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On the First Eigenvalues of Free Vibrating Membranes in Conformal Regular Domains

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Communicated by P. Rabinowitz

Abstract

In 1961 G. Polya published a paper about the eigenvalues of vibrating membranes. The "free vibrating membrane" corresponds to the Neumann–Laplace operator in bounded plane domains. In this paper we obtain estimates for the first non-trivial eigenvalue of this operator in a large class of domains that we call conformal regular domains. This class includes convex domains, John domains etc. On the basis of our estimates we conjecture that the eigenvalues of the Neumann–Laplace operator depend on the hyperbolic metrics of plane domains. We propose a new method for the estimates which is based on weighted Poincaré–Sobolev inequalities, obtained by the authors recently.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected plane domain with a smooth boundary $\partial\Omega$. We consider the Neumann–Laplace spectral problem (the free membrane problem)

$$- \Delta u = \lambda u \quad \text{in } \Omega, \tag{1}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{2}$$

The weak statement of the spectral problem of the Neumann–Laplace operator is as follows: a function u solves the previous problem iff $u \in W^{1,2}(\Omega)$ and

$$\iint_\Omega \nabla u(x, y) \cdot \nabla v(x, y) \, dxdy = \lambda \iint_\Omega u(x, y) v(x, y) \, dxdy$$

for all $v \in W^{1,2}(\Omega)$. The weak statement of the Neumann–Laplace spectral problem is correct for any domain.

Let $\psi : \mathbb{D} \to \Omega$ be the Riemann conformal mapping of the unit disc $\mathbb{D} \subset \mathbb{R}^2$ onto Ω that is a simply connected plane domain. We shall say that Ω is a conformal α-regular domain [7] if $\psi' \in L^\alpha(\mathbb{D})$ for some $\alpha > 2$. The degree α does not depends on choice of ψ (by the Riemann Mapping Theorem) and depends on the hyperbolic metric on Ω only.

The main result of the paper is:

Theorem A. *Let $\Omega \subset \mathbb{R}^2$ be a conformal α-regular domain. Then the spectrum of Neumann–Laplace operator in Ω is discrete, can be written in the form of a non-decreasing sequence*

$$0 = \lambda_0[\Omega] < \lambda_1[\Omega] \leqq \lambda_2[\Omega] \leqq \cdots \leqq \lambda_n[\Omega] \leqq \cdots ,$$

and

$$1/\lambda_1[\Omega] \leqq \frac{4}{\sqrt[\alpha]{\pi^2}} \left(\frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha-2}{\alpha}} \|\psi' \mid L^\alpha(\mathbb{D})\|^2 \tag{3}$$

where $\psi : \mathbb{D} \to \Omega$ is the Riemann conformal mapping of the unit disc $\mathbb{D} \subset \mathbb{R}^2$ onto Ω.

Note that Polya, in 1961 ([28]), obtained upper estimates for eigenvalues of the Neumann–Laplace operator in so-called plane-covering domains. Namely, for the first eigenvalue:

$$\lambda_1[\Omega] \leqq 4\pi |\Omega|^{-1}.$$

So, for the plane-covering conformal α-regular domains we have the two-side estimate:

$$\frac{\sqrt[\alpha]{\pi^2}}{4} \left(\frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2-2\alpha}{\alpha}} \frac{1}{\|\psi' \mid L^\alpha(\mathbb{D})\|^2} \leqq \lambda_1[\Omega] \leqq 4\pi \frac{1}{|\Omega|}.$$

The first non-trivial eigenvalue of the Neumann–Laplace operator is connected to the sharp constants in the isoperimetric inequalities [25]. Note that lower estimates of the first non-trivial eigenvalue of the Neumann–Laplace operator in terms of isoperimetric constants were considered in [3,4].

Theorem A is based on the existence of the universal weighted Poincaré–Sobolev inequality; that is to say, in any simply connected plane domain with non-empty boundary we have:

Theorem 1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a simply connected domain with non empty boundary; $\varphi : \Omega \to \mathbb{D}$ is a conformal homeomorphism and $h(x, y) = J_\varphi(x, y)$ is the conformal weight. Then for every function $f \in W^{1,2}(\Omega, h, 1)$, the weighted Poincaré–Sobolev inequality*

$$\left(\iint_\Omega |f(x, y) - f_{\Omega,h}|^r h(x, y) \, dx dy \right)^{\frac{1}{r}} \leqq B_{r,2}[\Omega, h] \left(\iint_\Omega |\nabla f(x, y)|^2 \, dx dy \right)^{\frac{1}{2}} \tag{4}$$

holds for every $r \in [1, \infty)$ with the exact constant

$$B_{r,2}[\Omega, h] = B_{r,2}[\mathbb{D}] \leqq 2\pi^{\frac{2-r}{2r}} \left((r+2)/2\right)^{\frac{r+2}{2r}}$$

where $B_{r,2}[\mathbb{D}]$ is the exact constant in the Poincaré inequality for the unit disk $\mathbb{D}$

$$\left(\int_{\mathbb{D}} |g(x, y) - g_{\mathbb{D}}|^r \, dxdy\right)^{\frac{1}{r}} \leqq B_{r,2}[\mathbb{D}] \left(\int_{\mathbb{D}} |\nabla g(x, y)|^2 \, dxdy\right)^{\frac{1}{2}}.$$

The paper is organized as follows. In Section 2 we discuss the notion of conformal regular domains, formulate the Poincaré–Sobolev inequality for conformal regular domains and discuss its connection with the Neumann–Laplace operator. The main point is an estimate for the constant in this inequality. We also discuss connection between the Poincaré–Sobolev inequality and composition operators on Sobolev spaces. Section 2 can be seen as an extension of the introduction. In Section 3 we prove main facts about composition operators in the conformal regular domains. In Section 4 we prove the Poincaré–Sobolev inequality for conformal regular domains. In Sections 5 and 6 we apply the results of Section 4 to lower estimates of the first non-trivial eigenvalue for the Neumann–Laplace operator in the conformal regular domains. In Section 7 we compare the lower estimates with previous results available in the literature.

2. The Neumann–Laplace Problem in Conformal Regular Domains

Let $\Omega \subset \mathbb{R}^2$ be a simply connected plane domain of finite area and $\psi : \mathbb{D} \to \Omega$ be a conformal mapping. Then

$$\iint_{\mathbb{D}} |\psi'(u, v)|^2 \, dudv = \iint_{\mathbb{D}} J_{\psi}(u, v) \, dudv = |\Omega| < \infty.$$

Integrability of the derivative in the power $\alpha > 2$ is impossible without additional assumptions on the geometry of Ω. We proved in [19] that the integrability in the power $\alpha > 2$ leads to finiteness of the geodesic diameter of Ω and as result it is a bounded domain.

A domain Ω is a conformal regular domain if it is a conformal α-regular domain for some $\alpha > 2$. Note that any C^2-smooth simply connected bounded domain domain is ∞-regular (see, for example, [24]).

The notion of conformal regular domains was introduced in [7] and was applied to the stability problem for eigenvalues of the Dirichlet–Laplace operator. It does not depend on choice of a conformal mapping $\psi : \mathbb{D} \to \Omega$ and can be reformulated in terms of the hyperbolic metrics [7]. That is,

$$\iint_{\mathbb{D}} |\psi'(u, v)|^\alpha \, dudv = \iint_{\mathbb{D}} \left(\frac{\lambda_{\mathbb{D}}(u, v)}{\lambda_{\Omega}(\psi(u, v))}\right)^\alpha \, dudv$$

when $\lambda_{\mathbb{D}}$ and λ_{Ω} are hyperbolic metrics in $\mathbb{D}$ and Ω [5].

Note that a boundary $\partial \Omega$ of a conformal regular domain can have any Hausdorff dimension between one and two, but cannot be equal to two [23].

