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Structure of Solutions of Multidimensional Conservation Laws with Discontinuous Flux and Applications to Uniqueness

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Abstract

We investigate the structure of solutions of conservation laws with discontinuous flux under quite general assumption on the flux. We show that any entropy solution admits traces on the discontinuity set of the coefficients and we use this to prove the validity of a generalized Kato inequality for any pair of solutions. Applications to uniqueness of solutions are then given.

1. Introduction

The aim of this paper is to study the structure of solutions of conservation laws with discontinuous flux of the form

$$\operatorname{div}_z \boldsymbol{A}(z, u) = 0 \tag{1}$$

in order to establish a general framework for studying the uniqueness of solutions of the Cauchy problem associated with the evolutionary equation

$$u_t + \operatorname{div}_x \boldsymbol{F}(t, x, u) = 0, \quad \text{in } (0, +\infty) \times \mathbb{R}^N.^{1} \tag{2}$$

Here $\boldsymbol{A}$ (respectively $\boldsymbol{F}$) is discontinuous in its first variable z (respectively (t, x)). More precisely we will assume that $\boldsymbol{A}(z, \cdot) \in C^1(\mathbb{R}, \mathbb{R}^n)$, $\boldsymbol{A}(\cdot, v) \in SBV(\mathbb{R}^n, \mathbb{R}^n)$ where SBV is the space of special function of bounded variation, see [5, Chapter 4], and that $\boldsymbol{A}$ satisfies some mild structural assumptions listed in Section 2.

In recent years, the study of conservation laws with discontinuous flux has attracted the attention of many authors since they naturally arise in many models, see [1,6,8–10,14,16,17,20,21,23] and the references therein.

[1] Note that (2) is a particular case of (1) with $\boldsymbol{A}(z, u) = (u, \boldsymbol{F}(z, u))$ and $z = (t, x)$.

Even in the case that the flux $\boldsymbol{F}$ is smooth it is well known that the Cauchy problem associated with (2) is not well posed and some additional *entropy conditions* have to be imposed in order to recover the uniqueness of the solution, see [18]. In the case of a discontinuous flux, these conditions are still not sufficient to select a unique solution to (2), and further dissipation conditions, involving the traces of the solutions on the set of discontinuities of the flux, must be imposed in order to ensure uniqueness.

The problem of existence and uniqueness for solutions of (2) has been mainly studied in the case of one space variable and of fluxes with just one point of discontinuity (but the analysis can be easily extended to the case of finitely many discontinuity points). Assuming that the discontinuity is located at $x = 0$, and imposing the validity of Kruzhkov entropy inequalities separately on $(-\infty, 0)$ and $(0, +\infty)$, one can show that every pair of u, v bounded solutions satisfies

$$\int |u(T,x) - v(T,x)| \, \mathrm{d}x \leqq \int |u(0,x) - v(0,x)| \, \mathrm{d}x + \int_0^T W(u^{\pm}(t,0), v^{\pm}(t,0)) \, \mathrm{d}t, \tag{3}$$

where W is a quantity that depends only on the traces $u^{\pm}$, $v^{\pm}$ of u and v at $x = 0$. The L^1-contractivity of the semigroup associated with (2) is then obtained if $W(u^{\pm}, v^{\pm}) \leqq 0$ for every pair of solutions. Several conditions have been proposed in the literature in order to have that $W \leqq 0$, and different conditions lead to different physically relevant semigroups of solutions, see [6,16].

In [6], Andreianov, Karlsen and Risebro proposed a general framework in order to study uniqueness for (2) in the model case of one space variable and for fluxes with finitely many discontinuity points. The validity of the inequality $W \leqq 0$ is axiomatized in the notion of L^1-dissipative germ and given a germ $\mathcal{G}$ they show the uniqueness of $\mathcal{G}$-entropy solutions, see Definition 3.8 in [6] and Definition 2.8 below.

Loosely speaking, at a point of discontinuity of the flux $\boldsymbol{F}$, a germ $\mathcal{G}$ is a set of pairs (u^-, u^+) satisfying the Rankine–Hugoniot condition such that

$$W(u^{\pm}, v^{\pm}) \leqq 0 \qquad \forall (u^-, u^+), \ (v^-, v^+) \in \mathcal{G}$$

and, in the model case of flux with one single discontinuity at $x = 0$, a $\mathcal{G}$-entropy solution is a solution of (2) satisfying Kruzkov's conditions outside the origin and whose traces at 0 belong to $\mathcal{G}$. A similar analysis has been performed—in the model case of one dimensional fluxes with one discontinuity point—by Garavello, Natalini, Piccoli and Terracina in [16] in terms of the notion of dissipative Riemannian solvers. Let us also mention that this analysis can be extended to the multidimensional case by assuming that the set of discontinuity of the flux is a regular submanifold, see [8], or by assuming a priori BV regularity of the solution, see [11].

The main purpose of this paper is to provide a general framework to extend this analysis to solutions of (2) under quite general assumptions on the flux. In order to do this we introduce a rather weak notion of entropy solution, see Definition 2.3

below, and under a suitable genuine nonlinearity assumption on the flux we show that these solutions admits traces on the discontinuity set of the coefficients, see Theorem 1.1 below. Once the existence of traces has been established we prove that any pair of weak entropy solutions of (1) satisfies a generalized Kato inequality with a reminder term concentrated on the discontinuity set of the flux, see Theorem 1.2 below. It is then classical to show that this Kato type inequality leads to a quasi contractivity inequality for solutions of (2) of the form (3). Once this inequality has been established, the analysis in [6] in terms of germs and of $\mathcal{G}$-entropy solutions can be straightforwardly extended to (2), see Theorem 2.9 below. As a byproduct of our results we can also obtain the existence and uniqueness of solutions of (2) assuming Sobolev dependence of the flux $\boldsymbol{F}$ with respect to (t, x), see Theorem 2.11 below.

Let us now describe in a more detailed way our main results. First of all, the structural assumptions on $\boldsymbol{A}$ and the results in [4] guarantee the existence of a $\mathcal{H}^{n-1}$-rectifiable set $\mathcal{N}$ (defined in (11) below) that represents a universal jump set of $\boldsymbol{A}(\cdot, v)$, independent of v.

We say that a distributional solution $u \in L^\infty(\mathbb{R}^n)$ of (1) is a *weak entropy solution* (WES) of (1) if there exists a non-negative Radon measure μ such that $\mu(\mathbb{R}^n \setminus \mathcal{N}) = 0$ and, for every $k \in \mathbb{R}$,

$$\operatorname{div}_z(\operatorname{sign}(u-k)[\boldsymbol{A}(z,u) - \boldsymbol{A}(z,k)]) + \operatorname{sign}(u-k)\operatorname{div}_z^a \boldsymbol{A}(z,k) \leqq \mu, \tag{4}$$

see Definition 2.3 below. Here $\operatorname{div}_z^a \boldsymbol{A}(\cdot, k)$ denotes, for every $k \in \mathbb{R}$, the absolutely continuous part of the measure $\operatorname{div}_z \boldsymbol{A}(\cdot, k)$.

As we shall see in a moment, the notion of a weak entropy solution is strong enough to guarantee that such solutions possess a reasonable structure. On the other hand, it is weak enough to include, essentially, all solutions of (1) obtained by approximation schemes. In particular, under our assumptions on the flux, the solutions constructed by Panov in [23] are weak entropy solutions.

Assuming the genuine nonlinearity of the flux, and adapting to our setting the techniques developed by De Lellis, Otto and Westdickenberg in [13] (see also [22,24]), our first result ensures the existence of traces on $\mathcal{N}$ for weak entropy solutions. Loosely speaking, we have the following result (see Theorem 2.5 below for the precise statement):

Theorem 1.1. (Existence of traces) *If u is a bounded weak entropy solution of* (1), *then u admits traces $u^\pm$ on $\mathcal{N}$* (*in a generalized sense*, *see Definition* 2.4).

The existence of generalized traces of weak entropy solutions allows us to prove the validity of the following *generalized Kato inequality* (see Theorem 2.6 below for the precise statement):

Theorem 1.2. (Generalized Kato inequality) *Let u and v be weak entropy solutions. Then*

$$\operatorname{div}_z(\operatorname{sign}(u-v)[\boldsymbol{A}(z,u) - \boldsymbol{A}(z,v)]) \leqq W(u^\pm, v^\pm)\, \mathcal{H}^{n-1} \llcorner \mathcal{N}, \tag{5}$$

where

$$W(u^\pm, v^\pm) = \{\mathrm{sign}(u^+ - v^+)[\boldsymbol{A}^+(z, u^+) - \boldsymbol{A}^+(z, v^+)] \\ - (\mathrm{sign}(u^- - v^-)[\boldsymbol{A}^-(z, u^-) - \boldsymbol{A}^-(z, v^-)]\} \cdot \nu_{\mathcal{N}}, \qquad (6)$$

where $\nu_{\mathcal{N}}$ *is the measure-theoretic normal to the* $\mathcal{H}^{n-1}$ *rectifiable set* $\mathcal{N}$, *and* $\boldsymbol{A}^\pm(z, v)$ *are the traces at* z *of the SBV function* $z \mapsto \boldsymbol{A}(z, v)$.

In order to prove the above theorem, we combine Kruzkov's doubling of variables technique (see [18]) with Ambrosio's lemma on incremental quotients of BV functions (see [3]) to show that the left-hand side of (5) is a measure whose positive part is concentrated on $\mathcal{N}$. Once this result has been established, the representation formula (6) is an easy consequence of the existence of traces.

Our main application concerns the study of uniqueness conditions for the Cauchy problem associated to the multidimensional evolutionary equation (2). In this case, a bounded distributional solution $u \in L^\infty((0, +\infty) \times \mathbb{R}^N)$ of (2) is a weak entropy solution to (2) if, for every $k \in \mathbb{R}$,

$$\partial_t |u - k| + \mathrm{div}_x(\mathrm{sign}(u - k)[\boldsymbol{F}(t, x, u) - \boldsymbol{F}(t, x, k)]) \\ + \mathrm{sign}(u - k)\,\mathrm{div}_x^a\,\boldsymbol{F}(t, x, k) \leqq \mu, \qquad (7)$$

where μ is, as before, a non-negative measure concentrated on $\mathcal{N}$.

