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Well-Posedness for a Class of Thin-Film Equations with General Mobility in the Regime of Partial Wetting

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Abstract

We establish well-posedness for the family of thin-film equations

$$\begin{cases} h_t + (h^n h_{xxx})_x \;=\; 0 & \text{in } \{h > 0\}, \\ h \;=\; 0,\; |h_x| \;=\; 1 & \text{on } \partial\{h > 0\} \end{cases} \tag{1}$$

with $n \in (0, \frac{14}{5})\backslash\{1, 2\}$. The model (1) with $n \in (0, 3]$ has been used to describe the evolution of a capillary driven thin liquid droplet on a solid substrate in terms of its height profile $h \geqq 0$. The family of thin-film equations (1) provides a model problem to investigate contact line propagation in fluid dynamics under relaxed slip conditions. The parabolicity of the fourth order parabolic problem degenerates at the free boundary, which leads to a loss of regularity at the moving contact point. Our solutions are regular in terms of the two variables $d(x)$ and $d(x)^{3-n}$, where $d(x)$ is the distance to the free boundary. The main technical difficulty in the analysis of (1) is related to the loss of regularity at the contact points.

Contents

1. Introduction

In this paper, we establish well-posedness for the family of thin-film equations

$$h_t + (h^n h_{xxx})_x \;=\; 0 \quad \text{in } \{h > 0\} \tag{2}$$

for the range of parameters $n \in (0, \frac{14}{5})\backslash\{1, 2\}$ and with boundary conditions

$$h \;=\; 0, \qquad |h_x| \;=\; 1, \quad \text{on } \partial\{h > 0\}. \tag{3}$$

The model (2)–(3) is a fourth order parabolic evolution problem, defined on the free domain $\{h > 0\}$. In the case $n \in (0, 3]$, this model has been used to describe the evolution of a capillary driven thin liquid droplet on a solid substrate in terms of its height profile $h \geqq 0$, see Fig. 1. It can be derived from Navier–Stokes equations in the regime of *lubrication approximation*, see for example [20,26,40]. The parameter n represents the slip condition imposed at the liquid-solid interface. In particular, $n = 3$ corresponds to the no slip condition, while $n = 2$ is related to Navier slip; more general slip conditions corresponding to $n \in (0, 3)$ have also been considered, see [5,10,26,36]. The condition (3) of a fixed non-zero contact angle is a consequence of Young's Law in the so called partial wetting regime, see for example [14,39]. We only consider the situation when the contact angle is 45°, however all the results in this paper can be easily adapted to the situation when the contact angle takes another non-zero value.

Contact point movement in fluid dynamics is an ongoing challenge from the modelling as well as the analytical view point, see for example [19,41,42]. Huh and Scriven have discovered that movement of the contact for a viscous fluid leads to infinite dissipation if the no slip condition is assumed (this is the so called *no slip paradox*) [29]. This singularity of the dissipation can be regularized by allowing for slip at the liquid-solid interface. The correct regularization of the no-slip paradox is still disputed, see for example [14,19]. The family of thin-film equations (2) provide a model problem to investigate contact point propagation under relaxed slip conditions. Indeed, formal calculation suggests that if $n \geqq 3$, then the contact point stays fixed for solutions of (2)–(3). On the other hand, if $n = 0$, then (2) turns into a linear parabolic fourth order equation with infinite speed of propagation of the contact line. This is the motivation to investigate well-posedness and regularity for solutions of (2) in the range $n \in (0, 3)$.

In order to see which generic behaviour for solutions of (2)–(3) can be expected at the contact point, let us first discuss the behaviour of travelling wave solutions. With the ansatz $h(t, x) = h_{\mathrm{TW}}(x - \sigma t)$, (2) implies

$$h_{\mathrm{TW}}^{n-1}(y) h_{\mathrm{TW}}'''(y) = \sigma \quad \text{for } y > 0 \tag{4}$$

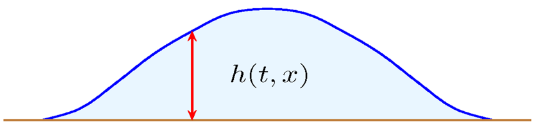

Fig. 1. Sketch of liquid droplet spreading on solid substrate. In our model, the contact angle θ is constant and determined by Young's Law

with $h_{\mathrm{TW}}(0) = 0$, $h'_{\mathrm{TW}}(0) = 1$. A power series ansatz for the solution of (4) yields

$$h_{\mathrm{TW}}(y) = y + \begin{cases} \dfrac{\sigma}{(4-n)(3-n)(2-n)} y^{4-n} \; + \; l.o.t. & \text{for } n \in (0,3)\backslash\{2\}, \\ \dfrac{\sigma}{2} y^2 \ln y \; + \; \text{l.o.t.} & \text{for } n = 2. \end{cases} \tag{5}$$