In Section 4 we prove the following Poincaré–Sobolev inequality for conformal regular domains:

Theorem 2. *Suppose that $\Omega \subset \mathbb{R}^2$ is a conformal α-regular domain. Then for every function $f \in W^{1,2}(\Omega)$, the inequality*

$$\inf_{c \in \mathbb{R}} \left(\iint_\Omega |f(x,y) - c|^s \, dx dy \right)^{\frac{1}{s}} \leqq B_{s,2}[\Omega] \left(\iint_\Omega |\nabla f(x,y)|^2 \, dx dy \right)^{\frac{1}{2}} \tag{5}$$

holds with the constant

$$B_{s,2}[\Omega] \leqq \|J_{\varphi^{-1}}|L^{\frac{r}{r-s}}(\mathbb{D})\|^{\frac{1}{s}} B_{r,2}[\Omega, h]$$
$$\leqq 2\pi^{\frac{2-r}{2r}} ((r+2)/2)^{\frac{r+2}{2r}} \cdot \|J_{\varphi^{-1}}|L^{\frac{r}{r-s}}(\mathbb{D})\|^{\frac{1}{s}}$$

for any $s \in [1, \infty)$, where $r = \alpha s/(\alpha - 2)$.

Remark 1. The conformal regular domains have an equivalent description in the terms of the β-(quasi)hyperbolic boundary condition [6,20]. In [20] it was proved (without estimates of constants) that domains with β-(quasi)hyperbolic boundary conditions support the (s, p)-Poincaré–Sobolev inequalities for p that depends on β.

The existence of the Poincaré–Sobolev inequality is an essential property of the conformal regular domains. In [19] we proved but did not formulate the following fact about conformal regular domains:

Theorem 3. *If a simply connected domain $\Omega \subset \mathbb{R}^2$ of finite measure does not support the (s,2)-Poincaré–Sobolev inequality*

$$\left(\int_\Omega |f(x,y) - f_\Omega|^s \, dx dy \right)^{\frac{1}{s}} \leqq B_{s,2}[\Omega] \left(\int_\Omega |\nabla f(x,y)|^2 \, dx dy \right)^{\frac{1}{2}}$$

for some $s \geq 2$, then Ω is not a conformal regular domain.

Remark 2. Conformal regular domains allow narrow gaps which can destroy the John condition [22].

It is well known that solvability of the Neumann–Laplace problem and its spectrum discreteness depends on the regularity of Ω (see, for example, [25]).

In the present work we suggest a new method for the study of the Poincaré–Sobolev inequality in the conformal regular domains. This method is based on the composition operators theory on Sobolev spaces, and allows us to estimate constants in the Poincaré–Sobolev inequalities.

As an application we study the eigenvalues problem for the Neumann–Laplace operator. A detailed survey of this eigenvalue problem can be found in [26] (for example). A global lower bound of the non-trivial first eigenvalue $\lambda_1[\Omega]$ for convex domains was obtained in [27]. We obtain a global lower bound for the first eigenvalue $\lambda_1[\Omega]$ in conformal α-regular domains, which are not necessary convex domains.

The suggested method is based on the composition operators theory [31,32] and its application to the Sobolev type embedding theorems [14,15]. The following diagram illustrate this idea:

$$
\begin{array}{ccc}
W^{1,p}(\Omega) & \xrightarrow{(\varphi^{-1})^*} & W^{1,q}(\mathbb{D}) \\
\downarrow & & \downarrow \\
L^s(\Omega) & \xleftarrow{\varphi^*} & L^r(\mathbb{D}).
\end{array}
$$

Here the operator φ^* defined by the composition rule $\varphi^*(f) = f \circ \varphi$ is a bounded composition operator on Lebesgue spaces induced by a homeomorphism φ of Ω and $\mathbb{D}$ and the operator $(\varphi^{-1})^*$ defined by the composition rule $(\varphi^{-1})^*(f) = f \circ \varphi^{-1}$ is a bounded composition operator on Sobolev spaces.

Remark 3. In recent works we studied composition operators on Sobolev spaces defined on planar domains in connection with the conformal mappings theory [16]. This connection leads to weighted Sobolev embeddings [17,18] with the universal conformal weights. Another application of conformal composition operators was given in [7], where the spectral stability problem for conformal regular domains was considered.

3. Composition Operators

3.1. Composition Operators on Lebesgue Spaces

For any domain $\Omega \subset \mathbb{R}^2$ and any $1 \leqq p < \infty$ we consider the Lebesgue space

$$
L^p(\Omega) := \left\{ f : \Omega \to R : \|f \mid L^p(\Omega)\| := \left(\iint_\Omega |f(x,y)|^p \, dxdy \right)^{1/p} < \infty \right\}.
$$

The following theorem about the composition operator on Lebesgue spaces is well known (see, for example [32]):

Theorem 4. *A diffeomorphism $\varphi : \Omega \to \Omega'$ between two plane domains Ω and Ω' induces a bounded composition operator*

$$
\varphi^* : L^r(\Omega') \to L^s(\Omega), \quad 1 \leqq s \leqq r < \infty,
$$

(by the chain rule $\varphi^*(f) := f \circ \varphi$) if and only if

$$
\left(\iint_{\Omega'} \left(J_{\varphi^{-1}}(u,v) \right)^{\frac{r}{r-s}} \, dudv \right)^{\frac{r-s}{rs}} = K < \infty, \quad 1 \leqq s < r < \infty,
$$

$$
\left(\mathrm{esssup}_{(u,v) \in \Omega'} \, J_{\varphi^{-1}}(u,v) \right)^{\frac{1}{s}} = K < \infty, \quad 1 \leqq s = r < \infty.
$$

The norm of the composition operator $\|\varphi^*\| = K$.

3.2. Composition Operators on Sobolev Spaces

We define the Sobolev space $W^{1,p}(\Omega)$, $1 \leqq p < \infty$ as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^{1,p}(\Omega)\| = \left(\iint_{\Omega} |f(x,y)|^p \, dxdy\right)^{\frac{1}{p}} + \left(\iint_{\Omega} |\nabla f(x,y)|^p \, dxdy\right)^{\frac{1}{p}}.$$

We also define the homogeneous seminormed Sobolev space $L^{1,p}(\Omega)$ of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^{1,p}(\Omega)\| = \left(\iint_{\Omega} |\nabla f(x,y)|^p \, dxdy\right)^{\frac{1}{p}}.$$

Recall that the embedding operator $i : L^{1,p}(\Omega) \to L^1_{\mathrm{loc}}(\Omega)$ is continuous.

Remark 4. By the standard definition functions of $L^{1,p}(\Omega)$ are defined only up to a set of measure zero, but they can be redefined quasi-everywhere that is, up to a set of p-capacity zero (see, for example [21,25]).

Let Ω and Ω' be domains in $\mathbb{R}^2$. We say that a diffeomorphism $\varphi : \Omega \to \Omega'$ induces a bounded composition operator

$$\varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega), \quad 1 \leqq q \leqq p \leqq \infty$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if for any $f \in L^{1,p}(\Omega')$ the composition $\varphi^*(f) \in L^{1,q}(\Omega)$ and there exists a constant $K < \infty$ such that

$$\|\varphi^*(f) \mid L^{1,q}(\Omega)\| \leqq K \|f \mid L^{1,p}(\Omega')\|.$$

The main result of [31,32] gives an analytic description of composition operators on Sobolev spaces $L^{1,p}$. We reproduce it here for diffeomorphisms.

Theorem 5. [31] *A diffeomorphism $\varphi : \Omega \to \Omega'$ between two domains Ω and Ω' induces a bounded composition operator*

$$\varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega), \quad 1 \leqq q < p < \infty,$$

if and only if

$$K_{p,q}(\varphi; \Omega) = \left(\iint_{\Omega} \left(\frac{|\varphi'(x,y)|^p}{|J_\varphi(x,y)|}\right)^{\frac{q}{p-q}} dxdy\right)^{\frac{p-q}{pq}} < \infty.$$

The norm of the composition operator $\|\varphi^*\| \leqq K_{p,q}(\varphi; \Omega)$.

