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Global Well-Posedness in Spatially Critical Besov Space for the Boltzmann Equation

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Abstract

The unique global strong solution in the Chemin–Lerner type space to the Cauchy problem on the Boltzmann equation for hard potentials is constructed in a perturbation framework. Such a solution space is of critical regularity with respect to the spatial variable, and it can capture the intrinsic properties of the Boltzmann equation. For the proof of global well-posedness, we develop some new estimates on the nonlinear collision term through the Littlewood–Paley theory.

Contents

1. Introduction

There have been extensive studies of the global well-posedness for the Cauchy problem on the Boltzmann equation. Basically, two kinds of global solutions can be established in terms of different approaches. One kind is the renormalized solution by the weak stability method [8], where initial data can be of large size and of no regularity. The uniqueness for such solutions still remains open. The other kind is the perturbative solution, by either the spectrum method [22,24,28,29,31] or the energy method [17,18,20,21], where initial data is assumed to be sufficiently close to Maxwellians. In general, solutions exist uniquely in the perturbation framework.

It is a fundamental problem in the theory of the Boltzmann equation to find a function space with minimal regularity for the global existence and uniqueness of solutions. This paper aims at presenting a function space whose spatial variable belongs to the critical Besov space $B^{3/2}_{2,1}$ in dimension three. The motivation to consider function spaces with spatially critical regularity is inspired by their many existing applications in the study of the fluid dynamical equations [4,7,27], see also the recent work [26] on the general hyperbolic symmetric conservation laws with relaxations. Indeed it will be seen in the paper that the Boltzmann equation at the kinetic level shares a similar dissipative structure in the so-called critical Chemin–Lerner type space (cf. [6]) with those at the fluid level. Specifically, using the Littlewood–Paley theory, we establish the global well-posedness of solutions in such function spaces for the angular cutoff hard potentials. It remains an interesting and challenging problem to extend the current result to other situations, such as soft potentials [15,23], angular non cutoff [2,13] and the appearance of self-consistent force [11].

The Boltzmann equation in dimension three (cf. [5,32]), which is used to describe the time evolution of the unknown velocity distribution function $F = F(t, x, \xi) \geqq 0$ of particles with position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t \geqq 0$, reads

$$\partial_t F + \xi \cdot \nabla_x F = Q(F, F). \tag{1.1}$$

Initial data $F(0, x, \xi) = F_0(x, \xi)$ is given. $Q(\cdot, \cdot)$ is the bilinear Boltzmann collision operator, defined by

$$Q(F, H) = \int_{\mathbb{R}^3} \mathrm{d}\xi_* \int_{\mathbb{S}^2} \mathrm{d}\omega \, |\xi - \xi_*|^\gamma B_0(\theta)(F'_* H' - F_* H),$$

where

$$F'_* = F(t, x, \xi'_*), \quad H' = H(t, x, \xi'), \quad F_* = F(t, x, \xi_*), \quad H = H(t, x, \xi),$$

with

$$\xi' = \xi - ((\xi - \xi_*) \cdot \omega)\, \omega, \qquad \xi'_* = \xi_* + ((\xi - \xi_*) \cdot \omega)\, \omega,$$

and θ is given by $\cos\theta = \omega \cdot (\xi - \xi_*)/|\xi - \xi_*|$. The collision kernel $|\xi - \xi_*|^\gamma B_0(\theta)$ is determined by the interaction law between particles. Throughout this paper, we assume $0 \leqq \gamma \leqq 1$ and $0 \leqq B_0(\theta) \leqq C|\cos\theta|$ for a constant C, and this includes the hard potentials with angular cutoff as an example, cf. [12].

In the paper, we study the solution of the Boltzmann equation (1.1) around the global Maxwellian

$$\mu = \mu(\xi) = (2\pi)^{-3/2} \mathrm{e}^{-|\xi|^2/2},$$

which has been normalized to have zero bulk velocity and unit density and temperature. For this purpose, we set the perturbation $f = f(t, x, \xi)$ by $F = \mu + \mu^{1/2} f$. Then (1.1) can be reformulated as

$$\partial_t f + \xi \cdot \nabla_x f + Lf = \Gamma(f, f), \tag{1.2}$$

with initial data $f(0,x,\xi) = f_0(x,\xi)$ given by $F_0 = \mu + \mu^{1/2} f_0$. Here Lf, $\Gamma(f,f)$ are the linearized and nonlinear collision terms, respectively, and their precise expressions will be given later on. Recall that L is nonnegative-definite on L^2_ξ, and $\ker L$ is spanned by five elements $\sqrt{\mu}$, $\xi_i\sqrt{\mu}(1 \leqq i \leqq 3)$, and $|\xi|^2\sqrt{\mu}$ in L^2_ξ. For later use, define the macroscopic projection of $f(t,x,\xi)$ by

$$\mathbf{P}f = \{a(t,x) + \xi \cdot b(t,x) + (|\xi|^2 - 3)c(t,x)\}\sqrt{\mu}, \tag{1.3}$$

where for the notational brevity we have skipped the dependence of coefficient functions on f. Then, the function $f(t,x,\xi)$ can be decomposed as $f = \mathbf{P}f + \{\mathbf{I} - \mathbf{P}\}f$.

We now state the main result of the paper. The norms and other notations below will be made precise in the next section. We define the energy functional and energy dissipation rate, respectively, as

$$\mathcal{E}_T(f) \sim \|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi (B^{3/2}_x)}, \tag{1.4}$$

and

$$\mathcal{D}_T(f) = \|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{1/2}_x)} + \|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}. \tag{1.5}$$

Theorem 1.1. *There are $\varepsilon_0 > 0$, $C > 0$ such that if*

$$\|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)} \leqq \varepsilon_0,$$

then there exists a unique global strong solution $f(t,x,\xi)$ to the Boltzmann equation (1.2) with initial data $f(0,x,\xi) = f_0(x,\xi)$, satisfying

$$\mathcal{E}_T(f) + \mathcal{D}_T(f) \leqq C\|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)}, \tag{1.6}$$

for any $T > 0$. Moreover, if $F_0(x,\xi) = \mu + \mu^{1/2} f_0(x,\xi) \geqq 0$, then $F(t,x,\xi) = \mu + \mu^{1/2} f(t,x,\xi) \geqq 0$.

We now give a few comments on Theorem 1.1. Those function spaces appearing in the key inequality (1.6) are called the Chemin–Lerner type spaces. When the velocity variable is not taken into account, the usual Chemin–Lerner space was first introduced in [6] to study the existence of solutions to the incompressible Navier–Stokes equations in $\mathbb{R}^3$. To the best of our knowledge, Theorem 1.1 is the first result for the application of such a space to the well-posedness theory of the Cauchy problem on the Boltzmann equation. Moreover, noticing for $s = 3/2$ that the Besov space is $B^s_x = B^s_{2,1} \hookrightarrow L^\infty_x$, but the Sobolev space H^s_x is not embedded into L^∞_x, the regularity with respect to the x variable that we consider here is critical. In most of the previous work [2,10,13,15,20,24], the Sobolev space $H^s_x (s > 3/2)$ obeying the Banach algebra property is typically used for global well-posedness of strong or classical solutions. Therefore, Theorem 1.1 presents a global result not only in a larger class of function spaces but also in a space with spatially critical regularity. Note that so far it is unknown how to obtain any blow-up result in the space $L^2_\xi(H^s_x)$ with $s < 3/2$ for the Boltzmann equation in the perturbation framework, and thus the criticality of spatial regularity in such a function space is only restricted in the sense that the critical embedding $H_x^{3/2} \hookrightarrow L_x^\infty$ is false in three dimensions. Here, we would mention DANCHIN [7], where the homogeneous critical space of Besov type was used for the study of the well posedness of the Cauchy problem on the compressible Navier–Stokes equations near constant equilibrium states. In that work, the critical regularity in a spatial variable is closely linked to the scaling invariance of the equations under consideration, and the Sobolev imbedding $B_x^{3/2} \hookrightarrow L_x^\infty$ was also used to control the fluid density.

We would also point out another two kinds of applications of the Besov space to the Boltzmann equation. In fact, in ARSÉNIO and MASMOUDI [3], a new approach to velocity averaging lemmas in some Besov spaces is developed based on the dispersive property of the kinetic transport equation, and in SOHINGER and STRAIN [25], the optimal time decay rates in the whole space are investigated in the framework of [13] under the additional assumption that initial data belongs to a negative power Besov space $B^s_{2,\infty}$ for some $s < 0$ with respect to x variable.

In what follows let us recall, in a little bit of detail, some related works as far as the choice of different function spaces for the well-posedness of the Boltzmann equation near Maxwellians is concerned. The first global existence theorem for the mild solution is given by UKAI [28,29] in the space

$$L^\infty(0,\infty; L^\infty_\beta(\mathbb{R}^3_\xi; H^N(\mathbb{R}^3_x))), \quad \beta > \frac{5}{2},\ N \geqq 2,$$

by using the spectrum method as well as the contraction mapping principle, see also NISHIDA-IMAI [22] and KAWASHIMA [19]. Here $L^\infty_\beta(\mathbb{R}^3_\xi)$ denotes a space of all functions f with $(1+|\xi|)^\beta f$ uniformly bounded. Note that in the above space, H^N_x with the integer $N \geqq 2$ can be reduced to H^ℓ_x with $\ell > 3/2$; see UKAI and ASANO [30] for the study of the soft potential case in such a space. Using a similar approach, SHIZUTA [24] obtains the global existence of the classical solution $f(t,x,\xi) \in C^{1,1,0}((0,\infty) \times \mathbb{T}^3_x \times \mathbb{R}^3_\xi)$ on a torus, with the uniform bound in the space

$$L^\infty(0,\infty; L^\infty_\beta(\mathbb{R}^3_\xi; C^s(\mathbb{T}^3_x))), \quad \beta > \frac{5}{2},\ s > \frac{3}{2}.$$

The spectrum method was later improved in UKAI and YANG [31] for the existence of the mild solution in the space

$$L^\infty(0,\infty; L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi) \cap L^\infty_\beta(\mathbb{R}^3_\xi; L^\infty(\mathbb{R}^3_x))), \quad \beta > \frac{3}{2},$$

without any regularity conditions, where some L^∞–L^2 estimates in terms of the Duhamel's principle are developed. Note that the $L^2 \cap L^\infty$ theory has been also developed by GUO [14,16] to treat the Boltzmann equation on the bounded domain.

On the other hand, by means of the robust energy method, for instance, GUO [17], LIU et al. [20] and LIU and YU [21], the well-posedness of classical solutions is also established in the space

$$C(0,\infty; H^N_{t,x,\xi}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)), \quad N \geqq 4,$$

where the Sobolev space $H^N_{t,x,\xi}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ denotes a set of all functions whose derivatives with respect to all variables t, x and ξ up to N order are integrable in $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$. It turns out that if only the strong solution with the uniqueness property is considered then the time differentiation can be disregarded in the above Sobolev space. Indeed the first author of this paper studied, in [10], the existence of such strong solution in the space

$$C(0,\infty; L^2(\mathbb{R}^3_\xi; H^N(\mathbb{R}^3_x))), \quad N \geqq 4,$$

where $N \geqq 4$ can actually be straightforwardly extended to $N \geqq 2$, cf. [13]. We here remark that the techniques used in [10] lead to an extensive application of the Fourier energy method to the linearized Boltzmann equation as well as the related collision kinetic equations in plasma physics (cf. [9,11]), in order to provide the time-decay properties of the linearized solution operator instead of using the spectrum approach, and further give the optimal time-decay rates of solutions for the nonlinear problem.

The previously mentioned works are mainly focused on the angular cutoff Boltzmann equation. Recently, Amuxy [2] and Gressman and Strain [13] independently prove a the global existence of small-amplitude solutions in the angular non-cutoff case for general hard and soft potentials. In particular, the function space for the energy that [13] used in the hard potential case can take the form of

$$L^\infty(0,\infty; L^2(\mathbb{R}^3_\xi; H^N(\mathbb{R}^3_x))), \quad N \geqq 2.$$

Notice that the energy dissipation rate in the non-cutoff case becomes more complicated compared to the cutoff situation, see (3.3), and the key issue in [2,13] is to provide a good characterization of the Dirichlet form of the linearized Boltzmann operator so as to control the nonlinear dynamics. Very recently, AMUXY [1] presents a result for local existence in a function space significantly larger than those used in the existing works, where in the non cutoff case the index of Sobolev spaces for the solution is related to the parameter of the angular singularity, and in the cutoff case the solution space may take

$$L^\infty(0,T_0; L^2(\mathbb{R}^3_\xi; H^s(\mathbb{R}^3_x))), \quad s > \frac{3}{2},$$

where $T_0 > 0$ is a finite time.

As mentioned before, whenever $s = 3/2$, since H^s_x is not a Banach algebra, it seems impossible to expect to obtain a global result in $L^\infty(0,\infty; X)$ with $X = L^2_\xi H^s_x$. A natural idea is to replace H^s_x by $B^s_x = B^s_{2,1}$, which is a Banach algebra and is of the critical regularity. In fact, instead of directly using $L^\infty(0,\infty; B^s_x L^2_\xi)$ we will consider the Chemin–Lerner type function space $\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_x)$, which has stronger topology than $L^\infty(0,\infty; B^s_x L^2_\xi)$. Here, for $T \geqq 0$, $f(t,x,\xi) \in \widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_x)$ means that the norm

$$\sum_{q \geqq -1} 2^{qs} \sup_{0 \leqq t \leqq T} \|\Delta_q f(t,\cdot,\cdot)\|_{L^2_{x,\xi}}$$

is finite. The reason why the supremum with respect to time is put after the summation is that one has to use a stronger norm to control the nonlinear term, for instance, see the estimate on I_1 given by (3.7).

