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Some Uniform Estimates and Large-Time Behavior of Solutions to One-Dimensional Compressible Navier–Stokes System in Unbounded Domains with Large Data

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Abstract

We study the large-time behavior of solutions to the initial and initial boundary value problems with large initial data for the compressible Navier–Stokes system describing the one-dimensional motion of a viscous heat-conducting perfect polytropic gas in unbounded domains. The temperature is proved to be bounded from below and above, independent of both time and space. Moreover, it is shown that the global solution is asymptotically stable as time tends to infinity. Note that the initial data can be arbitrarily large. This result is proved by using elementary energy methods.

1. Introduction

The compressible Navier–Stokes system describing the one-dimensional motion of a viscous heat-conducting perfect polytropic gas can be written in the Lagrange variables in the following form (see [4,24]):

$$v_t = u_x, \tag{1.1}$$

$$u_t + P_x = \mu \left(\frac{u_x}{v}\right)_x, \tag{1.2}$$

$$\left(e + \frac{u^2}{2}\right)_t + (Pu)_x = \left(\kappa \frac{\theta_x}{v} + \mu \frac{u u_x}{v}\right)_x, \tag{1.3}$$

$$P = R\theta/v, \quad e = c_v\theta + \text{ const.}, \tag{1.4}$$

where $t > 0$ is time, $x \in \Omega \subset \mathbb{R} = (-\infty, \infty)$ denotes the Lagrange mass coordinate, the unknown functions $v > 0$, u, $\theta > 0$, $e > 0$, and P are, respectively,

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the specific volume of the gas, fluid velocity, internal energy, absolute temperature, and pressure, μ is the viscosity coefficient, κ is the heat conductivity one, $R > 0$ is the gas constant, and c_v is heat capacity at constant volume. We assume that μ, κ, and c_v are positive constants.

The system (1.1)–(1.4) is supplemented with the initial condition

$$(v(x,0), u(x,0), \theta(x,0)) = (v_0(x), u_0(x), \theta_0(x)), \quad x \in \Omega, \tag{1.5}$$

and three types of far-field and boundary conditions:

(1) Cauchy problem

$$\Omega = \mathbb{R}, \lim_{|x|\to\infty} (v(x,t), u(x,t), \theta(x,t)) = (1, 0, 1), \quad t > 0; \tag{1.6}$$

(2) boundary and far-field conditions for $\Omega = (0, \infty)$,

$$u(0,t) = 0, \ \theta_x(0,t) = 0, \ \lim_{x\to\infty} (v(x,t), u(x,t), \theta(x,t)) = (1, 0, 1), \quad t>0; \tag{1.7}$$

(3) boundary and far-field conditions for $\Omega = (0, \infty)$,

$$u(0,t) = 0, \ \theta(0,t) = 1, \ \lim_{x\to\infty} (v(x,t), u(x,t), \theta(x,t)) = (1, 0, 1), \quad t>0. \tag{1.8}$$

Kanel [11] considered the Cauchy problem of the model system of equations (1.1) (1.2) with $P = Rv^{-\gamma} (\gamma > 0)$ and obtained both the existence and the large-time asymptotic behavior of the global solutions for large initial data. For system (1.1)–(1.4), Kazhikhov and Shelukhin [16] first obtained the global existence of solutions in bounded domains for large initial data. From that time, significant progress has been made on the mathematical aspect of the initial and initial boundary value problems. For initial boundary value problems in bounded domains, the existence and uniqueness of global (generalized) solutions and the regularity have been known. Moreover, the global solution is asymptotically stable as time tends to infinity; see [1–3,18–21,23] among others. For the Cauchy problem (1.1)–(1.6) and the initial boundary value problems (1.1)–(1.5), (1.7) and (1.1)–(1.5), (1.8) (in unbounded domains), Kazhikhov [15] (also cf. [3,7]) proved the following:

Lemma 1.1. *Assume that the initial data* (v_0, u_0, θ_0) *satisfy*

$$v_0 - 1, u_0, \theta_0 - 1 \in H^1(\Omega), \quad \inf_{x\in\Omega} v_0(x) > 0, \ \inf_{x\in\Omega} \theta_0(x) > 0, \tag{1.9}$$

and are compatible with (1.7), (1.8). *Then there exists a unique global* (*large*) *generalized solution* (v, u, θ) *with positive* $v(x,t)$ *and* $\theta(x,t)$ *to* (1.1)–(1.6), *or* (1.1)–(1.5), (1.7), *or* (1.1)–(1.5), (1.8) *satisfying that for any* $T > 0$,

$$\begin{cases} v-1, \ u, \ \theta - 1 \in L^\infty(0,T; H^1(\Omega)), \ \ v_t \in L^\infty(0,T; L^2(\Omega)), \\ u_t, \ \theta_t, \ v_{xt}, \ u_{xx}, \ \theta_{xx} \ \in L^2(0,T; L^2(\Omega)). \end{cases} \tag{1.10}$$