Definition 1. We call a bounded domain $\Omega \subset \mathbb{C}$ as (r, q)-Poincaré domain, $1 \leqq q, r \leqq \infty$, if the Poincaré–Sobolev inequality

$$\inf_{c \in \mathbb{R}} \|g - c \mid L^r(\Omega)\| \leqq B_{r,q}[\Omega] \|g \mid L^{1,q}(\Omega)\|$$

holds for any $g \in L^{1,q}(\Omega)$ with the constant $B_{r,q}[\Omega] < \infty$. The unit disc $\mathbb{D} \subset \mathbb{R}^2$ is an example of the $(r, 2)$-embedding domain for all $r \geq 1$.

The following theorem gives a characterization of composition operators in the classical Sobolev spaces $W^{1,p}$ (this theorem was proved, but not formulated, in [14,15]): For readers convenience we reproduce here the proof.

Theorem 6. *Let $\Omega \subset \mathbb{R}^n$ be an (r, q)-Poincaré domain for some $1 \leq q \leq r \leq \infty$ and a domain Ω' has finite measure. Suppose that a diffeomorphism $\varphi : \Omega \to \Omega'$ induces a bounded composition operator*

$$\varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega), \quad 1 \leqq q \leqq p < \infty,$$

and the inverse diffeomorphism $\varphi^{-1} : \Omega' \to \Omega$ induces a bounded composition operator

$$(\varphi^{-1})^* : L^r(\Omega) \to L^s(\Omega'), \quad 1 \leqq s \leqq r < \infty,$$

for some $p \leqq s \leqq r$.
Then $\varphi : \Omega \to \Omega'$ induces a bounded composition operator

$$\varphi^* : W^{1,p}(\Omega') \to W^{1,q}(\Omega), \quad 1 \leqq q \leqq p < \infty.$$

Proof. Let $f \in W^{1,p}(\Omega)$ and $g = \varphi^*(f)$. Because $p \leqq r$ and the composition operator $(\varphi^{-1})^* : L^r(\Omega) \to L^s(\Omega')$ is bounded, the inequality

$$\|(\varphi^{-1})^* g \mid L^s(\Omega')\| \leqq A_{r,s}(\Omega) \|g \mid L^r(\Omega)\|$$

is correct for a positive constant $A_{r,s}(\Omega)$.
Since domain Ω is a (r, q)-Poincaré domain

$$\inf_{c \in \mathbb{R}} \|g - c \mid L^r(\Omega)\| \leqq B_{r,q}[\Omega] \|g \mid L^{1,q}(\Omega)\|.$$

and the composition operator

$$\varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega)$$

is bounded, we obtain the following inequalities:

$$\inf_{c \in \mathbb{R}} \|f - c \mid L^s(\Omega')\| \leqq A_{r,s}(\Omega) \inf_{c \in \mathbb{R}} \|g - c \mid L^r(\Omega)\|$$
$$\leqq A_{r,s}(\Omega) B_{r,q}[\Omega] \|g \mid L^{1,q}(\Omega)\| \leqq A_{r,s}(\Omega) K_{p,q}(\Omega) B_{r,q}[\Omega] \|f \mid L^{1,p}(\Omega')\|.$$

Here $K_{p,q}(\Omega)$ is the upper bound of the norm of the composition operator $\varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega)$.

The Hölder inequality implies the following estimate:

$$|c| = |\Omega'|^{-\frac{1}{p}}\|c \mid L^p(\Omega')\| \leqq |\Omega'|^{-\frac{1}{p}}\left(\|f \mid L^p(\Omega')\| + \|f - c \mid L^p(\Omega')\|\right)$$
$$\leqq |\Omega'|^{-\frac{1}{p}}\|f \mid L^p(\Omega')\| + |\Omega'|^{-\frac{1}{s}}\|f - c \mid L^s(\Omega')\|.$$

Because $q \leqq r$, we have

$$\|g \mid L^q(\Omega)\| \leqq \|c \mid L^q(\Omega)\| + \|g - c \mid L^q(\Omega)\| \leqq |c||\Omega|^{\frac{1}{q}}$$
$$+ |\Omega|^{\frac{r-q}{r}}\|g - c \mid L^r(\Omega)\|$$
$$\leqq \left(|\Omega'|^{-\frac{1}{p}}\|f \mid L^p(\Omega')\| + |\Omega'|^{-\frac{1}{s}}\|f - c \mid L_s(\Omega')\|\right)|\Omega|^{\frac{1}{q}}$$
$$+ |\Omega|^{\frac{r-q}{r}}\|g - c \mid L^r(\Omega)\|.$$

Combining previous inequalities we finally obtain

$$\|g \mid L^q(\Omega)\| \leqq |\Omega|^{\frac{1}{q}}|\Omega'|^{-\frac{1}{p}}\|f \mid L^p(\Omega')\|$$
$$+ A_{r,s}(\Omega)K_{p,q}(\Omega)B_{r,q}[\Omega]|\Omega|^{\frac{1}{q}}|\Omega'|^{-\frac{1}{p}}\|f \mid L^{1,p}(\Omega')\|$$
$$+ K_{p,q}(\Omega)B_{r,q}[\Omega]|\Omega|^{\frac{r-q}{r}}\|f \mid L^{1,p}(\Omega)\|.$$

Therefore the composition operator

$$\varphi^* : W^{1,p}(\Omega') \to W^{1,q}(\Omega)$$

is bounded. $\square$

4. Poincaré–Sobolev Inequalities for Functions of $W^{1,2}(\Omega)$

4.1. Weighted Lebesgue Spaces

We follow [21] for notation and basic facts about weighted Lebesgue spaces.

Let $\Omega \subset \mathbb{R}^2$ be a domain and let $v : \Omega \to \mathbb{R}$ be a locally integrable almost everywhere positive real valued function in Ω (that is $v > 0$ almost everywhere). Then a Radon measure ν is canonically associated with the weight function v:

$$\nu(E) := \iint_E v(x, y)\, dxdy.$$

By the local integrability of v, the measure ν and the Lebesgue measure are absolutely continuous with respect one to another, that is,

$$d\nu = v(x, y)dxdy.$$

In what follows, the weight v and the measure ν will be identified. The sets of measure zero are the same for the Lebesgue measure and for the measure ν. That means that, almost everywhere, we do not need to specify the convergence.

Denote by

$$\mathcal{V}(\Omega) := \{v \in L^1_{\text{loc}}(\Omega) : v > 0 \text{ almost everywhere on } \Omega\}$$

the set of all such weights. Here $L^1_{\text{loc}}(\Omega)$ is the space of locally integrable functions in Ω.

For $1 \leqq p < \infty$ and $v \in \mathcal{V}(\Omega)$, consider the weighted Lebesgue space

$$L^p(\Omega, v)$$
$$:= \left\{ f : \Omega \to R : \|f \mid L^p(\Omega, v)\| := \left(\iint_\Omega |f(x, y)|^p v(x, y) \, dxdy \right)^{1/p} < \infty \right\}.$$

That is a Banach space for the norm $\|f \mid L^p(\Omega, v)\|$.

The space $L^p(\Omega, v)$ may fail to embed into $L^1_{\text{loc}}(\Omega)$.

Proposition 1. [21] *If* $v^{\frac{1}{1-p}} \in L^1_{\text{loc}}(\Omega)$ *and* $1 < p < \infty$, *then the embedding operator* $i : L^p(\Omega, v) \to L^1_{\text{loc}}(\Omega)$, *is continuous.*

If $v^{-1} \in L^\infty_{\text{loc}}(\Omega)$, *then the embedding operator* $i : L^1(\Omega, v) \to L^1_{\text{loc}}(\Omega)$ *is continuous.*

For $1 < p < \infty$, we put

$$\mathcal{V}_p(\Omega) := \left\{ v \in \mathcal{V}(\Omega) : v^{\frac{1}{1-p}} \in L^1_{\text{loc}}(\Omega) \right\},$$

and for $p = 1$,

$$\mathcal{V}_1(\Omega) := \left\{ v \in \mathcal{V}(\Omega) : v^{-1} \in L^\infty_{\text{loc}}(\Omega) \right\}.$$

Corollary 1. *If a weight v is continuous and positive then* $i : L^p(\Omega, v) \to L^1_{\text{loc}}(\Omega)$ *is continuous.*

This follows immediately from Proposition 1 because a continuous and positive weight belongs to $\mathcal{V}_p(\Omega)$ and also to $\mathcal{V}_1(\Omega)$.