The generalized Kato inequality (5) implies the *quasi*-contractivity of the L^1 norm of the difference of solutions in the following sense: if $u, v \in C^0([0, +\infty); L^1(\mathbb{R}^N)) \cap L^\infty((0, +\infty) \times \mathbb{R}^N)$ are weak entropy solutions of (2), then for every $T > 0$ and every $R > 0$

$$\int_{B_R} |u(T, x) - v(T, x)|\,\mathrm{d}x \\ \leqq \int_{B_{R+VT}} |u(0, x) - v(0, x)|\,\mathrm{d}x + \int_{\mathcal{N} \cap ([0,T] \times B_{R+VT})} W(u^\pm, v^\pm)\,\mathrm{d}\mathcal{H}^N, \quad (8)$$

where $B_r := \{x \in \mathbb{R}^N : |x| < r\}$ and $V := \|\boldsymbol{A}\|_\infty$. As a consequence, if one prescribes an entropy condition stronger than (4) and implying the inequality $W \leqq 0$, then the generalized Kato inequality would give the standard contractivity inequality

$$\int_{B_R} |u(T, x) - v(T, x)|\,\mathrm{d}x \leqq \int_{B_{R+VT}} |u(0, x) - v(0, x)|\,\mathrm{d}x, \qquad (9)$$

and hence the uniqueness of solutions to the Cauchy problems associated with (2), see Definition 2.7 and Theorem 2.9 below.

Let us also stress that the existence of solutions satisfying these additional entropy conditions is not trivial and currently not known in the general setting here considered. Existence results are available assuming additional conditions on the structure of the flux field, see Remark 2.10 for a more detailed discussion.

In case $\boldsymbol{F}(\cdot, u) \in W^{1,1}$ satisfies the assumptions listed in Section 2, it is straightforward to check that $\mathcal{N} = \emptyset$, so that (5) implies *contractivity* of the semigroup associated to (2). In particular, also using the results of Panov [23], we can generalize to this situation the classical Kruzkov results concerning the existence and uniqueness of solutions of (2), see Theorem 2.11 and Remark 2.10 below.

Let us conclude this Introduction by presenting the structure of the paper. In Section 2 below we state our main assumption on the flux $\boldsymbol{A}$, we recall some of its consequence and we provide the precise statements of our main results. In Section 3 we prove Theorem 2.5, in Section 4 we prove Theorem 2.6 and eventually in Section 5 we provide the proofs of Theorems 2.9 and 2.11.

2. Assumptions on the Vector Field and Main Results

In this section we state our main structural hypotheses on the vector field (assumptions (H1)–(H5) below) and prove some consequences of these assumptions.

2.1. Structural Assumptions on the Vector Field

Let $\boldsymbol{A} \in L^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$ be such that:

(H1) There exists a set $\mathcal{C}_{\boldsymbol{A}}$ with $\mathcal{L}^n(\mathcal{C}_{\boldsymbol{A}}) = 0$ such that $\boldsymbol{A}(z, \cdot) \in C^1(\mathbb{R}, \mathbb{R}^n)$ for every $z \in \mathbb{R}^n \setminus \mathcal{C}_{\boldsymbol{A}}$ and $\boldsymbol{A}(\cdot, v) \in SBV(\mathbb{R}^n, \mathbb{R}^n)$ for every $v \in \mathbb{R}^n$.

(H2) There exists a constant M such that

$$|\partial_v \boldsymbol{A}(z, v)| \leqq M \qquad \forall\, z \in \mathbb{R}^n \setminus \mathcal{C}_{\boldsymbol{A}}, \quad v \in \mathbb{R}.$$

(H3) There exists a modulus of continuity ω such that

$$|\partial_v \boldsymbol{A}(z, u) - \partial_v \boldsymbol{A}(z, w)| \leqq \omega(|u - w|) \qquad \forall\, z \in \mathbb{R}^n \setminus \mathcal{C}_{\boldsymbol{A}}, \quad u, w \in \mathbb{R}.$$

(H4) There exists a function $g \in L^1(\mathbb{R}^n)$ such that

$$|\nabla_z \boldsymbol{A}(z, u) - \nabla_z \boldsymbol{A}(z, w)| \leqq g(z)|u - w| \qquad \forall\, z \in \mathbb{R}^n \setminus \mathcal{C}_{\boldsymbol{A}}, \quad u, w \in \mathbb{R},$$

where $\nabla_z \boldsymbol{A}(z, v)$ denotes the approximate gradient of the map $z \mapsto \boldsymbol{A}(z, v)$.

(H5) The measure

$$\sigma := \bigvee_{u \in \mathbb{R}} |D_z \boldsymbol{A}(\cdot, u)| \tag{10}$$

satisfies $\sigma(\mathbb{R}^n) < \infty$. Here $D_z \boldsymbol{A}(\cdot, u)$ is the distributional gradient of the map $z \mapsto \boldsymbol{A}(z, u)$ (which is a measure since $\boldsymbol{A}(\cdot, u) \in BV$) and $\bigvee$ denotes the least upper bound in the space of non-negative Borel measures, see [5, Definition 1.68].

Assumptions (H1)–(H5) imply that $\boldsymbol{A}$ satisfies the hypotheses of [4]. Let us summarize some consequences of this fact. First of all, from the definition of σ, we deduce that

$$\bigvee_{v \in \mathbb{R}} |\nabla \boldsymbol{A}(\cdot, v)| \mathcal{L}^n \leqq \sigma^a \mathcal{L}^n \qquad \bigvee_{v \in \mathbb{R}} |D^s \boldsymbol{A}(\cdot, v)| \leqq \sigma^s,$$

where $\nabla \boldsymbol{A}(\cdot, v)$ and $D^s \boldsymbol{A}(\cdot, v)$ are the approximate differential of $\boldsymbol{A}(\cdot, v)$ and the singular part of the measure $D\boldsymbol{A}(\cdot, v)$, respectively, and $\sigma^a \mathcal{L}^n$ and σ^s are the absolutely continuous and singular parts of σ. Moreover if we define

$$\mathcal{N} := \left\{ z \in \mathbb{R}^n : \liminf_{r \to 0} \frac{\sigma(B_r(z))}{r^{n-1}} > 0 \right\}, \tag{11}$$

then $\mathcal{N}$ is a $\mathcal{H}^{n-1}$ rectifiable set, see Section 3 in [4]. [2] Furthermore, for $\mathcal{H}^{n-1}$-almost every point in $\mathbb{R}^n \setminus \mathcal{N}$ and *every* $v \in \mathbb{R}$ there exists the limit

$$\tilde{\boldsymbol{A}}(z, v) := \lim_{r \to 0} \fint_{B_r(z)} \boldsymbol{A}(y, v)\, \mathrm{d}y,$$

and for $\mathcal{H}^{n-1}$ almost every $z \in \mathcal{N}$ and *every* $v \in \mathbb{R}$ there exist the traces of $\boldsymbol{A}$ on $\mathcal{N}$ defined as:

$$\boldsymbol{A}^{\pm}(z, v) := \lim_{r \to 0} \fint_{B_r^{\pm}(z)} \boldsymbol{A}(y, v)\, \mathrm{d}y, \tag{12}$$

where we denoted $B_r^{\pm}(z) = \{w \in B_r(z) : \pm\langle w - z, \nu(z)\rangle \geqq 0\}$. In addition, the functions $v \mapsto \tilde{\boldsymbol{A}}(z, v)$, $\boldsymbol{A}^{\pm}(z, v)$ are C^1 with derivatives given by $\partial_v \widetilde{\boldsymbol{A}}(z, v) = \widetilde{\partial_v \boldsymbol{A}}(z, v)$ and $\partial_v \boldsymbol{A}^{\pm}(z, v) = (\partial_v \boldsymbol{A}(z, v))^{\pm}$ respectively, see [4, Proposition 3.2]. Hence, if we denote by $\boldsymbol{a}$ the vector field

$$\boldsymbol{a}(z, v) := \partial_v \boldsymbol{A}(z, v), \tag{13}$$

then $\boldsymbol{a}$ admits a precise representative for $\mathcal{H}^{n-1}$-almost every $z \in \mathbb{R}^n \setminus \mathcal{N}$ as well as one sided traces on $\mathcal{N}$ that agree with $\partial_v \boldsymbol{A}$ (respectively with $\partial_v \boldsymbol{A}^{\pm}$).

In the sequel we shall assume the following genuine nonlinearity hypothesis:

$$\begin{aligned} &\mathcal{L}^1(\{v : \boldsymbol{a}^{\pm}(z, v) \cdot \xi = 0\}) = 0 \\ &\quad \text{for every } \xi \in S^{n-1} \text{ and for } \mathcal{H}^{n-1} \text{ almost every } z \in \mathcal{N}. \end{aligned} \tag{GNL}$$

Remark 2.1. Let us point out that our hypotheses include (and actually are modeled on) the case $\boldsymbol{A}(z, v) = \widehat{\boldsymbol{A}}(w(z), v)$ where $w \in SBV(\mathbb{R}^n; \mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^n; \mathbb{R}^d)$, $\widehat{\boldsymbol{A}} \in C^1(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^n)$, and

$$\mathcal{L}^1(\{v : \partial_v \widehat{\boldsymbol{A}}(w, v) \cdot \xi = 0\}) = 0 \quad \text{for every } \xi \in S^{n-1} \text{ and for every } w \in \mathbb{R}^d.$$

Remark 2.2. Since we are dealing with bounded solutions, all our assumptions can be localized in the v variable. Moreover, it is not difficult to modify the proofs in order to also localize in the z variable, see Remark 3.5 in [4].

[2] Recall that a set $\mathcal{N} \subset \mathbb{R}^n$ is said $\mathcal{H}^{n-1}$-rectifiable (shortened: rectifiable) if there are countably many C^1 submanifolds M_i of dimension $n - 1$ such that $\mathcal{H}^{n-1}(\mathcal{N} \setminus \bigcup_i M_i) = 0$.

2.2. Main Results

We consider the following scalar conservation law

$$\operatorname{div}_z \boldsymbol{A}(z, u(z)) = 0, \tag{14}$$

where $\boldsymbol{A}\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfies the structural assumption (H1)–(H5) and (GNL).

Definition 2.3. (*Weak entropy solutions*) A function $u \in L^\infty(\mathbb{R}^n)$ is a *weak entropy solution* (WES shortened) of (14) if u is a distributional solution of (14) and for every $k \in \mathbb{R}$ it holds

$$\operatorname{div}_z(\operatorname{sign}(u-k)[\boldsymbol{A}(z,u) - \boldsymbol{A}(z,k)]) + \operatorname{sign}(u-k)\operatorname{div}_z^a \boldsymbol{A}(z,k) \leqq \mu, \tag{15}$$

where μ is a non-negative Radon measure *independent* of k and such that $\mu(\mathbb{R}^n \backslash \mathcal{N}) = 0$.

Here $\operatorname{div}_z^a \boldsymbol{A}(z,k) = \operatorname{tr}\nabla \boldsymbol{A}(z,k)$ is the absolutely continuous part of $\operatorname{div}_z \boldsymbol{A}(\cdot, k)$.

Definition 2.4. (*Traces*) Let $u \in L^\infty(\mathbb{R}^n)$ and let $\mathcal{J} \subset \mathbb{R}^n$ be an $\mathcal{H}^{n-1}$-rectifiable set oriented by a normal vector field ν. We let the set of traces of u at $z_0 \in \mathcal{J}$ be

$$\Gamma_{u,\mathcal{J}}(z_0) := \left\{(c^-, c^+) : \exists r_k \downarrow 0 : u_{z_0,r_k} \to c^- \mathbf{1}_{H^-} + c^+ \mathbf{1}_{H^+} \text{ in } L^1_{\mathrm{loc}}\right\},$$

where $u_{z_0,r_k}(z) := u(z_0 + r_k(z - z_0))$, $H^\pm := \{z \in \mathbb{R}^n : \pm\langle z - z_0, \nu\rangle \geqq 0\}$ and $\mathbf{1}_A$ denotes the characteristic function of a set A.