The asymptotic behavior as $t \to \infty$ of the solution has been studied under some smallness conditions on the initial data; see [5,8,12,14,17,22,23] and the references therein. However, there are few results on the large-time behavior of the solution in the case of large data. Jiang [9,10] first obtained some interesting results on the large-time behavior of solutions for large initial data by proving that the specific volume is pointwise bounded from below and above independent of both time and space, and that for all $t \geqq 0$ the temperature is bounded from below and above locally in x. In particular, Jiang [9,10] showed:

Lemma 1.2. [9,10] *Under the conditions of Lemma 1.1, let (v, u, θ) be a generalized solution to (1.1)–(1.6), or (1.1)–(1.5), (1.7), or (1.1)–(1.5), (1.8) satisfying (1.10) for any $T > 0$. Then there exists a positive constant C_1 depending only on $\mu, \kappa, R, c_v, \|(v_0 - 1, u_0, \theta_0 - 1)\|_{H^1(\Omega)}, \inf_{x\in\Omega} v_0(x)$, and $\inf_{x\in\Omega}\theta_0(x)$ such that*

$$C_1^{-1} \leqq v(x,t) \leqq C_1, \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,\infty). \tag{1.11}$$

From then on, for large initial data, whether the temperature is pointwise bounded from below and above independent of both time and space or not remains completely open. This is an interesting problem partially because it is the key to studying the large-time dynamical behavior of the global generalized solutions to (1.1)–(1.6), (1.1)–(1.5), (1.7), and (1.1)–(1.5), (1.8). In this paper, we will give a positive answer and further prove that the global solution is asymptotically stable as time tends to infinity for large initial data. Our main result is as follows:

Theorem 1.1. *Under the conditions of Lemma 1.1, let (v, u, θ) be the (unique) generalized solution to (1.1)–(1.6), or (1.1)–(1.5), (1.7), or (1.1)–(1.5), (1.8) satisfying (1.10) for any $T > 0$. Then there exists a positive constant C_0 depending only on $\mu, \kappa, R, c_v, \|(v_0 - 1, u_0, \theta_0 - 1)\|_{H^1(\Omega)}, \inf_{x\in\Omega} v_0(x)$, and $\inf_{x\in\Omega}\theta_0(x)$ such that*

$$C_0^{-1} \leqq \theta(x,t) \leqq C_0, \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,\infty), \tag{1.12}$$

$$\sup_{0\leqq t<\infty} \|(v-1, u, \theta-1)\|_{H^1(\Omega)} + \int_0^\infty \left(\|v_x\|_{L^2(\Omega)}^2 + \|(u_x, \theta_x)\|_{H^1(\Omega)}^2\right) \mathrm{d}t \leqq C_0. \tag{1.13}$$

Moreover, the following large-time behavior holds

$$\lim_{t\to\infty} \left(\|(v-1, u, \theta-1)(t)\|_{L^p(\Omega)} + \|(v_x, u_x, \theta_x)(t)\|_{L^2(\Omega)}\right) = 0, \tag{1.14}$$

for any $p \in (2, \infty]$.

Remark 1.1. In Theorem 1.1, we only assume that the initial data satisfy the conditions which are needed for the global existence of generalized solutions (see Lemma 1.1). Therefore, our results greatly improve the previous ones due to [5,8,12,14,17,22,23] where some additional smallness conditions on the initial data are needed.

Remark 1.2. For large initial data, Theorem 1.1 shows that the temperature is bounded from below and above independent of both time and space and that the global solution converges to the constant steady state uniformly with respect to the spatial variable as time goes to infinity. Therefore, our results improve those due to Jiang [9,10] where he proved that the temperature is uniformly (in time) bounded from below and above locally in x and that global solutions are convergent locally in space as time goes to infinity.

We now make some comments on the analysis of this paper. The key step to studying the large-time behavior of the global generalized solutions is to get the L^2-norm (in both space and time) bound of θ_x (see (2.3)). In fact, (2.3) has also been obtained under some additional smallness conditions on the initial data; see [8,12,14,17,22,23] and the references therein. However, in our case, since the initial data may be arbitrarily large, to obtain (2.3), some new ideas are needed. The key observations are as follows: the combination of the standard energetic estimate (see (2.1)) with (1.11) shows that for $\Omega_2(t) \triangleq \{x \in \Omega | \, \theta(x,t) > 2\}$,

$$\int_0^\infty \int_{\Omega\setminus\Omega_2(t)} \theta_x^2 \mathrm{d}x\mathrm{d}t$$

is bounded. Hence, it suffices to estimate the integral

$$A \triangleq \int_0^\infty \int_{\Omega_2(t)} \theta_x^2 \mathrm{d}x\mathrm{d}t.$$