In what follows we explain the technical part in the proof of Theorem 1.1. First, compared to the case of the fluid dynamic equations mentioned before, the corresponding estimates in the space $\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)$ for the Boltzmann equation are much more complicated, not only because of the additional velocity variable ξ but also because of the nonlinear integral operator $\Gamma(f, f)$. In fact, applying the energy estimates in $L^2_{x,\xi}$ to (1.2) for each $\Delta_q f$ and due to the choice of the solution space $\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)$, one has to take the time integral, take the square root on both sides of the resulting estimate and then take the summation over all $q \geqq -1$, so that we are formally forced to make the trilinear estimate in the form of

$$\sum_{q\geqq -1} 2^{qs}\left[\int_0^T |(\Delta_q\Gamma(f, g), \Delta_q h)|\,\mathrm{d}t\right]^{1/2}.$$

We need to control the above trilinear term in terms of the norms $\mathcal{E}_T(\cdot)$ and $\mathcal{D}_T(\cdot)$ which correspond to the linearized dynamics in the same space $\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)$ and play the usual role of the energy and the energy dissipation, respectively. Thus, Lemma 3.1 becomes the key step in the proof of the global existence. Due to the nonlinearity of $\Gamma(f, g)$, we need to use the Bony decomposition, for instance, for the loss term,

$$f_* g = \mathcal{T}_{f_*} g + \mathcal{T}_g f_* + \mathcal{R}(f_*, g),$$

where notions $\mathcal{T}$ and $\mathcal{R}$ are to be given later on. For each term on the right of the above decomposition, we will use the boundedness property in L^p $(1 \leqq p \leqq \infty)$ for the operators Δ_q and S_q see (9.2) in the Appendix and further make use of the techniques in [6] developed by Chemin and Lerner to close the estimate on the above trilinear term.

Second, it seems very standard (cf. [10,11]) to estimate the macroscopic component (a, b, c) in the space $\widetilde{L}^2_T(B^{1/2}_x)$ on the basis of the fluid-type system (5.2). The only new difficulty that we have to overcome lies on the estimates on

$$\sum_{q\geqq -1} 2^{qs}(\|\Lambda_i(\Delta_q \mathbb{h})\|_{L^2_T L^2_x} + \|\Theta_{im}(\Delta_q \mathbb{h})\|_{L^2_T L^2_x}), \quad s = 1/2,$$

where Λ_i and Θ_{im} with $1 \leqq i, m \leqq 3$ are velocity moment functions, defined later on, and $\mathbb{h} = -L\{\mathbf{I} - \mathbf{P}\}f + \Gamma(f, f)$. As in [15,18], this can actually be done in the general situation given by Lemma 4.2. We remark that the proof of Lemma 4.2 is based on the key Lemma 3.1, related to the trilinear estimate. The global a priori estimate (6.1) can then be obtained by combining those trilinear estimates and the estimate on the macroscopic dissipations.

Third, since the local existence in the space $\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)$ is not obvious, we also provide the complete proof of that by using the idea of, for instance, [23]. The main goal in this part is to obtain the uniform bound of an approximate solution sequence in the norm $\widetilde{Y}_T(\cdot)$ given in (7.2), and also show the uniqueness and continuity with respect to T of solutions satisfying such a uniform bound. Here, once again, the result of Lemma 3.1 is needed. We also point out that the proof of continuity for $T \mapsto \widetilde{Y}_T(f)$ is essentially reduced to prove that

$$t \mapsto \sum_{q \geqq -1} 2^{3q/2} \|\Delta_q f(t,\cdot,\cdot)\|_{L^2_{x,\xi}}$$

is continuous, with the help of the boundedness of the norm $\widetilde{\mathcal{D}}_T(f)$, see Theorem 7.1 for more details.

The rest of the paper is arranged as follows. In Section 2, we explain some notations and present definitions of some function spaces. In Sections 3 and 4, we deduce the key estimates for the collision operators Γ and L. The estimate for the macroscopic dissipation is given in Section 5. Section 6 is devoted to obtaining the global a priori estimates for the Boltzmann equation in the Chemin–Lerner type space. In Section 7, we construct the local solutions of the Boltzmann equation and further show qualitative properties of the constructed local solutions. In Section 8 we give the proof of Theorem 1.1. Finally, an appendix is given for some preliminary lemmas which will be used in the previous sections.

2. Notations and Function Spaces

Throughout the paper, C denotes some generic positive (generally large) constant and λ denotes some generic positive (generally small) constant, where both C and λ may take different values in different places. For two quantities A and B, $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leqslant CB$, and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. For simplicity, $(\cdot,\cdot)$ stands for the inner product in either $L^2_x = L^2(\mathbb{R}^3_x)$, $L^2_\xi = L^2(\mathbb{R}^3_\xi)$ or $L^2_{x,\xi} = L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$. We use $\mathcal{S}(\mathbb{R}^3)$ to denote the Schwartz function space on $\mathbb{R}^3$, and use $\mathcal{S}'(\mathbb{R}^3)$ to denote the dual space of $\mathcal{S}(\mathbb{R}^3)$, that is the tempered function space.

Since the crucial nonlinear estimates require a dyadic decomposition of the Fourier variable, in what follows we recall briefly the Littlewood–Paley decomposition theory and some function spaces, such as the Besov space and the Chemin–Lerner space. Readers may refer to [4] for more details. Let us start with the Fourier transform. Here and below, the Fourier transform is taken with respect to the variable x only, not variables ξ and t. Given (t,ξ), the Fourier transform $\hat{f}(t,k,\xi) = \mathcal{F}_x f(t,k,\xi)$ of a Schwartz function $f(t,x,\xi) \in \mathcal{S}(\mathbb{R}^3_x)$ is given by

$$\hat{f}(t,k,\xi) := \int_{\mathbb{R}^3} \mathrm{d}x\, \mathrm{e}^{-ix\cdot k} f(t,x,\xi),$$

and the Fourier transform of a tempered function $f(t,x,\xi) \in \mathcal{S}'(\mathbb{R}^3_x)$ is defined by the dual argument in the standard way.

We now introduce a dyadic partition of $\mathbb{R}^3_x$. Let (φ,χ) be a couple of smooth functions valued in the closed interval $[0,1]$ such that φ is supported in the shell

$\mathbb{C}(0, \frac{3}{4}, \frac{8}{3}) = \{k \in \mathbb{R}^3 : \frac{3}{4} \leqq |k| \leqq \frac{8}{3}\}$ and χ is supported in the ball $\mathbb{B}(0, \frac{4}{3}) = \{k \in \mathbb{R}^3 : |k| \leqq \frac{4}{3}\}$, with

$$\chi(k) + \sum_{q \geqq 0} \varphi(2^{-q}k) = 1, \quad \forall\, k \in \mathbb{R}^3,$$
$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}k) = 1, \quad \forall\, k \in \mathbb{R}^3 \backslash \{0\}.$$

The nonhomogeneous dyadic blocks of $f = f(x) \in \mathcal{S}'(\mathbb{R}^3_x)$ are defined as follows:

$$\Delta_{-1} f := \chi(D) f = \tilde{\psi} * f = \int_{\mathbb{R}^3} \tilde{\psi}(y) f(x-y)\,\mathrm{d}y, \quad \text{with } \tilde{\psi} = \mathcal{F}^{-1}\chi;$$
$$\Delta_q f := \varphi(2^{-q}D) f = 2^{3q} \int_{\mathbb{R}^3} \psi(2^q y) f(x-y)\,\mathrm{d}y \quad \text{with } \psi = \mathcal{F}^{-1}\varphi, \quad q \geqq 0,$$

where $*$ is the convolution operator with respect to the variable x and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Define the low frequency cut-off operator S_q ($q \geqq -1$) by

$$S_q f := \sum_{j \leqq q-1} \Delta_j f.$$

It is a convention that $S_0 f = \Delta_{-1} f$ for $q = 0$, and $S_{-1} f = 0$ in the case of $q = -1$. Moreover, the homogeneous dyadic blocks are defined by

$$\dot{\Delta}_q f := \varphi(2^{-q}D) f = 2^{3q} \int_{\mathbb{R}^3} \psi(2^q y) f(x-y)\,\mathrm{d}y, \quad \forall\, q \in \mathbb{Z}.$$

With these notions, the nonhomogeneous Littlewood–Paley decomposition of $f \in \mathcal{S}'(\mathbb{R}^3_x)$ is given by

$$f = \sum_{q \geqq -1} \Delta_q f.$$

For $f \in \mathcal{S}'$, one also has

$$f = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q f$$

modulo a polynomial only. Recall that the above Littlewood–Paley decomposition is almost orthogonal in L^2_x.

Having defined the linear operators Δ_q for $q \geqq -1$ (or $\dot{\Delta}_q$ for $q \in \mathbb{Z}$), we give the definition of nonhomogeneous (or homogeneous) Besov spaces as follows.

Definition 2.1. Let $1 \leqq p \leqq \infty$ and $s \in \mathbb{R}$. For $1 \leqq r \leqq \infty$, the nonhomogeneous Besov space $B^s_{p,r}$ is defined by

$$B^s_{p,r} := \Bigg\{ f \in \mathcal{S}'(\mathbb{R}^3_x) : \ f = \sum_{q \geqq -1} \Delta_q f \text{ in } \mathcal{S}', \text{ with}$$

$$\|f\|_{B^s_{p,r}} =: \left(\sum_{q \geqq -1} (2^{qs} \|\Delta_q f\|_{L^p_x})^r \right)^{\frac{1}{r}} < \infty \Bigg\},$$

where in the case $r = \infty$ we set

$$\|f\|_{B^s_{p,\infty}} = \sup_{q \geqq -1} 2^{qs} \|\Delta_q f\|_{L^p_x}.$$

Let $\mathcal{P}$ denote the class of all polynomials on $\mathbb{R}^3_x$ and let $\mathcal{S}'/\mathcal{P}$ denote the tempered distributions on $\mathbb{R}^3_x$ modulo polynomials. The corresponding definition for the homogeneous Besov space is given as follows.

Definition 2.2. Let $1 \leqq p \leqq \infty$ and $s \in \mathbb{R}$. For $1 \leqq r < \infty$, the homogeneous Besov space is defined by

$$\dot{B}^s_{p,r} := \Bigg\{ f \in \mathcal{S}'/\mathcal{P} : \ f = \sum_{q \in \mathbb{Z}} \Delta_q f \text{ in } \mathcal{S}'/\mathcal{P}, \text{ with}$$

$$\|f\|_{\dot{B}^s_{p,r}} := \left(\sum_{q \in \mathbb{Z}} (2^{qs} \|\dot{\Delta}_q f\|_{L^p_x})^r \right)^{\frac{1}{r}} < \infty \Bigg\},$$

where in the case $r = \infty$ we set

$$\|f\|_{\dot{B}^s_{p,\infty}} = \sup_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q f\|_{L^p_x} < \infty.$$

To the end, for brevity of presentation, we denote $B^s_{2,1}$ by B^s, and $\dot{B}^s_{2,1}$ by $\dot{B}^s$, respectively, and we also write B^s, $\dot{B}^s$ as B^s_x, $\dot{B}^s_x$ to emphasize the x variable.

Since the velocity distribution function $f = f(t, x, \xi)$ involves the velocity variable ξ and the time variable t, it is natural to define the Banach space valued function space

$$L^{p_1}_T L^{p_2}_\xi L^{p_3}_x := L^{p_1}(0, T; L^{p_2}(\mathbb{R}^3_\xi; L^{p_3}(\mathbb{R}^3_x))),$$

for $0 < T \leqq \infty$, $1 \leqq p_1,\ p_2,\ p_3 \leqq \infty$, with the norm

$$\|f\|_{L^{p_1}_T L^{p_2}_\xi L^{p_3}_x} = \left(\int_0^T \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |f(t, x, \xi)|^{p_3} \, \mathrm{d}x \right)^{p_2/p_3} \mathrm{d}\xi \right)^{p_1/p_2} \mathrm{d}t \right)^{1/p_1},$$

where we have used the normal convention in the case when $p_1 = \infty$, $p_2 = \infty$ or $p_3 = \infty$. Moreover, in order to characterize the Boltzmann dissipation rate, we also define the following velocity weighted norm

$$\|f\|_{L^{p_1}_T L^{p_2}_{\xi,\nu} L^{p_3}_x} = \left(\int_0^T \left(\int_{\mathbb{R}^3} \nu(\xi) \left(\int_{\mathbb{R}^3} |f(t,x,\xi)|^{p_3} \, \mathrm{d}x \right)^{p_2/p_3} \mathrm{d}\xi \right)^{p_1/p_2} \mathrm{d}t \right)^{1/p_1}$$

for $0 < T \leqq \infty$, $1 \leqq p_1,\ p_2,\ p_3 \leqq \infty$, where the normal convention in the case when $p_1 = \infty$, $p_2 = \infty$ or $p_3 = \infty$ has been used.

In what follows, we present the definition of the Chemin–Lerner type spaces

$$\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r}), \quad \widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_{\xi,\nu} (B^s_{p,r}),$$

and

$$\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (\dot{B}^s_{p,r}), \quad \widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_{\xi,\nu} (\dot{B}^s_{p,r}),$$

which are initiated by the work [6].