The very same definition can be given component-wise for a vector field $\boldsymbol{B} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Moreover it is immediately apparent from the definition that if $u \in L^\infty(\mathbb{R}^n)$ and $f \in C^0(\mathbb{R})$ then

$$\Gamma_{f(u),\mathcal{J}}(z_0) = \left\{(f(c^-), f(c^+)) : (c^-, c^+) \in \Gamma_{u,\mathcal{J}}(z_0)\right\}.$$

Theorem 2.5. (Existence of generalized traces) *If u is a WES, then for $\mathcal{H}^{n-1}$ almost every $z_0 \in \mathcal{N}$*

$$\Gamma_{u,\mathcal{N}}(z_0) \neq \emptyset.$$

Moreover if $(c^-, c^+) \in \Gamma_{u,\mathcal{N}}(z_0)$ satisfies $c^- \neq c^+$ then the traces are unique: $\Gamma_{u,\mathcal{N}}(z_0) = \{(c^-, c^+)\}$. Otherwise there exist $a, b \in \mathbb{R}$ such that

$$\Gamma_{u,\mathcal{N}}(z_0) = \{(v, v) : v \in [a, b]\}.$$

Finally, the Rankine–Hugoniot condition holds:

$$\boldsymbol{A}^-(z_0, c^-) \cdot \nu(z_0) = \boldsymbol{A}^+(z_0, c^+) \cdot \nu(z_0) \qquad \forall\, (c^-, c^+) \in \Gamma_{u,\mathcal{N}}(z_0).$$

Theorem 2.6. (Generalized Kato inequality) *Let u and v be WES. Then there exists a Borel function* $w\colon \mathcal{N} \to \mathbb{R}$ *such that the following Kato inequality holds true*:

$$\operatorname{div}_z(\operatorname{sign}(u - v)[\boldsymbol{A}(z, u) - \boldsymbol{A}(z, v)]) \leqq w\, \mathcal{H}^{n-1} \llcorner \mathcal{N}. \tag{16}$$

Furthermore, for $\mathcal{H}^{n-1}$ *almost every* $z \in \mathcal{N}_1 := \{z \in \mathcal{N} : w(z) \neq 0\}$, *the functions u and v admit unique traces at z and the following representation formula holds*:

$$\begin{aligned} w = W(u^{\pm}, v^{\pm}) = \{&\operatorname{sign}(u^+ - v^+)[\boldsymbol{A}^+(z, u^+) - \boldsymbol{A}^+(z, v^+)] \\ &- (\operatorname{sign}(u^- - v^-)[\boldsymbol{A}^-(z, u^-) - \boldsymbol{A}^-(z, v^-)]\} \cdot \nu. \end{aligned} \tag{17}$$

The generalized Kato inequality yields a uniqueness result for the Cauchy problem for the evolutionary equation

$$\begin{cases} u_t + \operatorname{div}_x \boldsymbol{A}(t, x, u) = 0, & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{18}$$

More precisely, if we prescribe an entropy condition stronger than (15) and implying the inequality $w \leqq 0$, then the generalized Kato inequality gives the uniqueness of solutions to (18). To this end let us recall the definition of *dissipative germ* introduced in [6], see Definition 3.1 there.

Definition 2.7. (*Germ*) Given two functions $f^{\pm} \in C^0(\mathbb{R})$, a set $\mathcal{G} \subset \mathbb{R}^2$ is said to be a dissipative germ associated to $f^{\pm}$ if the following two conditions hold true:

(i) Every $(u^-, u^+) \in \mathcal{G}$ satisfies the Rankine–Hugoniot condition $f^+(u^+) = f^-(u^-)$.
(ii) For every two pairs $(u^-, u^+), (v^-, v^+) \in \mathcal{G}$ we have

$$\begin{aligned} W_{f^{\pm}}(u^{\pm}, v^{\pm}) := \{&\operatorname{sign}(u^+ - v^+)[f^+(u^+) - f^+(v^+)] \\ &- (\operatorname{sign}(u^- - v^-)[f^-(u^-) - f^-(v^-)]\} \leqq 0. \end{aligned}$$

Following [7] we now define $\boldsymbol{\mathcal{G}}$-entropy solutions associated with germs; compare with Definition 3 there and Definition 3.8 in [6].

Definition 2.8. ($\boldsymbol{\mathcal{G}}$-*entropy solutions*) Let $\boldsymbol{F} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ be such that $\boldsymbol{A} := (u, \boldsymbol{F})$ satisfies (H1)–(H5) and (GNL) above. Let $\mathcal{N} \subset \mathbb{R} \times \mathbb{R}^N$ be the rectifiable set defined in (11). Assume that for every $z = (t, x) \in \mathcal{N}$ such that the traces $\boldsymbol{A}^{\pm}(z, u)$ exist it is given a dissipative germ $\mathcal{G}_z$ associated to $f^{\pm}(u) := \boldsymbol{A}^{\pm}(z, u) \cdot \nu(z)$ and let us set $\boldsymbol{\mathcal{G}} = \{\mathcal{G}_z\}_{z \in \mathcal{N}}$. We say that a bounded function $u \in C^0([0, +\infty); L^1(\mathbb{R}^N))$ is a $\boldsymbol{\mathcal{G}}$-entropy solution of (18) if

(i) u is a weak entropy solution of (18) according to Definition 2.3.
(ii) For $\mathcal{H}^N$-almost every $x \in \mathcal{N}$ any $(u^-, u^+) \in \Gamma_{u,\mathcal{N}}(z)$ belongs to the germ $\mathcal{G}_z$.

A straightforward consequence of Theorem 2.6 is then the following:

Theorem 2.9. (Uniqueness of $\boldsymbol{\mathcal{G}}$-entropy solutions) *Let* $\boldsymbol{F} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ *be such that* $\boldsymbol{A} := (u, \boldsymbol{F})$ *satisfies* (H1)–(H5) *and* (GNL) *above. Then for any choice of* $\boldsymbol{\mathcal{G}}$ *there exists* at most *one* $\boldsymbol{\mathcal{G}}$*-entropy solution of* (18).

Remark 2.10. Under mild requirements on the flux, the existence of weak entropy solutions can be obtained by the results of Panov, see [23]. On the other hand, the existence of $\mathcal{G}$-entropy solutions, that is, solutions additionally satisfying condition (ii) in Definition 2.8, is far from trivial and known only in some special cases. Positive results in this direction are available either in one space dimension for a flux with a finite number of discontinuity points, see for instance [6,16] and the references therein, or in many space dimensions and for the particular case of the *vanishing viscosity germ*, assuming that the jump set of the $\boldsymbol{F}$ is a C^2 submanifold [7], see also [8] where a more general situation is considered.

If $\boldsymbol{F}(\cdot, u)$ is a Sobolev function one can easily obtain from the above analysis the uniqueness of (weak) entropy solutions.

Theorem 2.11. *Let* $\boldsymbol{F} : \mathbb{R}\times\mathbb{R}^N\times\mathbb{R} \to \mathbb{R}^N$ *be such that* $\boldsymbol{A} := (u, \boldsymbol{F})$ *satisfies* (H1)–(H5) *and* (GNL) *above and assume that* $\boldsymbol{F}(\cdot, u) \in W^{1,1}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ *for every* $u \in \mathbb{R}$. *Then any two* (*weak*) *entropy solutions* $u, v \in C^0([0, +\infty); L^1(\mathbb{R}^N)) \cap L^\infty((0, +\infty) \times \mathbb{R}^N)$ *of* (2) *satisfy*

$$\int_{\mathbb{R}^N} |u(T, x) - v(T, x)|\, \mathrm{d}x \leqq \int_{\mathbb{R}^N} |u(0, x) - v(0, x)|\, \mathrm{d}x.$$

3. Proof of Theorem 2.5

In this section we prove Theorem 2.5. We start with the following well known lemma:

Lemma 3.1. *Let* $\boldsymbol{B} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *and assume that* $\mu = \operatorname{div} \boldsymbol{B}$ *is a Radon measure. Then* $|\mu| \ll \mathcal{H}^{n-1}$. *Furthermore if* $\mathcal{J}$ *is a rectifiable set and* $\Gamma_{\boldsymbol{B},\mathcal{J}}(z) \neq \emptyset$ *for* $\mathcal{H}^{n-1}$ *almost every* $z \in \mathcal{J}$ *then it holds*

$$\operatorname{div}(\boldsymbol{B}) \llcorner \mathcal{J} = (\boldsymbol{B}^+ - \boldsymbol{B}^-) \cdot \nu\, \mathcal{H}^{n-1} \llcorner \mathcal{J}$$

where $(\boldsymbol{B}^-(z), \boldsymbol{B}^+(z)) \in \Gamma_{\boldsymbol{B},\mathcal{J}}(z)$. *In particular, for every two pairs in* $\Gamma_{\boldsymbol{B},\mathcal{J}}(z)$, *their projections along* $\nu(z)$ *have the same difference.*

Proof. The fact that $|\mu| \ll \mathcal{H}^{n-1}$ is proved, for instance, in [12, Lemma 2.4]. To show the second part we decompose μ as

$$\mu = \mu \llcorner \mathcal{J} + \mu \llcorner (\mathbb{R}^n \setminus \mathcal{J}) =: \mu_1 + \mu_2$$

with $\mu_1 \perp \mu_2$. Since μ_1 is a Radon measure and $\mathcal{H}^{n-1} \llcorner J$ is σ-finite we can apply the Radon–Nikodym Theorem to get that

$$\mu_1 = \operatorname{div}(\boldsymbol{B}) \llcorner \mathcal{J} = h \mathcal{H}^{n-1} \llcorner \mathcal{J}$$

for some $h \in L^1(\mathcal{H}^{n-1} \llcorner \mathcal{J})$. Let now z_0 be a point such that $\Gamma_{\boldsymbol{B},\mathcal{J}}(z_0) \neq \emptyset$,

$$h(z_0 + rz)\mathcal{H}^{n-1} \llcorner \frac{\mathcal{J} - z_0}{r} \overset{*}{\rightharpoonup} h(z_0)\mathcal{H}^{n-1} \llcorner \{z \cdot \nu(z_0) = 0\}$$

and

$$\lim_{r\to 0}\frac{|\mu_2|(B_r(z_0))}{r^{n-1}}=0.$$

Note that $\mathcal{H}^{n-1}$ almost every point z satisfies the above properties. Indeed, the first one follows by our assumptions, while the second and the third ones follow, respectively, from [5, Theorem 2.83] and [5, Equation 2.41].

