Arch. Rational Mech. Anal. 221 (2016) 1317–1344
Digital Object Identifier (DOI) 10.1007/s00205-016-0985-z

CrossMark

Ill-Posedness of the Hydrostatic Euler and Singular Vlasov Equations

DANIEL HAN-KWAN & TOAN T. NGUYEN

Communicated by P. CONSTANTIN

This paper is dedicated to Claude Bardos on the occasion of his 75th birthday, as a token of friendship and admiration

Abstract

In this paper, we develop an abstract framework to establish ill-posedness, in the sense of Hadamard, for some nonlocal PDEs displaying unbounded unstable spectra. We apply this to prove the ill-posedness for the hydrostatic Euler equations as well as for the kinetic incompressible Euler equations and the Vlasov–Dirac–Benney system.

1. Introduction

In this paper, we develop an abstract framework to establish ill-posedness, in the sense of Hadamard, for some nonlocal PDEs displaying unbounded unstable spectra; this phenomenon is reminiscent of Lax–Mizohata ill-posedness for first-order systems violating the hyperbolicity condition (that is, when the spectrum of the operator's principal symbol is not included in the real line).
By (local-in-time) well-posedness of the Cauchy problem for a PDE, we mean:

- given initial data, there exists a time $T > 0$ so that a solution exists for all times $t \in [0, T]$;
- the solution is unique;
- the solution map is (Hölder) continuous with respect to initial data.

This notion of well-posedness for PDEs was introduced by HADAMARD [22]. The Lax–Mizahota ill-posedness result is concerned with this definition. *In this paper, we shall describe situations in which the third well-posedness condition breaks down* (*for data in Sobolev spaces*).

In the context of systems of first-order partial differential equations, Hadamard's well-posedness was extensively studied by many authors, including Friedrichs, Gårding, Hörmander and Lax, among others; see, for instance, [4,22,32–34,37,38,42] and the references therein. For linear equations, it was LAX [32] and MIZOHATA

[38] who first showed that hyperbolicity is a necessary condition for well-posedness of the Cauchy problem for C^∞ initial data. The result was later extended to quasilinear systems by WAKABAYASHI [42], and recently by MÉTIVIER [37]. The violation of hyperbolicity creates an unbounded unstable spectrum of the underlying principal differential operators. MÉTIVIER [37] showed that for first order systems that are not hyperbolic, the solution map is not α-Hölder continuous (for all $\alpha \in (0, 1]$) from any Sobolev space to L^2, or more precisely, that it does not belong to $C^\alpha(H^s, L^2)$, for all $s \geqq 0$ and $\alpha \in (0, 1]$, within arbitrarily short time; that is, the Cauchy problem is ill-posed, violating the above third condition for well-posedness.

In this paper, we prove an analogue of Métivier's result for some nonlocal PDEs, namely, the hydrostatic Euler equations as well as for some singular Vlasov equations: the kinetic incompressible Euler equations and the Vlasov–Dirac–Benney system. The purpose of this introduction is to briefly discuss these equations and to present the main results. Our abstract framework for proving the ill-posedness is inspired by the analysis of MÉTIVIER [37] and DESJARDINS and GRENIER [13].

1.1. Hydrostatic Euler Equations

The Hydrostatic Euler equations arise in the context of two-dimensional incompressible ideal flows in a narrow channel (see for example [35]). They read:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_z u + p_x = 0, \\ \partial_x u + \partial_z v = 0, \end{cases} \tag{1.1}$$

for $(x, z) \in \mathbb{T} \times [-1, 1]$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The torus $\mathbb{T}$ is equipped with the normalized Lebesgue measure, so that $\mathrm{Leb}(\mathbb{T}) = 1$. Here $(u(t, x, z), v(t, x, z)) \in \mathbb{R}^2$, and $p(t, x)$ are the unknowns in the equation. We impose the zero boundary conditions:

$$v_{|z=\pm 1} = 0.$$

The vorticity $\omega := \partial_z u$ satisfies the equation

$$\partial_t \omega + u\partial_x \omega + v\partial_z \omega = 0, \tag{1.2}$$

in which $u := \partial_z \varphi$, $v := -\partial_x \varphi$, and the stream function φ solves the elliptic problem:

$$\partial_z^2 \varphi = \omega, \quad \varphi_{|z=\pm 1} = 0. \tag{1.3}$$

Thus, one can observe a loss of one x-derivative in the Equation (1.2) through $v = -\partial_x \varphi$, as compared to ω. This indicates that a standard Cauchy theory cannot be expected for this equation.

Brenier was the first to develop a Cauchy theory in Sobolev spaces for data with convex profiles [7]; this was revisited and extended recently in MASMOUDI and WONG [36] and KUKAVICA et al. [30]. In [31], KUKAVICA et al. also provide an existence result for data with analytic regularity.

The derivation of hydrostatic Euler from the incompressible Euler equations set in a narrow channel, for data with convex profiles, was first performed by GRENIER

[18], then by Brenier [9] with different methods (see also [36]). One key idea in these works is the use of the convexity to build a suitable energy which is not degenerate in the hydrostatic limit.

In [41], Renardy showed that for arbitrary odd shear flows $U(z)$ so that $\frac{1}{U(z)^2}$ is integrable, the linearized hydrostatic Euler Equations (1.2) and (1.3) around U' have unbounded unstable spectrum. Such profiles do not satisfy the convexity condition. Following an argument of [21], this property for the spectrum can be used to straightforwardly prove some ill-posedness for the nonlinear equations (see also [15,16,20] for the ill-posedness of the Prandtl equations or [14] for the SQG equations); loosely speaking, it asserts that the flow of solutions, if it exists, cannot be $C^1(H^s, H^1)$, for all $s \geqq 0$, within a fixed positive time. In this work, we shall construct a family of solutions to show that the solution map from H^s to L^2 has unbounded Hölder norm, within arbitrarily short time. In the proof, we shall take an unstable shear flow that is analytic. Such a shear flow exists; for instance, $U(z) = \tanh(\frac{z}{d_1})$ for small d_1 yields unstable spectrum as shown by [12].

We prove the following ill-posedness result:

Theorem 1.1. (Ill-posedness for the hydrostatic Euler equations) *There exists a stationary shear flow $U(z)$ such that the following holds. For all $s \in \mathbb{N}$, $\alpha \in (0, 1]$, and $k \in \mathbb{N}$, there are families of solutions $(\omega_\varepsilon)_{\varepsilon>0}$ of* (1.2) *and* (1.3)*, times $t_\varepsilon = \mathcal{O}(\varepsilon|\log \varepsilon|)$, and $(x_0, z_0) \in \mathbb{T} \times (-1, 1)$ such that*

$$\lim_{\varepsilon\to 0} \frac{\|\omega_\varepsilon - U'\|_{L^2([0,t_\varepsilon]\times\Omega_\varepsilon)}}{\|\omega_\varepsilon|_{t=0} - U'\|^{\alpha}_{H^s(\mathbb{T}\times(-1,1))}} = +\infty \tag{1.4}$$

with $\Omega_\varepsilon = B(x_0, \varepsilon^k) \times B(z_0, \varepsilon^k)$.

We remark that the instability is strong enough so that it occurs within a vanishing spatial domain Ω_ε and a vanishing time t_ε, as $\varepsilon \to 0$. As will be seen in the proof, (x_0, z_0) can actually be taken arbitrarily in $\mathbb{T} \times (-1, 1)$.

1.2. Kinetic Incompressible Euler and Vlasov–Dirac–Benney Equations

The so-called kinetic incompressible Euler and Vlasov–Dirac–Benney systems are kinetic models from plasma physics, arising in the context of *small Debye lengths regimes*. Although our results will be stated in the three-dimensional framework, they can be adapted to any dimension.

Consider first the kinetic incompressible Euler equations, which read

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \varphi \cdot \nabla_v f = 0, \tag{1.5}$$

$$\rho(t, x) := \int_{\mathbb{R}^3} f(t, x, v)\, \mathrm{d}v = 1, \tag{1.6}$$

for $(t, x, v) \in \mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3$, in which $f(t, x, v)$ is the distribution function at time $t \geqq 0$, position $x \in \mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$, and velocity $v \in \mathbb{R}^3$ of electrons in a plasma. The torus $\mathbb{T}^3$ is equipped with the normalized Lebesgue measure, so that $\mathrm{Leb}(\mathbb{T}^3) = 1$. The potential φ stands for a Lagrange multiplier (or, from the physical point of view, a pressure) related to the constraint $\rho = 1$. It is possible to obtain an explicit formula for the potential φ, arguing as follows. Introduce the current density $j(t,x) := \int_{\mathbb{R}^3} f(t,x,v)v\,\mathrm{d}v$. We start by writing the local conservation of charge and current from the Vlasov equation:

$$\partial_t \rho + \nabla \cdot j = 0,$$
$$\partial_t j + \nabla \cdot \int f v \otimes v \,\mathrm{d}v = -\nabla\varphi.$$

By using the constraint (1.6), it follows that $\nabla \cdot j = 0$. Plugging this into the conservation of current, one gets the law

$$-\Delta\varphi = \nabla \cdot \left(\nabla \cdot \int f v \otimes v \,\mathrm{d}v\right). \tag{1.7}$$

Looking for solutions to (1.5) and (1.6) of the form $f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)}$ turns out to be equivalent to finding solutions (ρ, u) of the classical incompressible Euler equations. This therefore justifies the name we have chosen for (1.5) and (1.6), as suggested by BRENIER [6].

The Vlasov–Dirac–Benney system is closely related, reading:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \varphi \cdot \nabla_v f = 0, \tag{1.8}$$
$$\varphi = \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v - 1. \tag{1.9}$$

This model appears to be a kinetic analogue of the compressible isentropic Euler equations with parameter $\gamma = 2$. The name Vlasov–Dirac–Benney was coined by BARDOS [1], due to connections with the Benney model for Water Waves.

Both kinetic incompressible Euler and Vlasov–Dirac–Benney equations can be formally derived in the *quasineutral limit* of the Vlasov–Poisson system, that is in the small Debye length regime. This corresponds to the singular limit $\varepsilon \to 0$ in the following scaled equations:

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon - \nabla_x \varphi_\varepsilon \cdot \nabla_v f_\varepsilon = 0,$$

where φ_ε solves a Poisson equation given,

1. *for the case of electron dynamics*, by

$$-\varepsilon^2 \Delta_x \varphi_\varepsilon = \rho_\varepsilon - 1, \quad \rho_\varepsilon := \int_{\mathbb{R}^3} f_\varepsilon \,\mathrm{d}v,$$

which yields the kinetic incompressible Euler equations in the formal limit $\varepsilon \to 0$ (see [6]);

2. *for the case of ion dynamics*, by

$$-\varepsilon^2 \Delta_x \varphi_\varepsilon = \rho_\varepsilon - \varphi_\varepsilon - 1, \quad \rho_\varepsilon := \int_{\mathbb{R}^3} f_\varepsilon \,\mathrm{d}v,$$

which yields the Vlasov–Dirac–Benney system in the formal limit $\varepsilon \to 0$ (see [27]).

Directly from the laws (1.7) and (1.9) for the potential φ, one sees that there is a loss of one x-derivative through the force $-\nabla_x \varphi$, as compared to the distribution function f. This explains why a standard Cauchy theory cannot be expected for these equations. What is known though is the existence of *analytic solutions* (see [27], Jabin and Nouri [29], Bossy et al. [5]), as well as an H^s theory for *stable* data (see Bardos and Besse [2] and the recent work of the first author and Rousset [26]).

As for the rigorous justification of the quasineutral limit, we first refer to the work of Grenier [17] in the case of data with analytic regularity in x (see also [23,24] where it is shown that exponentially small but rough perturbations of the data considered by Grenier are admissible). In [8], Brenier introduced the so-called modulated energy method and derived the incompressible Euler equations in the limiting case of monokinetic distributions (see [27] for what concerns the case of the compressible isentropic Euler system). In the work [28], the first author and Hauray showed that the formal limit (to (1.5), (1.6) or (1.8), (1.9)) is in general not true in Sobolev spaces, because of instabilities of the Vlasov–Poisson system (see also [25]). The rigorous derivation of (1.8) and (1.9) for initial data with a Penrose stability condition was completed only recently by the first author and Rousset [26].