4.2. Weighted Poincaré–Sobolev Inequalities

Let $\varphi : \Omega \to \Omega'$ be a conformal homeomorphism. The following fact is well-known:

Lemma 1. *Let Ω and Ω' be two plane domains. Any conformal homeomorphism $w = \varphi(z) : \Omega \to \Omega'$ induces an isometry of spaces $L^{1,2}(\Omega')$ and $L^{1,2}(\Omega)$.*

Proof. Let $f \in L^{1,2}(\Omega')$ be a smooth function. The smooth function $g = f \circ \varphi$ belongs to $L^{1,2}(\Omega)$ because

$$\|\nabla g \mid L^2(\Omega)\| = \left(\iint_\Omega |\nabla(f \circ \varphi(x, y))|^2 \, dxdy\right)^{\frac{1}{2}}$$

$$= \left(\iint_\Omega |\nabla f|^2(\varphi(x, y))|\varphi'(x, y))|^2 \, dxdy\right)^{\frac{1}{2}}$$

$$= \left(\iint_\Omega |\nabla f|^2(\varphi(x, y))J_\varphi(x, y) \, dxdy\right)^{\frac{1}{2}}$$

$$= \left(\iint_{\Omega'} |\nabla f|^2(u, v) \, dudv\right) = \|\nabla f \mid L^2(\Omega')\|.$$

We used the equality: $|\varphi'(x, y))|^2 = J_\varphi(x, y)$, which is correct for any conformal homeomorphism.

Approximating an arbitrary function $f \in L^{1,2}(\Omega')$ by smooth functions, we obtain an isometry between $L^{1,2}(\Omega')$ and $L^{1,2}(\Omega)$. $\square$

We define the weighted Sobolev space $W^{1,p}(\Omega, h, 1)$, $1 \leqq p < \infty$ as the normed space of all locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ with the finite norm given by

$$\|f \mid W^{1,p}(\Omega, h, 1)\| = \|f \mid L^p(\Omega, h)\| + \|\nabla f \mid L^p(\Omega)\|.$$

The existence of the Poincaré–Sobolev inequalities depends on a conformal (hyperbolic) geometry of Ω. For any conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$ define the conformal (hyperbolic) weight $h(x, y) := J_\varphi(x, y)$.

We denote

$$f_{\Omega,h} := \frac{1}{m_h(\Omega)} \iint_{\mathbb{D}} f(z)h(z) \, dxdy = g_{\mathbb{D}} = \frac{1}{|\mathbb{D}|} \iint_{\mathbb{D}} g(w) \, dudv,$$
$$f(z) = g(\varphi(z)), \quad w = \varphi(z).$$

Here

$$m_h(\Omega) = \iint_\Omega h(z) \, dxdy = \iint_\Omega J_\varphi(z) \, dxdy = \iint_{\mathbb{D}} dudv = |\mathbb{D}|.$$

The following "universal" weighted Poincaré–Sobolev inequality is correct for any simply connected plane domain with non-empty boundary.

Theorem 7. *Let Ω be a simply connected plane domain with non-empty boundary. Then for any function $f \in W^{1,2}(\Omega, h, 1)$ the weighted Poincaré–Sobolev inequality*

$$\left(\iint_\Omega |f(x, y) - f_{\Omega,h}|^r h(x, y) \, dxdy\right)^{\frac{1}{r}} \leqq B_{r,2}[\Omega, h]\left(\iint_\Omega |\nabla f(x, y)|^2 \, dxdy\right)^{\frac{1}{2}}$$

holds for any $r \geqq 1$ with the exact constant $B_{r,2}[\Omega, h] = B_{r,2}[\mathbb{D}]$ where $B_{r,2}[\mathbb{D}]$ is the exact constant of the Poincaré inequality in the unit disk

$$\left(\int_{\mathbb{D}} |g(x, y) - g_{\mathbb{D}}|^r \, dxdy\right)^{\frac{1}{r}} \leqq B_{r,2}[\mathbb{D}]\left(\int_{\mathbb{D}} |\nabla g(x, y)|^2 \, dxdy\right)^{\frac{1}{2}}.$$

Proof. Let $r \geq 1$. By the Riemann Mapping Theorem there exists a conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$. Using the change of variable formula for conformal homeomorphism, the Poincaré–Sobolev inequality in the unit disc $\mathbb{D} \subset \mathbb{R}^2$ and Lemma 1 we get

$$\left(\iint_\Omega |f(x, y) - f_{\Omega,h}|^r h(x, y) \, dxdy \right)^{\frac{1}{r}} = \left(\iint_\Omega |f(x, y) - g_\mathbb{D}|^r h(x, y) \, dxdy \right)^{\frac{1}{r}}$$

$$= \left(\iint_\Omega |f(x, y) - g_\mathbb{D}|^r J_\varphi(x, y) \, dxdy \right)^{\frac{1}{r}} = \left(\iint_\mathbb{D} |g(u, v) - g_\mathbb{D}|^r \, dudv \right)^{\frac{1}{r}}$$

$$\leq B_{r,2}[\mathbb{D}] \left(\iint_\mathbb{D} |\nabla g(u, v)|^2 \, dudv \right)^{\frac{1}{2}} = B_{r,2}[\mathbb{D}] \left(\iint_\Omega |\nabla f(x, y)|^2 \, dxdy \right)^{\frac{1}{2}}$$

for any function $f \in W^{1,2}(\Omega, h, 1)$. $\square$

Let us estimate $B_{r,2}[\mathbb{D}]$ using the following n-dimensional inequalities [12]. For any $\mu \in (0, 1)$ and any domain $\Omega \subset \mathbb{R}^n$ define the operator V_μ acting on $L^1(\Omega)$ by the expression

$$(V_\mu f)(x) = \int_\Omega |x - y|^{n(\mu-1)} f(y) \, dy.$$

Here $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $dy = dy_1...dy_n$.

Lemma 2. [12] *The operator V_μ maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ for any q, $1 \leq q \leq \infty$, satisfying*

$$0 \leq \delta = \delta(p, q) = p^{-1} - q^{-1} < \mu.$$

Furthermore, for any $f \in L^p(\Omega)$,

$$\|V_\mu f \mid L^q(\Omega)\| \leq \left(\frac{1 - \delta}{\mu - \delta} \right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta} \|f \mid L^p(\Omega)\|.$$

Here $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$.

In the convex domains there are the following point-wise estimates:

Lemma 3. [12] *Let $\Omega \subset \mathbb{R}^n$ be a convex domain and $f \in W^{1,p}(\Omega)$. Then*

$$|f(x) - f_\Omega| \leq \frac{d^n}{n|\Omega|} \int_\Omega |x - y|^{1-n} |\nabla f(y)| \, dy \quad \text{almost everywhere in } \Omega,$$

where d is the diameter of Ω.