Definition 2.3. Let $1 \leqq \varrho_1, \varrho_2, p, r \leqq \infty$ and $s \in \mathbb{R}$. For $0 < T \leqq \infty$, the space $\widetilde{L}^{\varrho_1}_t \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r})$ is defined by

$$\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r}) = \{ f(t,x,\xi) \in \mathcal{S}' : \ \|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r})} < \infty \},$$

where

$$\|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r})} = \left(\sum_{q \geqq -1} 2^{qsr} \left(\int_0^T \left(\int_{\mathbb{R}^3} \|\Delta_q f\|^{\varrho_2}_{L^p_x} \, \mathrm{d}\xi \right)^{\varrho_1/\varrho_2} \mathrm{d}t \right)^{r/\varrho_1} \right)^{\frac{1}{r}}$$

that is,

$$\|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r})} = \left(\sum_{q \geqq -1} 2^{qsr} \|\Delta_q f\|^r_{L^{\varrho_1}_T L^{\varrho_2}_\xi L^p_x} \right)^{1/r},$$

with the usual convention for $\varrho_1, \varrho_2, p, r = \infty$. Similarly, one also denotes

$$\|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_{\xi,\nu} (B^s_{p,r})} = \left(\sum_{q \geqq -1} 2^{qsr} \|\Delta_q f\|^r_{L^{\varrho_1}_T L^{\varrho_2}_{\xi,\nu} L^p_x} \right)^{1/r},$$

and

$$\|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (\dot{B}^s_{p,r})} = \left(\sum_{q \in \mathbb{Z}} 2^{qsr} \|\dot{\Delta}_q f\|^r_{L^{\varrho_1}_T L^{\varrho_2}_\xi L^p_x} \right)^{1/r},$$

$$\|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_{\xi,\nu} (\dot{B}^s_{p,r})} = \left(\sum_{q \in \mathbb{Z}} 2^{qsr} \|\dot{\Delta}_q f\|^r_{L^{\varrho_1}_T L^{\varrho_2}_{\xi,\nu} L^p_x} \right)^{1/r},$$

with the usual convention for $\varrho_1, \varrho_2, p, r = \infty$.

We conclude this section with a few remarks. First, since the goal of the paper is to establish the well-posedness in the spatially critical Besov space for the Boltzmann equation, we mainly consider the above norms in the case that $p = 2$, $r = 1$ and $\varrho_2 = 2$. Thus, the spaces $\widetilde{L}^{\varrho}_T \widetilde{L}^2_\xi(B^s_x)$, $\widetilde{L}^{\varrho}_T \widetilde{L}^2_{\xi,\nu}(B^s_x)$, $\widetilde{L}^{\varrho}_T \widetilde{L}^2_\xi(\dot{B}^s_x)$ and $\widetilde{L}^{\varrho}_T \widetilde{L}^2_{\xi,\nu}(\dot{B}^s_x)$ with $\varrho = 1$ or ∞ will be frequently used. Next, whenever a function $f = f(t, x, \xi)$ is independent of t or ξ, the corresponding norms defined above are modified in the usual way by omitting the t-variable or ξ-variable, respectively. Finally, it should be pointed out that the Chemin–Lerner type norm $\|\cdot\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi(B^s_{p,r})}$ is the refinement of the usual norm $\|\cdot\|_{L^{\varrho_1}_T L^{\varrho_2}_\xi(B^s_{p,r})}$ given by

$$\|f\|_{L^{\varrho_1}_T L^{\varrho_2}_\xi(B^s_{p,r})} = \left(\int_0^T \left(\int_{\mathbb{R}^3} \left(\sum_{q \geqq -1} 2^{qsr} \|\Delta_q f\|^r_{L^p_x}\right)^{\varrho_2/r} \mathrm{d}\xi\right)^{\varrho_1/\varrho_2} \mathrm{d}t\right)^{1/\varrho_1},$$

for $0 < T \leqq \infty$, $1 \leqq r, p, \varrho_1, \varrho_2 \leqq \infty$.

3. Trilinear Estimates

Recall the Boltzmann equation (1.2). The linearized collision operator L can be written as $L = \nu - K$. Here the multiplier $\nu = \nu(\xi)$, called the collision frequency, is given by

$$\nu(\xi) = \int_{\mathbb{R}^3} \mathrm{d}\xi_* \int_{\mathbb{S}^2} \mathrm{d}\omega\, |\xi - \xi_*|^\gamma B_0(\theta)\mu(\xi_*).$$

It holds that $\nu(\xi) \sim (1+|\xi|)^\gamma$, cf. [5,31], and the integral operator $K = K_2 - K_1$ is defined as

$$[K_1 f](\xi) = \int_{\mathbb{R}^3} \mathrm{d}\xi_* \int_{\mathbb{S}^2} \mathrm{d}\omega\, |\xi - \xi_*|^\gamma B_0(\theta)\mu^{1/2}(\xi_*)\mu^{1/2}(\xi) f(\xi_*), \quad (3.1)$$

$$[K_2 f](\xi) = \int_{\mathbb{R}^3} \mathrm{d}\xi_* \int_{\mathbb{S}^2} \mathrm{d}\omega\, |\xi - \xi_*|^\gamma B_0(\theta)\mu^{1/2}(\xi_*) \times\{\mu^{1/2}(\xi'_*) f(\xi') + \mu^{1/2}(\xi') f(\xi'_*)\}. \quad (3.2)$$

L is coercive in the sense that there is $\lambda_0 > 0$ such that

$$\int_{\mathbb{R}^3} f L f\, \mathrm{d}\xi \geqq \lambda_0 \int_{\mathbb{R}^3} \nu(\xi)|\{\mathbf{I} - \mathbf{P}\} f|^2. \quad (3.3)$$

Moreover, the nonlinear collision operator $\Gamma(f, g)$ is written as

$$\begin{aligned}\Gamma(f, g) &= \mu^{-1/2}(\xi) Q[\mu^{-1/2} f, \mu^{-1/2} g] = \Gamma_{\text{gain}}(f, g) - \Gamma_{\text{loss}}(f, g) \\ &= \int_{\mathbb{R}^3} \mathrm{d}\xi_* \int_{\mathbb{S}^2} \mathrm{d}\omega\, |\xi - \xi_*|^\gamma B_0(\theta)\mu^{1/2}(\xi_*) f(\xi'_*) g(\xi') \\ &\quad - g(\xi) \int_{\mathbb{R}^3} \mathrm{d}\xi_* \int_{\mathbb{S}^2} \mathrm{d}\omega\, |\xi - \xi_*|^\gamma B_0(\theta)\mu^{1/2}(\xi_*) f(\xi_*). \end{aligned} \quad (3.4)$$

In this section, we intend to give the key estimates for the nonlinear Bolztmann collision operator $\Gamma(\cdot,\cdot)$ defined by (3.4) in terms of the spatially critical Besov space. It should be pointed out that the following lemmas are new and they play a crucial role in the proof of the global existence of solutions to the Boltzmann equation (1.2). First we show the trilinear estimate in the following:

Lemma 3.1. *Assume $s>0$, $0<T\leqq\infty$. Let $f=f(t,x,\xi)$, $g=g(t,x,\xi)$, and $h=h(t,x,\xi)$ be three suitably smooth distribution functions such that all the norms on the right of the following inequalities are well defined, then it holds that*

$$\begin{aligned}\sum_{q\geqq-1}2^{qs}\left[\int_0^T\left|(\Delta_q\Gamma(f,g),\Delta_q h)\right|\,\mathrm{d}t\right]^{1/2}&\lesssim\|h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\\ &\times\Big[\|g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|f\|^{1/2}_{L^\infty_T L^2_\xi L^\infty_x}+\|f\|^{1/2}_{L^2_T L^2_{\xi,\nu}L^\infty_x}\|g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)}\\ &+\|f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|g\|^{1/2}_{L^\infty_T L^2_\xi L^\infty_x}+\|g\|^{1/2}_{L^2_T L^2_{\xi,\nu}L^\infty_x}\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)}\Big],\end{aligned}\tag{3.5}$$

where the inner product $(\cdot,\cdot)$ is taken with respect to variables (x,ξ).

Proof. Recalling (3.4) and using the inequality $(A+B)^{1/2}\leqq A^{1/2}+B^{1/2}$ for $A\geqq0$ and $B\geqq0$, one has

$$\begin{aligned}\left[\int_0^T|(\Delta_q\Gamma(f,g),\Delta_q h)|\,\mathrm{d}t\right]^{\frac12}&\leqq\left[\int_0^T|(\Delta_q\Gamma_{\mathrm{gain}}(f,g),\Delta_q h)|\,\mathrm{d}t\right]^{\frac12}\\ &+\left[\int_0^T|(\Delta_q\Gamma_{\mathrm{loss}}(f,g),\Delta_q h)|\,\mathrm{d}t\right]^{\frac12}.\end{aligned}\tag{3.6}$$

Here, notice, that since the collision integral acts on the ξ variable only and that Δ_q acts on x variable only, one can write

$$\Delta_q\Gamma_{\mathrm{gain}}(f,g)=\int_{\mathbb{R}^3}\mathrm{d}\xi_*\int_{\mathbb{S}^2}\mathrm{d}\omega\,|\xi-\xi_*|^\gamma B_0(\theta)\mu^{1/2}(\xi_*)\Delta_q[f(\xi'_*)g(\xi')],$$
$$\Delta_q\Gamma_{\mathrm{loss}}(f,g)=\int_{\mathbb{R}^3}\mathrm{d}\xi_*\int_{\mathbb{S}^2}\mathrm{d}\omega\,|\xi-\xi_*|^\gamma B_0(\theta)\mu^{1/2}(\xi_*)\Delta_q[f(\xi_*)g(\xi)].$$

By applying the Cauchy–Schwarz inequality to both integrals on the right of (3.6) with respect to all variables (t,x,ξ,ξ_*,ω), making the change of variables $(\xi,\xi_*)\to(\xi',\xi'_*)$ in the gain term, and then taking the summation over $q\geqq-1$ after multiplying it by 2^{qs}, we see that

$$
\begin{aligned}
&\sum_{q\geqq -1} 2^{qs}\left[\int_0^T \left|(\Delta_q\Gamma(f,g),\Delta_q h)\right|\,\mathrm{d}t\right]^{1/2}\\
&\quad\lesssim \sum_{q\geqq -1} 2^{qs}\left[\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^9\times\mathbb{S}^2}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\,\mathrm{d}\omega\,|\xi'-\xi_*'|^\gamma\mu^{1/2}(\xi_*')\left|\Delta_q[f_*g]\right|^2\right)^{1/2}\right]^{1/2}\\
&\quad\times\left[\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^9}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\,\mathrm{d}\omega\,|\xi-\xi_*|^\gamma\mu^{1/2}(\xi_*)\left|\Delta_q h\right|^2\right)^{1/2}\right]^{1/2}\\
&\quad+\sum_{q\geqq -1} 2^{qs}\left[\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^9\times\mathbb{S}^2}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\,\mathrm{d}\omega\,|\xi-\xi_*|^\gamma\mu^{1/2}(\xi_*)\left|\Delta_q[f_*g]\right|^2\right)^{1/2}\right]^{1/2}\\
&\quad\times\left[\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^9}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\,\mathrm{d}\omega\,|\xi-\xi_*|^\gamma\mu^{1/2}(\xi_*)\left|\Delta_q h\right|^2\right)^{1/2}\right]^{1/2}:=I_0,
\end{aligned}
$$

where $0\leqq B_0(\theta)\leqq C|\cos\theta|\leqq C$ has been used. Further, by using the discrete version of Cauchy–Schwarz inequality to the two summations $\sum_{q\geqq -1}$ above, one obtains that

$$
\begin{aligned}
I_0&\lesssim\left[\sum_{q\geqq -1} 2^{qs}\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^9}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi-\xi_*|^\gamma\left|\Delta_q[f_*g]\right|^2\right)^{1/2}\right]^{1/2}\\
&\quad\times\left[\sum_{q\geqq -1} 2^{qs}\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^9}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi-\xi_*|^\gamma\mu^{1/2}(\xi_*)\left|\Delta_q h\right|^2\right)^{1/2}\right]^{1/2}\\
&:=I^{1/2}\times II^{1/2},
\end{aligned}
$$

where $|\xi'-\xi_*'|=|\xi-\xi_*|$, $\mu^{1/2}(\xi_*')\leqq 1$ and $\int_{\mathbb{S}^2}\mathrm{d}\omega=4\pi$ have been used. It is straightforward to see that

$$
II\leqq \|h\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)},
$$

due to

$$
\int_{\mathbb{R}^3}\mathrm{d}\xi_*\,|\xi-\xi_*|^\gamma\mu^{1/2}(\xi_*)\sim(1+|\xi|)^\gamma\sim\nu(\xi).
$$

We now turn to compute I. Recalling Bony's decomposition, one can write $\Delta_q[f_*g]$ as

$$
\Delta_q[f_*g]=\Delta_q[\mathcal{T}_{f_*}g+\mathcal{T}_g f_*+\mathcal{R}(f_*,g)].
$$

Here $\mathcal{T}.\cdot$, and $\mathcal{R}(\cdot,\cdot)$ are the usual paraproduct operators. They are defined as follows. For suitable smooth distribution functions u and v,

$$
\mathcal{T}_u v=\sum_j S_{j-1}u\Delta_j v,\quad \mathcal{R}(u,v)=\sum_{|j'-j|\leqq 1}\Delta_{j'}u\Delta_j v.
$$

We therefore get from Minkowski's inequality that

$$
\begin{aligned}
I &\leqq \sum_{q\geqq -1} 2^{qs}\left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^9} \mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_* |\xi-\xi_*|^\gamma \left|\sum_j \Delta_q[S_{j-1}f_*\Delta_j g]\right|^2\right)^{1/2}\\
&\quad+\sum_{q\geqq -1} 2^{qs}\left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^9} \mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_* |\xi-\xi_*|^\gamma \left|\sum_j \Delta_q[S_{j-1}g\Delta_j f_*]\right|^2\right)^{1/2}\\
&\quad+\sum_{q\geqq -1} 2^{qs}\left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^9} \mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_* |\xi-\xi_*|^\gamma \left|\sum_{|j-j'|\leqq 1} \Delta_q[\Delta_j f_*\Delta_{j'} g]\right|^2\right)^{1/2}\\
&:= I_1+I_2+I_3.
\end{aligned}
$$

Now we estimate I_1, I_2 and I_3 term by term.
<u>*Estimates on I_1:*</u> Notice that

$$
\sum_j \Delta_q[S_{j-1}f_*\Delta_j g]=\sum_{|j-q|\leqq 4}\Delta_q[S_{j-1}f_*\Delta_j g].
$$

