control the second part of (1.24) and $\varepsilon\|\nabla\partial_t^{m-2}\mathrm{div}u^\varepsilon(t)\|^2 + \kappa(\varepsilon)\|\Delta\partial_t^{m-2}\theta^\varepsilon(t)\|^2$, see Lemma 3.8 below. Therefore, combining the above estimates, one can obtain the conormal energy estimates of (1.22), except for $\|\nabla\partial_t^{m-1}u^\varepsilon\|^2$, see also (3.162) below.

Next, we try to establish the pointwise estimates part. It is difficult to obtain such estimates however, because the equations of $\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon$ are strongly coupled and the viscosity and heat conductivity are not of the same order. Actually, if one estimates $\|\nabla\rho^\varepsilon\|_{\mathcal{H}^{1,\infty}}$ directly, then one has to deal with the high order derivative term $\int_0^t \|\nabla\theta^\varepsilon\|_{\mathcal{H}^{2,\infty}}^2\, d\tau$, however it is hard to control this term in our functional space. Instead, we try to control the pointwise estimates of $\nabla(\rho^\varepsilon\theta^\varepsilon)$, $\nabla\theta^\varepsilon$ and ∇u^ε. This is key to overcoming the difficulty. Indeed, we can obtain (see Lemma 3.15 below)

$$\|\nabla(\rho^\varepsilon\theta^\varepsilon)(t)\|^2_{\mathcal{H}^{1,\infty}} + \varepsilon\|\nabla(\rho^\varepsilon\theta^\varepsilon)(t)\|^2_{\mathcal{H}^{2,\infty}} + \int_0^t \|\nabla(\rho^\varepsilon\theta^\varepsilon)(\tau)\|^2_{\mathcal{H}^{2,\infty}}$$
$$\leqq \varepsilon\int_0^t \|\nabla\theta^\varepsilon(\tau)\|^2_{\mathcal{H}^{3,\infty}}\,\mathrm{d}\tau + \ldots, \tag{1.25}$$

where ... means that terms can be controlled. Since the strength and width of the thermal boundary layers is connected with κ, then the first term on the RHS above is actually the interaction of viscous and thermal boundary layers. If the heat conductivity κ vanishes too fast, it will be very hard to control such an interaction term. To overcome this difficulty, we assume that the decay rate of κ satisfies (1.20). Then the interaction term can be controlled as follows:

$$\varepsilon\int_0^t \|\nabla\theta^\varepsilon(\tau)\|^2_{\mathcal{H}^{3,\infty}}\,\mathrm{d}\tau \leqq \varepsilon^4\int_0^t \|\Delta\theta^\varepsilon(\tau)\|^2_{\mathcal{H}^4}\,\mathrm{d}\tau + \cdots$$
$$\leqq \kappa\int_0^t \|\Delta\theta^\varepsilon(\tau)\|^2_{\mathcal{H}^4}\,\mathrm{d}\tau + \cdots, \tag{1.26}$$

where the last term of the above has already been controlled in the conormal energy estimates part. It is worth pointing out that the above interaction estimate will be employed repeatedly throughout the pointwise estimates part. On the other hand, to control the pointwise estimate of $\nabla\theta^\varepsilon$, the most difficult part is to deal with the term $p^\varepsilon\nabla\mathrm{div}u^\varepsilon$ which comes from the term $p^\varepsilon\mathrm{div}u^\varepsilon$ on the LHS energy equation (see also (3.223) below). Actually, if $p^\varepsilon\nabla\mathrm{div}u^\varepsilon$ is regarded as a source term, it will be very hard to control $\int_0^t \|p^\varepsilon\nabla\mathrm{div}u^\varepsilon(\tau)\|^2_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau$ because the derivatives are too high. We remark that such difficulty does not arise in the isentropic case [22]. To overcome the difficulty, some new ideas are needed. Fortunately, we find that the term $p^\varepsilon\nabla\mathrm{div}u^\varepsilon$ can be represented as follows:

$$p^\varepsilon\nabla\mathrm{div}u^\varepsilon = R\rho^\varepsilon[\nabla\theta^\varepsilon_t + (u^\varepsilon\cdot\nabla)\nabla\theta^\varepsilon]$$
$$-R[\nabla(\rho^\varepsilon\theta^\varepsilon)_t + (u^\varepsilon\cdot\nabla)\nabla(\rho^\varepsilon\theta^\varepsilon)] + \text{lower order terms}. \tag{1.27}$$

It is noted that the first part (that is, the hardest part) on the RHS of (1.27) can be absorbed into the main part of the equation (see (3.223)–(3.226) below), while the second part on the RHS of (1.27) can be regarded as a source term because the term $\int_0^t \|\nabla(\rho^\varepsilon\theta^\varepsilon)(\tau)\|^2_{\mathcal{H}^{2,\infty}}$ has already been controlled above. This observation is key to closing the pointwise estimates. It is also one of the main reasons to include $\varepsilon\|\nabla(\rho^\varepsilon\theta^\varepsilon)\|^2_{\mathcal{H}^{2,\infty}}$ in our functional space. Based on the above observation, one can obtain the control of $\|\nabla\theta^\varepsilon\|^2_{\mathcal{H}^{1,\infty}}$. Later, by arguments similar to [22], one can control $\|\nabla u^\varepsilon\|^2_{\mathcal{H}^{1,\infty}} + \varepsilon\|\nabla^2 u^\varepsilon\|^2_{L^\infty}$. Finally, in order to estimate $\|\nabla\partial_t^{m-1}u^\varepsilon\|^2$, we still need to obtain the uniform bound of $\|\partial_t^{m-1}\mathrm{div}u^\varepsilon\|^2$; this is hard to get by the conormal energy estimate because some boundary terms are hard to control. By employing the mass equation, it is found that $\|\partial_t^{m-1}\mathrm{div}u^\varepsilon\|^2$ can be controlled by the conormal energy estimates and pointwise estimates obtained above. Therefore, combining all the above estimates, one proves (1.22).

Based on the uniform estimates in Theorem 1.1, by strong compactness arguments similar to those in [14], one can justify the vanishing dissipation limit of solutions of the full compressible Navier–Stokes system (1.1) to the solutions of the full Euler equations (1.5) in the L^∞-norm, but without a convergence rate. In the present paper, we are interested in the vanishing dissipation limit with rates of convergence.

We supplement the full Euler equations (1.5) and the full compressible Navier–Stokes equations (1.1) with the same initial data (ρ_0, u_0, θ_0) satisfying

$$0 < \hat{C}_0^{-1} \leqq \rho_0, \theta_0 \leqq \hat{C}_0 \quad \text{and} \quad (\rho_0, u_0, \theta_0) \in H^3 \cap X_{NS}^{\varepsilon,m} \quad \text{with } m \geqq 6. \quad (1.28)$$

It is well known that there exists a unique smooth solution $(\rho, u, \theta) \in H^3$ for the problem (1.5) and (1.6) with initial data (ρ_0, u_0, θ_0) at least locally in time $[0, T_1]$ where $T_1 > 0$ depends only on $\|(\rho_0, u_0, \theta_0)\|_{H^3}$. On the other hand, it follows from Theorem 1.1 that there exists a time T_0 and $\tilde{C}_1$ independent of $\varepsilon \in (0, 1]$, such that there exists a unique solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ of (1.1) and (1.3) with the initial data (ρ_0, u_0, θ_0) and that satisfies (1.21) and (1.22).

Then we justify the vanishing dissipation limit as follows:

Theorem 1.5. (Vanishing dissipation limit) *Based on the above preparations, under the assumptions of Theorem 1.1 and $\kappa(\varepsilon) \to 0+$ as $\varepsilon \to 0+$, there exists $T_2 = \min\{T_0, T_1\} > 0$, which is independent of $\varepsilon > 0$, such that*

$$\begin{aligned}&\|(\rho^\varepsilon - \rho, u^\varepsilon - u, \theta^\varepsilon - \theta)(t)\|_{L^2}^2 + \int_0^t \varepsilon\|(u^\varepsilon - u)(\tau)\|_{H^1}^2 + \kappa(\varepsilon)\|(\theta^\varepsilon - \theta)(\tau)\|_{H^1}^2 \, d\tau \\ &\quad \leqq C \max\{\varepsilon^{\frac{3}{2}}, \kappa(\varepsilon)^{\frac{3}{2}}\}, \quad t \in [0, T_2],\end{aligned} \quad (1.29)$$

$$\begin{aligned}&\|(\rho^\varepsilon - \rho, u^\varepsilon - u, \theta^\varepsilon - \theta)(t)\|_{H^1}^2 + \int_0^t \varepsilon\|(u^\varepsilon - u)(\tau)\|_{H^2}^2 + \kappa(\varepsilon)\|(\theta^\varepsilon - \theta)(\tau)\|_{H^2}^2 \, d\tau \\ &\quad \leqq C \max\{\varepsilon^{\frac{1}{2}}, \kappa(\varepsilon)^{\frac{1}{3}}\}, \quad t \in [0, T_2],\end{aligned} \quad (1.30)$$

and

$$\begin{aligned}&\|(\rho^\varepsilon - \rho, u^\varepsilon - u, \theta^\varepsilon - \theta)\|_{L^\infty(\Omega\times[0,T_2])} \\ &\quad \leqq \|(\rho^\varepsilon - \rho, u^\varepsilon - u, \theta^\varepsilon - \theta)\|_{L^2}^{\frac{2}{5}} \cdot \|(\rho^\varepsilon - \rho, u^\varepsilon - u, \theta^\varepsilon - \theta)\|_{W^{1,\infty}}^{\frac{3}{5}} \\ &\quad \leqq C \max\{\varepsilon, \kappa(\varepsilon)\}^{\frac{3}{10}},\end{aligned} \quad (1.31)$$

where C depends only on the norm $\|(\rho_0, u_0, \theta_0)\|_{H^3} + \|(\rho_0, u_0, \theta_0)\|_{X_m^\varepsilon}$.

Remark 1.6. It is easy to see that $k(\varepsilon) = \varepsilon^a$ with $0 < a \leqq 4$ satisfies the condition of Theorem 1.5.

Remark 1.7. Compared to the isentropic case [22], one can see that the convergence rates of the vanishing dissipation limit are influenced by the decay rate of heat conductivity. In particular, for the case $k(\varepsilon) = \varepsilon$, Theorem 1.5 implies that the convergence rate in $L^\infty(0, T_2; H^1)$ is $\varepsilon^{\frac{1}{3}}$, which is slower than the isentropic case [22] whose corresponding rate is $\varepsilon^{\frac{1}{2}}$. This is mainly due to the influence of thermal boundary layers, see Lemma 5.2 below. If one can prove that $\kappa(\varepsilon)^{\frac{3}{2}} \int_0^t \|\nabla\Delta\theta^\varepsilon\|^2 \, d\tau$ is uniformly bounded, then the convergence rate of (1.30) could be improved to be $\max(\varepsilon^{\frac{1}{2}}, \kappa(\varepsilon)^{\frac{1}{2}})$, however it is very hard to obtain such a uniform estimate in our framework.

Remark 1.8. By the same arguments as Theorem 1.5, one can also prove the dissipation limit of the full compressible Navier–Stokes system to the following system:

$$\begin{cases} \rho_t + \text{div}(\rho u) = 0, \\ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p = 0, \\ [\rho(\theta + \frac{1}{2}|u|^2)]_t + \text{div}[\rho u(\theta + \frac{1}{2}|u|^2) + pu] = \kappa_0 \Delta\theta, \end{cases} \quad \text{if } \kappa(\varepsilon) \to \kappa_0 > 0 \quad \text{as } \varepsilon \to 0+ \tag{1.32}$$

with boundary conditions

$$u \cdot n|_{\partial\Omega} = 0, \quad n \cdot \nabla\theta|_{\partial\Omega} = \nu\theta.$$

The rest of the paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. We prove the *a priori* estimates of Theorem 3.1 in Section 3, which is the main part of this paper. By using the *a priori* estimates, we prove Theorem 1.1 in Section 4. Section 5 is devoted to the proof of Theorem 1.5.

Notations. Throughout this paper, the positive generic constants that are independent of ε are denoted by c, C (through may depend on μ, λ). $\|\cdot\|$ denotes the standard $L^2(\Omega;\ dx)$ norm, and $\|\cdot\|_{H^m}$ $(m = 1, 2, 3, \ldots)$ denotes the Sobolev $H^m(\Omega;\ dx)$ norm. The notation $|\cdot|_{H^m}$ will be used for the standard Sobolev norm of functions defined on $\partial\Omega$. Note that this norm involves only tangential derivatives. $P(\cdot)$ denotes a polynomial function.

2. Preliminaries

The following lemma [20,23] allows one to control the $H^m(\Omega)$-norm of a vector valued function u by its $H^{m-1}(\Omega)$-norm of $\nabla \times u$ and div$u$, together with the $H^{m-\frac{1}{2}}(\partial\Omega)$-norm of $u \cdot n$:

Proposition 2.1. *Let $m \in \mathbb{N}_+$ be an integer. Let $u \in H^m$ be a vector-valued function. Then there exists a constant $C > 0$ independent u such that*

$$\|u\|_{H^m} \leqq C\left(\|\nabla \times u\|_{H^{m-1}} + \|\text{div}u\|_{H^{m-1}} + \|u\|_{H^{m-1}} + |u \cdot n|_{H^{m-\frac{1}{2}}}\right) \tag{2.1}$$

and

$$\|u\|_{H^m} \leqq C\left(\|\nabla \times u\|_{H^{m-1}} + \|\text{div}u\|_{H^{m-1}} + \|u\|_{H^{m-1}} + |n \times u|_{H^{m-\frac{1}{2}}}\right). \tag{2.2}$$

In this paper, we shall use repeatedly the Gagliardo–Nirenberg–Morser type inequality, whose proof can be found in [9]. First, define the space

$$\mathcal{W}^m(\Omega\times[0,T]) = \{f(x,t) \in L^2(\Omega\times[0,T]) | \mathcal{Z}^\alpha f \in L^2(\Omega\times[0,T]), \quad |\alpha| \leqq m\}. \tag{2.3}$$

Then the Gagliardo–Nirenberg–Morser type inequality is as follows:

Proposition 2.2. *For $u, v \in L^\infty(\Omega \times [0,T]) \cap \mathcal{W}^m(\Omega \times [0,T])$ with $m \in \mathbb{N}_+$ be an integer. It holds that*

$$\int_0^t \|(\mathcal{Z}^\beta u \mathcal{Z}^\gamma v)(\tau)\|^2 \, d\tau \lesssim \|u\|_{L^\infty_{t,x}}^2 \int_0^t \|v(\tau)\|_{\mathcal{H}^m}^2 \, d\tau + \|v\|_{L^\infty_{t,x}}^2 \int_0^t \|u(\tau)\|_{\mathcal{H}^m}^2 \, d\tau, \quad |\beta| + |\gamma| = m. \tag{2.4}$$

We also need the following anisotropic Sobolev embedding and trace estimates:

Proposition 2.3. *Let $m_1 \geqq 0$ and $m_2 \geqq 0$ be integers; $f \in H_{co}^{m_1}(\Omega) \cap H_{co}^{m_2}(\Omega)$ and $\nabla f \in H_{co}^{m_2}(\Omega)$. Then:*

(1) *The anisotropic Sobolev embedding*

$$\|f\|_{L^\infty}^2 \leqq C(\|\nabla f\|_{H_{co}^{m_2}} + \|f\|_{H_{co}^{m_2}}) \cdot \|f\|_{H_{co}^{m_1}}, \tag{2.5}$$

holds, provided that $m_1 + m_2 \geqq 3$.

(2) *The following trace estimate holds:*

$$|f|_{H^s(\partial\Omega)}^2 \leqq C(\|\nabla f\|_{H_{co}^{m_2}} + \|f\|_{H_{co}^{m_2}}) \cdot \|f\|_{H_{co}^{m_1}}, \tag{2.6}$$

provided that $m_1 + m_2 \geqq 2s \geqq 0$.

Proof. The proof just uses the covering $\Omega \subset \Omega_0 \cup_{k=1}^n \Omega_k$ and Proposition 2.2 in [15], the details are omitted here. □

3. A Priori Estimates

The aim of this section is to prove some *a priori* estimates, which is a crucial step to proving Theorem 1.1. For notational convenience, we drop the superscript ε throughout this section.

Theorem 3.1. *Let m be an integer satisfying $m \geqq 6$, let $\kappa(\varepsilon)$ satisfy (1.20), let Ω be a $\mathcal{C}^{m+2}$ domain and $A \in \mathcal{C}^{m+1}(\partial\Omega)$. For a very sufficiently smooth solution defined on $[0,T]$ of (1.1) and (1.3) in Ω, we then have*

$$|\rho(x,0)| \exp\left(-\int_0^t \|\text{div} u\|_{L^\infty} \, d\tau\right) \leqq \rho(x,t) \leqq |\rho(x,0)| \exp\left(\int_0^t \|\text{div} u\|_{L^\infty} \, d\tau\right), \quad \forall t \in [0,T] \tag{3.1}$$

and

$$\theta_0 - \int_0^t \|\theta(\tau)\|_{L^\infty} \, d\tau \leqq \theta(x,t) \leqq \theta_0 + \int_0^t \|\theta(\tau)\|_{L^\infty} \, d\tau, \quad \forall t \in [0,T]. \tag{3.2}$$

In addition, if it holds that

$$0 < c_0 \leqq \rho(x,t), \theta(x,t) \leqq \frac{1}{c_0} < \infty, \quad \forall t \in [0,T], \tag{3.3}$$

where c_0 is any given small positive constant. Then we have the a priori estimate

$$\begin{aligned}
&\Upsilon_m(\rho,u,\theta)\\
&\triangleq \mathcal{N}_m(t) + \int_0^t \|\nabla\partial_t^{m-1}(\rho,\ \theta)(\tau)\|^2 \mathrm{d}\tau + \int_0^t \|\nabla(\rho\theta)(\tau)\|^2_{\mathcal{H}^{2,\infty}}\mathrm{d}\tau\\
&\quad + \int_0^t \varepsilon\|\nabla u(\tau)\|^2_{\mathcal{H}^m}\mathrm{d}\tau + \kappa(\varepsilon)\int_0^t \|\nabla\theta(\tau)\|^2_{\mathcal{H}^m}\mathrm{d}\tau\\
&\quad + \sum_{k=0}^{m-2}\int_0^t \varepsilon\|\nabla^2\partial_t^k u(\tau)\|^2_{m-1-k} + \kappa(\varepsilon)\|\Delta\partial_t^k\theta(\tau)\|^2_{m-1-k}\mathrm{d}\tau\\
&\quad + \kappa(\varepsilon)^2\int_0^t \|\Delta\partial_t^{m-1}\theta(\tau)\|^2 + \|\nabla\mathcal{Z}^{m-2}\Delta\theta(\tau)\|^2\mathrm{d}\tau\\
&\quad + \varepsilon^2\int_0^t \|\nabla^2\partial_t^{m-1}u(\tau)\|^2\mathrm{d}\tau\\
&\leqq \tilde{C}_2 C_{m+2}\Big\{P(\mathcal{N}_m(0)) + tP(\mathcal{N}_m(t))\Big\}.
\end{aligned} \tag{3.4}$$

where $\tilde{C}_2$ depends only on $\frac{1}{c_0}$, $P(\cdot)$ is a polynomial and

$$\mathcal{N}_m(t) \triangleq \mathcal{N}_m(\rho,u,\theta)(t) = \sup_{0\leqq\tau\leqq t}\{1 + \|(\rho,u,\theta)(\tau)\|^2_{X_m^\varepsilon}\}. \tag{3.5}$$

Throughout this section, we shall work on the interval of time $[0,T]$ such that $c_0 \leqq \rho,\theta \leqq \frac{1}{c_0}$. We point out that the generic constant C may depend on $\frac{1}{c_0}$ in this section. Since the proof of Theorem 3.1 is very complicated, we shall divide the proof into several subsections.

3.1. Conormal Energy Estimates for ρ, u and θ

Notice that

$$\Delta u = \nabla \mathrm{div} u - \nabla\times\nabla\times u, \tag{3.6}$$

then (1.1) is rewritten as

$$\begin{cases}
\rho_t + \mathrm{div}(\rho u) = 0,\\
\rho u_t + \rho u\cdot\nabla u + \nabla p = -\mu\varepsilon\nabla\times\omega + (2\mu+\lambda)\varepsilon\nabla\mathrm{div}u,\\
\rho\theta_t + \rho u\cdot\nabla\theta + p\mathrm{div}u = \kappa(\varepsilon)\Delta\theta + 2\mu\varepsilon|Su|^2 + \lambda\varepsilon|\mathrm{div}u|^2,
\end{cases} \tag{3.7}$$

where $\omega = \nabla\times u$ is the vorticity.

Lemma 3.2. *For a smooth solution of* (1.1) *and* (1.3)*, it holds that for* $\varepsilon \in (0, 1]$*,*

$$\sup_{0\leqq\tau\leqq t}\left(\int Rf_1(\rho)+\rho f_2(\theta)+\frac{1}{2}\rho|u|^2\,\mathrm{d}x\right)+c_1\int_0^t \varepsilon\|\nabla u(\tau)\|^2+\kappa(\varepsilon)\|\nabla\theta(\tau)\|^2\,\mathrm{d}\tau$$
$$\leqq \int Rf(\rho_0)+\rho_0 h(\theta_0)+\frac{1}{2}\rho_0|u_0|^2\,\mathrm{d}x + C\int_0^t \|u(\tau)\|^2\,\mathrm{d}\tau + Ct, \tag{3.8}$$

where $c_1 > 0$ *and* $f_1(t) = t\ln t - t + 1$ *and* $f_2(t) = t - \ln t + 1$ *for* $t > 0$.

Proof. Multiplying (3.7) by $\frac{1}{\theta}$, one gets that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\ln\theta\,\mathrm{d}x + \int R\rho\mathrm{div}u\,\mathrm{d}x$$
$$= \kappa(\varepsilon)\int \frac{|\nabla\theta|^2}{\theta^2}\,\mathrm{d}x + \varepsilon\int \frac{1}{\theta}(2\mu|Su|^2+\lambda|\mathrm{div}u|^2)\,\mathrm{d}x + \kappa(\varepsilon)\int_{\partial\Omega}\frac{\nabla\theta\cdot n}{\theta}\,\mathrm{d}\sigma. \tag{3.9}$$

It follows from the boundary condition (1.3) that

$$\kappa(\varepsilon)\int_{\partial\Omega}\frac{\nabla\theta\cdot n}{\theta}\,\mathrm{d}\sigma = \nu\kappa(\varepsilon)\int_{\partial\Omega}\mathrm{d}\sigma \leqq C\kappa(\varepsilon)\leqq C. \tag{3.10}$$

We rewrite the mass equation $(3.7)_1$ to be

$$\rho_t + u\cdot\nabla\rho + \rho\mathrm{div}u = 0, \tag{3.11}$$

which yields immediately that

$$\int \rho\mathrm{div}u\,\mathrm{d}x = -\int \rho_t + u\cdot\nabla\rho\,\mathrm{d}x$$
$$= -\int \rho((\ln\rho)_t + u\cdot\nabla\ln\rho)\,\mathrm{d}x = -\frac{d}{dt}\int \rho\ln\rho\,\mathrm{d}x. \tag{3.12}$$

Substituting (3.12) and (3.10) into (3.9), one obtains that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (R\rho\ln\rho - \rho\ln\theta)\,\mathrm{d}x + \int \kappa(\varepsilon)\frac{|\nabla\theta|^2}{\theta^2} + \frac{\varepsilon}{\theta}(2\mu|Su|^2+\lambda|\mathrm{div}u|^2)\,\mathrm{d}x \leqq C. \tag{3.13}$$

On the other hand, it follows from $(1.1)_1$ and $(1.1)_3$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\,\mathrm{d}x = 0, \tag{3.14}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\theta + \frac{1}{2}\rho|u|^2\,\mathrm{d}x = \kappa(\varepsilon)\int_{\partial\Omega} n\cdot\nabla\theta\,\mathrm{d}\sigma + 2\mu\varepsilon\int_{\partial\Omega}((Su)u)n\,\mathrm{d}\sigma$$
$$\leqq C + \delta\varepsilon\|\nabla u\|^2 + C_\delta\|u\|^2, \tag{3.15}$$

where we have used the following facts in the estimates of (3.15)

$$\begin{aligned} &\kappa(\varepsilon)\Big|\int_{\partial\Omega} n\cdot\nabla\theta \,\mathrm{d}\sigma\Big| + 2\mu\varepsilon\Big|\int_{\partial\Omega}((Su)u)n\,\mathrm{d}\sigma\Big| \\ &= \nu\kappa(\varepsilon)\Big|\int_{\partial\Omega}\theta\,\mathrm{d}\sigma\Big| + 2\mu\varepsilon\Big|\int_{\partial\Omega}((Su)n)_\tau u_\tau\,\mathrm{d}\sigma\Big| \\ &\leqq C + C\varepsilon|u|_{L^2}^2 \leqq C + \delta\varepsilon\|\nabla u\|^2 + C_\delta\|u\|^2. \end{aligned}$$

Then, combining (3.13)–(3.15), one obtains that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int R(\rho\ln\rho-\rho+1)+\rho(\theta-\ln\theta+1)+\frac{1}{2}\rho|u|^2\,\mathrm{d}x\Big) \\ &\quad+\int\kappa(\varepsilon)\frac{|\nabla\theta|^2}{\theta^2}+\frac{\varepsilon}{\theta}(2\mu|Su|^2+\lambda|\mathrm{div}u|^2)\,\mathrm{d}x \\ &\leqq \delta\varepsilon^2\|\nabla u\|^2 + C_\delta\|u\|^2 + C. \end{aligned} \tag{3.16}$$

It follows from Korn's inequality and the fact $2\mu+\lambda>0$ that

$$\int\kappa(\varepsilon)\frac{|\nabla\theta|^2}{\theta^2}+\frac{\varepsilon}{\theta}(2\mu|Su|^2+\lambda|\mathrm{div}u|^2)\,\mathrm{d}x \geqq 2c_1[\kappa(\varepsilon)\|\nabla\theta\|^2+\varepsilon\|\nabla u\|^2]-C\|u\|^2, \tag{3.17}$$

where the fact that $c_1>0$ is a given positive constant depends on c_0,μ,λ,κ. Thus, choosing δ small, it holds that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int R(\rho\ln\rho-\rho+1)+\rho(\theta-\ln\theta+1)+\frac{1}{2}\rho|u|^2\,\mathrm{d}x\Big) \\ &+c_1\kappa(\varepsilon)\|\nabla\theta\|^2 + c_1\varepsilon\|\nabla u\|^2 \leqq C + C\|u\|^2. \end{aligned} \tag{3.18}$$

Integrating (3.18) over $[0,t]$, one obtains (3.8). Thus the proof of Lemma 3.2 is completed. □

However, the above basic energy estimates are far from enough to get the vanishing dissipation limit; one needs to get some conormal derivative estimates. Set

$$Q(t) \triangleq \sup_{0\leqq\tau\leqq t}\{\|\nabla(\rho,u,\theta)(t)\|_{\mathcal{H}^{1,\infty}}^2 + \|(\theta,u,\theta,\rho_t,u_t,\theta_t)(t)\|_{L_x^\infty}^2 + \varepsilon\|\nabla^2 u\|_{L^\infty}^2\}. \tag{3.19}$$

It follows from Proposition 2.3 that

$$Q(t) \leqq CP(\mathcal{N}_m(t)) \quad \text{for } m \geqq 3. \tag{3.20}$$

Lemma 3.3. *For $m\geqq 3$, it holds that*

$$\sup_{0\leqq\tau\leqq t}\|(\rho,\theta,u)(\tau)\|_{\mathcal{H}^m}^2 + \varepsilon\int_0^t\|\nabla u(\tau)\|_{\mathcal{H}^m}^2\,\mathrm{d}\tau + \kappa(\varepsilon)\int_0^t\|\nabla\theta(\tau)\|_{\mathcal{H}^m}^2\,\mathrm{d}\tau$$

$$\leqq CC_{m+2}\Big\{1+\|(\rho_0,u_0,\theta_0)\|_{\mathcal{H}^m}^2+\delta\varepsilon^2\int_0^t\|\nabla^2u(\tau)\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau+C_\delta tP(\mathcal{N}_m(t))$$
$$+\delta\kappa(\varepsilon)^2\int_0^t\|\Delta\theta(\tau)\|_{\mathcal{H}^{m-1}}^2\mathrm{d}\tau+\delta\int_0^t\|\nabla\partial_t^{m-1}(\rho,\theta)(\tau)\|^2\mathrm{d}\tau\Big\}, \tag{3.21}$$

where δ is a small amount which will be chosen later and C_δ is a polynomial function of $\frac{1}{\delta}$ which may vary from line to line.

Proof. The estimate for $k=0$ is already given in Lemma 3.2. Assuming that it is proven for $k\leqq m-1$, we shall prove it for $k=m\geqq 1$. Applying $\mathcal{Z}^\alpha$ with $|\alpha|=m$ to (3.7), one obtains that

$$\begin{cases}\rho\mathcal{Z}^\alpha u_t+\rho u\cdot\nabla\mathcal{Z}^\alpha u+\mathcal{Z}^\alpha\nabla p\\ \quad=-\mu\varepsilon\mathcal{Z}^\alpha\nabla\times\omega+(2\mu+\lambda)\varepsilon\mathcal{Z}^\alpha\nabla\mathrm{div}u+\mathcal{C}_1^\alpha+\mathcal{C}_2^\alpha,\\ \rho\mathcal{Z}^\alpha\theta_t+\rho u\cdot\nabla\mathcal{Z}^\alpha\theta+p\mathcal{Z}^\alpha\mathrm{div}u-\kappa(\varepsilon)\mathcal{Z}^\alpha\Delta\theta\\ \quad=2\mu\varepsilon\mathcal{Z}^\alpha(|Su|^2)+\lambda\varepsilon\mathcal{Z}^\alpha(|\mathrm{div}u|^2)+\mathcal{C}_3^\alpha+\mathcal{C}_4^\alpha+\mathcal{C}_5^\alpha,\end{cases} \tag{3.22}$$

with

$$\begin{cases}\mathcal{C}_1^\alpha=-[\mathcal{Z}^\alpha,\rho]u_t=-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta\rho\mathcal{Z}^\gamma u_t,\\ \mathcal{C}_2^\alpha=-[\mathcal{Z}^\alpha,\rho u\cdot\nabla]u=-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta(\rho u)\mathcal{Z}^\gamma\nabla u-\rho u\cdot[\mathcal{Z}^\alpha,\nabla]u,\\ \mathcal{C}_3^\alpha=-[\mathcal{Z}^\alpha,\rho]\theta_t=-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta\rho\mathcal{Z}^\gamma\theta_t,\\ \mathcal{C}_4^\alpha=-[\mathcal{Z}^\alpha,\rho u\cdot\nabla]\theta=-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta(\rho u)\mathcal{Z}^\gamma\nabla\theta-\rho u\cdot[\mathcal{Z}^\alpha,\nabla]\theta,\\ \mathcal{C}_5^\alpha=-[\mathcal{Z}^\alpha,p]\mathrm{div}u=-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta p\mathcal{Z}^\gamma\mathrm{div}u,\end{cases} \tag{3.23}$$

where $C_{\alpha,\beta}$ are the corresponding binomial coefficients. Multiplying (3.22) by $\mathcal{Z}^\alpha u$ and integrating by parts, one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}\rho|\mathcal{Z}^\alpha u|^2\mathrm{d}x+\int\mathcal{Z}^\alpha\nabla p\mathcal{Z}^\alpha u\mathrm{d}x$$
$$=-\mu\varepsilon\int\mathcal{Z}^\alpha\nabla\times\omega\cdot\mathcal{Z}^\alpha u\mathrm{d}x+(2\mu+\lambda)\varepsilon\int\mathcal{Z}^\alpha\nabla\mathrm{div}u\cdot\mathcal{Z}^\alpha u\mathrm{d}x$$
$$+\int(\mathcal{C}_1^\alpha+\mathcal{C}_2^\alpha)\mathcal{Z}^\alpha u\mathrm{d}x \tag{3.24}$$

Using the same arguments as Lemma 3.3 of [22], one can get that

$$-\varepsilon\int\mathcal{Z}^\alpha\nabla\times\omega\cdot\mathcal{Z}^\alpha u\,\mathrm{d}x\leqq-\frac{3\varepsilon}{4}\|\nabla\times\mathcal{Z}^\alpha u\|^2+\delta\varepsilon\|\nabla u\|_{\mathcal{H}^m}^2+\delta\varepsilon^2\|\nabla^2u\|_{\mathcal{H}^{m-1}}^2$$
$$+C_\delta C_{m+2}(\|\nabla u\|_{\mathcal{H}^{m-1}}^2+\|u\|_{\mathcal{H}^m}^2) \tag{3.25}$$

and

$$\varepsilon\int\mathcal{Z}^\alpha\nabla\mathrm{div}u\cdot\mathcal{Z}^\alpha u\,\mathrm{d}x\leqq-\frac{3\varepsilon}{4}\|\mathrm{div}\mathcal{Z}^\alpha u\|^2+\delta\varepsilon\|\nabla u\|_{\mathcal{H}^m}^2+\delta\varepsilon^2\|\nabla^2u\|_{\mathcal{H}^{m-1}}^2$$
$$+C_\delta C_{m+2}(\|\nabla u\|_{\mathcal{H}^{m-1}}^2+\|u\|_{\mathcal{H}^m}^2). \tag{3.26}$$

It follows from Proposition 2.1 that

$$
\begin{aligned}
2c_1\|\nabla \mathcal{Z}^\alpha u\|_{L^2} &\leqq \left(\|\nabla\times\mathcal{Z}^\alpha u\|_{L^2}+\|\mathrm{div}\mathcal{Z}^\alpha u\|_{L^2}+\|\mathcal{Z}^\alpha u\|_{L^2}+|\mathcal{Z}^\alpha u\cdot n|_{H^{\frac{1}{2}}(\partial\Omega)}\right)\\
&\leqq (\|\nabla\times\mathcal{Z}^\alpha u\|_{L^2}+\|\mathrm{div}\mathcal{Z}^\alpha u\|_{L^2})+C_{m+2}(\|u\|^2_{\mathcal{H}^m}+\|\nabla u\|^2_{\mathcal{H}^{m-1}}).
\end{aligned}
\tag{3.27}
$$