In fact, to estimate A, we multiply the equation for the temperature by $(\theta - 2)_+$ (see (2.5)). Then, to control the most difficult term appearing in (2.5), motivated by [6], we multiply the equation for the velocity by $2u(\theta-2)_+$ (see (2.6)). After some careful analysis on the integration by parts over $\Omega_2(t)$ (see (2.12)) and multiplying the equation for the velocity by u^3, we finally find that A can be controlled by (see (2.21))

$$\int_0^\infty \sup_{x\in\Omega} (\theta - 3/2)_+^2(x,t)\mathrm{d}t,$$

which in fact is bounded by $C(\varepsilon) + \varepsilon A$ for any $\varepsilon > 0$ (see (2.22)). These are the key to the proof of (2.3), and once that is obtained, the proof follows in the same way as in [8,12,14,17,22,23]. The whole procedure will be carried out in the next section.

2. Proof of Theorem 1.1

We begin with the standard energetic estimate, which is motivated by the second law of thermodynamics and embodies the dissipative effects of viscosity and thermal diffusion.

Lemma 2.1. *It holds that*

$$\begin{aligned}&\sup_{0\leqq t<\infty}\int_\Omega\left(\frac{1}{2}u^2+R(v-\ln v-1)+c_v(\theta-\ln\theta-1)\right)\\&+\mu\int_0^\infty\int_\Omega\frac{u_x^2}{v\theta}+\kappa\int_0^\infty\int_\Omega\frac{\theta_x^2}{v\theta^2}\leqq C,\end{aligned}\tag{2.1}$$

where (and in what follows) C and $C_i\,(i=2,\ldots,5)$ denote generic positive constants depending only on $\mu,\kappa,R,c_v,\|(v_0-1,u_0,\theta_0-1)\|_{H^1(\Omega)}$, $\inf_{x\in\Omega}v_0(x)$, and $\inf_{x\in\Omega}\theta_0(x)$.

Proof. Using (1.1), (1.2), and (1.4), we rewrite (1.3) as

$$c_v\theta_t+R\frac{\theta}{v}u_x=\kappa\left(\frac{\theta_x}{v}\right)_x+\mu\frac{u_x^2}{v}.\tag{2.2}$$

Multiplying (1.1) by $R(1-v^{-1})$, (1.2) by u, (2.2) by $1-\theta^{-1}$, and adding them altogether, we obtain

$$\begin{aligned}&(u^2/2+R(v-\ln v-1)+c_v(\theta-\ln\theta-1))_t+\mu\frac{u_x^2}{v\theta}+\kappa\frac{\theta_x^2}{v\theta^2}\\&=\left(\frac{\mu uu_x}{v}-\frac{Ru\theta}{v}\right)_x+Ru_x+\kappa\left((1-\theta^{-1})\frac{\theta_x}{v}\right)_x,\end{aligned}$$

which together with (1.6) or (1.7) or (1.8) yields (2.1). We finish the proof of Lemma 2.1.

Next, we derive the following L^2-norm (in both space and time) bound of θu_x and θ_x, which is essential in our analysis.

Lemma 2.2. *There exists some positive constant C such that for any $T>0$,*

$$\sup_{0\leqq t\leqq T}\int_\Omega\left[(\theta-1)^2+u^4\right]+\int_0^T\int_\Omega\left[(1+\theta+u^2)u_x^2+\theta_x^2\right]\leqq C.\tag{2.3}$$

Proof. The proof of Lemma 2.2 will be divided into three steps.

Step 1. First, for $t\geqq 0$ and $a>1$, denoting

$$\Omega_a(t)\triangleq\{x\in\Omega|\,\theta(x,t)>a\},$$

we derive from (2.1) that

$$\sup_{0\leqq t<\infty}\int_{\Omega_a(t)}\theta\leqq C(a)\sup_{0\leqq t<\infty}\int_\Omega(\theta-\ln\theta-1)\leqq C(a).\tag{2.4}$$

Next, integrating (2.2) multiplied by $(\theta-2)_+ \triangleq \max\{\theta-2, 0\}$ over $\Omega\times(0,T)$ gives

$$\begin{aligned}&\frac{c_v}{2}\int_\Omega(\theta-2)_+^2+\kappa\int_0^T\int_{\Omega_2(t)}\frac{\theta_x^2}{v}\\&=\frac{c_v}{2}\int_\Omega(\theta_0-2)_+^2-R\int_0^T\int_\Omega\frac{\theta}{v}u_x(\theta-2)_++\mu\int_0^T\int_\Omega\frac{u_x^2}{v}(\theta-2)_+.\end{aligned}\tag{2.5}$$