From these two lemmas follows:

Proposition 2. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain and $f \in W^{1,p}(\Omega)$. Then*

$$\left(\int_\Omega |f(x) - f_\Omega|^q \, dx \right)^{\frac{1}{q}}$$

$$\leqq \frac{d^n}{n|\Omega|} \left(\frac{1 - \frac{1}{p} + \frac{1}{q}}{\frac{1}{n} - \frac{1}{p} + \frac{1}{q}} \right)^{1 - \frac{1}{p} + \frac{1}{q}} \omega_n^{-1 - \frac{1}{n}} |\Omega|^{\frac{1}{n} - \frac{1}{p} + \frac{1}{q}} \||\nabla f| \mid L^p(\Omega)\|.$$

Proof. We take $\mu = 1/n$. Then for function $f \in W^{1,p}(\Omega)$ we have

$$\left(\int_\Omega |f(x) - f_\Omega|^q \, dx \right)^{\frac{1}{q}} \leqq \frac{d^n}{n|\Omega|} \left(\int_\Omega \left| \int_\Omega |x - y|^{1-n} |\nabla f(y)| \, dy \right|^q dx \right)^{\frac{1}{q}}$$

$$= \frac{d^n}{n|\Omega|} \|V_{\frac{1}{n}} |\nabla f| \mid L^q(\Omega)\|$$

$$\leqq \frac{d^n}{n|\Omega|} \left(\frac{1 - 1/p + 1/q}{1/n - 1/p + 1/q} \right)^{1 - 1/p + 1/q} \omega_n^{1 - 1/n} |\Omega|^{1/n - 1/p + 1/q} \||\nabla f \mid L^p(\Omega)\|.$$

$$\square$$

Proposition 2 give us the upper estimate of the constant in the weighted Poincaré–Sobolev inequality in any simply connected plane domain $\Omega \subset \mathbb{R}^2$ with non-empty boundary:

$$B_{r,2}[\Omega, h] = B_{r,2}[\mathbb{D}] \leqq 2\pi^{\frac{2-r}{2r}} ((r + 2)/2)^{\frac{r+2}{2r}}.$$

We are ready to prove the main technical result of this work:

Theorem 8. *If Ω is a conformal α-regular domain then:*

(1) *The embedding operator*

$$i : W^{1,2}(\Omega) \hookrightarrow L^s(\Omega),$$

is compact for any $s \geqq 1$.

(2) *For any function $f \in W^{1,2}(\Omega)$ the Poincaré–Sobolev inequality*

$$\inf_{c \in \mathbb{R}} \left(\int_\Omega |f(x, y) - f_\Omega|^s \, dxdy \right)^{\frac{1}{s}} \leqq B_{s,2}[\Omega] \left(\int_\Omega |\nabla f(x, y)|^2 \, dxdy \right)^{\frac{1}{2}}$$

holds for any $s \geqq 1$.

(3) *The following estimate is correct $B_{s,2}[\Omega] \leqq B_{r,2}[\mathbb{D}] \cdot \|\psi'|L^\alpha(\mathbb{D})\|^{\frac{2}{s}}$. Here $B_{r,2}[\mathbb{D}]$ is the exact constant for the Poincaré inequality in the unit disc, $r = \alpha s/(\alpha - 2)$.*

Proof. Let $s \geqq 1$. Since Ω is a conformal α-regular domain, then for any conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$, its inverse conformal homeomorphism $\psi = \varphi^{-1}$ satisfies the following condition of α-regularity:

$$\iint_{\mathbb{D}} |\psi'(u, v)|^\alpha \, dudv = \iint_{\mathbb{D}} |J_{\varphi^{-1}}(u, v)|^{\alpha/2} \, dudv < \infty.$$

For the unit disc $\mathbb{D}$, the embedding operator

$$i_{\mathbb{D}} : W^{1,2}(\mathbb{D}) \hookrightarrow L^r(\mathbb{D})$$

is compact (see, for example, [25]) for any $r \geqq 1$.

By Theorem 4 the composition operator

$$\varphi^* : L^r(\mathbb{D}) \to L^s(\Omega)$$

is bounded if

$$\iint_{\mathbb{D}} |J_{\varphi^{-1}}(u, v)|^{\frac{r}{r-s}} \, du \, dv < \infty.$$

Because Ω is a conformal α-regular domain this condition holds for $\frac{r}{r-s} = \alpha/2$ i.e for $r = s\alpha/(\alpha - 2)$.

Since a conformal homeomorphism φ^{-1} induces a bounded composition operator

$$(\varphi^{-1})^* : L^{1,2}(\Omega) \to L^{1,2}(\mathbb{D}),$$

then by Theorem 6 the composition operator

$$(\varphi^{-1})^* : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{D})$$

is bounded.

Therefore the imbedding operator

$$i_\Omega : W^{1,2}(\Omega) \hookrightarrow L^s(\Omega)$$

is compact as a composition of bounded composition operators φ^*, $(\varphi^{-1})^*$ and the compact embedding operator $i_{\mathbb{D}}$, where

$$i_{\mathbb{D}} : W^{1,2}(\mathbb{D}) \hookrightarrow L^r(\mathbb{D}).$$

The first part of this theorem is proved.

For any function $f \in W^{1,2}(\Omega)$ and $r = s\alpha/(\alpha - 2)$ the following inequalities are correct:

$$\inf_{c \in \mathbb{R}} \left(\iint_\Omega |f(x, y) - c|^s \, dx dy \right)^{\frac{1}{s}} \leqq \left(\iint_\Omega |f(x, y) - f_{\Omega,h}|^s \, dx dy \right)^{\frac{1}{s}}$$

$$\leqq \left(\iint_\Omega |J_\varphi(x, y)|^{-\frac{s}{r-s}} \, dx dy \right)^{\frac{r-s}{rs}} \cdot \left(\iint_\Omega |f(x, y) - f_{\Omega,h}|^r h(x, y) \, dx dy \right)^{\frac{1}{r}}$$

$$= \left(\iint_{\mathbb{D}} |J_{\varphi^{-1}}(u, v)|^{\frac{r}{r-s}} \, du dv \right)^{\frac{r-s}{rs}} \cdot \left(\iint_\Omega |f(x, y) - f_{\Omega,h}|^r h(x, y) \, dx dy \right)^{\frac{1}{r}}$$

$$= \left(\iint_{\mathbb{D}} |J_{\varphi^{-1}}(u, v)|^{\frac{\alpha}{2}} \, du dv \right)^{\frac{2}{\alpha s}} \cdot \left(\iint_\Omega |f(x, y) - f_{\Omega,h}|^r h(x, y) \, dx dy \right)^{\frac{1}{r}}.$$

$$(6)$$

Using Theorem 7 we obtain

$$\inf_{c \in \mathbb{R}} \left(\iint_{\Omega} |f(x, y) - f_{\Omega}|^s \, dxdy \right)^{\frac{1}{s}}$$

$$\leqq B_{r,2}[\mathbb{D}] \cdot \|\psi'|L^{\alpha}(\mathbb{D})\|^{\frac{2}{s}} \left(\iint_{\Omega} |\nabla f(x, y)|^2 \, dxdy \right)^{\frac{1}{2}}.$$

$\square$

For $\alpha = \infty$, the following analog of the previous theorem is correct:

Theorem 9. *If Ω is a conformal ∞-regular domain then:*

(1) *The embedding operator*

$$i : W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$$

is compact.

(2) *For any function $f \in W^{1,2}(\Omega)$, the Poincaré–Sobolev inequality*

$$\left(\iint_{\Omega} |f(x, y) - f_{\Omega}|^2 \, dxdy \right)^{\frac{1}{2}} \leqq B_{2,2}[\Omega] \left(\iint_{\Omega} |\nabla f(x, y)|^2 \, dxdy \right)^{\frac{1}{2}}$$

holds.

(3) *The following estimate is correct: $B_{2,2}[\Omega] \leqq B_{2,2}[\mathbb{D}] \cdot \|\psi'|L^{\infty}(\mathbb{D})\|$. Here $B_{2,2}[\mathbb{D}] = 1/\sqrt{\lambda_1[\Omega]}$ is the exact constant for the Poincaré inequality in the unit disk.*

Proof. Since Ω is a conformal ∞-regular domain then for any conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$ its inverse conformal homeomorphism $\psi = \varphi^{-1}$ satisfies the following condition:

$$\|\psi' \mid L^{\infty}(\mathbb{D})\|^2 = \|J_{\varphi^{-1}} \mid L^{\infty}(\mathbb{D})\| < \infty.$$

For the unit disc $\mathbb{D}$ the embedding operator

$$i_{\mathbb{D}} : W^{1,2}(\mathbb{D}) \hookrightarrow L^2(\mathbb{D})$$

is compact (see, for example, [25]).