Substituting (3.25) and (3.26) into (3.24) and using (3.27), then integrating the resultant inequality over $[0,t]$, one obtains that

$$
\begin{aligned}
&\frac{1}{2}\int\rho|\mathcal{Z}^\alpha u|^2\,\mathrm{d}x+\int_0^t\int\mathcal{Z}^\alpha\nabla p\cdot\mathcal{Z}^\alpha u\,\mathrm{d}x\,\mathrm{d}\tau+2c_1\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha u(\tau)\|^2_{L^2}\,\mathrm{d}\tau\\
&\quad\leqq\frac{1}{2}\int\rho_0|\mathcal{Z}^\alpha u_0|^2\,\mathrm{d}x+C\delta\varepsilon^2\int_0^t\|\nabla^2u(\tau)\|^2_{\mathcal{H}^{m-1}}\,\mathrm{d}\tau+C\delta\varepsilon\int_0^t\|\nabla u(\tau)\|^2_{\mathcal{H}^m}\,\mathrm{d}\tau\\
&\qquad+\int_0^t\int(\mathcal{C}_1^\alpha+\mathcal{C}_2^\alpha)\cdot\mathcal{Z}^\alpha u\,\mathrm{d}x\,\mathrm{d}\tau+C_{m+2}C_\delta\int_0^t\|\nabla u(\tau)\|^2_{\mathcal{H}^{m-1}}+\|u(\tau)\|^2_{\mathcal{H}^m}\,\mathrm{d}\tau.
\end{aligned}
\tag{3.28}
$$

On the other hand, multiplying (3.22) by $\frac{\mathcal{Z}^\alpha\theta}{\theta}$ and integrating by parts, one has that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\rho}{2\theta}|\mathcal{Z}^\alpha\theta|^2\,\mathrm{d}x+\int R\rho\mathcal{Z}^\alpha\mathrm{div}u\mathcal{Z}^\alpha\theta\,\mathrm{d}x-\kappa(\varepsilon)\int\mathcal{Z}^\alpha\Delta\theta\cdot\frac{\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\\
&\quad\leqq\varepsilon\int(2\mu\mathcal{Z}^\alpha(|Su|^2)+\lambda\mathcal{Z}^\alpha(|\mathrm{div}u|^2))\frac{\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x+\int(\mathcal{C}_3^\alpha+\mathcal{C}_4^\alpha+\mathcal{C}_5^\alpha)\frac{\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\\
&\qquad+CP(\mathcal{N}_m(t)).
\end{aligned}
\tag{3.29}
$$

It follows from the boundary condition $(1.3)_3$ that

$$
\begin{aligned}
|n\cdot\mathcal{Z}^\alpha\nabla\theta|_{L^2}&\leqq C_{m+1}(|\mathcal{Z}^\alpha\theta|_{L^2}+|\mathcal{Z}^{m-1}\nabla\theta|_{L^2})\\
&\leqq C_{m+1}\Big(\|\mathcal{Z}^m\theta\|^{\frac{1}{2}}\|\nabla\mathcal{Z}^m\theta\|^{\frac{1}{2}}+\|\mathcal{Z}^m\theta\|\\
&\qquad+\|\nabla^2\theta\|^{\frac{1}{2}}_{\mathcal{H}^{m-1}}\|\nabla\theta\|^{\frac{1}{2}}_{\mathcal{H}^{m-1}}+\|\nabla\theta\|^{\frac{1}{2}}_{\mathcal{H}^{m-1}}\Big),
\end{aligned}
\tag{3.30}
$$

which, together with integrating by parts, yields that

$$
\begin{aligned}
&-\kappa(\varepsilon)\int\mathcal{Z}^\alpha\Delta\theta\cdot\frac{\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x=-\kappa(\varepsilon)\int\mathcal{Z}^\alpha\mathrm{div}\nabla\theta\frac{\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\\
&\quad=-\kappa(\varepsilon)\int\mathrm{div}\mathcal{Z}^\alpha\nabla\theta\frac{\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x-\kappa(\varepsilon)\int[\mathrm{div},\mathcal{Z}^\alpha]\nabla\theta\frac{\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\\
&\quad=\kappa(\varepsilon)\int\mathcal{Z}^\alpha\nabla\theta\frac{\nabla\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x-\kappa(\varepsilon)\int[\mathrm{div},\mathcal{Z}^\alpha]\nabla\theta\frac{\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\\
&\qquad-\kappa(\varepsilon)\int_{\partial\Omega}\frac{\mathcal{Z}^\alpha\theta}{\theta}\mathcal{Z}^\alpha\nabla\theta\cdot n\,\mathrm{d}\sigma
\end{aligned}
$$

$$
\begin{aligned}
&\geqq \kappa(\varepsilon)\int \frac{|\nabla \mathcal{Z}^\alpha \theta|^2}{\theta}\,\mathrm{d}x - C\varepsilon^a\Big(\|\nabla\theta\|_{\mathcal{H}^{m-1}}\|\nabla\mathcal{Z}^\alpha\theta\| \\
&\quad + \|\theta\|_{\mathcal{H}^m}\|\nabla^2\theta\|_{\mathcal{H}^{m-1}} + \|\mathcal{Z}^\alpha\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\mathcal{Z}^\alpha\theta\|^{\frac{1}{2}}|\mathcal{Z}^\alpha\nabla\theta\cdot n|_{L^2}\Big) \\
&\geqq \frac{3\kappa(\varepsilon)}{4}\int\frac{|\nabla\mathcal{Z}^\alpha\theta|^2}{\theta}\,\mathrm{d}x - \delta\kappa(\varepsilon)^2\|\Delta\theta\|_{\mathcal{H}^{m-1}}^2 - \delta\kappa(\varepsilon)\|\nabla\mathcal{Z}^m\theta\|^2 - \delta\|\nabla\partial_t^{m-1}\theta\|^2 \\
&\quad - C_\delta C_{m+1}P(\mathcal{N}_m(t)),
\end{aligned}
\tag{3.31}
$$

where we have used Young's inequality in the last inequality of (3.31). It is easy to calculate that

$$
\begin{aligned}
&\varepsilon\int_0^t\int\frac{\mathcal{Z}^\alpha\theta}{\theta}[2\mu\mathcal{Z}^\alpha(|Su|^2)+\lambda\mathcal{Z}^\alpha(|\mathrm{div}u|^2)]\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad\leqq \delta\varepsilon^2\int_0^t\|\nabla u\|_{\mathcal{H}^m}^2\,\mathrm{d}\tau + C_\delta t P(\mathcal{N}_m(t)).
\end{aligned}
\tag{3.32}
$$

Substituting (3.31) and (3.32) into (3.29) and integrating the resultant inequality over $[0,t]$, one gets that

$$
\begin{aligned}
&\int\frac{\rho}{2\theta}|\mathcal{Z}^\alpha\theta|^2\,\mathrm{d}x + \int_0^t\int R\rho\mathcal{Z}^\alpha\mathrm{div}u\mathcal{Z}^\alpha\theta\,\mathrm{d}x\,\mathrm{d}\tau + \frac{3\kappa(\varepsilon)}{4}\int_0^t\int\frac{|\nabla\mathcal{Z}^\alpha\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqq \int\frac{\rho_0}{2\theta_0}|\mathcal{Z}^\alpha\theta_0|^2\,\mathrm{d}x + C\delta\int_0^t\kappa(\varepsilon)\|\nabla\mathcal{Z}^m\theta(\tau)\|^2 + \kappa(\varepsilon)^2\|\Delta\theta(\tau)\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau \\
&\quad + \delta\varepsilon\int_0^t\|\nabla u(\tau)\|_{\mathcal{H}^m}^2\,\mathrm{d}\tau + \delta\int_0^t\|\nabla\partial_t^{m-1}\theta(\tau)\|^2\,\mathrm{d}\tau + C_\delta tP(\mathcal{N}_m(t)) \\
&\quad + \int_0^t\int(\mathcal{C}_3^\alpha+\mathcal{C}_4^\alpha+\mathcal{C}_5^\alpha)\frac{\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
\tag{3.33}
$$

Combining (3.28) and (3.33), one obtains that

$$
\begin{aligned}
&\frac{1}{2}\int\rho|\mathcal{Z}^\alpha u|^2 + \frac{\rho}{2\theta}|\mathcal{Z}^\alpha\theta|^2\mathrm{d}x + 2c_1\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha u(\tau)\|_{L^2}^2\mathrm{d}\tau \\
&\quad + \frac{3\kappa(\varepsilon)}{4}\int_0^t\int\frac{|\nabla\mathcal{Z}^\alpha\theta|^2}{\theta}\mathrm{d}x\mathrm{d}\tau \\
&\quad + R\int_0^t\int\mathcal{Z}^\alpha\nabla(\rho\theta)\cdot\mathcal{Z}^\alpha u + \rho\mathcal{Z}^\alpha\mathrm{div}u\cdot\mathcal{Z}^\alpha\theta\mathrm{d}x\mathrm{d}\tau \\
&\leqq \frac{1}{2}\int\rho_0|\mathcal{Z}^\alpha u_0|^2 + \frac{\rho_0}{2\theta_0}|\mathcal{Z}^\alpha\theta_0|^2\mathrm{d}x + C\delta\int_0^t\|\nabla\partial_t^{m-1}\theta(\tau)\|^2\mathrm{d}\tau \\
&\quad + C\delta\int_0^t\varepsilon\|\nabla u(\tau)\|_{\mathcal{H}^m}^2 + \varepsilon^2\|\nabla^2u(\tau)\|_{\mathcal{H}^{m-1}}^2\mathrm{d}\tau + C_\delta C_{m+2}tP(\mathcal{N}_m(t)) \\
&\quad + C\delta\int_0^t\kappa(\varepsilon)\|\nabla\theta(\tau)\|_{\mathcal{H}^m}^2 + \kappa(\varepsilon)^2\|\Delta\theta(\tau)\|_{\mathcal{H}^{m-1}}^2\mathrm{d}\tau \\
&\quad + C\Big(\int_0^t\|(\mathcal{C}_1^\alpha,\mathcal{C}_2^\alpha,\mathcal{C}_3^\alpha,\mathcal{C}_4^\alpha,\mathcal{C}_5^\alpha)\|^2\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(int_0^t\|\mathcal{Z}^\alpha(u,\theta)\|^2\mathrm{d}\tau\Big)^{\frac{1}{2}}.
\end{aligned}
\tag{3.34}
$$

Now we estimate the fourth term on the LHS of (3.34). Note that

$$\mathcal{Z}^\alpha \nabla(\rho\theta) = \theta \cdot \nabla \mathcal{Z}^\alpha \rho + \rho \cdot \nabla \mathcal{Z}^\alpha \theta + [\mathcal{Z}^\alpha, \nabla\rho]\theta + [\mathcal{Z}^\alpha, \nabla\theta]\rho + \theta \cdot [\mathcal{Z}^\alpha, \nabla]\rho \\ + \rho \cdot [\mathcal{Z}^\alpha, \nabla]\theta, \tag{3.35}$$

which, together with Proposition 2.2 and Hölder inequality, yields that

$$\begin{aligned} I &= \int_0^t \int \mathcal{Z}^\alpha \nabla(\rho\theta) \cdot \mathcal{Z}^\alpha u + \rho \mathcal{Z}^\alpha \mathrm{div} u \cdot \mathcal{Z}^\alpha \theta \, \mathrm{d}x \, \mathrm{d}\tau \\ &\geqq \int_0^t \int (\theta \cdot \nabla \mathcal{Z}^\alpha \rho + \rho \cdot \nabla \mathcal{Z}^\alpha \theta) \mathcal{Z}^\alpha u \, \mathrm{d}x \, \mathrm{d}\tau + \int_0^t \int \rho \mathcal{Z}^\alpha \mathrm{div} u \cdot \mathcal{Z}^\alpha \theta \, \mathrm{d}x \, \mathrm{d}\tau \\ &\quad - C\delta \int_0^t \|\nabla \partial_t^{m-1}(\rho,\theta)\|^2 \, \mathrm{d}\tau - C_\delta (1 + Q(t)) \int_0^t \mathcal{N}_m(\tau) \, \mathrm{d}\tau \\ &\geqq - \int_0^t \int (\theta \mathcal{Z}^\alpha \rho + \rho \mathcal{Z}^\alpha \theta) \mathrm{div} \mathcal{Z}^\alpha u \, \mathrm{d}x \, \mathrm{d}\tau + \int_0^t \int \rho \mathcal{Z}^\alpha \mathrm{div} u \cdot \mathcal{Z}^\alpha \theta \, \mathrm{d}x \, \mathrm{d}\tau \\ &\quad - C\delta \int_0^t \|\nabla \partial_t^{m-1}(\rho,\theta)\|^2 \, \mathrm{d}\tau + \int_0^t \int_{\partial\Omega} (\theta \mathcal{Z}^\alpha \rho + \rho \mathcal{Z}^\alpha \theta) \mathcal{Z}^\alpha u \cdot n \, \mathrm{d}\sigma \, \mathrm{d}\tau \\ &\quad - C_\delta P(\mathcal{N}_m(t)) \\ &\geqq - \int_0^t \int \theta \mathcal{Z}^\alpha \rho \cdot \mathrm{div} \mathcal{Z}^\alpha u \, \mathrm{d}x \, \mathrm{d}\tau + \int_0^t \int_{\partial\Omega} (\theta \mathcal{Z}^\alpha \rho + \rho \mathcal{Z}^\alpha \theta) \mathcal{Z}^\alpha u \cdot n \, \mathrm{d}\sigma \, \mathrm{d}\tau \\ &\quad - C\delta \int_0^t \|\nabla \partial_t^{m-1}(\rho,\theta)\|^2 \, \mathrm{d}\tau - C_\delta t P(\mathcal{N}_m(t)). \end{aligned} \tag{3.36}$$

We shall calculate the boundary term in (3.36) when $\alpha_{13} = 0$ (for $\alpha_{13} \neq 0$, we have that $\mathcal{Z}^\alpha u = 0$ on the boundary) in the right hand side of (3.36). It follows from (1.3) and (2.6), for $k \leqq m$, that

$$|Z_y^{m-k} \partial_t^k u \cdot n|_{H^{\frac{1}{2}}} \leqq \begin{cases} 0, & \text{if } k = m, \\ C_{m+2}\{\|\nabla u\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}\}, & \text{if } k \leqq m-1. \end{cases} \tag{3.37}$$

If $|\alpha_0| = |\alpha|$, it follows from (3.37) that

$$\int_0^t \int_{\partial\Omega} (\theta \mathcal{Z}^\alpha \rho + \rho \mathcal{Z}^\alpha \theta) \mathcal{Z}^\alpha u \cdot n \, \mathrm{d}\sigma \, \mathrm{d}\tau = 0. \tag{3.38}$$

If $|\alpha_1| \geqq 1$, integrating by parts along the boundary and using (3.38), one has that

$$\begin{aligned} &\left| \int_0^t \int_{\partial\Omega} (\theta \mathcal{Z}^\alpha \rho + \rho \mathcal{Z}^\alpha \theta) \mathcal{Z}^\alpha u \cdot n \, \mathrm{d}\sigma \, \mathrm{d}\tau \right| \\ &\quad = \left| \int_0^t \int_{\partial\Omega} (\theta Z_y^{\alpha_1} \partial_t^{\alpha_0} \rho + \rho Z_y^{\alpha_1} \partial_t^{\alpha_0} \theta) Z^\alpha u \cdot n \, \mathrm{d}\sigma \, \mathrm{d}\tau \right| \\ &\quad \leqq C Q(t) \int_0^t \left(|Z_y^{\alpha_1 - 1} \partial_t^{\alpha_0} \rho|_{H^{\frac{1}{2}}} + |Z_y^{\alpha_1 - 1} \partial_t^{\alpha_0} \theta|_{H^{\frac{1}{2}}} \right) |Z^\alpha u \cdot n|_{H^{\frac{1}{2}}} \, \mathrm{d}\sigma \, \mathrm{d}\tau \\ &\quad \leqq \delta \int_0^t \|\nabla(\rho,\theta)\|_{\mathcal{H}^{m-1}}^2 \, \mathrm{d}\tau + C_\delta C_{m+2} t P(\mathcal{N}_m(t)). \end{aligned} \tag{3.39}$$

Therefore, from (3.39) and (3.38), we obtain that

$$\begin{aligned}&\left|\int_0^t\int_{\partial\Omega}(\theta\mathcal{Z}^\alpha\rho+\rho\mathcal{Z}^\alpha\theta)\mathcal{Z}^\alpha u\cdot n\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|\\&\quad\leqq\delta\int_0^t\|\nabla\partial_t^{m-1}(\rho,\theta)\|^2\,\mathrm{d}\tau+C_\delta C_{m+2}tP(\mathcal{N}_m(t)).\end{aligned}\tag{3.40}$$

In order to estimate the first term on the RHS of (3.36), we use the following equation, which is derived from $(3.7)_1$:

$$\mathrm{div}u=-\frac{\rho_t}{\rho}-\frac{u}{\rho}\cdot\nabla\rho.\tag{3.41}$$

Applying $\mathcal{Z}^\alpha$ to (3.41), one immediately obtains that

$$\begin{aligned}\mathcal{Z}^\alpha\mathrm{div}u=&-\frac{1}{\rho}\mathcal{Z}^\alpha\rho_t-\frac{u}{\rho}\cdot\mathcal{Z}^\alpha\nabla\rho-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta\left(\frac{1}{\rho}\right)\cdot\mathcal{Z}^\gamma\rho_t\\&-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta\left(\frac{u}{\rho}\right)\cdot\mathcal{Z}^\gamma\nabla\rho.\end{aligned}\tag{3.42}$$

It is easy to get that

$$\int_0^t\int\frac{\theta}{\rho}\mathcal{Z}^\alpha\rho\cdot\mathcal{Z}^\alpha\rho_t\,\mathrm{d}x\,\mathrm{d}\tau\geqq\int\frac{\theta}{2\rho}|\mathcal{Z}^\alpha\rho|^2\,\mathrm{d}x-\int\frac{\theta_0}{2\rho_0}|\mathcal{Z}^\alpha\rho_0|^2\,\mathrm{d}x-CtP(\mathcal{N}_m(t)).\tag{3.43}$$

Integrating by parts and using boundary condition (1.3), one has

$$\begin{aligned}\int_0^t\int\frac{\theta}{\rho}u\mathcal{Z}^\alpha\rho\cdot\mathcal{Z}^\alpha\nabla\rho\,\mathrm{d}x\,\mathrm{d}\tau&=\int_0^t\int\frac{\theta}{\rho}u\mathcal{Z}^\alpha\rho(\nabla\mathcal{Z}^\alpha\rho+[\mathcal{Z}^\alpha,\nabla]\rho)\,\mathrm{d}x\,\mathrm{d}\tau\\&\geqq-\delta\int_0^t\|\nabla\partial_t^{m-1}\rho(\tau)\|^2\,\mathrm{d}\tau-C_\delta tP(\mathcal{N}_m(t)).\end{aligned}\tag{3.44}$$

It follows from Proposition 2.2 that

$$\begin{aligned}&\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}\left|\int_0^t\int C_{\alpha,\beta}\theta\mathcal{Z}^\alpha\rho\cdot\left(\mathcal{Z}^\beta\left(\frac{1}{\rho}\right)\cdot\mathcal{Z}^\gamma\rho_t+\mathcal{Z}^\beta\left(\frac{u}{\rho}\right)\mathcal{Z}^\gamma\nabla\rho\right)\mathrm{d}x\,\mathrm{d}\tau\right|\\&\quad\leqq\delta\int_0^t\|\nabla\partial_t^{m-1}\rho(\tau)\|^2\,\mathrm{d}\tau+C_\delta tP(\mathcal{N}_m(t)).\end{aligned}\tag{3.45}$$

Combining (3.42)–(3.45), one obtains that

$$\begin{aligned}&-\int_0^t\int\theta\mathcal{Z}^\alpha\rho\cdot\mathcal{Z}^\alpha\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\geqq\int\frac{1}{2\rho}|\mathcal{Z}^\alpha\rho|^2\,\mathrm{d}x\\&\quad-\int\frac{1}{2\rho_0}|\mathcal{Z}^\alpha\rho_0|^2\,\mathrm{d}x-C\delta\int_0^t\|\nabla\partial_t^{m-1}\rho(\tau)\|^2\,\mathrm{d}\tau-C_\delta tP(\mathcal{N}_m(t)).\end{aligned}\tag{3.46}$$

Substituting (3.40) and (3.46) into (3.36), one gets that

$$I = \int_0^t \int \mathcal{Z}^\alpha \nabla(\rho\theta) \cdot \mathcal{Z}^\alpha u + \rho \mathcal{Z}^\alpha \mathrm{div} u \cdot \mathcal{Z}^\alpha \theta \,\mathrm{d}x\,\mathrm{d}\tau$$
$$\geqq \int \frac{1}{2\rho} |\mathcal{Z}^\alpha \rho|^2 \,\mathrm{d}x - \int \frac{1}{2\rho_0} |\mathcal{Z}^\alpha \rho_0|^2 \,\mathrm{d}x - C\delta \int_0^t \|\nabla \partial_t^{m-1}(\rho, \theta)\|^2 \,\mathrm{d}\tau$$
$$- C_{m+2} C_\delta t P(\mathcal{N}_m(t)). \tag{3.47}$$

In order to complete the estimates in (3.34), it remains to estimate the terms involving $\mathcal{C}_i^\alpha$, $i = 1 \ldots 5$. It follows from Proposition 2.2 and (3.27) that

$$\sum_{i=1}^5 \int_0^t \|\mathcal{C}_i^\alpha\|^2 \,\mathrm{d}x\,\mathrm{d}\tau \leqq C[1 + P(Q(t))] \int_0^t \|\nabla \partial_t^{m-1} \theta\|^2 \,\mathrm{d}\tau + CtP(\mathcal{N}_m(t)). \tag{3.48}$$

Therefore, substituting (3.47) and (3.48) into (3.34) and using Cauchy inequality, one proves (3.11). Thus, the proof of Lemma 3.3 is completed. □

3.2. Conormal Estimates for div$u$*,* $\nabla\rho$ *and* $\nabla\theta$

In order to use the compactness argument in the proof of the vanishing dissipation limit, one needs some uniform spatial derivative estimates. In this subsection, we shall get some uniform estimates on div$u$, $\nabla\rho$ and $\nabla\theta$. In fact, in order to get the uniform estimate of $\|\nabla u\|_{\mathcal{H}^{m-1}}$, one needs the uniform estimate of $\|\mathrm{div} u\|_{\mathcal{H}^{m-1}}$, since in this paper we consider the compressible flow.

Lemma 3.4. *For $m \geqq 3$, it holds that*

$$\sup_{0 \leqq \tau \leqq t} \|(\mathrm{div} u, \nabla\rho, \nabla\theta)(\tau)\|^2 + \varepsilon \int_0^t \|\nabla \mathrm{div} u(\tau)\|^2 \,\mathrm{d}\tau + \kappa(\varepsilon) \int_0^t \|\Delta\theta(\tau)\|^2 \,\mathrm{d}\tau$$
$$\leqq C\Big\{ \|(\mathrm{div} u_0, \nabla\rho_0, \nabla\theta_0)\|^2 + \delta\varepsilon^2 \int_0^t \|\nabla^2 u(\tau)\|^2 \,\mathrm{d}\tau$$
$$+ \delta\kappa(\varepsilon)^2 \int_0^t \|\nabla\Delta\theta(\tau)\|^2 \,\mathrm{d}\tau + C_3 t P(\mathcal{N}_m(t)) \Big\}. \tag{3.49}$$

Proof. Multiplying $(3.7)_2$ by ∇div$u$, one has that

$$\int_0^t \int (\rho u_t + \rho u \cdot \nabla u) \cdot \nabla \mathrm{div} u \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int \nabla p \cdot \nabla \mathrm{div} u \,\mathrm{d}x\,\mathrm{d}\tau$$
$$= -\mu\varepsilon \int_0^t \int \nabla \times \omega \cdot \nabla \mathrm{div} u \,\mathrm{d}x\,\mathrm{d}\tau + (2\mu + \lambda)\varepsilon \int_0^t \|\nabla \mathrm{div} u\|^2 \,\mathrm{d}\tau. \tag{3.50}$$

It follows from integrating by parts and the boundary conditions (1.3) that

$$\begin{aligned}
&\int_0^t \int (\rho u_t+\rho u\cdot\nabla u)\cdot\nabla \mathrm{div} u \,\mathrm{d}x\,\mathrm{d}\tau=-\int_0^t\int(\rho \mathrm{div} u_t+\rho u\cdot\nabla \mathrm{div} u)\mathrm{div} u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad-\int_0^t\int(\nabla\rho\cdot u_t+\nabla(\rho u)^t\cdot\nabla u)\mathrm{div} u\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^t\int_{\partial\Omega}\rho(u\cdot\nabla)u\cdot n\mathrm{div} u\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&\leqq-\frac{1}{2}\int\rho|\mathrm{div} u|^2\,\mathrm{d}x+\frac{1}{2}\int\rho_0|\mathrm{div} u_0|^2\,\mathrm{d}x+C[1+P(Q(t))]\int_0^t\|(u_t,\nabla u)\|^2\,\mathrm{d}\tau\\
&\quad+\left|\int_0^t\int_{\partial\Omega}\rho(u\cdot\nabla)n\cdot u\mathrm{div} u\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|\\
&\leqq-\frac{1}{2}\int\rho|\mathrm{div} u|^2\,\mathrm{d}x+\frac{1}{2}\int\rho_0|\mathrm{div} u_0|^2\,\mathrm{d}x\\
&\quad+C[1+P(Q(t))]\int_0^t\|(u_t,\nabla u)\|^2+|u|_{L^2}^2\,\mathrm{d}\tau\\
&\leqq-\frac{1}{2}\int\rho|\mathrm{div} u|^2\,\mathrm{d}x+\frac{1}{2}\int\rho_0|\mathrm{div} u_0|^2\,\mathrm{d}x+CtP(\mathcal{N}_m(t)).
\end{aligned}\tag{3.51}$$

By using the boundary condition (1.3) and integrating by parts along the boundary, one has that

$$\begin{aligned}
&\varepsilon\int_0^t\int\nabla\times\omega\cdot\nabla\mathrm{div} u\,\mathrm{d}x\,\mathrm{d}\tau=\varepsilon\int_0^t\int_{\partial\Omega}n\times\omega\cdot\nabla\mathrm{div} u\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&\quad=\varepsilon\int_0^t\int_{\partial\Omega}(Bu)\cdot\Pi(\nabla\mathrm{div} u)\,\mathrm{d}\sigma\,\mathrm{d}\tau=\varepsilon\int_0^t\int_{\partial\Omega}(Bu)\cdot Z_y\mathrm{div} u\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&\quad\leqq C_3\varepsilon\int_0^t|u|_{H^{\frac12}}|\mathrm{div} u|_{H^{\frac12}}\,\mathrm{d}\tau\leqq C_3\varepsilon\int_0^t\|u\|_{H^1}\|\mathrm{div} u\|_{H^1}\,\mathrm{d}\tau\\
&\quad\leqq\frac{\varepsilon}{4}\int_0^t\|\nabla\mathrm{div} u\|^2\,\mathrm{d}\tau+C_3\varepsilon\int_0^t\|(\nabla u,u)\|^2\,\mathrm{d}\tau.
\end{aligned}\tag{3.52}$$

Substituting (3.51) and (3.52) into (3.50), one obtains that

$$\begin{aligned}
&\frac{1}{2}\int\rho|\mathrm{div} u|^2\,\mathrm{d}x+\frac{7}{8}(2\mu+\lambda)\varepsilon\int_0^t\|\nabla\mathrm{div} u\|^2\,\mathrm{d}\tau-R\int_0^t\int\nabla(\rho\theta)\cdot\nabla\mathrm{div} u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad\leqq\frac{1}{2}\int\rho_0|\mathrm{div} u_0|^2\,\mathrm{d}x+CtP(\mathcal{N}_m(t)).
\end{aligned}\tag{3.53}$$

On the other hand, applying ∇ to $(3.7)_3$ yields that

$$\begin{aligned}
&\rho\nabla\theta_t+\rho(u\cdot\nabla)\nabla\theta+p\nabla\mathrm{div} u=\kappa(\varepsilon)\Delta\nabla\theta+\varepsilon\nabla(2\mu|Su|^2+\lambda|\mathrm{div} u|^2)\\
&-[\nabla\rho\cdot\theta_t+\nabla p\cdot\mathrm{div} u+\nabla(\rho u)^t\nabla\theta].
\end{aligned}\tag{3.54}$$

Multiplying (3.54) by $\frac{\nabla\theta}{\theta}$, one obtains that

$$\begin{aligned}
&\int_0^t\int [\rho\nabla\theta_t+\rho(u\cdot\nabla)\nabla\theta]\cdot\frac{\nabla\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau+R\int_0^t\int\rho\nabla\theta\cdot\nabla\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad=\kappa(\varepsilon)\int_0^t\int\Delta\nabla\theta\cdot\frac{\nabla\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau+\varepsilon\int_0^t\int\nabla(2\mu|Su|^2+\lambda|\mathrm{div}u|^2)\cdot\frac{\nabla\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad-\int_0^t\int(\nabla\rho\cdot\theta_t+\nabla p\cdot\mathrm{div}u+\nabla(\rho u)^t\nabla\theta)\cdot\frac{\nabla\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau. \qquad (3.55)
\end{aligned}$$

For the first terms on the LHS of (3.55), it follows from integrating by parts that

$$\begin{aligned}
&\int_0^t\int[\rho\nabla\rho_t+\rho(u\cdot\nabla)\nabla\theta]\cdot\frac{\nabla\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad\geqq\int\frac{\rho}{2\theta}|\nabla\theta|^2\,\mathrm{d}x-\int\frac{\rho_0}{2\theta_0}|\nabla\theta_0|^2\,\mathrm{d}x-CtP(\mathcal{N}_m(t)). \qquad (3.56)
\end{aligned}$$

For the last two terms on the right hand side of (3.55), it follows from the Cauchy inequality that

$$\begin{aligned}
&\varepsilon\left|\int_0^t\int\nabla(2\mu|Su|^2+\lambda|\mathrm{div}u|^2)\cdot\frac{\nabla\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\quad+\left|\int_0^t\int(\nabla\rho\cdot\theta_t+\nabla p\cdot\mathrm{div}u+\nabla(\rho u)^t\nabla\theta)\cdot\frac{\nabla\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\leqq\delta\varepsilon^2\int_0^t\|\nabla^2u\|^2\,\mathrm{d}\tau+C_\delta tP(\mathcal{N}_m(t)). \qquad (3.57)
\end{aligned}$$

Using the boundary condition $(1.3)_3$, one gets that

$$\begin{aligned}
&\kappa(\varepsilon)\int_0^t\int\nabla\Delta\theta\cdot\frac{\nabla\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad=-\kappa(\varepsilon)\int_0^t\int\frac{|\Delta\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau+\kappa(\varepsilon)\int_0^t\int\Delta\theta\frac{|\nabla\theta|^2}{\theta^2}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+\kappa(\varepsilon)\int_0^t\int_{\partial\Omega}\Delta\theta\frac{\nabla\theta\cdot n}{\theta}\,\mathrm{d}\sigma\,\mathrm{d}\tau\leqq-\frac{7}{8}\kappa(\varepsilon)\int_0^t\int\frac{|\Delta\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+C\kappa(\varepsilon)\int_0^t|\Delta\theta|_{L^2}\,\mathrm{d}\tau+CtP(\mathcal{N}_m(t)) \qquad (3.58)\\
&\quad\leqq-\frac{7}{8}\kappa(\varepsilon)\int_0^t\int\frac{|\Delta\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau+C\kappa(\varepsilon)\int_0^t\|\Delta\theta\|_{H^1}\,\mathrm{d}\tau+CtP(\mathcal{N}_m(t))\\
&\quad\leqq-\frac{3}{4}\kappa(\varepsilon)\int_0^t\int\frac{|\Delta\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau+\delta\kappa(\varepsilon)^2\int_0^t\|\nabla\Delta\theta\|^2\,\mathrm{d}\tau+C_\delta tP(\mathcal{N}_m(t)).
\end{aligned}$$

Substituting (3.57) and (3.58) into (3.55), one obtains that

$$\int \frac{\rho}{2\theta}|\nabla\theta|^2\,\mathrm{d}x + R\int_0^t\int \rho\nabla\theta\cdot\nabla\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau + \frac{3}{4}\kappa(\varepsilon)\int_0^t\int \frac{|\Delta\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leqq \int \frac{\rho_0}{2\theta_0}|\nabla\theta_0|^2\,\mathrm{d}x + C\delta\int_0^t \varepsilon^2\|\nabla^2u\|^2 + \kappa(\varepsilon)^2\|\nabla\Delta\theta\|^2\,\mathrm{d}\tau + C_\delta t P(\mathcal{N}_m(t)). \tag{3.59}$$

Combining (3.53) and (3.59), it holds that

$$\frac{1}{2}\int \frac{\rho}{\theta}|\nabla\theta(\tau)|^2 + \rho|\mathrm{div}u(\tau)|^2\,\mathrm{d}x - R\int_0^t\int \theta\nabla\rho\cdot\nabla\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ \frac{3\kappa(\varepsilon)}{4}\int_0^t\int \frac{|\Delta\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{3}{4}(2\mu+\lambda)\varepsilon\int_0^t \|\nabla\mathrm{div}u\|^2\,\mathrm{d}\tau$$
$$\leqq C\Big\{\|(\nabla\theta_0, \mathrm{div}u_0)\|^2 + \delta\int_0^t \varepsilon^2\|\nabla^2u\|^2 + \kappa(\varepsilon)^2\|\nabla\Delta\theta\|^2\,\mathrm{d}\tau + C_\delta t P(\mathcal{N}_m(t))\Big\}. \tag{3.60}$$

Finally, it follows from (3.41) that

$$I = -\int_0^t\int \theta\nabla\rho\cdot\nabla\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\geqq \int \frac{\theta}{2\rho}|\nabla\rho|^2\,\mathrm{d}x - \int \frac{\theta_0}{2\rho_0}|\nabla\rho_0|^2\,\mathrm{d}x - CtP(\mathcal{N}_m(t)). \tag{3.61}$$

Substituting (3.61) into (3.60), one proves (3.49). Thus, the proof of Lemma 3.4 is completed. □

Next we consider the higher order estimates. Firstly, we estimate $\mathcal{Z}^\alpha\mathrm{div}u$ for $|\alpha_0| \leqq m-2$ with $|\alpha| = m-1$.