In [3], Bardos and Nouri show that around *unstable* homogeneous equilibria, the linearized equations of (1.8) and (1.9) have unbounded unstable spectrum. This property was used to prove some ill-posedness, using the above-mentioned argument of [21], see [3, Theorem 4.1]; loosely speaking they show that the flow of solutions, if it exists, cannot be $C^1(H^s, H^1)$, for all $s \geqq 0$. What we shall prove in this paper is that the flow cannot be $C^\alpha(H^s_{\text{weight}}, L^2)$, for all $s \geqq 0$, $\alpha \in (0, 1]$, and any polynomial weight in v. In the proof we shall take unstable homogeneous equilibria that are analytic and decaying sufficiently fast at infinity: typical examples are double-bump equilibria satisfying these constraints.

More precisely, we prove the following ill-posedness result:

Theorem 1.2. (Ill-posedness for the kinetic incompressible Euler and Vlasov–Dirac–Benney equations) *There exists a stationary solution $\mu(v)$ such that the following holds. For all $m, s \in \mathbb{N}$, $\alpha \in (0, 1]$, and $k \in \mathbb{N}$, there are families of solutions $(f_\varepsilon)_{\varepsilon>0}$ of (1.5) and (1.6) (respectively, of the system (1.8), (1.9)), times $t_\varepsilon = \mathcal{O}(\varepsilon |\log \varepsilon|)$, and $(x_0, v_0) \in \mathbb{T}^3 \times \mathbb{R}^3$, such that*

$$\lim_{\varepsilon \to 0} \frac{\| f_\varepsilon - \mu \|_{L^2([0,t_\varepsilon] \times \Omega_\varepsilon)}}{\| \langle v \rangle^m (f_\varepsilon|_{t=0} - \mu) \|^\alpha_{H^s(\mathbb{T}^3 \times \mathbb{R}^3)}} = +\infty \tag{1.10}$$

with $\Omega_\varepsilon = B(x_0, \varepsilon^k) \times B(v_0, \varepsilon^k)$. Here, $\langle v \rangle := \sqrt{1 + |v|^2}$.

In the proof, we shall focus only on the system (1.5)–(1.7), since the analysis is almost identical for what concerns the Vlasov–Dirac–Benney equations. Furthermore, this result also holds for (1.5) and (1.6) and (1.5)–(1.7) in any dimension $d \in \mathbb{N}^*$.

The abstract ill-posedness framework will be presented in Section 2. The ill-posedness of the hydrostatic Euler and kinetic incompressible Euler equations is then proved in Sections 3 and 4, respectively.

2. An Abstract Framework for Ill-Posedness

In this section, we present a framework to study the ill-posedness of the following abstract PDE for $U = U(t, x, z)$:

$$\partial_t U - \mathscr{L}U = \mathcal{Q}(U, U), \quad t \geqq 0, \quad x \in \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d, \quad z \in \Omega, \tag{2.1}$$

in which $\mathscr{L}$ (resp. $\mathcal{Q}$) is a linear (resp. bilinear) integro-differential operator in (x, z), $d \in \mathbb{N}^*$ and Ω is an open subset of $\mathbb{R}^{d'}$, $d' \in \mathbb{N}^*$. If $\Omega \neq \mathbb{R}^{d'}$, then some suitable boundary conditions on $\partial\Omega$ are enforced for U. The choice of $\mathbb{T}^d$ is made for simplicity, and other settings are possible.

Consider the sequence $\varepsilon_k = \frac{1}{k}$, for $k \in \mathbb{N}^*$. In the following, we forget the subscript k for readability. Following MÉTIVIER [37], we look for solutions U under the form

$$U(t, x, z) \equiv u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, z\right), \tag{2.2}$$

where $u(s, y, z)$ is 1-periodic in $y_1, \ldots, y_d$. Assume that one can write

$$\mathscr{L}U = \left[\frac{1}{\varepsilon}Lu + R_1(u)\right]\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, z\right),$$

$$\mathcal{Q}(U, U) = \left[\frac{1}{\varepsilon}Q(u, u) + R_2(u, u)\right]\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, z\right),$$

where L is independent of ε; on the other hand, although we do not write it explicitly, the operators R_1, Q, R_2 may depend on ε. This leads to the study of the following abstract PDE:

$$\partial_s u - Lu = Q(u, u) + \varepsilon(R_1(u) + R_2(u, u)), \quad s \geqq 0, \quad y \in \mathbb{T}^d, \quad z \in \Omega. \tag{2.3}$$

We finally assume the existence of a family of norms $(\|\cdot\|_{\delta,\delta'})_{\delta,\delta'>0}$ satisfying the following properties. For all $\delta, \delta' > 0$, the corresponding function space

$$X_{\delta,\delta'} := \{u(y, z),\ \|u\|_{\delta,\delta'} < +\infty\}$$

(with possible boundary conditions in z, if $\Omega \neq \mathbb{R}^{d'}$) is complete and compactly embedded into $\langle v\rangle^m$-weighted Sobolev spaces H^s (with $\langle z\rangle = \sqrt{1 + |z|^2}$), for all $m, s > 0$.

Moreover, for all $0 \leqq \delta < \delta_1$, $0 \leqq \delta' < \delta'_1$, the following inequalities hold for all u:

$$\|u\|_{\delta,\delta'} \leqq \|u\|_{\delta_1,\delta'_1}, \quad \|\partial_y u\|_{\delta,\delta'} \leqq \frac{\delta_1}{\delta_1 - \delta}\|u\|_{\delta_1,\delta'}, \quad \|\partial_z u\|_{\delta,\delta'} \leqq \frac{\delta'_1}{\delta'_1 - \delta'}\|u\|_{\delta,\delta'_1}.$$

In what follows, we shall carry our analysis on the scaled system (2.3).
We make the following structural assumptions on the abstract PDE (2.3).

(H.1) (*Spectral instability for L*) There exists an eigenfunction g associated to an eigenvalue λ_0, with $\Re\lambda_0 > 0$, for L. In other words, the set

$$\Sigma^+ := \{\lambda_0 \in \mathbb{C}, \Re\lambda_0 > 0,\ \exists g \neq 0,\ Lg = \lambda_0 g\}$$

is not empty.

(H.2) [*Loss of analyticity for the semigroup* (*in y only*)] The semigroup e^{Ls}, associated to L, is well-defined in $X_{\delta,\delta'}$, for $s > 0$ and $\delta' > 0$ small enough. Furthermore, there is $\gamma_0 > 0$, such that the following holds:

- For all $\eta > 0$, there exist $k_0 \in [1, +\infty)$ and $\lambda_0 \in \Sigma^+$ such that

$$\left|\frac{\Re\lambda_0}{k_0} - \gamma_0\right| \leqq \eta. \tag{2.4}$$

Moreover there exists an eigenfunction g for L, associated to λ_0, such that $\|g\|_{\delta,\delta'_0} < +\infty$, for all $\delta > 0$ and *some* $\delta'_0 > 0$.

- For any $\Lambda > \gamma_0$, there are $C_\Lambda > 0$, $\delta'_1 > 0$ such that for all $\delta - \Lambda s \geqq 0$, $\delta' \in (0, \delta'_1]$, and all $\varepsilon > 0$,

$$\|e^{Ls}u\|_{\delta-\Lambda s,\delta'} \leqq C_\Lambda \|u\|_{\delta,\delta'}, \quad \forall u \in X_{\delta,\delta'}. \tag{2.5}$$

(H.3) (*Commutator identity*) We have the identity

$$[\partial_y, L] = 0.$$

(H.4) (*Structure of Q*) Q is bilinear and we have for all $\delta, \delta' > 0$, and all $\varepsilon > 0$,

$$\|Q(f,h)\|_{\delta,\delta'} \leqq C_0\|f\|_{\delta,\delta'}(\|\partial_y h\|_{\delta,\delta'} + \|\partial_z h\|_{\delta,\delta'}), \quad \forall f, h \in X_{\delta,\delta'},$$

for some $C_0 > 0$.

(H.5) (*Structure of* R_1, R_2) R_1 is linear and R_2 is bilinear. We have for all $\delta, \delta' > 0$, and all $\varepsilon > 0$,

$$\|R_1(f)\|_{\delta,\delta'} \leqq C_0(\|f\|_{\delta,\delta'} + \|\partial_y f\|_{\delta,\delta'} + \|\partial_z f\|_{\delta,\delta'}), \quad \forall f \in X_{\delta,\delta'},$$
$$\|R_2(f,h)\|_{\delta,\delta'} \leqq C_0\|f\|_{\delta,\delta'}(\|\partial_y h\|_{\delta,\delta'} + \|\partial_z h\|_{\delta,\delta'}), \quad \forall f, h \in X_{\delta,\delta'},$$

for some $C_0 > 0$.

Let us make a few comments about (H.1)–(H.5).

The norm $\|\cdot\|_{\delta,\delta'}$ has to be seen as an *analytic* norm used to build a solution to (2.3). The requested properties are classical in the context of spaces of real analytic functions, see for example [17,39]. Assumption (H.1) yields a violent *instability* for the operator $\mathscr{L}$. In the case $R_1 = 0$, the interpretation is clear: it reveals that $\mathscr{L}$ has an unbounded unstable spectrum. Indeed it means that for all $\varepsilon > 0$,

$$\mathscr{L} g\left(\frac{\cdot}{\varepsilon}, \cdot\right) = \frac{\lambda_0}{\varepsilon} g\left(\frac{\cdot}{\varepsilon}, \cdot\right),$$

that is, $g(\frac{\cdot}{\varepsilon}, \cdot)$ is an eigenfunction associated to the eigenvalue $\frac{\lambda_0}{\varepsilon}$, with $\Re\lambda_0 > 0$. In the case where $\mathscr{L} = L$, it means that L itself has these unstable features.

Assumption (H.2) reveals a *loss of analytic regularity* for the semigroup associated to L; we emphasize that the loss concerns only the y variable, and not the z variable. The constraint on the admissible losses $\Lambda > \gamma_0$ in (2.5) is *sharp*, in the sense that it is the best one can hope for, in view of (2.4) and its possible consequences on the growth of the spectrum. It means in practice that this number γ_0 has to be seen as the supremum of some rescaled functional; see Sections 3 and 4 for illustrations of these facts. Note also that the eigenfunction g has a very demanding regularity with respect to the first variable. In the context of real analyticity, it means in practice that g has to be very well localized in the Fourier space (with respect to the first variable). The assumption (H.2) is certainly the most technical to check in practice, while assumption (H.3) is a simple computation. Note finally that in (H.4) and (H.5), the losses of derivatives are only of order 1, as usual for Cauchy–Kowalevsky type results.

The main result of this section is the following abstract ill-posedness theorem:

Theorem 2.1. *Assume* (H.1)–(H.5). *For all* $m, s \in \mathbb{N}$, $\alpha \in (0,1]$, $k \in \mathbb{N}$, *there are families of solutions* $(U_\varepsilon)_{\varepsilon>0}$ *of* (2.1), *times* $t_\varepsilon = \mathcal{O}(\varepsilon|\log\varepsilon|)$ *and* $(x_0, z_0) \in \mathbb{T}^d \times \Omega$ *such that*

$$\lim_{\varepsilon\to 0} \frac{\|U_\varepsilon\|_{L^2([0,t_\varepsilon]\times\Omega_\varepsilon)}}{\|\langle z\rangle^m U_\varepsilon|_{t=0}\|_{H^s(\mathbb{T}^d\times\Omega)}} = +\infty \tag{2.6}$$

where $\Omega_\varepsilon = B(x_0, \varepsilon^k) \times B(z_0, \varepsilon^k)$.

In the case where the linear differential operator L has constant coefficients, the theorem is due to MÉTIVIER [37] using the power series approach. In applications to the equations we have in mind, the differential operator L typically depends on variables (x, z). Our functional framework is closer to that of DESJARDINS and GRENIER [13], who introduced an *analytic* framework for studying nonlinear (Rayleigh–Taylor) *instability*.

Remark 2.2. We expect that this abstract framework can be useful to prove ill-posedness for multi-phase Euler models, see for example [10]. These models can be (formally) derived in the context of the quasineutral limit of the Vlasov–Poisson equation, see GRENIER [17]. Whereas the one-dimensional model surely fits the local framework by MÉTIVIER [37], the multi-dimensional analogue appears to be nonlocal due to the pressure.