By Minkowski's inequality again, one can see that

$$
\begin{aligned}
I_1 &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^9} \mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\, |\xi|^\gamma |\Delta_q[S_{j-1}f_*\Delta_j g]|^2\right)^{1/2}\\
&\quad+\sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^9} \mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\, |\xi_*|^\gamma |\Delta_q[S_{j-1}f_*\Delta_j g]|^2\right)^{1/2}\\
&:= I_{1,1}+I_{1,2}.
\end{aligned}
\tag{3.7}
$$

Applying (9.2) in the appendix, one can deduce that

$$
\begin{aligned}
I_{1,1} &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^3} \|f_*\|^2_{L^\infty_x}\,\mathrm{d}\xi_* \int_{\mathbb{R}^3} |\xi|^\gamma \left\|\Delta_j g\right\|^2_{L^2_x}\,\mathrm{d}\xi\right)^{1/2}\\
&\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\sup_{0\leqq t\leqq T}\int_{\mathbb{R}^3} \|f_*\|^2_{L^\infty_x}\,\mathrm{d}\xi_* \int_0^T \mathrm{d}t \int_{\mathbb{R}^3} |\xi|^\gamma \left\|\Delta_j g\right\|^2_{L^2_x}\,\mathrm{d}\xi\right)^{1/2}\\
&\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^3} |\xi|^\gamma \left\|\Delta_j g\right\|^2_{L^2_x}\,\mathrm{d}\xi\right)^{1/2} \|f\|_{L^\infty_T L^2_\xi L^\infty_x}\\
&\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{(q-j)s} c_1(j)\|g\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|f\|_{L^\infty_T L^2_\xi L^\infty_x},
\end{aligned}
$$

where $c_1(j)$ is defined as

$$c_1(j) = \frac{2^{js}\left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^3} |\xi|^\gamma \left\|\Delta_j g\right\|_{L_x^2}^2 \,\mathrm{d}\xi\right)^{1/2}}{\|g\|_{\widetilde{L}_T^2 \widetilde{L}_{\xi,\nu}^2(B_x^s)}}, \tag{3.8}$$

which satisfies $\|c_1(j)\|_{\ell^1} \leqq 1$. From the above estimate on $I_{1,1}$, using the following convolution inequality for series

$$\begin{aligned} \sum_{q \geqq -1} \sum_{|j-q| \leqq 4} 2^{(q-j)s} c_1(j) &= \sum_{q \geqq -1} [(\mathbf{1}_{|j| \leqq 4} 2^{js}) * c_1(j)](q) \\ &\leqq \|\mathbf{1}_{|j| \leqq 4} 2^{js}\|_{\ell^1} \|c_1(j)\|_{\ell^1} < +\infty, \end{aligned} \tag{3.9}$$

we further get that

$$I_{1,1} \lesssim \|g\|_{\widetilde{L}_T^2 \widetilde{L}_{\xi,\nu}^2(B_x^s)} \|f\|_{L_T^\infty L_\xi^2 L_x^\infty}. \tag{3.10}$$

The estimate for $I_{1,2}$ is slightly different from $I_{1,1}$. In fact, we may compute it as

$$\begin{aligned} I_{1,2} &\leqq \sum_{q \geqq -1} \sum_{|j-q| \leqq 4} 2^{qs} \left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^3} |\xi_*|^\gamma \|f_*\|_{L_x^\infty}^2 \,\mathrm{d}\xi_* \int_{\mathbb{R}^3} \left\|\Delta_j g\right\|_{L_x^2}^2 \,\mathrm{d}\xi\right)^{1/2} \\ &\leqq \sum_{q \geqq -1} \sum_{|j-q| \leqq 4} 2^{qs} \left(\sup_{0 \leqq t \leqq T} \int_{\mathbb{R}^3} \left\|\Delta_j g\right\|_{L_x^2}^2 \,\mathrm{d}\xi \int_0^T \mathrm{d}t \int_{\mathbb{R}^3} |\xi_*|^\gamma \|f_*\|_{L_x^\infty}^2 \,\mathrm{d}\xi_*\right)^{1/2} \\ &\leqq \sum_{q \geqq -1} \sum_{|j-q| \leqq 4} 2^{qs} \left(\sup_{0 \leqq t \leqq T} \int_{\mathbb{R}^3} \left\|\Delta_j g\right\|_{L_x^2}^2 \,\mathrm{d}\xi\right)^{1/2} \|f\|_{L_T^2 L_{\xi,\nu}^2 L_x^\infty} \\ &\leqq \sum_{q \geqq -1} \sum_{|j-q| \leqq 4} 2^{(q-j)s} c_2(j) \|g\|_{\widetilde{L}_T^\infty \widetilde{L}_\xi^2(B_x^s)} \|f\|_{L_T^2 L_{\xi,\nu}^2 L_x^\infty}, \end{aligned}$$

where $c_2(j)$ is defined as

$$c_2(j) = \frac{2^{js}\left(\sup_{0 \leqq t \leqq T} \int_{\mathbb{R}^3} \left\|\Delta_j g\right\|_{L_x^2}^2 \,\mathrm{d}\xi\right)^{1/2}}{\|g\|_{\widetilde{L}_T^\infty \widetilde{L}_\xi^2(B_x^s)}}.$$

Since $\|c_2(j)\|_{\ell^1} = 1$, then in a similar way as for obtaining (3.10), we have

$$I_{1,2} \lesssim \|f\|_{L_T^2 L_{\xi,\nu}^2 L_x^\infty} \|g\|_{\widetilde{L}_T^\infty \widetilde{L}_\xi^2(B^s)}. \tag{3.11}$$

Now substituting (3.10) and (3.11) into (3.7), one has

$$I_1 \lesssim \|g\|_{\widetilde{L}_T^2 \widetilde{L}_{\xi,\nu}^2(B_x^s)} \|f\|_{L_T^\infty L_\xi^2 L_x^\infty} + \|f\|_{L_T^2 L_{\xi,\nu}^2 L_x^\infty} \|g\|_{\widetilde{L}_T^\infty \widetilde{L}_\xi^2(B_x^s)}.$$

This gives the estimate for I_1.

<u>*Estimates on I_2:*</u> From (3.14), we have

$$\begin{aligned}I_2 &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^9}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi|^\gamma|\Delta_q[S_{j-1}g\Delta_j f_*]|^2\right)^{1/2}\\ &\quad+\sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^9}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi_*|^\gamma|\Delta_q[S_{j-1}g\Delta_j f_*]|^2\right)^{1/2}\\ &:= I_{2,1}+I_{2,2}.\end{aligned}\tag{3.12}$$

As before, it follows from (9.2) in the appendix that

$$\begin{aligned}I_{2,1} &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\|\Delta_j f_*\|_{L^2_x}^2\,\mathrm{d}\xi_*\int_{\mathbb{R}^3}|\xi|^\gamma\|g\|_{L^\infty_x}^2\,\mathrm{d}\xi\right)^{1/2}\\ &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\sup_{0\leqq t\leqq T}\int_{\mathbb{R}^3}\|\Delta_j f_*\|_{L^2_x}^2\,\mathrm{d}\xi_*\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}|\xi|^\gamma\|g\|_{L^\infty_x}^2\,\mathrm{d}\xi\right)^{1/2}\\ &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\sup_{0\leqq t\leqq T}\int_{\mathbb{R}^3}\|\Delta_j f_*\|_{L^2_x}^2\,\mathrm{d}\xi_*\right)^{1/2}\|g\|_{L^2_T L^2_{\xi,\nu}L^\infty_x}\\ &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{(q-j)s}c_3(j)\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi((B^s_x))}\|g\|_{L^2_T L^2_{\xi,\nu}L^\infty_x}\end{aligned}\tag{3.13}$$

with

$$c_3(j)=\frac{2^{js}\left(\sup_{0\leqq t\leqq T}\int_{\mathbb{R}^3}\|\Delta_j f\|_{L^2_x}^2\,\mathrm{d}\xi\right)^{1/2}}{\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi((B^s_x))}}.$$

Similarly, it holds that

$$\begin{aligned}I_{2,2} &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}|\xi_*|^\gamma\|\Delta_j f_*\|_{L^2_x}^2\,\mathrm{d}\xi_*\int_{\mathbb{R}^3}\|g\|_{L^\infty_x}^2\,\mathrm{d}\xi\right)^{1/2}\\ &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\sup_{0\leqq t\leqq T}\int_{\mathbb{R}^3}\|g\|_{L^\infty_x}^2\,\mathrm{d}\xi\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}|\xi_*|^\gamma\|\Delta_j f_*\|_{L^2_x}^2\,\mathrm{d}\xi_*\right)^{1/2}\\ &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{qs}\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}|\xi_*|^\gamma\|\Delta_j f_*\|_{L^2_x}^2\,\mathrm{d}\xi_*\right)^{1/2}\|g\|_{L^\infty_T L^2_\xi L^\infty_x}\\ &\leqq \sum_{q\geqq -1}\sum_{|j-q|\leqq 4} 2^{(q-j)s}c_4(j)\|f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}((B^s_x))}\|g\|_{L^\infty_T L^2_\xi L^\infty_x}\end{aligned}\tag{3.14}$$

with

$$c_4(j) = \frac{2^{js}\left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^3} |\xi|^\gamma \|\Delta_j f\|_{L_x^2}^2 \,\mathrm{d}\xi\right)^{1/2}}{\|f\|_{\widetilde{L}_T^2 \widetilde{L}_{\xi,\nu}^2(B_x^s)}}.$$

Noticing $\|(c_3)\|_{\ell^1} = 1$ and $\|c_4(j)\|_{\ell^1} \leqq 1$, it follows from (3.12), (3.13) and (3.14) that

$$I_2 \lesssim \|f\|_{\widetilde{L}_T^\infty \widetilde{L}_\xi^2((B_x^s))} \|g\|_{L_T^2 L_{\xi,\nu}^2 L_x^\infty} + \|f\|_{\widetilde{L}_T^2 \widetilde{L}_{\xi,\nu}^2((B_x^s))} \|g\|_{L_T^\infty L_\xi^2 L_x^\infty}.$$

Estimates on I_3: We start from the fact that

$$\sum_j \sum_{|j-j'|\leqq 1} \Delta_q[\Delta_{j'} f_* \Delta_j g] = \sum_{\max\{j,j'\}\geqq q-2} \sum_{|j-j'|\leqq 1} \Delta_q[\Delta_{j'} f_* \Delta_j g].$$

With this, one can see that

$$\begin{aligned}
I_3 &\leqq \sum_{q\geqq -1} \sum_{\max\{j,j'\}\geqq q-2} \sum_{|j-j'|\leqq 1} 2^{qs} \left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^9} \mathrm{d}x\, \mathrm{d}\xi\, \mathrm{d}\xi_*\, |\xi| |\Delta_q[\Delta_{j'} f_* \Delta_j g]|^2\right)^{1/2} \\
&\quad + \sum_{q\geqq -1} \sum_{\max\{j,j'\}\geqq q-2} \sum_{|j-j'|\leqq 1} 2^{qs} \left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^9} \mathrm{d}x\, \mathrm{d}\xi\, \mathrm{d}\xi_*\, |\xi_*| |\Delta_q[\Delta_{j'} f_* \Delta_j g]|^2\right)^{1/2} \\
&:= I_{3,1} + I_{3,2}.
\end{aligned}$$

As before, applying (9.2), we get that

$$\begin{aligned}
I_{3,1} &\leqq \sum_{q\geqq -1} \sum_{j\geqq q-3} 2^{qs} \left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^3} \|f_*\|_{L_x^\infty}^2 \,\mathrm{d}\xi_* \int_{\mathbb{R}^3} |\xi| \|\Delta_j g\|_{L_x^2}^2 \,\mathrm{d}\xi\right)^{1/2} \\
&\leqq \sum_{q\geqq -1} \sum_{j\geqq q-3} 2^{qs} \left(\sup_{0\leqq t\leqq T} \int_{\mathbb{R}^3} \|f_*\|_{L_x^\infty}^2 \,\mathrm{d}\xi_* \int_0^T \mathrm{d}t \int_{\mathbb{R}^3} |\xi| \|\Delta_j g\|_{L_x^2}^2 \,\mathrm{d}\xi\right)^{1/2} \\
&\leqq \sum_{q\geqq -1} \sum_{j\geqq q-3} 2^{qs-js} 2^{js} \left(\int_0^T \mathrm{d}t \int_{\mathbb{R}^3} |\xi| \|\Delta_j g\|_{L_x^2}^2 \,\mathrm{d}\xi\right)^{1/2} \|f\|_{L_T^\infty L_\xi^2 L_x^\infty} \\
&\leqq \sum_{q\geqq -1} \sum_{j\geqq q-3} 2^{qs-js} c_1(j) \|g\|_{\widetilde{L}_T^2 \widetilde{L}_{\xi,\nu}^2(B_x^s)} \|f\|_{L_T^\infty L_\xi^2 L_x^\infty} \\
&\lesssim \|g\|_{\widetilde{L}_T^2 \widetilde{L}_{\xi,\nu}^2(B_x^s)} \|f\|_{L_T^\infty L_\xi^2 L_x^\infty},
\end{aligned}$$

where $c_1(j)$ is defined in (3.8), and the same type of convolution inequality for the series as in (3.9) has been used in the last inequality. Similarly, one can see that $I_{3,2}$ is also bounded as

$$I_{3,2} \lesssim \|f\|_{\widetilde{L}_T^2 \widetilde{L}_{\xi,\nu}^2(B_x^s)} \|g\|_{L_T^\infty L_\xi^2 L_x^\infty}.$$

Combing all of the above estimates on I_1, I_2 and I_3, we obtain the inequality (3.5). Hence, the proof of Lemma 3.1 is completed. □

Having Lemma 3.1, one can see that the following result also holds true.