In particular, the expansion (5) indicates that solutions of (2)–(3) generically are not smooth at the contact point. Indeed, if $\sigma \neq 0$ then solutions of (5) can only be smooth for $n = 1$. Also note that the velocity is not determined by the leading order term in (5). Consequently, the evolution can include both propagation as well as recession of the contact point. This is different from the situation of the related second order equation, the porous media equation, where the support of the solution can only propagate, see for example [15]. The application of a power series ansatz on (4) for $n \geqq 3$ shows that the linear term in the expansion can only be the leading order term of the expansion at the contact point if the velocity σ is zero. This suggests that movement of the contact point can only occur if $n < 3$ for solutions that satisfy the boundary condition (3).

Note that the evolution in (2)–(3) can be understood as a gradient flow with energy given by the total interfacial energy between liquid, air and solid. The decrease of interfacial energy by viscous dissipation is described by the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} h_x^2 \,\mathrm{d}x + \mathscr{L}^1\{h > 0\}\right) = -\int_{\mathbb{R}} h^n h_{xxx}^2 \,\mathrm{d}x \;\leqq\; 0, \tag{6}$$

which is valid for any sufficiently regular solution of (2), see also [13] for a more detailed view on the energetics of the thin-film equation in the partial wetting regime. Indeed, in the regime of lubrication approximation, the left hand side of (6) describes the net interfacial energy of air, liquid and solid in the lubrication approximation regime, while the term on the right hand side of (6) is the viscous dissipation.

There exists a well-developed theory for the existence, but not uniqueness, of weak solutions for (2) in the range $n \in (0, 3)$. Most of this theory is concerned with the case of complete wetting when the boundary condition (3) is replaced by a zero contact angle condition at the free boundary, that is $h = h_x = 0$. The existence of weak solutions in the complete wetting regime in the case of one space dimension has been shown in [6,9,11], for the higher dimensional case and further references see [12,16,28]. The existence of weak solutions for (2) with boundary condition (3) has been proved for $n = 1$ in [37]. The existence of weak solutions for general mobility $n \in (0, 3)$ using a different method has been addressed in [13]. However, the uniqueness of weak solutions is an open problem. Moreover, the notion of a weak solution is not strong enough to state (2) explicitly as a free boundary problem since neither boundary condition nor the speed of the free boundary are well-defined in the solution space. For this reason, the position of the free boundary is only implicitly given in the definition of weak solutions. However, some qualitative properties of weak solutions have been shown such as a weak notion for finite speed of propagation [7,8,27,30] and the existence of a waiting time [17,25].

The existence and uniqueness of classical solutions for (2) have previously been known only for $n = 1$ and $n = 2$. Well-posedness for (2) in the case of complete wetting and for $n = 1$ has been derived in [24]. For initial data near the stationary solution and in the situation of complete wetting, the authors show long-time existence of solutions. Moreover, smoothness of the solutions for all positive times is shown. Well-posedness in weighted Hölder spaces has been shown in [23] using a different proof based on Safonov's method. A well-posedness result for $n = 1$ in the case of partial wetting is included in [32,33]. Note that there is also related work for the second order porous media equation, see [2–4,18,34]. The arguments used for the porous media equation, however, rely strongly on the maximum principle which does not hold for (2).

The first well-posedness result for solutions of (2) with singular expansions has been given in [31]. In this paper, the author proves well-posedness for (2) with $n = 2$ in a function space which allows for the logarithmic expansions in (5). The result is obtained by a decomposition of the solution into a part which captures the logarithmic singular expansion and a homogeneous remainder. The homogeneous part is analyzed using the tool of Mellin transform. The case of complete wetting with $n = 2$ is addressed in [21,22]. The first paper is concerned with the analysis of source type solutions. Using a sophisticated shooting argument, the authors are able to show analyticity for a source type solution with singular expansion at the moving contact point. The result in [21] is a corresponding well-posedness theory. The methods used in [21] differ from the ones used in [31]. In particular, the methods are based on the real space formulation rather than the Mellin transform. Furthermore, in [21], a systematic analysis for coercivity and ellipticity for degenerate linear operators is developed. Note that a decomposition of functions into a homogeneous part and a polynomial remainder has been used in the analysis of elliptic equations on non-smooth domains, see for example [32,33,35]. A related decomposition has been applied in the analysis of the porous media equation in by Angenent [3,4].