Lemma 3.5. *For every $m \geqq 3$ and $|\alpha| \leqq m-1$ with $|\alpha_0| \leqq m-2$, it holds that*

$$\sup_{0\leqq\tau\leqq t}\|(\mathcal{Z}^\alpha\mathrm{div}u, \mathcal{Z}^\alpha\nabla\rho, \nabla\mathcal{Z}^\alpha\theta)(\tau)\|^2 + \varepsilon\int_0^t \|\nabla\mathcal{Z}^\alpha\mathrm{div}u(\tau)\|^2\mathrm{d}\tau$$
$$+ \kappa(\varepsilon)\int_0^t \|\mathcal{Z}^\alpha\Delta\theta(\tau)\|^2\mathrm{d}\tau$$
$$\leqq CC_{m+2}\Big\{\mathcal{N}_m(0) + (\delta+\varepsilon)\int_0^t \|\nabla\mathcal{Z}^{m-2}\mathrm{div}u(\tau)\|^2\mathrm{d}\tau + C_\delta t P(\mathcal{N}_m(t))$$
$$+ \delta\int_0^t \|\nabla\partial_t^{m-1}(\rho,\theta)(\tau)\|^2\mathrm{d}\tau + \varepsilon\int_0^t \|\nabla^2\mathcal{Z}^{m-2}u(\tau)\|^2\mathrm{d}\tau$$
$$+ \delta\int_0^t \varepsilon^2\|\nabla^2u(\tau)\|^2_{\mathcal{H}^{m-1}} + \kappa(\varepsilon)^2\|\nabla\mathcal{Z}^{m-2}\Delta\theta(\tau)\|^2\mathrm{d}\tau\Big\}. \tag{3.62}$$

Proof. The estimate for $|\alpha| = 0$ is already given in Lemma 3.4. Assuming that it is proven for $|\alpha| \leqq m-2$. We shall prove it for $|\alpha| = m-1 \geqq 1$ with $|\alpha_0| \leqq m-2$. Multiplying (3.22) by $\nabla \mathcal{Z}^\alpha \mathrm{div} u$ yields that

$$\begin{aligned}
&\int_0^t \int (\rho \mathcal{Z}^\alpha u_t + \rho u \cdot \nabla \mathcal{Z}^\alpha u) \cdot \nabla \mathcal{Z}^\alpha \mathrm{div} u \mathrm{d}x \mathrm{d}\tau + \int_0^t \int \mathcal{Z}^\alpha \nabla p \cdot \nabla \mathcal{Z}^\alpha \mathrm{div} u \mathrm{d}x \mathrm{d}\tau \\
&\quad = -\mu\varepsilon \int_0^t \int \mathcal{Z}^\alpha \nabla \times \omega \cdot \nabla \mathcal{Z}^\alpha \mathrm{d}\,\mathrm{iv} u \mathrm{d}x \mathrm{d}\tau + \int_0^t \int (\mathcal{C}_1^\alpha + \mathcal{C}_2^\alpha) \cdot \nabla \mathcal{Z}^\alpha \mathrm{div} u \mathrm{d}x \mathrm{d}\tau \\
&\qquad + (2\mu + \lambda)\varepsilon \int_0^t \int \mathcal{Z}^\alpha \nabla \mathrm{div} u \cdot \nabla\, M Z^\alpha \mathrm{div} u \mathrm{d}x \mathrm{d}\tau.
\end{aligned} \tag{3.63}$$

Since

$$\begin{aligned}
&\int_0^t \int (\rho \mathcal{Z}^\alpha u_t + \rho u \cdot \nabla \mathcal{Z}^\alpha u) \cdot \nabla \mathcal{Z}^\alpha \mathrm{div} u \,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad = -\int_0^t \int (\rho \mathrm{div} \mathcal{Z}^\alpha u_t + \rho u \cdot \nabla \mathrm{div} \mathcal{Z}^\alpha u) \mathcal{Z}^\alpha \mathrm{div} u \,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad - \int_0^t \int (\nabla \rho \cdot \mathcal{Z}^\alpha u_t + \nabla (\rho u)^t \cdot \nabla \mathcal{Z}^\alpha u) \mathcal{Z}^\alpha \mathrm{div} u \,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad + \int_0^t \int_{\partial\Omega} (\rho \mathcal{Z}^\alpha u_t \cdot n + \rho (u \cdot \nabla) \mathcal{Z}^\alpha u \cdot n) \mathcal{Z}^\alpha \mathrm{div} u \,\mathrm{d}\sigma\,\mathrm{d}\tau \triangleq I_1 + I_2 + I_3,
\end{aligned} \tag{3.64}$$

for I_1 and I_2, one easily obtains that

$$\begin{aligned}
I_1 &= -\int_0^t \int (\rho \mathcal{Z}^\alpha \mathrm{div} u_t + \rho u \cdot \nabla \mathcal{Z}^\alpha \mathrm{div} u) \mathcal{Z}^\alpha \mathrm{div} u \mathrm{d}x \mathrm{d}\tau \\
&\quad - \int_0^t \int \Big(\rho [\mathrm{div}, \mathcal{Z}^\alpha] u_t + \rho (u_1 Z_{y^1} + u_2 Z_{y^2} + \frac{u \cdot n}{\varphi(z)} Z_3) [\mathrm{div}, \mathcal{Z}^\alpha] u \Big) \\
&\qquad \times \mathcal{Z}^\alpha \mathrm{div} u \mathrm{d}x \mathrm{d}\tau \\
&\leqq -\int \frac{\rho}{2} |\mathcal{Z}^\alpha \mathrm{div} u(t)|^2 \mathrm{d}x + \int \frac{\rho_0}{2} |\mathcal{Z}^\alpha \mathrm{div} u_0|^2 \mathrm{d}x \\
&\quad + C_2 [1 + P(Q(t))] \int_0^t \|\nabla u\|_{\mathcal{H}^{m-1}}^2 \mathrm{d}\tau,
\end{aligned} \tag{3.65}$$

and

$$I_2 \leqq C[1 + P(Q(t))] \int_0^t \|\nabla u\|_{\mathcal{H}^{m-1}}^2 \,\mathrm{d}\tau, \tag{3.66}$$

where $\varphi(z) = \frac{z}{1+z}$. Noting that $\mathcal{Z}^\alpha$ contains at least one tangential derivative Z_y, integrating by parts along the boundary and using (2.6) and (3.37), one has that

$$\begin{aligned}
I_3 &= \int_0^t \int_{\partial\Omega} [\rho \mathcal{Z}^\alpha u_t \cdot n - \rho (u \cdot \nabla) n \cdot \mathcal{Z}^\alpha u + \rho (u \cdot \nabla)(\mathcal{Z}^\alpha u \cdot n)] \mathcal{Z}^\alpha \mathrm{div} u \mathrm{d}\sigma \mathrm{d}\tau \\
&= \int_0^t \int_{\partial\Omega} [\rho \mathcal{Z}^\alpha u_t \cdot n - \rho (u \cdot \nabla) n \cdot \mathcal{Z}^\alpha u
\end{aligned}$$

$$
\begin{aligned}
&+\rho(u_1\partial_{y^1}+u_2\partial_{y^2})(\mathcal{Z}^\alpha u\cdot n)]\mathcal{Z}^\alpha \mathrm{div} u \mathrm{d}\sigma \mathrm{d}\tau\\
&\leqq C[1+P(Q(t))]\int_0^t \left(|\mathcal{Z}^\alpha u_t\cdot n|_{H^{\frac12}}+|\mathcal{Z}^\alpha u\cdot n|_{H^{\frac32}}+|\mathcal{Z}^\alpha u|_{H^{\frac12}}\right)\\
&\qquad\cdot|\mathcal{Z}^{m-2}\mathrm{div} u|_{H^{\frac12}}\mathrm{d}\tau\\
&\leqq \delta\int_0^t \|\nabla\mathcal{Z}^{m-2}\mathrm{div} u\|^2\mathrm{d}\tau + C_{m+2}C_\delta t P(\mathcal{N}_m(t)).
\end{aligned}\tag{3.67}
$$

Substituting (3.65)–(3.67) into (3.64), one gets that

$$
\begin{aligned}
&\int_0^t\int(\rho\mathcal{Z}^\alpha u_t+\rho u\cdot\nabla\mathcal{Z}^\alpha u)\nabla\mathcal{Z}^\alpha \mathrm{div} u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad\leqq -\int\frac{\rho}{2}|\mathcal{Z}^\alpha \mathrm{div} u(t)|^2\,\mathrm{d}x+\int\frac{\rho_0}{2}|\mathcal{Z}^\alpha \mathrm{div} u_0|^2\,\mathrm{d}x\\
&\qquad+\delta\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div} u\|^2\,\mathrm{d}\tau+C_\delta C_{m+2}tP(\mathcal{N}_m(t)).
\end{aligned}\tag{3.68}
$$

By the same argument as Lemma 3.6 of [22], one can obtain that

$$
\begin{aligned}
&\varepsilon\int_0^t\int\mathcal{Z}^\alpha\nabla \mathrm{div} u\cdot\nabla\mathcal{Z}^\alpha \mathrm{div} u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad\geqq\frac34\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha \mathrm{div} u\|^2\,\mathrm{d}\tau-C\varepsilon\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div} u\|^2\,\mathrm{d}\tau
\end{aligned}\tag{3.69}
$$

and

$$
\begin{aligned}
&-\varepsilon\int_0^t\int\mathcal{Z}^\alpha\nabla\times\omega\cdot\nabla\mathcal{Z}^\alpha \mathrm{div} u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad\geqq-\frac{\varepsilon}{4}\int_0^t\|\nabla\mathcal{Z}^\alpha \mathrm{div} u\|^2\,\mathrm{d}\tau-C\varepsilon\int_0^t\|\nabla\mathcal{Z}^{m-2}\omega\|^2\,\mathrm{d}\tau-C_{m+2}tP(\mathcal{N}_m(t)).
\end{aligned}\tag{3.70}
$$

It follows from Proposition 2.2 that

$$
\int_0^t\|\mathcal{C}_1^\alpha\|_1^2+\|\mathcal{C}_2^\alpha\|_1^2\,\mathrm{d}\tau\leqq C(1+P(Q(t)))\int_0^t P(\mathcal{N}_m(\tau))\,\mathrm{d}\tau\leqq CtP(\mathcal{N}_m(t)),\tag{3.71}
$$

which, together with integrating by parts, yields that

$$
\begin{aligned}
&\left|\int_0^t\int(\mathcal{C}_1^\alpha+\mathcal{C}_2^\alpha)\nabla\mathcal{Z}^\alpha \mathrm{div} u\,\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\quad\leqq\left|\int_0^t\int(\mathcal{C}_1^\alpha+\mathcal{C}_2^\alpha)Z\nabla\mathcal{Z}^{\alpha-1}\mathrm{div} u\,\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\qquad+C\int_0^t\int(|\mathcal{C}_1^\alpha|+|\mathcal{C}_2^\alpha|)|\nabla\mathcal{Z}^{\alpha-1}\mathrm{div} u|\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad\leqq\left|\int_0^t\int(|Z\mathcal{C}_1^\alpha|+|Z\mathcal{C}_2^\alpha|+|\mathcal{C}_1^\alpha|+|\mathcal{C}_2^\alpha|)\cdot|\nabla\mathcal{Z}^{\alpha-1}\mathrm{div} u|\right|\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
$$

$$\leqq \delta \int_0^t \|\nabla \mathcal{Z}^{m-2}\mathrm{div}u\|^2\,\mathrm{d}\tau + C\int_0^t \|\mathcal{C}_1^\alpha\|_{H^1_{co}}^2 + \|\mathcal{C}_2^\alpha\|_{H^1_{co}}^2\,\mathrm{d}\tau$$
$$\leqq \delta \int_0^t \|\nabla \mathcal{Z}^{m-2}\mathrm{div}u\|^2\,\mathrm{d}\tau + CtP(\mathcal{N}_m(t)). \tag{3.72}$$

Substituting (3.68)–(3.72) into (3.63), one obtains that

$$\begin{aligned}&\int \frac{\rho}{2}|\mathcal{Z}^\alpha \mathrm{div}u(t)|^2\,\mathrm{d}x - R\int_0^t\int \mathcal{Z}^\alpha\nabla(\rho\theta)\cdot\nabla\mathcal{Z}^\alpha\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\\&\quad+\frac{3}{4}(2\mu+\lambda)\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha\mathrm{div}u\|^2\,\mathrm{d}\tau\\&\quad\leqq\int\frac{\rho_0}{2}|\mathcal{Z}^\alpha\mathrm{div}u_0|^2\,\mathrm{d}x + C\Big\{(\delta+\varepsilon)\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div}u\|^2\,\mathrm{d}\tau\\&\quad+\varepsilon\int_0^t\|\nabla\mathcal{Z}^{m-2}\omega\|^2\,\mathrm{d}\tau + C_{m+2}C_\delta tP(\mathcal{N}_m(t))\Big\}.\end{aligned}\tag{3.73}$$

Next we shall estimate the temperature part. Applying ∇ to $(3.22)_2$, one obtains that

$$\begin{aligned}&\rho\nabla\mathcal{Z}^\alpha\theta_t+\rho(u\cdot\nabla)\nabla\mathcal{Z}^\alpha\theta+p\nabla\mathcal{Z}^\alpha\mathrm{div}u-\kappa(\varepsilon)\nabla\mathcal{Z}^\alpha\Delta\theta\\&\quad=\nabla\rho\mathcal{Z}^\alpha\theta_t+\nabla(\rho u)^t\nabla\mathcal{Z}^\alpha\theta+\nabla p\mathcal{Z}^\alpha\mathrm{div}u+\varepsilon\nabla\mathcal{Z}^\alpha(2\mu|Su|^2\\&\quad+\lambda|\mathrm{div}u|^2)+\nabla(\mathcal{C}_3^\alpha+\mathcal{C}_4^\alpha+\mathcal{C}_4^\alpha),\end{aligned}\tag{3.74}$$

with $|\alpha|=m-1$ and $|\alpha_0|\leqq m-2$. Multiplying (3.74) by $\frac{\nabla\mathcal{Z}^\alpha\theta}{\theta}$ yields that

$$\begin{aligned}&\int\frac{\rho}{2\theta}|\nabla\mathcal{Z}^\alpha\theta|^2\,\mathrm{d}x+R\int_0^t\int\rho\nabla\mathcal{Z}^\alpha\theta\cdot\nabla\mathcal{Z}^\alpha\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\\&\quad-\kappa(\varepsilon)\int_0^t\int\nabla\mathcal{Z}^\alpha\Delta\theta\frac{\nabla\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\&\quad\leqq\varepsilon\int_0^t\int\nabla\mathcal{Z}^\alpha(2\mu|Su|^2+\lambda|\mathrm{div}u|^2)\frac{\nabla\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\&\quad+\int_0^t\int\nabla(\mathcal{C}_3^\alpha+\mathcal{C}_4^\alpha+\mathcal{C}_5^\alpha)\frac{\nabla\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau+CtP(\mathcal{N}_m).\end{aligned}\tag{3.75}$$

It follows from integrating by parts that

$$\begin{aligned}&\kappa(\varepsilon)\int_0^t\int\nabla\mathcal{Z}^\alpha\Delta\theta\frac{\nabla\mathcal{Z}^\alpha\theta}{\theta}\mathrm{d}x\mathrm{d}\tau\\&\quad=-\kappa(\varepsilon)\int_0^t\int\mathcal{Z}^\alpha\Delta\theta\frac{\Delta\mathcal{Z}^\alpha\theta}{\theta}\mathrm{d}x\mathrm{d}\tau+\kappa(\varepsilon)\int_0^t\int\mathcal{Z}^\alpha\Delta\theta\frac{\nabla\mathcal{Z}^\alpha\theta}{\theta^2}\nabla\theta\mathrm{d}x\mathrm{d}\tau\\&\quad+\kappa(\varepsilon)\int_0^t\int_{\partial\Omega}\mathcal{Z}^\alpha\Delta\theta\frac{n\cdot\nabla\mathcal{Z}^\alpha\theta}{\theta}\mathrm{d}\sigma\mathrm{d}\tau\end{aligned}$$

$$\leqq -\frac{3\kappa(\varepsilon)}{4}\int_0^t\int \frac{|\mathcal{Z}^\alpha \Delta\theta|^2}{\theta}\mathrm{d}x\mathrm{d}\tau + \kappa(\varepsilon)\int_0^t\int_{\partial\Omega} \mathcal{Z}^\alpha\Delta\theta\frac{n\cdot\nabla\mathcal{Z}^\alpha\theta}{\theta}\mathrm{d}\sigma\mathrm{d}\tau$$
$$+ C\kappa(\varepsilon)\int_0^t \|\mathcal{Z}^{m-2}\Delta\theta\|^2\mathrm{d}\tau + CtP(\mathcal{N}_m(t)). \tag{3.76}$$

Noting that $\mathcal{Z}^\alpha$ contains at least one tangential derivative Z_y, integrating by parts along the boundary yields that

$$\kappa(\varepsilon)\left|\int_0^t\int_{\partial\Omega} \mathcal{Z}^\alpha\Delta\theta\frac{n\cdot\nabla\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|$$
$$= \kappa(\varepsilon)\left|\int_0^t\int_{\partial\Omega} \mathcal{Z}^{\alpha-1}\Delta\theta\cdot\left(\frac{1}{\theta}Z_y(n\cdot\nabla\mathcal{Z}^\alpha\theta) + n\cdot\nabla\mathcal{Z}^\alpha\theta Z_y\left(\frac{1}{\theta}\right)\right)\mathrm{d}\sigma\,\mathrm{d}\tau\right|$$
$$\leqq CP(Q(t))\kappa(\varepsilon)\int_0^t\left[\|\mathcal{Z}^{m-2}\Delta\theta\|_{L^2}^{\frac12}\|\nabla\mathcal{Z}^{m-2}\Delta\theta\|_{L^2}^{\frac12} + \|\mathcal{Z}^{m-2}\Delta\theta\|_{L^2}\right]$$
$$\times\left(|n\cdot\nabla\mathcal{Z}^\alpha\theta|_{H^1} + |n\cdot\nabla\mathcal{Z}^\alpha\theta|_{L^2}\right)\mathrm{d}\tau$$
$$\leqq \delta\kappa(\varepsilon)^2\int_0^t\|\nabla\mathcal{Z}^{m-2}\Delta\theta\|^2\,\mathrm{d}\tau + \sum_{|\beta|\leqq m-1}^{|\beta_0|\leqq m-2}\delta\kappa(\varepsilon)^2\int_0^t\|\mathcal{Z}^\beta\Delta\theta\|^2\,\mathrm{d}\tau$$
$$+ \delta\kappa(\varepsilon)\int_0^t\|\nabla\mathcal{Z}^m\theta\|^2\,\mathrm{d}\tau + C\kappa(\varepsilon)\int_0^t\|\mathcal{Z}^{m-2}\Delta\theta\|^2\,\mathrm{d}\tau$$
$$+ C_\delta C_{m+1}tP(\mathcal{N}_m(t)), \tag{3.77}$$

where we have used the following boundary estimates, for $|\alpha| \leqq m-1$ with $|\alpha_0| \leqq m-2$,

$$\begin{cases} |n\cdot\nabla\mathcal{Z}^\alpha\theta|_{L^2(\partial\Omega)} \leqq CC_m\left(\mathcal{N}_m^{\frac12} + \|\mathcal{Z}^{m-2}\Delta\theta\|^{\frac12}\mathcal{N}_m^{\frac14}\right), \\ |n\cdot\nabla\mathcal{Z}^\alpha\theta|_{H^1(\partial\Omega)} \leqq CC_{m+1}\left(\|\nabla\mathcal{Z}^m\theta\|^{\frac12} + \sum_{|\beta|\leqq m-1}^{|\beta_0|\leqq m-2}\|\mathcal{Z}^\beta\Delta\theta\|^{\frac12}\right)\mathcal{N}_m^{\frac14} + \mathcal{N}_m^{\frac12}, \end{cases} \tag{3.78}$$

which follows from the boundary condition (1.3) and (2.6). Substituting (3.77) into (3.76), one obtains that

$$-\kappa(\varepsilon)\int_0^t\int \nabla\mathcal{Z}^\alpha\Delta\theta\frac{\nabla\mathcal{Z}^\alpha\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\geqq \frac{\kappa(\varepsilon)}{2}\int_0^t\int\frac{|\mathcal{Z}^\alpha\Delta\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau - \delta\kappa(\varepsilon)^2\int_0^t\|\nabla\mathcal{Z}^{m-2}\Delta\theta\|^2\,\mathrm{d}\tau$$
$$- \sum_{|\beta|\leqq m-1}^{|\beta_0|\leqq m-2}\delta\kappa(\varepsilon)^2\int_0^t\|\mathcal{Z}^\beta\Delta\theta\|^2\,\mathrm{d}\tau - \delta\kappa(\varepsilon)\int_0^t\|\nabla\mathcal{Z}^m\theta\|^2\,\mathrm{d}\tau$$
$$- C\kappa(\varepsilon)\int_0^t\|\mathcal{Z}^{m-2}\Delta\theta\|^2\,\mathrm{d}\tau - C_\delta C_{m+1}tP(\mathcal{N}_m(t)). \tag{3.79}$$

For the terms on the RHS of (3.75), it follows from Proposition 2.2 and Hölder inequality that

$$\begin{aligned}
&\varepsilon\Big|\int_0^t\int \nabla\mathcal{Z}^\alpha\Big(2\mu|Su|^2+\lambda|\mathrm{div}u|^2\Big)\cdot\frac{\nabla\mathcal{Z}^\alpha\theta}{\theta}\mathrm{d}x\mathrm{d}\tau\Big| + \int_0^t \|(\nabla C_3^\alpha,\nabla C_5^\alpha)\|^2\mathrm{d}\tau\\
&\leqq \delta\varepsilon^2\int_0^t \|\nabla^2 u\|^2_{\mathcal{H}^{m-1}}\mathrm{d}\tau + C[P(\mathcal{N}_m(t))+\|\nabla\mathrm{div}u\|^2_{L^\infty}]\int_0^t\Big\{P(\mathcal{N}_m(\tau))\\
&\quad+\|\nabla\mathcal{Z}^{m-2}\mathrm{div}u,\nabla\partial_t^{m-1}(\rho,\theta)\|^2\Big\}\mathrm{d}\tau + C_\delta C_{m+1}tP(\mathcal{N}_m(t))\\
&\leqq \delta\varepsilon^2\int_0^t \|\nabla^2 u\|^2_{\mathcal{H}^{m-1}} + CP(\mathcal{N}_m(t))\int_0^t \|\nabla\mathcal{Z}^{m-2}\mathrm{div}u,\nabla\partial_t^{m-1}(\rho,\theta)\|^2\mathrm{d}\tau\\
&\quad+C_\delta C_{m+1}tP(\mathcal{N}_m(t)),
\end{aligned}\tag{3.80}$$

where we have used the fact that $\|\nabla\mathrm{div}u\|^2_{L^\infty}\leqq CP(\mathcal{N}_m(t))$, which will be proved in Lemma 3.12 below. For the term ∇C_4^α, one needs to be more careful. First, one notices that

$$\begin{aligned}
C_4^\alpha =& \sum_{|\beta|\geqq 1,\beta+\gamma=\alpha}\sum_{i=1}^2 C_{\alpha,\beta}\mathcal{Z}^\beta(\rho u_i)\mathcal{Z}^\gamma\partial_{y^i}\theta\\
&+\sum_{|\beta|\geqq 1,\beta+\gamma=\alpha} C_{\alpha,\beta}\mathcal{Z}^\beta(\rho u\cdot N)\mathcal{Z}^\gamma\partial_z\theta+\rho u\cdot N[\mathcal{Z}^\alpha,\partial_z]\theta,
\end{aligned}\tag{3.81}$$

then, from Proposition 2.2, it holds that

$$\begin{aligned}
&\int_0^t \|\nabla(\rho u\cdot N[\mathcal{Z}^\alpha,\partial_z]\theta)\|^2\\
&\quad+\sum_{|\beta|\geqq 1,\beta+\gamma=\alpha}\left[\sum_{i=1}^2\|\nabla(\mathcal{Z}^\beta(\rho u_i)\mathcal{Z}^\gamma\partial_{y^i}\theta)\|^2+\|\mathcal{Z}^\gamma\partial_z\theta\cdot\nabla\mathcal{Z}^\beta(\rho u\cdot N)\|^2\right]\mathrm{d}\tau\\
&\leqq C(1+P(Q(t)))\int_0^t P(\mathcal{N}_m(\tau))+\|\nabla\partial_t^{m-1}\rho\|^2\,\mathrm{d}\tau.
\end{aligned}\tag{3.82}$$

For $|\beta|\geqq 1$, $\beta+\gamma=\alpha$, and $|\alpha|=m-1$, one notices that

$$\mathcal{Z}^\beta(\rho u\cdot N)\nabla\mathcal{Z}^\gamma\partial_z\theta=\sum_{\tilde\beta\leqq\beta}C_{\tilde\beta}(z)\mathcal{Z}^{\tilde\beta}\left(\frac{\rho u\cdot N}{\varphi(z)}\right)\cdot\varphi(z)\nabla\mathcal{Z}^\gamma\partial_z\theta,\tag{3.83}$$

where $C_{\tilde\beta}(z)$ is a bounded smooth function of z. If $\tilde\beta=0$ and $|\gamma|\leqq m-2$, one gets that

$$\begin{aligned}
\int_0^t\|\mathcal{Z}^{\tilde\beta}\left(\frac{\rho u\cdot N}{\varphi(z)}\right)\cdot\varphi(z)\nabla\mathcal{Z}^\gamma\partial_z\theta\|^2\,\mathrm{d}\tau &\leqq \left\|\frac{\rho u\cdot N}{\varphi(z)}\right\|^2_{L^\infty}\int_0^t\|\varphi(z)\nabla\mathcal{Z}^\gamma\partial_z\theta\|^2\,\mathrm{d}\tau\\
&\leqq CtP(\mathcal{N}_m(t)).
\end{aligned}\tag{3.84}$$

If $|\tilde{\beta}| \neq 0$, from Proposition 2.2, one obtains that

$$\begin{aligned}&\int_0^t \|\mathcal{Z}^{\beta}\left(\frac{\rho u\cdot N}{\varphi(z)}\right)\nabla\mathcal{Z}^{\gamma}\partial_z\theta\|^2 \mathrm{d}\tau\\&\leqq C(1+P(Q(t)))\int_0^t P(\mathcal{N}_m)+\left\|\frac{\rho u\cdot N}{\varphi(z)}\right\|_{\mathcal{H}^{m-1}}^2 d\tau \leqq CtP(\mathcal{N}_m(t)),\end{aligned} \tag{3.85}$$

where in the last inequality, we have used the Hardy inequality

$$\left\|\frac{u\cdot N}{\varphi(z)}\right\|_{\mathcal{H}^{m-1}}^2 \leqq C_{m+1}\|\nabla u\|_{\mathcal{H}^{m-1}}^2, \tag{3.86}$$

which has already been proved on p. 543 of [14]. Then, combining (3.83)–(3.85), one obtains that

$$\sum_{|\beta|\geqq 1,\beta+\gamma=\alpha}\int_0^t \|\mathcal{Z}^{\beta}(\rho u\cdot N)\nabla\mathcal{Z}^{\gamma}\partial_z\theta\|^2\,\mathrm{d}\tau \leqq CtP(\mathcal{N}_m(t)). \tag{3.87}$$

Thus, it follows from (3.81), (3.82) and (3.87) that

$$\int_0^t \|\nabla C_4^{\alpha}\|^2\,\mathrm{d}\tau \leqq CP(\mathcal{N}_m(\tau))\int_0^t \|\nabla\partial_t^{m-1}(\rho,\theta)\|^2\,\mathrm{d}\tau + CtP(\mathcal{N}_m(t)). \tag{3.88}$$

Substituting (3.79), (3.80) and (3.88) into (3.75) and using Hölder inequality, one obtains that

$$\begin{aligned}&\int \frac{\rho}{2\theta}|\nabla\mathcal{Z}^{\alpha}\theta|^2\,\mathrm{d}x + R\int_0^t\int \rho\nabla\mathcal{Z}^{\alpha}\theta\cdot\nabla\mathcal{Z}^{\alpha}\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\\&\quad+\frac{\kappa(\varepsilon)}{2}\int_0^t\int\frac{|\mathcal{Z}^{\alpha}\Delta\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\&\leqq CC_{m+1}\Big\{\|\nabla\mathcal{Z}^{\alpha}\theta_0\|^2+\delta\varepsilon^2\int_0^t\|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau\\&\quad+\delta\kappa(\varepsilon)^2\int_0^t\|\nabla\mathcal{Z}^{m-2}\Delta\theta\|^2+\sum_{|\beta|\leqq m-1}^{|\beta_0|\leqq m-2}\|\mathcal{Z}^{\beta}\Delta\theta\|^2\,\mathrm{d}\tau\\&\quad+\delta\kappa(\varepsilon)\int_0^t\|\nabla\mathcal{Z}^m\theta\|^2\,\mathrm{d}\tau+\kappa(\varepsilon)\int_0^t\|\mathcal{Z}^{m-2}\Delta\theta\|^2\,\mathrm{d}\tau\\&\quad+\delta\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div}u\|^2+\|\nabla\partial_t^{m-1}(\rho,\theta)\|^2\,\mathrm{d}\tau+C_{\delta}tP(\mathcal{N}_m(t))\Big\}.\end{aligned} \tag{3.89}$$

It follows from (3.73) and (3.89) that

$$\begin{aligned}&\int\frac{\rho}{2}|\mathcal{Z}^{\alpha}\mathrm{div}u(t)|^2+\frac{\rho}{2\theta}|\nabla\mathcal{Z}^{\alpha}\theta|^2\mathrm{d}x\\&\quad+R\int_0^t\int(\rho\nabla\mathcal{Z}^{\alpha}\theta-\mathcal{Z}^{\alpha}\nabla(\rho\theta))\cdot\nabla\mathcal{Z}^{\alpha}\mathrm{div}u\mathrm{d}x\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{4}(2\mu+\lambda)\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha\mathrm{div}u\|^2\mathrm{d}\tau+\frac{\kappa(\varepsilon)}{2}\int_0^t\int\frac{|\mathcal{Z}^\alpha\Delta\theta|^2}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq CC_{m+1}\Big\{\mathcal{N}_m(0)+\delta\varepsilon^2\int_0^t\|\nabla^2u\|^2_{\mathcal{H}^{m-1}}\mathrm{d}\tau+\delta\kappa(\varepsilon)^2\int_0^t\|\nabla\mathcal{Z}^{m-2}\Delta\theta\|^2\mathrm{d}\tau\\
&+\delta\kappa(\varepsilon)^2\sum_{|\beta|\leqq m-1}^{|\beta_0|\leqq m-2}\int_0^t\|\mathcal{Z}^\beta\Delta\theta\|^2\mathrm{d}\tau+\delta\kappa(\varepsilon)\int_0^t\|\nabla\mathcal{Z}^m\theta\|^2\mathrm{d}\tau+C_\delta tP(\mathcal{N}_m(t))\\
&+C\int_0^t\kappa(\varepsilon)\|\mathcal{Z}^{m-2}\Delta\theta\|^2+\varepsilon\|\nabla^2\mathcal{Z}^{m-2}u\|^2\mathrm{d}\tau\\
&+(\delta+\varepsilon)\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div}u\|^2\mathrm{d}\tau+\delta\int_0^t\|\nabla\partial_t^{m-1}(\rho,\theta)\|^2\mathrm{d}\tau\Big\}.
\end{aligned}\tag{3.90}
$$

In order to close the estimate of (3.90), one notes that

$$
\begin{aligned}
\rho\nabla\mathcal{Z}^\alpha\theta-\mathcal{Z}^\alpha\nabla(\rho\theta)&=-\theta\mathcal{Z}^\alpha\nabla\rho-\rho[\mathcal{Z}^\alpha,\nabla]\theta\\
&\quad-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}(\mathcal{Z}^\beta\theta\cdot\mathcal{Z}^\gamma\nabla\rho+\mathcal{Z}^\beta\rho\cdot\mathcal{Z}^\gamma\nabla\theta).
\end{aligned}\tag{3.91}
$$

Since $\mathcal{Z}^\alpha\neq\partial_t^{m-1}$, it follows from (2.4), (3.91) and integrating by parts that

$$
\begin{aligned}
&\int_0^t\int(\rho\nabla\mathcal{Z}^\alpha\theta-\mathcal{Z}^\alpha\nabla(\rho\theta))\cdot\nabla\mathcal{Z}^\alpha\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\\
&=\int_0^t\int(\rho\nabla\mathcal{Z}^\alpha\theta-\mathcal{Z}^\alpha\nabla(\rho\theta))\cdot(\mathcal{Z}^\alpha\nabla\mathrm{div}u+[\nabla,\mathcal{Z}^\alpha]\mathrm{div}u)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\geqq\int_0^t\int(\rho\nabla\mathcal{Z}^\alpha\theta-\mathcal{Z}^\alpha\nabla(\rho\theta))\cdot\mathcal{Z}^\alpha\nabla\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau-\delta\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div}u\|^2\,\mathrm{d}\tau\\
&\quad-C_\delta tP(\mathcal{N}_m(t))\\
&\geqq-\int_0^t\int\theta\mathcal{Z}^\alpha\nabla\rho\cdot\mathcal{Z}^\alpha\nabla\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau-\delta\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div}u\|^2\,\mathrm{d}\tau\\
&\quad-C_\delta tP(\mathcal{N}_m(t))-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\int_0^t\int|Z(\mathcal{Z}^\beta\theta\cdot\mathcal{Z}^\gamma\nabla\rho+\mathcal{Z}^\beta\rho\cdot\mathcal{Z}^\gamma\nabla\theta)\\
&\qquad\cdot\mathcal{Z}^{\alpha-1}\nabla\mathrm{div}u|\,\mathrm{d}x\,\mathrm{d}\tau\\
&\geqq\int_0^t\int\theta\mathcal{Z}^\alpha\nabla\rho\cdot\mathcal{Z}^\alpha\nabla\left(\frac{\rho_t}{\rho}\right)+\int_0^t\int\theta\mathcal{Z}^\alpha\nabla\rho\cdot\mathcal{Z}^\alpha\nabla\left(\frac{u\cdot\nabla\rho}{\rho}\right)\\
&\quad-C\delta\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div}u\|^2\,\mathrm{d}\tau-C_\delta tP(\mathcal{N}_m(t)),
\end{aligned}\tag{3.92}
$$

where we have used (3.41) in the last inequality. For the first term on the right hand side of (3.101), one notices that

$$\mathcal{Z}^\alpha \nabla \left(\frac{\rho_t}{\rho}\right) = \frac{1}{\rho}\mathcal{Z}^\alpha \nabla \rho_t + \sum_{|\beta|\geqq 1, \beta+\gamma=\alpha} C_{\alpha,\beta}\mathcal{Z}^\beta \left(\frac{1}{\rho}\right)\cdot \mathcal{Z}^\gamma \nabla \rho_t + \sum_{\beta+\gamma=\alpha} C_{\alpha,\beta}\mathcal{Z}^\beta \rho_t \cdot \mathcal{Z}^\gamma \nabla \left(\frac{1}{\rho}\right). \tag{3.93}$$