To estimate the last term on the right hand side of (2.5), motivated by [6], we multiply (1.2) by $2u(\theta-2)_+$ and integrate the resulting equality over $\Omega\times(0,T)$ to get

$$\begin{aligned}&\int_\Omega u^2(\theta-2)_++2\mu\int_0^T\int_\Omega\frac{u_x^2}{v}(\theta-2)_+\\&=\int_\Omega u_0^2(\theta_0-2)_++2R\int_0^T\int_\Omega\frac{\theta}{v}u_x(\theta-2)_++2R\int_0^T\int_{\Omega_2(t)}\frac{\theta}{v}u\theta_x\\&\quad-2\mu\int_0^T\int_{\Omega_2(t)}\frac{u_x}{v}u\theta_x+\int_0^T\int_{\Omega_2(t)}u^2\theta_t.\end{aligned}\tag{2.6}$$

Adding (2.6) to (2.5), we obtain, after using (2.2), that

$$\begin{aligned}&\int_\Omega\left[\frac{c_v}{2}(\theta-2)_+^2+u^2(\theta-2)_+\right]+\kappa\int_0^T\int_{\Omega_2(t)}\frac{\theta_x^2}{v}+\mu\int_0^T\int_\Omega\frac{u_x^2}{v}(\theta-2)_+\\&=\int_\Omega\left[\frac{c_v}{2}(\theta_0-2)_+^2+u_0^2(\theta_0-2)_+\right]+R\int_0^T\int_\Omega\frac{\theta}{v}u_x(\theta-2)_+\\&\quad+2R\int_0^T\int_{\Omega_2(t)}\frac{\theta}{v}u\theta_x-2\mu\int_0^T\int_{\Omega_2(t)}\frac{u_x}{v}u\theta_x\\&\quad+\frac{1}{c_v}\int_0^T\int_{\Omega_2(t)}u^2\left(\mu\frac{u_x^2}{v}-R\frac{\theta}{v}u_x\right)+\frac{\kappa}{c_v}\int_0^T\int_{\Omega_2(t)}u^2\left(\frac{\theta_x}{v}\right)_x\\&\triangleq\int_\Omega\left[\frac{c_v}{2}(\theta_0-2)_+^2+u_0^2(\theta_0-2)_+\right]+\sum_{i=1}^5 I_i.\end{aligned}\tag{2.7}$$

We estimate each $I_i\,(i=1,\dots,5)$ in the following way.

First, it follows from Cauchy's inequality and (1.11) that

$$\begin{aligned}|I_1|&=R\left|\int_0^T\int_\Omega\frac{\theta}{v}u_x(\theta-2)_+\right|\\&\leqq\frac{\mu}{2}\int_0^T\int_\Omega\frac{u_x^2}{v}(\theta-2)_++C\int_0^T\int_\Omega\theta^2(\theta-2)_+\\&\leqq\frac{\mu}{2}\int_0^T\int_\Omega\frac{u_x^2}{v}(\theta-2)_++C\int_0^T\int_\Omega\theta(\theta-3/2)_+^2\\&\leqq\frac{\mu}{2}\int_0^T\int_\Omega\frac{u_x^2}{v}(\theta-2)_++C\int_0^T\sup_{x\in\Omega}(\theta-3/2)_+^2(x,t),\end{aligned}\tag{2.8}$$

where in the last inequality we have used (2.4).

Next, Cauchy's inequality and (1.11) yield that for any $\varepsilon > 0$,

$$
\begin{aligned}
|I_2| + |I_3| &= 2R\left|\int_0^T \int_{\Omega_2(t)} \frac{\theta}{v} u\theta_x\right| + 2\mu\left|\int_0^T \int_{\Omega_2(t)} \frac{u_x}{v} u\theta_x\right| \\
&\leqq \varepsilon \int_0^T \int_\Omega \theta_x^2 + C(\varepsilon)\int_0^T \int_{\Omega_2(t)} u^2\theta^2 + C(\varepsilon)\int_0^T \int_\Omega u^2u_x^2 \\
&\leqq \varepsilon \int_0^T \int_\Omega \theta_x^2 + C(\varepsilon)\int_0^T \sup_{x\in\Omega}(\theta-3/2)_+^2(x,t) + C(\varepsilon)\int_0^T \int_\Omega u^2u_x^2,
\end{aligned}
\tag{2.9}
$$

where in the last inequality we have used

$$
\int_0^T \int_{\Omega_2(t)} u^2\theta^2 \leqq 16\int_0^T \int_\Omega u^2(\theta - 3/2)_+^2 \leqq C\int_0^T \sup_{x\in\Omega}(\theta - 3/2)_+^2(x,t),
\tag{2.10}
$$

due to (2.1).