In this paper, we show well-posedness in a class of solutions which are not regular in x, but instead regular as a function of the two variables $d(x)$ and $d(x)^{3-n}$, where $d(x)$ is the distance to the free boundary. For this, we adapt and extend the methods used in [31] to the general equation of type (2) in the range $n \in (0, \frac{14}{5})$. There are some differences with respect to the case $n = 2$. First, note that the solution in (5) has a different expansion at the contact point for $n = 2$ and in particular, the function is regular as a function of $\log d(x)$. On the other hand, the solutions of the problem considered in this paper cannot be expressed as a regular function of a single variable (5). Also the treatment of the nonlinear operator is more involved, since we have to deal with a general mobility, while the nonlinear operator is bilinear in [31]. One particular issue is the loss of regularity for $n \to 3$ and the loss of control on the velocity for $n \to 0$. It would be interesting to extend the estimates to the full range $n \in (0, 3)$. However, with the L^2-based approach we used, an extension of the arguments to the case $n \in [\frac{14}{5}, 3)$ is not straightforward. This is related to the fact that the nonlinear operator gets more singular in this regime and no dissipative estimate is available which is compatible with the singular expansions of type (5). In the course of the analysis, we also derive corresponding interpolation inequalities which might be of more general interest.

The structure of the paper is as follows. In Section 2, we transform the equation onto a fixed domain. In Section 3, we introduce the norms and spaces for the solution. In Section 4, we state the main results of the paper. The proofs of these results are given in the following Sections 6–10. In the appendix, we give some results about the consistency of the expansion of the solution at the free boundary.

2. Reformulation on Fixed Domain

We first consider the situation when h has a single free boundary, that is

$$\operatorname{supp} h \;=\; [s(t), \infty). \tag{7}$$

We first reformulate the model such that the support of the fluid is fixed in the new coordinates. We follow an approach which has been used for example in [2,24,31], using a reference frame that moves with the contact point: We introduce $\hat{x}$ by

$$\hat{x} \;=\; x - s(t). \tag{8}$$

Expressed in terms of the new variable $\hat{x}$, (2) turns into

$$h_t - \dot{s}(t) h_{\hat{x}} + (h^n h_{\hat{x}\hat{x}\hat{x}})_{\hat{x}} \;=\; 0 \quad \text{for } \hat{x} \in (0, \infty). \tag{9}$$

For better readability, we omit the hats on top of the coordinates in the sequel. Note that other transformations have been used as well to fix the free boundary, in particular, a Van Mises transformation has been used for the related porous media equation, see [18,34], while volumetric coordinates have been used in [21].

For a fourth order parabolic evolution equation with free boundary, one would usually expect three boundary conditions: two boundary conditions since it is a fourth order equation, and one additional boundary condition which describes the velocity of the free boundary. Interestingly, *two boundary conditions* are sufficient for equation (2) since the velocity of the free boundary is implicitly determined by (2)–(3). Indeed, taking the limit $x \to 0$ in (9) and in view of (3), the speed of propagation can be expressed in terms of the profile as

$$\dot{s}(t) \;=\; \lim_{x\to 0} (h^n h_{xxx})_x. \tag{10}$$

By application of l'Hôspital's rule and in view of (3), this implies

$$\dot{s}(t) \;=\; \lim_{x\to 0} h^{n-1} h_{xxx} \;=\; \lim_{x\to 0} x^{n-1} h_{xxx}. \tag{11}$$

Inserting (11) into (9), we hence arrive at the following model

$$\begin{cases} h_t - h_x (x^{n-1} h_{xxx})_{|x=0} + (h^n h_{xxx})_x = 0 & \text{for } x \in (0,\infty), \\ h = 0, \quad h_x = 1 & \text{for } x = 0, \\ h = h_{\text{in}}, & \text{for } t = 0, \end{cases} \tag{12}$$

where here and in the sequel we use the notation $\cdot_{|x=0}$ for the trace operator at $x = 0$. We first consider the situation, where the wetted region approximates a cone, in particular $h_x \to 1$ for $x \to \infty$. We introduce the new dependent variable

$$f(t, x) \;=\; h_x(t, x) - 1. \tag{13}$$

Derivating (12) in x, we get

$$f_t - f_x \left(x^{n-1} f_{xx}\right)_{|x=0} + \left(h^n f_{xx}\right)_{xx} = 0. \tag{14}$$