Let us choose $r_k \downarrow 0$ with

$$\boldsymbol{B}_{r_k}\to \boldsymbol{B}^-(z_0)\mathbf{1}_{H^-}+\boldsymbol{B}^+(z_0)\mathbf{1}_{H^+},$$

where $H^\pm=\{\pm\langle z,\nu(z_0)\rangle \geqq 0\}$. Let $\varphi\in C^1_c(\mathbb{R}^n)$ and define $\varphi_{r_k}(z)=r_k^{1-n}\varphi((z-z_0)/r_k)$. Integrating by parts we get

$$\langle \mu,\varphi_{r_k}\rangle=\frac{1}{r_k^n}\int \boldsymbol{B}(z)\cdot\nabla\varphi\left(\frac{z-z_0}{r_k}\right)\mathrm{d}z=\int \boldsymbol{B}_{r_k}(z)\cdot\nabla\varphi(z)\,\mathrm{d}z. \tag{19}$$

Moreover

$$\begin{aligned}\langle \mu,\varphi_{r_k}\rangle&=\frac{1}{r_k^{n-1}}\left\langle \mu_1,\varphi\left(\frac{\cdot-z_0}{r_k}\right)\right\rangle+\frac{1}{r_k^{n-1}}\left\langle \mu_2,\varphi\left(\frac{\cdot-z_0}{r_k}\right)\right\rangle\\&=\int_{\frac{\mathcal{J}-z_0}{r_k}} h(r_kz+z_0)\varphi(z)\,\mathrm{d}\mathcal{H}^{n-1}(z)+O\left(\frac{|\mu_2|(B_{r_k}(z_0))}{{r_k}^{n-1}}\right).\end{aligned}$$

Hence, passing to the limit as k goes to infinity in (19), we get

$$\begin{aligned}h(z_0)\int_{\{z\cdot\nu(z_0)=0\}}\varphi(z)\,\mathrm{d}\mathcal{H}^{n-1}(z)=\boldsymbol{B}^-(z_0)\cdot\int_{H^-}\nabla\varphi(z)\,\mathrm{d}z\\+\boldsymbol{B}^+(z_0)\cdot\int_{H^+}\nabla\varphi(z)\,\mathrm{d}z.\end{aligned}$$

Integrating by parts we obtain that $h(z_0)=(\boldsymbol{B}^+(z_0)-\boldsymbol{B}^-(z_0))\cdot\nu(z_0)$, and this concludes the proof. □

Proof of Theorem 2.5. We divide the proof in several steps.
Step 1 (*Definition of the measure for the kinetic equation*) Let u be a WES, according to (15) for every $k\in\mathbb{R}$ the distribution

$$\eta_k:=\operatorname{div}_z(\operatorname{sign}(u-k)[\boldsymbol{A}(z,u)-\boldsymbol{A}(z,k)])+\operatorname{sign}(u-k)\operatorname{div}_z^a\boldsymbol{A}(z,k) \tag{20}$$

is a Radon measure. We now claim that for every $K\Subset\mathbb{R}$ and for every $R>0$

$$\sup_{k\in K}|\eta_k|(B_R)\leqq C(K,R). \tag{21}$$

To see this, note that $\mu-\eta_k\geqq 0$ for every $k\in K$. Therefore, if $\phi\in C^1_c(B_R)$ and $\chi\in C^1_c(B_{R+1})$ satisfies $\chi\geqq 0$, $\chi\equiv 1$ in B_R, we have

$$\langle \mu-\eta_k,(\|\phi\|_\infty\pm\phi)\chi\rangle\geqq 0,$$

hence, since $\chi\phi = \phi$,

$$\pm\langle\eta_k, \phi\rangle \leqq -\langle\eta_k, \chi\rangle\|\phi\|_\infty + 2\langle\mu, \chi\rangle\|\phi\|_\infty.$$

The above inequality implies the validity of (21), since, by the very definition of η_k, one has $\sup_{k\in K} |\langle\eta_k, \chi\rangle| \leqq C(K, R)$. In particular, the map

$$C_c^\infty(\mathbb{R}^n \times \mathbb{R}) \ni \Phi \mapsto \langle\eta, \Phi\rangle := \iint_{\mathbb{R}^n\times\mathbb{R}} \Phi(z, k)\,\mathrm{d}\eta_k(z)\,\mathrm{d}k$$

defines a Radon measure η in $\mathbb{R}^n \times \mathbb{R}$. Moreover if we define $\overline{\nu} := \pi_\#(|\eta|)$, where $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the projection on the first factor, then $\overline{\nu} \ll \mathcal{H}^{n-1}$.[3] Indeed by Lemma 3.1, $|\eta_k| \ll \mathcal{H}^{n-1}$, so that if $\mathcal{H}^{n-1}(A) = 0$, then

$$\overline{\nu}(A) \leqq \int_{\mathbb{R}} |\eta_k|(A)\,\mathrm{d}k = 0.$$

Step 2 (*Kinetic formulation*) The function $(k, z) \mapsto \chi(k, u(z)) := \operatorname{sign}(u(z) - k)$ is a solution of the kinetic equation, see [19]

$$\operatorname{div}_z [\chi(k, u)\partial_v \boldsymbol{A}(z, k)] - \partial_k \left[\chi(k, u) \operatorname{div}_z^a \boldsymbol{A}(z, k)\right] = -\partial_k\eta \quad \text{in } \mathcal{D}'(\mathbb{R}^{n+1}), \quad (22)$$

where $\eta(k, A) := \eta_k(A)$. Indeed, let us consider in Equation (20) a test function of the form $\Phi(k, z) := \varphi(z)\partial_k\psi(k)$. Recalling the definition of the measure η and of $\chi(k, u)$, integrating in k we get

$$-\iint \nabla\varphi(z)\partial_k\psi(k)\chi(k, u)[\boldsymbol{A}(z, u) - \boldsymbol{A}(z, k)]\,\mathrm{d}z\,\mathrm{d}k$$
$$+\iint \varphi(z)\partial_k\psi(k)\chi(k, u) \operatorname{div}_z^a \boldsymbol{A}(z, k)\,\mathrm{d}z\,\mathrm{d}k = \int \varphi(z)\partial_k\psi(k)\,\mathrm{d}\eta(k, z),$$

so that

$$\iint \nabla\varphi(z)\psi(k)\partial_k(\chi(k, u)[\boldsymbol{A}(z, u) - \boldsymbol{A}(z, k)])\,\mathrm{d}z\,\mathrm{d}k$$
$$-\iint \varphi(z)\psi(k)\partial_k(\chi(k, u) \operatorname{div}_z^a \boldsymbol{A}(z, k))\,\mathrm{d}z\,\mathrm{d}k = -\int \varphi(z)\psi(k)\,\mathrm{d}\partial_k\eta(k, z).$$

Since the function $k \mapsto \chi(k, u)[\boldsymbol{A}(z, u) - \boldsymbol{A}(z, k)]$ is Lipschitz, it is straightforward to check that $\partial_k(\chi(k, u)[\boldsymbol{A}(z, u) - \boldsymbol{A}(z, k)]) = -\chi(k, u)\partial_v\boldsymbol{A}(z, k)$, hence (22) holds.

Step 3 (*Blow-up*) Let $\eta(k, z) = \overline{\nu}(z) \otimes \lambda_z(k)$ be the disintegration of the measure η with respect to $\overline{\nu}$, see [5, Sect. 2.5]. Since $\mathcal{H}^{n-1} \llcorner \mathcal{N}$ is σ-finite by the Radon–Nikodym Theorem we can write

$$\overline{\nu} = h\,\mathcal{H}^{n-1} \llcorner \mathcal{N} + \overline{\nu} \llcorner (\mathbb{R}^n \setminus \mathcal{N}) \quad (23)$$

[3] Recall that given a Borel measure η on a space X and a Borel map $\pi : X \to Y$ the measure $\pi_\#\eta$ on Y is defined as $\pi_\#\eta(U) = \eta(\pi^{-1}(U))$ for every Borel set $U \subset Y$.

with $h \in L^1(\mathcal{H}^{n-1} \llcorner \mathcal{N})$. Let us now fix a point $z_0 \in \mathcal{N}$ and for $r > 0$ let us consider the following rescalings in the variable z:

$$
\begin{aligned}
&u_r(z) := u(z_0 + rz), \quad \boldsymbol{A}_r(z, v) := \boldsymbol{A}(z_0 + rz, v), \\
&\eta_{k,r}(V) := \frac{\eta_k(z_0 + rV)}{r^{n-1}}, \quad \eta_r(U \times V) := \frac{\eta(U \times (z_0 + rV))}{r^{n-1}}, \\
&\quad U \subset \mathbb{R}, \ V \subset \mathbb{R}^n \text{ Borel.}
\end{aligned} \tag{24}
$$

Recall the proof of Lemma 3.1: for $\mathcal{H}^{n-1}$ almost every z_0 in $\mathcal{N}$ we have

$$
h(z_0 + rz)\mathcal{H}^{n-1} \llcorner \frac{J - z_0}{r} \overset{*}{\rightharpoonup} h(z_0)\mathcal{H}^{n-1} \llcorner \{\nu(z_0) \cdot z = 0\}. \tag{25}
$$

We now claim that for $\mathcal{H}^{n-1}$ almost every such z_0 and for every $k \in \mathbb{R}$

$$
\begin{aligned}
&\boldsymbol{A}_r(z, k) \to \overline{\boldsymbol{A}}_{z_0}(z, k) := \boldsymbol{A}^+(z_0, k)\mathbf{1}_{H^+}(z) + \boldsymbol{A}^-(z_0, k)\mathbf{1}_{H^-}(z), \\
&\partial_v \boldsymbol{A}_r(z, k) \to \partial_v \overline{\boldsymbol{A}}_{z_0}(z, k) := \partial_v \boldsymbol{A}^+(z_0, k)\mathbf{1}_{H^+}(z) + \partial_v \boldsymbol{A}^-(z_0, k)\mathbf{1}_{H^-}(z), \\
&\operatorname{div}_z^a \boldsymbol{A}_r(z, k) \to 0,
\end{aligned} \tag{26}
$$

locally in $L^1(\mathbb{R}^n)$, with $H^\pm = \{z : \pm z \cdot \nu(z_0) > 0\}$. Indeed the first two equations follow directly from the hypotheses on $\boldsymbol{A}$, see [4, Proposition 3.2], while the last limit in (26) is a consequence of the fact that $\sup_k |\operatorname{div}^a \boldsymbol{A}(z, k)| \leqq \sigma^a(z)$ and that

$$
\lim_{r \to 0} \frac{1}{r^{n-1}} \int_{B_r(z_0)} |\sigma^a(z)| \, \mathrm{d}z = 0,
$$

for $\mathcal{H}^{n-1}$ almost every point in $\mathcal{N}$, see [5, Equation 2.41]. We now prove that, up to $\mathcal{H}^{n-1}$—a negligible subset of $z_0 \in \mathcal{N}$—it holds that:

$$
\eta_r \overset{*}{\rightharpoonup} h(z_0)\, \lambda_{z_0}(k) \otimes \mathcal{H}^{n-1} \llcorner \partial H^+. \tag{27}
$$