By Theorem 4 the composition operator

$$\varphi^* : L^2(\mathbb{D}) \to L^2(\Omega)$$

is bounded if

$$\|\psi' \mid L^{\infty}(\mathbb{D})\|^2 = \|J_{\varphi^{-1}} \mid L^{\infty}(\mathbb{D})\| < \infty.$$

Since a conformal homeomorphism φ^{-1} induces a bounded composition operator

$$(\varphi^{-1})^* : L^{1,2}(\Omega) \to L^{1,2}(\mathbb{D}),$$

then by Theorem 6 the composition operator

$$(\varphi^{-1})^* : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{D})$$

is bounded.

Therefore the embedding operator

$$i_\Omega : W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$$

is compact as a composition of bounded composition operators φ^*, $(\varphi^{-1})^*$ and the compact embedding operator $i_\mathbb{D}$:

$$i_\mathbb{D} : W^{1,2}(\mathbb{D}) \hookrightarrow L^2(\mathbb{D}).$$

The first part of this theorem is proved.

For any function $f \in W^{1,2}(\Omega)$ and $g = f \circ \varphi^{-1} \in W^{1,2}(\mathbb{D})$, the following inequalities are correct:

$$\left(\iint_\Omega |f(x,y) - f_\Omega|^2 \, dxdy \right)^{\frac{1}{2}}$$

$$= \inf_{c \in \mathbb{R}} \left(\iint_\Omega |f(x,y) - c|^2 \, dxdy \right)^{\frac{1}{2}} \leqq \left(\iint_\Omega |f(x,y) - f_{\Omega,h}|^2 \, dxdy \right)^{\frac{1}{2}}$$

$$= \left(\iint_\Omega |f(x,y) - f_{\Omega,h}|^2 J_\varphi^{-1}(x,y) J_\varphi(x,y) \, dxdy \right)^{\frac{1}{2}}$$

$$\leqq \| J_\varphi \mid L^\infty(\Omega) \|^{-\frac{1}{2}} \cdot \left(\iint_\Omega |f(x,y) - f_{\Omega,h}|^2 J_\varphi(x,y) \, dxdy \right)^{\frac{1}{2}}$$

$$= \| J_{\varphi^{-1}} \mid L^\infty(\mathbb{D}) \|^{\frac{1}{2}} \cdot \left(\iint_\Omega |f(x,y) - f_{\Omega,h}|^2 J_\varphi(x,y) \, dxdy \right)^{\frac{1}{2}}$$

$$= \| J_{\varphi^{-1}} \mid L^\infty(\mathbb{D}) \|^{\frac{1}{2}} \cdot \left(\iint_\Omega |f(x,y) - g_\mathbb{D}|^2 J_\varphi(x,y) \, dxdy \right)^{\frac{1}{2}}.$$

Using the change of variable formula and the Poincaré–Sobolev inequality in the unit disc we have:

$$\left(\iint_\Omega |f(x,y) - f_\Omega|^2 \, dxdy \right)^{\frac{1}{2}}$$

$$\leqq \| J_{\varphi^{-1}} \mid L^\infty(\mathbb{D}) \|^{\frac{1}{2}} \cdot \left(\iint_\mathbb{D} |g(u,v) - g_\mathbb{D}|^2 \, dudv \right)^{\frac{1}{2}}$$

$$= \| J_{\varphi^{-1}} \mid L^\infty(\mathbb{D}) \|^{\frac{1}{2}} \cdot \left(\iint_\mathbb{D} |\nabla g(u,v)|^2 \, dudv \right)^{\frac{1}{2}}$$

$$= \| \psi' \mid L^\infty(\mathbb{D}) \| \cdot \left(\iint_\Omega |\nabla f(x,y)|^2 \, dudv \right)^{\frac{1}{2}}.$$

$\square$

5. Eigenvalue Problem for Free Vibrating Membranes

The eigenvalue problem for free vibrating membranes is equivalent to the corresponding problem for the Neumann–Laplace operator. The classical formulation for smooth domains is the following:

$$-\Delta u = \lambda u \quad \text{in } \Omega, \tag{7}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{8}$$

Because conformal regular domains are not necessarily smooth, the weak statement of the spectral problem for the Neumann–Laplace operator is convenient: a function u solves the previous problem if ($u \in W^{1,2}(\Omega)$) and

$$\iint_\Omega \nabla u(x, y) \cdot \nabla v(x, y) \, dxdy = \lambda \iint_\Omega u(x, y)v(x, y) \, dxdy$$

for all $v \in W^{1,2}(\Omega)$.

By the Min–Max Principle [8], the inverse to the first eigenvalue is equal to the exact constant in the Poincaré inequality:

$$\int_\Omega |f(x, y) - f_\Omega|^2 \, dxdy \leqq B_{2,2}^2[\Omega] \int_\Omega |\nabla f(x, y)|^2 \, dxdy.$$

We are ready to prove the main result about the spectrum (Theorem A). For the reader's convenience we repeat its formulation:

Theorem A. *Let $\Omega \subset \mathbb{R}^2$ be a conformal α-regular domain. Then the spectrum of the Neumann–Laplace operator in Ω is discrete, and can be written in the form of a non-decreasing sequence:*

$$0 = \lambda_0[\Omega] < \lambda_1[\Omega] \leqq \lambda_2[\Omega] \leqq \cdots \leqq \lambda_n[\Omega] \leqq \cdots ,$$

and

$$1/\lambda_1[\Omega] \leqq B_{2\alpha/(\alpha-2),2}^2[\mathbb{D}] \left(\int_\mathbb{D} |\varphi'(x, y)|^\alpha \, dxdy \right)^{\frac{2}{\alpha}}$$

$$\leqq 4\pi^{-\frac{2}{\alpha}} \left(\frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha-2}{\alpha}} \|\psi' \mid L^\alpha(\mathbb{D})\|^2,$$

where $\psi : \mathbb{D} \to \Omega$ is the Riemann conformal mapping of the unit disc $\mathbb{D} \subset \mathbb{R}^2$ onto Ω.

Proof. By Theorem 8, in the case $s = 2$, the embedding operator

$$i : W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$$

is compact.

Therefore the spectrum of the Neumann–Laplace operator is discrete and can be written in the form of a non-decreasing sequence.

By the same theorem and the Min–Max principle we have

$$\int_\Omega |f(x,y) - f_\Omega|^2 \, dxdy = \inf_{c\in\mathbb{R}} \int_\Omega |f(x,y) - f_\Omega|^2 \, dxdy$$
$$\leqq B_{2,2}^2[\Omega] \int_\Omega |\nabla f(x,y)|^2 \, dxdy,$$

where $B_{2,2}[\Omega] \leqq B_{r,2}[\mathbb{D}] \cdot \|\psi'|L^\alpha(\mathbb{D})\|$.