The choice of parameters we make below follows MÉTIVIER [37]. Let $s \in \mathbb{N}$, $\alpha \in (0,1]$, $k \in \mathbb{N}$ be all arbitrary, but fixed. We take M large enough and $\beta > 0$ small enough such that

$$\alpha' := \frac{M-s}{M}\alpha - \frac{1+2dk}{2M} > 0, \tag{2.7}$$

$$\beta M < \frac{1}{2}, \quad \frac{2\beta}{1+\beta} < \alpha'. \tag{2.8}$$

Let $\gamma_0 > 0$ satisfying all requested properties in (H.2). Let $\eta \in \left(0, \min\left(\frac{\gamma_0}{2}, \frac{\beta\gamma_0}{4}\right)\right)$. We obtain a pair of eigenvalue and eigenfunction (λ_0, g), and $k_0 \in [1, +\infty)$, $\delta_0' > 0$, such that the first point of (H.2) is satisfied. We note that defining

$$\gamma_1 = (1+\beta)\frac{\Re\lambda_0}{k_0}, \tag{2.9}$$

we have $\gamma_1 > \gamma_0$ as well as $\frac{1}{2}\left(\frac{\Re\lambda_0}{k_0} + \gamma_1\right) > \gamma_0$. We also define

$$\delta_0 = \frac{(1-\beta)M}{k_0}|\log\varepsilon|.$$

Let (x_0, z_0) be such that $g(x_0, z_0) \neq 0$. By continuity, there is $c > 0$ such that for all small enough ε,

$$|g(x, z)| \geqq c, \quad \forall (x, z) \in B(x_0, \varepsilon^k) \times B(z_0, \varepsilon^k) \subset \Omega_\varepsilon. \tag{2.10}$$

Let us assume that λ_0 is real (we will explain the general case at the end of the proof).

Unlike MÉTIVIER [37], our analysis relies on weighted in time analytic type norms, introduced by Caflisch in his proof of the Cauchy–Kowalevsky theorem ([11]; see also [13]). To be precise, let us introduce the norm

$$\begin{aligned}\|w\| = \sup_{0 \leqq \delta \leqq \delta_0} \sup_{0 \leqq s \leqq \frac{1}{\gamma_1}(\delta_0 - \delta)} &[\|w(s)\|_{\delta,\delta'} + (\delta_0 - \delta - \gamma_1 s)^\gamma (\|\partial_y w(s)\|_{\delta,\delta'} \\ &+ M^{-\gamma} |\log \varepsilon|^{-\gamma} \|\partial_z w(s)\|_{\delta,\delta'})],\end{aligned} \tag{2.11}$$

in which δ' is a shorthand for $\frac{\delta_0' k_0 \delta}{M|\log \varepsilon|}$ and γ is an arbitrary fixed number in $(0, 1)$. We denote by X the space of functions w such that $\|w\| < +\infty$; it is well known that $(X, \|\cdot\|)$ is a Banach space.

Theorem 2.1 is a consequence of the following lemma, where we construct solutions in X that capture the instability for the scaled system (2.3).

Lemma 2.3. *Under the assumptions* (H.1)–(H.5)*, there is* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$*, there exists a solution* u *of* (2.3) *of the form*

$$u(s) = \varepsilon^M e^{\lambda_0 s} g + w(s), \quad \forall s \in [0, s_\varepsilon], \tag{2.12}$$

where (λ_0, g) *is defined as in* (H.2)*,*

$$s_\varepsilon = \frac{(1-\beta)M}{k_0 \gamma_1} |\log \varepsilon| = \frac{1-\beta}{1+\beta} \frac{M}{\lambda_0} |\log \varepsilon|$$

and w *is a remainder satisfying*

$$\|w\| \lesssim \varepsilon^{\frac{2\beta}{1+\beta} M}.$$

Proof. We set

$$u_{\text{app}}(s) := \varepsilon^M e^{Ls} g = \varepsilon^M e^{\lambda_0 s} g.$$

It follows directly from the definition and the assumption $\|g\|_{\delta_0,\delta_0'} \lesssim 1$ that

$$\|u_{\text{app}}\| \leqq C \varepsilon^M |\log \varepsilon|^\gamma e^{\frac{\lambda_0 \delta_0}{\gamma_1}} \leqq C \varepsilon^\kappa |\log \varepsilon|^\gamma, \quad \kappa := \frac{2\beta}{1+\beta} M,$$

in which the logarithmic loss is due to the weight in time in the norm $\|\cdot\|$. Next, we observe that by definition, u_{app} is a solution of

$$\partial_s u_{\text{app}} - L u_{\text{app}} = Q(u_{\text{app}}, u_{\text{app}}) + \varepsilon(R_1(u_{\text{app}}) + R_2(u_{\text{app}}, u_{\text{app}})) + R_{\text{app}}$$

where, thanks to the assumptions (H.4)–(H.5), the remainder $R_{\rm app}$ satisfies the estimate

$$\|R_{\rm app}\| \leqq C_{\rm app}\varepsilon^{2\kappa}|\log\varepsilon|^{2\gamma}.$$

(Here, we have used the fact that $\beta M < 1/2$, so that $\varepsilon \leqq \varepsilon^{\kappa}$.)

Our goal is now to solve the equation for the difference $w = u - u_{\rm app}$:

$$\begin{aligned}\partial_s w - Lw = Q(w,w) - Q(u_{\rm app},w) - Q(w,u_{\rm app}) + \varepsilon(R_1(w) + R_2(w,w))\\ - \varepsilon(R_2(u_{\rm app},w) + R_2(w,u_{\rm app})) - R_{\rm app}\end{aligned} \tag{2.13}$$

with $w|_{s=0} = 0$. Then $u = w + u_{\rm app}$ solves the Equation (2.3) as desired. To that purpose, let us study the following approximation scheme:

$$\begin{aligned}\partial_s w_{n+1} - Lw_{n+1} =& Q(w_n,w_n) - Q(u_{\rm app},w_n) - Q(w_n,u_{\rm app}) + \varepsilon(R_1(w_n)\\ &+ R_2(w_n,w_n)) - \varepsilon(R_2(u_{\rm app},w_n) + R_2(w_n,u_{\rm app})) - R_{\rm app}\end{aligned}$$

with $w_{n+1}|_{s=0} = 0$. Set $w_0 = 0$.

We shall prove that for $\varepsilon > 0$ small enough, we can make the scheme converge, and that the following estimates hold for all $n \geqq 0$:

$$\|w_n\| \leqq C\varepsilon^{2\kappa}|\log\varepsilon|^{1+2\gamma}, \tag{2.14}$$

for some universal constant C (independent of n). In addition, for all $n \geqq 1$,

$$\|w_{n+1} - w_n\| \leqq \frac{1}{2}\|w_n - w_{n-1}\|. \tag{2.15}$$

By induction, assume that (2.14) is true for all $k \leqq n$. We have

$$\begin{aligned}w_{n+1}(s) = \int_0^s & e^{(s-\tau)L}[Q(w_n,w_n) - (Q(u_{\rm app},w_n) + Q(w_n,u_{\rm app}))\\ &+ \varepsilon(R_1(w_n) + R_2(w_n,w_n) - R_2(u_{\rm app},w_n) - R_2(w_n,u_{\rm app}))\\ &- R_{\rm app}]\,d\tau\\ =:&\ I_1(s) + I_2(s) + I_3(s) + I_4(s).\end{aligned}$$

We claim that there are $C_0 > 0$, $\gamma' > 0$ and $\varepsilon_0 > 0$ (independent of n) such that for all $\varepsilon \in (0,\varepsilon_0]$, the following estimates hold:

- For the non-linear term:

$$\|I_1\| \leqq C_0\|w_n\|^2|\log\varepsilon|^{1+\gamma} \leqq C_0\varepsilon^{4\kappa}|\log\varepsilon|^{1+2\gamma+\gamma'}.$$

- For the linear term:

$$\|I_2\| \leqq C_0'\|w_n\|\|u_{\rm app}\||\log\varepsilon|^{1+\gamma} \leqq C_0\varepsilon^{1+2\kappa}|\log\varepsilon|^{1+2\gamma+\gamma'}.$$

- For the first remainder:

$$\|I_3\| \leqq C_0'\varepsilon\|w_n\|(1 + \|w_n\|)|\log\varepsilon|^{1+\gamma} \leqq C_0\varepsilon^{1+\kappa}|\log|^{1+2\gamma+\gamma'}.$$

- For the second remainder:

$$\|I_4\| \leqq C_{\text{app}}\varepsilon^{2\kappa}|\log\varepsilon|^{1+2\gamma}.$$

This shows that imposing ε small enough (but independently of n), (2.14) is satisfied at rank $n+1$, and thus closes the induction argument.

It remains to justify the above estimates. We shall only provide details of the computations for I_1, the other ones being similar. For all $\delta \in [1, \delta_0 - \gamma_1 s)$, we set $\delta' = \frac{\delta_0' k_0 \delta}{M|\log\varepsilon|}$ and $\Lambda := \frac{\lambda_0}{2k_0} + \frac{\gamma_1}{2}$. Note that $\Lambda > \gamma_0$. We write $\gamma_1 = \frac{\lambda_0}{k_0} + 2\nu$ with $\nu := \frac{\beta\lambda_0}{2k_0}$. We get, using (H.2) and (H.4),

$$\begin{aligned}
\|I_1(s)\|_{\delta,\delta'} &\lesssim \int_0^s \|e^{(s-\tau)L}Q(w_n,w_n)\|_{\delta,\delta'}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|Q(w_n,w_n)\|_{\delta+\Lambda(s-\tau),\delta'}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|w_n(\tau)\|_{\delta+\Lambda(s-\tau),\delta'}\left(\|\partial_y w_n(\tau)\|_{\delta+\Lambda(s-\tau),\delta'}\right. \\
&\quad \left.+\|\partial_z w_n(\tau)\|_{\delta+\Lambda(s-\tau),\delta'}\right)\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|w_n\|^2\,(\delta_0-\delta-\Lambda s-\nu\tau)^{-\gamma}\,(1+M^\gamma|\log\varepsilon|^\gamma)\,\mathrm{d}\tau,
\end{aligned}$$

since $\delta' \leqq \delta' + \frac{\delta_0' k_0 \Lambda(s-\tau)}{M|\log\varepsilon|}$. We thus get

$$\begin{aligned}
\|I_1(s)\|_{\delta,\delta'} &\lesssim (1+M^\gamma|\log\varepsilon|^\gamma)\|w_n\|^2 \int_0^s (\delta_0-\delta-\Lambda s-\nu\tau)^{-\gamma}\,\mathrm{d}\tau \\
&\lesssim (1+M^\gamma|\log\varepsilon|^\gamma)\|w_n\|^2 \frac{1}{1-\gamma}\frac{1}{\nu}\,(\delta_0-\delta-\Lambda s)^{1-\gamma} \\
&\lesssim (1+M^\gamma|\log\varepsilon|^\gamma)|\log\varepsilon|^{1-\gamma}\|w_n\|^2,
\end{aligned}$$

recalling that $\delta_0 = \frac{(1-\beta)M}{k_0}|\log\varepsilon|$. Likewise, by (H.3), (H.2) and (H.4), we obtain

$$\begin{aligned}
\|\partial_y I_1(s)\|_{\delta,\delta'} &\lesssim \int_0^s \|\partial_y e^{(s-\tau)L}Q(w_n,w_n)\|_{\delta,\delta'}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|e^{(s-\tau)L}\partial_y Q(w_n,w_n)\|_{\delta,\delta'}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|\partial_y[Q(w_n,w_n)]\|_{\delta+\Lambda(s-\tau),\delta'}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \frac{2\delta_0}{(\delta_0-\delta-\Lambda s-\nu\tau)}\|Q(w_n,w_n)\|_{\frac{\delta_0-\gamma_1\tau}{2}+\frac{\delta+\Lambda(s-\tau)}{2},\delta'}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|w_n\|^2\delta_0\,(\delta_0-\delta-\Lambda s-\nu\tau)^{-1-\gamma}\,(1+M^\gamma|\log\varepsilon|^\gamma)\,\mathrm{d}\tau \\
&\lesssim \|w_n\|^2|\log\varepsilon|^{1+\gamma}\,(\delta_0-\delta-\Lambda s-\nu s)^{-\gamma}.
\end{aligned}$$

Note that $\Lambda + \nu = \gamma_1$. Consequently, we deduce

$$\|\partial_y I_1(s)\|_{\delta,\delta'} \lesssim \|w_n\|^2 |\log\varepsilon|^{1+\gamma} (\delta_0 - \delta - \gamma_1 s)^{-\gamma}.$$