Therefore, using (3.93) and Proposition 2.2, one has that

$$\int_0^t \int \theta \mathcal{Z}^\alpha \nabla \rho \cdot \mathcal{Z}^\alpha \nabla \left(\frac{\rho_t}{\rho}\right) \mathrm{d}x\, \mathrm{d}\tau \leqq \int \frac{\theta}{2\rho}|\mathcal{Z}^\alpha \nabla \rho|^2 \,\mathrm{d}x - \int \frac{\theta_0}{2\rho_0}|\mathcal{Z}^\alpha \nabla \rho_0|^2 \,\mathrm{d}x + \delta \int_0^t \|\nabla \partial_t^{m-1}\rho\|^2 \,\mathrm{d}\tau + C_\delta t P(\mathcal{N}_m(t)). \tag{3.94}$$

To estimate the second term on the RHS of (3.92), note that

$$\begin{aligned}\mathcal{Z}^\alpha \nabla \left(\frac{u}{\rho}\cdot \nabla \rho\right) &= \sum_{i=1,2} \frac{u_i}{\rho}\mathcal{Z}^\alpha \nabla \partial_{y^i}\rho + \frac{u\cdot N}{\rho}\mathcal{Z}^\alpha \partial_z \nabla \rho \\ &+ \sum_{i=1,2}\sum_{|\beta|\geqq 1, \beta+\gamma=\alpha} C_{\alpha,\beta}\mathcal{Z}^\beta \left(\frac{u_i}{\rho}\right)\cdot \mathcal{Z}^\gamma \nabla \partial_{y^i}\rho \\ &+ \sum_{|\beta|\geqq 1, \beta+\gamma=\alpha} C_{\alpha,\beta}\mathcal{Z}^\beta \left(\frac{u\cdot N}{\rho}\right)\cdot \mathcal{Z}^\gamma \partial_z \nabla \rho \\ &+ \sum_{i=1,2}\sum_{\beta+\gamma=\alpha} C_{\alpha,\beta}\mathcal{Z}^\beta \nabla \left(\frac{u_i}{\rho}\right)\cdot \mathcal{Z}^\gamma \partial_{y^i}\rho \\ &+ \sum_{\beta+\gamma=\alpha} C_{\alpha,\beta}\mathcal{Z}^\beta \nabla \left(\frac{u\cdot N}{\rho}\right)\cdot \mathcal{Z}^\gamma \partial_z \rho.\end{aligned} \tag{3.95}$$

Integrating by parts immediately yields that

$$\begin{aligned}&\int_0^t \int \theta \mathcal{Z}^\alpha \nabla \rho \cdot \left(\sum_{i=1,2}\frac{u_i}{\rho}\mathcal{Z}^\alpha \nabla \partial_{y^i}\rho + \frac{u\cdot N}{\rho}\mathcal{Z}^\alpha \partial_z \nabla \rho\right) \mathrm{d}x\, \mathrm{d}\tau \\ &= -\int_0^t \int \theta \mathcal{Z}^\alpha \nabla \rho \cdot \left(\sum_{i=1,2}\frac{u_i}{\rho}\partial_{y^i}\mathcal{Z}^\alpha \nabla \rho + \frac{u\cdot N}{\rho}\partial_z \mathcal{Z}^\alpha \nabla \rho\right) \mathrm{d}x\, \mathrm{d}\tau \\ &\quad -\int_0^t \int \theta \mathcal{Z}^\alpha \nabla \rho \cdot \left(\sum_{i=1,2}\frac{u_i}{\rho}[\mathcal{Z}^\alpha \nabla, \partial_{y^i}]\rho + \frac{u\cdot N}{\rho\varphi(z)}\varphi(z)[\mathcal{Z}^\alpha, \partial_z]\nabla \rho\right) \mathrm{d}x\, \mathrm{d}\tau \\ &\leqq C_2 C_\delta t P(\mathcal{N}_m(t)).\end{aligned} \tag{3.96}$$

It follows from Proposition 2.2 that

$$\sum_{i=1,2}\sum_{\beta+\gamma=\alpha} C_{\alpha,\beta}\int_0^t\int \theta\mathcal{Z}^\alpha\nabla\rho\cdot\Big(\mathcal{Z}^\beta\nabla\left(\frac{u_i}{\rho}\right)\cdot\mathcal{Z}^\gamma\partial_{y^i}\rho$$
$$+\mathcal{Z}^\beta\left(\frac{u_i}{\rho}\right)\cdot\mathcal{Z}^\gamma\nabla\partial_{y^i}\rho\Big)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$-\sum_{\beta+\gamma=\alpha} C_{\alpha,\beta}\int_0^t\int\theta\mathcal{Z}^\alpha\nabla\rho\cdot\mathcal{Z}^\beta\nabla\left(\frac{u\cdot N}{\rho}\right)\cdot\mathcal{Z}^\gamma\partial_z\rho\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leqq \delta\int_0^t\|\nabla\partial_t^{m-1}\rho\|^2\,\mathrm{d}\tau + C_\delta t P(\mathcal{N}_m(t)). \tag{3.97}$$

On the other hand, note that for $|\beta|\geqq 1$, $\beta+\gamma=\alpha$, and $|\alpha|=m-1$

$$\mathcal{Z}^\beta\left(\frac{u\cdot N}{\rho}\right)\cdot\mathcal{Z}^\gamma\partial_z\nabla\rho=\sum_{\tilde\beta\leqq\beta,\tilde\gamma\leqq\gamma} C_{\alpha,\tilde\beta,\tilde\gamma}(z)\mathcal{Z}^{\tilde\beta}\left(\frac{u\cdot N}{\rho\varphi(z)}\right)\cdot\mathcal{Z}^{\tilde\gamma}(Z_3\nabla\rho), \tag{3.98}$$

where $|\tilde\beta|+|\tilde\gamma|\leqq m-1$, $|\tilde\gamma|\leqq m-2$ and $C_{\alpha,\tilde\beta,\tilde\gamma}(z)$ is some smooth bounded function of z. Using (3.98) and similar arguments as in the proof of (3.87), one has that

$$\sum_{\tilde\beta\leqq\beta,\tilde\gamma\leqq\gamma}\int_0^t\int C_{\alpha,\tilde\beta,\tilde\gamma}(z)\theta\mathcal{Z}^\alpha\nabla\rho\cdot\mathcal{Z}^\beta\left(\frac{u\cdot N}{\rho}\right)\cdot\mathcal{Z}^\gamma\partial_z\nabla\rho\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leqq C\left(\int_0^t\|\mathcal{Z}^\alpha\nabla\rho\|^2\,\mathrm{d}\tau\right)^{\frac12}\cdot\left\{\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}\int_0^t\left\|\mathcal{Z}^\beta\left(\frac{u\cdot N}{\rho}\right)\cdot\mathcal{Z}^\gamma\partial_z\nabla\rho\right\|^2\right\}^{\frac12}$$
$$\leqq CC_\delta t P(\mathcal{N}_m(t)). \tag{3.99}$$

Combining (3.95)–(3.99), one obtains that

$$\int_0^t\int\theta\mathcal{Z}^\alpha\nabla\rho\cdot\mathcal{Z}^\alpha\nabla\left(\frac{u}{\rho}\cdot\nabla\rho\right)\mathrm{d}x\,\mathrm{d}\tau\leqq C\delta\int_0^t\|\nabla\partial_t^{m-1}\rho\|^2\,\mathrm{d}\tau+C_\delta t P(\mathcal{N}_m(t)). \tag{3.100}$$

Then, substituting (3.94) and (3.100) into (3.92), one obtains that

$$\int_0^t\int(\rho\nabla\mathcal{Z}^\alpha\theta-\mathcal{Z}^\alpha\nabla(\rho\theta))\cdot\nabla\mathcal{Z}^\alpha\mathrm{div}u\mathrm{d}x\mathrm{d}\tau$$
$$\leqq\int\frac{\theta}{2\rho}|\mathcal{Z}^\alpha\nabla\rho|^2\mathrm{d}x\mathrm{d}\tau-\int\frac{\theta_0}{2\rho_0}|\mathcal{Z}^\alpha\nabla\rho_0|^2\mathrm{d}x\mathrm{d}\tau-C_\delta t P(\mathcal{N}_m(t))$$
$$-C\delta\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div}u\|^2\mathrm{d}\tau-C\delta\int_0^t\|\nabla\partial_t^{m-1}\rho\|^2\mathrm{d}\tau. \tag{3.101}$$

Substituting (3.101) into (3.90), we have proved (3.62). Thus, the proof of Lemma 3.5 is completed. □

In the proof of Lemma 3.5, we have used the fact that $|\alpha_0| \leqq m-2$ in (3.67) and (3.77). However, for the case $\mathcal{Z}^\alpha = \partial_t^{m-1}$, arguments such as (3.67) and (3.77) are not available anymore, and we can obtain only the following weak estimate $\varepsilon\|\partial_t^{m-1}(\mathrm{div}u, \nabla\rho)\|^2$, but the control of $\varepsilon^2\int_0^t \|\nabla\partial_t^{m-1}\mathrm{div}u(\tau)\|^2\,\mathrm{d}\tau$ is crucial for us to close the *a priori* estimation.

Lemma 3.6. *For every $m \geqq 1$, it holds that*

$$\begin{aligned}&\sup_{0\leqq\tau\leqq t}(\varepsilon\|(\partial_t^{m-1}\mathrm{div}u, \nabla\partial_t^{m-1}\rho)(\tau)\|^2) + \frac{1}{2}(2\mu+\lambda)\varepsilon^2\int_0^t \|\nabla\partial_t^{m-1}\mathrm{div}u(\tau)\|^2\,\mathrm{d}\tau\\ &\leqq C\left\{\varepsilon\|(\partial_t^{m-1}\mathrm{div}u_0, \nabla\partial_t^{m-1}\rho_0)\|^2 + \int_0^t \|\partial_t^{m-1}\nabla\theta\|^2\,\mathrm{d}\tau\right.\\ &\quad\left.+C_\delta C_{m+1}tP(\mathcal{N}_m(t))\right\}. \qquad (3.102)\end{aligned}$$

Proof. First, it follows from the boundary condition (1.3) that

$$n\cdot\partial_t^{m-1}u = 0, \quad n\times\partial_t^{m-1}\omega = [B\partial_t^{m-1}u]_\tau, \quad n\cdot\nabla\partial_t^{m-1}\theta = \nu\partial_t^{m-1}\theta. \qquad (3.103)$$

Multiplying $(3.22)_1$ (with $\mathcal{Z}^\alpha = \partial_t^m$) by $\varepsilon\nabla\mathrm{div}\partial_t^{m-1}u$, one obtains that

$$\begin{aligned}&\varepsilon\int_0^t\int(\rho\partial_t^{m-1}u_t + \rho u\cdot\nabla\partial_t^{m-1}u)\nabla\mathrm{div}\partial_t^{m-1}u\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad+\varepsilon\int_0^t\int\partial_t^{m-1}\nabla p\cdot\nabla\mathrm{div}\partial_t^{m-1}u\,\mathrm{d}x\,\mathrm{d}\tau\\ &= -\mu\varepsilon^2\int_0^t\int\nabla\times\partial_t^{m-1}\omega\cdot\nabla\mathrm{div}\partial_t^{m-1}u\,\mathrm{d}x\,\mathrm{d}\tau \qquad (3.104)\\ &\quad+(2\mu+\lambda)\varepsilon^2\int_0^t\|\nabla\partial_t^{m-1}\mathrm{div}u\|^2\,\mathrm{d}\tau + \varepsilon\int_0^t\int(\mathcal{C}_1^\alpha + \mathcal{C}_2^\alpha)\nabla\mathrm{div}\partial_t^{m-1}u.\end{aligned}$$

It follows from (3.103) and integrating by parts that

$$\begin{aligned}&\varepsilon^2\left|\int_0^t\int\nabla\times\partial_t^{m-1}\omega\cdot\nabla\mathrm{div}\partial_t^{m-1}u\,\mathrm{d}x\,\mathrm{d}\tau\right|\\ &= \varepsilon^2\left|\int_0^t\int_{\partial\Omega} n\times\partial_t^{m-1}\omega\cdot\Pi(\nabla\mathrm{div}\partial_t^{m-1}u)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|\\ &\leqq C\varepsilon^2\int_0^t |n\times\partial_t^{m-1}\omega|_{H^{\frac12}}\cdot|\partial_t^{m-1}\mathrm{div}u|_{H^{\frac12}}\,\mathrm{d}\tau\\ &\leqq CC_3\varepsilon^2\int_0^t |\partial_t^{m-1}u|_{H^{\frac12}}\cdot|\partial_t^{m-1}\mathrm{div}u|_{H^{\frac12}}\,\mathrm{d}\tau\\ &\leqq \frac{1}{16}\varepsilon^2\int_0^t\|\nabla\partial_t^{m-1}\mathrm{div}u\|^2\,\mathrm{d}\tau + CC_3tP(\mathcal{N}_m(t)) \qquad (3.105)\end{aligned}$$

and

$$
\begin{aligned}
&\varepsilon\int_0^t\int(\rho\partial_t^{m-1}u_t+\rho u\cdot\nabla\partial_t^{m-1}u)\nabla\mathrm{div}\partial_t^{m-1}u\\
&=-\varepsilon\int_0^t\int(\rho\partial_t^{m-1}\mathrm{div}u_t+\rho u\cdot\nabla\partial_t^{m-1}\mathrm{div}u)\mathrm{div}\partial_t^{m-1}u\\
&\quad-\varepsilon\int_0^t\int(\nabla\rho\cdot\partial_t^{m-1}u_t+\nabla(\rho u)^t\cdot\nabla\partial_t^{m-1}u)\mathrm{div}\partial_t^{m-1}u\mathrm{d}x\mathrm{d}\tau\\
&\quad-\varepsilon\int_0^t\int_{\partial\Omega}\rho(u\cdot\nabla)n\cdot\partial_t^{m-1}u\partial_t^{m-1}\mathrm{div}u\mathrm{d}\sigma\mathrm{d}\tau\\
&\leqq-\varepsilon\int\frac{\rho}{2}|\partial_t^{m-1}\mathrm{div}u(t)|^2\mathrm{d}x+\varepsilon\int\frac{\rho_0}{2}|\partial_t^{m-1}\mathrm{div}u_0|^2\mathrm{d}x+CtP(\mathcal{N}_m(t))\\
&\quad+C[1+P(Q(t))]\varepsilon\int_0^t\|\partial_t^{m-1}u\|^{\frac12}\|\partial_t^{m-1}u\|_{H^1}^{\frac12}\\
&\qquad\qquad\times\|\partial_t^{m-1}\mathrm{div}u\|_{H^1}^{\frac12}\|\partial_t^{m-1}\mathrm{div}u\|^{\frac12}\mathrm{d}\tau \qquad (3.106)\\
&\leqq-\varepsilon\int\rho|\partial_t^{m-1}\mathrm{div}u(t)|^2\mathrm{d}x+\varepsilon\int\rho_0|\partial_t^{m-1}\mathrm{div}u_0|^2\mathrm{d}x\\
&\quad+\frac{1}{16}\varepsilon^2\int_0^t\|\nabla\partial_t^{m-1}\mathrm{div}u\|^2\mathrm{d}\tau+CtP(\mathcal{N}_m(t)).
\end{aligned}
$$

Using (3.71), one obtains that

$$
\begin{aligned}
&\varepsilon\left|\int_0^t\int(\mathcal{C}_1^\alpha+\mathcal{C}_2^\alpha)\nabla\mathrm{div}\partial_t^{m-1}u\,\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\quad\leqq\frac{1}{16}\varepsilon^2\int_0^t\|\nabla\partial_t^{m-1}\mathrm{div}u\|^2\,\mathrm{d}\tau+C\int_0^t\|(\mathcal{C}_1^\alpha,\mathcal{C}_2^\alpha)\|^2\,\mathrm{d}\tau\\
&\quad\leqq\frac{1}{16}\varepsilon^2\int_0^t\|\nabla\partial_t^{m-1}\mathrm{div}u\|^2\,\mathrm{d}\tau+CtP(\mathcal{N}_m(t)).
\end{aligned}\tag{3.107}
$$

Substituting (3.105)–(3.107) into (3.104), one gets that

$$
\begin{aligned}
&\varepsilon\int\frac{\rho}{2}|\partial_t^{m-1}\mathrm{div}u(t)|^2\,\mathrm{d}x-\varepsilon R\int_0^t\int\partial_t^{m-1}\nabla(\rho\theta)\cdot\nabla\mathrm{div}\partial_t^{m-1}u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\frac34(2\mu+\lambda)\varepsilon^2\int_0^t\|\nabla\partial_t^{m-1}\mathrm{div}u\|^2\,\mathrm{d}\tau\\
&\leqq\varepsilon\int\frac{\rho_0}{2}|\partial_t^{m-1}\mathrm{div}u_0|^2\,\mathrm{d}x+C_3tP(\mathcal{N}_m(t)).
\end{aligned}\tag{3.108}
$$

Finally, it follows from (3.41) and the Cauchy inequality that

$$
\begin{aligned}
&-\varepsilon\int_0^t\int\partial_t^{m-1}\nabla(\rho\theta)\cdot\nabla\mathrm{div}\partial_t^{m-1}u\mathrm{d}x\mathrm{d}\tau\\
&=-\varepsilon\int_0^t\int\left\{\theta\partial_t^{m-1}\nabla\rho+[\partial_t^{m-1}\nabla(\rho\theta)-\theta\partial_t^{m-1}\nabla\rho]\right\}\nabla\mathrm{div}\partial_t^{m-1}u\mathrm{d}x\mathrm{d}\tau
\end{aligned}
$$

$$\geqq \varepsilon \int \frac{\theta}{2\rho}|\partial_t^{m-1}\nabla\rho|^2 \mathrm{d}x - \varepsilon \int \frac{\theta_0}{2\rho_0}|\partial_t^{m-1}\nabla\rho_0|^2 \mathrm{d}x - C_{m+1} t P(\mathcal{N}_m(t))$$
$$- \frac{1}{8}(2\mu+\lambda)\varepsilon^2 \int_0^t \|\nabla\partial_t^{m-1}\mathrm{div}u\|^2 \mathrm{d}\tau - C\int_0^t \|\partial_t^{m-1}\nabla\theta\|^2 \mathrm{d}\tau. \qquad (3.109)$$

Substituting (3.109) into (3.108), one proves (3.102). Thus, the proof of Lemma 3.6 is completed. □

Usually, it is hard to obtain the uniform estimate for the term $\int_0^t \|\nabla\partial_t^{m-2}\mathrm{div}u\|^2 \mathrm{d}\tau$, since it involves two times standard space derivatives. We observe, however, that $\mathrm{div}u$ can be expressed by some good terms by using the mass conservation law.

Lemma 3.7. *For every $m \geqq 3$, it holds that*

$$\int_0^t \|\nabla\mathcal{Z}^{m-2}\mathrm{div}u(\tau)\|^2 \,\mathrm{d}\tau \leqq C\int_0^t \|\nabla\partial_t^{m-1}\rho(\tau)\|^2 \,\mathrm{d}\tau + C_m t P(\mathcal{N}_m(t)), \qquad (3.110)$$

$$\varepsilon^2 \int_0^t \|\nabla^2\mathcal{Z}^{m-2}u(\tau)\|^2 \,\mathrm{d}\tau \leqq C_m t P(\mathcal{N}_m(t)), \qquad (3.111)$$

$$\kappa(\varepsilon)^2 \int_0^t \|\nabla\mathcal{Z}^{m-2}\Delta\theta(\tau)\|^2 \,\mathrm{d}\tau \leqq C P(\mathcal{N}_m(t)) \int_0^t 1 + \|\nabla\partial_t^{m-1}(\rho,\theta)\|^2 + \varepsilon^2\|\nabla^2 u\|^2_{\mathcal{H}^{m-2}} \,\mathrm{d}\tau, \qquad (3.112)$$

$$\kappa(\varepsilon)^2 \int_0^t \|\partial_t^{m-1}\Delta\theta(\tau)\|^2 \,\mathrm{d}\tau \leqq C t P(\mathcal{N}_m(t)). \qquad (3.113)$$

Proof. Applying $\nabla\mathcal{Z}^\alpha$ to (3.41) with $|\alpha| \leqq m-2$, one has

$$\nabla\mathcal{Z}^\alpha \mathrm{div}u = -\nabla\mathcal{Z}^\alpha(\ln\rho)_t - \nabla\mathcal{Z}^\alpha(u_i\partial_{y^i}\ln\rho) - \nabla\mathcal{Z}^\alpha(u\cdot N\partial_z\ln\rho). \qquad (3.114)$$

By using Proposition 2.2, it is easy to obtain

$$\int_0^t \|\nabla\mathcal{Z}^\alpha(\ln\rho)_t\|^2 + \|\nabla\mathcal{Z}^\alpha(u_i\partial_{y^i}\ln\rho)\|^2 \,\mathrm{d}\tau \leqq C\int_0^t \|\nabla\partial_t^{m-1}p\|^2 \,\mathrm{d}\tau + C t P(\mathcal{N}_m(t)), \qquad (3.115)$$

and

$$\int_0^t \|\nabla\mathcal{Z}^\alpha(u\cdot N\partial_z\ln\rho)\|^2 \,\mathrm{d}\tau$$
$$\leqq \int_0^t \|\nabla(u\cdot N)\cdot\partial_z\ln\rho)\|^2_{\mathcal{H}^{m-2}} \,\mathrm{d}\tau + \int_0^t \left\|\left(\frac{u\cdot N}{\varphi(z)}\right)\cdot Z_3\nabla\ln\rho)\right\|^2_{\mathcal{H}^{m-2}} \mathrm{d}\tau$$
$$\leqq C\left[P(\mathcal{N}_m(t)) + \sup_{0\leqq\tau\leqq t}\left\|\frac{u\cdot N}{\varphi(z)}\right\|^2_{L^\infty}\right]\int_0^t \left(P(\mathcal{N}_m(\tau)) + \left\|\frac{u\cdot N}{\varphi(z)}\right\|^2_{\mathcal{H}^{m-2}}\right)\mathrm{d}\tau$$
$$\leqq C_m t P(\mathcal{N}_m(t)), \qquad (3.116)$$

where the hardy inequality was used in the last inequality of (3.116). Combining (3.114)–(3.116), we obtain (3.110).

Note that

$$\Delta = (1 + |\nabla\psi|^2)\partial_{zz} + \sum_{i=1,2} (\partial^2_{y^i} - \partial_{y^i}(\partial_i\psi\partial_z) - \partial_i\psi\partial_z\partial_{y^i}), \tag{3.117}$$

which, together with $(3.7)_2$, yields that

$$\begin{aligned}
\varepsilon^2 \int_0^t \|\nabla^2 \mathcal{Z}^{m-2} u(\tau)\|^2 \, d\tau &\leqq C\varepsilon^2 \int_0^t \|\mathcal{Z}^{m-2}\partial_z^2 u(\tau)\|^2 \, d\tau + C_m t P(\mathcal{N}_m(t)) \\
&\leqq C\varepsilon^2 \int_0^t \|\mathcal{Z}^{m-2}\Delta u(\tau)\|^2 \, d\tau + C_m t P(\mathcal{N}_m(t)) \\
&\leqq C\varepsilon^2 \int_0^t \|\mathcal{Z}^{m-2}\nabla \mathrm{div} u(\tau)\|^2 \, d\tau + C_m t P(\mathcal{N}_m(t)) \\
&\leqq C_m t P(\mathcal{N}_m(t)).
\end{aligned} \tag{3.118}$$

Thus, one obtains (3.111). It follows from (2.4), (3.110), (3.111) and (3.74)(with $|\alpha| = m - 2$) that

$$\begin{aligned}
&\kappa(\varepsilon)^2 \int_0^t \|\nabla \mathcal{Z}^{m-2}\Delta\theta\|^2 \, d\tau \\
&\leqq C(1 + P(Q(t))) \int_0^t \|\nabla \mathcal{Z}^{m-2} \mathrm{div} u\|^2 + \|\nabla\partial_t^{m-1}\theta(\tau)\|^2 + P(\mathcal{N}_m(\tau)) \, d\tau \\
&\quad + C(1 + P(Q(t))) \int_0^t \varepsilon^2 \|\nabla^2 u(\tau)\|^2_{\mathcal{H}^{m-2}} \, d\tau + \int_0^t \|\nabla(\mathcal{C}_3^\alpha, \mathcal{C}_4^\alpha, \mathcal{C}_5^\alpha)(\tau)\|^2 \, d\tau \\
&\leqq C P(\mathcal{N}_m(t)) \int_0^t \|\nabla\partial_t^{m-1}(\rho, \theta)(\tau)\|^2 \, d\tau + Ct P(\mathcal{N}_m(t)),
\end{aligned} \tag{3.119}$$

which yields (3.112). Finally, it follows from (2.4) and (3.74) with $\mathcal{Z}^\alpha = \partial_t^{m-1}$ that

$$\kappa(\varepsilon)^2 \int_0^t \|\partial_t^{m-1}\Delta\theta\|^2 \, d\tau \leqq C(1 + P(Q(t))) \int_0^t P(\mathcal{N}_m(\tau)) \, d\tau \leqq Ct P(\mathcal{N}_m(t)), \tag{3.120}$$

which yields (3.113). Therefore, the proof of this lemma is completed. □

Due to the difficulty on the boundary estimates, it is hard to get the uniform estimates on $\sup_{0\leqq\tau\leqq t} \|\partial_t^{m-1}\nabla(\rho, \theta)\|$. However, the uniform estimate on $\int_0^t \|\nabla\partial_t^{m-1}(\rho, \theta)\|^2 \, d\tau$ is possible, which is crucial for us to close the *a priori* estimates.

Lemma 3.8. *It holds, for $m \geqq 3$, that*

$$\begin{aligned}
&\sup_{0\leqq\tau\leqq t} (\varepsilon\|\nabla\partial_t^{m-2}\mathrm{div} u(\tau)\|^2 + \kappa(\varepsilon)\|\partial_t^{m-2}\Delta\theta(\tau)\|^2) \\
&\quad + \int_0^t \|(\nabla\partial_t^{m-1}\theta, \nabla\partial_t^{m-1}\rho)(\tau)\|^2 \, d\tau \\
&\leqq C\{\mathcal{N}_m(0) + t P(\mathcal{N}_m(t))\}.
\end{aligned} \tag{3.121}$$

Proof. Multiplying $(3.22)_1$(with $\mathcal{Z}^\alpha = \partial_t^{m-1}$) by $\nabla\partial_t^{m-2}\mathrm{div} u$, one obtains, by using (2.4), that

$$\frac{1}{2}(2\mu+\lambda)\varepsilon\|\nabla\partial_t^{m-2}\mathrm{div}u(t)\|^2 - R\int_0^t\int \nabla\partial_t^{m-1}(\rho\theta)\cdot\nabla\partial_t^{m-2}\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau \leqq \frac{1}{2}(2\mu+\lambda)\varepsilon\|\nabla\partial_t^{m-2}\mathrm{div}u_0\|^2 + CtP(\mathcal{N}_m(t)), \tag{3.122}$$

where we have used the following facts:

$$\varepsilon\int_0^t\int \nabla\partial_t^{m-1}\mathrm{div}u\cdot\nabla\partial_t^{m-2}\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau = \frac{1}{2}\varepsilon\|\nabla\partial_t^{m-2}\mathrm{div}u(t)\|^2 - \frac{1}{2}\varepsilon\|\nabla\partial_t^{m-2}\mathrm{div}u_0\|^2$$

and

$$\begin{aligned}
&-\mu\varepsilon\int_0^t\int \nabla\times\partial_t^{m-1}\omega\cdot\nabla\partial_t^{m-2}\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\\
&= -\mu\varepsilon\int_0^t\int n\times\partial_t^{m-1}\omega\cdot\Pi(\nabla\partial_t^{m-2}\mathrm{div}u)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq C\varepsilon\int_0^t |\partial_t^{m-1}u|_{H^{\frac{1}{2}}}|\partial_t^{m-2}\mathrm{div}u|_{H^{\frac{1}{2}}}\,\mathrm{d}\tau\\
&\leqq C\varepsilon\int_0^t \|\partial_t^{m-1}u\|_{H^1}\|\partial_t^{m-2}\mathrm{div}u\|_{H^1}\,\mathrm{d}\tau \leqq CtP(\mathcal{N}_m(t)).
\end{aligned}$$

On the other hand, multiplying (3.74) (with $\mathcal{Z}^\alpha=\partial_t^{m-2}$) by $\frac{\nabla\partial_t^{m-1}\theta}{\theta}$ and using (2.4), one obtains immediately that

$$\begin{aligned}
&\int_0^t\int\frac{\rho}{\theta}|\nabla\partial_t^{m-1}\theta|^2\,\mathrm{d}x\,\mathrm{d}\tau + R\int_0^t\int\rho\nabla\partial_t^{m-1}\theta\cdot\nabla\partial_t^{m-2}\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad -\kappa(\varepsilon)\int_0^t\int\nabla\partial_t^{m-2}\Delta\theta\cdot\frac{\nabla\partial_t^{m-1}\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq \frac{1}{8}\int_0^t\int\frac{\rho}{\theta}|\nabla\partial_t^{m-1}\theta|^2\,\mathrm{d}x\,\mathrm{d}\tau + CP(\mathcal{N}_m(t))\int_0^t\varepsilon^2\|\nabla^2u\|^2_{\mathcal{H}^{m-2}}\,\mathrm{d}\tau\\
&\quad + CtP(\mathcal{N}_m(t)).
\end{aligned}\tag{3.123}$$

Combining (3.122) and (3.123) and using (3.111), one gets that

$$\begin{aligned}
&\frac{1}{2}(2\mu+\lambda)\varepsilon\|\nabla\partial_t^{m-2}\mathrm{div}u(t)\|^2 + \frac{7}{8}\int_0^t\int\frac{\rho}{\theta}|\nabla\partial_t^{m-1}\theta|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad -\kappa(\varepsilon)\int_0^t\int\nabla\partial_t^{m-2}\Delta\theta\cdot\frac{\nabla\partial_t^{m-1}\theta}{\theta}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad + R\int_0^t\int[\rho\nabla\partial_t^{m-1}\theta-\nabla\partial_t^{m-1}(\rho\theta)]\cdot\nabla\partial_t^{m-2}\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq \mathcal{N}_m(0)+CtP(\mathcal{N}_m(t)).
\end{aligned}\tag{3.124}$$

In order to estimate the terms on the LHS of (3.124), we first note that

$$\rho\nabla\partial_t^{m-1}\theta-\nabla\partial_t^{m-1}(\rho\theta) = -\theta\nabla\partial_t^{m-1}\rho-[\partial_t^{m-1},\rho]\nabla\theta-[\partial_t^{m-1},\theta]\nabla\rho, \tag{3.125}$$

and

$$\nabla\partial_t^{m-2}\mathrm{div}u = -\partial_t^{m-2}\nabla\left(\frac{\rho_t}{\rho}\right)-\sum_{i=1,2}\partial_t^{m-2}\nabla\left(\frac{u_i\cdot\partial_{y^i}\rho}{\rho}\right)-\partial_t^{m-2}\nabla\left(\frac{u\cdot N\partial_z\rho}{\rho}\right). \tag{3.126}$$

Then, using (2.4), (3.125) and (3.126), and after some tedious calculation, one obtains that

$$\begin{aligned}
&R\int_0^t\int[\rho\nabla\partial_t^{m-1}\theta-\nabla\partial_t^{m-1}(\rho\theta)]\cdot\nabla\partial_t^{m-2}\mathrm{div}u\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad\geqq\frac{7}{8}R\int_0^t\int\frac{\theta}{\rho}|\nabla\partial_t^{m-1}\rho|^2\,\mathrm{d}x\,\mathrm{d}\tau-\delta_1\int_0^t\|\nabla\partial_t^{m-2}\mathrm{div}u\|^2\,\mathrm{d}\tau-C_{\delta_1}tP(\mathcal{N}_m(t)).
\end{aligned} \tag{3.127}$$

It follows from the trace theorem, (3.103) and integrating by parts that

$$\begin{aligned}
&-\kappa(\varepsilon)\int_0^t\int\nabla\partial_t^{m-2}\Delta\theta\cdot\frac{\nabla\partial_t^{m-1}\theta}{\theta}\mathrm{d}x\mathrm{d}\tau\\
&\quad=\kappa(\varepsilon)\int_0^t\int\partial_t^{m-2}\Delta\theta\cdot\Big(\frac{\Delta\partial_t^{m-1}\theta}{\theta}-\frac{\nabla\theta\cdot\nabla\partial_t^{m-1}\theta}{\theta^2}\Big)\mathrm{d}x\mathrm{d}\tau\\
&\qquad-\nu\kappa(\varepsilon)\int_0^t\int_{\partial\Omega}\partial_t^{m-2}\Delta\theta\cdot\frac{\partial_t^{m-1}\theta}{\theta}\mathrm{d}\sigma\mathrm{d}\tau\\
&\quad\geqq\kappa(\varepsilon)\int\frac{|\partial_t^{m-2}\Delta\theta|^2}{2\theta}\mathrm{d}x-\kappa(\varepsilon)\int\frac{|\partial_t^{m-2}\Delta\theta_0|^2}{2\theta_0}\mathrm{d}x\\
&\qquad-CtP(\mathcal{N}_m(t))-\frac{1}{16}\int_0^t\int\frac{\rho}{\theta}|\nabla\partial_t^{m-1}\theta|^2\mathrm{d}x\mathrm{d}\tau\\
&\qquad-C\kappa(\varepsilon)\int_0^t\Big(\|\nabla\partial_t^{m-2}\Delta\theta\|^{\frac12}\|\Delta\partial_t^{m-2}\theta\|^{\frac12}+\|\Delta\partial_t^{m-2}\theta\|\Big)\\
&\qquad\qquad\times\Big(\|\nabla\partial_t^{m-1}\theta\|^{\frac12}\|\partial_t^{m-1}\theta\|^{\frac12}+\|\partial_t^{m-1}\theta\|\Big)\mathrm{d}\tau\\
&\quad\geqq\kappa(\varepsilon)\int\frac{|\partial_t^{m-2}\Delta\theta|^2}{2\theta}\mathrm{d}x-\kappa(\varepsilon)\int\frac{|\partial_t^{m-2}\Delta\theta_0|^2}{2\theta_0}\mathrm{d}x-C_\delta tP(\mathcal{N}_m(t))\\
&\qquad-\frac{1}{8}\int_0^t\int\frac{\rho}{\theta}|\nabla\partial_t^{m-1}\theta|^2\mathrm{d}x\mathrm{d}\tau-\delta\kappa(\varepsilon)^2\int_0^t\|\nabla\partial_t^{m-2}\Delta\theta\|^2\mathrm{d}\tau.
\end{aligned} \tag{3.128}$$

Substituting (3.127) and (3.128) into (3.124), one obtains immediately that

$$\begin{aligned}
&\varepsilon\|\nabla\partial_t^{m-2}\mathrm{div}u(t)\|^2+\kappa(\varepsilon)\|\partial_t^{m-2}\Delta\theta(t)\|^2+\int_0^t\|\nabla\partial_t^{m-1}(\theta,\rho)\|^2\,\mathrm{d}\tau\\
&\quad\leqq C\Big\{\mathcal{N}_m(0)+\delta\kappa(\varepsilon)^2\int_0^t\|\nabla\partial_t^{m-2}\Delta\theta\|^2\,\mathrm{d}\tau+C\delta_1\int_0^t\|\nabla\partial_t^{m-2}\mathrm{div}u\|^2\,\mathrm{d}\tau\\
&\qquad+C_{\delta_1,\delta}tP(\mathcal{N}_m(t))\Big\}
\end{aligned}$$

$$\leqq C\Big\{\mathcal{N}_m(0)+\delta_1\int_0^t\|\nabla\partial_t^{m-1}\rho\|^2\,\mathrm{d}\tau+\delta P(\mathcal{N}_m(t))\int_0^t\|\nabla\partial_t^{m-1}(\rho,\theta)\|^2\,\mathrm{d}\tau$$
$$+C_{\delta_1,\delta}tP(\mathcal{N}_m(t))\Big\}, \tag{3.129}$$

where we have used (3.110) and (3.112) in the last inequality. Setting $\delta=\delta_1 P(\mathcal{N}_m(t))^{-1}$ and taking δ_1 suitably small, one proves (3.121). Thus, the proof of Lemma 3.8 is completed. □

Since the estimate in Lemma 3.6 is not enough for us to get the uniform estimate for $\nabla\partial_t^{m-1}u$, we need some new estimate on $\|\partial_t^{m-1}\mathrm{div}u\|$. Fortunately, we have the following subtle control regarding $\|\partial_t^{m-1}\mathrm{div}u\|$:

Lemma 3.9. *Let us define*

$$\Lambda_m(t)\triangleq\|(\rho,u,\theta)\|_{\mathcal{H}^m}^2+\sum_{|\beta|\leqq m-2}\|\mathcal{Z}^\beta\nabla(\rho,\theta)\|_{H^1_{co}}^2+\sum_{|\beta|\leqq m-2}\|\mathcal{Z}^\beta\nabla u\|_{H^1_{co}}^2. \tag{3.130}$$

Then, for every $m\geqq 3$, it holds that

$$\|\partial_t^{m-1}\mathrm{div}u(t)\|^2\leqq C_2\{P(\Lambda_m(t))+P(Q(t))\}. \tag{3.131}$$

Proof. Since the proof is the same as the one in [22], we omit the details for the side of brevity. □

Remark 3.10. We point out that it does not contain the terms $\|\nabla\partial_t^{m-1}(\rho,u,\theta)\|$ in the right hand side of (3.131). The estimation of $Q(t)$ will be given in Section 3.4 below. This key observation allows us to obtain the uniform estimates for $\|\nabla\partial_t^{m-1}u\|$.