Then, it follows from Cauchy's inequality and (2.10) that

$$
|I_4| \leqq C\int_0^T \int_\Omega u^2u_x^2 + C\int_0^T \sup_{x\in\Omega}(\theta - 3/2)_+^2(x,t).
\tag{2.11}
$$

Finally, for $\eta > 0$ and

$$
\varphi_\eta(\theta) \triangleq \begin{cases} 1, & \theta - 2 > \eta, \\ (\theta - 2)/\eta, & 0 \leqq \theta - 2 \leqq \eta, \\ 0, & \theta - 2 \leqq 0, \end{cases}
$$

Lebesgue's dominated convergence theorem shows that for any $\varepsilon > 0$,

$$
\begin{aligned}
I_5 &= \frac{\kappa}{c_v}\lim_{\eta\to 0+}\int_0^T \int_\Omega \varphi_\eta(\theta)u^2\left(\frac{\theta_x}{v}\right)_x \\
&= \frac{\kappa}{c_v}\lim_{\eta\to 0+}\int_0^T \int_\Omega \left(-2\varphi_\eta(\theta)uu_x\frac{\theta_x}{v} - \varphi_\eta'(\theta)u^2\frac{\theta_x^2}{v}\right) \\
&\leqq -\frac{2\kappa}{c_v}\int_0^T \int_{\Omega_2(t)} uu_x\frac{\theta_x}{v} \\
&\leqq \varepsilon\int_0^T \int_\Omega \theta_x^2 + C(\varepsilon)\int_0^T \int_\Omega u^2u_x^2,
\end{aligned}
\tag{2.12}
$$

where in the third inequality we have used $\varphi_\eta'(\theta) \geqq 0$.

Noticing that

$$\begin{aligned}
&\int_0^T \int_\Omega \left(u_x^2\theta + \theta_x^2\right) \\
&= \int_0^T \int_{\Omega_3(t)} \left(u_x^2\theta + \theta_x^2\right) + \int_0^T \int_{\Omega\backslash\Omega_3(t)} \left(u_x^2\theta + \theta_x^2\right) \\
&\leqq 3 \int_0^T \int_{\Omega_3(t)} \left(u_x^2(\theta-2)_+ + \theta_x^2\right) + C \int_0^T \int_{\Omega\backslash\Omega_3(t)} \left(\mu \frac{u_x^2}{v\theta} + \kappa \frac{\theta_x^2}{v\theta^2}\right) \\
&\leqq C \int_0^T \int_{\Omega_2(t)} \frac{1}{v} \left(\mu u_x^2(\theta-2)_+ + \kappa\theta_x^2\right) + C,
\end{aligned}$$

where in the last inequality we have used $\Omega_3(t) \subset \Omega_2(t)$, (1.11), and (2.1), we substitute (2.8), (2.9), (2.11), and (2.12) into (2.7) and choose ε suitably small to obtain

$$\begin{aligned}
&\sup_{0\leqq t\leqq T} \int_\Omega (\theta-2)_+^2 + \int_0^T \int_\Omega \left(u_x^2\theta + \theta_x^2\right) \\
&\qquad \leqq C + C \int_0^T \sup_{x\in\Omega} (\theta - 3/2)_+^2(x,t) + C_2 \int_0^T \int_\Omega u^2 u_x^2. \qquad (2.13)
\end{aligned}$$

Step 2. To estimate the last term on the right hand side of (2.13), we multiply (1.2) by u^3 and integrate the resulting equality over $\Omega \times (0, T)$ to get

$$\begin{aligned}
&\frac{1}{4} \int_\Omega u^4 + 3\mu \int_0^T \int_\Omega \frac{u^2 u_x^2}{v} \\
&= \frac{1}{4} \int_\Omega u_0^4 + 3R \int_0^T \int_\Omega \frac{1-v}{v} u^2 u_x + 3R \int_0^T \int_{\Omega\backslash\Omega_2(t)} \frac{\theta - 1}{v} u^2 u_x \\
&\quad +3R \int_0^T \int_{\Omega_2(t)} \frac{\theta-1}{v} u^2 u_x \triangleq \frac{1}{4} \int_\Omega u_0^4 + \sum_{i=1}^3 J_i. \qquad (2.14)
\end{aligned}$$

It follows from (2.1) and (1.11) that for any $\alpha \in [2, 3]$,

$$\begin{aligned}
&\sup_{0\leqq t\leqq T} \int_\Omega (v-1)^2 + \sup_{0\leqq t\leqq T} \int_{\Omega\backslash\Omega_\alpha(t)} (\theta-1)^2 \\
&\qquad \leqq C \sup_{0\leqq t\leqq T} \int_\Omega (v - \ln v - 1) + C \sup_{0\leqq t\leqq T} \int_\Omega (\theta - \ln\theta - 1) \leqq C, \quad (2.15)
\end{aligned}$$

which together with Holder's inequality yields that

$$\begin{aligned}
|J_1| + |J_2| &\leqq C \int_0^T \sup_{x\in\Omega} u^2(x,t) \|u_x\|_{L^2(\Omega)} \left(\int_\Omega (v-1)^2 + \int_{\Omega\backslash\Omega_2(t)} (\theta-1)^2\right)^{1/2} \\
&\leqq C \int_0^T \int_\Omega u_x^2, \qquad (2.16)
\end{aligned}$$

where in the second inequality we have used (2.1) and the following simple fact that for any $w \in H^1(\Omega)$,

$$\sup_{x\in\Omega} w^2(x) = \sup_{x\in\Omega}\left(-2\int_x^{\infty} w(y)w_x(y)dy\right) \leqq 2\|w\|_{L^2(\Omega)}\|w_x\|_{L^2(\Omega)}. \tag{2.17}$$