Now we use that there is no loss in the z variable for the semigroup. Recalling $\delta' = \frac{\delta_0' k_0 \delta}{M|\log\varepsilon|}$, we get with similar computations

$$\begin{aligned}
\|\partial_z I_1(s)\|_{\delta,\delta'} &\lesssim \int_0^s \|\partial_z e^{(s-\tau)L} Q(w_n,w_n)\|_{\delta,\delta'}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|e^{(s-\tau)L}[Q(w_n,w_n)]\|_{\delta,\frac{\delta_0' k_0}{M|\log\varepsilon|}\left[\frac{\delta_0-\gamma_1\tau}{2}+\frac{\delta}{2}\right]} \delta_0\,(\delta_0-\delta-\gamma_1\tau)^{-1}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|Q(w_n,w_n)\|_{\delta+\Lambda(s-\tau),\frac{\delta_0' k_0}{M|\log\varepsilon|}\left[\frac{\delta_0-\gamma_1\tau}{2}+\frac{\delta}{2}\right]} \delta_0\,(\delta_0-\delta-\Lambda\tau-\nu\tau)^{-1}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|Q(w_n,w_n)\|_{\delta+\Lambda(s-\tau),\frac{\delta_0' k_0}{M|\log\varepsilon|}\left[\frac{\delta_0-\gamma_1\tau}{2}+\frac{\delta}{2}\right]} \delta_0\,(\delta_0-\delta-\Lambda s-\nu\tau)^{-1}\,\mathrm{d}\tau \\
&\lesssim \int_0^s \|w_n\|^2 \frac{\delta_0(1+M^\gamma|\log\varepsilon|^\gamma)}{(\delta_0-\delta-\Lambda s-\nu\tau)^{\gamma+1}}\,\mathrm{d}\tau \\
&\lesssim \|w_n\|^2|\log\varepsilon|^{1+\gamma}(\delta_0-\delta-\gamma_1 s)^{-\gamma}.
\end{aligned}$$

We end up with the claimed estimate for $\|I_1\|$, using the induction assumption on $\|w_n\|$. The contraction estimates are now straightforward; we have indeed for all $n \geqq 2$,

$$\begin{aligned}
\|w_{n+1}-w_n\| \leqq C_1|\log\varepsilon|^{1+\gamma}(\|w_{n-1}\|+\|w_n\|)\|\|w_n-w_{n-1}\| \\
+C_2\varepsilon^\kappa|\log\varepsilon|^{1+\gamma}\|\|w_n-w_{n-1}\|.
\end{aligned}$$

By (2.14), we have

$$\|w_{n-1}\|+\|w_n\| \lesssim \varepsilon^{2\kappa}|\log\varepsilon|^{1+2\gamma},$$

so that by imposing $\varepsilon > 0$ small, we obtain (2.15).

Since $(X,\|\cdot\|)$ is a Banach space, we get a solution w of (2.13), satisfying in addition the estimate

$$\|w\| \leqq C\varepsilon^{2\kappa}|\log\varepsilon|^{1+2\gamma}.$$

□

Finally, we complete the proof of Theorem 2.1:

Proof of Theorem 2.1. Consider the initial condition $U_\varepsilon|_{t=0}(x,z) = \varepsilon^M g\left(\frac{x}{\varepsilon},z\right)$ for (2.1). We note that

$$\|\langle z\rangle^m U_\varepsilon|_{t=0}\|^\alpha_{H^s} \leqq C'\varepsilon^{\alpha(M-s)}$$

By Lemma 2.3, we obtain a solution $u_\varepsilon(s,y,z)$ of (2.3) with initial condition $\varepsilon^M g(y,z)$, satisfying (2.12). Thus we get a solution $U_\varepsilon(t,x,z) = u_\varepsilon\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},z\right)$ for (2.1). Consider the time

$$s_\varepsilon = \frac{1-\beta}{1+\beta}\frac{M}{\lambda_0}|\log\varepsilon|$$

and let $t_\varepsilon = \varepsilon s_\varepsilon$. Using the embedding in L^2, (2.12), and the lower bound (2.10), we have, for some constants $\theta_1, \theta_1' > 0$ that are independent of ε:

$$\begin{aligned}
\|U_\varepsilon\|_{L^2([0,t_\varepsilon]\times\Omega_\varepsilon)} &= \|\varepsilon^M e^{Lt/\varepsilon} g\left(\frac{\cdot}{\varepsilon},\cdot\right) + w\left(\frac{\cdot}{\varepsilon},\cdot\right)\|_{L^2([0,t_\varepsilon]\times\Omega_\varepsilon)} \\
&\geqq \|\varepsilon^M e^{Lt/\varepsilon} g\left(\frac{\cdot}{\varepsilon},\cdot\right) + w\left(\frac{\cdot}{\varepsilon},\cdot\right)\|_{L^2([t_\varepsilon-\varepsilon,t_\varepsilon]\times B(x_0,\varepsilon^k)\times B(z_0,\varepsilon^k))} \\
&\geqq \theta_1 \varepsilon^\kappa \varepsilon^{dk+\frac12} - \|w\left(\frac{\cdot}{\varepsilon},\cdot\right)\|_{L^2([t_\varepsilon-\varepsilon,t_\varepsilon]\times B(x_0,\varepsilon^k)\times B(z_0,\varepsilon^k))} \\
&\geqq \theta_1 \varepsilon^\kappa \varepsilon^{dk+\frac12} - C\varepsilon^{dk+\frac12}\varepsilon^{2\kappa}|\log\varepsilon|^{\gamma_1} \\
&\geqq \theta_1' \varepsilon^\kappa \varepsilon^{dk+\frac12}
\end{aligned}$$

for sufficiently small ε. We recall that $\kappa = \frac{2\beta}{1+\beta}M$. By a view of the choice on the parameters in (2.7) and (2.8), we get

$$\frac{\|U_\varepsilon\|_{L^2([0,t_\varepsilon]\times\Omega_\varepsilon)}}{\|\langle z\rangle^m U_\varepsilon|_{t=0}\|^\alpha_{H^s}} \geqq \frac{\theta_1'}{C'}\varepsilon^{M\left(\frac{2\beta}{1+\beta}-\alpha'\right)},$$

which tends to infinity as $\varepsilon \to 0$. This concludes the proof of the theorem when λ_0 is real. In the general case, the modifications are standard, see for example Métivier [37]. In Lemma 2.3, one has to replace (2.12) by

$$u(s) = \varepsilon^M \Re(e^{\lambda_0 s} g) + w(s), \quad \forall s \in [0, s_\varepsilon], \tag{2.16}$$

meaning that instead of comparing to a pure exponentially growing mode, we have to compare u to an exponentially growing mode multiplied by an oscillating function. The above analysis can be performed again, making sure to avoid the (discrete) times when this oscillating function cancels. □

3. Ill-Posedness of the Hydrostatic Euler Equations

In this section, we give the proof of Theorem 1.1, establishing the ill-posedness of the hydrostatic Euler Equation (1.1), which we write in the stream-vorticity formulation:

$$\partial_t \Omega + U\partial_x \Omega + V\partial_z \Omega = 0, \tag{3.1}$$

in which $U := \partial_z \Phi$, $V := -\partial_x \Phi$, and the stream function Φ solves the elliptic problem:

$$\partial_z^2 \Phi = \Omega, \quad \Phi_{|z=\pm1} = 0. \tag{3.2}$$

We shall prove in this section how Theorem 1.1 follows from our abstract ill-posedness framework. We work with the analytic function space $X_{\delta,\delta'}$, equipped with the following norm:

$$\|\omega\|_{\delta,\delta'} := \sum_{n\in\mathbb{Z}}\sum_{k\geqq0} \|\partial_z^k \omega_n\|_{L^2([-1,1])} \frac{|\delta'|^k}{k!} e^{\delta|n|}, \tag{3.3}$$

for any $\delta, \delta' > 0$, in which $\omega_n = \langle \omega, e^{inx} \rangle_{L^2(\mathbb{T})}$ stands for the Fourier coefficients of ω with respect to x-variable. We also denote by $X_{\delta'}$ the z-analytic function space, equipped with the following norm:

$$\|\omega\|_{\delta'} := \sum_{k \geqq 0} \|\partial_z^k \omega\|_{L^2([-1,1])} \frac{|\delta'|^k}{k!}. \tag{3.4}$$

We study ill-posedness near a well-chosen shear flow, that is $\Omega = U'(z)$. We assume that U is real analytic so that the norm

$$|||U|||_{\delta'} := \sum_{k \geqq 0} \|\partial_z^k U'\|_{L^\infty([-1,1])} \frac{|\delta'|^k}{k!} \tag{3.5}$$

is finite for some $\delta' > 0$.

By viewing the analytic norms (3.3) and (3.4), we have for all $0 < \delta' < \delta_1'$ that

$$\|\partial_z \omega\|_{\delta'} \leqq \frac{\delta_1'}{\delta_1' - \delta'} \|\omega\|_{\delta_1'},$$

and so

$$\|\partial_z \omega\|_{\delta,\delta'} \leqq \frac{\delta_1'}{\delta_1' - \delta'} \|\omega\|_{\delta,\delta_1'}.$$

We can argue similarly for $\|\partial_y \omega\|_{\delta,\delta'}$.

We write the perturbed solution in the fast variables as follows:

$$\Omega(t,x,z) = U'(z) + \omega\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, z\right), \quad \Phi(t,x,z) = \int_{-1}^{z} U(\theta)\, d\theta + \varphi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, z\right).$$

Let $(s, y) = (\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$. The function $\omega(s, y, z)$ solves

$$\partial_s \omega - \mathcal{L}\omega = -\partial_z \varphi \partial_y \omega + \partial_y \varphi \partial_z \omega, \tag{3.6}$$

in which the linearized operator is defined by

$$\mathcal{L}\omega := -U\partial_y \omega + \partial_y \varphi\, U'', \quad \partial_z^2 \varphi = \omega, \quad \varphi_{|z=\pm 1} = 0. \tag{3.7}$$

To treat the loss of derivatives from each quantity in the quadratic term $\partial_y \varphi \partial_z \omega$, we further write the above equation in a matrix form. Set

$$L := \begin{pmatrix} \mathcal{L} & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & -U' + U''\partial_z \varphi(\cdot) + U'''\varphi(\cdot) & -U\partial_y \end{pmatrix}, \tag{3.8}$$

in which by convention, $\varphi(W)$ solves $\partial_z^2 \varphi(W) = W$ with $\varphi_{|z=\pm 1} = 0$. Similarly, for any two vector fields $V = (v_1, v_2, v_3)$ and $W = (w_1, w_2, w_3)$, we set

$$Q(V,W) := \begin{pmatrix} -\partial_z \varphi(v_1) w_2 + \varphi(v_2) w_3 \\ -\partial_z \varphi(v_2) w_2 - \partial_z \varphi(v_1) \partial_y w_2 + \partial_y \varphi(v_2) w_3 + \varphi(v_2) \partial_y w_3 \\ -v_1 w_2 - \partial_z \varphi(v_1) \partial_z w_2 + \partial_z \varphi(v_2) w_3 + \varphi(v_2) \partial_z w_3 \end{pmatrix}.$$

It follows that ω solves (3.6) if and only if the function

$$W := \begin{pmatrix} \omega \\ \partial_y \omega \\ \partial_z \omega \end{pmatrix}$$

solves

$$\partial_s W - LW = Q(W, W). \tag{3.9}$$

We shall show the ill-posedness of (3.9) by directly checking the assumptions (H.1)–(H.5) made in our abstract ill-posedness framework.