3.3. Normal Derivatives Estimates

Similar to the corresponding part in [22], in order to estimate $\|\nabla u\|_{\mathcal{H}^{m-1}}$, it remains to estimate $\|\chi\partial_n u\|_{\mathcal{H}^{m-1}}$, where χ is supported compactly in one of the Ω_j and with value one in a neighborhood of the boundary. Indeed, it follows from the definition of the norm that $\|\chi\partial_{y^i}u\|_{\mathcal{H}^{m-1}}\leqq C\|u\|_{\mathcal{H}^m}$ for $i=1,2$. Thus, it suffices to estimate $\|\chi\partial_n u\|_{\mathcal{H}^{m-1}}$.

Note that

$$\mathrm{div}u=\partial_n u\cdot n+(\Pi\partial_{y^1}u)_1+(\Pi\partial_{y^2}u)_2 \tag{3.132}$$

and

$$\partial_n u=[\partial_n u\cdot n]n+\Pi(\partial_n u), \tag{3.133}$$

where Π is defined in (1.11). Thus it follows from (3.132) and (3.133) that

$$\begin{aligned}\|\chi\partial_n u\|_{\mathcal{H}^{m-1}}&\leqq\|(\chi\partial_n u\cdot n,\chi\Pi(\partial_n u))\|_{\mathcal{H}^{m-1}}\\&\leqq C_m\{\|\chi\mathrm{div}u\|_{\mathcal{H}^{m-1}}+\|\chi\Pi(\partial_n u)\|_{\mathcal{H}^{m-1}}+\|u\|_{\mathcal{H}^m}\}.\end{aligned} \tag{3.134}$$

Thus it suffices to estimate $\|\chi\Pi(\partial_n u)\|_{\mathcal{H}^{m-1}}$, since $\|u\|_{\mathcal{H}^m}$ and $\|\chi \operatorname{div} u\|_{\mathcal{H}^{m-1}}$ have been estimated in Sections 3.1 and 3.2, respectively. We extend the smooth symmetric matrix A in (1.3) to be

$$A(y,z) = A(y).$$

Define

$$\eta \triangleq \chi(\omega \times n + \Pi(Bu)) = \chi(\Pi(\omega \times n) + \Pi(Bu)). \tag{3.135}$$

The η defined here, which enables one to avoid to estimate $\nabla^2 p$, is slightly different from the one in [14]. Then, in view of the Navier-slip boundary condition (1.3), η satisfies:

$$\eta|_{\partial\Omega} = 0. \tag{3.136}$$

Since $\omega \times n = (\nabla u - (\nabla u)^t) \cdot n$, η can be rewritten as

$$\eta = \chi\{\Pi(\partial_n u) - \Pi(\nabla(u \cdot n)) + \Pi((\nabla n)^t \cdot u) + \Pi(Bu)\}, \tag{3.137}$$

which immediately yields that

$$\|\chi\Pi(\partial_n u)\|_{\mathcal{H}^{m-1}} \leqq C_{m+1}(\|\eta\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}). \tag{3.138}$$

Hence, it remains to estimate $\|\eta\|_{\mathcal{H}^{m-1}}$. In fact, one can get the following conormal estimates for η:

Lemma 3.11. *For every $m \geqq 3$, it holds that*

$$\begin{aligned}
&\sup_{0\leqq\tau\leqq t} \|\eta(\tau)\|^2_{\mathcal{H}^{m-1}} + \varepsilon\int_0^t \|\nabla\eta\|^2_{\mathcal{H}^{m-1}}\,\mathrm{d}\tau \\
&\quad\leqq CC_{m+2}\left\{P(\mathcal{N}_m(0)) + \delta\varepsilon^2\int_0^t \|\nabla^2 u\|^2_{\mathcal{H}^{m-1}}\,\mathrm{d}\tau + C_\delta t P(\mathcal{N}_m(t))\right\}.
\end{aligned} \tag{3.139}$$

Proof. Notice that

$$\nabla \times ((u\cdot\nabla)u) = (u\cdot\nabla)\omega - (\omega\cdot\nabla)u + \operatorname{div} u \cdot \omega,$$

so ω solves the following vorticity equation:

$$\rho\omega_t + \rho(u\cdot\nabla)\omega = \mu\varepsilon\Delta\omega + F_1, \tag{3.140}$$

with

$$F_1 \triangleq -\nabla\rho \times u_t - \nabla\rho \times (u\cdot\nabla)u + \rho(\omega\cdot\nabla)u - \rho \operatorname{div} u\omega. \tag{3.141}$$

Consequently, we obtain that η solves the equation

$$\begin{aligned}
&\rho\eta_t + \rho u_1\partial_{y^1}\eta + \rho u_2\partial_{y^2}\eta + \rho u\cdot N\partial_z\eta - \mu\varepsilon\Delta\eta \\
&\quad= \chi[F_1 \times n + \Pi(BF_2)] + \chi(F_3 + F_4) + F_5 + \varepsilon\Delta(\Pi B)\cdot u \triangleq F,
\end{aligned} \tag{3.142}$$

where

$$\begin{cases} F_2 = (2\mu+\lambda)\varepsilon\nabla \mathrm{div} u - \nabla p, \\ F_3 = -2\mu \sum_{j=1}^3 \varepsilon \partial_j \omega \times \partial_j n - \mu\varepsilon\omega\times\Delta n + \sum_{i=1}^2 \rho u_i \omega \times \partial_{y^i} n + \rho u\cdot N\omega\times\partial_z n \\ \qquad - \left[\sum_{i=1}^2 \rho u_i \Pi(\partial_{y^i} B\cdot u) + \rho u\cdot N\Pi(\partial_z B\cdot u)\right] + \mu\sum_{j=1}^3 \varepsilon\Pi(\partial_j B\partial_j u), \\ F_4 = -\sum_{i=1}^2 \rho u_i(\partial_{y^i}\Pi)(Bu) - \rho u\cdot N(\partial_z\Pi)(Bu), \\ F_5 = \sum_{i=1}^2 \rho u_i \partial_{y^i}\chi\cdot(\omega\times n + \Pi(Bu)) + \rho u\cdot N\partial_z\chi\cdot(\omega\times n+\Pi(Bu)) \\ \qquad -2\mu\sum_{j=1}^3 \varepsilon\partial_j\chi\partial_j(\omega\times n+\Pi(Bu)) + \varepsilon\mu\Delta\chi\cdot(\omega\times n+\Pi(Bu)). \end{cases} \tag{3.143}$$

Let us begin with the proof of the L^2-energy estimate. Multiplying (3.142) by η yields that

$$\sup_{0\leqq\tau\leqq t}\int \rho|\eta|^2\,\mathrm{d}x + 2\varepsilon\int_0^t \|\nabla\eta\|^2\,\mathrm{d}\tau \leqq \int \rho_0|\eta_0|^2\,\mathrm{d}x + \int_0^t\int F\eta\,\mathrm{d}x\,\mathrm{d}\tau. \tag{3.144}$$

To estimate the terms on the RHS, note that

$$\int_0^t \|\chi\Pi(F_1\times n)\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau \leqq C_m P(\mathcal{N}_m(t))\int_0^t \|\nabla\partial_t^{m-1}(\rho,\theta)(\tau)\|^2\,\mathrm{d}\tau + CP(\mathcal{N}_m(t)), \tag{3.145}$$

$$\int_0^t \|\chi\Pi(BF_2)\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau \leqq C_{m+1}\left\{\int_0^t P(\mathcal{N}_m) + \|\nabla\partial_t^{m-1}(\rho,\theta)\|^2\,\mathrm{d}\tau + \varepsilon^2\int_0^t \|\chi\nabla\mathrm{div}u\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau\right\}, \tag{3.146}$$

$$\int_0^t \|\chi F_3\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau \leqq C_{m+2}\left\{\varepsilon^2\int_0^t \|\chi\nabla^2 u\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau + tP(\mathcal{N}_m(t))\right\}, \tag{3.147}$$

$$\int_0^t \|\chi F_4\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau \leqq C_{m+2}tP(\mathcal{N}_m(t)). \tag{3.148}$$

Since all the terms in F_5 are supported away from the boundary, one can estimate all the derivatives by the $\|\cdot\|_{\mathcal{H}^m}$ norms. Therefore, it is easy to obtain

$$\int_0^t \|F_5\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau \leqq C_{m+1}\left\{\varepsilon^2\int_0^t \|\chi\nabla^2 u\|_{\mathcal{H}^{m-1}}^2\,\mathrm{d}\tau + tP(\mathcal{N}_m(t))\right\}. \tag{3.149}$$

Finally, by integrating by parts, it is easy to obtain, for $|\alpha|\leqq m-1$, that

$$\int_0^t\int \varepsilon\mathcal{Z}^\alpha(\Delta(\Pi B)\cdot u)\cdot\mathcal{Z}^\alpha\eta\,\mathrm{d}\tau \leqq \delta\varepsilon\int_0^t \|\nabla\mathcal{Z}\eta\|^2\,\mathrm{d}\tau + C_{m+2}tP(\mathcal{N}_m(t)). \tag{3.150}$$

Consequently, substituting these estimates into (3.144) and using the Cauchy inequality, one has that

$$\sup_{0\leqq\tau\leqq t}\int\rho|\eta|^2\,\mathrm{d}x+2\varepsilon\int_0^t\|\nabla\eta\|^2\,\mathrm{d}\tau$$
$$\leqq\int\rho_0|\eta_0|^2\,\mathrm{d}x+C_{m+2}\bigg\{\delta\int_0^t\|\nabla\partial_t^{m-1}(\rho,\theta)(\tau)\|^2\,\mathrm{d}\tau$$
$$+\delta\varepsilon^2\int_0^t\|\chi\nabla^2u\|^2_{\mathcal{H}^{m-1}}\,\mathrm{d}\tau+C_\delta tP(\mathcal{N}_m(t))\bigg\}. \tag{3.151}$$

Thus, we proved (3.139) for $k=0$ by using Lemma 3.8.

To prove the general case, let us assume that (3.139) is proved for $k\leqq m-2$. We apply $\mathcal{Z}^\alpha$ to (3.142) for $|\alpha|=m-1$ to obtain that

$$\rho\mathcal{Z}^\alpha\eta+\rho(u\cdot\nabla)\mathcal{Z}^\alpha\eta-\mu\varepsilon\mathcal{Z}^\alpha\Delta\eta=\mathcal{Z}^\alpha F+\tilde{\mathcal{C}}_1^\alpha+\tilde{\mathcal{C}}_2^\alpha, \tag{3.152}$$

where

$$\begin{cases}\tilde{\mathcal{C}}_1^\alpha=-[\mathcal{Z}^\alpha,\rho]\eta_t=\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta\rho\mathcal{Z}^\gamma\eta_t,\\ \tilde{\mathcal{C}}_2^\alpha=-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta(\rho u_i)\mathcal{Z}^\gamma\partial_{y^i}\eta\\ \qquad-\sum_{|\beta|\geqq1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta(\rho u\cdot N)\mathcal{Z}^\gamma\partial_z\eta\\ \qquad-\rho(u\cdot N)\sum_{|\beta|\leqq m-2}C_\beta(z)\partial_z\mathcal{Z}^\beta\eta,\end{cases} \tag{3.153}$$

where $C_\beta(z)$ is bounded smooth function of z. Multiplying (3.152) by $\mathcal{Z}^\alpha\eta$ and using (3.145)–(3.149), one obtains that

$$\sup_{0\leqq\tau\leqq t}\int\frac{1}{2}\rho|\mathcal{Z}^\alpha\eta|^2\,\mathrm{d}x$$
$$\leqq\mu\varepsilon\int_0^t\int\mathcal{Z}^\alpha\Delta\eta\mathcal{Z}^\alpha\eta\,\mathrm{d}x\,\mathrm{d}\tau+\int\frac{1}{2}\rho_0|\mathcal{Z}^\alpha\eta_0|^2\,\mathrm{d}x$$
$$+\int_0^t\int(\tilde{\mathcal{C}}_1^\alpha+\tilde{\mathcal{C}}_2^\alpha)\mathcal{Z}^\alpha\eta\,\mathrm{d}x\,\mathrm{d}\tau+C_{m+1}\bigg\{\delta\varepsilon^2\int_0^t\|\chi\nabla^2u\|^2_{\mathcal{H}^{m-1}}\,\mathrm{d}\tau$$
$$+\delta\int_0^t\|\nabla\partial_t^{m-1}(\rho,\theta)\|^2\,\mathrm{d}\tau+C_\delta tP(\mathcal{N}_m(t))\bigg\}. \tag{3.154}$$

By the same argument as Lemma 3.12 of [22], one gets that

$$\varepsilon\int_0^t\int\mathcal{Z}^\alpha\Delta\eta\mathcal{Z}^\alpha\eta\,\mathrm{d}x\,\mathrm{d}\tau\leqq-\frac{3}{4}\mu\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha\eta\|^2\,\mathrm{d}\tau+C\varepsilon\int_0^t\|\nabla\eta\|^2_{\mathcal{H}^{m-2}}\,\mathrm{d}\tau$$
$$+C_{m+2}tP(\mathcal{N}_m(t)) \tag{3.155}$$

and

$$\int_0^t\|(\tilde{\mathcal{C}}_1^\alpha,\tilde{\mathcal{C}}_2^\alpha)\|^2\,\mathrm{d}\tau\leqq C_{m+2}[1+P(Q(t))]\int_0^t P(\mathcal{N}_m(\tau))\,\mathrm{d}\tau\leqq C_{m+2}tP(\mathcal{N}_m(t)). \tag{3.156}$$

Substituting (3.155) and (3.156) into (3.154) and using Lemma 3.8, one obtains that

$$\begin{aligned}\sup_{0\leqq\tau\leqq t}&\int\rho|\mathcal{Z}^\alpha\eta|^2\,\mathrm{d}x+\mu\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha\eta\|^2\,\mathrm{d}\tau\\&\leqq C_{m+2}\Big\{\int\rho_0|\mathcal{Z}^\alpha\eta_0|^2\,\mathrm{d}x+\varepsilon\int_0^t\|\nabla\eta\|^2_{\mathcal{H}^{m-2}}\,\mathrm{d}\tau\\&\quad+\delta\varepsilon^2\int_0^t\|\nabla^2u\|^2_{\mathcal{H}^{m-1}}\,\mathrm{d}\tau+C_\delta tP(\mathcal{N}_m(t))\Big\}.\end{aligned}\tag{3.157}$$

By using the induction assumption, one can eliminate the term $\varepsilon\int_0^t\|\nabla\eta\|^2_{\mathcal{H}^{m-2}}\,\mathrm{d}\tau$. Therefore, the proof Lemma 3.11 is completed. □

From (3.132), (3.133) and (3.138), it holds that

$$\sum_{|\beta|\leqq m-2}\|\mathcal{Z}^\beta\nabla u\|^2_{H^1_{co}}\leqq C_{m+1}\Big(\|u\|^2_{\mathcal{H}^m}+\|\eta\|^2_{\mathcal{H}^{m-1}}+\sum_{k=0}^{m-2}\|\partial_t^k\mathrm{div}u(t)\|^2_{m-1-k}\Big),\tag{3.158}$$

$$\begin{aligned}\int_0^t\|\nabla^2u\|^2_{\mathcal{H}^{m-1}}\,\mathrm{d}\tau\leqq C_{m+2}\int_0^t(&\|\nabla u\|^2_{\mathcal{H}^m}+\|\nabla\eta\|^2_{\mathcal{H}^{m-1}}+\|\nabla\mathrm{div}u\|^2_{\mathcal{H}^{m-1}}\\&+P(\mathcal{N}_m))\,\mathrm{d}\tau,\end{aligned}\tag{3.159}$$

$$\begin{aligned}\sum_{k=0}^{m-2}\int_0^t\|\nabla^2\partial_t^ku\|^2_{m-1-k}\,\mathrm{d}\tau\leqq C_{m+2}\Big\{&\int_0^t\|\nabla u\|^2_{\mathcal{H}^m}+\|\nabla\eta\|^2_{\mathcal{H}^{m-1}}\,\mathrm{d}\tau\\&+\sum_{k=0}^{m-2}\int_0^t\|\partial_t^k\nabla\mathrm{div}u\|^2_{m-1-k}\,\mathrm{d}\tau+tP(\mathcal{N}_m)\Big\},\end{aligned}\tag{3.160}$$

$$\varepsilon\int_0^t\|\nabla^2\mathcal{Z}^{m-2}u\|^2\,\mathrm{d}\tau\leqq C_{m+1}\varepsilon\int_0^t\|\nabla\eta\|^2_{\mathcal{H}^{m-2}}\,\mathrm{d}\tau+C_{m+1}tP(\mathcal{N}_m),\tag{3.161}$$

where (3.110) is used in the estimate of (3.161). Setting $\delta=\delta_1P(\mathcal{N}_m(t))^{-1}$ and taking δ_1 suitably small, then it follows from (3.21), (3.62), (3.102), (3.121), (3.139), (3.158)–(3.161) and Lemma 3.7 that

$$\begin{aligned}\sup_{0\leqq\tau\leqq t}&\{\Lambda_m(\tau)+\|\eta(\tau)\|^2_{\mathcal{H}^{m-1}}+\varepsilon\|\partial_t^{m-1}(\nabla\rho,\mathrm{div}u)(\tau)\|^2\\&+\varepsilon\|\nabla\partial_t^{m-2}\mathrm{div}u(\tau)\|^2+\kappa(\varepsilon)\|\partial_t^{m-2}\Delta\theta(\tau)\|^2\}\\&+\int_0^t\|\nabla\partial_t^{m-1}(\rho,\theta)(\tau)\|^2\,\mathrm{d}\tau+\int_0^t\varepsilon\|\nabla u(\tau)\|^2_{\mathcal{H}^m}+\kappa(\varepsilon)\|\nabla\theta(\tau)\|^2_{\mathcal{H}^m}\,\mathrm{d}\tau\\&+\varepsilon\int_0^t\sum_{k=0}^{m-2}\|\partial_t^k\nabla^2u(\tau)\|^2_{m-1-k}\,\mathrm{d}\tau+\kappa(\varepsilon)\int_0^t\sum_{k=0}^{m-2}\|\partial_t^k\Delta\theta(\tau)\|^2_{m-1-k}\,\mathrm{d}\tau\end{aligned}$$

$$+\int_0^t \varepsilon^2\|\nabla^2\partial_t^{m-1}u(\tau)\|^2+\kappa(\varepsilon)^2\|\partial_t^{m-1}\Delta\theta(\tau)\|^2\,\mathrm{d}\tau$$
$$\leqq CC_{m+2}\{\mathcal{N}_m(0)+tP(\mathcal{N}_m(t))\},\quad \text{for } m\geqq 3. \tag{3.162}$$

3.4. L^∞-Estimates

This section is devoted to estimating L^∞-norm parts contained in (1.15). However, it is not easy to get such estimates because the equations of ρ, u, θ are strongly coupled and the viscosity and heat conductivity are not at the same order. Actually, if one estimates $\|\nabla\rho\|_{\mathcal{H}^{1,\infty}}$ directly, then the high order term $\int_0^t\|\nabla\theta\|_{\mathcal{H}^{2,\infty}}\,\mathrm{d}\tau$ must appear, and it is very hard to control such a term. To overcome this difficulty, we try to estimate the L^∞-norm of $\nabla(\rho\theta)$,∇u and $\nabla\theta$. Firstly, we have the following useful Lemma:

Lemma 3.12. *For every $|\alpha|\geqq 0$, it holds that*

$$\|\mathcal{Z}^\alpha(\rho,\theta,u)\|_{L^\infty}^2\leqq CP(\Lambda_m(t)),\quad \textit{for } m\geqq 2+|\alpha|, \tag{3.163}$$

$$\|\nabla\rho\|_{\mathcal{H}^{1,\infty}}^2\leqq C_3(\|\nabla(\rho\theta)\|_{\mathcal{H}^{1,\infty}}^2+\|\nabla\theta\|_{\mathcal{H}^{1,\infty}}^2)\cdot P(\Lambda_m(t)),\quad \textit{for } m\geqq 5, \tag{3.164}$$

$$Q(t)\leqq C_3\{\|\nabla u\|_{\mathcal{H}^{1,\infty}}^2+\varepsilon\|\partial_z^2u\|_{L^\infty}^2+\|\nabla(\rho\theta)\|_{\mathcal{H}^{1,\infty}}^4$$
$$+\|\nabla\theta\|_{\mathcal{H}^{1,\infty}}^4+P(\Lambda_m(t))\}\quad \textit{for } m\geqq 5. \tag{3.165}$$

$$\|\mathrm{div}u\|_{\mathcal{H}^{1,\infty}}^2\leqq C_3P(\Lambda_m(t))[1+\|\nabla(\rho\theta)\|_{\mathcal{H}^{1,\infty}}^2+\|\nabla\theta\|_{\mathcal{H}^{1,\infty}}^2],\quad \textit{for } m\geqq 4, \tag{3.166}$$

$$\|\nabla\mathrm{div}u\|_{L^\infty}^2\leqq\begin{cases}C_3P(Q(t)),\\ C_3[P(\|\nabla\rho\|_{\mathcal{H}^{1,\infty}}^2)+P(\|\nabla u\|_{\mathcal{H}^{1,\infty}}^2)+P(\Lambda_m)],\end{cases}\quad \textit{for } m\geqq 3, \tag{3.167}$$

$$\|\nabla\mathrm{div}u\|_{\mathcal{H}^{1,\infty}}^2\leqq C\{\|\nabla(\rho\theta)\|_{\mathcal{H}^{2,\infty}}^2+\|\nabla\theta\|_{\mathcal{H}^{2,\infty}}^2$$
$$+P(\Lambda_m(t))+P(Q(t))\},\quad \textit{for } m\geqq 6. \tag{3.168}$$

Proof. The proof of (3.163) is an immediate consequence of (2.5), we omit the details here. Notice that

$$\nabla\rho=\frac{\nabla(\rho\theta)}{\theta}-\frac{\rho}{\theta}\nabla\theta, \tag{3.169}$$

which immediately implies (3.164). Using (3.163) and (3.164), we immediately obtain (3.165).

Using $(1.4)_1$, (3.163), (3.164), (3.41), and the facts that

$$\mathrm{div}u=\theta^{-1}[\theta_t+(u\cdot\nabla)\theta]-p^{-1}[p_t+(u\cdot\nabla)p], \tag{3.170}$$
$$\nabla\mathrm{div}u=\theta^{-1}[\nabla\theta_t+(u\cdot\nabla)\nabla\theta]-p^{-1}[\nabla p_t+(u\cdot\nabla)\nabla p]-p^{-1}\nabla p\,\mathrm{div}u$$
$$+Rp^{-1}\nabla\rho[\theta_t+(u\cdot\nabla)\theta]+\theta^{-1}\nabla u\cdot\nabla\theta-p^{-1}\nabla u\cdot\nabla p, \tag{3.171}$$

it is easy to prove (3.166)–(3.168). For simplicity, we omit the details here. Therefore, we complete the proof of Lemma 3.12. □

Remark 3.13. Lemma 3.12 implies that one needs only to estimate $\|\nabla(\rho\theta)\|_{\mathcal{H}^{1,\infty}}^2$, $\varepsilon\|\nabla(\rho\theta)\|_{\mathcal{H}^{2,\infty}}^2$, $\|\nabla\theta\|_{\mathcal{H}^{1,\infty}}^2$, $\|\nabla u\|_{\mathcal{H}^{1,\infty}}^2$ and $\varepsilon\|\partial_z^2 u\|_{L^\infty}^2$. Indeed, it is crucial to estimate $\|\nabla(\rho\theta)\|_{\mathcal{H}^{1,\infty}}$, but not $\|\nabla\rho\|_{\mathcal{H}^{1,\infty}}$. We also point out that the condition (1.20) will be used repeatedly to control the possible interaction between the viscous and the thermal boundary layers in the following analysis.

Uniform Estimate for $\|\nabla p\|_{\mathcal{H}^{1,\infty}}^2$ and $\varepsilon\|\nabla p\|_{\mathcal{H}^{2,\infty}}^2$

Firstly, we have the following lemma which will be used to prove (3.174) below:

Lemma 3.14. *Assume that (1.20) holds, then one has, for $m \geqq 6$, that*

$$\varepsilon\int_0^t \|\partial_{zz}u\|_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau \leqq C_{m+2}\{\Lambda_m(0)+tP(\mathcal{N}_m(t))\}. \tag{3.172}$$

Proof. It follows from the momentum Equations $(1.1)_2$, (2.5), (3.117), (3.162) and (3.168) that

$$\begin{aligned}
\varepsilon\int_0^t \|\partial_{zz}u\|_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau &\leqq C_3\int_0^t P(\mathcal{N}_m)+\varepsilon\|\nabla \mathrm{div} u\|_{\mathcal{H}^{1,\infty}}+\varepsilon\|\nabla u\|_{\mathcal{H}^{2,\infty}}\,\mathrm{d}\tau\\
&\leqq C_3tP(\mathcal{N}_m)+C_3\varepsilon\int_0^t\|\nabla(\theta,u)\|_{\mathcal{H}^{2,\infty}}\,\mathrm{d}\tau \leqq C_3tP(\mathcal{N}_m)\\
&\quad + C_3\varepsilon\int_0^t \|\nabla^2\mathcal{Z}^2(\theta,u)\|_{H^1_{co}}^{\frac12}\|\nabla\mathcal{Z}^2(\theta,u)\|_{H^2_{co}}^{\frac12}\,\mathrm{d}\tau\\
&\leqq C_3tP(\mathcal{N}_m)+C_3\varepsilon^4\int_0^t\|(\nabla^2u,\Delta\theta)\|_{\mathcal{H}^3}^2\,\mathrm{d}\tau\\
&\leqq CC_{m+2}\{\mathcal{N}_m(0)+tP(\mathcal{N}_m(t))\},
\end{aligned} \tag{3.173}$$

where we have used (1.20) in the last inequality. Thus, the proof of Lemma 3.14 is completed. □

Lemma 3.15. *Assume that (1.20) holds, then one has, for $m \geqq 6$, that*

$$\|\nabla p\|_{\mathcal{H}^{1,\infty}}^2+\varepsilon\|\nabla p\|_{\mathcal{H}^{2,\infty}}^2+\int_0^t\|\nabla p\|_{\mathcal{H}^{2,\infty}}^2\,\mathrm{d}\tau \leqq C_{m+2}\{P(\mathcal{N}_m(0))+tP(\mathcal{N}_m(t))\}. \tag{3.174}$$

Proof. Note that

$$\begin{aligned}
&\|\nabla p\|_{\mathcal{H}^{1,\infty}}^2+\varepsilon\|\nabla p\|_{\mathcal{H}^{2,\infty}}^2+\int_0^t\|\nabla p\|_{\mathcal{H}^{2,\infty}}^2\,\mathrm{d}\tau\\
&\quad\leqq CC_{m+2}\left\{P(\Lambda_m(t))+tP(\mathcal{N}_m(t))+\varepsilon\|\partial_n p\|_{\mathcal{H}^{2,\infty}}^2+\int_0^t\|\partial_n p\|_{\mathcal{H}^{2,\infty}}^2\,\mathrm{d}\tau\right\}.
\end{aligned} \tag{3.175}$$

Therefore, one needs only to control $\varepsilon\|\partial_n p\|_{\mathcal{H}^{2,\infty}}^2+\int_0^t\|\partial_n p\|_{\mathcal{H}^{2,\infty}}^2\,\mathrm{d}\tau$. Substituting (3.41) into $(3.7)_2$, one can obtain that

$$
\begin{aligned}
(2\mu+\lambda)\varepsilon[\nabla\rho_t+(u\cdot\nabla)\nabla\rho]+\rho\nabla p=-\rho^2\dot{u}-\mu\varepsilon\rho\nabla\times\omega\\
-(2\mu+\lambda)\varepsilon\rho\left[\nabla\left(\frac{1}{\rho}\right)\rho_t+\nabla\left(\frac{u}{\rho}\right)\cdot\nabla\rho\right],
\end{aligned}
\tag{3.176}
$$

where $\dot{u}=u_t+(u\cdot\nabla)u$. It follows from (3.169) that

$$
\nabla\rho_t=\frac{1}{\theta}\nabla(\rho\theta)_t-\frac{1}{\theta}\left(\frac{1}{\theta}\theta_t\nabla(\rho\theta)+\rho_t\nabla\theta+\rho\nabla\theta_t-\frac{\rho}{\theta}\nabla\theta\theta_t\right)
\tag{3.177}
$$

and

$$
\begin{aligned}
(u\cdot\nabla)\nabla\rho=\frac{1}{\theta}(u\cdot\nabla)\nabla(\rho\theta)-\frac{1}{\theta}\Bigg(\nabla(\rho\theta)\left(\frac{u}{\theta}\cdot\nabla\right)\theta\\
+\nabla\theta(u\cdot\nabla)\rho+\rho(u\cdot\nabla)\nabla\theta-\frac{\rho}{\theta}\nabla\theta(u\cdot\nabla)\theta\Bigg).
\end{aligned}
\tag{3.178}
$$

Substituting (3.177) and (3.178) into (3.176), one can obtain that

$$
\begin{aligned}
&(2\mu+\lambda)\varepsilon[\nabla p_t+(u\cdot\nabla)\nabla p]+p\nabla p\\
&=-p\rho\dot{u}-\mu\varepsilon p\nabla\times\omega-(2\mu+\lambda)\varepsilon\Bigg\{p\left[\nabla\left(\frac{1}{\rho}\right)\rho_t+\nabla\left(\frac{u}{\rho}\right)\cdot\nabla\rho\right]\\
&\quad-\left(\frac{1}{\theta}\theta_t\nabla p+R\rho_t\nabla\theta+R\rho\nabla\theta_t-R\frac{\rho}{\theta}\nabla\theta\theta_t\right)\\
&\quad-\left(\nabla p\left(\frac{u}{\theta}\cdot\nabla\right)\theta+R\nabla\theta(u\cdot\nabla)\rho+R\rho(u\cdot\nabla)\nabla\theta-R\frac{\rho}{\theta}\nabla\theta(u\cdot\nabla)\theta\right)\Bigg\}\\
&\triangleq p\rho\dot{u}-\mu\varepsilon p\nabla\times\omega+I.
\end{aligned}
\tag{3.179}
$$