The combination of Cauchy's inequality with (2.10) leads to

$$\begin{aligned}|J_3| &\leqq \mu\int_0^T\int_{\Omega_2(t)}\frac{u^2u_x^2}{v} + C\int_0^T\int_{\Omega_2(t)}\theta^2u^2\\ &\leqq \mu\int_0^T\int_\Omega\frac{u^2u_x^2}{v} + C\int_0^T\sup_{x\in\Omega}(\theta-3/2)_+^2(x,t).\end{aligned} \tag{2.18}$$

Putting (2.16) and (2.18) into (2.14) gives

$$\begin{aligned}&\sup_{0\leqq t\leqq T}\int_\Omega u^4 + \int_0^T\int_\Omega u^2u_x^2\\ &\leqq C + C\int_0^T\int_\Omega u_x^2 + C\int_0^T\sup_{x\in\Omega}(\theta-3/2)_+^2(x,t)\\ &\leqq C(\delta) + C\delta\int_0^T\int_\Omega\theta u_x^2 + C\int_0^T\sup_{x\in\Omega}(\theta-3/2)_+^2(x,t),\end{aligned} \tag{2.19}$$

where in the last inequality we have used the following simple fact that for any $\delta > 0$,

$$2\int_0^T\int_\Omega u_x^2 \leqq \delta\int_0^T\int_\Omega\theta u_x^2 + \delta^{-1}\int_0^T\int_\Omega\theta^{-1}u_x^2 \leqq \delta\int_0^T\int_\Omega\theta u_x^2 + C(\delta), \tag{2.20}$$

due to Cauchy's inequality, (2.1), and (1.11).

Adding (2.19) multiplied by $C_2 + 1$ to (2.13), then choosing δ suitably small, we have

$$\begin{aligned}&\sup_{0\leqq t\leqq T}\int_\Omega\left[(\theta-2)_+^2 + u^4\right] + \int_0^T\int_\Omega\left[(\theta+u^2)u_x^2 + \theta_x^2\right]\\ &\leqq C + C\int_0^T\sup_{x\in\Omega}(\theta-3/2)_+^2(x,t).\end{aligned} \tag{2.21}$$

Step 3. It remains to estimate the last term on the right hand side of (2.21). In fact, standard calculations yield that for any $\varepsilon > 0$,

$$\begin{aligned}\int_0^T \sup_{x\in\Omega}(\theta(x,t)-3/2)_+^2 &= \int_0^T \sup_{x\in\Omega}\left(\int_x^\infty \partial_x(\theta-3/2)_+\right)^2 \\ &\leqq \int_0^T\left(\int_{\Omega_{3/2}(t)}|\theta_x|\right)^2 \\ &\leqq \int_0^T\left(\int_{\Omega_{3/2}(t)}\frac{\theta_x^2}{\theta}\int_{\Omega_{3/2}(t)}\theta\right) \\ &\leqq C\int_0^T\int_\Omega \frac{\theta_x^2}{\theta} \\ &\leqq C(\varepsilon)\int_0^T\int_\Omega\frac{\theta_x^2}{v\theta^2}+\varepsilon\int_0^T\int_\Omega\theta_x^2 \\ &\leqq C(\varepsilon)+\varepsilon\int_0^T\int_\Omega\theta_x^2, \end{aligned}\tag{2.22}$$

where in the fourth and last inequalities we have used (2.4) and (2.1) respectively. Putting (2.22) into (2.21) and choosing ε suitably small leads to

$$\sup_{0\leqq t\leqq T}\int_\Omega\left[(\theta-2)_+^2+u^4\right]+\int_0^T\int_\Omega\left[(\theta+u^2)u_x^2+\theta_x^2\right]\leqq C,$$

which combined with (2.15) and (2.20) immediately gives (2.3). The proof of Lemma 2.2 is completed.

Next, we will derive some necessary uniform estimates on the spatial derivatives of the global generalized solution (v,u,θ).