3.1. Unbounded Unstable Spectrum of the Linearized Operator

Our starting point is the work by Renardy [41], in which he showed that the linearized hydrostatic Euler system near certain shear flows $U(z)$ possesses ellipticity or unbounded unstable spectrum. Indeed, let us study the linearization near $U(z)$:

$$\partial_t \omega + U \partial_y \omega - \partial_y \varphi \, U'' = 0, \quad \partial_z^2 \varphi = \omega, \quad \varphi_{|z=\pm 1} = 0, \tag{3.10}$$

and search for a growing mode of the form

$$\omega = e^{in(y-ct)} \hat{\omega}(z), \tag{3.11}$$

with $\Im c \neq 0$ and $\hat{\omega} = \partial_z^2 \hat{\varphi}$. The stream function $\hat{\varphi}$ then solves the Rayleigh problem:

$$(U - c)\partial_z^2 \hat{\varphi} - U'' \hat{\varphi} = 0, \quad \hat{\varphi}_{|z=\pm 1} = 0. \tag{3.12}$$

This is a very classical problem in fluid mechanics (see for instance the recent work [19]). There are two independent solutions of the Rayleigh problem:

$$\hat{\varphi}_1 = U - c, \quad \hat{\varphi}_2 = (U - c) \int_{-1}^{z} \frac{1}{(U(z') - c)^2} \, \mathrm{d}z' \tag{3.13}$$

whose Wronskian determinant is $W[\hat{\varphi}_1, \hat{\varphi}_2] = \partial_z \hat{\varphi}_2 \hat{\varphi}_1 - \partial_z \hat{\varphi}_1 \hat{\varphi}_2 = 1$. The pair $(\hat{\varphi}, c)$ solves the Rayleigh problem if and only if c is a zero of the (Evans) function

$$D(c) := \hat{\varphi}_1(-1)\hat{\varphi}_2(1) - \hat{\varphi}_1(1)\hat{\varphi}_2(-1). \tag{3.14}$$

This means, precisely, that c has to solve the equation

$$\int_{-1}^{1} \frac{1}{(U(z) - c)^2} \, \mathrm{d}z = 0 \tag{3.15}$$

(see also [41, Theorem 1]). As an explicit example, one can take $U(z) = \tanh(\frac{z}{d_1})$ for small $d_1 > 0$ as shown in [12]. Since c does not depend on n, the unstable spectrum is unbounded. Let us summarize this discussion in the following statement:

Lemma 3.1. *The linearized operator $\mathcal{L}$ possesses a growing mode of the form* (3.11) *if and only if c is a zero of the Evans function* (3.14). *If such a growing mode exists, the unstable spectrum is unbounded, containing all the points $\lambda = -inc$, with $n \in \mathbb{Z}$ such that $n\Im c > 0$ and with corresponding eigenfunctions of the form*

$$\omega = e^{iny} \partial_z^2 \hat{\varphi}_2. \tag{3.16}$$

3.2. Sharp Semigroup Bounds

Let L be the matrix operator defined as in (3.8). From now on, we consider a shear flow $U(z)$ such that $|||U|||_{\delta_1'} < +\infty$ for some $\delta_1' > 0$, with which the unstable spectrum of the linearization (3.10) is unbounded; see the previous section. We set γ_0 to be defined by

$$\gamma_0 := \max_{\Im c \neq 0} \{\Im c \colon D(c) = 0\}. \tag{3.17}$$

The above maximum exists and is positive, since by a view of (3.15) the Rayleigh problem has no solution as $|c| \to \infty$ and $D(c)$ is continuous (in fact, analytic) in $\{\Im c \neq 0\}$. Let c_0 be the solution of $D(c_0) = 0$ so that $\gamma_0 = \Im c_0$, and let ω be the corresponding eigenfunction as in (3.16). By Lemma 3.1, $\lambda_0 = -inc_0$ is an unstable eigenvalue of $\mathcal{L}$, for all $n \in \mathbb{Z}$, so that $\Re\lambda_0 = n\Im c_0 > 0$. Since $\omega = e^{iny}\partial_z^2\hat{\varphi}_2$, as defined in (3.13), the regularity of ω follows from that of the given shear flow $U(z)$.

The main goal of this section is to prove the following proposition.

Proposition 3.2. *Let $\delta, \delta' > 0$ and let γ_0 be defined as in (3.17). The semigroup e^{Ls}, associated to L, is well-defined in $X_{\delta,\delta'}$, for small $s > 0$ and small δ'. More precisely, for any $\gamma > \gamma_0$, there is a positive constant C_γ so that*

$$\|e^{Ls}h\|_{\delta-\gamma s,\delta'} \leqq C_\gamma \|h\|_{\delta,\delta'}$$

for all $h \in X_{\delta,\delta'}$, $0 < \delta' \ll \gamma_0$, and for all s so that $\delta - \gamma s > 0$.

We start the proof of Proposition 3.2 by proving the same bounds for the semigroup $e^{\mathcal{L}s}$. In Fourier variables, we first study the semigroup $e^{\mathcal{L}_n s}$, that is to say we study $\omega = e^{\mathcal{L}_n s}h$, solving

$$\partial_s\omega - \mathcal{L}_n\omega = 0, \quad \omega_{|s=0} = h,$$

with

$$\mathcal{L}_n\omega := -inU\omega + in\varphi U'', \quad \partial_z^2\varphi = \omega, \quad \varphi_{|z=\pm 1} = 0.$$

Lemma 3.3. *For each $n \in \mathbb{Z}$, let ω solve the transport equation $(\partial_s + inU)\omega = g$. Then,*

$$\|\omega(s)\|_{\delta'} \leqq e^{\delta'|n||||U|||_{\delta'}s}\|\omega(0)\|_{\delta'} + \int_0^s e^{\delta'|n||||U|||_{\delta'}(s-\tau)}\|g(\tau)\|_{\delta'}\,\mathrm{d}\tau,$$

for all $\delta' \in (0, \delta_1']$.

Proof. The proof follows by L^2 energy estimates. Indeed, differentiating the equation for ω, we have

$$\partial_s\partial_z^k\omega = -in\sum_{j=0}^k \frac{k!}{j!(k-j)!}\partial_z^{k-j}U\partial_z^j\omega + \partial_z^k g.$$

Now taking the L^2 product against $\partial_z^k\omega$ and taking the real part, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\partial_z^k\omega\|_{L^2(-1,1)}^2 \leqq \left[|n|\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\|\partial_z^{k-j}U\partial_z^j\omega\|_{L^2(-1,1)} + \|\partial_z^k g\|_{L^2(-1,1)}\right]\|\partial_z^k\omega\|_{L^2(-1,1)},$$

upon noting that the real part of $i\langle U\partial_z^k\omega, \partial_z^k\omega\rangle_{L^2(-1,1)}$ is equal to zero. This yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\partial_z^k\omega\|_{L^2(-1,1)} \leqq |n|\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\|\partial_z^{k-j}U\|_{L^\infty(-1,1)}\|\partial_z^j\omega\|_{L^2(-1,1)} + \|\partial_z^k g\|_{L^2(-1,1)}.$$

By definition of the analytic norm, we get from the above estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|\omega\|_{\delta'} &\leqq |n|\sum_{k\geqq 0}\frac{|\delta'|^k}{k!}\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}\|\partial_z^{k-j}U\|_{L^\infty(-1,1)}\|\partial_z^j\omega\|_{L^2(-1,1)} + \|g\|_{\delta'}\\ &\leqq \delta'|n|\sum_{k\geqq 0}\sum_{j=0}^{k-1}\frac{|\delta'|^j}{j!}\frac{|\delta'|^{k-j-1}}{(k-j-1)!}\|\partial_z^{k-j-1}U'\|_{L^\infty(-1,1)}\|\partial_z^j\omega\|_{L^2(-1,1)} + \|g\|_{\delta'}\\ &\leqq \delta'|n|\sum_{\ell\geqq 0}\sum_{j\geqq 0}\frac{|\delta'|^j}{j!}\frac{|\delta'|^\ell}{\ell!}\|\partial_z^\ell U'\|_{L^\infty(-1,1)}\|\partial_z^j\omega\|_{L^2(-1,1)} + \|g\|_{\delta'}\\ &\leqq \delta'|n|\,|||U'|||_{\delta'}\|\omega\|_{\delta'} + \|g\|_{\delta'}.\end{aligned} \tag{3.18}$$

The lemma follows from the Gronwall inequality. □

Lemma 3.4. *Let γ_0 be defined as in (3.17). For each $n \in \mathbb{Z}$, the operator $\mathcal{L}_n$ generates a continuous semigroup $e^{\mathcal{L}_n s}$ from $X_{\delta'}$ to itself, for small $\delta' > 0$. In addition, for any $\gamma > \gamma_0$, there is a positive constant C_γ so that*

$$\|e^{\mathcal{L}_n s}h\|_{\delta'} \leqq C_\gamma e^{\gamma|n|s}\|h\|_{\delta'}, \quad \forall s \geqq 0,$$

for all $h \in X_{\delta'}$, and $0 < \delta' \ll \gamma_0$.

Proof. By the time rescaling $s \mapsto s|n|$, it suffices to study the semigroup for $n = \pm 1$. Let us focus on $n = 1$, the case $n = -1$ being identical. We obtain the sharp bound via the inverse Laplace transform for the semigroup ([40] or [43, Appendix A]):

$$e^{\mathcal{L}_1 s}h = \mathrm{PV}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda s}(\lambda - \mathcal{L}_1)^{-1}h\,\mathrm{d}\lambda,$$

for sufficiently large γ, where PV denotes the Cauchy principal value. Set $\hat{\omega} := (\lambda - \mathcal{L}_1)^{-1} h$. We shall solve the resolvent equation for all $\lambda = -ic$ with sufficiently large values of $|\Im c|$. It follows that

$$\hat{\omega} = \partial_z^2 \hat{\varphi}, \quad (U - c)\partial_z^2 \hat{\varphi} - U'' \hat{\varphi} = \frac{h}{i}, \quad \hat{\varphi}_{|z=\pm 1} = 0.$$

This is a nonhomogenous Rayleigh problem with unknown $\hat{\varphi}$. By the definition of γ_0, see (3.17), for all c such that $|\Im c| > \gamma_0$, the Rayleigh operator $(U - c)\partial_z^2 - U''$ is invertible, and so the Rayleigh problem has an unique solution $\hat{\varphi}$ (in fact, one can derive an explicit representation involving the homogenous solutions $\hat{\varphi}_1, \hat{\varphi}_2$ defined as in (3.13), see for example [19]). In addition, together with zero boundary conditions, there holds

$$\|\hat{\varphi}\|_{H^2} \leqq \frac{C}{1 + |\Re c|} \|h\|_{L^2}, \quad \forall c \in \mathbb{C} \colon |\Im c| > \gamma_0.$$

Recalling that $\hat{\omega} = (\lambda - \mathcal{L}_1)^{-1} h$ is the resolvent solution with $\hat{\omega} = \partial_z^2 \hat{\varphi}$, we then get

$$\begin{aligned} e^{\mathcal{L}_1 s} h &= \mathrm{PV} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda s} (\lambda - \mathcal{L}_1)^{-1} h \, \mathrm{d}\lambda, \\ &= \mathrm{PV} \frac{1}{2\pi} \int_{i\gamma - \infty}^{i\gamma + \infty} e^{-ics} \Big[\frac{U''}{U - c} \hat{\varphi} + \frac{h}{i(U - c)} \Big] \, \mathrm{d}c \\ &= \mathrm{PV} \frac{1}{2\pi} \int_{i\gamma - \infty}^{i\gamma + \infty} e^{-ics} \frac{U''}{U - c} \hat{\varphi} \, \mathrm{d}c + e^{-iU(z)s} h. \end{aligned}$$

This identity, together with the elliptic bound on $\hat{\varphi}$, and the fact that we can take any c such that $\Im c = \gamma > \gamma_0$ yield

$$\begin{aligned} \|e^{\mathcal{L}_1 s} h\|_{L^2} &\leqq \frac{C}{2\pi} \int_{\mathbb{R}} e^{\gamma s} \frac{\|U''\|_{L^\infty}}{\sqrt{|U - \Re c|^2 + \gamma^2}} \|\hat{\varphi}\|_{L^2} \, d(\Re c) + \|h\|_{L^2} \\ &\leqq \frac{C}{2\pi} \int_{\mathbb{R}} e^{\gamma s} \frac{\|U''\|_{L^\infty}}{\sqrt{|U - \Re c|^2 + \gamma^2}} \frac{1}{1 + |\Re c|} \|h\|_{L^2} \, d(\Re c) + \|h\|_{L^2} \\ &\leqq C_\gamma e^{\gamma s} \|h\|_{L^2}, \end{aligned}$$

for all $\gamma > \gamma_0$.