To this end observe that by [5, Equation 2.41], for $\mathcal{H}^{n-1}$ almost every $z_0 \in \mathcal{N}$,

$$
\lim_{r \to 0} \frac{|\overline{\nu} \llcorner (\mathbb{R}^n \backslash \mathcal{N})|(B_r(z_0))}{r^{n-1}} = 0.
$$

Now it is easy to see that, up to negligible sets,

$$
\left\{ z \in \mathcal{N} : 0 < \limsup_{r \to 0} \frac{|\overline{\nu} \llcorner \mathcal{N}|(B_r(z))}{r^{n-1}} < \infty \right\} = \{z \in \mathcal{N} : h(z) > 0\}. \tag{28}
$$

Since $\mathcal{H}^{n-1} \llcorner (\mathcal{N} \cap \{h > 0\}) \ll \overline{\nu} \llcorner \mathcal{N}$, $\mathcal{H}^{n-1}$ almost every $z_0 \in \mathcal{N} \cap \{h > 0\}$ is a Lebesgue point for the measure valued map $z \mapsto \lambda_z$ with respect to ν. By combining this with (25) one can argue as in Lemma 3.1 to deduce (27) on $\{h > 0\}$, see for instance [13, Proposition 9]. Finally by (28) we have that $\mathcal{H}^{n-1}$ almost every $z_0 \in \mathcal{N}$ satisfies (27), since this convergence trivially holds for $\mathcal{H}^{n-1}$ almost every $z_0 \in \{h = 0\} \cap \mathcal{N}$.

Step 4 (*Limiting equation and existence of traces*) Let us take a point z_0 such that (26) and (27) hold true. According to Lemma 3.2 below, the sequence $(u_r)_r$ is relatively compact in $L^1_{\rm loc}(\mathbb{R}^n)$. Let us now compute the equation satisfied by any cluster point u_∞ of $(u_r)_r$. To this end, note that u_r solves

$$\mathrm{div}_z(\mathrm{sign}(u_r-k)\partial_v[\boldsymbol{A}_r(z,u_r)-\boldsymbol{A}_r(z,k)])+\mathrm{sign}(u_r-k)\,\mathrm{div}_z^a\,\boldsymbol{A}_r(z,k)=\eta_{k,r}.$$

Let (r_j) be a sequence converging to 0 such that $u_{r_j}\to u^\infty$ in $L^1(B_1)$. Passing to the limit in the kinetic equation satisfied by the function $(k,z)\mapsto\chi(k,u_{r_j}(z))$,

$$\mathrm{div}_z\left[\chi(k,u_{r_j})\partial_v\boldsymbol{A}_{r_j}(z,k)\right]-\partial_k\left[\chi(k,u_{r_j})\,\mathrm{div}_z^a\,\boldsymbol{A}_{r_j}(z,k)\right]=-\partial_k\eta_{r_j}\quad\text{in }\mathcal{D}'(\mathbb{R}^{n+1}),$$

and taking into account (26) and (27), we obtain

$$\mathrm{div}_z\left[\chi(k,u^\infty)\partial_v\overline{\boldsymbol{A}}_{z_0}(z,k)\right]=-\partial_k\big(h(z_0)\,\lambda_{z_0}(k)\,\mathcal{H}^{n-1}\llcorner\partial H^\pm\big)\quad\text{in }\mathcal{D}'(\mathbb{R}^{n+1}).\tag{29}$$

In particular, due to the special form (26) of $\overline{\boldsymbol{A}}_{z_0}$, in the half-space H^+ (resp. H^-), Equation (29) is a transport equation of the form

$$\boldsymbol{a}^+(k)\cdot\nabla_z\chi(k,u^\infty)=0\quad(\text{resp. }\boldsymbol{a}^-(k)\cdot\nabla_z\chi(k,u^\infty)=0),\tag{30}$$

where

$$\boldsymbol{a}^\pm(k):=\partial_v\boldsymbol{A}^\pm(z_0,k).$$

Since, by (GLN), these vector fields are genuinely nonlinear, we conclude that u^∞ must be constant on H^+ and on H^-, that is there exist $u^-,u^+\in\mathbb{R}$ such that

$$u^\infty=u^+\mathbf{1}_{H^+}+u^-\mathbf{1}_{H^-}\tag{31}$$

compare [13, Proposition 7(b)]. Indeed let $\bar z\in H^+$ be a Lebesgue point of u^∞ and $-\|u\|_\infty-1<\bar k<u^\infty(\bar z)$ such that $\mathcal{L}^n(\{u^\infty=\bar k\})=0$. Fix $\tau>0$ and convolve with a nonnegative smooth kernel δ_ε supported in B_ε: for $\varepsilon<\varepsilon(\tau,\bar z)$ sufficiently small

$$\delta_\varepsilon*\chi(\bar k,u^\infty)(\bar z)\geqq 1-\tau.$$

Thanks to (GLN) we can choose n values $k_1,\ldots,k_n$ (depending on τ, ε and $\bar k$) with $|k_n-\bar k|$ sufficiently small and such that $\bar k<k_1<\cdots<k_n$, $\{\boldsymbol{a}^+(k_i)\}$ are linearly independent and

$$\delta_\varepsilon*\chi(k_n,u^\infty)(\bar z)\geqq 1-2\tau.\tag{32}$$

For every z the function $k\mapsto\chi(k,u^\infty(z))$ is decreasing, and so it remains when we convolve it with δ_ε, in particular,

$$\delta_\varepsilon*\chi(\bar k,u^\infty)(z)\geqq\delta_\varepsilon*\chi(k_1,u^\infty)(z)\geqq\cdots\geqq\delta_\varepsilon*\chi(k_n,u^\infty)(z)\qquad\forall z\in H^+_\varepsilon:=\{z\cdot\nu(z_0)>\varepsilon\}.\tag{33}$$

Equation (30), which holds also for $\delta_\varepsilon*\chi(k,u^\infty)$, implies that $\delta_\varepsilon*\chi(k_i,u^\infty)$ is constant along lines parallel to $\boldsymbol{a}^+(k_i)$. Since the $\{\boldsymbol{a}^+(k_i)\}$ are linearly independent, starting from (32) and exploiting (33) we obtain $\rho_\varepsilon * \chi(\bar{k}, u^\infty) \geqq 1 - 2\tau$ in H_ε^+. Letting $\tau \downarrow 0$ we get

$$\chi(\bar{k}, u^\infty) \geqq 1 \quad \text{in } H^+.$$

Since $\bar{k}$ can be taken arbitrarily close to $u^\infty(\bar{z})$, u^∞ is constantly equal to $u^\infty(\bar{z})$. A completely analogous argument holds for H^-. In particular $\Gamma_{u,\mathcal{N}}(z_0) \neq \emptyset$.
Step 5 (*Characterization of traces*) By (29) and the special form (31) of u^∞, we deduce that

$$\chi(k, u^+)\, \boldsymbol{a}^+(k) \cdot \nu(z_0) - \chi(k, u^-)\, \boldsymbol{a}^-(k) \cdot \nu(z_0) = -\partial_k (h(z_0)\, \lambda_y(k)) \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{34}$$

Let us now show that as the above equality uniquely determines $u^\pm$ whenever $u^+ \neq u^-$, hence in particular the traces do not depend on the choice of the subsequence (r_j). To this end, let (u_{ρ_j}) be another converging subsequence of (u_r); by Step 4 we have

$$u_{\rho_j} \to v^\infty := v^+ \mathbf{1}_{H^+} + v^- \mathbf{1}_{H^-} \quad \text{in } L^1(B_1),$$

so that the pair (v^-, v^+) also satisfies (34). Subtracting the equation satisfied by the pair (u^-, u^+), we get, for almost every $k \in \mathbb{R}$,

$$[\chi(k, u^+) - \chi(k, v^+)]\, \boldsymbol{a}^+(k) \cdot \nu(z_0) = [\chi(k, u^-) - \chi(k, v^-)]\, \boldsymbol{a}^-(k) \cdot \nu(z_0),$$

that is:

$$\operatorname{sign}(u^+ - v^+) \mathbf{1}_{(u^+, v^+)}(k)\, \boldsymbol{a}^+(k) \cdot \nu(z_0) = \operatorname{sign}(u^- - v^-) \mathbf{1}_{(u^-, v^-)}(k)\, \boldsymbol{a}^-(k) \cdot \nu(z_0).$$

Since, again by the assumption (GNL) of genuine nonlinearity, the functions

$$k \mapsto \boldsymbol{a}^\pm(k) \cdot \nu(z_0)$$

cannot vanish on any interval, the two intervals $I(u^-, u^+)$ and $I(v^-, v^+)$ must coincide.[4] If $u^- \neq u^+$, the condition $I(u^-, u^+) = I(v^-, v^+)$ can be satisfied either in the case $v^- = u^-$, $v^+ = u^+$ or in the case $v^- = u^+$, $v^+ = u^-$. On the other hand, this second possibility is excluded by the fact that the map $r \mapsto u(y + rz)$ is continuous from $(0, 1]$ to $L^1(B_1)$. Indeed, since

$$u_{r_j} \to u^\infty = u^+ \mathbf{1}_{H^+} + u^- \mathbf{1}_{H^-},$$

and

$$u_{\rho_j} \to v^\infty = u^- \mathbf{1}_{H^+} + u^+ \mathbf{1}_{H^-},$$

we have

$$\int_{B_1} |u_{r_j} - u^\infty| \to 0, \quad \int_{B_1} |u_{\rho_j} - u^\infty| \to \int_{B_1} |v^\infty - u^\infty| =: m \neq 0.$$

---

[4] Here, $I(a, b)$ denotes the interval $[a, b]$ if $a \leqq b$ or the interval $[b, a]$ if $b < a$.

By the continuity of the map

$$(0,1] \ni r \mapsto \int_{B_1} u_r$$

and the relative compactness of the family $(u_r)_r$, we can find a third sequence (u_{s_j}) such that

$$\begin{aligned} u_{s_j} &\to w^\infty := w^+ \mathbf{1}_{H^+} + w^- \mathbf{1}_{H^-}, \\ \int_{B_1} |w^\infty - u^\infty| &= \frac{m}{2} \leqq \int_{B_1} |w^\infty - v^\infty|, \end{aligned} \tag{35}$$

but then we must have $I(w^-, w^+) = I(u^-, u^+) = I(v^-, v^+)$ so that either $w^- = u^-$ and $w^+ = u^+$, or $w^- = u^+$ and $w^+ = u^-$, and in each case we get a contradiction with (35).

In conclusion, if $u^- \neq u^+$ then all subsequences of (u_r) must converge to the same function u^∞, hence the traces are uniquely determined.