Lemma 3.2. *Assume that $s > 0$, $0 \leqq T \leqq +\infty$, and let $f = f(t,x,\xi)$, $g = g(t,x,\xi)$, and $h = h(t,x,\xi)$ be some suitable smooth distribution functions such that the following norms are well defined, then it holds that*

$$\begin{aligned}
&\sum_{q\geqq -1} 2^{qs}\left[\int_0^T |(\Delta_q\Gamma(f,g),\Delta_q h)|\,\mathrm{d}t\right]^{1/2} \lesssim \|h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\\
&\quad\times\Big[\|g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(\mathbf{X})}+\|f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(\mathbf{X})}\|g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)}\\
&\quad+\|f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(\mathbf{X})}+\|g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(\mathbf{X})}\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)}\Big], \qquad (3.15)
\end{aligned}$$

where $\mathbf{X}$ denotes either the inhomogeneous critical Besov space $B^{3/2}_x$ or the homogeneous critical Besov space $\dot{B}^{3/2}_x$.

Proof. Noticing that $B^{3/2}_x \subset L^\infty_x$ and $\dot{B}^{3/2}_x \subset L^\infty_x$, (3.15) follows from (3.5) in Lemma 3.1 and (9.5) in Lemma 9.5. This ends the proof of Lemma 3.2. □

The following is an immediate corollary of Lemmas 3.1 and 3.2.

Corollary 3.1. *Assume $s > 0$, $0 < T \leqq \infty$. Let $f = f(t,x,\xi)$, $g = g(t,x,\xi)$, and $h = h(t,x,\xi)$ be three suitably smooth distribution functions such that all the norms on the right of the following inequalities are well defined, then it holds that*

$$\begin{aligned}
&\sum_{q\geqq -1} 2^{qs}\left[\int_0^T |(\Delta_q\Gamma(\mathbf{P}f,g),\Delta_q h)|\,\mathrm{d}t\right]^{1/2} \lesssim \|h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\\
&\quad\times\left[\|g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|\mathbf{P}f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(\mathbf{X})}+\|g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(\mathbf{X})}\|\mathbf{P}f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)}\right], \qquad (3.16)
\end{aligned}$$

$$\begin{aligned}
&\sum_{q\geqq -1} 2^{qs}\left[\int_0^T |(\Delta_q\Gamma(f,\mathbf{P}g),\Delta_q h)|\,\mathrm{d}t\right]^{1/2} \lesssim \|h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\\
&\quad\times\left[\|f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(\mathbf{X})}+\|f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(\mathbf{X})}\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)}\right], \qquad (3.17)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{q\geqq -1} 2^{qs}\left[\int_0^T |(\Delta_q\Gamma(\mathbf{P}f,\mathbf{P}g),\Delta_q h)|\,\mathrm{d}t\right]^{1/2} \lesssim \|h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\\
&\quad\times\left[\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)}\|\mathbf{P}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(\mathbf{X})}+\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(\mathbf{X})}\|\mathbf{P}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(B^s_x)}\right], \qquad (3.18)
\end{aligned}$$

where $\mathbf{X}$ denotes either $B^{3/2}_x$ or $\dot{B}^{3/2}_x$ as in Lemma 3.2.

Proof. We prove the first estimate (3.16) only, since (3.17) and (3.18) can be obtained in the same way. In fact, (3.16) follows directly from Lemma 3.1 with a slightly modification. Applying I from Lemma 3.1 with $f = \mathbf{P}f$, and noticing that

$$\|\Delta_q \mathbf{P}f\|_{L^2_{\xi,\nu}L^2_x} \sim \|\Delta_q \mathbf{P}f\|_{L^2_\xi L^2_x}, \quad \|S_j \mathbf{P}f\|_{L^2_{\xi,\nu}L^2_x} \sim \|S_j \mathbf{P}f\|_{L^2_\xi L^2_x},$$

one can always take the L^∞_t-norm of the terms involving $\mathbf{P}f$, so that it is not necessary to exchange the L^∞_t-norm or L^2_t-norm of $\mathbf{P}f$ or g. By this means, (3.16) can be verified through a tedious calculation, we omit the details for brevity. This completes the proof of Corollary 3.1. □

4. Estimate on Nonlinear Term

In this section we give the estimates on the nonlinear term $\Gamma(f, f)$ and an estimate on the upper bound of Lf. Recall (1.4) and (1.5).

Lemma 4.1. *It holds that*

$$\sum_{q\geqq -1} 2^{\frac{3q}{2}} \left[\int_0^T \left|(\Delta_q \Gamma(f, f), \Delta_q\{\mathbf{I}-\mathbf{P}\}f)\right| \mathrm{d}t\right]^{1/2} \lesssim \sqrt{\mathcal{E}_T(f)}\mathcal{D}_T(f), \quad (4.1)$$

for any $T > 0$.

Proof. By the splitting $f = \mathbf{P}f + \{\mathbf{I}-\mathbf{P}\}f$, we have

$$\begin{aligned}\Gamma(f, f) = {} & \Gamma(\mathbf{P}f, \mathbf{P}f) + \Gamma(\mathbf{P}f, \{\mathbf{I}-\mathbf{P}\}f) + \Gamma(\{\mathbf{I}-\mathbf{P}\}f, \mathbf{P}f) \\ & + \Gamma(\{\mathbf{I}-\mathbf{P}\}f, \{\mathbf{I}-\mathbf{P}\}f).\end{aligned} \quad (4.2)$$

It now suffices to compute the left hand of (4.1) in terms of the corresponding four terms on the right of (4.2). In light of Corollary 3.1, one can see that

$$\begin{aligned}&\sum_{q\geqq -1} 2^{\frac{3q}{2}} \left[\int_0^T \left|(\Delta_q \Gamma(\mathbf{P}f, \mathbf{P}f), \Delta_q\{\mathbf{I}-\mathbf{P}\}f)\right| \mathrm{d}t\right]^{1/2} \\ &\quad \lesssim \|\mathbf{P}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(\dot{B}^{3/2}_x)} \|\mathbf{P}f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^{3/2}_x)} \|\{\mathbf{I}-\mathbf{P}\}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)} \lesssim \sqrt{\mathcal{E}_T(f)}\mathcal{D}_T(f),\end{aligned}$$

where we have used Lemma 9.3 to ensure

$$\begin{aligned}\|\mathbf{P}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(\dot{B}^{3/2}_x)} &\lesssim \|(a, b, c)\|^{1/2}_{\widetilde{L}^2_T(\dot{B}^{3/2}_x)} \sim \|\nabla_x(a, b, c)\|^{1/2}_{\widetilde{L}^2_T(\dot{B}^{1/2}_x)} \\ &\lesssim \|\nabla_x(a, b, c)\|^{1/2}_{\widetilde{L}^2_T(B^{1/2}_x)} \lesssim \sqrt{\mathcal{D}_T(f)}.\end{aligned}$$

In a similar way, we next get from Corollary 3.1 that

$$\sum_{q\geqq -1} 2^{\frac{3q}{2}}\left[\int_0^T |(\Delta_q\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f),\Delta_q\{\mathbf{I}-\mathbf{P}\}f)|\,\mathrm{d}t\right]^{1/2}$$
$$\lesssim \|\{\mathbf{I}-\mathbf{P}\}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}\|\mathbf{P}f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_{\xi}(B^{3/2}_x)}\|\{\mathbf{I}-\mathbf{P}\}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)},$$
$$\sum_{q\geqq -1} 2^{\frac{3q}{2}}\left[\int_0^T |(\Delta_q\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f),\Delta_q\{\mathbf{I}-\mathbf{P}\}f)|\,\mathrm{d}t\right]^{1/2}$$
$$\lesssim \|\mathbf{P}f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_{\xi}(B^{3/2}_x)}\|\{\mathbf{I}-\mathbf{P}\}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}\|\{\mathbf{I}-\mathbf{P}\}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)},$$

and

$$\sum_{q\geqq -1} 2^{\frac{3q}{2}}\left[\int_0^T |(\Delta_q\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f),\Delta_q\{\mathbf{I}-\mathbf{P}\}f)|\,\mathrm{d}t\right]^{1/2}$$
$$\lesssim \|\{\mathbf{I}-\mathbf{P}\}f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_{\xi}(B^{3/2}_x)}\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}.$$

Furthermore, it is straightforward to see that the above three estimates can be further bounded by $\sqrt{\mathcal{E}_T(f)}\mathcal{D}_T(f)$ up to a generic constant. Therefore (4.1) follows from all the above estimates, and this completes the proof of Lemma 4.1. □

The following lemma will be used in the process of deducing the macroscopic dissipation rates in the next section.

Lemma 4.2. *Let $\zeta=\zeta(\xi)\in\mathcal{S}(\mathbb{R}^3_x)$ and $0<s\leqq 3/2$. Then it holds that*

$$\sum_{q\geqq -1} 2^{qs}\left[\int_0^T \|\Delta_q(\Gamma(f,f),\zeta)\|^2_{L^2_x}\,\mathrm{d}t\right]^{1/2}\lesssim \mathcal{E}_T(f)\mathcal{D}_T(f), \tag{4.3}$$

for any $T>0$, where the inner product $(\cdot,\cdot)$ on the left is taken with respect to velocity variable ξ only.

Proof. We first consider the general case of $\Gamma(f,g)$ instead of $\Gamma(f,f)$. By Hölder's inequality and the change of variable $(\xi,\xi_*)\to(\xi',\xi'_*)$, it follows that

$$\begin{aligned}|\Delta_q(\Gamma(f,g),\zeta)| &\leqq \left(\int_{\mathbb{R}^6}\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi-\xi_*|^\gamma\mu^{1/2}(\xi'_*)|\Delta_q[f_*g]|^2\right)^{1/2}\\ &\quad\times\left(\int_{\mathbb{R}^6}\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi-\xi_*|^\gamma\mu^{1/2}(\xi_*)|\zeta(\xi)|^2\right)^{1/2}\\ &\quad+\left(\int_{\mathbb{R}^6}\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi-\xi_*|^\gamma\mu^{1/2}(\xi_*)|\Delta_q[f_*g]|^2\right)^{1/2}\\ &\quad\times\left(\int_{\mathbb{R}^6}\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi-\xi_*|^\gamma\mu^{1/2}(\xi_*)|\zeta(\xi)|^2\right)^{1/2}\\ &\lesssim\left(\int_{\mathbb{R}^6}\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi-\xi_*|^\gamma|\Delta_q[f_*g]|^2\right)^{1/2}.\end{aligned}\tag{4.4}$$

With (4.4) in hand, one can further deduce

$$\begin{aligned}
&\sum_{q\geqq -1} 2^{qs}\left[\int_0^T \|\Delta_q(\Gamma(f,g),\zeta)\|_{L^2_x}^2\,\mathrm{d}t\right]^{1/2}\\
&\quad\lesssim \sum_{q\geqq -1} 2^{qs}\left(\int_0^T \mathrm{d}t\int_{\mathbb{R}^9}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\xi_*\,|\xi-\xi_*||\Delta_q[f_*g]|^2\right)^{1/2} := I.
\end{aligned}$$

Recalling that we have obtained the estimates for I in the proof of Lemmas 3.1 and 3.2,

$$\begin{aligned}
I \lesssim &\Big[\|g\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(\mathbf{X})} + \|f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(\mathbf{X})}\|g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)}\\
&+\|g\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(\mathbf{X})}\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)} + \|f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(\mathbf{X})}\Big],
\end{aligned}\tag{4.5}$$

where $\mathbf{X}$ denotes either $B^{3/2}_x$ or $\dot{B}^{3/2}_x$.

In particular, if $\Gamma(\mathbf{P}f,\mathbf{P}f)$ is considered, it follows from Corollary 3.1 that

$$I \lesssim \|\mathbf{P}f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)}\|\mathbf{P}f\|_{\widetilde{L}^2_T\widetilde{L}^2_\xi(\dot{B}^{3/2}_x)} \lesssim \mathcal{E}_T(f)\mathcal{D}_T(f),$$

where we have used $0 < s \leqq 3/2$. Recalling the splitting (4.2) and applying (4.5) and Corollary 3.1, the other three terms corresponding to the splitting (4.2) can be computed as:

$$\begin{aligned}
&\sum_{q\geqq -1} 2^{qs}\left[\int_0^T \|\Delta_q(\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f),\zeta)\|_{L^2_x}^2\,\mathrm{d}t\right]^{1/2}\\
&\quad+\sum_{q\geqq -1} 2^{qs}\left[\int_0^T \|\Delta_q(\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f),\zeta)\|_{L^2_x}^2\,\mathrm{d}t\right]^{1/2}\\
&\quad\lesssim \|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_t\widetilde{L}^2_{\xi,\nu}(B^s_x)}\|\mathbf{P}f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^{3/2}_x)}\\
&\quad\quad+\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_t\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}\|\mathbf{P}f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_x)},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{q\geqq -1} 2^{qs}\left[\int_0^T \|\Delta_q(\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f),\zeta)\|_{L^2_x}^2\,\mathrm{d}t\right]^{1/2}\\
&\quad\lesssim \|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T\widetilde{L}^2_\xi(B^s_x)}\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}\\
&\quad\quad+\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T\widetilde{L}^2_\xi(B^{3/2}_x)}\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_{\xi,\nu}(B^s_x)},
\end{aligned}$$

which can both be further bounded by $\mathcal{E}_T(f)\mathcal{D}_T(f)$ up to a generic constant. Combing all the above estimates, we obtain (4.3). This completes the proof of Lemma 4.2. □

Finally we give an estimate on the upper bound of the linear term Lf.

Lemma 4.3. *Let $\zeta = \zeta(\xi) \in \mathcal{S}(\mathbb{R}^3_\xi)$ and $s > 0$. Then it holds that*

$$\sum_{q \geqq -1} 2^{qs} \left[\int_0^T \|\Delta_q(L\{\mathbf{I}-\mathbf{P}\}f, \zeta)\|^2_{L^2_x} \, \mathrm{d}t \right]^{1/2} \lesssim \|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_x)} \tag{4.6}$$

for any $T > 0$, where the inner product $(\cdot,\cdot)$ on the left is taken with respect to velocity variable ξ only.