Then it follows from (3.179) that

$$
\begin{aligned}
&(2\mu+\lambda)\varepsilon[\partial_n p_t+(u\cdot\nabla)\partial_n p]+p\partial_n p\\
&\quad=-\rho p\dot{u}\cdot n-\mu\varepsilon p(n\cdot\nabla\times\omega)+I\cdot n+(2\mu+\lambda)\varepsilon\nabla p\cdot(u\cdot\nabla)n\triangleq J.
\end{aligned}
\tag{3.180}
$$

Define

$$
h\triangleq\mathcal{Z}^\alpha\partial_n p,
\tag{3.181}
$$

then, applying $\mathcal{Z}^\alpha$ with $|\alpha|\leqq 2$, one obtains that

$$
\begin{aligned}
&(2\mu+\lambda)\varepsilon[h_t+(u\cdot\nabla)h]+ph\\
&\quad=\mathcal{Z}^\alpha J-(2\mu+\lambda)\varepsilon[\mathcal{Z}^\alpha,u\cdot\nabla]\partial_n p-[\mathcal{Z}^\alpha,p]\partial_n p\triangleq K.
\end{aligned}
\tag{3.182}
$$

It is convenient to consider the above equation in the Lagrangian coordinates:

$$
\tilde{h}(t,\xi)=h(t,X(t,\xi)),\quad \tilde{p}(t,\xi)=p(t,X(t,\xi)),\quad \tilde{K}(t,\xi)=K(t,X(t,\xi)),
\tag{3.183}
$$

where

$$\begin{cases} \frac{\mathrm{d}X(t,\xi)}{\mathrm{d}t} = u(t, X(t,\xi)), \\ X(0,\xi) = \xi \in \Omega. \end{cases}$$

Then (3.182) is rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{h} + \frac{\tilde{p}}{(2\mu+\lambda)\varepsilon}\tilde{h} = \frac{1}{(2\mu+\lambda)\varepsilon}\tilde{K}, \tag{3.184}$$

which immediately yields the following solution:

$$\tilde{h}(t,\xi) = \tilde{h}(0,\xi)\exp\left(-\int_0^t \frac{\tilde{p}(\tau,\xi)}{(2\mu+\lambda)\varepsilon}\,\mathrm{d}\tau\right) + \frac{1}{(2\mu+\lambda)\varepsilon}\int_0^t \tilde{K}(\tau,\xi)\exp\left(-\int_\tau^t \frac{\tilde{p}(s,\xi)}{(2\mu+\lambda)\varepsilon}\mathrm{d}s\right)\mathrm{d}\tau. \tag{3.185}$$

Notice that $\frac{\tilde{p}}{2\mu+\lambda} \geqq c > 0$, with c independent of ε, together with (3.185), yields that

$$\|\tilde{h}(t)\|_{L^\infty} \leqq \|\tilde{h}(0)\|_{L^\infty} e^{-\frac{ct}{\varepsilon}} + C\int_0^t \frac{1}{\varepsilon}\|\tilde{K}(\tau)\|_{L^\infty} e^{-c\varepsilon^{-1}(t-\tau)}\,\mathrm{d}\tau. \tag{3.186}$$

It follows from (3.186) and Hölder inequality that

$$\begin{aligned} \|\tilde{h}(t)\|_{L^\infty} &\leqq \|\tilde{h}(0)\|_{L^\infty} e^{-\frac{ct}{\varepsilon}} \\ &\quad + C\left(\int_0^t \|\tilde{K}(\tau)\|_{L^\infty}^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t \frac{1}{\varepsilon^2}\exp\left(-\frac{2c(t-\tau)}{\varepsilon}\right)\mathrm{d}\tau\right)^{\frac{1}{2}} \\ &\leqq \|\tilde{h}(0)\|_{L^\infty} e^{-\frac{ct}{\varepsilon}} + C\frac{1}{\sqrt{\varepsilon}}\left(\int_0^t \|\tilde{K}(\tau)\|_{L^\infty}^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}, \end{aligned}$$

that is,

$$\varepsilon\|\tilde{h}(t)\|_{L^\infty}^2 \leqq \varepsilon\|\tilde{h}(0)\|_{L^\infty}^2 + C\int_0^t \|\tilde{K}(\tau)\|_{L^\infty}^2\,\mathrm{d}\tau. \tag{3.187}$$

On the other hand, integrating (3.186) over $[0, t]$, one can obtain that

$$\begin{aligned} &\int_0^t \|\tilde{h}(\tau)\|_{L^\infty}^2\,\mathrm{d}\tau \\ &\leqq C\varepsilon\|\tilde{h}(0)\|_{L^\infty}^2 + C\int_0^t \left(\int_0^\tau \frac{1}{\varepsilon}\|\tilde{K}(s)\|_{L^\infty} e^{-c\varepsilon^{-1}(\tau-s)}\mathrm{d}s\right)^2\mathrm{d}\tau \\ &\leqq C\varepsilon\|\tilde{h}(0)\|_{L^\infty}^2 + C\int_0^t \left(\int_{\mathbb{R}} \frac{1}{\varepsilon}\|\tilde{K}(s)\|_{L^\infty} I_{(0,t)}(s) e^{-c\varepsilon^{-1}(\tau-s)} I_{(0,t)}(\tau-s)\mathrm{d}s\right)^2\mathrm{d}\tau \\ &\leqq C\varepsilon\|\tilde{h}(0)\|_{L^\infty}^2 + C\int_0^t \left|\left(\|\tilde{K}(\cdot)\|_{L^\infty} I_{(0,t)}(\cdot)\right) * \left(\frac{1}{\varepsilon}e^{-c\varepsilon^{-1}(\cdot)} I_{(0,t)}(\cdot)\right)\right|^2\mathrm{d}\tau \end{aligned}$$

$$\leqq C\varepsilon\|\tilde{h}(0)\|_{L^\infty}^2 + C\left\|\left(\|\tilde{K}(\cdot)\|_{L^\infty} I_{(0,t)}(\cdot)\right)\right\|_{L^2}^2 \cdot \left\|\left(\frac{1}{\varepsilon}e^{-c\varepsilon^{-1}(\cdot)} I_{(0,t)}(\cdot)\right)\right\|_{L^1}^2$$
$$\leqq C\varepsilon\|\tilde{h}(0)\|_{L^\infty}^2 + C\int_0^t \|\tilde{K}(\tau)\|_{L^\infty}^2 \,\mathrm{d}\tau. \tag{3.188}$$

Then it follows from (3.188) and (3.187) that

$$\varepsilon\|\tilde{h}(t)\|_{L^\infty}^2 + \int_0^t \|\tilde{h}(\tau)\|_{L^\infty}^2 \,\mathrm{d}\tau \leqq C\varepsilon\|h(0)\|_{L^\infty}^2 + C\int_0^t \|\tilde{K}(\tau)\|_{L^\infty}^2 \,\mathrm{d}\tau. \tag{3.189}$$

In order to close the estimates of (3.189), one still needs to control the second term on the RHS of (3.189). Direct calculation implies that

$$(\nabla\times\omega)\cdot N = -\partial_{y^1}\omega_2 + \partial_{y^2}\omega_1 + \partial_1\psi\cdot\partial_{y^2}\omega_3 - \partial_2\psi\cdot\partial_{y^1}\omega_3, \tag{3.190}$$

then it follows from (3.162), (3.190), (3.169) and (3.163) that

$$\int_0^t \|J\|_{\mathcal{H}^{2,\infty}}^2 \,\mathrm{d}\tau \leqq C\int_0^t P(\mathcal{N}_m(\tau))\cdot(1+\varepsilon^2\|\nabla p\|_{\mathcal{H}^{2,\infty}}^2 + \varepsilon^2\|(\nabla\times\omega)\cdot N\|_{\mathcal{H}^{2,\infty}}^2$$
$$+\varepsilon^2\|\nabla\theta\|_{\mathcal{H}^{3,\infty}}^2 + \varepsilon^2\|\nabla u\|_{\mathcal{H}^{2,\infty}}^2)\,\mathrm{d}\tau$$
$$\leqq C\int_0^t P(\mathcal{N}_m(\tau))\cdot(1+\varepsilon^2\|\nabla\theta\|_{\mathcal{H}^{3,\infty}}^2 + \varepsilon^2\|\nabla u\|_{\mathcal{H}^{3,\infty}}^2)\,\mathrm{d}\tau$$
$$\leqq \varepsilon^4\int_0^t \|\Delta\theta\|_{\mathcal{H}^4}^2 \,\mathrm{d}\tau + \varepsilon^2\int_0^t \|\nabla^2 u\|_{\mathcal{H}^4}^2 \,\mathrm{d}\tau + CtP(\mathcal{N}_m(t))$$
$$\leqq CC_{m+2}\{\mathcal{N}_m(0) + tP(\mathcal{N}_m(t))\}, \tag{3.191}$$

where we have used condition (1.20) in the last inequality. It follows from (3.162) that

$$\int_0^t \|(2\mu+\lambda)\varepsilon[\mathcal{Z}^\alpha, u\cdot\nabla]\partial_n p\|_{L^\infty}^2 + \|[\mathcal{Z}^\alpha, p]\partial_n p\|_{L^\infty}^2 \,\mathrm{d}\tau$$
$$\leqq C\varepsilon^2\int_0^t \|\nabla^2 u\|_{\mathcal{H}^4}^2 \,\mathrm{d}\tau + CtP(\mathcal{N}_m(t)) \leqq CC_{m+2}\{\mathcal{N}_m(0) + tP(\mathcal{N}_m(t))\}, \tag{3.192}$$

which, together with (3.191), yields that

$$\int_0^t \|\tilde{K}(\tau)\|_{L^\infty}^2 \,\mathrm{d}\tau \leqq CC_{m+2}\{\mathcal{N}_m(0) + tP(\mathcal{N}_m(t))\}. \tag{3.193}$$

Combining (3.162), (3.175), (3.181), (3.183), (3.189) and (3.193), one proves (3.174). Thus, the proof of Lemma 3.15 is completed. □

Estimates for $\|\nabla u\|_{\mathcal{H}^{1,\infty}}^2$ *and* $\varepsilon\|\nabla u\|_{\mathcal{H}^{1,\infty}}^2$

Lemma 3.16. *Assume that* (1.20) *holds, then one has, for* $m \geqq 6$*, that*

$$\|\nabla u\|_{\mathcal{H}^{1,\infty}}^2 \leqq C_{m+2}\{\mathcal{N}_m(0) + P(\|\nabla\theta\|_{\mathcal{H}^{1,\infty}}^2) + tP(\mathcal{N}_m(t))\}. \tag{3.194}$$

Proof. The proof of this lemma is similar to the corresponding lemma [22], except the pressure is a function of density and temperature. Away from the boundary, one clearly has, by the classical isotropic Sobolev embedding theorem, that

$$\|\chi \mathcal{Z}\nabla u\|_{L^\infty}^2 + \|\chi \nabla u\|_{L^\infty}^2 \leqq C\|u\|_{\mathcal{H}^m}^2 \leqq \Lambda_m(t), \quad \text{for } m \geqq 4, \tag{3.195}$$

where the support of χ is away from the boundary. Therefore, by using a partition of unity subordinated to the covering (1.7), we only need to estimate $\|\chi_j \mathcal{Z}\nabla u\|_{L^\infty} + \|\chi_j \nabla u\|_{L^\infty}$ for $j \geqq 1$. For notational convenience, we shall denote χ_j by χ. Similarly to [14], we use the local parametrization in the neighborhood of the boundary given by a normal geodesic system which makes the Laplacian have a convenient form. Let us denote

$$\Psi^n(y,z) = \begin{pmatrix} y \\ \psi(y) \end{pmatrix} - zn(y) = x,$$

where

$$n(y) = \frac{1}{\sqrt{1+|\nabla\psi(y)|^2}} \begin{pmatrix} \partial_1\psi(y) \\ \partial_1\psi(y) \\ -1 \end{pmatrix}$$

is the unit outward normal. As before, we can extend n and Π in the interior by setting

$$n(\Psi^n(y,z)) = n(y), \quad \Pi(\Psi^n(y,z)) = \Pi(y).$$

Note that $n(y,z)$ and $\Pi(y,z)$ have different definitions from the ones used before. The interest of this parametrization is that in the associated local basis (e_{y^1}, e_{y^2}, e_z) of $\mathbb{R}^3$, it holds that $\partial_z = \partial_n$ and

$$\left(e_{y^i}\right)\Big|_{\Psi^n(y,z)} \cdot \left(e_z\right)\Big|_{\Psi^n(y,z)} = 0.$$

The scalar product on $\mathbb{R}^3$ induces in this coordinate system the Riemannian metric g under the form

$$g(y,z) = \begin{pmatrix} \tilde{g}(y,z) & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the Laplacian in this coordinate system reads

$$\Delta f = \partial_{zz} f + \frac{1}{2}\partial_z(\ln|g|)\partial_z f + \Delta_{\tilde{g}} f, \tag{3.196}$$

where $|g|$ denotes the determinant of the matrix g, and $\Delta_{\tilde{g}}$ is given by

$$\Delta_{\tilde{g}} f = \frac{1}{\sqrt{|\tilde{g}|}} \sum_{i,j=1,2} \partial_{y^i}(\tilde{g}_{ij}|\tilde{g}|^{\frac{1}{2}}\partial_{y^j} f),$$

which involves only the tangential derivatives.

It follows from (3.132) (n and Π in the coordinate system we have just defined) and Lemma 3.12, for $m \geqq 5$, that

$$
\begin{aligned}
&\|\chi \nabla u\|_{L^\infty}^2 + \|\chi \mathcal{Z} \nabla u\|_{L^\infty}^2 \\
&\quad \leqq C_2(\|\chi \Pi \partial_n u\|_{L^\infty}^2 + \|\mathcal{Z}(\chi \Pi \partial_n u)\|_{L^\infty}^2 + \|\chi \operatorname{div} u\|_{L^\infty}^2 \\
&\qquad + \|\mathcal{Z} \operatorname{div} u\|_{L^\infty}^2 + \|\mathcal{Z} Z_y u\|_{L^\infty}^2 + \|Z_y u\|_{L^\infty}^2) \\
&\quad \leqq C_3\{\|\chi \Pi \partial_n u\|_{L^\infty}^2 + \|\mathcal{Z}(\chi \Pi \partial_n u)\|_{L^\infty}^2 + P(\Lambda_m) + P(\|(\nabla(\rho\theta), \nabla\theta)\|_{\mathcal{H}^{1,\infty}}^2)\}.
\end{aligned}
\tag{3.197}
$$

Consequently, one needs only to estimate $\|\chi \Pi \partial_n u\|_{L^\infty}^2 + \|\mathcal{Z}(\chi \Pi \partial_n u)\|_{L^\infty}^2$. To estimate this quantity, it is useful to use the vorticity ω. Indeed, we have

$$
\Pi(\omega \times n) = \Pi((\nabla u - \nabla u^t) \cdot n) = \Pi(\partial_n u - \nabla(u \cdot n) - \nabla n^t \cdot u).
$$

Therefore, one obtains that

$$
\begin{aligned}
&\|\chi \Pi \partial_n u\|_{L^\infty}^2 + \|\mathcal{Z}(\chi \Pi \partial_n u)\|_{L^\infty}^2 \\
&\quad \leqq C_3\{\|\chi \Pi(\omega \times n)\|_{L^\infty}^2 + \|\mathcal{Z}(\chi \Pi(\omega \times n))\|_{L^\infty}^2 + \Lambda_m(t)\},
\end{aligned}
\tag{3.198}
$$

which yields that we only need to estimate $\|\chi \Pi(\omega \times n)\|_{L^\infty}^2$ and $\|\mathcal{Z}(\chi \Pi(\omega \times n))\|_{L^\infty}^2$.

By setting things in the support of χ:

$$
\tilde{\omega}(y, z) = \omega(\Psi^n(y, z)), \quad (\tilde{\rho}, \tilde{u})(y, z) = (\rho, u)(\Psi^n(y, z)).
$$

Then it follows from (3.140) and (3.196) that

$$
\tilde{\rho}\tilde{\omega}_t + \tilde{\rho}\tilde{u}^1 \partial_{y^1}\tilde{\omega} + \tilde{\rho}\tilde{u}^2 \partial_{y^2}\tilde{\omega} + \tilde{\rho}\tilde{u} \cdot n \partial_z \tilde{\omega} = \mu\varepsilon\left(\partial_{zz}\tilde{\omega} + \frac{1}{2}\partial_z(\ln|g|)\partial_z\tilde{\omega} + \Delta_{\tilde{g}}\tilde{\omega}\right) + \tilde{F}_1
\tag{3.199}
$$

and

$$
\tilde{\rho}\tilde{u}_t + \tilde{\rho}\tilde{u}^1 \partial_{y^1}\tilde{u} + \tilde{\rho}\tilde{u}^2 \partial_{y^2}\tilde{u} + \tilde{\rho}\tilde{u} \cdot n \partial_z \tilde{u} = \mu\varepsilon\left(\partial_{zz}\tilde{u} + \frac{1}{2}\partial_z(\ln|g|)\partial_z\tilde{u} + \Delta_{\tilde{g}}\tilde{u}\right) + \tilde{F}_2,
\tag{3.200}
$$

where

$$
\tilde{F}_1(y, z) = F_1(\Psi^n(y, z)), \quad \tilde{F}_2(y, z) = F_2(\Psi^n(y, z)),
$$

where F_1 and F_2 are defined in (3.141) and (3.143), respectively. Note that we use the same convention as before for a vector u, and u^j denotes the components of u in the local basis (e_{y^1}, e_{y^2}, e_z) just defined in this section, whereas u_j denotes its components in the standard basis of $\mathbb{R}^3$. The vectorial equation of (3.199) and (3.200) have to be understood component by component in the standard basis of $\mathbb{R}^3$.

Similarly to (3.137), we define that

$$
\tilde{\eta}(y, z) = \chi(\tilde{\omega} \times n + \Pi(B\tilde{u})),
\tag{3.201}
$$

where A is extended into the interior domain by $B(y, z) = B(y)$. Thus, from the boundary conditions (1.3), one has

$$
\tilde{\eta}(y, 0) = 0.
\tag{3.202}
$$

By using (3.199) and (3.200), $\tilde{\eta}$ solves the equations

$$\begin{aligned}&\tilde{\rho}\tilde{\eta}_t + \tilde{\rho}\tilde{u}^1\partial_{y^1}\tilde{\eta} + \tilde{\rho}\tilde{u}^2\partial_{y^2}\tilde{\eta} + \tilde{\rho}\tilde{u}\cdot n\partial_z\tilde{\eta}\\ &\quad = \mu\varepsilon\left(\partial_{zz}\tilde{\eta} + \frac{1}{2}\partial_z(\ln|g|)\partial_z\tilde{\eta}\right) + \chi(\tilde{F}_1\times n) + \chi\Pi(B\tilde{F}_2) + F_\chi + \chi F_\kappa,\end{aligned} \tag{3.203}$$

where the source terms are given by

$$\begin{cases} F_\chi = (\tilde{\rho}\tilde{u}^1\partial_{y^1} + \tilde{\rho}\tilde{u}^2\partial_{y^2} + \tilde{\rho}\tilde{u}\cdot n\partial_z)\chi\cdot(\tilde{\omega}\times n + \Pi(B\tilde{u}))\\ \qquad -\mu\varepsilon(\partial_{zz}\chi + 2\partial_z\chi\partial_z + \frac{1}{2}\partial_z(\ln|g|)\cdot\partial_z\chi)\cdot(\tilde{\omega}\times n + \Pi(B\tilde{u})),\\ F_\kappa = (\tilde{\rho}\tilde{u}^1\partial_{y^1}\Pi + \tilde{\rho}\tilde{u}^2\partial_{y^2}\Pi)\cdot(B\tilde{u}) + \tilde{\omega}\times(\tilde{\rho}\tilde{u}^1\partial_{y^1}n + \tilde{\rho}\tilde{u}^2\partial_{y^2}n)\\ \qquad +\Pi((\tilde{\rho}\tilde{u}^1\partial_{y^1} + \tilde{\rho}\tilde{u}^2\partial_{y^2})B\cdot\tilde{u}) + \mu\varepsilon\Delta_{\tilde{g}}\tilde{\omega}\times n + \mu\varepsilon\Pi(B\Delta_{\tilde{g}}\tilde{u}). \end{cases} \tag{3.204}$$

Note that in the calculating of the source terms, in particular F_κ, which contains all the commutators coming from the fact that n and Π are not constant, we have used the idea that in the coordinate system that we have just defined, B, n and Π do not depend on the normal variable. By using that $\Delta_{\tilde{g}}$ involves only the tangential derivatives and that the derivatives of χ are compactly supported away from the boundary, one obtains the following estimates, for $m \geqq 6$:

$$\begin{cases} \|\chi\Pi(F_1\times n)\|^2_{\mathcal{H}^{1,\infty}} \leqq C_2P(\mathcal{N}_m(t)),\\ \|F_\chi\|^2_{\mathcal{H}^{1,\infty}} \leqq C_3(\|u\|^2_{\mathcal{H}^{1,\infty}}\cdot\|u\|^2_{\mathcal{H}^{2,\infty}} + \varepsilon^2\|u\|^2_{\mathcal{H}^{3,\infty}}) \leqq C_3P(\mathcal{N}_m(t)),\\ \|\chi F_\kappa\|^2_{\mathcal{H}^{1,\infty}} \leqq C_4\{\|u\|^4_{\mathcal{H}^{1,\infty}} + \|u\|^2_{\mathcal{H}^{1,\infty}}\|\nabla u\|^2_{\mathcal{H}^{1,\infty}} + \varepsilon^2(\|u\|^2_{\mathcal{H}^{3,\infty}} + \|\nabla u\|^2_{\mathcal{H}^{3,\infty}})\}\\ \qquad\qquad \leqq C_4\{P(\mathcal{N}_m(t)) + \varepsilon^2\|\nabla^2 u\|^2_{\mathcal{H}^4}\}, \end{cases} \tag{3.205}$$

and from (3.168), it holds that

$$\begin{aligned}\|\chi\Pi(B\tilde{F}_2)\|^2_{\mathcal{H}^{1,\infty}} &\leqq C_3\{\varepsilon^2\|\nabla\mathrm{div}u\|^2_{\mathcal{H}^{1,\infty}} + \|\nabla p\|^2_{\mathcal{H}^{1,\infty}}\}\\ &\leqq C_4\{P(\mathcal{N}_m(t)) + \varepsilon^2\|(\nabla\theta,\nabla(\rho\theta))\|^2_{\mathcal{H}^{2,\infty}}\}.\end{aligned} \tag{3.206}$$

Consequently, it follows from (3.205) and (3.206), for $m \geqq 6$, that

$$\|\tilde{F}\|^2_{\mathcal{H}^{1,\infty}} \leqq C_4\{\varepsilon^2\|\nabla^2 u\|^2_{\mathcal{H}^4} + \varepsilon^2\|(\nabla\theta,\nabla(\rho\theta))\|^2_{\mathcal{H}^{2,\infty}} + P(\mathcal{N}_m(t))\}, \tag{3.207}$$

where $\tilde{F} = \chi(\tilde{F}_1\times n) + \chi\Pi(B\tilde{F}_2) + F_\chi + \chi F_\kappa$.

In order to eliminate the term $\frac{1}{2}\partial_z(\ln|g|)\partial_z\tilde{\eta}$, one defines

$$\tilde{\eta} = \frac{1}{|g|^{\frac{1}{4}}}\eta = \bar{\gamma}\bar{\eta}. \tag{3.208}$$

Note that

$$\|\tilde{\eta}\|_{\mathcal{H}^{1,\infty}} \leqq C_3\|\bar{\eta}\|_{\mathcal{H}^{1,\infty}}, \quad \text{and} \quad \|\bar{\eta}\|_{\mathcal{H}^{1,\infty}} \leqq C_3\|\tilde{\eta}\|_{\mathcal{H}^{1,\infty}}, \tag{3.209}$$

and $\bar{\eta}$ solves the equations

$$\begin{aligned}
&\tilde{\rho}\bar{\eta}_t + \tilde{\rho}\tilde{u}^1\partial_{y^1}\bar{\eta} + \tilde{\rho}\tilde{u}^2\partial_{y^2}\bar{\eta} + \tilde{\rho}\tilde{u}\cdot n\partial_z\bar{\eta} - \varepsilon\partial_{zz}\bar{\eta} \\
&\quad = \frac{1}{\bar{\gamma}}\left(\tilde{F} + \varepsilon\partial_{zz}\bar{\gamma}\cdot\bar{\eta} + \frac{1}{2}\varepsilon\partial_z(\ln|g|)\partial_z\gamma\cdot\bar{\eta} - \tilde{\rho}(\tilde{u}\cdot\nabla\bar{\gamma})\bar{\eta}\right) \triangleq \mathcal{S}. \quad (3.210)
\end{aligned}$$

It is difficult to directly obtain the explicit solution formula of (3.210), so one rewrites it as

$$\begin{aligned}
&\tilde{\rho}(t,y,0)[\bar{\eta}_t + \tilde{u}^1(t,y,0)\partial_{y^1}\bar{\eta} + \tilde{u}^2(t,y,0)\partial_{y^2}\bar{\eta} + z\partial_z(\tilde{u}\cdot n)(t,y,0)\partial_z\bar{\eta}] - \varepsilon\partial_{zz}\bar{\eta} \\
&\quad = \mathcal{S} + [\tilde{\rho}(t,y,0) - \rho(t,y,z)]\bar{\eta}_t + \sum_{i=1,2}[(\tilde{\rho}\tilde{u}^i)(t,y,0) - (\rho\tilde{u}^i)(t,y,z)]\partial_{y^i}\bar{\eta} \\
&\qquad - \rho(t,y,z)[(\tilde{u}\cdot n)(t,y,z) - z\partial_z(\tilde{u}\cdot n)(t,y,0)]\partial_z\bar{\eta} \\
&\qquad + [\rho(t,y,z) - \rho(t,y,0)]\cdot z\partial_z(\tilde{u}\cdot n)(t,y,0)\partial_z\bar{\eta} \triangleq M \quad \text{for } z>0, \quad (3.211)
\end{aligned}$$

with the boundary condition $\bar{\eta}(t,y,0) = 0$. By using Lemma 6.1 in the "Appendix", one has that

$$\begin{aligned}
\|\bar{\eta}\|_{\mathcal{H}^{1,\infty}} &\lesssim \|\bar{\eta}_0\|_{\mathcal{H}^{1,\infty}} + \int_0^t \|\tilde{\rho}^{-1}\|_{L^\infty}\|M\|_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau \\
&\quad + \int_0^t (1+\|\tilde{\rho}^{-1}\|_{L^\infty})(1+\|\mathcal{Z}(\rho,u,\nabla u)\|_{L^\infty}^2)\|\bar{\eta}\|_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau \\
&\lesssim \|\bar{\eta}_0\|_{\mathcal{H}^{1,\infty}} + C\int_0^t \|M\|_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau + CtP(\mathcal{N}_m(t)). \quad (3.212)
\end{aligned}$$

It remains to estimate the right hand side of (3.212). Firstly, by using (3.207), one has that

$$\|\mathcal{S}\|_{\mathcal{H}^{1,\infty}}^2 \leqq C_4\{\varepsilon^2\|\nabla^2 u\|_{\mathcal{H}^4}^2 + \varepsilon^2\|\nabla\theta\|_{\mathcal{H}^{2,\infty}}^2 + CP(\mathcal{N}_m(t))\}, \quad \text{for } m \geqq 6. \quad (3.213)$$

Next, using the Taylor formula and the fact that $\bar{\eta}$ is compactly supported in z, and by the same argument as that of Lemma 3.14 in [22], one can obtain, for $m \geqq 5$, that

$$\begin{aligned}
&\|[\tilde{\rho}(t,y,0) - \rho(t,y,z)]\bar{\eta}_t\|_{\mathcal{H}^{1,\infty}}^2 \|[(\tilde{\rho}\tilde{u}^1)(t,y,0) - (\tilde{\rho}\tilde{u}^1)(t,y,z)]\partial_{y^1}\bar{\eta}\|_{\mathcal{H}^{1,\infty}}^2 \\
&\quad + \|[(\tilde{\rho}\tilde{u}^2)(t,y,0) - (\tilde{\rho}\tilde{u}^2)(t,y,z)]\partial_{y^2}\bar{\eta}\|_{\mathcal{H}^{1,\infty}}^2 \\
&\quad \times \|[\rho(t,y,z) - \rho(t,y,0)]\cdot z\partial_z(\tilde{u}\cdot n)(t,y,0)\partial_z\bar{\eta}\|_{\mathcal{H}^{1,\infty}}^2 \\
&\quad + \|\rho(t,y,z)[(\tilde{u}\cdot n)(t,y,z) - z\partial_z(\tilde{u}\cdot n)(t,y,0)]\partial_z\bar{\eta}\|_{\mathcal{H}^{1,\infty}}^2 \leqq CP(\mathcal{N}_m(t)), \quad (3.214)
\end{aligned}$$

Then it follows from (3.213) and (3.214) that

$$\|M\|_{\mathcal{H}^{1,\infty}}^2 \leqq C_4\{\varepsilon^2\|\nabla^2 u\|_{\mathcal{H}^4}^2 + \varepsilon^2\|\nabla\theta\|_{\mathcal{H}^{2,\infty}}^2 + P(\mathcal{N}_m(t))\}, \quad \text{for } m \geqq 6. \quad (3.215)$$

Substituting (3.215) into (3.212), we have, for $m \geqq 6$, that

$$\begin{aligned}\|\bar{\eta}\|_{\mathcal{H}^{1,\infty}}^2 &\lesssim \|\bar{\eta}_0\|_{\mathcal{H}^{1,\infty}}^2 + C_4 t P(\mathcal{N}_m(t)) + C_4 t \int_0^t \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^4}^2 + \varepsilon^2 \|\nabla\theta\|_{\mathcal{H}^{2,\infty}}^2 \,\mathrm{d}\tau \\ &\leqq C C_{m+2}\{\mathcal{N}_m(0) + tP(\mathcal{N}_m(t))\}, \end{aligned} \tag{3.216}$$

where we have used (1.20), (2.5) and the Hölder inequality in the last inequality. Therefore, combining (3.162), (3.195), (3.197), (3.198), (3.201), (3.209) and (3.216), one obtains (3.194). Therefore, the proof of Lemma 3.16 is completed. □

Lemma 3.17. *Assume that* (1.20) *holds, then one has, for* $m \geqq 6$*, that*

$$\varepsilon\|\partial_{zz}u\|_{L^\infty}^2 \leqq C C_{m+2}\{P(\mathcal{N}_m(0)) + P(\|\nabla\theta\|_{\mathcal{H}^{1,\infty}}^2) + tP(\mathcal{N}_m(t))\}. \tag{3.217}$$

Proof. By an argument similar to the one in Lemma 3.16, one firstly has that

$$\varepsilon\|\partial_{zz}u\|_{L^\infty}^2 \leqq C_2\{P(\Lambda_m) + P(\|(\nabla u, \nabla\theta, \nabla(\rho\theta))\|_{\mathcal{H}^{1,\infty}}^2) + \varepsilon\|\partial_z\bar{\eta}\|_{L^\infty}^2\}. \tag{3.218}$$

Thus, one needs only to estimate $\varepsilon\|\partial_z\bar{\eta}\|_{L^\infty}^2$. One rewrites (3.210) as

$$\bar{\eta}_t - \varepsilon\partial_{zz}\bar{\eta} = -(\tilde{\rho}-1)\bar{\eta}_t - \tilde{\rho}\tilde{u}^1\partial_{y^1}\bar{\eta} - \tilde{\rho}\tilde{u}^2\partial_{y^2}\bar{\eta} - \tilde{\rho}\tilde{u}\cdot n\partial_z\bar{\eta} + \mathcal{S} =: \Xi, \tag{3.219}$$

where $\bar{\eta}$ satisfies the homogenous Dirichlet boundary condition $\bar{\eta}|_{z=0} = 0$. Then $\bar{\eta}$ has the following expression:

$$\begin{aligned}\bar{\eta}(t,y,z) &= \int_0^{+\infty} G(t,z,z')\eta_0(y,z')\,\mathrm{d}z' \\ &\quad + \int_0^t\int_0^{+\infty} G(t-\tau,z,z')\Xi(\tau,y,z')\,\mathrm{d}z'\,\mathrm{d}\tau,\end{aligned}$$

where

$$G(t,z,z') = \frac{1}{\sqrt{4\pi\mu\varepsilon t}}\left[\exp\left(-\frac{|z-z'|^2}{4\mu\varepsilon t}\right) - \exp\left(-\frac{|z+z'|^2}{4\mu\varepsilon t}\right)\right].$$

Then one obtains that

$$\begin{aligned}\sqrt{\varepsilon}\partial_z\bar{\eta}(t,y,z) &= \sqrt{\varepsilon}\int_0^{+\infty} \partial_z G(t,z,z')\eta_0(y,z')\,\mathrm{d}z' \\ &\quad + \sqrt{\varepsilon}\int_0^t\int_0^{+\infty} \partial_z G(t-\tau,z,z')\Xi(\tau,y,z')\,\mathrm{d}z'\,\mathrm{d}\tau.\end{aligned}$$

Since $\eta_0(y,z)$ vanishes on the boundary due to the compatibility condition, it follows from integrating by parts to the first term that

$$\sqrt{\varepsilon}\|\partial_z\bar{\eta}\|_{L^\infty} \leqq \sqrt{\varepsilon}\|\partial_z\eta_0\|_{L^\infty} + \int_0^t \frac{1}{\sqrt{t-\tau}}\|\Xi(\tau)\|_{L^\infty}\,\mathrm{d}\tau. \tag{3.220}$$

Direct calculation shows that

$$\left(\int_0^t \frac{1}{\sqrt{t-\tau}}\|\Xi(\tau)\|_{L^\infty}\,\mathrm{d}\tau\right)^2 \leqq C\varepsilon^2\int_0^t \|\nabla^2 u\|_{\mathcal{H}^3}^2 + CtP(\mathcal{N}_m(t)), \quad \text{for } m \geqq 5. \tag{3.221}$$

Substituting (3.220) and (3.221) into (3.218) and using (3.162), (3.174) and (3.194), one proves (3.217). Therefore, the proof of this lemma is completed. □

Estimate for $\|\nabla\theta\|_{\mathcal{H}^{1,\infty}}$

In order to estimate $\|\nabla\theta\|_{\mathcal{H}^{1,\infty}}$, the most difficult part is to control the term $p\nabla\mathrm{div}u$, which comes from the term $p\mathrm{div}u$ that appears on the LHS of the energy equation $(3.7)_3$. Actually, if $p\nabla\mathrm{div}u$ is regarded as the source term, it is very difficult to bound the term $\int_0^t \|p\nabla\mathrm{div}u(\tau)\|_{\mathcal{H}^{1,\infty}}^2\,\mathrm{d}\tau$, since the derivative is too high. It is noted that such difficulty does not arise in the isentropic case [22]. To overcome the difficulty, a new idea is needed. Fortunately, we find that the term $p\nabla\mathrm{div}u$ can be decomposed into two parts, that is, $\nabla(\rho\theta)_t$ and $\nabla\theta_t$. The most difficult term, $\nabla\theta_t$, can be absorbed into the main part of equation, while $\nabla(\rho\theta)_t$ is regarded as the source term that has already been controlled in Lemma 3.15. This observation is key to closing the pointwise estimates.