In terms of f, the model (12), hence can be expressed as

$$\begin{cases} f_t + Af \;=\; N(f) \quad \text{for } x \in (0, \infty), \\ f \;=\; 0 \quad \text{for } x = 0, \\ f \;=\; f_{\text{in}} \quad \text{for } t = 0, \end{cases} \tag{15}$$

where the linear operator A and the nonlinear operator $N(f, f)$ are given by

$$Af \;=\; \left(x^n f_{xx}\right)_{xx}, \tag{16}$$

$$N(f, \tilde{f}) \;=\; \left((x^n - h^n)\tilde{f}_{xx}\right)_{xx} + f_x \left(x^{n-1} \tilde{f}_{xx}\right)_{|x=0} \tag{17}$$

with $h = x + \int_0^x f \,\mathrm{d}\tilde{x}$. We use the notation $N(f) := N(f, f)$. The analysis in this paper relies on the dissipative structure of (15). The energy estimate (6) in terms of f takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty f^2 \,\mathrm{d}x + \int_0^\infty h^n f_{xx}^2 \,\mathrm{d}x \;=\; 0. \tag{18}$$

Indeed, (18) follows by using f as a test function on (14) and integration by parts. Note that the second term in (6), related to the interfacial energy between liquid and solid is not seen in this dissipation relation. Indeed, this term is related to the size of the support of the droplet which is infinite in the situation of (18).
We next consider the situation, where the initial profile represents a "single droplet". More precisely, we assume that supp h_{in} is compact and simply connected. Furthermore, we assume that $h_{\text{in}} > 0$ in the interior of the support. By the scaling invariance (50) of the model, it is enough to consider the case when supp $h_{\text{in}} = (0, 1)$. We apply a local variant of the transformation (8). The following coordinate transform has been for example used by Angenent [2] for the 1-d porous medium equation, see also [23] for an analogous transformation. We define $M(t)$ as the center of the wetted region and $D(t)$ as its diameter, that is

$$M(t) \;:=\; \frac{1}{2}(s_+(t) + s_-(t)), \quad D(t) \;:=\; s_+(t) - s_-(t). \tag{19}$$

The time-dependent footprint $(s_-(t), s_+(t))$ of the droplet is then transformed onto a fixed set by the coordinate transformation

$$\hat{x} \;=\; \frac{1}{D(t)}(x - M(t)) + \frac{1}{2}, \quad \hat{t} \;=\; t, \quad \hat{h}(\hat{t}, \hat{x}) = \frac{1}{D(t)} h(t, x). \tag{20}$$

In particular, we have $h_t = \dot{D}\hat{h} + D\hat{h}_{\hat{t}} + D\hat{h}_{\hat{x}}(\frac{x-M}{D})_t$ and $(h^n h_{xxx})_x = D^{n-3}(\hat{h}^n \hat{h}_{\hat{x}\hat{x}\hat{x}})_{\hat{x}}$. It follows that, in the new variables, equation (2) is transformed onto the fixed domain $(\hat{t}, \hat{x}) \in (0, \infty) \times (0, 1)$ and takes the form

$$\mathscr{L}(\hat{h}) := (D\hat{h})_{\hat{t}} - \left(\hat{x}\partial_{\hat{t}} s_+ + (1-\hat{x})\partial_{\hat{t}} s_-\right)\hat{h}_{\hat{x}} + D^{n-3}\left(\hat{h}^n \hat{h}_{\hat{x}\hat{x}\hat{x}}\right)_{\hat{x}} = 0. \quad (21)$$

The corresponding boundary conditions are

$$\hat{h} = 0, \quad |\partial_{\hat{x}}\hat{h}| = 1 \quad \text{for } \hat{x} = 0, 1. \quad (22)$$

We will skip the hats in our notation in the sequel.