Hence

$$1/\lambda_1[\Omega] \leqq B_{r,2}^2[\mathbb{D}] \left(\int_\mathbb{D} |\psi'(u,v)|^\alpha \, dudv \right)^{\frac{2}{\alpha}}.$$

By Proposition 2,

$$B_{r,2}[\Omega, h] = B_{r,2}[\mathbb{D}] \leqq 2\pi^{\frac{2-r}{2r}} \left((r+2)/2 \right)^{\frac{r+2}{2r}}.$$

Recall that in Theorem 8, $r = 2\alpha/(\alpha - 2)$. In this case

$$B_{2\alpha/(\alpha-2),2}[\mathbb{D}] \leqq 2\pi^{-\frac{1}{\alpha}} \left(\frac{2\alpha-2}{\alpha-2} \right)^{\frac{\alpha-1}{\alpha}}.$$

Therefore,

$$1/\lambda_1[\Omega] \leqq B_{2\alpha/(\alpha-2),2}^2[\mathbb{D}] \left(\int_\mathbb{D} |\varphi'(x,y)|^\alpha \, dxdy \right)^{\frac{2}{\alpha}}$$
$$\leqq 4\pi^{-\frac{2}{\alpha}} \left(\frac{2\alpha-2}{\alpha-2} \right)^{\frac{2\alpha-2}{\alpha}} \|\psi' \mid L^\alpha(\mathbb{D})\|^2.$$

$\square$

In the case of conformal α-regular domains for $\alpha = \infty$, by Theorem 9 we immediately have:

Theorem B. *Let $\Omega \subset \mathbb{R}^2$ be a conformal α-regular domain for $\alpha = \infty$. Then the spectrum of the Neumann–Laplace operator in Ω is discrete, and can be written in the form of a non-decreasing sequence*

$$0 = \lambda_0[\Omega] < \lambda_1[\Omega] \leqq \lambda_2[\Omega] \leqq \cdots \leqq \lambda_n[\Omega] \leqq \cdots,$$

and

$$1/\lambda_1[\Omega] \leqq B_{2,2}^2[\mathbb{D}]\|\psi' \mid L^\infty(\mathbb{D})\|^2 = \frac{\|\psi' \mid L^\infty(\mathbb{D})\|^2}{j_{1,1}^2}, \tag{9}$$

where $j_{1,1}$ is the first positive zero of the Bessel function J_1, and $\psi : \mathbb{D} \to \Omega$ is the Riemann conformal mapping of the unit disc $\mathbb{D} \subset \mathbb{R}^2$ onto Ω.

5.1. Examples

Now we describe a rather wide class of plane domains for which there exist conformal mappings with Jacobians of the class $L^p(\mathbb{D})$ for some $p > 1$, that is, with complex derivatives of the class $L^p(\mathbb{D})$ for some $p > 2$.

Definition 2. A homeomorphism $\varphi : \Omega \to \Omega_1$ between planar domains is called K-quasiconformal if it preserves orientation, belongs to the Sobolev class $W^{1,2}_{\text{loc}}(\Omega)$ and its directional derivatives ∂_α satisfy the distortion inequality

$$\max_\alpha |\partial_\alpha \varphi| \leqq K \min_\alpha |\partial_\alpha \varphi| \quad \text{almost everywhere in } \Omega.$$

Infinitesimally, quasiconformal homeomorphisms transform circles to ellipses with eccentricity uniformly bounded by K. If $K = 1$ we recover conformal homeomorphisms, while for $K > 1$, the plane quasiconformal mappings need not be smooth.

Definition 3. A domain Ω is called a K-quasidisc if it is the image of the unit disc $\mathbb{D}$ under a K-quasiconformal homeomorphism of the plane onto itself.

It is well known that the boundary of any K-quasidisc Ω admits a K^2-quasiconformal reflection, and thus, for example, any conformal homeomorphism $\varphi : \mathbb{D} \to \Omega$ can be extended to a K^2-quasiconformal homeomorphism of the whole plane to itself.

The boundaries of quasidiscs are called quasicircles. It is known that there are quasicircles for which no segment has finite length. The Hausdorff dimension of quasicircles was first investigated by Gehring and Väisälä [11], who proved that it can take all values in the interval $[1, 2)$. Smirnov proved recently [30] that the Hausdorff dimension of any K-quasicircle is at most $1 + k^2$, where $k = (K - 1)/(K + 1)$.

Ahlfors's 3-point condition [1] gives a complete geometric characterization of quasicircles: a Jordan curve γ in the plane is a quasicircle if and only if for each two points a, b in γ the (smaller) arc between them has the diameter comparable with $|a - b|$. This condition is easily checked for the snowflake. On the other hand, every quasicircle can be obtained by an explicit snowflake-type construction (see [29]).

For any planar K-quasiconformal homeomorphism $\varphi : \Omega \to \Omega_1$, the following sharp result is known: $J(z, \varphi) \in L^p_{\text{loc}}(\Omega_1)$ for any $p < \frac{K}{K-1}$ ([2,13]).

Proposition 3. *Any conformal homeomorphism $\varphi : \mathbb{D} \to \Omega$ of the unit disc $\mathbb{D}$ onto a K-quasidisc Ω belongs to $L^{1,p}(\mathbb{D})$ for any $1 \leqq p < \frac{2K^2}{K^2-1}$.*

Proof. Any conformal homeomorphism $\varphi : \mathbb{D} \to \Omega$ can be extended to a K^2 quasiconformal homeomorphism ψ of the whole plane to the whole plane by reflection. Since the domain Ω is bounded, ψ belongs to the class $L^p(\Omega)$ for any $1 \leqq p < \frac{2K^2}{K^2-1}$ ([2,13]). Therefore φ belongs to the same class.   □

For quasidiscs, the following estimate readily follows from Theorem A:

Proposition 4. *Suppose a conformal homeomorphism* $\varphi : \mathbb{D} \to \Omega$ *maps the unit disc* $\mathbb{D}$ *onto a* K-*quasidisc* Ω. *Then*

$$1/\lambda_1[\Omega] \leqq B_{2\alpha/(\alpha-2),2}^2[\mathbb{D}] \left(\int_{\mathbb{D}} |\varphi'(x, y)|^\alpha \, dxdy \right)^{\frac{2}{\alpha}}$$

$$\leqq 4\pi^{-\frac{2}{\alpha}} \left(\frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha-2}{\alpha}} \|\psi' \mid L^\alpha(\mathbb{D})\|^2$$

for any $2 < \alpha < \frac{2K^2}{K^2-1}$.

As the second example, we consider the interior of the cardioid. By Alhfor's condition the cardioid is not a quasidisc. Because the cardioid is a conformal ∞-regular domain, we have the following example:

Example 1. Let Ω_c be the interior of the cardioid. The diffeomorphism

$$z = \psi(w) = (w + 1)^2, \ z = x + iy,$$

is conformal and maps the unit disc $\mathbb{D}$ onto Ω_c. Then, by Theorem B,

$$\|\psi' \mid L^\infty(\mathbb{D})\| = \max_{w \in \mathbb{D}} 2|w + 1| \leqq 4.$$

Hence,

$$\lambda_1[\Omega_c] \geqq \frac{j_{1,1}^2}{16}.$$

Here $j_{1,1}$ is the first positive zero of the Bessel function J_1.

The third example is a m-polygon P_m.

Consider a m-polygon P_m with vertices z_k on the unit circle and the angles α_k are measured in fractions of π. Then the conformal mapping of the unit disc $\mathbb{D}$ onto the n-polygon P_n is given by the Schwarz–Christoffel formula:

$$\psi(z) = C \int_{z_0}^z (z - z_1)^{\alpha_1-1}(z - z_2)^{\alpha_2-1} \cdots \cdot (z - z_m)^{\alpha_m-1} dz + C_1$$

and

$$\psi'(z) = C(z - z_1)^{\alpha_1-1}(z - z_2)^{\alpha_2-1} \cdots \cdot (z - z_m)^{\alpha_m-1}.$$

Example 2. Q_m is a regular m-polygon with vertices z_k on the unit circle and the angles $\alpha_k = 1 - 2/m$ are measured in fractions of π. The diffeomorphism

$$\psi(z) = C \int_{z_0}^z (z^m - 1)^{-\frac{2}{m}} dz + C_1, \quad z = x + iy$$

is conformal and maps the unit disc $\mathbb{D}$ onto the regular m-polygon Q_m. If we have $\varphi(0) = 0$ and $|\varphi'(z)| = 1$, then $C_1 = 0$ and $C = 1$. Then

$$\frac{1}{\lambda_1[\Omega]} \leqq \pi^{-\frac{2}{\alpha}} \inf_{2 < \alpha < m} \left(\frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha-2}{\alpha}} \iint_{\mathbb{D}} |z^m - 1|^{-\frac{2\alpha}{m}} \, dxdy.$$