In the case $u^- = u^+$, reasoning as above, we can always conclude that $w^- = w^+$ for every $(w^-, w^+) \in \Gamma_{u,\mathcal{N}}$. Moreover, exploiting again the continuity of the map $r \mapsto \int_{B_1} u_r$, we get that $\Gamma_{u,\mathcal{N}}$ is a compact connected set. Finally the Rankine–Hugoniot condition follows from Lemma 3.1, thus concluding the proof. □

The following lemma has been used in the proof of Theorem 2.5:

Lemma 3.2. (Strong pre-compactness of blow-ups) *The family* (u_r) *defined in* (24) *is pre-compact in* $L^1(B_1)$.

Proof of Lemma 3.2. For every $r > 0$, the function u_r is a solution to

$$\operatorname{div}_z \boldsymbol{A}_r(z, u_r(z)) = 0,$$

hence

$$\operatorname{div}_z \overline{\boldsymbol{A}}_{z_0}(z, u_r(z)) = -\operatorname{div}_z \left[\boldsymbol{A}_r(z, u_r(z)) - \overline{\boldsymbol{A}}_{z_0}(u_r(z))\right].$$

We claim that the family of functions

$$q_r(z) := \boldsymbol{A}_r(z, u_r(z)) - \overline{\boldsymbol{A}}_{z_0}(z, u_r(z))$$

is pre-compact in $L^2(B_1)$, so that $(\operatorname{div}_z \overline{\boldsymbol{A}}_{z_0}(z, u_r(z)))_r$ is pre-compact in the negative Sobolev space $W^{-1,2}(B_1)$. If this condition is satisfied, then by [23, Thm. 6] we can conclude that (u_r) is pre-compact in the strong $L^1(B_1)$ topology.

Let us consider the functions

$$f_{r,z}(v) := |\boldsymbol{A}_r(z, v) - \overline{\boldsymbol{A}}_{z_0}(z, v)|, \quad r > 0,\ z \in B_1,\ v \in \mathbb{R}.$$

By (26)

$$\lim_{r \downarrow 0} f_{r,z}(v) = 0 \quad \text{for every } z \in B_1 \setminus D_0 \text{ and } \forall v \in \mathbb{R}, \tag{36}$$

where $D_0 \subset B_1$ is a set of Lebesgue measure 0. Moreover,

$$|f_{r,z}(v) - f_{r,z}(v')| \leqq |\boldsymbol{A}_r(z,v) - \boldsymbol{A}_r(z,v')| + |\overline{\boldsymbol{A}}_{z_0}(z,v) - \overline{\boldsymbol{A}}_{z_0}(z,v')| \\ \leqq 2\|\partial_v \boldsymbol{A}\|_\infty |v - v'|,$$

hence $(f_{r,z})_r$ is an equi-Lipschitz family of functions converging pointwise to 0 for every $z \in B_1 \setminus D_0$.

Let $L = \|u\|_\infty$, and let $(v_k) \subset [-L, L]$ be a countable dense set in $[-L, L]$. Using a diagonal argument, we can construct a sequence (r_j) converging to 0 such that

$$\lim_{j \to +\infty} f_{r_j,z}(v_k) = 0 \qquad \forall z \in B_1 \setminus D,\ \forall k \in \mathbb{N},$$

where $D \supseteq D_0$ is a set of Lebesgue measure 0.

Using the classical argument in the proof of the Ascoli–Arzelà compactness theorem, we have that, for every $z \in B_1 \setminus D$, the sequence $(f_{r_j,z})_j$ converges uniformly to 0 in $[-L, L]$. In other words,

$$g_j(z) := \sup_{|v| \leqq L} |\boldsymbol{A}_{r_j}(z,v) - \overline{\boldsymbol{A}}_{z_0}(z,v)| \to 0. \qquad \forall z \in B_1 \setminus D.$$

Since the functions g_j are equi-bounded, they converge to 0 in $L^2(B_1)$. Moreover,

$$|q_{r_j}(z)|^2 := |\boldsymbol{A}_r(z, u_{r_j}(z)) - \overline{\boldsymbol{A}}_{z_0}(z, u_{r_j}(z))|^2 \leqq g_j(z)^2,$$

so that the sequence $(q_{r_j})_j$ converges to 0 in $L^2(B_1)$ and the claim is proved. □

4. Proof of Theorem 2.6

In this section we prove Theorem 2.6. To this end we will need two technical lemmas: the first one is a slight generalization of classical arguments used in [18]. The second one allows us to study the limiting behavior of the incremental quotient of $\boldsymbol{A}$ in the spirit of [3, Thm. 2.4] and [15, Lemma II.1], and it is crucial in the proof of Theorem 2.6. For the sake of exposition we postpone the proofs of both lemmas at the end of the section.

Lemma 4.1. *Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ satisfy the following assumptions*:

- $z \mapsto \sup_v |f(z,v)| \in L^1_{\text{loc}}(\mathbb{R}^n)$;
- $|f(z,v) - f(z,v')| \leqq g(z)\omega(|v - v'|)$ *for some* $g \in L^1_{\text{loc}}$ *and some modulus of continuity* ω.

Then for every $u, v \in L^\infty_{\text{loc}}(\mathbb{R}^n)$

$$|f(z+\tau, u(z)) - f(z, u(z))| \to 0$$
$$\operatorname{sign}(u(z+\tau) - v(z))[f(z+\tau, u(z+\tau)) - f(z, v(z))] \\ \to \operatorname{sign}(u(z) - v(z))[f(z, u(z)) - f(z, v(z))]$$

in L^1_{loc} *as* $\tau \to 0$.

Lemma 4.2. (Uniform differential quotients) *Let* $\boldsymbol{A}$ *satisfy* (H1)–(H5) *and let* $w \in \mathbb{R}^n$. *Then there exists a measurable set* $D = D_w \subset \mathbb{R}^n$, *with* $\mathcal{L}^n(D) = 0$, *such that the difference quotients for* $\boldsymbol{A}$ *can be canonically written as*

$$\frac{\boldsymbol{A}(z+\varepsilon w, v) - \boldsymbol{A}(z, v)}{\varepsilon} = \boldsymbol{A}^1_\varepsilon(z, v) + \boldsymbol{A}^2_\varepsilon(z, v)$$

where $\boldsymbol{A}^1_\varepsilon$ *and* $\boldsymbol{A}^2_\varepsilon$ *satisfy the following properties*:

(i) $\lim_{\varepsilon \downarrow 0} \boldsymbol{A}^1_\varepsilon(z, v) = \nabla_z \boldsymbol{A}(z, v) \cdot w, \qquad \forall v \in \mathbb{R}$ *and* $z \in \mathbb{R}^n \backslash D$;

(ii) *The family of functions* $h_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ *defined by*

$$h_\varepsilon(z) := |w| \sup_{v \in \mathbb{R}} |\boldsymbol{A}^1_\varepsilon(z, v)|$$

is equi-integrable;

(iii) *For every compact set* $K \subset \mathbb{R}^n$ *we have*

$$\int_K \sup_{v \in \mathbb{R}} |\boldsymbol{A}^2_\varepsilon(z, v)| \, \mathrm{d}z \leqq \sigma^s(K_{\varepsilon|w|})|w|,$$

where $K_\tau := K + B_\tau(0)$.

Proof Theorem 2.6. We divide the proof into several steps:

Step 1: Doubling of variables. We follow the classical technique of Kruzhkov [18]. Let $u(z)$ and $v(z')$ be WES: let us set $k = v(z')$ in (15) for u and $k = u(z)$ in (15) for v. Let us also choose a test function $\Phi(z, z') = \varphi(z + z')\delta_\varepsilon(z - z')$ where $\varphi \in C^1_c(\mathbb{R}^n)$ is nonnegative and δ_ε is the usual smooth approximation of the identity in 0:

$$\delta_\varepsilon(\zeta) = \frac{1}{\varepsilon^n}\psi\big(\zeta/\varepsilon\big) \qquad \psi \in C^1_c(B_1), \quad \int \psi = 1, \quad \psi(z) = \psi(-z).$$

Multiplying both equations by Φ, integrating in z and z' and subtracting the corresponding inequalities, we obtain

$$\begin{aligned}
&\iint \{\delta_\varepsilon(z - z')\nabla\varphi(z + z') + \varphi(z + z')\nabla\delta_\varepsilon(z - z')\} \operatorname{sign}(u(z) - v(z'))(\boldsymbol{A}(z, u(z)) \\
&\quad - \boldsymbol{A}(z, v(z'))) - \operatorname{sign}(u(z) - v(z')) \operatorname{div}^a_z \boldsymbol{A}(z, v(z'))\varphi(z + z')\delta_\varepsilon(z - z') \\
&\quad + \{\delta_\varepsilon(z - z')\nabla\varphi(z + z') - \varphi(z + z')\nabla\delta_\varepsilon(z - z')\} \operatorname{sign}(v(z') \\
&\quad - u(z))(\boldsymbol{A}(z', v(z')) - \boldsymbol{A}(z', u(z))) \\
&\quad - \operatorname{sign}(v(z') - u(z)) \operatorname{div}^a_{z'} \boldsymbol{A}(z', u(z))\varphi(z + z')\delta_\varepsilon(z - z') \, \mathrm{d}z \, \mathrm{d}z' \\
&\geqq -2 \iint \delta_\varepsilon(z - z')\varphi(z + z') \, \mathrm{d}z' \, \mathrm{d}\mu(z).
\end{aligned}$$

This can be written as

$$I^\varepsilon_1 - I^\varepsilon_2 + I^\varepsilon_3 \geqq -2 \iint \delta_\varepsilon(z - z')\varphi(z + z') \, \mathrm{d}z' \, \mathrm{d}\mu(z), \tag{37}$$

where

$$
\begin{aligned}
I_1^\varepsilon = & \iint \psi(w) \nabla\varphi(2z-\varepsilon w)\,\mathrm{sign}((u(z)-v(z-\varepsilon w)) \\
& \times \{\boldsymbol{A}(z,u(z)) + \boldsymbol{A}(z-\varepsilon w, u(z)) - \boldsymbol{A}(z, v(z-\varepsilon w)) \\
& - \boldsymbol{A}(z-\varepsilon w, v(z-\varepsilon w))\}\,\mathrm{d}w\,\mathrm{d}z, \\
I_2^\varepsilon = & \iint \varphi(2z-\varepsilon w)\,\mathrm{sign}(u(z)-v(z-\varepsilon w)) \\
& \times \left\{ \nabla\psi(w) \frac{\boldsymbol{A}(z-\varepsilon w, u(z)) - \boldsymbol{A}(z,u(z))}{\varepsilon} \right. \\
& \left. -\psi(w)\,\mathrm{div}_z^a\,\boldsymbol{A}(z-\varepsilon w, u(z)) \right\} \mathrm{d}w\,\mathrm{d}z, \\
I_3^\varepsilon = & \iint \varphi(2z+\varepsilon w)\,\mathrm{sign}(u(z+\varepsilon w)-v(z)) \\
& \times \left\{ \nabla\psi(w) \frac{\boldsymbol{A}(z, v(z)) - \boldsymbol{A}(z+\varepsilon w,v(z))}{\varepsilon} \right. \\
& \left. -\psi(w)\,\mathrm{div}_z^a\,\boldsymbol{A}(z+\varepsilon w, v(z)) \right\} \mathrm{d}w\,\mathrm{d}z.
\end{aligned}
$$