3. Spaces

By (5), solutions of (2) are singular at the moving contact points. In particular, by (5), we expect that $h - (x - s(t)) \simeq (x - s(t))^{4-n}$ and thus $f \simeq (x - s(t))^{3-n} = (x - s(t))^{\alpha}$ near the contact point $x = s(t)$, where for $n \in (0, 3)\backslash\{1, 2\}$ we define

$$\alpha := 3 - n \in (0, 3)\backslash\{1, 2\}. \quad (23)$$

Note that the power x^{α} is also an element of the kernel of A. Indeed, one can calculate that the kernel of A for $n \in (0, 3)\backslash\{1, 2\}$ is given by

$$\text{kern}\, A = \text{span}\langle x^{\alpha-1},\ 1,\ x^{\alpha}, x\rangle. \quad (24)$$

The four dimensional space kern A is hence spanned by the *two regular functions* 1 and x (which are smooth up to the boundary) and by the *two singular functions* x^{α} and $x^{\alpha-1}$. In particular, repeated action of ∂_x on either x^{α} or $x^{\alpha-1}$ eventually leads to a blow up of the derivated function at $x = 0$ which leads to some technical issues in the analysis of (2). The power series ansatz (5) suggests that solutions of the nonlinear equation (2) have a certain power series expansion in the two variables x and x^{α} at the moving contact point. In the following, we define norms and spaces which allow for such expansions at the contact point, see the appendix for more details.

3.1. Weighted Sobolev Spaces

We will use weighted Sobolev spaces: Let $n \in (0, 3)$, $\ell \in \mathbb{N}_0$ and let $\varepsilon \in \mathbb{R}$ with $\varepsilon + \frac{\ell n}{2} > -1$. We then define for $f, g \in C_c^{\infty}([0, \infty))$

$$\langle f, g\rangle_{H_\ell} := \int_0^{\infty} x^{\frac{\ell n}{2}+\varepsilon}(\partial_x^{\ell} f)(\partial_x^{\ell} g)\, \mathrm{d}x \quad \text{and} \quad [f]_{H_\ell} := \langle f, f\rangle_{H_\ell}^{1/2}. \quad (25)$$

The corresponding norms for $k \in \mathbb{N}_0$ are defined by

$$\|f\|_{H_k} := \left(\sum_{\ell=0}^{k} [f]_{H_\ell}^2\right)^{1/2}. \quad (26)$$

We define H_k as the completion of $C_c^\infty([0,\infty))$ with respect to the norm $\|\cdot\|_{H_k}$. Furthermore, $\mathring{H}_k$ is the completion of $C_c^\infty((0,\infty))$ with respect to the norm $\|\cdot\|_{H_k}$. In Lemma 5.4, we will show that, under certain assumptions on ε, $\mathring{H}_k$ is the subspace of functions $f \in H_k$ which satisfy $f = o(x^{\delta_k})$ near $x = 0$. Note that for $\varepsilon = 0$ the homogeneous $[\cdot]_{H_0}$-norm, which coincides with the L^2-norm, represents the physical energy of the model, cf. (18).

Most of the results in this paper hold in the case $\varepsilon = 0$. We have chosen to work with more general weights for a number of reasons. One reason is that there is a failure of the elliptic estimates and Hardy's inequality for a certain discrete set of values $n \in (0,3)$ if $\varepsilon = 0$. This is analogous to a similar phenomenon for elliptic equations on singular domains where there is a loss of regularity for certain angles. The second reason is that we can use ε to extend the range of values n where the estimates apply. Indeed, with the assumption $\varepsilon = 0$, with our methods we could only prove well-posedness in the set $n \in (0, \frac{5}{2})\backslash E$ for some finite set E. With the introduction of ε we are able to extend our results to the larger set $n \in (0, \frac{14}{5})$. We will impose certain conditions on ε, which are stated in Section 3.3.

For $n > \frac{1}{2}$, we also need to use homogeneous norms of type (36), related to the negative index -2: For $n \in (0,3)$, we say that $f \in H_{-2}$, if

$$[f]_{H_{-2}} := \left(\int_0^\infty x^{-n+\varepsilon} F^2 \, \mathrm{d}x,\right)^{1/2} < \infty \tag{27}$$

for some F with $F_{xx} = f$. The inner product $\langle f, g\rangle_{H_{-2}}$ is defined correspondingly. The definition of norms and spaces can be generalized to the case of compact domains. We consider the particular case $\Omega := (0,1)$. Suppose that $f, g \in C^\infty(\overline{\Omega})$. With the notation $d(x) := \mathrm{dist}(x, \partial\Omega)$, the local variant of (25) is

$$\langle f, g\rangle_{H_\ell(\Omega)} := \int_\Omega d(x)^{\frac{\ell n}{2}+\varepsilon} (\partial_x^\ell f)(\partial_x^\ell g) \, \mathrm{d}x. \tag{28}$$

The corresponding norms $[\cdot]_{H_k(\Omega)}$ and $\|\cdot\|_{H_k(\Omega)}$ are defined analogously. As before, the corresponding spaces are defined by completion. Most of our analysis will, however, be concerned with the global setting (25)–(26).