6. Estimates for Domains Conformally Equivalent to a Rectangle

We take the unit ball as the basic domain for our estimates. In many applications it is more convenient to take a rectangle as the basic domain:

$$\mathbb{Q}_{ab} = \{(x, y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\}.$$

In this case we have the following assertion:

Theorem C. *Let $\Omega \subset \mathbb{R}^2$ be a plane domain with non-empty boundary. Suppose that there exists a conformal mapping $\psi : \mathbb{Q}_{ab} \to \Omega$ such that $\psi' \in L^\alpha(\mathbb{Q}_{ab})$ for some $\alpha > 2$. Then the spectrum of Neumann–Laplace operator in Ω is discrete and can be written in the form of a non-decreasing sequence:*

$$0 = \lambda_0[\Omega] < \lambda_1[\Omega] \leqq \lambda_2[\Omega] \leqq \cdots \leqq \lambda_n[\Omega] \leqq \cdots ,$$

$$1/\lambda_1[\Omega] \leqq B_{2\alpha/(\alpha-2),2}^2[\mathbb{Q}_{ab}] \left(\int_{\mathbb{Q}_{ab}} |\varphi'(x, y)|^\alpha \, dxdy \right)^{\frac{2}{\alpha}}$$

$$\leqq \left(\frac{a^2 + b^2}{(ab)^{\frac{r-1}{r}}} \right)^2 \left(\frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha-2}{\alpha}} \|\psi' \mid L^\alpha(\mathbb{Q}_{ab})\|^2 \quad \text{for } \alpha < \infty$$

and

$$1/\lambda_1[\Omega] \leqq B_{2,2}^2[\mathbb{Q}_{ab}]\|\psi' \mid L^\infty(\mathbb{Q}_{ab})\|^2$$

$$= \left(\frac{\max\{a, b\}}{\pi} \right)^2 \|\psi' \mid L^\infty(\mathbb{Q}_{ab})\|^2, \quad \text{for } \alpha = \infty.$$

Proof. Because $\mathbb{Q}_{ab}$ is the convex domain, then by Proposition 2 we have that

$$B_{r,2}(\mathbb{Q}_{ab}) \leqq \frac{d^2}{2|\mathbb{Q}_{ab}|} \left(\frac{1 - 1/2 + 1/r}{1/2 - 1/2 + 1/r} \right)^{1-1/2+1/r} \omega_n^{1-1/2} |\mathbb{Q}_{ab}|^{1/2-1/2+1/2}$$

$$= \left(\frac{a^2 + b^2}{(ab)^{\frac{r-1}{r}}} \right)^2 \left(\frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha-2}{\alpha}}.$$

For $B_{2,2}(\mathbb{Q}_{ab})$ there is the exact calculation (see, for example, [26])

$$B_{2,2}(\mathbb{Q}_{ab}) = \frac{\max\{a, b\}}{\pi}.$$

Now, replacing in the proof of Theorems 8 and 9 the unit disc $\mathbb{D}$ by the rectangle $\mathbb{Q}_{ab}$ and using the well known fact that $\mathbb{Q}_{ab}$ is a Poincaré domain, we prove Theorem C in a fashion similar to the proofs of Theorems A and B. $\square$

Example 3. Let $a = 1$ and $b = 2\pi$. The conformal mapping $\psi = e^w : \mathbb{Q}_{ab} \to R_e$ maps the rectangle $\mathbb{Q}_{ab}$ onto the split ring R_e. Then, by Theorem C

$$\|\psi' \mid L^\infty(\mathbb{D})\| = \max_{u \in (0,1)} e^u \leqq e.$$

Hence

$$\lambda_1[R_e] \geqq \left(\frac{\pi}{2\pi} \right)^2 \frac{1}{\|\psi' \mid L^\infty(\mathbb{D})\|^2} = \frac{1}{4e^2}.$$

Finally, we note that using the elementary conformal functions like sin, tan and so on, it is possible to construct many non-trivial examples of domains in which Theorem B gives estimates of the first non-trivial eigenvalue of the Neumann–Laplace operator.

7. Comparison of the Estimates with Previous Results

Let $\mathbb{D}_a$ denote the disc of the radius $a > 0$. Then, for $\mathbb{D}_a$, our estimate is exact. In this case,

$$\mathbb{D}_a = \psi(\mathbb{D}),$$

where $\psi(z) = az$. By Theorem B:

$$\lambda_1[\mathbb{D}_r] \geqq \frac{j_{1,1}^2}{\|\psi' \mid L^\infty(\mathbb{D})\|^2} = \frac{j_{1,1}^2}{a^2}.$$

In the paper [27] the authors proved: if Ω is a convex domain with diameter $d(\Omega)$ (see, also [9,10]), then

$$\lambda_1[\Omega] \geqq \frac{\pi^2}{d(\Omega)^2}. \tag{10}$$

Definition 4. Let $\Omega = \psi(\mathbb{D})$ be a conformal α-regular domain for $\alpha = \infty$. We call a domain Ω *a conformal uniform domain* if

$$\|\psi' \mid L^\infty(\mathbb{D})\| \leqq d(\Omega).$$

For the class of conformal uniform domains the estimate (9) (Theorem B) improves the estimate (10):

$$\lambda_1[\Omega] \geqq \frac{j_{1,1}^2}{\|\psi' \mid L^\infty(\mathbb{D})\|^2} > \frac{\pi^2}{d(\Omega)^2}.$$

The class of conformal uniform domains is not empty. Consider domains

$$\Omega_n = \psi_n(\mathbb{D}), \quad \psi_n(z) = (z+n)^2, \quad n \geqq 1.$$

Then,

$$\|\psi_n' \mid L^\infty(\mathbb{D})\| = \max_{|z| \leqq 1} |2(z+n)| = 2(n+1).$$

From another side,

$$d(\Omega) \geqq |\psi_n(1) - \psi_n(-1)| = |(n+1)^2 - (n-1)^2| = 4n.$$

Hence $\|\psi_n' \mid L^\infty(\mathbb{D})\| \leqq d(\Omega_n)$ and Ω_n are conformal uniform domains. For $n > 3$ the domains Ω_n are convex, because

$$Re\left\{1 + z\frac{\psi_n''}{\psi_n'}\right\} > 0, \quad n > 3.$$

The conformal uniform domains can be characterized in geometric terms: a conformal ∞-regular domain Ω is conformal uniform if

$$R\left(\psi(z), \Omega\right) \leqq (1 - |z|^2)d(\Omega),$$

where $R\left(\psi(z), \Omega\right)$ is a conformal radius of Ω.

Another example of conformal uniform domains is the domain $\Omega_\alpha = \psi_\alpha(\mathbb{D})$, where $\psi(z) = e^{\alpha z}$, for $0 = \alpha_0 < \alpha < \alpha_1$, where α_0 and α_1 are the zeros of the function $f(\alpha) = 1 - \alpha - e^{-2\alpha}$ $(1/2 < \alpha_1 < 1)$. For these mappings ψ_α:

$$\|\psi_\alpha' \mid L^\infty(\mathbb{D})\| = \max_{|z|\leqq 1} |\alpha e^{\alpha z}| = \alpha e^\alpha,$$

and

$$d(\Omega) \geqq |\psi_\alpha(1) - \psi_\alpha(-1)| = e^\alpha - e^{-\alpha}.$$

Hence the domains Ω_α are conformal uniform domains if

$$\alpha e^\alpha \leqq e^\alpha - e^{-\alpha}$$

or

$$f(\alpha) = 1 - \alpha - e^{-2\alpha} \geqq 0.$$

For these α the domains Ω_α are convex, because

$$Re\left\{1 + z\frac{\psi_\alpha''}{\psi_\alpha'}\right\} = Re\left\{1 + \alpha z\right\} > 0, \quad for\ 0 = \alpha_0 < \alpha < \alpha_1.$$

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