Regarding I_1^ε, Lemma 4.1 implies that

$$
I_1^\varepsilon \to 2 \int \nabla\varphi(2z)\,\mathrm{sign}((u(z)-v(z))\,\mathrm{d}z(\boldsymbol{A}(z,u(z)) - \boldsymbol{A}(z,v(z))). \tag{38}
$$

We will now show that

$$
\limsup_{\varepsilon\to 0} |I_2^\varepsilon - I_3^\varepsilon| \leqq C\|\varphi\|_\infty |\sigma^s|(\mathrm{spt}\,\varphi). \tag{39}
$$

This, together with (37) and (38), will then give that, in the sense of distributions,

$$
\mathrm{div}_z(\mathrm{sign}((u(z)-v(z))(\boldsymbol{A}(z,u(z)) - \boldsymbol{A}(z,v(z)))) \leqq 2\mu + C|\sigma^s| =: \beta, \tag{40}
$$

where $(\mu + C|\sigma^s|)(\mathbb{R}^n \backslash \mathcal{N}) = 0$. In turn, the left hand side of (40) is a signed measure, which we denote by α, for which:

$$
\alpha \leqq \alpha^+ = \alpha^+ \llcorner \mathcal{N} = (\alpha \llcorner \mathcal{N})^+ \leqq \beta.
$$

Since the map

$$
(u,v) \mapsto \mathrm{sign}(u-v)(\boldsymbol{A}(z,u) - \boldsymbol{A}(z,v))
$$

is Lipschitz and

$$
\boldsymbol{A}(z_0+\varepsilon z, v) \to \boldsymbol{A}^\pm(z_0,v) \quad \text{in } L^1_{\mathrm{loc}} \text{ for every } v \in \mathbb{R},
$$

by arguing as in Lemma 4.1, the traces of the vector field

$$
z \mapsto \mathrm{sign}((u(z)-v(z))(\boldsymbol{A}(z,u(z)) - \boldsymbol{A}(z,v(z)))
$$

exist for $\mathcal{H}^{n-1}$ almost every $z \in \mathcal{N}$ and are given by

$$
\mathrm{sign}((u^\pm(z)-v^\pm(z))(\boldsymbol{A}^\pm(z,u^\pm(z)) - \boldsymbol{A}^\pm(z,v^\pm(z))).
$$

A direct application of Lemma 3.1 yields the desired representation (17).

To show the uniqueness of the traces at points where $w(z) \neq 0$, we note that we only have to discuss the case when (say) $v^-(z) = v^+(z) = v$ and $u^-(z) \neq u^+(z)$, otherwise either the traces are unique by Theorem 2.5 or $w = 0$. The Rankine–Hugoniot condition gives

$$w(z) = [\operatorname{sign}(u^+(z) - v) - \operatorname{sign}(u^-(z) - v)][\boldsymbol{A}^+(z, u^+(z)) - \boldsymbol{A}^+(z, v)] \cdot \nu(z). \tag{41}$$

Moreover we know by Theorem 2.5 that if $\Gamma_{v,\mathcal{N}}$ is not a singleton it contains pairs (v', v') with v' ranging in a non trivial interval $[a, b]$. With w being uniquely determined and non zero we have that (41) holds for any such $v' \in [a, b]$ and that $v' \in I(u^-, u^+)$. This implies that

$$[\boldsymbol{A}^+(z, v') - \boldsymbol{A}^+(z, v)] \cdot \nu(z) = 0 \qquad \forall\, v, v' \in [a, b],$$

contradicting the genuine nonlinearity assumption (GLN).

In order to conclude the proof of the Theorem we only have to show the validity of (39). According to Lemma 4.2 above we can write

$$\frac{\boldsymbol{A}(z - \varepsilon w, u(z)) - \boldsymbol{A}(z, u(z))}{\varepsilon} = \boldsymbol{A}^1_{\varepsilon,w}(z) + \boldsymbol{A}^2_{\varepsilon,w}(z),$$

where

$$\boldsymbol{A}^1_{\varepsilon,w}(z) \xrightarrow{L^1_{\text{loc}}} -\nabla \boldsymbol{A}(z, u(z)) \cdot w$$

and

$$\int \mathrm{d}z |\boldsymbol{A}^2_{\varepsilon,w}(z)| \varphi(z) \leqq |w| \|\varphi\|_\infty |\sigma^s|((\operatorname{spt} \varphi)_{\varepsilon|w|}).$$

Hence, by also using Lemma 4.1, we obtain that

$$\begin{aligned} I_2^\varepsilon = & \iint \varphi(2z - \varepsilon w) \operatorname{sign}(u(z) - v(z - \varepsilon w)) \\ & \times \{-\nabla\psi(w) \nabla \boldsymbol{A}(z, u(z)) \cdot w - \psi(w) \operatorname{div}_z^a \boldsymbol{A}(z, u(z))\} \,\mathrm{d}w\, \mathrm{d}z + R_1^\varepsilon + R_2^\varepsilon, \end{aligned}$$

where

$$\limsup_{\varepsilon \to 0} |R_1^\varepsilon| \leqq C(\psi) \|\varphi\|_\infty |\sigma^s|(\operatorname{spt} \varphi) \quad \text{and} \quad \lim_{\varepsilon \to 0} |R_2^\varepsilon| = 0.$$

By applying the same decomposition to I_3^ε we obtain, after a change of variable, that

$$\begin{aligned} \limsup_{\varepsilon \to 0} |I_2^\varepsilon - I_3^\varepsilon| \leqq & \; 2C(\psi) \|\varphi\|_\infty |\sigma^s|(\operatorname{spt} \varphi) \\ & + \limsup_{\varepsilon \to 0} \Bigg| \iint \varphi(2z - \varepsilon w) \operatorname{sign}(u(z) - v(z - \varepsilon w)) \\ & \times \{\nabla\psi(w) \big[\nabla \boldsymbol{A}(z, u(z)) \cdot w - \nabla \boldsymbol{A}(z - \varepsilon w, v(z - \varepsilon w)) \cdot w\big] \\ & + \psi(w) [\operatorname{div}_z^a \boldsymbol{A}(z, u(z)) - \operatorname{div}_z^a \boldsymbol{A}(z - \varepsilon w, v(z - \varepsilon w))]\} \mathrm{d}w\, \mathrm{d}z \Bigg|. \end{aligned}$$

By Lemma 4.1 the latter integral converges to

$$\iint \varphi(2z)\,\mathrm{sign}(u(z)-v(z))\{\nabla\psi(w)\nabla \boldsymbol{A}(z,u(z))\cdot w + \psi(w)\,\mathrm{div}_z^a\,\boldsymbol{A}(z,u(z))\}\,\mathrm{d}w\,\mathrm{d}z - \iint \varphi(2z)\,\mathrm{sign}(u(z)-v(z))\{\nabla\psi(w)\nabla \boldsymbol{A}(z,v(z))\cdot w + \psi(w)\,\mathrm{div}_z^a\,\boldsymbol{A}(z,v(z))\}\,\mathrm{d}w\,\mathrm{d}z.$$

Integrating by parts with respect to the w variable, we get that both integrals are zero, thus concluding the proof of (39). □

We conclude the section by proving Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. Let $Q\subset\mathbb{R}$ be a countable dense set: by the continuity of translations in L^1

$$|f(z+\tau,u)-f(z,u)|\to 0 \quad \text{in } L^1_{\mathrm{loc}} \text{ for every } u\in Q.$$

It is not restrictive to assume that $u\in L^\infty$. Therefore there exists $u_k=\sum_{i=1}^{N_k}u_k^i\mathbf{1}_{A_k^i}$ with $u_k^i\in Q$ such that $\|u-u_k\|_\infty\to 0$. Hence for every compact set $K\subset\mathbb{R}^n$,

$$\begin{aligned}\int_K |f(z+\tau,u(z))-f(z,u(z))|\,\mathrm{d}z &\leqq \omega(\|u-u_k\|_\infty)\int_K (g(z)+g(z+\tau))\,\mathrm{d}z \\ &\quad + \int_K |f(z+\tau,u_k(z))-f(z,u_k(z))|\,\mathrm{d}z \\ &\leqq o_k(1)+\sum_{i=1}^{N_k}\int_{K\cap A_k^i}|f(z+\tau,u_k^i)-f(z,u_k^i)|\,\mathrm{d}z,\end{aligned}$$

where $o_k(1)\to 0$ independently on τ as $k\to\infty$. Passing to the limit first on τ and then on k proves the first claim. To prove the second claim note that thanks to what we have proved it is enough to show that

$$\mathrm{sign}(u(z+\tau)-v(z))[f(z,u(z+\tau))-f(z,v(z))]\to \mathrm{sign}(u(z)-v(z))[f(z,u(z))-f(z,v(z))]$$

in L^1_{loc} as $\tau\to 0$. Since the map

$$(u,v)\mapsto \mathrm{sign}(u-v)[f(z,u)-f(z,v)]$$

has modulus of continuity 2ω independently on z this plainly follows by the continuity of translations in L^1. □

Proof of Lemma 4.2. Up to dilating and rotating we can assume that $w = e_n$. We will write $z = (z', z_n)$ with $z' \in \mathbb{R}^{n-1}$ and $z_n \in \mathbb{R}$.