Lemma 3.18. *Assume that* (1.20) *holds, then one has, for* $m \geqq 6$, *that*

$$\|\nabla\theta\|_{\mathcal{H}^{1,\infty}}^2 \leqq CC_{m+2}\{P(\mathcal{N}_m(0)) + tP(\mathcal{N}_m(t))\}. \tag{3.222}$$

Proof. Due to (3.163), one needs only to estimate $\|\partial_z\theta\|_{\mathcal{H}^{1,\infty}}$ or $\|\partial_n\theta\|_{\mathcal{H}^{1,\infty}}$. It follows from $(3.7)_3$ that

$$\begin{aligned}&\rho[\nabla\theta_t + (u\cdot\nabla)\nabla\theta] + p\nabla\mathrm{div}u - \kappa(\varepsilon)\Delta\nabla\theta\\ &\quad = -\nabla p\mathrm{div}u - [\nabla\rho(\theta_t + u\cdot\nabla\theta) + \rho\nabla u\cdot\nabla\theta] + \varepsilon\nabla[2\mu|Su|^2 + \lambda|\mathrm{div}u|^2].\end{aligned} \tag{3.223}$$

In order to deal with the term $p\nabla\mathrm{div}u$, using the mass equation $(3.7)_1$, one first notices that

$$p\mathrm{div}u = R\rho[\theta_t + (u\cdot\nabla)\theta] - [p_t + (u\cdot\nabla)p], \tag{3.224}$$

and

$$\begin{aligned}p\nabla\mathrm{div}u = {}& R\rho[\nabla\theta_t + (u\cdot\nabla)\nabla\theta] - [\nabla p_t + (u\cdot\nabla)\nabla p] - \nabla p\mathrm{div}u\\ &+ R\nabla\rho[\theta_t + (u\cdot\nabla)\theta] + R\rho\nabla u\cdot\nabla\theta - \nabla u\cdot\nabla p.\end{aligned} \tag{3.225}$$

Then it follows from $(3.7)_3$ and (3.223)–(3.225) that

$$\begin{aligned}&(R+1)\rho[\nabla\theta_t + (u\cdot\nabla)\nabla\theta] - \kappa(\varepsilon)\Delta\nabla\theta\\ &\quad = [\nabla p_t + (u\cdot\nabla)\nabla p] + \nabla u\cdot\nabla p - (1+R)[\nabla\rho(\theta_t + u\cdot\nabla\theta) + \rho\nabla u\cdot\nabla\theta]\\ &\qquad + \varepsilon\nabla[2\mu|Su|^2 + \lambda|\mathrm{div}u|^2] \triangleq B_1,\end{aligned} \tag{3.226}$$

and

$$
\begin{aligned}
&(R+1)\rho[\theta_t+(u\cdot\nabla)\theta]-\kappa(\varepsilon)\Delta\theta \\
&\quad=[p_t+(u\cdot\nabla)p]+\varepsilon[2\mu|Su|^2+\lambda|\mathrm{div}u|^2]\triangleq B_2.
\end{aligned}
\tag{3.227}
$$

We use the local coordinates (y,z) defined in Lemma 3.16 in the neighborhood of the boundary, which makes the Laplacian have a convenient form. The functions $\tilde\rho,\tilde u,\chi,n,\Pi$ are the same as the ones defined in Lemma 3.16. By setting this in the support of χ:

$$
\tilde\theta(y,z,t)=\theta(\Psi^n(y,z),t),\quad \widetilde{\nabla\theta}(y,z,t)=(\nabla\theta)(\Psi^n(y,z),t). \tag{3.228}
$$

Then it follows from (3.226) and (3.227) that

$$
\begin{aligned}
&(R+1)[\tilde\rho\widetilde{\nabla\theta}_t+\tilde\rho\tilde u^1\partial_{y^1}\widetilde{\nabla\theta}+\tilde\rho\tilde u^2\partial_{y^2}\widetilde{\nabla\theta}+\tilde\rho\tilde u\cdot n\partial_z\widetilde{\nabla\theta}] \\
&\quad=\kappa(\varepsilon)(\partial_{zz}\widetilde{\nabla\theta}+\frac12\partial_z(\ln|g|)\partial_z\widetilde{\nabla\theta}+\Delta_{\tilde g}\widetilde{\nabla\theta})+\tilde B_1,
\end{aligned}
\tag{3.229}
$$

and

$$
\begin{aligned}
&(R+1)[\tilde\rho\tilde\theta_t+\tilde\rho\tilde u^1\partial_{y^1}\tilde\theta+\tilde\rho\tilde u^2\partial_{y^2}\tilde\theta+\tilde\rho\tilde u\cdot n\partial_z\tilde\theta] \\
&\quad=\kappa(\varepsilon)\left(\partial_{zz}\tilde\theta+\frac12\partial_z(\ln|g|)\partial_z\tilde\theta+\Delta_{\tilde g}\tilde\theta\right)+\tilde B_2,
\end{aligned}
\tag{3.230}
$$

where

$$
\tilde B_1=B_1(\Psi^n(y,z),t),\quad \tilde B_2=B_2(\Psi^n(y,z),t). \tag{3.231}
$$

Define

$$
\zeta(y,z,t)=\chi(n\cdot\widetilde{\nabla\theta}-\nu\tilde\theta), \tag{3.232}
$$

then, in view of the boundary condition $(1.3)_3$, ζ satisfies $\zeta=0$ on $\partial\Omega$. Considering $(3.229)\cdot n+\nu\cdot(3.230)$, it is easy to know that ζ satisfies

$$
\begin{aligned}
&\tilde\rho\zeta_t+\tilde\rho\tilde u^1\partial_{y^1}\zeta+\tilde\rho\tilde u^2\partial_{y^2}\zeta+\tilde\rho\tilde u\cdot n\partial_z\zeta \\
&\quad=\kappa(\varepsilon)\big(\partial_{zz}\zeta+\frac12\partial_z(\ln|g|)\partial_z\zeta\big)+\chi(\tilde B_1\cdot n)+\nu\chi\tilde B_2+F^\theta_\chi+\chi F^\theta_\kappa,
\end{aligned}
\tag{3.233}
$$

where the source terms are given by

$$
\begin{cases}
F^\theta_\chi=(R+1)(\tilde\rho\tilde u^1\partial_{y^1}+\tilde\rho\tilde u^2\partial_{y^2}+\tilde\rho\tilde u\cdot n\partial_z)\chi\cdot(n\cdot\widetilde{\nabla\theta}-\nu\tilde\theta) \\
\qquad-\kappa(\varepsilon)\left(\partial_{zz}\chi+2\partial_z\chi\partial_z+\frac12\partial_z(\ln|g|)\cdot\partial_z\chi\right)\cdot(n\cdot\widetilde{\nabla\theta}-\nu\tilde\theta), \\
F^\theta_\kappa=(R+1)\widetilde{\nabla\theta}\cdot(\tilde\rho\tilde u^1\partial_{y^1}n+\tilde\rho\tilde u^2\partial_{y^2}n)+\kappa(\varepsilon)n\cdot\Delta_{\tilde g}\widetilde{\nabla\theta}+\nu\kappa(\varepsilon)\Delta_{\tilde g}\tilde\theta.
\end{cases}
\tag{3.234}
$$

Then, using arguments similar to those in Lemma 3.16, one can obtain that

$$
\begin{aligned}
\|\zeta(t)\|_{\mathcal{H}^{1,\infty}}^2 \leqq C\Big\{ & P(\mathcal{N}_m(0)) + P(\Lambda_m) + tP(\mathcal{N}_m(t)) \\
& + P(\mathcal{N}_m(t))\left(\int_0^t \kappa(\varepsilon)\|\Delta\theta\|_{\mathcal{H}^4}^{\frac{1}{2}} + \|\nabla p\|_{\mathcal{H}^{2,\infty}}\,\mathrm{d}\tau\right)^2 \\
& + \left(\int_0^t \varepsilon\|\nabla u\|_{L^\infty}\|\nabla^2 u\|_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau\right)^2\Big\} \leqq C\Big\{P(\mathcal{N}_m(0)) + tP(\mathcal{N}_m(t)) \\
& + (\|\nabla u_0\|_{L^\infty}^2 + t\mathcal{N}_m(t))\left(\int_0^t \varepsilon\|\nabla^2 u\|_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau\right)^2\Big\} \\
\leqq\ & CC_{m+2}\{P(\mathcal{N}_m(0)) + tP(\mathcal{N}_m(t))\},
\end{aligned} \tag{3.235}
$$

where we have used (3.162), (3.172), (3.174), and the Hölder inequality above. Then (3.222) follows from (3.162), (3.163), (3.232) and (3.235). Therefore, the proof of Lemma 3.18 is completed. □

Combining Lemmas 3.15–3.18, one can obtain:

Proposition 3.19. *Assume that (1.20) holds, then one has, for $m \geqq 6$, that*

$$
\begin{aligned}
& \|\nabla(\rho\theta)\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla u\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla\theta\|_{\mathcal{H}^{1,\infty}}^2 + \varepsilon\|\nabla(\rho\theta)\|_{\mathcal{H}^{2,\infty}}^2 + \varepsilon\|\partial_{zz}u\|_{L^\infty}^2 \\
& + \int_0^t \|\nabla(\rho\theta)\|_{\mathcal{H}^{2,\infty}}^2\,\mathrm{d}\tau \leqq CC_{m+2}\{P(\mathcal{N}_m(0)) + tP(\mathcal{N}_m(t))\}.
\end{aligned} \tag{3.236}
$$

3.5. Proof of Theorem 3.1

Firstly, it follows from (3.162), (3.165) and (3.236) that

$$
Q(t) \leqq CC_{m+2}\{P(\mathcal{N}_m(0)) + tP(\mathcal{N}_m(t))\}. \tag{3.237}
$$

In order to close the *a priori* estimate, one still needs to get the uniform estimate for $\|\nabla\partial_t^{m-1}u\|$. It follows from Lemma 3.9, (3.162) and (3.237) that

$$
\begin{aligned}
\|\nabla\partial_t^{m-1}u\|^2 & \lesssim C_{m+1}\{\|u\|_{\mathcal{H}^m}^2 + \|\eta\|_{\mathcal{H}^m}^2 + \|\partial_t^{m-1}\mathrm{div}u\|_{L^2}^2\} \\
& \lesssim C_{m+2}\{P(\mathcal{N}_m(0)) + tP(\mathcal{N}_m(t))\}.
\end{aligned} \tag{3.238}
$$

Combining (3.162), (3.236) and (3.238), one gets (3.4). Finally, it follows from (1.1) that

$$
|\rho(x,0)|\exp\left(-\int_0^t \|\mathrm{div}u\|_{L^\infty}\,\mathrm{d}\tau\right) \leqq \rho(x,t) \leqq |\rho(x,0)|\exp\left(\int_0^t \|\mathrm{div}u\|_{L^\infty}\,\mathrm{d}\tau\right),
$$

so we have proved (3.1). The Newton-Leibniz formula yields immediately that (3.2). Thus the proof of Theorem 3.1 is completed. □

4. Proof of Theorem 1.1

Proof of Theorem 1.1. In this section, we shall show how we can combine our *a priori* estimates to obtain the uniform existence result. Let us fix $m \geqq 6$, and consider initial data such that

$$\mathcal{I}_m(0) = \sup_{\varepsilon\in(0,1]} \|(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)\|_{X_m^\varepsilon}^2 \leqq \tilde{C}_0, \quad \text{and} \quad \hat{C}_0^{-1} \leqq \rho_0^\varepsilon, \theta_0^\varepsilon \leqq \hat{C}_0, \tag{4.1}$$

For such initial data, we are not aware of local existence results for (1.1) and (1.3), so we first need to prove the local existence results for (1.1) and (1.3) with initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon) \in X_{NS}^{\varepsilon,m}$. For such initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)$, it follows from the definition of $X_{NS}^{\varepsilon,m}$ that there exists a sequence of smooth approximate initial data $(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, \theta_0^{\varepsilon,\delta}) \in X_{NS,ap}^{\varepsilon,m}$($\delta$ being a regularization parameter) which has enough space regularity that the time derivatives at the initial time can be defined by Navier–Stokes equations and the boundary compatibility conditions can be satisfied. For fixed $\varepsilon \in (0, 1]$, we construct approximate solutions, inductively, as follows:

(1) Define $u^0 = u_0^{\varepsilon,\delta}$, and
(2) Assuming that u^{k-1} was defined for $k \geqq 1$, let (ρ^k, u^k, θ^k) be the unique solution to the following linearized initial boundary value problem:

$$\begin{cases} \rho_t^k + \mathrm{div}(\rho^k u^{k-1}) = 0 \quad \text{in } (0,T)\times\Omega, \\ \rho^k u_t^k + \rho^k u^{k-1}\cdot\nabla u^k + R\nabla(\rho^k\theta^k) = \varepsilon\Delta u^k + \varepsilon\nabla\mathrm{div}u^k, \quad \text{in } (0,T)\times\Omega, \\ \rho^k\theta_t^k + \rho^k u^{k-1}\cdot\nabla\theta^k + R\rho^k\theta^k\mathrm{div}u^{k-1} \\ \quad = \kappa(\varepsilon)\Delta\theta^k + 2\mu\varepsilon|Su^{k-1}|^2 + \lambda\varepsilon|\mathrm{div}u^{k-1}|^2, \quad \text{in } (0,T)\times\Omega, \\ (\rho^k, u^k, \theta^k)|_{t=0} = (\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, \theta_0^{\varepsilon,\delta}), \quad \text{with } \frac{2}{3C_0} \leqq \rho_0^{\varepsilon,\delta},\ \theta_0^{\varepsilon,\delta} \leqq \frac{3}{2}C_0, \\ \text{with boundary conditions (1.3).} \end{cases} \tag{4.2}$$

Since ρ^k, θ^k and u^k are decoupled, the existence of a global unique smooth solution $(\rho^k, u^k, \theta^k)(t)$ of (4.2) with $0 < \rho^k(t), \theta^k(t) < \infty$ can be obtained by using classical methods, for example, the same argument as that of in CHO and KIM [4], and the standard elliptic regularity results as those in AGMON et al. [1]. On the other hand, since $(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, \theta_0^{\varepsilon,\delta}) \in H^{3m}$, one proves that there exists a positive time $\tilde{T}_1 = \tilde{T}_1(\varepsilon)$ such that, for $0 \leqq t \leqq \tilde{T}_1$,

$$\|(\rho^k, u^k, \theta^k)(t)\|_{H^{3m}}^2 \leqq \hat{C}_1, \quad \text{and} \quad (2\hat{C}_0)^{-1} \leqq \rho^k(t),\ \theta^k(t) \leqq 2\hat{C}_0, \tag{4.3}$$

where $\tilde{T}_1$ and $\hat{C}_1$ depend on $\hat{C}_0, \varepsilon^{-1}$ and $\|(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, \theta_0^{\varepsilon,\delta})\|_{H^{3m}}$. Based on the above uniform estimates for (ρ^k, u^k, θ^k), by the same arguments as those of Section 3 of [4], there exists a uniform time $\hat{T}_1(\leqq \tilde{T}_1)$ (independent of k) such that (ρ^k, u^k, θ^k) converges to a limit $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \rho^{\varepsilon,\delta})$ as $k \to +\infty$ in the following strong sense:

$$(\rho^k, u^k, \theta^k) \to (\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta}) \quad \text{in } L^\infty(0, \hat{T}_1; L^2),$$
$$\text{and} \quad \nabla u^k \to \nabla u^{\varepsilon,\delta} \quad \text{in } L^2(0, \hat{T}_1, L^2).$$

It is easy to check that $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})(t)$ is a classical solution to the problem (1.1) and (1.3) with initial data $(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, \theta_0^{\varepsilon,\delta})$. Then, by virtue of the lower semi-continuity of norms, one can deduce from (4.3) that, for $0 \leqq t \leqq \hat{T}_1$,

$$\|(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \rho^{\varepsilon,\delta})(t)\|_{H^{3m}}^2 \leqq \hat{C}_1 \quad \text{and} \quad (2\hat{C}_0)^{-1} \leqq \rho^{\varepsilon,\delta}(t),\ \theta^{\varepsilon,\delta}(t) \leqq 2\hat{C}_0, \tag{4.4}$$

Applying the *a priori* estimates given in Theorem 3.1 to $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})(t)$, we obtain a uniform time $T_0 > 0$ and positive constant $\tilde{C}_3$ (independent of ε and δ) such that it holds for $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})(t)$ that

$$\Upsilon_m(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})(t) \leqq \tilde{C}_3 \quad \text{and} \quad (2\hat{C}_0)^{-1} \leqq \rho^{\varepsilon,\delta}(t),\ \theta^{\varepsilon,\delta}(t) \leqq 2\hat{C}_0,\ \forall t \in [0, \tilde{T}_2], \tag{4.5}$$

where $\tilde{T}_2 \triangleq \min\{T_0, \hat{T}_1\}$ and the uniform constants $T_0, \tilde{C}_3$ (independent of ε, δ) depend only on $\hat{C}_0$ and $\mathcal{I}_m(0)$. Based on the uniform estimates (4.5) for $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})$, one can pass the limit $\delta \to 0$ to get a strong solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ of (1.1) and (1.3) with initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)$ satisfying (4.1) by using a strong compactness arguments. It follows from (4.5) that $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})$ is uniformly bounded in $L^\infty([0, T_0]; H^m_{co})$, $\nabla(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})$ is uniformly bounded in $L^\infty([0, T_0]; H^{m-1}_{co})$, and $\partial_t(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})$ is uniformly bounded in $L^\infty([0, T_0]; H^{m-1}_{co})$. Then, it follows from the compactness argument [19] that $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})$ is compact in $\mathcal{C}([0, T_0]; H^{m-1}_{co})$. In particular, there exists a sequence $\delta_n \to 0+$ and $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon) \in \mathcal{C}([0, T_0]; H^{m-1}_{co})$ such that

$$(\rho^{\varepsilon,\delta_n}, u^{\varepsilon,\delta_n}, \theta^{\varepsilon,\delta_n}) \to (\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon) \quad \text{in } \mathcal{C}([0, T]; H^{m-1}_{co}) \quad \text{as } \delta_n \to 0+. \tag{4.6}$$

Moreover, applying the lower semi-continuity of norms to the bounds (4.5), one obtains the bounds for $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ such that

$$\Upsilon_m(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)(t) \leqq \tilde{C}_3 \quad \text{and} \quad (2\hat{C}_0)^{-1} \leqq \rho^\varepsilon(t),\ \theta^\varepsilon(t) \leqq 2\hat{C}_0, \quad \forall t \in [0, \tilde{T}_2]. \tag{4.7}$$

It follows from (4.7) and the anisotropic Sobolev inequality (2.5) that

$$\begin{aligned}
&\sup_{t\in[0,T_0]} \|(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, \theta^{\varepsilon,\delta_n} - \theta^\varepsilon)\|_{L^\infty}^2 \\
&\quad \leqq \sup_{t\in[0,T_0]} \|(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, \theta^{\varepsilon,\delta_n} - \theta^\varepsilon)\|_{H^2_{co}}^2 \\
&\qquad + \sup_{t\in[0,T_0]} (\|\nabla(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, \theta^{\varepsilon,\delta_n} - \theta^\varepsilon)\|_{H^1_{co}} \\
&\qquad \cdot \|(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, \theta^{\varepsilon,\delta_n} - \theta^\varepsilon)\|_{H^2_{co}}) \to 0, \\
&\qquad \text{as } \delta_n \to 0+.
\end{aligned} \tag{4.8}$$

Then it is easy to check that $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ is a strong solution of the Navier–Stokes system. The uniqueness of the solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ is easy, since we work on functions with Lipschitz regularity. Thus the whole family of $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})$ converges to $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ as $\delta \to 0+$. Therefore, for initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon) \in X^{\varepsilon,m}_{NS}$, we have established the local existence result for (1.1) and (1.3) such that $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)(t) \in X_{NS}^{\varepsilon,m}, \quad t \in [0, \tilde{T}_2]$.

We shall use the above local existence results to prove Theorem 1.1. If $T_0 \leqq \hat{T}_1$, then Theorem 1.1 follows from (4.7) with $\tilde{C}_1 \triangleq \tilde{C}_3$. On the other hand, if $\hat{T}_1 \leqq T_0$, based on the uniform estimates (4.7), we can use the local existence results established above to extend our solution step by step to the uniform time interval $t \in [0, T_0]$. Therefore, the proof of Theorem 1.1 is completed. □

5. Proof of Theorem 1.5: Vanishing Dissipation Limit

In this section, we study the vanishing dissipation limit of the solutions of the full compressible Navier–Stokes system (1.1) to the solutions of the full compressible Euler system with a rate of convergence. It is well known that the solution $(\rho, u, \theta)(t) \in H^3$ of the Euler system (1.5), (1.6) and (1.28) satisfies

$$\sum_{k=0}^{3} \|(\rho, u, \theta)\|_{C^k(0,T_1;H^{3-k})} \leqq \tilde{C}_4, \quad \frac{1}{2\hat{C}_0} \leqq \rho(t), \theta(t) \leqq 2\hat{C}_0, \tag{5.1}$$

where $\tilde{C}_4$ depends only on $\|(\rho_0, u_0, \theta_0)\|_{H^3}$. On the other hand, it follows from Theorem 1.1 that the solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)(t)$ of (1.1), (1.3) and (1.28) satisfies

$$\|(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)(t)\|_{X_m^\varepsilon} \leqq \tilde{C}_1, \quad \frac{1}{2\hat{C}_0} \leqq \rho^\varepsilon(t), \theta^\varepsilon(t) \leqq 2\hat{C}_0, \quad \forall t \in [0, T_0], \tag{5.2}$$

where T_0, $\hat{C}_0$, and $\tilde{C}_1$ are defined in Theorem 1.1. In particular, this uniform regularity implies the following bound:

$$\|(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)\|_{W^{1,\infty}} + \|\partial_t(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)\|_{L^\infty} \leqq \tilde{C}_1, \tag{5.3}$$

which plays an important role in the proof of Theorem 1.5.

Define

$$\phi^\varepsilon = \rho^\varepsilon - \rho, \quad \psi^\varepsilon = u^\varepsilon - u, \; \xi^\varepsilon = \theta^\varepsilon - \theta. \tag{5.4}$$

It then follows from (1.1) and (1.5) that

$$\begin{cases} \phi_t^\varepsilon + \rho \mathrm{div}\psi^\varepsilon + u \cdot \nabla\phi^\varepsilon = R_1^\varepsilon, \\ \rho\psi_t^\varepsilon + \rho u \cdot \nabla\psi^\varepsilon + \nabla(p^\varepsilon - p) + \Phi^\varepsilon \\ \qquad = -\mu\varepsilon\nabla \times (\nabla \times \psi^\varepsilon) + (2\mu + \lambda)\varepsilon\nabla\mathrm{div}\psi^\varepsilon + R_2^\varepsilon, \\ \rho\xi_t^\varepsilon + \rho u \cdot \nabla\xi^\varepsilon + p\mathrm{div}\psi^\varepsilon + \Psi^\varepsilon \\ \qquad = \kappa(\varepsilon)\Delta\xi^\varepsilon + 2\mu\varepsilon|Su^\varepsilon|^2 + \lambda\varepsilon|\mathrm{div}u^\varepsilon|^2 + R_3^\varepsilon, \end{cases} \tag{5.5}$$

where

$$\begin{cases} R_1^\varepsilon = -\phi^\varepsilon \mathrm{div}\psi^\varepsilon - \psi^\varepsilon \cdot \nabla\phi^\varepsilon - \phi^\varepsilon \mathrm{div}u - \nabla\rho \cdot \psi^\varepsilon, \\ R_2^\varepsilon = -\phi^\varepsilon \psi_t^\varepsilon - \phi^\varepsilon u_t + \mu\varepsilon\Delta u + (\mu + \lambda)\varepsilon\nabla\mathrm{div}u, \\ R_3^\varepsilon = -\phi^\varepsilon \xi_t^\varepsilon - (p^\varepsilon - p)\mathrm{div}\psi^\varepsilon - (p^\varepsilon - p)\mathrm{div}\psi + \kappa(\varepsilon)\Delta\theta, \end{cases} \tag{5.6}$$

and

$$\begin{cases} \Phi^\varepsilon = (\rho^\varepsilon u^\varepsilon - \rho u) \cdot \nabla u^\varepsilon = (\rho^\varepsilon u^\varepsilon - \rho u) \cdot \nabla \psi^\varepsilon + (\rho^\varepsilon u^\varepsilon - \rho u) \cdot \nabla u, \\ \Psi^\varepsilon = (\rho^\varepsilon u^\varepsilon - \rho u) \cdot \nabla \theta^\varepsilon = (\rho^\varepsilon u^\varepsilon - \rho u) \cdot \nabla \xi^\varepsilon + (\rho^\varepsilon u^\varepsilon - \rho u) \cdot \nabla \theta. \end{cases} \tag{5.7}$$

The boundary conditions to (5.5) are

$$\psi^\varepsilon \cdot n = 0, \quad n \times (\nabla \times \psi^\varepsilon) = [B\psi^\varepsilon]_\tau + [Bu]_\tau - n \times \omega,$$
$$\text{and } \nabla \xi^\varepsilon \cdot n = \nu \xi^\varepsilon + \nu\theta - \nabla\theta \cdot n \quad \text{on } \partial\Omega. \tag{5.8}$$

Lemma 5.1. *It holds that*

$$\|(\phi^\varepsilon, \psi^\varepsilon, \xi^\varepsilon)(t)\|^2 + \int_0^t \varepsilon \|\psi^\varepsilon\|_{H^1}^2 + \kappa(\varepsilon)\|\xi^\varepsilon\|_{H^1}^2 \,\mathrm{d}\tau$$
$$\leqq C(\varepsilon^{\frac{3}{2}} + \kappa(\varepsilon)^{\frac{3}{2}}), \quad t \in [0, T_2], \tag{5.9}$$

where $T_2 = \min\{T_0, T_1\}$*, and* $C > 0$ *depends only on* $\hat{C}_0$, $\tilde{C}_1$ *and* $\tilde{C}_4$.

Proof. Multiplying $(5.5)_2$ by ψ^ε, one obtains that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho|\psi^\varepsilon|^2\,\mathrm{d}x + \int_\Omega \Phi^\varepsilon \cdot \psi^\varepsilon\,\mathrm{d}x + \int_\Omega \nabla(p^\varepsilon - p)\cdot\psi^\varepsilon\,\mathrm{d}x$$
$$= -\mu\varepsilon\int_\Omega \nabla\times(\nabla\times\psi^\varepsilon)\cdot\psi^\varepsilon\,\mathrm{d}x + (2\mu+\lambda)\varepsilon\int_\Omega \nabla\mathrm{div}\psi^\varepsilon\cdot\psi^\varepsilon\,\mathrm{d}x$$
$$+ \int_\Omega R_2^\varepsilon\cdot\psi^\varepsilon\,\mathrm{d}x. \tag{5.10}$$

It follows from integrating by parts and $(5.5)_1$ that

$$\int_\Omega \nabla(p^\varepsilon - p)\cdot\psi^\varepsilon\,\mathrm{d}x = -\int_\Omega (p^\varepsilon - p)\mathrm{div}\psi^\varepsilon\,\mathrm{d}x$$
$$\geqq -R\int_\Omega \theta\phi^\varepsilon\mathrm{div}\psi^\varepsilon\,\mathrm{d}x - R\int_\Omega \rho\xi^\varepsilon\mathrm{div}\psi^\varepsilon\,\mathrm{d}x - C\|(\phi^\varepsilon,\xi^\varepsilon)\|^2$$
$$\geqq R\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\theta}{2\rho}|\phi^\varepsilon|^2\,\mathrm{d}x - R\int_\Omega \rho\xi^\varepsilon\mathrm{div}\psi^\varepsilon\,\mathrm{d}x - C\|(\phi^\varepsilon,\xi^\varepsilon)\|^2. \tag{5.11}$$

It is easy to check that

$$-\mu\varepsilon\int_\Omega \nabla\times(\nabla\times\psi^\varepsilon)\cdot\psi^\varepsilon\,\mathrm{d}x$$
$$= -\mu\varepsilon\int_\Omega |\nabla\times\psi^\varepsilon|^2\,\mathrm{d}x - \mu\varepsilon\int_{\partial\Omega} n\times(\nabla\times\psi^\varepsilon)\cdot\psi^\varepsilon\,\mathrm{d}x$$
$$\leqq -\mu\varepsilon\|\nabla\times\psi^\varepsilon\|^2 + C\varepsilon\left|\int_{\partial\Omega}[B\psi^\varepsilon + Bu - n\times\omega]\cdot\psi^\varepsilon\,\mathrm{d}x\right|$$
$$\leqq -\mu\varepsilon\|\nabla\times\psi^\varepsilon\|^2 + C\varepsilon\left(|\psi^\varepsilon|_{L^2(\partial\Omega)}^2 + |\psi^\varepsilon|_{L^2(\partial\Omega)}\right), \tag{5.12}$$

$$\varepsilon\int_\Omega \nabla\mathrm{div}\psi^\varepsilon\cdot\psi^\varepsilon\,\mathrm{d}x = -\varepsilon\|\mathrm{div}\psi^\varepsilon\|^2, \tag{5.13}$$

$$\left|\int_\Omega \Phi^\varepsilon\cdot\psi^\varepsilon\,\mathrm{d}x\right| = \left|\int_\Omega ((\rho^\varepsilon u^\varepsilon-\rho u)\cdot\nabla)u^\varepsilon\cdot\psi^\varepsilon\,\mathrm{d}x\right|$$
$$\leqq C(1+\|(\rho^\varepsilon,u^\varepsilon,\nabla u^\varepsilon)\|_{L^\infty})\|(\phi^\varepsilon,\psi^\varepsilon)\|_{L^2}^2, \tag{5.14}$$

and

$$\left|\int_\Omega R_2^\varepsilon\cdot\psi^\varepsilon\,\mathrm{d}x\right| \leqq C\|(\phi^\varepsilon,\psi^\varepsilon)\|_{L^2}^2 + C\varepsilon^2. \tag{5.15}$$

Collecting all the above estimates, one gets that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\psi^\varepsilon|^2 + R\frac{\theta}{2\rho}|\phi^\varepsilon|^2\,\mathrm{d}x - R\int_\Omega \rho\xi^\varepsilon\mathrm{div}\psi^\varepsilon\,\mathrm{d}x$$
$$+\mu\varepsilon\|\nabla\times\psi^\varepsilon\|^2 + (2\mu+\lambda)\varepsilon\|\mathrm{div}\psi^\varepsilon\|^2$$
$$\leqq C\|(\phi^\varepsilon,\psi^\varepsilon)\|_{L^2}^2 + C\varepsilon^2 + C\varepsilon(|\psi^\varepsilon|_{L^2}^2 + |\psi^\varepsilon|_{L^2}). \tag{5.16}$$

On the other hand, multiplying $\frac{\xi^\varepsilon}{\theta}$, one can obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2\theta}|\xi^\varepsilon|^2\,\mathrm{d}x + R\int_\Omega \rho\xi^\varepsilon\mathrm{div}\psi^\varepsilon\,\mathrm{d}x + \frac{3\kappa(\varepsilon)}{4}\int_\Omega \frac{|\nabla\xi^\varepsilon|^2}{\theta}\,\mathrm{d}x$$
$$\leqq C\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|^2 + C\kappa(\varepsilon)(|\xi^\varepsilon|_{L^2}^2 + |\xi^\varepsilon|_{L^2}) + C(\varepsilon^2+\kappa(\varepsilon)^2), \tag{5.17}$$

where we have used the facts that

$$\kappa(\varepsilon)\int_\Omega \Delta\xi^\varepsilon\frac{\xi^\varepsilon}{\theta}\,\mathrm{d}x = -\kappa(\varepsilon)\int_\Omega \frac{|\nabla\xi^\varepsilon|^2}{\theta}\,\mathrm{d}x + \kappa(\varepsilon)\int_\Omega \frac{\xi^\varepsilon}{\theta}\nabla\theta\cdot\nabla\xi^\varepsilon\,\mathrm{d}x$$
$$+\kappa(\varepsilon)\int_{\partial\Omega}\frac{\xi^\varepsilon}{\theta}n\cdot\nabla\xi^\varepsilon\,\mathrm{d}\sigma$$
$$\leqq -\frac{3\kappa(\varepsilon)}{4}\int_\Omega \frac{|\nabla\xi^\varepsilon|^2}{\theta}\,\mathrm{d}x + C\kappa(\varepsilon)\|\xi^\varepsilon\|^2$$
$$+C\kappa(\varepsilon)\left|\int_{\partial\Omega}\frac{\xi^\varepsilon}{\theta}[\nu\xi^\varepsilon+\varepsilon\theta-n\cdot\nabla\theta]\,\mathrm{d}\sigma\right|$$
$$\leqq -\frac{3\kappa(\varepsilon)}{4}\int_\Omega \frac{|\nabla\xi^\varepsilon|^2}{\theta}\,\mathrm{d}x + C\|\xi^\varepsilon\|^2 + C\kappa(\varepsilon)(|\xi^\varepsilon|_{L^2}^2 + |\xi^\varepsilon|_{L^2}) \tag{5.18}$$

and

$$\left|\int_\Omega \frac{\xi^\varepsilon}{\theta}\Psi^\varepsilon\,\mathrm{d}x\right| + \left|\int_\Omega \frac{\xi^\varepsilon}{\theta}R_3^\varepsilon\,\mathrm{d}x\right| + \left|\int_\Omega \frac{\xi^\varepsilon}{\theta}(2\mu\varepsilon|Su^\varepsilon|^2 + \lambda\varepsilon|\mathrm{div}u^\varepsilon|^2)\,\mathrm{d}x\right|$$
$$\leqq C\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|^2 + C(\varepsilon+\kappa(\varepsilon))\|\xi^\varepsilon\| \leqq C\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|^2 + C(\varepsilon^2+\kappa(\varepsilon)^2). \tag{5.19}$$

It follows from (2.1) that

$$\|\psi^\varepsilon\|_{H^1}^2 \leqq C_1(\|\nabla\times\psi^\varepsilon\|^2 + \|\mathrm{div}\psi^\varepsilon\|^2 + \|\psi^\varepsilon\|^2). \tag{5.20}$$

The trace theorem yields that

$$|\psi^\varepsilon|_{L^2}^2 \leqq \delta\|\nabla\psi^\varepsilon\|^2 + C_\delta\|\psi^\varepsilon\|^2, \text{ and } |\xi^\varepsilon|_{L^2}^2 \leqq \delta\|\nabla\xi^\varepsilon\|^2 + C_\delta\|\xi^\varepsilon\|^2 \tag{5.21}$$

and

$$\begin{cases} \varepsilon|\psi^\varepsilon|_{L^2} \leqq \delta\varepsilon\|\nabla\psi^\varepsilon\|^2 + C_\delta\varepsilon\|\psi^\varepsilon\|^{\frac{2}{3}} \leqq \delta\varepsilon\|\nabla\psi^\varepsilon\|^2 + \|\psi^\varepsilon\|^2 + C_\delta\varepsilon^{\frac{3}{2}}, \\ \kappa(\varepsilon)|\xi^\varepsilon|_{L^2} \leqq \delta\kappa(\varepsilon)\|\nabla\xi^\varepsilon\|^2 + C_\delta\kappa(\varepsilon)\|\xi^\varepsilon\|^{\frac{2}{3}} \\ \qquad\qquad\qquad \leqq \delta\kappa(\varepsilon)\|\nabla\xi^\varepsilon\|^2 + \|\xi^\varepsilon\|^2 + C_\delta\kappa(\varepsilon)^{\frac{3}{2}}. \end{cases} \tag{5.22}$$

Adding (5.16) and (5.17) together, using (5.20)–(5.22), and choosing δ suitably small, one obtains that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{\rho}{2}|\psi^\varepsilon|^2 + R\frac{\theta}{2\rho}|\phi^\varepsilon|^2 + \frac{\rho}{2\theta}|\xi^\varepsilon|^2\,\mathrm{d}x\right) + c_1(\varepsilon\|\psi^\varepsilon\|_{H^1}^2 + \kappa(\varepsilon)\|\xi^\varepsilon\|_{H^1}^2) \\ &\quad \leqq C\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{L^2}^2 + C(\varepsilon^{\frac{3}{2}} + \kappa(\varepsilon)^{\frac{3}{2}}), \end{aligned} \tag{5.23}$$

where $c_1 > 0$ is a positive constant independent of ε. Then Gronwall's inequality yields immediately that (5.9). Therefore, the proof of Lemma 5.1 is completed. □

Lemma 5.2. *It holds that*

$$\begin{aligned} &\|(\mathrm{div}\psi^\varepsilon, \nabla\phi^\varepsilon, \nabla\xi^\varepsilon)(t)\|^2 + \varepsilon\int_0^t \|\nabla\mathrm{div}\psi^\varepsilon(\tau)\|^2\,\mathrm{d}\tau + \kappa(\varepsilon)\int_0^t \|\Delta\xi^\varepsilon\|^2\,\mathrm{d}\tau \\ &\quad \leqq C\delta\int_0^t \|(\psi_t^\varepsilon,\xi_t^\varepsilon)\|^2 + \varepsilon\|\psi^\varepsilon\|_{H^2}^2\,\mathrm{d}\tau \\ &\qquad + C_\delta\int_0^t \|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}^2\,\mathrm{d}\tau + C_\delta[\varepsilon^{\frac{1}{2}} + \kappa(\varepsilon)^{\frac{1}{3}}], \quad t\in[0,T_2], \end{aligned} \tag{5.24}$$

where $\delta > 0$ will be chosen later.