Next, we derive analytic estimates for $\omega := e^{\mathcal{L}_1 s} h$, solving

$$(\partial_s + iU)\omega - iU''\varphi = 0, \quad \partial_z^2 \varphi = \omega,$$

with zero boundary conditions on φ. Using (3.18), we have

$$\frac{\mathrm{d}}{\mathrm{d}s} \|\omega\|_{\delta'} \leqq \delta' |||U|||_{\delta'} \|\omega\|_{\delta'} + \|U''\varphi\|_{\delta'}.$$

By definition, $\|U''\varphi\|_{\delta'} \leqq \|U''\|_{\delta'}\|\varphi\|_{\delta'}$ and $\|\varphi\|_{\delta'} \leqq C_0\|\varphi\|_{H^1(-1,1)}+|\delta'|^2\|\partial_z^2\varphi\|_{\delta'}$. We thus obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|\omega\|_{\delta'} &\leqq C_0\|\varphi\|_{H^1(-1,1)} + \delta'(|||U|||_{\delta'} + \delta'\|U''\|_{\delta'})\|\omega\|_{\delta'}\\ &\leqq C_0\|\omega\|_{L^2} + \delta'(|||U|||_{\delta'} + \delta'\|U''\|_{\delta'})\|\omega\|_{\delta'},\end{aligned}$$

in which the last estimate is due to the Poincaré inequality. Hence, for δ' sufficiently small so that $\delta'(|||U'|||_{\delta'} + \delta'\|U''\|_{\delta'}) \leqq \gamma_0 < \gamma$, the Gronwall inequality and the previous L^2 bound give

$$\begin{aligned}\|\omega(s)\|_{\delta'} &\leqq e^{\delta'(|||U|||_{\delta'}+\delta'\|U''\|_{\delta'})s}\|h\|_{\delta'}+C_0\int_0^s e^{\delta'(|||U|||_{\delta'}+\delta'\|U''\|_{\delta'})(s-\tau)}\|\omega(\tau)\|_{L^2}\,\mathrm{d}s\\ &\leqq e^{\gamma s}\|h\|_{\delta'} + C_\gamma e^{\gamma s}\|h\|_{L^2},\end{aligned}$$

which proves the claimed bound for $e^{\mathcal{L}_1 s}$. □

We are now in position to prove Proposition 3.2.

Proof of Proposition 3.2. Let us prove the bound for the semigroup $(W_1, W_2, W_3) = e^{Ls}H$, with L being the matrix operator defined as in (3.8). Let $W_{j,n}(z) = \langle W_j(\cdot, z), e^{iny}\rangle_{L^2(\mathbb{T})}$, for $j = 1, 2, 3$. By the structure of the matrix operator L, Lemma 3.4 gives the bound for $W_{1,n}$, $W_{2,n}$:

$$\|W_{j,n}\|_{\delta'} \leqq C_\gamma e^{\gamma|n|s}\|H_{j,n}\|_{\delta'}, \quad j = 1, 2, \tag{3.19}$$

for all $\gamma > \gamma_0$, $\delta' \leqq \gamma_0$, and all $n \in \mathbb{Z}$. Hence, by definition of the analytic norm, we get

$$\|W_j(s)\|_{\delta-\gamma s,\delta'} \leqq C_\gamma \sum_{n\in\mathbb{Z}} e^{\gamma|n|s}\|H_{j,n}\|_{\delta'}e^{(\delta-\gamma s)|n|} \leqq C_\gamma\|H_j\|_{\delta,\delta'}.$$

for any s so that $\delta - \gamma s > 0$.

As for W_3, we write

$$(\partial_s + U\partial_y)W_3 + U'W_2 - U''\partial_z\varphi(W_2) - U'''\varphi(W_2) = 0.$$

Again, in Fourier variables, we have

$$(\partial_s + inU)W_{3,n} + U'W_{2,n} - U''\partial_z\varphi(W_{2,n}) - U'''\varphi(W_{2,n}) = 0.$$

Again, using (3.18), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|W_{3,n}\|_{\delta'} &\leqq \delta'|n||||U'|||_{\delta'}\|W_{3,n}\|_{\delta'}+\|U'W_{2,n}-U''\partial_z\varphi(W_{2,n})-U'''\varphi(W_{2,n})\|_{\delta'}\\ &\leqq \delta'|n||||U|||_{\delta'}\|W_{3,n}\|_{\delta'} + C_0\|W_{2,n}\|_{\delta'}.\end{aligned}$$

Using the Gronwall inequality, the bound (3.19) on $W_{2,n}$, and the assumption that $\delta'|||U|||_{\delta'} \leqq \gamma_0$, we obtain

$$\|W_{3,n}(s)\|_{\delta'} \leqq C_\gamma e^{\gamma|n|s}\|H_n\|_{\delta'}.$$

Summing the estimate over $n \in \mathbb{Z}$ yields the claimed bound for W_3. This completes the proof of the semigroup estimate and thus of the proposition. □

3.3. Conclusion

We now show that system (3.9) fits our abstract framework.
For (H.1), we note that if ω is an eigenfunction for $\mathcal{L}$ with an eigenvalue λ, that is $\mathcal{L}\omega = \lambda\omega$, then $W = \begin{pmatrix} \omega \\ \partial_y \omega \\ \partial_z \omega \end{pmatrix}$ is an eigenfunction for L with the same eigenvalue λ. In addition, the growing mode must be of the form given by Lemma 3.1. The regularity of W follows from that of the given shear flow $U(z)$, which is real analytic; see (3.5). The definition of γ_0 in (3.17), Lemma 3.1, and Proposition 3.2 finally prove that (H.2) holds.

Assumption (H.3) follows directly from the definition of L, whereas (H.4) and (H.5) are clear, thanks to the structure of the quadratic nonlinearity $Q(W, W)$ and the fact that there are no R_1, R_2 generated from the system (3.9).

4. Ill-Posedness of the Kinetic Incompressible Euler Equations

In this section, we give the proof of Theorem 1.2, establishing ill-posedness for the kinetic incompressible Euler equations (1.5) and (1.6), which we recall below for convenience:

$$\partial_t g + v \cdot \nabla_x g - \nabla_x \Phi \cdot \nabla_v g = 0 \tag{4.1}$$

with

$$-\Delta_x \Phi = \nabla_x \cdot \left(\nabla_x \cdot \left(\int v \otimes v g \, \mathrm{d}v \right) \right), \quad \int_{\mathbb{T}^3} \Phi \, dx = 0. \tag{4.2}$$

We shall prove in this section how Theorem 1.2 follows from the abstract Theorem 2.1. We work with the analytic function space $X_{\delta,\delta'}$, equipped with the following norm:

$$\|f\|_{\delta,\delta'} := \sum_{n \in \mathbb{Z}^3} \sum_{|\alpha| \geqq 0} \|\langle v \rangle^m \partial_v^\alpha f_n\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \frac{|\delta'|^{|\alpha|}}{|\alpha|!} e^{\delta |n|}, \tag{4.3}$$

for any $\delta, \delta' > 0$, in which $f_n = \langle f, e^{in \cdot y} \rangle_{L^2(\mathbb{T}^3)}$. Here, m is a fixed number, $m \geqq 4$. We also introduce the v-analytic function space $X_{\delta'}$, equipped with the norm

$$\|f\|_{\delta'} := \sum_{|\alpha| \geqq 0} \|\langle v \rangle^m \partial_v^\alpha f\|_{L^2(\mathbb{R}^3)} \frac{|\delta'|^{|\alpha|}}{|\alpha|!} \tag{4.4}$$

for any $\delta' > 0$. We study ill-posedness near radial homogeneous equilibria of the form $g = \mu(v) \equiv \mu(|v|^2)$ and $\Phi = 0$, for real analytic functions μ satisfying $\int \mu(v) \, \mathrm{d}v = 1$ and $\|\mu\|_{\delta'} < +\infty$ for some $\delta' > 0$.

We write the perturbed solution in the fast variables as follows:

$$g(t, x, v) = \mu(v) + f(s, y, v), \quad \Phi(t, x) = \varphi(s, y), \quad s = t/\varepsilon, \quad y = x/\varepsilon.$$

The new pair (f, φ) then solves

$$\begin{cases} \partial_s f + v \cdot \nabla_y f - \nabla_y \varphi \cdot \nabla_v \mu - \nabla_y \varphi \cdot \nabla_v f = 0, \\ -\Delta_y \varphi = \nabla_y \cdot \left(\nabla_y \cdot \left(\int v \otimes v f \, \mathrm{d}v \right) \right), \quad \int_{\mathbb{T}^3} \varphi \, \mathrm{d}y = 0. \end{cases} \tag{4.5}$$

We shall show that the problem (4.5) is ill-posed due to the unbounded unstable spectrum of the linearized operator

$$\mathcal{L} f := -v \cdot \nabla_y f + \nabla_y \varphi \cdot \nabla_v \mu, \quad -\Delta_y \varphi = \nabla_y \cdot \left(\nabla_y \cdot \left(\int v \otimes v f \, \mathrm{d}v \right) \right),$$
$$\int_{\mathbb{T}^3} \varphi \, \mathrm{d}y = 0,$$

which we shall study in details in the next section. Next, since the nonlinearity $\nabla_y \varphi \cdot \nabla_v f$ is quadratic with respect to the partial derivatives of f, we are led to write the problem (4.5) in the matrix form for the vector:

$$F := \begin{pmatrix} f \\ \nabla_y f \\ \nabla_v f \end{pmatrix}. \tag{4.6}$$

We write $F = (F_1, F_2, F_3) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$. We introduce the matrix operator

$$L := \begin{pmatrix} \mathcal{L} & 0 & 0 \\ 0 & \mathcal{L}_3 & 0 \\ 0 & \mathcal{M} & \mathcal{T} \end{pmatrix}, \tag{4.7}$$

in which

$$\mathcal{L}_3 = \begin{pmatrix} \mathcal{L} & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & \mathcal{L} \end{pmatrix}, \quad \mathcal{T} = -v \cdot \nabla_y, \quad \mathcal{M} G := -G + \nabla_v (\varphi(G) \cdot \nabla_v \mu).$$

Here, the vector $\varphi(G) = (\varphi(G_1), \varphi(G_2), \varphi(G_3))$ is understood as the unique solution to the elliptic problem

$$-\Delta_y \varphi(G_k) = \nabla_y \cdot \left(\nabla_y \cdot \left(\int v \otimes v G_k(v) \, \mathrm{d}v \right) \right),$$

with zero average over $\mathbb{T}^3$, for each vector field $G(v)$.

It follows that f solves (4.5) if and only if the vector field $F = (F_1, F_2, F_3)$ solves

$$\partial_s F - LF = Q(F, F), \tag{4.8}$$

in which direct calculations show

$$Q(F, F) = \begin{pmatrix} \varphi(F_2) \cdot F_3 \\ \nabla_y (\varphi(F_2) \cdot F_3) \\ \nabla_v (\varphi(F_2) \cdot F_3) \end{pmatrix}.$$

We shall show the ill-posedness of (4.8) by directly checking the assumptions (H.1)–(H.5) made in our abstract ill-posedness framework.

4.1. Unbounded Unstable Spectrum of the Linearized Operator

In this section, we study the linearized problem:

$$\partial_s f + v \cdot \nabla_y f - \nabla_y \varphi \cdot \nabla_v \mu = 0, \quad -\Delta_y \varphi = \nabla_y \cdot \left(\nabla_y \cdot \left(\int v \otimes v f \, \mathrm{d}v \right) \right). \tag{4.9}$$

We search for a possible growing mode of the form:

$$(f, \varphi) = (e^{in\cdot(y-\omega t)} \hat{f}(v), e^{in\cdot(y-\omega t)} \hat{\varphi}) \tag{4.10}$$

for some complex constant $\hat{\varphi}$ and complex function $\hat{f}(v)$, with $\Im(n \cdot \omega) > 0$, for some complex vector ω. Plugging the above ansatz into the Vlasov equation in (4.9), we get

$$in \cdot (v - \omega) \hat{f} - in\hat{\varphi} \cdot \nabla_v \mu = 0$$

which gives

$$\hat{f} = \frac{\nabla_v \mu \cdot n}{n \cdot (v - \omega)} \hat{\varphi}, \quad \hat{\varphi} = \frac{-1}{|n|^2} \sum_{j,\ell} n_j n_\ell \int v_j v_\ell \hat{f}(v) \, \mathrm{d}v. \tag{4.11}$$

This yields the existence of a growing mode if and only if there is a pair (n, ω), with $\Im(n \cdot \omega) > 0$, so that the following dispersion relation holds:

$$\frac{1}{|n|^2} \sum_{j,\ell} n_j n_\ell \int v_j v_\ell \frac{\nabla_v \mu \cdot n}{n \cdot (v - \omega)} \, \mathrm{d}v = -1. \tag{4.12}$$

We shall call this property the Penrose instability condition.
We summarize this statement in the following lemma:

Lemma 4.1. *The linearized operator $\mathcal{L}$ possesses a growing mode in the form* (4.10) *if and only if the Penrose instability condition* (4.12) *holds for some complex number* $\Im\omega \neq 0$. *In case of instability, the unstable spectrum is unbounded, containing all the points $\lambda = -in \cdot \omega$, with $n \in \mathbb{Z}^3$ so that $\Im(n \cdot \omega) > 0$ and with corresponding eigenfunctions given by* (4.10) *and* (4.11).