Proof. Since $L\{\mathbf{I}-\mathbf{P}\}f$ can be rewritten as

$$L\{\mathbf{I}-\mathbf{P}\}f = -\{\Gamma(\{\mathbf{I}-\mathbf{P}\}f, \mu^{1/2}) + \Gamma(\mu^{1/2}, \{\mathbf{I}-\mathbf{P}\}f)\},$$

then (4.6) follows directly from similar estimates as in the proof of Lemma 4.2. This ends the proof of Lemma 4.3. □

5. Estimate on Macroscopic Dissipation

In this section, we obtain the macroscopic dissipation rate basing on the Equation (1.2).

Lemma 5.1. *It holds that*

$$\begin{aligned}\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{1/2}_x)} \lesssim \|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)} + \mathcal{E}_T(f) \\ + \|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)} + \mathcal{E}_T(f)\mathcal{D}_T(f),\end{aligned} \tag{5.1}$$

for any $T > 0$.

Proof. First, as in [9,11], by taking the following velocity moments

$$\mu^{1/2}, \xi_i\mu^{1/2}, \quad \frac{1}{6}(|\xi|^2-3)\mu^{1/2}, \quad (\xi_i\xi_m - 1)\mu^{1/2}, \quad \frac{1}{10}(|\xi|^2-5)\xi_i\mu^{1/2}$$

with $1 \leqq i, m \leqq 3$ for the Equation (1.2), the coefficient functions (a, b, c) of the macroscopic component $\mathbf{P}f$ given by (1.3) satisfy the fluid-type system

$$\begin{cases} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b + \nabla_x(a+2c) + \nabla_x \cdot \Theta(\{\mathbf{I}-\mathbf{P}\}f) = 0, \\ \partial_t c + \dfrac{1}{3}\nabla_x \cdot b + \dfrac{1}{6}\nabla_x \cdot \Lambda(\{\mathbf{I}-\mathbf{P}\}f) = 0, \\ \partial_t[\Theta_{im}(\{\mathbf{I}-\mathbf{P}\}f) + 2c\delta_{im}] + \partial_i b_m + \partial_m b_i = \Theta_{im}(\mathbb{r}+\mathbb{h}), \\ \partial_t \Lambda_i(\{\mathbf{I}-\mathbf{P}\}f) + \partial_i c = \Lambda_i(\mathbb{r}+\mathbb{h}), \end{cases} \tag{5.2}$$

where the high-order moment functions $\Theta = (\Theta_{im}(\cdot))_{3\times 3}$ and $\Lambda = (\Lambda_i(\cdot))_{1\leqq i\leqq 3}$ are defined by

$$\Theta_{im}(f) = ((\xi_i\xi_m - 1)\mu^{1/2}, f), \quad \Lambda_i(f) = \frac{1}{10}((|\xi|^2-5)\xi_i\mu^{1/2}, f),$$

with the inner product taken with respect to velocity variable ξ only, and the terms $\mathbb{r}$ and $\mathbb{h}$ are given by

$$\mathbb{r} = -\xi \cdot \nabla_x \{\mathbf{I}-\mathbf{P}\}f, \quad \mathbb{h} = -L\{\mathbf{I}-\mathbf{P}\}f + \Gamma(f, f).$$

Applying Δ_q with $q \geqq -1$ to the system (5.2), we obtain

$$\begin{cases} \partial_t \Delta_q a + \nabla_x \cdot \Delta_q b = 0, \\ \partial_t \Delta_q b + \nabla_x \Delta_q (a+2c) + \nabla_x \cdot \Theta(\Delta_q\{\mathbf{I}-\mathbf{P}\}f) = 0, \\ \partial_t \Delta_q c + \frac{1}{3}\nabla_x \cdot \Delta_q b + \frac{1}{6}\nabla_x \cdot \Lambda(\Delta_q\{\mathbf{I}-\mathbf{P}\}f) = 0, \\ \partial_t[\Theta_{im}(\Delta_q\{\mathbf{I}-\mathbf{P}\}f) + 2\Delta_q c\delta_{im}] + \partial_i \Delta_q b_m + \partial_m \Delta_q b_i = \Theta_{im}(\Delta_q \mathbb{r} + \Delta_q \mathbb{h}), \\ \partial_t \Lambda_i(\Delta_q\{\mathbf{I}-\mathbf{P}\}f) + \partial_i \Delta_q c = \Lambda_i(\Delta_q \mathbb{r} + \Delta_q \mathbb{h}). \end{cases}$$

Now, in a similar way to [10], one can prove from the above system that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_q^{\mathrm{int}}(f(t)) + \lambda \|\Delta_q \nabla_x (a,b,c)\|_{L_x^2}^2 \\ &\lesssim \|\Delta_q \nabla_x \{\mathbf{I}-\mathbf{P}\}f\|_{L_\xi^2 L_x^2}^2 \\ &\quad + C\sum_{i=1}^3 \left\|\Lambda_i\left(\Delta_q \mathbb{h}\right)\right\|_{L_x^2}^2 + C\sum_{i,m=1}^3 \|\Theta_{im}(\Delta_q \mathbb{h})\|_{L_x^2}^2 \end{aligned} \tag{5.3}$$

for any $t \geqq 0$, where the temporal interactive functional $\mathcal{E}_q^{\mathrm{int}}(f(t))$ for each $q \geqq -1$ is defined by

$$\begin{aligned} \mathcal{E}_q^{\mathrm{int}}(f(t)) = &\sum_{i=1}^3 (\Delta_q \partial_i c, \Lambda_i(\Delta_q\{\mathbf{I}-\mathbf{P}\}f)) \\ &+ \kappa_1 \sum_{i,m=1}^3 (\partial_i \Delta_q b_m + \partial_m \Delta_q b_i, \Theta_{im}(\{\mathbf{I}-\mathbf{P}\}\Delta_q f) \\ &+ \kappa_2 \sum_{i=1}^3 (\Delta_q \partial_i a, \Delta_q b_i), \end{aligned} \tag{5.4}$$

with suitably chosen constants $0 < \kappa_2 \ll \kappa_1 \ll 1$. Integrating (5.3) with respect to t over $[0, T]$ and taking the square roots of both sides of the resulting inequality, one has

$$\begin{aligned} &\left(\int_0^T \|\Delta_q \nabla_x (a,b,c)\|_{L_x^2}^2 \,\mathrm{d}t\right)^{1/2} \\ &\lesssim \sqrt{|\mathcal{E}_q^{\mathrm{int}}(f(T))|} + C\sqrt{|\mathcal{E}_q^{\mathrm{int}}(f(0))|} + \|\Delta_q \nabla_x \{\mathbf{I}-\mathbf{P}\}f\|_{L_T^2 L_\xi^2 L_x^2} \\ &\quad + \sum_{i=1}^3 \left\|\Lambda_i(\Delta_q \mathbb{h})\right\|_{L_T^2 L_x^2} + \sum_{i,m=1}^3 \|\Theta_{im}(\Delta_q \mathbb{h})\|_{L_T^2 L_x^2}. \end{aligned} \tag{5.5}$$

Now multiplying (5.5) by $2^{q/2}$ and taking the summation over $q \geqq -1$ gives

$$\begin{aligned}\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{1/2}_x)} \lesssim &\sum_{q\geqq -1} 2^{q/2}\sqrt{|\mathcal{E}^{\text{int}}_q(f(T))|}\\ &+\sum_{q\geqq -1} 2^{q/2}\sqrt{|\mathcal{E}^{\text{int}}_q(f(0))|}\\ &+\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}\\ &+\sum_{i=1}^{3}\sum_{q\geqq -1} 2^{q/2}\left\|\Lambda_i\left(\Delta_q \mathbb{h}\right)\right\|_{L^2_T L^2_x}\\ &+\sum_{i,m=1}^{3}\sum_{q\geqq -1} 2^{q/2}\|\Theta_{im}(\Delta_q \mathbb{h})\|_{L^2_T L^2_x}. \end{aligned} \tag{5.6}$$

Furthermore, it follows from (5.4), Lemma 9.4 and the Cauchy–Schwarz inequality that

$$\begin{aligned}\sum_{q\geqq -1} 2^{q/2}\sqrt{|\mathcal{E}^{\text{int}}_q(f(t))|} \lesssim &\sum_{q\geqq -1} 2^{q/2}\{\|\Delta_q\nabla_x(a,b,c)(t)\|_{L^2_x}+\|\Delta_q b(t)\|_{L^2_x}\\ &+\|\Delta_q\nabla_x\{\mathbf{I}-\mathbf{P}\}f(t)\|_{L^2_\xi L^2_x}\},\end{aligned}$$

for any $0 \leqq t \leqq T$, which implies

$$\sum_{q\geqq -1} 2^{q/2}\sqrt{|\mathcal{E}^{\text{int}}_q(f(T))|} \lesssim \mathcal{E}_T(f), \quad \sum_{q\geqq -1} 2^{q/2}\sqrt{|\mathcal{E}^{\text{int}}_q(f(0))|} \lesssim \|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)}. \tag{5.7}$$

It further follows that Lemmas 4.2 and 4.3 imply

$$\begin{aligned}&\sum_{q\geqq -1} 2^{q/2}\|\Lambda_i(\Delta_q \mathbb{h})\|_{L^2_T L^2_x}+\sum_{q\geqq -1} 2^{q/2}\|\Theta_{im}(\Delta_q \mathbb{h})\|_{L^2_T L^2_x}\\ &\quad\lesssim \|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}+\mathcal{E}_T(f)\mathcal{D}_T(f).\end{aligned} \tag{5.8}$$

Therefore (5.1) follows from (5.6) with the help of (5.7) and (5.8). This completes the proof of Lemma 5.1. □

6. Global a Priori Estimate

This section is devoted to deducing the global a priori estimate for the Boltzmann equation (1.2).

Lemma 6.1. *There is indeed an energy functional $\mathcal{E}_T(f)$ satisfying (1.4) such that*

$$\mathcal{E}_T(f)+\mathcal{D}_T(f) \leqq C\|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)}+C\{\sqrt{\mathcal{E}_T(f)}+\mathcal{E}_T(f)\}\mathcal{D}_T(f) \tag{6.1}$$

for any $T>0$, where C is a constant independent of T.

Proof. Applying the operator Δ_q ($q \geqq -1$) to (1.2), taking the inner product with $2^{3q}\Delta_q f$ over $\mathbb{R}^3_x \times \mathbb{R}^3_\xi$ and applying Lemma 9.1, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}2^{3q}\|\Delta_q f\|^2_{L^2_\xi L^2_x} + \lambda_0 2^{3q}\|\Delta_q\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_{\xi,\nu}L^2_x}\\ &\quad\leqq 2^{3q}|(\Delta_q\Gamma(f,f), \Delta_q\{\mathbf{I}-\mathbf{P}\}f)|. \end{aligned}\tag{6.2}$$

Integrating (6.2) over $[0, t]$ with $0 \leqq t \leqq T$ and taking the square root of both sides of the resulting inequality yields

$$\begin{aligned}&2^{\frac{3q}{2}}\|\Delta_q f(t)\|_{L^2_\xi L^2_x} + \sqrt{\lambda_0}2^{\frac{3q}{2}}\left(\int_0^t \|\Delta_q\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_{\xi,\nu}L^2_x}\,\mathrm{d}\tau\right)^{1/2}\\ &\quad\leqq 2^{\frac{3q}{2}}\|\Delta_q f_0\|_{L^2_\xi L^2_x} + 2^{\frac{3q}{2}}\left(\int_0^t |(\Delta_q\Gamma(f,f), \Delta_q\{\mathbf{I}-\mathbf{P}\}f)|\,\mathrm{d}\tau\right)^{1/2}\end{aligned}$$

for any $0 \leqq t \leqq T$. Further, by taking the summation over $q \geqq -1$, the above estimate implies

$$\begin{aligned}&\sum_{q\geqq -1} 2^{\frac{3q}{2}}\sup_{0\leqq t\leqq T}\|\Delta_q f(t)\|_{L^2_\xi L^2_x}+\sqrt{\lambda_0}\sum_{q\geqq -1}2^{\frac{3q}{2}}\left(\int_0^T \|\Delta_q\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_{\xi,\nu}L^2_x}\,\mathrm{d}t\right)^{1/2}\\ &\quad\leqq \sum_{q\geqq -1}2^{\frac{3q}{2}}\|\Delta_q f_0\|_{L^2_\xi L^2_x}+\sum_{q\geqq -1}2^{\frac{3q}{2}}\left(\int_0^T |(\Delta_q\Gamma(f,f), \Delta_q\{\mathbf{I}-\mathbf{P}\}f)|\,\mathrm{d}t\right)^{1/2}.\end{aligned}$$

Due to Lemma 4.1 it further follows that

$$\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^{3/2}_x)} + \sqrt{\lambda_0}\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)} \lesssim \|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)} + \sqrt{\mathcal{E}_T(f)}\mathcal{D}_T(f),\tag{6.3}$$

where $T > 0$ can be arbitrary. Furthermore, we recall Lemma 5.1. By letting $0 < \kappa_3 \ll 1$, we get from (5.1) $\times\, \kappa_3 +$ (6.3) that

$$\begin{aligned}&\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^{3/2}_x)} - \kappa_3\mathcal{E}_T(f)+\lambda\left\{\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{1/2}_x)}+\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}\right\}\\ &\lesssim \|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)} + \{\sqrt{\mathcal{E}_T(f)}+\mathcal{E}_T(f)\}\mathcal{D}_T(f).\end{aligned}\tag{6.4}$$

Therefore, (6.1) follows from (6.4) by noticing

$$\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^{3/2}_x)} - \kappa_3\mathcal{E}_T(f) \sim \mathcal{E}_T(f),$$

since $\kappa_3 > 0$ can be small enough. The proof of Lemma 6.1 is complete. □

7. Local Existence

In this section, we will establish the local-in-time existence of solutions to the Boltzmann equation (1.2) in the space $\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^{3/2}_x)$ for $T>0$ small enough. The construction of the local solution is based on a uniform energy estimate for the following sequence of iterating approximate solutions:

$$\begin{cases} \{\partial_t+\xi\cdot\nabla_x\}F^{n+1}+F^{n+1}(\xi)\displaystyle\int_{\mathbb{R}^3\times\mathbb{S}^2}|\xi-\xi_*|^\gamma B_0(\theta)F^n(\xi_*)\,\mathrm{d}\xi_*\mathrm{d}\omega \\ \qquad\qquad =\displaystyle\int_{\mathbb{R}^3\times\mathbb{S}^2}|\xi-\xi_*|^\gamma B_0(\theta)F^n(\xi_*')F^n(\xi')\,\mathrm{d}\xi_*\mathrm{d}\omega, \\ F^{n+1}(0,x,\xi)=F_0(x,\xi), \end{cases}$$

starting with $F^0(t,x,\xi)=F_0(x,\xi)$.