Lemma 2.3. *There exists some positive constant C such that for any $T>0$,*

$$\sup_{0\leqq t\leqq T}\int_\Omega\left(v_x^2+u_x^2+\theta_x^2\right)+\int_0^T\int_\Omega\left(\theta v_x^2+u_{xx}^2+\theta_{xx}^2\right)\leqq C. \tag{2.23}$$

Proof. First, integrating (1.2) multiplied by $\frac{v_x}{v}$ over Ω, we obtain after using (1.1) that

$$\begin{aligned}\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{v_x^2}{v^2} &= R\int_\Omega\left(\frac{\theta}{v}\right)_x\frac{v_x}{v}+\int_\Omega u_t\frac{v_x}{v} \\ &= R\int_\Omega\frac{\theta_x v_x}{v^2}-R\int_\Omega\frac{\theta v_x^2}{v^3}+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\frac{v_x}{v}+\int_\Omega u_x\frac{v_t}{v} \\ &\leqq C\int_\Omega\frac{\theta_x^2}{v\theta}-\frac{R}{2}\int_\Omega\frac{\theta v_x^2}{v^3}+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\frac{v_x}{v}+\int_\Omega\frac{u_x^2}{v},\end{aligned}$$

which together with Cauchy's inequality, (1.11), (2.1), (2.3), and (2.22) gives

$$\sup_{0\leqq t\leqq T}\int_\Omega v_x^2+\int_0^T\int_\Omega\theta v_x^2\leqq C. \tag{2.24}$$

Next, integrating (1.2) multiplied by u_{xx} over Ω yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_x^2 + \mu\int_\Omega \frac{u_{xx}^2}{v} = R\int_\Omega \left(\frac{\theta}{v}\right)_x u_{xx} + \mu\int_\Omega \frac{u_x}{v^2}v_x u_{xx}. \tag{2.25}$$

It follows from (2.3), (2.24), and (2.17) that

$$\begin{aligned}
&\int_0^T \left| R\int_\Omega \left(\frac{\theta}{v}\right)_x u_{xx} + \mu\int_\Omega \frac{u_x}{v^2}v_x u_{xx}\right| \\
&\leqq \frac{\mu}{4}\int_0^T\int_\Omega \frac{u_{xx}^2}{v} + C\int_0^T\int_\Omega \left(\theta^2 v_x^2 + \theta_x^2 + u_x^2 v_x^2\right) \\
&\leqq C + \frac{\mu}{4}\int_0^T\int_\Omega \frac{u_{xx}^2}{v} + C\max_{\overline{\Omega}\times[0,T]}\theta\int_0^T\int_\Omega \theta v_x^2 + C\int_0^T \|u_x(\cdot,t)\|_{L^\infty(\Omega)}^2 \\
&\leqq C + \frac{\mu}{2}\int_0^T\int_\Omega \frac{u_{xx}^2}{v} + C\max_{\overline{\Omega}\times[0,T]}\theta,
\end{aligned} \tag{2.26}$$

which combined with (2.25) shows

$$\sup_{0\leqq t\leqq T}\int_\Omega u_x^2 + \int_0^T\int_\Omega u_{xx}^2 \leqq C + C\max_{\overline{\Omega}\times[0,T]}\theta. \tag{2.27}$$

Next, integrating (2.2) multiplied by θ_{xx} over Ω leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta_x^2 + \kappa\int_\Omega \frac{\theta_{xx}^2}{v} = \kappa\int_\Omega \frac{\theta_x v_x}{v^2}\theta_{xx} - \mu\int_\Omega \frac{u_x^2}{v}\theta_{xx} + \int_\Omega R\frac{\theta}{v}u_x\theta_{xx}. \tag{2.28}$$

Cauchy's inequality and (2.17) give

$$\begin{aligned}
&\int_0^T \left|\kappa\int_\Omega \frac{\theta_x v_x}{v^2}\theta_{xx} - \mu\int_\Omega \frac{u_x^2}{v}\theta_{xx} + \int_\Omega R\frac{\theta}{v}u_x\theta_{xx}\right| \\
&\leqq C\int_0^T \|\theta_{xx}\|_{L^2(\Omega)}\|\theta_x\|_{L^\infty(\Omega)}\|v_x\|_{L^2(\Omega)} \\
&\quad + C\int_0^T \|\theta_{xx}\|_{L^2(\Omega)}\left(\|u_x\|_{L^\infty(\Omega)}\|u_x\|_{L^2(\Omega)} + \|\theta\|_{L^\infty(\Omega)}\|u_x\|_{L^2(\Omega)}\right) \\
&\leqq C\int_0^T \|\theta_{xx}\|_{L^2(\Omega)}\|\theta_{xx}\|_{L^2(\Omega)}^{1/2}\|\theta_x\|_{L^2(\Omega)}^{1/2}\|v_x\|_{L^2(\Omega)} \\
&\quad + C\int_0^T \|\theta_{xx}\|_{L^2(\Omega)}\|u_x\|_{H^1(\Omega)}\left(\|u_x\|_{L^2(\Omega)} + \|\theta\|_{L^\infty(\Omega)}\right) \\
&\leqq \frac{\kappa}{4}\int_0^T\int_\Omega \frac{\theta_{xx}^2}{v} + C + C\max_{\overline{\Omega}\times[0,T]}\theta^3,
\end{aligned} \tag{2.29}$$

where in the last inequality we have used (2.24), (2.3), and (2.27). Integrating (2.28) over $(0,T)$, we obtain after using (2.29) that

$$\sup_{0\leqq t\leqq T}\int_\Omega \theta_x^2 + \int_0^T\int_\Omega \theta_{xx}^2 \leqq C + C\max_{\overline{\Omega}\times[0,T]}\theta^3. \tag{2.30}$$