4.2. Sharp Semigroup Bounds

From now on, we consider a smooth radial equilibrium μ such that $\|\mu\|_{\delta_1'} < +\infty$ for some $\delta_1' > 0$ and which gives unstable spectrum for (4.9). Typical examples are analytic radial double-bump equilibria with fast decay at infinity. Let L be the matrix operator defined as in (4.7). We shall derive sharp bounds on the corresponding semigroup e^{Ls} in the analytic function space $X_{\delta,\delta'}$.

Let us introduce, for all $n \in \mathbb{S}^2$,

$$\mathcal{L}_{\hat{n}} f = -i\hat{n} \cdot (vf - \nabla_v \mu \varphi(f)), \quad \varphi(f) := -\sum_{j,\ell} \hat{n}_j \hat{n}_\ell \int v_j v_\ell f(v) \, \mathrm{d}v.$$

We set γ_0 to be defined by

$$\gamma_0 := \sup_{\hat{n}\in\mathbb{S}^2,\ \exists k\in\mathbb{N}^*,\ \sqrt{k}\hat{n}\in\mathbb{Z}^3} \{\Re\lambda_{\hat{n}} :\ \lambda_{\hat{n}} \in \sigma(\mathcal{L}_{\hat{n}})\}. \quad (4.13)$$

Let us first quickly show that γ_0 exists and is positive. Since $|\varphi(f)| \le C_0\|\langle v\rangle^4 f\|_{L^2}$ and μ decays sufficiently fast in v, it follows that $i\hat{n}\cdot\nabla_v\mu\varphi(f)$ is a compact perturbation of the multiplication operator $-i\hat{n}\cdot v$. As a consequence, the unstable spectrum of $\mathcal{L}_{\hat{n}}$ consists precisely of possible eigenvalues λ, solving the equation $(\lambda - \mathcal{L}_{\hat{n}})f = 0$. It follows directly that $\lambda = -i\hat{n}\cdot\omega$ is an eigenvalue of $\mathcal{L}_{\hat{n}}$ if and only if the function

$$D(\omega;\hat{n}) := 1 + \sum_{j,\ell}\hat{n}_j\hat{n}_\ell\int v_jv_\ell\frac{\nabla_v\mu\cdot\hat{n}}{\hat{n}\cdot(v-\omega)}\,\mathrm{d}v$$

has a zero $\omega\in\mathbb{C}^3$, for some $\hat{n} = \frac{n}{|n|}$. We observe that $\sup_{\hat{n}\in\mathbb{S}^2} D(\omega;\hat{n}) \to 1$ as $|\omega|\to\infty$, and thus possible eigenvalues must lie in a bounded domain in the complex domain. Since we assume the existence of unstable spectrum for (4.9), the above set is not empty, and γ_0 is well-defined and positive.

The main goal of this section is to prove the following proposition.

Proposition 4.2. *Let γ_0 be defined as in (4.13), $\delta > 0$. The semigroup e^{Ls}, associated to L, is well-defined in $X_{\delta,\delta'}$, for s and δ' small enough. More precisely, for any $\gamma > \gamma_0$, there is a positive constant C_γ so that*

$$\|e^{Ls}h\|_{\delta-\gamma s,\delta'} \leqq C_\gamma\|h\|_{\delta,\delta'},$$

for all $h\in X_{\delta,\delta'}$, and for all $\delta' \leqq \min(\delta'_{1/2},\gamma_0)$ and s so that $\delta - \gamma s > 0$.

We start the proof of Proposition 4.2 with the semigroup $e^{\mathcal{L}s}$. Here, we recall that

$$\mathcal{L}f = -v\cdot\nabla_y f + \nabla_y\varphi\cdot\nabla_v\mu, \quad -\Delta_y\varphi = \nabla_y\cdot\left(\nabla_y\cdot\left(\int v\otimes vf\,\mathrm{d}v\right)\right).$$

In Fourier variables (with respect to y), we study for all $n\in\mathbb{Z}^3$

$$\mathcal{L}_nf = -in\cdot(vf - \nabla_v\mu\varphi(f)), \quad \varphi(f) := \frac{-1}{|n|^2}\sum_{j,\ell}n_jn_\ell\int v_jv_\ell f(v)\,\mathrm{d}v$$

and solve the ODEs

$$(\partial_s - \mathcal{L}_n)f = 0, \quad f(0,v) = f_0(v).$$

Lemma 4.3. *Let γ_0 be defined as in (4.13). For each $n\in\mathbb{Z}^3$, the operator $\mathcal{L}_n$ generates a continuous semigroup $e^{\mathcal{L}_nt}$ from $X_{\delta'}$ to itself, for δ' small enough. In addition, for any $\gamma > \gamma_0$, there is a positive constant C_γ so that*

$$\|e^{\mathcal{L}_ns}h\|_{\delta'} \leqq C_\gamma e^{\gamma|n|s}\|h\|_{\delta'}, \quad \forall s \geqq 0,$$

for all $h\in X_{\delta'}$, for small $\delta' > 0$.

Proof. We introduce the time scaling $s \mapsto |n|s$. It suffices to study the semigroup $e^{\mathcal{L}_{\hat{n}} s}$ for the scaled operator

$$\mathcal{L}_{\hat{n}} f = -i\hat{n} \cdot (vf - \nabla_v \mu \varphi(f)), \quad \varphi(f) := -\sum_{j,\ell} \hat{n}_j \hat{n}_\ell \int v_j v_\ell f(v) \, \mathrm{d}v$$

for $\hat{n} = \frac{n}{|n|}$. Let R be the rotation matrix so that $R\hat{n} = \hat{n}_1 := (1, 0, 0)$. Since $\mu \equiv \mu(|v|^2)$, the operator $\mathcal{L}_{\hat{n}}$ is invariant under the change of variable: $\hat{n} \mapsto R\hat{n}$ and $v \mapsto Rv$. Hence, it suffices to derive estimates for $\mathcal{L}_1 := \mathcal{L}_{\hat{n}_1} = -iv_1 + \mu_{v_1}\varphi(\cdot)$. Since $|\varphi(f)| \leqq C_0 \|\langle v\rangle^4 f\|_{L^2}$ and μ_{v_1} decays sufficiently fast in v, $\mu_{v_1}\varphi(f)$ is a compact perturbation of the multiplication operator by $-iv_1$. Hence $\mathcal{L}_1$ generates a continuous semigroup $e^{\mathcal{L}_1 s}$ with respect to the weighted norm $\|\langle v\rangle^m \cdot \|_{L^2(\mathbb{R}^3)}$.

In addition, following the standard semigroup theory ([40] or [43, Appendix A]), we can write

$$e^{\mathcal{L}_1 s} h = \mathrm{PV} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda s} (\lambda - \mathcal{L}_1)^{-1} h \, \mathrm{d}\lambda \tag{4.14}$$

for any $\gamma > \gamma_0$. With $\mathcal{L}_1 = -iv_1 + \mu_{v_1}\varphi(\cdot)$, the resolvent equation $(\lambda - \mathcal{L}_1) f = h$ gives

$$f + \frac{i\mu_{v_1}}{\lambda + iv_1} \varphi(f) = \frac{h}{\lambda + iv_1}. \tag{4.15}$$

By a view of $\varphi(f)$, we can first solve $\varphi(f)$ in terms of the initial data h:

$$\varphi(f) = -\frac{1}{D(\lambda)} \int \frac{v_1^2 h}{\lambda + iv_1} \, \mathrm{d}v, \quad D(\lambda) := 1 - i \int \frac{v_1^2 \mu_{v_1}}{\lambda + iv_1} \, \mathrm{d}v.$$

We note that $D(\lambda) = 0$ if and only if λ is an eigenvalue of $\mathcal{L}_1$. It follows that

$$|\varphi(f)| \leqq \frac{C_\gamma}{1 + |\Im\lambda|} \|\langle v\rangle^4 h\|_{L^2(\mathbb{R}^3)} \tag{4.16}$$

uniformly for all $\lambda \in \gamma + i\mathbb{R}$, with any fixed number $\gamma > \gamma_0$. Thus, from (4.14) and (4.15), we compute

$$e^{\mathcal{L}_1 s} h = \mathrm{PV} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda s} \left[-\frac{i\mu_{v_1}}{\lambda + iv_1} \varphi(f) + \frac{h}{\lambda + iv_1} \right] \mathrm{d}\lambda,$$

in which the second integral is equal to $e^{-iv_1 s} h$, while the first integral can be estimated directly using the estimate (4.16) on $\varphi(f)$. This at once yields

$$\|\langle v\rangle^m e^{\mathcal{L}_1 s} h\|_{L^2} \leqq C_\gamma e^{\gamma s} \|\langle v\rangle^m h\|_{L^2} \tag{4.17}$$

for any $\gamma > \gamma_0$.

Next, for higher derivatives of $f = e^{\mathcal{L}_1 s} h$, we note that $\partial_v^\alpha f$ solves

$$\partial_s \partial_v^\alpha f + iv_1 \partial_v^\alpha f + i\partial_v^\alpha \mu_{v_1} \varphi(f) + i[\partial_v^\alpha, v_1] f = 0.$$

Standard L^2 estimates for $\partial_v^\alpha f$ yield, for all $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha' = (\alpha_1 - 1, \alpha_2, \alpha_3)$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\langle v\rangle^m \partial_v^\alpha f(s)\|_{L^2}^2 &\leqq [\|\partial_v^\alpha \mu_{v_1}\varphi(f)\|_{L^2} + \|[\partial_v^\alpha, v_1]f\|_{L^2}]\|\langle v\rangle^m \partial_v^\alpha f(s)\|_{L^2} \\ &\leqq [C_0\|\langle v\rangle^m f\|_{L^2}\|\partial_v^\alpha \mu_{v_1}\|_{L^2} \\ &\quad + |\alpha_1|\|\partial_v^{\alpha'} f\|_{L^2}]\|\langle v\rangle^m \partial_v^\alpha f(s)\|_{L^2},\end{aligned} \tag{4.18}$$

upon using the fact that the term $iv_1 f$ does not yield any contribution when taking the real part of the L^2 energy identities. By a view of the definition of the analytic norm, the above estimates give

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|f(s)\|_{\delta'} &= \sum_{|\alpha|\geqq 0}\frac{\mathrm{d}}{\mathrm{d}s}\|\langle v\rangle^m \partial_v^\alpha f(s)\|_{L^2(\mathbb{R}^3)}\frac{|\delta'|^{|\alpha|}}{|\alpha|!} \\ &\leqq C_0\|\langle v\rangle^m f\|_{L^2} + \sum_{|\alpha|\geqq 1}\frac{|\delta'|^{|\alpha|}}{|\alpha|!}[C_0\|\langle v\rangle^m f\|_{L^2}\|\partial_v^\alpha \mu_{v_1}\|_{L^2} \\ &\quad + |\alpha_1|\|\partial_v^{\alpha'} f\|_{L^2}] \\ &\leqq C_0(1 + \|\nabla_v \mu\|_{\delta'})\|\langle v\rangle^m f\|_{L^2} + \delta' \sum_{|\alpha'|\geqq 1}\frac{|\delta'|^{|\alpha'|}}{|\alpha'|!}\|\partial_v^{\alpha'} f\|_{L^2} \\ &\leqq C_0(1 + \|\nabla_v \mu\|_{\delta'})\|\langle v\rangle^m f\|_{L^2} + \delta'\|f\|_{\delta'},\end{aligned}$$

which entails

$$\|f(s)\|_{\delta'} \leqq e^{\delta' s}\|f(0)\|_{\delta'} + C_0(1 + \|\nabla_v \mu\|_{\delta'})\int_0^s e^{\delta'(s-\tau)}\|\langle v\rangle^m f(\tau)\|_{L^2}\,\mathrm{d}\tau, \tag{4.19}$$

for any $\delta' > 0$. Now, thanks to the L^2 bound (4.17) on the semigroup and the assumption that $\delta' \leqq \gamma_0$, we get

$$\|f(s)\|_{\delta'} \leqq \tilde{C}_\gamma(1 + \|\nabla_v \mu\|_{\delta'})e^{\gamma s}\|f(0)\|_{\delta'}. \tag{4.20}$$

The claimed bound in the lemma is therefore proved. □

We can finally end the proof of Proposition 4.2.