Noticing that $F^{n+1}=\mu+\mu^{1/2}f^{n+1}$, equivalently we need to solve f^{n+1} such that

$$\begin{aligned} \{\partial_t+\xi\cdot\nabla_x+\nu\}f^{n+1}-Kf^n&=\Gamma_{\text{gain}}(f^n,f^n)-\Gamma_{\text{loss}}(f^n,f^{n+1}),\\ f^{n+1}(0,x,\xi)&=f_0(x,\xi). \end{aligned} \tag{7.1}$$

Our discussion is based on the uniform bound in n for $\mathcal{E}_T(f^n)$ for a small time $T>0$. The crucial energy estimate is given as follows.

Lemma 7.1. *The solution sequence $\{f^n\}_{n=1}^\infty$ is well defined. For a sufficiently small constant $M_0>0$, there exists $T^*=T^*(M_0)>0$ such that if*

$$\|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)}\leqq M_0,$$

then for any n, it holds that

$$\widetilde{Y}_T(f^n):=\mathcal{E}_T(f^n)+\widetilde{\mathcal{D}}_T(f^n)\leqq 2M_0,\qquad \forall\, T\in[0,T^*), \tag{7.2}$$

where $\widetilde{\mathcal{D}}_T(f)$ is defined by

$$\widetilde{\mathcal{D}}_T(f)=\|f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)}.$$

Proof. To prove (7.2), we use induction on n. Namely, for each integer $l\geqq 0$, we are going to verify

$$\widetilde{Y}_T(f^l)\leqq 2M_0 \tag{7.3}$$

for $0\leqq T<T^*$, where M_0 and $T^*>0$ are to be suitably chosen later on. Clearly the case $l=0$ is valid. We assume (7.3) is true for $l=n$. Applying Δ_q $(q\geqq -1)$ to (7.1) and taking the inner product with $2^{3q}\Delta_q f^{n+1}$ over $\mathbb{R}^3_x\times\mathbb{R}^3_\xi$, one has

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}2^{3q}\|\Delta_q f^{n+1}\|^2_{L^2_\xi L^2_x}+2^{3q+1}\|\Delta_q f^{n+1}\|^2_{L^2_{\xi,\nu}L^2_x}\\ &\quad=2^{3q+1}(\Delta_q\Gamma_{\text{gain}}(f^n,f^n)-\Delta_q\Gamma_{\text{loss}}(f^n,f^{n+1})+K\Delta_q f^n,\Delta_q f^{n+1}), \end{aligned} \tag{7.4}$$

which further implies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}2^{3q}\|\Delta_q f^{n+1}\|^2_{L^2_\xi L^2_x}+2^{3q+1}\|\Delta_q f^{n+1}\|^2_{L^2_{\xi,\nu}L^2_x}\\ &\quad\leqq 2^{3q+1}|(\Delta_q\Gamma_{\mathrm{gain}}(f^n,f^n),\Delta_q f^{n+1})|\\ &\qquad+2^{3q+1}|(\Delta_q\Gamma_{\mathrm{loss}}(f^n,f^{n+1}),\Delta_q f^{n+1})|+2^{3q+1}|(K\Delta_q f^n,\Delta_q f^{n+1})|.\end{aligned}\tag{7.5}$$

Now by integrating (7.5) with respect to the time variable over $[0,t]$ with $0\leqq t\leqq T$, taking the square root of both sides of the resulting inequality and summing up over $q\geqq -1$, one has

$$\begin{aligned}&\sum_{q\geqq-1}2^{\frac{3q}{2}}\sup_{0\leqq t\leqq T}\|\Delta_q f^{n+1}(t)\|_{L^2_\xi L^2_x}+\sum_{q\geqq-1}2^{\frac{3q+1}{2}}\left(\int_0^T\left\|\Delta_q f^{n+1}\right\|^2_{L^2_{\xi,\nu}L^2_x}\,\mathrm{d}t\right)^{1/2}\\ &\quad\leqq\sum_{q\geqq-1}2^{\frac{3q}{2}}\|\Delta_q f_0\|_{L^2_\xi L^2_x}\\ &\qquad+\sum_{q\geqq-1}2^{\frac{3q+1}{2}}\left(\int_0^T|(\Delta_q\Gamma_{\mathrm{gain}}(f^n,f^n),\Delta_q f^{n+1})|\,\mathrm{d}t\right)^{1/2}\\ &\qquad+\sum_{q\geqq-1}2^{\frac{3q+1}{2}}\left(\int_0^T|(\Delta_q\Gamma_{\mathrm{loss}}(f^n,f^{n+1}),\Delta_q f^{n+1})|\,\mathrm{d}t\right)^{1/2}\\ &\qquad+\sum_{q\geqq-1}2^{\frac{3q+1}{2}}\left(\int_0^T|(K\Delta_q f^n,\Delta_q f^{n+1})|\,\mathrm{d}t\right)^{1/2}.\end{aligned}\tag{7.6}$$

From Lemmas 3.2 and 9.2, (7.6) implies

$$\begin{aligned}&\mathcal{E}_T(f^{n+1})+\widetilde{\mathcal{D}}_T(f^{n+1})\\ &\quad\leqq\|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)}+C\sqrt{\mathcal{E}_T(f^n)}\sqrt{\widetilde{\mathcal{D}}_T(f^n)}\sqrt{\widetilde{\mathcal{D}}_T(f^{n+1})}\\ &\qquad+C\sqrt{\mathcal{E}_T(f^n)}\widetilde{\mathcal{D}}_T(f^{n+1})+C\sqrt{\mathcal{E}_T(f^{n+1})}\sqrt{\widetilde{\mathcal{D}}_T(f^n)}\sqrt{\widetilde{\mathcal{D}}_T(f^{n+1})}\\ &\qquad+\sum_{q\geqq-1}2^{\frac{3q+1}{2}}\left(\int_0^T\|\Delta_q f^n\|_{L^2_\xi L^2_x}\|\Delta_q f^{n+1}\|_{L^2_\xi L^2_x}\,\mathrm{d}t\right)^{1/2}.\end{aligned}\tag{7.7}$$

The last term on the right hand side of (7.7) can be bounded by

$$\begin{aligned}
&\sqrt{T}\sum_{q\geqq -1} 2^{\frac{3q+1}{2}}\left(\sup_{0\leqq t\leqq T}\|\Delta_q f^n\|_{L^2_\xi L^2_x}\sup_{0\leqq t\leqq T}\|\Delta_q f^{n+1}\|_{L^2_\xi L^2_x}\right)^{1/2}\\
&\lesssim \sqrt{T}\left(\sum_{q\geqq -1} 2^{\frac{3q}{2}}\sup_{0\leqq t\leqq T}\|\Delta_q f^n\|_{L^2_\xi L^2_x}\right)^{1/2}\left(\sum_{q\geqq -1} 2^{\frac{3q}{2}}\sup_{0\leqq t\leqq T}\|\Delta_q f^{n+1}\|_{L^2_\xi L^2_x}\right)^{1/2}\\
&\lesssim \sqrt{T}\mathcal{E}_T(f^n)+\sqrt{T}\mathcal{E}_T(f^{n+1}),
\end{aligned} \tag{7.8}$$

where in the second line the discrete version of the Cauchy–Schwarz inequality has been used. Using the Cauchy–Schwarz inequality, the second and fourth terms on the right hand side of (7.7) can be dominated by

$$\eta\widetilde{\mathcal{D}}_T(f^{n+1})+\frac{C}{\eta}\mathcal{E}_T(f^n)\widetilde{\mathcal{D}}_T(f^n)+C\sqrt{\widetilde{\mathcal{D}}_T(f^n)}\{\mathcal{E}_T(f^{n+1})+\widetilde{\mathcal{D}}_T(f^{n+1})\}, \tag{7.9}$$

where η is an arbitrary small positive constant. By substituting (7.8) and (7.9) into (7.7), applying the inductive hypothesis and recalling $0\leqq T<T^*$, it follows that

$$\begin{aligned}
&(1-C\sqrt{T^*}-C\sqrt{M_0})\mathcal{E}_T(f^{n+1})+(1-\eta-2C\sqrt{M_0})\widetilde{\mathcal{D}}_T(f^{n+1})\\
&\quad\leqq M_0+C\sqrt{T^*}M_0+\frac{C}{\eta}M_0^2.
\end{aligned}$$

This then implies (7.3) for $l=n+1$, since $\eta>0$ can be small enough and both $T^*>0$ and $M_0>0$ are chosen to be suitably small. The proof of Lemma 7.1 is therefore complete. □

With the uniform bound on the iterative solution sequence in terms of (7.1) by Lemma 7.1, we can give the proof of the local existence of solutions in the following theorem. We remark that the approach used here is due to GUO [18].

Theorem 7.1. *Assume* $0\leqq\gamma\leqq 1$. *For a sufficiently small* $M_0>0$, *there exists* $T^*=T^*(M_0)>0$ *such that if*

$$\|f_0\|_{\widetilde{L}^2_\xi(B^{3/2}_x)}\leqq M_0,$$

then there is a unique strong solution $f(t,x,\xi)$ *to the Boltzmann equation* (1.2) *in* $(0,T^*)\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi$ *with initial data* $f(0,x,\xi)=f_0(x,\xi)$, *such that*

$$\widetilde{Y}_T(f)\leqq 2M_0,$$

for any $T\in[0,T^*)$, *where* $\widetilde{Y}_T(f)$ *is defined in* (7.2). *Moreover* $\widetilde{Y}_T(f)$ *is continuous in* T *over* $[0,T^*)$, *and if* $F_0(x,\xi)=\mu+\mu^{1/2}f_0\geqq 0$, *then* $F(t,x,\xi)=\mu+\mu^{1/2}f(t,x,\xi)\geqq 0$ *holds true.*

Proof. In terms of (7.2), the limit function $f(t, x, \xi)$ of the approximate solution sequence $\{f^n\}_{n=1}^{\infty}$ must be the solution to (1.2) with $f(0, x, \xi) = f_0(x, \xi)$ in the sense of distribution. The distribution solution turns out to be a strong solution because it can be shown to be unique as follows.

To prove the uniqueness, we assume that another solution g with the same initial data with f, that is $g(0, x, \xi) = f_0(x, \xi)$, exists such that

$$\widetilde{Y}_T(g) \leqq 2M_0,$$

on $T \in [0, T^*)$. Taking the difference of the Boltzmann equation (1.2) for f and g, one has

$$[\partial_t + \xi \cdot \nabla_x](f - g) + \nu(f - g) = \Gamma(f - g, f) + \Gamma(g, f - g) + K(f - g).$$

Then, by performing the completely same energy estimate as for obtaining (7.7), it follows that

$$\begin{aligned}
&\widetilde{Y}_T(f-g)\\
&\quad\lesssim \sqrt{\mathcal{E}_T(f) + \mathcal{E}_T(g)}\widetilde{\mathcal{D}}_T(f-g)\\
&\qquad + \sqrt{\mathcal{E}_T(f-g)}\sqrt{\widetilde{\mathcal{D}}_T(f) + \widetilde{\mathcal{D}}_T(g)}\sqrt{\widetilde{\mathcal{D}}_T(f-g)}\\
&\qquad + \sum_{q\geqq -1} 2^{\frac{3q+1}{2}} \left(\int_0^T \|\Delta_q(f-g)\|_{L^2_\xi L^2_x} \|\Delta_q(f-g)\|_{L^2_\xi L^2_x}\,\mathrm{d}t\right)^{1/2}\\
&\quad\lesssim \sqrt{\mathcal{E}_T(f) + \mathcal{E}_T(g)}\widetilde{\mathcal{D}}_T(f-g) + \sqrt{\mathcal{E}_T(f-g)}\sqrt{\widetilde{\mathcal{D}}_T(f) + \widetilde{\mathcal{D}}_T(g)}\sqrt{\widetilde{\mathcal{D}}_T(f-g)}\\
&\qquad + \sqrt{T}\mathcal{E}_T(f-g) + \sqrt{T}\mathcal{E}_T(f-g).
\end{aligned}$$

We therefore deduce $f \equiv g$ by letting $T < T^*$, because $\widetilde{Y}_T(f) \leqq 2M_0$, $\widetilde{Y}_T(g) \leqq 2M_0$, and M_0 and T^* can be chosen suitably small.