Finally, it follows from (2.17) and (2.3) that for all $t \geqq 0$,

$$\begin{aligned} \|(\theta-1)(\cdot,t)\|_{C(\overline{\Omega})}^2 &\leqq C\|(\theta-1)(\cdot,t)\|_{L^2(\Omega)}\|\theta_x(\cdot,t)\|_{L^2(\Omega)} \\ &\leqq C\|\theta_x(\cdot,t)\|_{L^2(\Omega)}, \end{aligned} \tag{2.31}$$

which combined with (2.30) yields

$$\max_{\overline{\Omega}\times[0,T]}(\theta-1)^2 \leqq C + C\max_{\overline{\Omega}\times[0,T]}\theta^{3/2}.$$

This implies that there exists a positive constant C_3 such that for any $(x,t) \in \overline{\Omega}\times[0,T]$,

$$\theta(x,t) \leqq C_3, \tag{2.32}$$

which together with (2.27), (2.30), and (2.24) gives (2.23) and finishes the proof of Lemma 2.3.

Finally, the following large-time behavior of global generalized solutions together with Lemmas 2.1-2.3 finishes the proof of Theorem 1.1.

Lemma 2.4. *It holds that*

$$\lim_{t\to\infty}\left(\|(v-1,u,\theta-1)(t)\|_{L^p(\Omega)} + \|(v_x,u_x,\theta_x)(t)\|_{L^2(\Omega)}\right) = 0, \tag{2.33}$$

for any $p \in (2,\infty]$. *Moreover, there exists a positive constant* C_4 *such that for all* $(x,t) \in \overline{\Omega}\times[0,\infty)$

$$C_4^{-1} \leqq \theta(x,t) \leqq C_4. \tag{2.34}$$

Proof. It follows from (2.3), (2.25), (2.26), (2.28), (2.29), (2.32), and (2.23) that

$$\begin{aligned} &\int_0^\infty\left(\|u_x(\cdot,t)\|_{L^2(\Omega)}^2 + \left|\frac{\mathrm{d}}{\mathrm{d}t}\|u_x(\cdot,t)\|_{L^2(\Omega)}^2\right|\right)\mathrm{d}t \\ &\quad + \int_0^\infty\left(\|\theta_x(\cdot,t)\|_{L^2(\Omega)}^2 + \left|\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_x(\cdot,t)\|_{L^2(\Omega)}^2\right|\right)\mathrm{d}t \leqq C, \end{aligned}$$

which directly gives

$$\lim_{t\to\infty}\left(\|u_x(\cdot,t)\|_{L^2(\Omega)} + \|\theta_x(\cdot,t)\|_{L^2(\Omega)}\right) = 0. \tag{2.35}$$

This, combined with (2.31), shows

$$\lim_{t\to\infty}\|\theta(\cdot,t)-1\|_{C(\overline{\Omega})} = 0.$$

Hence, there exists some $T_0 > 0$ such that for all $(x,t) \in \overline{\Omega}\times[T_0,\infty)$

$$1/2 \leqq \theta(x,t) \leqq 3/2, \tag{2.36}$$

which, along with (2.23), leads to

$$\int_{T_0}^\infty \|v_x(\cdot,t)\|_{L^2(\Omega)}^2 \leqq C. \tag{2.37}$$

This, combined with (1.1) and (2.23), yields

$$\int_{T_0}^{\infty}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|v_x(\cdot,t)\|_{L^2(\Omega)}^2\right| = 2\int_{T_0}^{\infty}\left|\int_\Omega u_{xx}v_x\right| \leqq \int_{T_0}^{\infty}\int_\Omega u_{xx}^2 + \int_{T_0}^{\infty}\int_\Omega v_x^2 \leqq C,$$

which together with (2.37) implies

$$\lim_{t\to\infty}\|v_x(\cdot,t)\|_{L^2(\Omega)} = 0. \tag{2.38}$$

The combination of (2.38), (2.35), (2.15), (2.1), and (2.3) directly yields (2.33).

Finally, it follows from the proof in [3,16] that there exists some constant $C_5 > 2$ such that for all $(x,t) \in \overline{\Omega} \times [0,\infty)$

$$C_5^{-1}e^{-C_5 t} \leqq \theta(x,t),$$

which together with (2.36) implies that for all $(x,t) \in \overline{\Omega} \times [0,\infty)$

$$C_5^{-1}e^{-C_5 T_0} \leqq \theta(x,t).$$

This combined with (2.32) gives (2.34) provided we choose $C_4 \triangleq \max\{C_3, C_5 e^{C_5 T_0}\}$. The proof of Lemma 2.4 is finished. □

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