Let $Q = (v_j) \subset \mathbb{R}$ be a countable dense set in $\mathbb{R}$. By slicing theory for BV functions, see [5, Chapter 3], for every $j \in \mathbb{N}$ there exists a set $D_j \subset \mathbb{R}^n$ with $\mathcal{L}^n(D_j) = 0$, such that, for every $z \in \mathbb{R}^n \setminus D_j$, the function $t \mapsto \boldsymbol{A}(z', z_n + t, v_j)$ belongs to $BV(\mathbb{R})$ and the absolutely continuous part of its derivative, denoted by $\frac{\partial \boldsymbol{A}}{\partial z_n}(z', z_n + t, v_j)$, coincides with $\nabla_z \boldsymbol{A}(z', z_n + t, v_j) \cdot e_n$. Hence for $j \in \mathbb{N}$ and $z \in \mathbb{R}^n \setminus D_j$ we define

$$\boldsymbol{A}^1_\varepsilon(z', z_n, v_j) = \int_0^1 \frac{\partial \boldsymbol{A}}{\partial z_n}(z', z_n + \varepsilon t, v_j)\, \mathrm{d}t = \int_0^1 \nabla_z \boldsymbol{A}(z', z_n + \varepsilon t, v_j) \cdot e_n \, \mathrm{d}t.$$

From [2, Thm. 2.4] there exists a measurable set $D \subset \mathbb{R}^n$, with $D \supset \mathcal{C}_{\boldsymbol{A}} \cup \bigcup_j D_j$ and $\mathcal{L}^n(D) = 0$, such that

$$\lim_{\varepsilon \downarrow 0} \boldsymbol{A}^1_\varepsilon(z, v_j) = \nabla_z \boldsymbol{A}(z, v_j) \cdot e_n, \quad \forall j \in \mathbb{N} \text{ and } z \in \mathbb{R}^n \setminus D. \tag{42}$$

Moreover, possibly adding to D a set of Lebesgue measure zero, we can assume that every z in $\mathbb{R}^n \setminus D$ is a Lebesgue point for the function g appearing in (H4) and that

$$G_\varepsilon(z) := \int_0^1 g(z', z_n + \varepsilon t)\, \mathrm{d}t \;\to g(z) \quad \text{as } \varepsilon \downarrow 0. \tag{43}$$

Let us now fix $z \in \mathbb{R}^n \setminus D$ and $j, k \in \mathbb{N}$; by (H4) we have that

$$\begin{aligned} |\boldsymbol{A}^1_\varepsilon(z, v_j) - \boldsymbol{A}^1_\varepsilon(z, v_k)| &\leqq \int_0^1 |\nabla_z \boldsymbol{A}(z', z_n + \varepsilon t, v_j) - \nabla_z \boldsymbol{A}(z', z_n + \varepsilon t, v_k)|\, \mathrm{d}t \\ &\leqq G_\varepsilon(z)\, \omega(|v_j - v_k|). \end{aligned} \tag{44}$$

Let us now take $v \in \mathbb{R}$ and $v_j \in Q$ with $v_j \to v$. By (44), $\left(\boldsymbol{A}^1_\varepsilon(z, v_j)\right)_j$ is a Cauchy sequence, hence it converges to a unique limit $\ell_\varepsilon(z, v)$. Let us define for $v \in \mathbb{R}$ and $z \in \mathbb{R}^n \setminus D$

$$\boldsymbol{A}^1_\varepsilon(z, v) = \ell_\varepsilon(z, v)$$

and

$$\boldsymbol{A}^2_\varepsilon(z, v) = \frac{\boldsymbol{A}(z + \varepsilon w, v) - \boldsymbol{A}(z, v)}{\varepsilon} - \boldsymbol{A}^1_\varepsilon(z, v).$$

We now verify the validity of (i)–(iii). First of all (44) implies

$$|\boldsymbol{A}^1_\varepsilon(z, v) - \boldsymbol{A}^1_\varepsilon(z, v')| \leqq G_\varepsilon(z)\, \omega(|v - v'|) \quad \forall v, v' \in \mathbb{R}. \tag{45}$$

Moreover, according to [4, Lemma 3.4], we can add to D a set of measure zero outside which $\nabla \boldsymbol{A}(z, v)$ is well defined and continuous in v. Hence for $z \in \mathbb{R}^n \setminus D$ and $v \in \mathbb{R}$, by (45) and (H4), we have

$$\begin{aligned}|\boldsymbol{A}^1_\varepsilon(z,v) - \nabla_z \boldsymbol{A}(z,v)\cdot e_n| &\leqq |\boldsymbol{A}^1_\varepsilon(z,v) - \boldsymbol{A}^1_\varepsilon(z,v_j)| \\ &\quad + |\boldsymbol{A}^1_\varepsilon(z,v_j) - \nabla_z \boldsymbol{A}(z,v_j)\cdot e_n| \\ &\quad + |\nabla_z \boldsymbol{A}(z,v_j)\cdot e_n - \nabla_z \boldsymbol{A}(z,v)\cdot e_n| \\ &\leqq G_\varepsilon(z)\,\omega(|v-v_j|) + |\boldsymbol{A}^1_\varepsilon(z,v_j) - \nabla_z \boldsymbol{A}(z,v_j)\cdot e_n| \\ &\quad + g(z)\,\omega(|v-v_j|).\end{aligned}$$

Taking the limsup as $\varepsilon \downarrow 0$ and taking into account (42) and (43) we get

$$\limsup_{\varepsilon\downarrow 0} |\boldsymbol{A}^1_\varepsilon(z,v) - \nabla_z \boldsymbol{A}(z,v)\cdot e_n| \leqq 2g(z)\,\omega(|v-v_j|).$$

Since (v_j) is dense in $\mathbb{R}$, we conclude that (i) holds.

Let us prove (ii). For almost every $z \in \mathbb{R}^n$ we have

$$h_\varepsilon(z) = \sup_{j\in\mathbb{N}} |\boldsymbol{A}^1_\varepsilon(z,v_j)| \leqq \int_0^1 \sigma^a(z', z_n + \varepsilon t)\,\mathrm{d}t.$$

Since $\sigma^a \in L^1(\mathbb{R}^n)$, there exists a superlinear, convex, increasing function $\psi\colon [0,+\infty) \to [0,+\infty)$ such that

$$\int_{\mathbb{R}^n} \psi(\sigma^a(z))\,\mathrm{d}z < +\infty.$$

Then, by Jensen's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} \psi(h_\varepsilon(z))\,\mathrm{d}z &\leqq \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \psi\left(\int_0^1 \sigma^a(z', z_n+\varepsilon t)\,\mathrm{d}t\right)\mathrm{d}z_n\,\mathrm{d}z' \\ &\leqq \int_0^1\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} \psi(\sigma^a(z', z_n+\varepsilon t))\,\mathrm{d}z'\,\mathrm{d}z_n\,\mathrm{d}t \\ &= \int_{\mathbb{R}^n} \psi(\sigma^a(z))\,\mathrm{d}z < +\infty.\end{aligned}$$

By the Dunford–Pettis compactness criterion we conclude that the family (h_ε) is equi–integrable in $L^1(\mathbb{R}^n)$, hence (ii) is proved.

To conclude (iii), for every $j \in \mathbb{N}$ we let $\varphi^j_{z'}(z_n) := \boldsymbol{A}(z', z_n, v_j)$, and we note that for almost every z_n

$$\begin{aligned}&\frac{\boldsymbol{A}(z+\varepsilon e_n, v_j) - \boldsymbol{A}(z,v_j)}{\varepsilon} \\ &= \frac{1}{\varepsilon} D\varphi^j_{z'}([z_n, z_n+\varepsilon]) \\ &= \int_0^1 \nabla_z \boldsymbol{A}(z+t\varepsilon e_n, v_j)\cdot e_n\,\mathrm{d}t + \frac{1}{\varepsilon} D^s\varphi^j_{z'}([z_n, z_n+\varepsilon]),\end{aligned}$$

hence

$$\left|\boldsymbol{A}^2_\varepsilon(z,v_j)\right| \leqq \frac{1}{\varepsilon}|D^s\varphi^j_{z'}|([z_n, z_n+\varepsilon]).$$

If K is a compact subset of $\mathbb{R}^n$ we get

$$\begin{aligned}\int_K |A^2_\varepsilon(z, v_j)|\,\mathrm{d}z &\leqq \int_{\mathbb{R}^{n-1}} \mathrm{d}z' \int_{\{z_n:\,(z',z_n)\in K\}} \mathrm{d}z_n \frac{1}{\varepsilon}\,|D^s\varphi^j_{z'}|([z_n, z_n+\varepsilon]) \\ &= \int_{\mathbb{R}^{n-1}} \mathrm{d}z' \int_{\{z_n:\,(z',z_n)\in K\}} \mathrm{d}z_n \int_{\mathbb{R}} \frac{1}{\varepsilon}\,\mathbf{1}_{[z_n,z_n+\varepsilon]}(t)\,\mathrm{d}|D^s\varphi^j_{z'}|(t) \\ &\leqq \int_{\mathbb{R}^{n-1}} \mathrm{d}z' \int_{\{t:\,(z',t)\in K_\varepsilon\}} \mathrm{d}|D^s\varphi^j_{z'}|(t) \int_{\mathbb{R}} \mathrm{d}z_n \frac{1}{\varepsilon}\,\mathbf{1}_{[z_n,z_n+\varepsilon]}(t) \\ &\leqq |D^s A(\cdot, v_j)|(K_\varepsilon) \leqq \sigma^s(K_\varepsilon).\end{aligned} \tag{46}$$

Now let $v \in \mathbb{R}$. From (45), (46) and (H1) we get

$$\begin{aligned}\int_K |A^2_\varepsilon(z, v)|\,\mathrm{d}z &\leqq \int_K |A^2_\varepsilon(z, v) - A^2_\varepsilon(z, v_j)|\,\mathrm{d}z + \int_K |A^2_\varepsilon(z, v_j)|\,\mathrm{d}z \\ &\leqq \int_K \left| \frac{A(z+\varepsilon w, v) - A(z, v)}{\varepsilon} - \frac{A(z+\varepsilon w, v_j) - A(z, v_j)}{\varepsilon} \right| \mathrm{d}z \\ &\quad + \int_K |A^1_\varepsilon(z, v) - A^1_\varepsilon(z, v_j)|\,\mathrm{d}z + \int_K |A^2_\varepsilon(z, v_j)|\,\mathrm{d}z \\ &\leqq \frac{2}{\varepsilon} M\,\mathcal{L}^n(K_\varepsilon)\,|v - v_j| + \left(\int_K G_\varepsilon(z)\,\mathrm{d}z\right) \omega(|v - v_j|) + \sigma^s(K_\varepsilon).\end{aligned}$$

Exploiting the density of (v_j), we get (iii). □

5. Proofs of Theorems 2.9 and 2.11

In this section we briefly sketch the proofs of Theorems 2.9 and 2.11.

Proof of Theorem 2.9. By Theorem 2.6 we have that any two $\mathcal{G}$-entropy solutions u, v satisfy the generalized Kato inequality (16). By the usual test function argument (see [18]) we then obtain that for every $T > 0$ and every $R > 0$

$$\begin{aligned}&\int_{B_R} |u(T, x) - v(T, x)|\,\mathrm{d}x \\ &\leqq \int_{B_{R+VT}} |u(0, x) - v(0, x)|\,\mathrm{d}x + \int_{\mathcal{N}\cap([0,T]\times B_{R+VT})} w(t, x)\,\mathrm{d}\mathcal{H}^N(t, x),\end{aligned}$$

where $w(t, x)$ is given by (17). Since u, v are $\mathcal{G}$-entropy solutions, $w \leqq 0$, from which uniqueness immediately follows. □

Proof of Theorm 2.11. If $F(\cdot, u)$ belongs to $W^{1,1}$, it easily follows from the definition of the supremum of measures that $\sigma^s = 0$, which implies that $\mathcal{H}^N(\mathcal{N}) = 0$. Theorem 2.6 then gives that any two entropy solutions satisfy a true Kato inequality:

$$\partial_t |u - v| + \mathrm{div}_x(\mathrm{sign}(u - v)[F(t, x, u) - F(t, x, v)]) \leqq 0,$$

from which the validity of the L^1 contraction inequality is then straightforward. □

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