To prove that $T \mapsto \widetilde{Y}_T(f)$ is continuous in $[0, T^*)$, we first show that

$$t \mapsto \mathcal{E}(f(t)) := \sum_{q\geqq -1} 2^{\frac{3q}{2}} \|\Delta_q f(t)\|_{L^2_{x,\xi}} \tag{7.10}$$

is continuous on $[0, T^*)$. Indeed, take t_1, t_2 with $0 \leqq t_1, t_2 < T^*$ and let $t_1 < t_2$ for brevity of presentation. By letting $f^{n+1} = f^n = f$ in (7.4), integrating the resulting inequality with respect to the time variable over $[t_1, t_2]$, taking the square root of both sides and then summing up over $q \geqq -1$, similar to obtaining (7.7), one has

$$|\mathcal{E}(f(t_2)) - \mathcal{E}(f(t_1))| \lesssim (\sqrt{M_0} + 1) \sum_{q\geqq -1} 2^{\frac{3q}{2}} \left(\int_{t_1}^{t_2} \|\Delta_q f\|^2_{L^2_{\xi,\nu} L^2_x}\,\mathrm{d}t\right)^{1/2}.$$

With this, it suffices to prove

$$\lim_{t_2\to t_1} \sum_{q\geqq -1} 2^{\frac{3q}{2}} \left(\int_{t_1}^{t_2} \|\Delta_q f\|^2_{L^2_{\xi,\nu} L^2_x}\,\mathrm{d}t\right)^{1/2} = 0. \tag{7.11}$$

Take $\varepsilon > 0$. Since $\sum_{q\geqq -1} 2^{\frac{3q}{2}} (\int_0^T \|\Delta_q f\|^2_{L^2_{\xi,\nu}L^2_x} \,\mathrm{d}t)^{1/2}$ is finite for a fixed time T with $\max\{t_1, t_2\} < T < T^*$, there is an integer N such that

$$\begin{aligned}\sum_{q\geqq N+1} 2^{\frac{3q}{2}} \left(\int_{t_1}^{t_2} \|\Delta_q f\|^2_{L^2_{\xi,\nu}L^2_x} \,\mathrm{d}t\right)^{1/2} \\ \leqq \sum_{q\geqq N+1} 2^{\frac{3q}{2}} \left(\int_0^T \|\Delta_q f\|^2_{L^2_{\xi,\nu}L^2_x} \,\mathrm{d}t\right)^{1/2} < \frac{\varepsilon}{2}.\end{aligned}$$

On the other hand, it is straightforward to see

$$\lim_{t_2\to t_1} \sum_{-1\leqq q\leqq N} 2^{\frac{3q}{2}} \left(\int_{t_1}^{t_2} \|\Delta_q f\|^2_{L^2_{\xi,\nu}L^2_x} \,\mathrm{d}t\right)^{1/2} = 0,$$

which implies that there is $\delta > 0$ such that

$$\sum_{-1\leqq q\leqq N} 2^{\frac{3q}{2}} \left(\int_{t_1}^{t_2} \|\Delta_q f\|^2_{L^2_{\xi,\nu}L^2_x} \,\mathrm{d}t\right)^{1/2} < \frac{\varepsilon}{2},$$

whenever $|t_2 - t_1| < \delta$. Hence, for $|t_2 - t_1| < \delta$,

$$\sum_{q\geqq -1} 2^{\frac{3q}{2}} \left(\int_{t_1}^{t_2} \|\Delta_q f\|^2_{L^2_{\xi,\nu}L^2_x} \,\mathrm{d}t\right)^{1/2} = \left(\sum_{-1\leqq q\leqq N} + \sum_{q\geqq N+1}\right) \cdots < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then (7.11) is proved, and thus $t \mapsto \mathcal{E}(f(t))$ is continuous on $[0, T^*)$. In particular, $\mathcal{E}(f(t))$ given in (7.10) is well defined for each $t \in [0, T^*)$. Notice that the same proof also yields that for each $q \geqq -1$, the function $t \mapsto \|\Delta_q f(t)\|_{L^2_{x,\xi}}$ is continuous on $[0, T^*)$.

We now show that $T \mapsto \widetilde{Y}_T(f)$ is continuous in $[0, T^*)$. Indeed, take T_1, T_2 with $0 \leqq T_1 < T_2 < T^*$. Recall (7.2). Notice that $\widetilde{Y}_T(f)$ is nondecreasing in T. Then,

$$\begin{aligned}0 &\leqq \widetilde{Y}_{T_2}(f) - \widetilde{Y}_{T_1}(f)\\ &= (\|f\|_{\widetilde{L}^\infty_{T_2}\widetilde{L}^2_\xi(B^{3/2}_x)} - \|f\|_{\widetilde{L}^\infty_{T_1}\widetilde{L}^2_\xi(B^{3/2}_x)}) + (\|f\|_{\widetilde{L}^2_{T_2}\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)} - \|f\|_{\widetilde{L}^2_{T_1}\widetilde{L}^2_{\xi,\nu}(B^{3/2}_x)})\\ &\leqq \sum_{q\geqq -1} 2^{\frac{3q}{2}} \left(\sup_{T_1\leqq t\leqq T_2} \|\Delta_q f(t)\|_{L^2_{x,\xi}} - \|\Delta_q f(T_1)\|_{L^2_{x,\xi}}\right)\\ &\quad + \sum_{q\geqq -1} 2^{\frac{3q}{2}} \left(\int_{T_1}^{T_2} \|\Delta_q f(t)\|^2_{L^2_{\xi,\nu}L^2_x} \,\mathrm{d}t\right)^{1/2}.\end{aligned}$$

Here, the second summation on the right tends to zero as $T_2 \to T_1$ by (7.11), and in completely the same way to proving (7.11), one can see that the first summation on the right also tends to zero as $T_2 \to T_1$, since $t \mapsto \|\Delta_q f(t)\|_{L^2_{x,\xi}}$ is continuous and $\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi (B^{3/2}_x)}$ is finite for some fixed time T with $T_2 < T < T^*$.

Finally, by using the iteration form (7.1), the proof of the positivity is quite standard, for instance see [15]. We now finish the proof of Lemma 7.1. □

8. Proof of Global Existence

In this section, we prove the main result Theorem 1.1 for the global existence of solutions to the Boltzmann equation (1.2) with initial data $f(0, x, \xi) = f_0(x, \xi)$. The approach is based on the local existence result Theorem 7.1 as well as the standard continuity argument.

Proof of Theorem 1.1. Recall (1.4) and (1.5). Define

$$Y_T(f) = \mathcal{E}_T(f) + \mathcal{D}_T(f).$$

Let us redefine the constant C on the right of (6.1) to be $C_1 \geqq 1$, and choose $M_1 > 0$ such that

$$C_1(\sqrt{M_1} + M_1) \leqq \frac{1}{2}.$$

Set

$$M = \min\{M_1, M_0\}.$$

Let initial data f_0 be chosen such that

$$\|f_0\|_{\widetilde{L}^2_\xi B^{3/2}_x} \leqq \frac{M}{4C_1} \leqq \frac{M_0}{2}.$$

Define

$$\widetilde{T} = \sup\{T : Y_T(f) \leqq M\}.$$

By Theorem 7.1, $\widetilde{T} > 0$ holds true, because the solution f exists locally in time and $T \mapsto Y_T(f)$ is continuous by the same proof as for $\widetilde{Y}_T(f)$. Moreover, Lemma 6.1 gives that for $0 \leqq T \leqq \widetilde{T}$,

$$Y_T(f) \leqq C_1 \|f_0\|_{\widetilde{L}^2_\xi (B^{3/2}_x)} + C_1(\sqrt{M_1} + M_1) Y_T(f).$$

That is, for $0 \leqq T \leqq \widetilde{T}$,

$$Y_T(f) \leqq 2C_1 \|f_0\|_{\widetilde{L}^2_\xi (B^{3/2}_x)} \leqq \frac{M}{2} < M.$$

This implies $\widetilde{T} = \infty$. The global existence and uniqueness are then proved. The proof of Theorem 1.1 is complete. □

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Appendix

In this appendix, we will state some basic estimates related to the Botlzmann equation and Besov space. First we point out that the coercivity property (3.3) of the linearized collision operator L implies:

Lemma 9.1. *Let* $(\cdot,\cdot)$ *be the inner product on* $L^2_{x,\xi}$. *It holds that*

$$(\Delta_q Lf, \Delta_q f) \geqq \lambda_0 \|\{\mathbf{I}-\mathbf{P}\}\Delta_q f\|^2_{L^2_{\xi,\nu}L^2_x},$$

for each $q \geqq -1$. *Moreover, for* $s \in \mathbb{R}$, *it holds that*

$$\sum_{q\geqq -1} 2^{qs}\left(\int_0^T (\Delta_q Lf, \Delta_q f)\,\mathrm{d}t\right)^{1/2} \geqq \sqrt{\lambda_0}\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_x)},$$

for any $T \geqq 0$.

It is known that $K = K_2 - K_1$ defined in (3.1) and (3.2) is a self-adjoint compact operator on L^2_ξ (cf. [5]) and it enjoys the following estimate.

Lemma 9.2. *Let* $(\cdot,\cdot)$ *be the inner product on* $L^2_{x,\xi}$. *It holds that*

$$(\Delta_q Kf, \Delta_q g) \leqq C\|\Delta_q f\|_{L^2_\xi L^2_x}\|\Delta_q g\|_{L^2_\xi L^2_x}, \tag{9.1}$$

for each $q \geqq -1$, *where* C *is a constant independent of* q, f *and* g.

Proof. Kg can be written as

$$Kg = \int_{\mathbb{R}^3} \mathcal{K}(\xi,\xi_*)g(\xi_*)\,\mathrm{d}\xi_*,$$

and $\mathcal{K}(\xi,\xi_*)$ is a bounded operator from L^2_ξ to L^2_ξ. Then (9.1) follows from the Cauchy–Schwarz inequality. □

In addition, for the convenience of readers we list some basic facts which are frequently used in the paper.

Lemma 9.3. *Let* $1 \leqq p \leqq \infty$, *then*

$$\|\Delta_q \cdot\|_{L^p_x} \leqq C\|\cdot\|_{L^p_x}, \quad \|S_q \cdot\|_{L^p_x} \leqq C\|\cdot\|_{L^p_x}, \tag{9.2}$$

where C *is a constant independent of* p *and* q.

Lemma 9.4. *Let* $1 \leqq \varrho, p, r \leqq \infty$, *if* $s > 0$, *then*

$$\|\nabla_x \cdot \|_{\widetilde{L}^{\varrho}_T(\dot{B}^s_{p,r})} \sim \| \cdot \|_{\widetilde{L}^{\varrho}_T(\dot{B}^{s+1}_{p,r})}, \quad \| \cdot \|_{\widetilde{L}^{\varrho}_T(\dot{B}^s_{p,r})} \lesssim \| \cdot \|_{\widetilde{L}^{\varrho}_T(B^s_{p,r})}. \tag{9.3}$$

We would like to mention that the first relation can be achieved by the classical Bernstein inequality (see, for example, [4]) and another follows from the recent fact in [26], which indicates the relation between homogeneous and inhomogeneous Chemin–Lerner spaces.

Finally, the Chemin–Lerner type spaces $\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r})$ may be linked with the classical spaces $L^{\varrho_1}_T L^{\varrho_2}_\xi (B^s_{p,r})$ in the following way:

Lemma 9.5. *Let* $1 \leqq \varrho_1, \varrho_2, p, r \leqq \infty$ *and* $s \in \mathbb{R}$.

(1) *If* $r \geqq \max\{\varrho_1, \varrho_2\}$, *then*

$$\|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r})} \leqq \|f\|_{L^{\varrho_1}_T L^{\varrho_2}_\xi (B^s_{p,r})}. \tag{9.4}$$

(2) *If* $r \leqq \min\{\varrho_1, \varrho_2\}$, *then*

$$\|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r})} \geqq \|f\|_{L^{\varrho_1}_T L^{\varrho_2}_\xi (B^s_{p,r})}. \tag{9.5}$$

Proof. We only prove (9.5) in terms of $1 \leqq r, \varrho_2, \varrho_1 < +\infty$, the other cases and (9.4) can be proved similarly.

Since $\varrho_2/r \geqq 1$ and $\varrho_1/r \geqq 1$, by applying the Generalized Minkowski's inequality twice, one can see that

$$\begin{aligned}
\|f\|_{L^{\varrho_1}_T L^{\varrho_2}_\xi (B^s_{p,r})} &= \left(\int_0^T \left(\int_{\mathbb{R}^3} \left(\sum_{q \geqq -1} 2^{qsr} \|\Delta_q f\|^r_{L^p_x} \right)^{\varrho_2/r} \mathrm{d}\xi \right)^{\varrho_1/\varrho_2} \mathrm{d}t \right)^{1/\varrho_1} \\
&= \left(\int_0^T \left(\int_{\mathbb{R}^3} \left(\sum_{q \geqq -1} 2^{qsr} \|\Delta_q f\|^r_{L^p_x} \right)^{\varrho_2/r} \mathrm{d}\xi \right)^{\frac{r}{\varrho_2} \cdot \frac{\varrho_1}{r}} \mathrm{d}t \right)^{1/\varrho_1} \\
&\leqq \left(\int_0^T \left(\sum_{q \geqq -1} 2^{qsr} \left(\int_{\mathbb{R}^3} \|\Delta_q f\|^{\varrho_2}_{L^p_x} \mathrm{d}\xi \right)^{r/\varrho_2} \right)^{\frac{\varrho_1}{r}} \mathrm{d}t \right)^{1/\varrho_1} \\
&= \left(\int_0^T \left(\sum_{q \geqq -1} 2^{qsr} \left(\int_{\mathbb{R}^3} \|\Delta_q f\|^{\varrho_2}_{L^p_x} \mathrm{d}\xi \right)^{r/\varrho_2} \right)^{\frac{\varrho_1}{r}} \mathrm{d}t \right)^{\frac{r}{\varrho_1} \cdot \frac{1}{r}} \\
&\leqq \left(\sum_{q \geqq -1} 2^{qsr} \left(\int_0^T \left(\int_{\mathbb{R}^3} \|\Delta_q f\|^{\varrho_2}_{L^p_x} \mathrm{d}\xi \right)^{\varrho_1/\varrho_2} \mathrm{d}t \right)^{r/\varrho_1} \right)^{1/r} \\
&= \|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_\xi (B^s_{p,r})}.
\end{aligned}$$

Thus Lemma 9.5 holds true. □

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