Proof. Multiplying $(5.5)_2$ by $\nabla\mathrm{div}\psi^\varepsilon$ leads to

$$\begin{aligned} &\int_\Omega (\rho\psi_t^\varepsilon + \rho u\cdot\nabla\psi^\varepsilon)\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x + \int_\Omega \nabla(p^\varepsilon - p)\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x \\ &\quad = -\mu\varepsilon\int_\Omega \nabla\times(\nabla\times\psi^\varepsilon)\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x + (2\mu+\lambda)\varepsilon\|\nabla\mathrm{div}\psi^\varepsilon\|^2 \\ &\qquad + \int_\Omega R_2^\varepsilon\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x - \int_\Omega \Phi^\varepsilon\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x. \end{aligned} \tag{5.25}$$

It follows from $(5.5)_1$ that

$$\begin{aligned} \nabla\mathrm{div}\psi^\varepsilon = &-\frac{1}{\rho^\varepsilon}[\nabla\phi_t^\varepsilon + (u^\varepsilon\cdot\nabla)\nabla\phi^\varepsilon] \\ &- \frac{1}{\rho^\varepsilon}[\nabla\rho^\varepsilon\mathrm{div}\psi^\varepsilon - \nabla u^\varepsilon\nabla\phi^\varepsilon + \nabla(\phi^\varepsilon\mathrm{div}u + \psi^\varepsilon\cdot\nabla\rho)], \end{aligned} \tag{5.26}$$

which, together with integrating by parts, yields that

$$
\begin{aligned}
&\int_\Omega \nabla(p^\varepsilon - p)\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x = -R\int_\Omega \nabla(\rho\xi^\varepsilon + \theta\phi^\varepsilon + \phi^\varepsilon\xi^\varepsilon)\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\\
&\quad\leqq R\int_\Omega \rho\nabla\xi^\varepsilon\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x + R\int_\Omega \theta\nabla\phi^\varepsilon\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\\
&\qquad + R\left|\int_\Omega \mathrm{div}(\xi^\varepsilon\nabla\rho + \phi^\varepsilon\nabla\theta)\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\right|\\
&\qquad + R\left|\int_{\partial\Omega}(\rho\xi^\varepsilon + \theta\phi^\varepsilon)n\cdot\mathrm{div}\psi^\varepsilon\,\mathrm{d}\sigma\right| + C\|(\phi^\varepsilon,\xi^\varepsilon)\|_{H^1}^2\\
&\quad\leqq -R\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\theta}{2\rho^\varepsilon}|\nabla\phi^\varepsilon|^2\,\mathrm{d}x + R\int_\Omega \rho\nabla\xi^\varepsilon\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x + C\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}^2\\
&\qquad + C|(\phi^\varepsilon,\xi^\varepsilon)|_{L^2}.
\end{aligned}
\tag{5.27}
$$

It follows from (1.22) and (3.167) that

$$
\|\nabla\mathrm{div}u^\varepsilon\|_{L^\infty} + \|\nabla\mathrm{div}u^\varepsilon\|_{L^2} \leqq C < \infty,
\tag{5.28}
$$

where $C > 0$ depends only on $\tilde{C}_1$. Integrating by parts and using the Hölder inequality, one has that

$$
\begin{aligned}
&\int_\Omega \left(\rho\psi_t^\varepsilon + \rho u\cdot\nabla\psi^\varepsilon\right)\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x \leqq -\int_\Omega \left(\rho\mathrm{div}\psi_t^\varepsilon + \rho u\cdot\nabla\mathrm{div}\psi^\varepsilon\right)\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\\
&\qquad + \left|\int_\Omega \left(\nabla\rho\psi_t^\varepsilon + \nabla(\rho u)^t\nabla\psi^\varepsilon\right)\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\right| + \left|\int_{\partial\Omega}\rho(u\cdot\nabla)\psi^\varepsilon\cdot n\mathrm{div}\psi^\varepsilon\,\mathrm{d}\sigma\right|\\
&\quad\leqq -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\mathrm{div}\psi^\varepsilon|^2\,\mathrm{d}x + \delta\|\psi_t^\varepsilon\|^2 + C_\delta\|\nabla\psi^\varepsilon\|^2 + \left|\int_{\partial\Omega}\rho(u\cdot\nabla)n\psi^\varepsilon\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\right|\\
&\quad\leqq -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\mathrm{div}\psi^\varepsilon|^2\,\mathrm{d}x + \delta\|\psi_t^\varepsilon\|^2 + C_\delta\|\nabla\psi^\varepsilon\|^2 + C|\psi^\varepsilon|_{L^2},
\end{aligned}
\tag{5.29}
$$

$$
\begin{aligned}
\left|\int_\Omega \Phi^\varepsilon\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\right| &= \left|\int_\Omega [(\rho^\varepsilon u^\varepsilon - \rho u)\cdot\nabla\psi^\varepsilon + (\rho^\varepsilon u^\varepsilon - \rho u)\cdot\nabla u]\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\right|\\
&\leqq C\|(\phi^\varepsilon,\psi^\varepsilon)\|_{H^1}^2 + \left|\int_{\partial\Omega}((\rho^\varepsilon u^\varepsilon - \rho u)\cdot\nabla u)\cdot n\mathrm{div}\psi^\varepsilon\,\mathrm{d}\sigma\right|\\
&\leqq C(\|(\phi^\varepsilon,\psi^\varepsilon)\|_{H^1}^2 + |(\phi^\varepsilon,\psi^\varepsilon)|_{L^2}),
\end{aligned}
\tag{5.30}
$$

and

$$
\begin{aligned}
&\varepsilon\left|\int \nabla\times(\nabla\times\psi^\varepsilon)\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\right| = \varepsilon\left|\int_{\partial\Omega} n\times(\nabla\times\psi^\varepsilon)\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}\sigma\right|\\
&\quad = \varepsilon\left|\int_{\partial\Omega}\left(B\psi^\varepsilon + Bu - n\times\omega\right)\cdot\Pi(\nabla\mathrm{div}\psi^\varepsilon)\,\mathrm{d}\sigma\right|
\end{aligned}
$$

$$\leqq C\varepsilon\left(1+|\psi^\varepsilon|_{H^{\frac{1}{2}}}\right)|\mathrm{div}\psi^\varepsilon|_{H^{\frac{1}{2}}}$$
$$\leqq C\varepsilon\|\mathrm{div}\psi^\varepsilon\|_{H^1}(1+\|\psi^\varepsilon\|_{H^1})\leqq C(\varepsilon+\|\psi^\varepsilon\|_{H^1}^2). \tag{5.31}$$

For the term involving R_2^ε it follows from (5.28) and integrating by parts that

$$\left|\int R_2^\varepsilon\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x\right|\leqq C(1+\|\nabla\mathrm{div}u^\varepsilon\|_{L^\infty})[\|\phi^\varepsilon\|\|\psi_t^\varepsilon\|$$
$$+\|(\phi^\varepsilon,\psi^\varepsilon)\|_{H^1}^2]+\varepsilon\|u\|_{H^2}\|\nabla\mathrm{div}\psi^\varepsilon\|$$
$$\leqq\delta\|\psi_t^\varepsilon\|^2+C_\delta[\|(\phi^\varepsilon,\psi^\varepsilon)\|_{H^1}^2+\varepsilon^2]+C\varepsilon. \tag{5.32}$$

Then the trace theorem implies that

$$|(\phi^\varepsilon,\psi^\varepsilon,\xi)|_{L^2}\leqq C\{\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}^2+\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{L^2}^{\frac{2}{3}}\}$$
$$\leqq C\{\|(\phi^\varepsilon,\psi^\varepsilon,\xi)\|_{H^1}^2+\kappa(\varepsilon)^{\frac{1}{2}}+\varepsilon^{\frac{1}{2}}\}. \tag{5.33}$$

Substituting (5.27) and (5.29)–(5.32) into (5.25) and using the (5.33), one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\frac{\rho}{2}|\mathrm{div}\psi^\varepsilon|^2+\frac{R\theta}{2\rho^\varepsilon}|\nabla\phi^\varepsilon|^2\,\mathrm{d}x\right)-R\int_\Omega\rho\nabla\xi^\varepsilon\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x$$
$$+\frac{3}{4}(2\mu+\lambda)\varepsilon\|\nabla\mathrm{div}\psi^\varepsilon\|^2$$
$$\leqq\delta\|\psi_t^\varepsilon\|^2+C_\delta[\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}^2+\kappa(\varepsilon)^{\frac{1}{2}}+\varepsilon^{\frac{1}{2}}]. \tag{5.34}$$

Applying ∇ to $(5.5)_3$, one can obtain that

$$\rho^\varepsilon\nabla\xi_t^\varepsilon+\rho^\varepsilon(u^\varepsilon\cdot\nabla)\nabla\xi^\varepsilon+p\nabla\mathrm{div}\psi^\varepsilon-\kappa(\varepsilon)\Delta\nabla\xi^\varepsilon$$
$$=\varepsilon\nabla(2\mu|Su^\varepsilon|^2+\lambda|\mathrm{div}u^\varepsilon|^2)+\nabla\tilde{R}_3^\varepsilon-\nabla\rho^\varepsilon\xi_t^\varepsilon-\nabla(\rho^\varepsilon u^\varepsilon)\nabla\xi^\varepsilon-\nabla p\mathrm{div}\psi^\varepsilon, \tag{5.35}$$

where

$$\tilde{R}_3^\varepsilon=-\xi^\varepsilon\theta_t-(\rho^\varepsilon u^\varepsilon-\rho u)\cdot\nabla\theta-(p^\varepsilon-p)\mathrm{div}u^\varepsilon+\kappa(\varepsilon)\Delta\theta. \tag{5.36}$$

Multiplying (5.35) by $\frac{\nabla\xi^\varepsilon}{\theta}$, one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\rho^\varepsilon}{2\theta}|\nabla\xi^\varepsilon|^2\,\mathrm{d}x+R\int_\Omega\rho\nabla\xi^\varepsilon\cdot\nabla\mathrm{div}\psi^\varepsilon\,\mathrm{d}x-\kappa(\varepsilon)\int_\Omega\Delta\nabla\xi^\varepsilon\cdot\frac{\nabla\xi^\varepsilon}{\theta}\,\mathrm{d}x$$
$$\leqq\delta\varepsilon\|\psi^\varepsilon\|_{H^2}^2+\delta\|\xi_t^\varepsilon\|^2+C_\delta\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}^2+C_\delta(\varepsilon^2+\kappa(\varepsilon)^2). \tag{5.37}$$

It follows from integrating by parts and the boundary condition (5.8) that

$$-\kappa(\varepsilon)\int_\Omega\Delta\nabla\xi^\varepsilon\cdot\frac{\nabla\xi^\varepsilon}{\theta}\,\mathrm{d}x$$
$$\geqq\kappa(\varepsilon)\int_\Omega\frac{1}{\theta}|\Delta\xi^\varepsilon|^2\,\mathrm{d}x-C\kappa(\varepsilon)\|\nabla\xi^\varepsilon\|\|\Delta\xi^\varepsilon\|-C\kappa(\varepsilon)\left|\int_{\partial\Omega}\Delta\xi^\varepsilon\frac{n\cdot\nabla\xi^\varepsilon}{\theta}\,\mathrm{d}\sigma\right|$$

$$
\begin{aligned}
&\geqq \frac{3}{4}\kappa(\varepsilon)\int_{\Omega}\frac{1}{\theta}|\Delta\xi^{\varepsilon}|^{2}\,\mathrm{d}x - C\kappa(\varepsilon)\|\nabla\xi^{\varepsilon}\|^{2} - C\kappa(\varepsilon)|\Delta\xi^{\varepsilon}|_{L^{2}}\\
&\geqq \frac{3}{4}\kappa(\varepsilon)\int_{\Omega}\frac{1}{\theta}|\Delta\xi^{\varepsilon}|^{2}\,\mathrm{d}x - C\kappa(\varepsilon)\|\nabla\xi^{\varepsilon}\|^{2} - C\kappa(\varepsilon)\|\Delta\xi^{\varepsilon}\|_{L^{2}}^{\frac{1}{2}}\|\Delta\xi^{\varepsilon}\|_{H^{1}}^{\frac{1}{2}}\\
&\geqq \frac{1}{2}\kappa(\varepsilon)\int_{\Omega}\frac{1}{\theta}|\Delta\xi^{\varepsilon}|^{2}\,\mathrm{d}x - C\|\xi^{\varepsilon}\|_{H^{1}}^{2} - C\kappa(\varepsilon)^{\frac{1}{3}}[1+\kappa(\varepsilon)^{2}\|\nabla\Delta\xi^{\varepsilon}\|^{2}]. \quad (5.38)
\end{aligned}
$$

Substituting (5.38) into (5.37), one obtains that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\frac{\rho^{\varepsilon}}{2\theta}|\nabla\xi^{\varepsilon}|^{2}\,\mathrm{d}x + R\int_{\Omega}\rho\nabla\xi^{\varepsilon}\cdot\nabla\mathrm{div}\psi^{\varepsilon}\,\mathrm{d}x + \frac{1}{2}\kappa(\varepsilon)\int_{\Omega}\frac{1}{\theta}|\Delta\xi^{\varepsilon}|^{2}\,\mathrm{d}x\\
&\quad\leqq \delta\varepsilon\|\psi^{\varepsilon}\|_{H^{2}}^{2} + \delta\|\xi_{t}^{\varepsilon}\|^{2} + C_{\delta}\|(\phi^{\varepsilon},\psi^{\varepsilon},\xi^{\varepsilon})\|_{H^{1}}^{2}\\
&\qquad + C\kappa(\varepsilon)^{\frac{1}{3}}[1+\kappa(\varepsilon)^{2}\|\nabla\Delta\xi^{\varepsilon}\|^{2}] + C_{\delta}(\varepsilon^{2}+\kappa(\varepsilon)^{2}). \quad (5.39)
\end{aligned}
$$

Combining (5.34) and (5.39), one has that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega}\frac{\rho}{2}|\mathrm{div}\psi^{\varepsilon}|^{2} + \frac{R\theta}{2\rho^{\varepsilon}}|\nabla\phi^{\varepsilon}|^{2} + \frac{\rho^{\varepsilon}}{2\theta}|\nabla\xi^{\varepsilon}|^{2}\,\mathrm{d}x\right) + \frac{3}{4}(2\mu+\lambda)\varepsilon\|\nabla\mathrm{div}\psi^{\varepsilon}\|^{2}\\
&\quad + \frac{1}{2}\kappa(\varepsilon)\int_{\Omega}\frac{1}{\theta}|\Delta\xi^{\varepsilon}|^{2}\,\mathrm{d}x\\
&\leqq \delta\varepsilon\|\psi^{\varepsilon}\|_{H^{2}}^{2} + \delta\|\xi_{t}^{\varepsilon}\|^{2} + C_{\delta}\left[\|(\phi^{\varepsilon},\psi^{\varepsilon},\xi^{\varepsilon})\|_{H^{1}}^{2} + \kappa(\varepsilon)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}\right]\\
&\quad + C\kappa(\varepsilon)^{\frac{1}{3}}[1+\kappa(\varepsilon)^{2}\|\nabla\Delta\xi^{\varepsilon}\|^{2}]. \quad (5.40)
\end{aligned}
$$

It follows from Theorem 1.1 and (5.1) that

$$
\kappa(\varepsilon)^{2}\int_{0}^{t}\|\nabla\Delta\xi^{\varepsilon}\|^{2}\,\mathrm{d}\tau \leqq C < \infty. \quad (5.41)
$$

Then, integrating (5.40) over $[0, T_2]$ and using (5.41), one gets (5.24). Thus, the proof of Lemma 5.2 is completed. □

Lemma 5.3. *It holds that*

$$
\begin{aligned}
&\|\nabla\times\psi^{\varepsilon}\|^{2} + \varepsilon\int_{0}^{t}\|(\nabla\times\psi^{\varepsilon})(\tau)\|_{H^{1}}^{2}\,\mathrm{d}\tau\\
&\quad\leqq \delta\|\nabla(\phi^{\varepsilon},\psi^{\varepsilon},\xi^{\varepsilon})\|_{L^{2}}^{2} + C\delta\int_{0}^{t}\|\psi_{t}^{\varepsilon}\|^{2} + \varepsilon\|\nabla^{2}\psi^{\varepsilon}\|^{2}\,\mathrm{d}\tau\\
&\qquad + C_{\delta}\int_{0}^{t}\|(\phi^{\varepsilon},\psi^{\varepsilon},\xi^{\varepsilon})\|_{H^{1}}^{2}\,\mathrm{d}\tau + C_{\delta}[\varepsilon^{\frac{1}{2}}+\kappa(\varepsilon)^{\frac{1}{2}}], \quad (5.42)
\end{aligned}
$$

where $\delta > 0$ *will be chosen later.*

Proof. Multiplying $(5.5)_2$ by $\nabla\times(\nabla\times\psi^{\varepsilon})$ gives that

$$
\begin{aligned}
&\int_{\Omega}\rho^{\varepsilon}\psi_{t}^{\varepsilon}\cdot\nabla\times(\nabla\times\psi^{\varepsilon})\,\mathrm{d}x + \int_{\Omega}\nabla(p^{\varepsilon}-p)\cdot\nabla\times(\nabla\times\psi^{\varepsilon})\,\mathrm{d}x\\
&\quad + \mu\varepsilon\|\nabla\times(\nabla\times\psi^{\varepsilon})\|^{2}
\end{aligned}
$$

$$
\begin{aligned}
&= (2\mu+\lambda)\varepsilon \int_\Omega \nabla\times(\nabla\times\psi^\varepsilon)\cdot\nabla \mathrm{div}\psi^\varepsilon\,\mathrm{d}x + \int_\Omega \tilde{\Phi}^\varepsilon\cdot\nabla\times(\nabla\times\psi^\varepsilon)\,\mathrm{d}x \\
&\quad + \int_\Omega \tilde{R}_2^\varepsilon\cdot\nabla\times(\nabla\times\psi^\varepsilon)\,\mathrm{d}x, \qquad (5.43)
\end{aligned}
$$

where one has rewritten $(5.5)_2$ and

$$
\begin{aligned}
\tilde{R}_2^\varepsilon &= -\phi^\varepsilon u_t + \mu\varepsilon\Delta u + (\mu+\lambda)\varepsilon\nabla\mathrm{div}u, \\
\tilde{\Phi}^\varepsilon &= \rho^\varepsilon u^\varepsilon\cdot\nabla\psi^\varepsilon + (\rho^\varepsilon u^\varepsilon - \rho u)\cdot\nabla u.
\end{aligned}
$$

Integrating along the boundary, one has that

$$
\begin{aligned}
&\left|\int_\Omega \nabla(p^\varepsilon - p)\cdot\nabla\times(\nabla\times\psi^\varepsilon)\,\mathrm{d}x\right| \\
&\quad = \left|\int_{\partial\Omega} \nabla(p^\varepsilon - p)\cdot(n\times(\nabla\times\psi^\varepsilon))\,\mathrm{d}\sigma\right| \\
&\quad = \left|\int_{\partial\Omega} \Pi(\nabla(p^\varepsilon - p))\cdot[B\psi^\varepsilon + Bu - n\times\omega]\,\mathrm{d}x\right| \\
&\quad \leqq C\left[|p^\varepsilon - p|_{H^{\frac{1}{2}}}|\psi^\varepsilon|_{H^{\frac{1}{2}}} + |p^\varepsilon - p|_{L^2}\right] \\
&\quad \leqq C[\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}^2 + |(\phi^\varepsilon,\xi^\varepsilon)|_{L^2}]. \qquad (5.44)
\end{aligned}
$$

Note that the first term on the right hand side of (5.43) has been estimated in (5.31). It remains to estimate the other terms of (5.43). By the same argument as that of Lemma 6.3 of [22], one obtains that

$$
\begin{aligned}
&\int_\Omega \rho^\varepsilon\psi_t^\varepsilon\cdot\nabla\times(\nabla\times\psi^\varepsilon)\,\mathrm{d}x \\
&\quad \geqq \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}\rho^\varepsilon|\nabla\times\psi^\varepsilon|^2\,\mathrm{d}x + \int_{\partial\Omega}\frac{1}{2}\rho^\varepsilon\psi^\varepsilon B\psi^\varepsilon + \rho^\varepsilon\psi^\varepsilon\cdot(Bu - n\times\omega)\,\mathrm{d}\sigma\right) \\
&\quad - \delta\|\psi_t^\varepsilon\|^2 - C_\delta(\|\psi^\varepsilon\|_{H^1}^2 + |\psi^\varepsilon|_{L^2}), \qquad (5.45)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_\Omega \tilde{R}_2^\varepsilon\cdot\nabla\times(\nabla\times\psi^\varepsilon)\,\mathrm{d}x\right| \\
&\leqq C\|(\phi^\varepsilon,\psi^\varepsilon)\|_{H^1}^2 + \left|\int_{\partial\Omega}\phi^\varepsilon u_t\cdot(n\times(\nabla\times\psi^\varepsilon))\right| \\
&\qquad + C\varepsilon\|u\|_{H^3}\|\psi^\varepsilon\|_{H^1} + C\varepsilon\|u\|_{H^3}|\nabla\times\psi^\varepsilon|_{L^2} \\
&\leqq \delta\varepsilon\|\nabla\times(\nabla\times\psi^\varepsilon)\|^2 + C_\delta(\|(\phi^\varepsilon,\psi^\varepsilon)\|_{H^1}^2 + |(\phi^\varepsilon,\psi^\varepsilon)|_{L^2} + \varepsilon^{\frac{3}{2}}), \qquad (5.46)
\end{aligned}
$$

and

$$
\left|\int_\Omega \tilde{\Phi}^\varepsilon\cdot\nabla\times(\nabla\times\psi^\varepsilon)\,\mathrm{d}x\right| \leqq C[\|(\phi^\varepsilon,\psi^\varepsilon)\|_{H^1}^2 + |(\phi^\varepsilon,\psi^\varepsilon)|_{L^2}]. \qquad (5.47)
$$

Then, combining (5.43) and (5.44)–(5.47), one obtains that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}\rho^\varepsilon|\nabla\times\psi^\varepsilon|^2\,\mathrm{d}x+\int_{\partial\Omega}\frac{1}{2}\rho^\varepsilon\psi^\varepsilon B\psi^\varepsilon+\rho^\varepsilon\psi^\varepsilon\cdot(Bu-n\times\omega)\,\mathrm{d}\sigma\right)$$
$$+\frac{1}{2}\mu\varepsilon\|\nabla\times(\nabla\times\psi^\varepsilon)\|^2$$
$$\leqq C\delta\|\psi_t^\varepsilon\|^2+C\delta\varepsilon\|\nabla^2\psi^\varepsilon\|^2+C_\delta\left(\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}^2+\varepsilon^{\frac{1}{2}}+\kappa(\varepsilon)^{\frac{1}{2}}\right), \tag{5.48}$$

where we have used

$$\begin{cases} |(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)|_{L^2}\leqq C\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}^{\frac{1}{2}}\cdot\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|^{\frac{1}{2}} \\ \qquad\leqq \delta\|\nabla(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|^2+C_\delta(\varepsilon^{\frac{1}{2}}+\kappa(\varepsilon)^{\frac{1}{2}}), \\ |(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)|_{L^2}^2\leqq C\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}\cdot\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\| \\ \qquad\leqq \delta\|\nabla(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|^2+C_\delta(\varepsilon^{\frac{3}{2}}+\kappa(\varepsilon)^{\frac{3}{2}}), \end{cases} \tag{5.49}$$

which are consequences of the trace theorem and (5.9). It follows from (2.2) that

$$\|\nabla\times\psi^\varepsilon\|_{H^1}^2$$
$$\leqq C_1\left(\|\nabla\times(\nabla\times\psi^\varepsilon)\|^2+\|\mathrm{div}(\nabla\times\psi^\varepsilon)\|^2+\|\nabla\times\psi^\varepsilon\|^2+|n\times(\nabla\times\psi^\varepsilon)|_{H^{\frac{1}{2}}}^2\right)$$
$$\leqq C_1(\|\nabla\times(\nabla\times\psi^\varepsilon)\|^2+\|\nabla\times\psi^\varepsilon\|^2+|B\psi^\varepsilon|_{H^{\frac{1}{2}}}^2+|(Bu)_\tau-n\times\omega|_{H^{\frac{1}{2}}}^2),$$
$$\leqq C_1(\|\nabla\times(\nabla\times\psi^\varepsilon)\|^2+\|\psi^\varepsilon\|_{H^1}^2+C). \tag{5.50}$$

Substituting (5.50) into (5.48) yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}\rho^\varepsilon|\nabla\times\psi^\varepsilon|^2\,\mathrm{d}x+\int_{\partial\Omega}\frac{1}{2}\rho^\varepsilon\psi^\varepsilon B\psi^\varepsilon+\rho^\varepsilon\psi^\varepsilon\cdot(Bu-n\times\omega)\,\mathrm{d}\sigma\right)$$
$$+c_1\varepsilon\|\nabla\times\psi^\varepsilon\|_{H^1}^2$$
$$\leqq C\delta\|\psi_t^\varepsilon\|^2+C\delta\varepsilon\|\nabla^2\psi^\varepsilon\|^2+C_\delta\left(\|(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)\|_{H^1}^2+\varepsilon^{\frac{1}{2}}+\kappa(\varepsilon)^{\frac{1}{2}}\right). \tag{5.51}$$

Integrating (5.51) over $[0,t]$ and using (5.49), one gets (5.42). Therefore, the proof of Lemma 5.3 is completed. □

Proof of Theorem 1.5. It follows from (2.1) that

$$\|\psi^\varepsilon\|_{H^1}^2\leqq C\left(\|\nabla\times\psi^\varepsilon\|^2+\|\mathrm{div}\psi^\varepsilon\|^2+\|\psi^\varepsilon\|^2+|\psi^\varepsilon\cdot n|_{H^{\frac{1}{2}}}\right)$$
$$\leqq C(\|\nabla\times\psi^\varepsilon\|^2+\|\mathrm{div}\psi^\varepsilon\|^2+\|\psi^\varepsilon\|^2), \tag{5.52}$$

and

$$\|\psi^\varepsilon\|_{H^2}^2 \leqq C\left(\|\nabla\times\psi^\varepsilon\|_{H^1}^2+\|\mathrm{div}\psi^\varepsilon\|_{H^1}^2+\|\psi^\varepsilon\|_{H^1}^2+|\psi^\varepsilon\cdot n|_{H^{\frac{3}{2}}}\right)$$
$$\leqq C(\|\nabla\times\psi^\varepsilon\|_{H^1}^2+\|\mathrm{div}\psi^\varepsilon\|_{H^1}^2+\|\psi^\varepsilon\|_{H^1}^2), \tag{5.53}$$

while $(5.5)_2$ and $(5.5)_3$ imply that

$$\begin{cases}\|\psi_t^\varepsilon\|_{L^2}^2 \leqq C(\|(\phi^\varepsilon,\psi^\varepsilon)\|_{H^1}^2+\varepsilon^2\|\nabla^2\psi^\varepsilon\|_{L^2}^2+\varepsilon^2),\\ \|\xi_t^\varepsilon\|_{L^2}^2 \leqq C(\|(\phi^\varepsilon,\psi^\varepsilon)\|_{H^1}^2+\kappa(\varepsilon)^2\|\Delta\xi^\varepsilon\|_{L^2}^2+\varepsilon^2+\kappa(\varepsilon)^2).\end{cases} \tag{5.54}$$

Then, collecting (5.9), (5.24), (5.42), (5.52)–(5.54), and choosing δ suitably small, one obtains that

$$\|\nabla(\psi^\varepsilon,\phi^\varepsilon,\xi^\varepsilon)\|^2+\varepsilon\int_0^t\|\psi^\varepsilon(\tau)\|_{H^2}^2\,\mathrm{d}\tau+\kappa(\varepsilon)\int_0^t\|\Delta\xi^\varepsilon(\tau)\|^2\,\mathrm{d}\tau$$
$$\leqq C\int_0^t\|\nabla(\phi^\varepsilon,\psi^\varepsilon,\xi^\varepsilon)(\tau)\|^2\,\mathrm{d}\tau+C[\varepsilon^{\frac{1}{2}}+\kappa(\varepsilon)^{\frac{1}{3}}],$$

which, together with Gronwall's inequality, yields immediately that

$$\|\nabla(\psi^\varepsilon,\phi^\varepsilon,\xi^\varepsilon)\|^2+\varepsilon\int_0^t\|\psi^\varepsilon(\tau)\|_{H^2}^2\,\mathrm{d}\tau+\kappa(\varepsilon)\int_0^t\|\Delta\xi^\varepsilon(\tau)\|^2\,\mathrm{d}\tau\leqq C[\varepsilon^{\frac{1}{2}}+\kappa(\varepsilon)^{\frac{1}{3}}]. \tag{5.55}$$

Then, (5.9) and (5.55) imply (1.29)–(1.30). On the other hand, (1.31) is an immediate consequence of (1.29), (5.1) and (5.3). Thus, the proof of Theorem 1.5 is completed. □

6. Appendix

We have the following Lemma whose proof can be found in the "Appendix" of [22]:

Lemma 6.1. *Consider h to be a smooth solution of*

$$\begin{cases}a(t,y)[\partial_t h+b_1(t,y)\partial_{y^1}h+b_2(t,y)\partial_{y^2}h+zb_3(t,y)\partial_z h]\\ \qquad\qquad -\varepsilon\partial_{zz}h=G,\ z>0,\\ h(0,y,z)=h_0(y,z),\quad h(t,y,0)=0,\end{cases} \tag{6.1}$$

for some smooth function $a(t,y)$ satisfying $c_1\leqq a(t,y)\leqq\frac{1}{c_1}$ and vector fields $b=(b_1,b_2,b_3)^t(t,y)$. Assume that h and G are compactly supported in z. Then, one has the estimate:

$$\|h\|_{\mathcal{H}^{1,\infty}}\lesssim\|h_0\|_{\mathcal{H}^{1,\infty}}+\int_0^t\left\|\frac{1}{a}\right\|_{L^\infty}\|G\|_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau+\int_0^t\left(1+\|\left(\frac{1}{a},b\right)\|_{L^\infty}\right)$$
$$\times\left(1+\sum_{i=0}^2\|Z_i(a,b)\|_{L^\infty}^2\right)\|h\|_{\mathcal{H}^{1,\infty}}\,\mathrm{d}\tau. \tag{6.2}$$

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