Proof of Proposition 4.2. We let $f = e^{\mathcal{L}s}h$. Lemma 4.3 yields

$$\|f_n\|_{\delta'} \leqq C_\gamma e^{\gamma|n|s}\|h_n\|_{\delta'},$$

for all $\gamma > \gamma_0$, and for small enough $\delta' > 0$. Hence, by definition of the norms, for any s so that $\delta - \gamma s > 0$, we get

$$\|f(s)\|_{\delta-\gamma s,\delta'} \leqq C_\gamma \sum_{n\in\mathbb{Z}^3} e^{\gamma|n|s}\|h_n\|_{\delta'}e^{(\delta-\gamma s)|n|} \leqq C_\gamma\|h\|_{\delta,\delta'}.$$

This proves the claimed bound for $e^{\mathcal{L}s}$. As for $F = e^{Ls}H$, by the structure of the matrix operator L (see (4.7)), it is clear that the above estimate holds for F_1, F_2. As for F_3, we write

$$(\partial_s + v \cdot \nabla_y)F_3 + F_2 - \nabla_v(\varphi(F_2) \cdot \nabla_v \mu) = 0.$$

Similarly to the estimate obtained in (4.19) via energy estimates, for $\delta' \leqq \gamma_0$, in the Fourier variable $n \in \mathbb{Z}^3$, we immediately get

$$\begin{aligned}
\|F_{3,n}(s)\|_{\delta'} &\leqq e^{\delta'|n|s}\|H_{3,n}(0)\|_{\delta'} + C_0(1+\|\nabla_v\mu\|_{\delta'})\int_0^s e^{\delta'|n|(s-\tau)}\|F_{2,n}(\tau)\|_{\delta'}\,\mathrm{d}\tau \\
&\leqq e^{\delta'|n|s}\|H_{3,n}(0)\|_{\delta'} + C_\gamma(1+\|\nabla_v\mu\|_{\delta'})\int_0^s e^{\delta'|n|(s-\tau)}e^{\gamma|n|\tau}\|H_{2,n}\|_{\delta'}\,\mathrm{d}\tau \\
&\leqq e^{\delta'|n|s}\|H_{3,n}(0)\|_{\delta'} + C_\gamma(1+\|\nabla_v\mu\|_{\delta'})e^{\gamma|n|s}\|H_{2,n}\|_{\delta'}.
\end{aligned}$$

Hence, as above, summing the norms for all $n \in \mathbb{Z}^3$, we obtain the claimed bound for F_3, and hence for e^{Ls}. This completes the proof of the proposition. □

4.3. Conclusion

We are now ready to check the assumptions (H.1)–(H.5) made in the abstract framework.

For (H.1), we note that if g is an eigenfunction for $\mathcal{L}$ with an eigenvalue λ, that is $\mathcal{L}g = \lambda g$, then $G = \begin{pmatrix} g \\ \nabla_y g \\ \nabla_v g \end{pmatrix}$ is an eigenfunction for L with the same eigenvalue λ. Thus, by construction of the growing mode of $\mathcal{L}$ in Lemma 4.1, the very definition of γ_0 in (4.13) and Proposition 4.2, (H.2) holds. Assumption (H.3) follows directly from the definition of L, whereas (H.4) and (H.5) are clear, thanks to the structure of the quadratic nonlinearity $Q(F, F)$ and the fact that there are no R_1, R_2 generated from the system (4.8).

Acknowledgments. We are grateful to Pennsylvania State University and the École polytechnique for hospitality during the preparation of this work. Toan T. Nguyen was partly supported by the NSF under Grant DMS-1405728 and by a 2-month visiting position at the École polytechnique during Spring 2015.

References

1. Bardos, C.: About a variant of the 1d Vlasov equation, dubbed “Vlasov–Dirac–Benney” equation. *Séminaire Laurent Schwartz—EDP et Appl.* **15**, 21 (2012–2013)
2. Bardos, C., Besse, N.: The Cauchy problem for the Vlasov–Dirac–Benney equation and related issues in fluid mechanics and semi-classical limits. *Kinet. Relat. Models* **6**(4), 893–917 (2013)
3. Bardos, C., Nouri, A.: A Vlasov equation with Dirac potential used in fusion plasmas. *J. Math. Phys.* **53**(11), 115621, 16 (2012)

4. Benzoni-Gavage, S., Coulombel, J.-F., Tzvetkov, N.: Ill-posedness of nonlocal Burgers equations. *Adv. Math.* **227**(6), 2220–2240 (2011)
5. Bossy, M., Fontbona, J., Jabin, P.-E., Jabir, J.-F.: Local existence of analytical solutions to an incompressible Lagrangian stochastic model in a periodic domain. *Commun. Partial Differ. Equ.* **38**(7), 1141–1182 (2013)
6. Brenier, Y.: A Vlasov–Poisson type formulation of the Euler equations for perfect incompressible fluids. *Rapp. de recherche INRIA* (1989)
7. Brenier, Y.: Homogeneous hydrostatic flows with convex velocity profiles. *Nonlinearity* **12**(3), 495–512 (1999)
8. Brenier, Y.: Convergence of the Vlasov–Poisson system to the incompressible Euler equations. *Commun. Partial Differ. Equ.* **25**(3–4), 737–754 (2000)
9. Brenier, Y.: Remarks on the derivation of the hydrostatic Euler equations. *Bull. Sci. Math.* **127**(7), 585–595 (2003)
10. Brenier, Y.: A homogenized model for vortex sheets. *Arch. Ration. Mech. Anal.* **138**(4), 319–353 (1997)
11. Caflisch, R.E.: A simplified version of the abstract Cauchy–Kowalewski theorem with weak singularities. *Bull. Am. Math. Soc.* (*N.S.*) **23**(2), 495–500 (1990)
12. Chen, X.L., Morrison, P.J.: A sufficient condition for the ideal instability of shear flow with parallel magnetic field. *Phys. Fluids B Plasma Phys.* (*1989–1993*) **3**(4), 863–865 (1991)
13. Desjardins, B., Grenier, E.: On nonlinear Rayleigh–Taylor instabilities. *Acta Math. Sin.* (*Engl. Ser.*) **22**(4), 1007–1016 (2006)
14. Friedlander, S., Vicol, V.: On the ill/well-posedness and nonlinear instability of the magneto-geostrophic equations. *Nonlinearity* **24**(11), 3019–3042 (2011)
15. Gérard-Varet, D., Nguyen, T.: Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.* **77**(1–2), 71–88 (2012)
16. Gérard-Varet, D., Dormy, E.: On the ill-posedness of the Prandtl equation. *J. Am. Math. Soc.* **23**(2), 591–609 (2010)
17. Grenier, E.: Defect measures of the Vlasov–Poisson system in the quasineutral regime. *Commun. Partial Differ. Equ.* **20**(7–8), 1189–1215 (1995)
18. Grenier, E.: On the derivation of homogeneous hydrostatic equations. *M2AN Math. Model. Numer. Anal.* **33**(5), 965–970 (1999)
19. Grenier, E., Guo, Y., Nguyen, T.: Spectral instability of symmetric shear flows in a two-dimensional channel. Adv. Math. **292**, 52–110 (2016)
20. Guo, Y., Nguyen, T.: A note on Prandtl boundary layers. *Commun. Pure Appl. Math.* **64**(10), 1416–1438 (2011)
21. Guo, Y., Tice, I.: Compressible, inviscid Rayleigh–Taylor instability. *Indiana Univ. Math. J.* **60**(2), 677–711 (2011)
22. Hadamard, J.: *Lectures on Cauchy's problem in linear partial differential equations*. Dover Publications, New York, 1953
23. Han-Kwan, D., Iacobelli, M.: The quasineutral limit of the Vlasov–Poisson equation in Wasserstein metric. *Commun. Math. Sci.* (2014). (To appear)
24. Han-Kwan, D., Iacobelli, M.: Quasineutral limit for Vlasov–Poisson via Wasserstein stability estimates in higher dimension (2015). (Submitted)
25. Han-Kwan, D., Nguyen, T.: Nonlinear instability of Vlasov–Maxwell systems in the classical and quasineutral limits (2015). (Preprint)
26. Han-Kwan, D., Rousset, F.: Quasineutral limit for Vlasov–Poisson with Penrose stable data. *Ann. Sci. Éc. Norm. Supér.* (2015). (To appear)
27. Han-Kwan, D.: Quasineutral limit of the Vlasov–Poisson system with massless electrons. *Commun. Partial Differ. Equ.* **36**(8), 1385–1425 (2011)
28. Han-Kwan, D., Hauray, M.: Stability issues in the quasineutral limit of the one-dimensional Vlasov–Poisson equation. *Commun. Math. Phys.* **334**(2), 1101–1152 (2015)
29. Jabin, P.-E., Nouri, A.: Analytic solutions to a strongly nonlinear Vlasov equation. *C. R. Math. Acad. Sci. Paris* **349**(9–10), 541–546 (2011)

30. Kukavica, I., Masmoudi, N., Vicol, V., Kwong Wong, T.: On the local well-posedness of the Prandtl and hydrostatic Euler equations with multiple monotonicity regions. *SIAM J. Math. Anal.* **46**(6), 3865–3890 (2014)
31. Kukavica, I., Temam, R., Vicol, V.C., Ziane, M.: Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain. *J. Differ. Equ.* **250**(3), 1719–1746 (2011)
32. Lax, P.D.: Nonlinear hyperbolic equations. *Commun. Pure Appl. Math.* **6**, 231–258 (1953)
33. Lebeau, G.: Régularité du problème de Kelvin–Helmholtz pour l'équation d'Euler 2d. *ESAIM Control Optim. Calc. Var.* **8**, 801–825 (2002). [A tribute to J. L. Lions (electronic)]
34. Lerner, N., Nguyen, T., Texier, B.: The onset of instability in first-order systems. arXiv:1504.04477
35. Lions, P.-L.: Mathematical topics in fluid mechanics. *Volume 3 of Oxford Lecture Series in Mathematics and its Applications*, Vol. 1. The Clarendon Press, Oxford University Press, New York, 1996. (Incompressible models, Oxford Science Publications)
36. Masmoudi, N., Wong, T.K.: On the H^s theory of hydrostatic Euler equations. *Arch. Ration. Mech. Anal.* **204**(1), 231–271 (2012)
37. Métivier, G.: Remarks on the well-posedness of the nonlinear Cauchy problem. *Geometric Analysis of PDE and Several Complex Variables*. *Contemp. Math.*, Vol. 368. Am. Math. Soc., Providence, 337–356, 2005
38. Mizohata, S.: Some remarks on the Cauchy problem. *J. Math. Kyoto Univ.* **1**, 109–127 (1961/1962)
39. Mouhot, C., Villani, C.: On Landau damping. *Acta Math.* **207**(1), 29–201 (2011)
40. Pazy, A.: Semigroups of linear operators and applications to partial differential equations. *Applied Mathematical Sciences*, Vol. 44. Springer, New York, 1983
41. Renardy, M.: Ill-posedness of the hydrostatic Euler and Navier–Stokes equations. *Arch. Ration. Mech. Anal.* **194**(3), 877–886 (2009)
42. Seiichiro W.: The Lax–Mizohata theorem for nonlinear Cauchy problems. *Commun. Partial Differ. Equ.* **26**(7–8), 1367–1384 (2001)
43. Zumbrun, K.: Planar stability criteria for viscous shock waves of systems with real viscosity. *Hyperbolic Systems of Balance Laws. Lecture Notes in Math.*, Vol. 1911. Springer, Berlin, 229–326, 2007

CMLS, École polytechnique, CNRS,
Université Paris-Saclay,
91128 Palaiseau Cedex,
France.
e-mail: daniel.han-kwan@polytechnique.edu

and

Department of Mathematics,
Pennsylvania State University,
State College, PA 16802,
USA.
e-mail: nguyen@math.psu.edu

(*Received July 7, 2015 / Accepted February 13, 2016*)
Published online February 29, 2016 – *© Springer-Verlag Berlin Heidelberg* (*2016*)