It is convenient to write the homogeneous norms in the form

$$[f]_{H_\ell}^2 = \sum_{j=0}^{\ell} \int_0^\infty |x^{-\delta_\ell + j} (\partial_x^j f)|^2 \, \frac{\mathrm{d}x}{x}, \tag{29}$$

where δ_ℓ is given by

$$\delta_\ell := \frac{\ell}{4}(4-n) - \frac{\varepsilon+1}{2}. \tag{30}$$

The parameter δ_ℓ characterizes the behaviour of the homogeneous norm $[\cdot]_{H_\ell}$ under rescaling. Indeed, with the change of coordinates $\tilde{f}(\lambda x) = f(x)$, we have

$$[f]_{H_\ell} = \lambda^{\delta_\ell} \left[\tilde{f}\right]_{H_\ell}. \tag{31}$$

Let $k \in \mathbb{N}$ and suppose that (44) holds. Then for any $f \in \mathring{H}_k$, we have

$$\|x^{-\gamma+j}\partial_x^j f\|_{L^\infty((0,\infty))} \leqq C\|f\|_{H_k} \quad \forall 0 \leqq j \leqq k-1 \text{ and } \gamma \in [\delta_j, \delta_k]. \tag{32}$$

Estimate (32) follows from Lemma 5.4.

3.2. Weighted Sobolev Spaces with Singular Expansion

In view of (5), solutions of (15) are, generally, not regular at the boundary and in particular $f(t,\cdot) \notin H_k$. In the following, we hence define a larger class of spaces X_k, where expansions of type (36) are possible.
We first specify the type of power series expansions in x and x^α which can appear at the contact point. The space of power series expansions is defined by

$$\mathscr{P} := \left\{ p = \sum_{i,j\in\mathbb{Z}} a_{ij} x^{i+j\alpha} \ : \ |a_{ij}| \text{ is summable} \right\} \tag{33}$$

and for coefficients $a_{ij} \in \mathbb{R}$ and with $x \geqq 0$. We define the corresponding norm by

$$\|p\|_{\mathscr{P}}^2 := \sum_{i,j\in\mathbb{Z}} |a_{ij}|^2. \tag{34}$$

This norm is finite for $p \in \mathscr{P}$ since $\ell^1 \subset \ell^2$. Notice that the space $\mathscr{P}$, equipped with the norm (34) is not complete. Our reason to choose the space $\mathscr{P}$ (instead of the larger space of polynomials which are bounded in the ℓ^2-norm) is that for polynomials in $\mathscr{P}$, we have absolute convergence of the power series and hence the order of summation does not change the sum. We set

$$\begin{aligned} I_X &:= \{\alpha, 1\} \cup \left\{ i + j\alpha \ : \ i, j \in \mathbb{Z}, i + j \geqq 3, i \geqq 0, j \geqq 1 \right\}, \\ I_Y &:= \left\{ i + j\alpha \ : \ i, j \in \mathbb{Z}, i + j \geqq 1, i \geqq -1, j \geqq 0 \right\}, \end{aligned} \tag{35}$$

see also Fig. 2. We define the subspaces $\mathscr{P}_X$, $\mathscr{P}_Y$ of $\mathscr{P}$ with $\mathscr{P}_X \subset \mathscr{P}_Y$ by

$$\begin{aligned} \mathscr{P}_X &:= \left\{ \sum_{i+j\alpha\in I_X} a_{ij} x^{i+j\alpha} \ : \ a_{ij} = 0 \text{ for } i, j \notin I_X \right\}, \\ \mathscr{P}_Y &:= \left\{ \sum_{i+j\alpha\in I_Y} a_{ij} x^{i+j\alpha} \ : \ a_{ij} = 0 \text{ for } i, j \notin I_Y \right\}. \end{aligned} \tag{36}$$

We also define $\mathscr{P}_{X,\delta_k}$, respectively $\mathscr{P}_{Y,\delta_k}$, as the subspace of $\mathscr{P}_X$, respectively $\mathscr{P}_{Y,\delta_k}$, where only powers of order δ_k or less appear, that is with the extra condition that $i + j\alpha \leqq \delta_k$. The choice of these spaces is motivated by the structure of the linear and nonlinear operator, cf. (16) and (17). We will show that the solution satisfies $f(t,\cdot) - p(t) \in C^k([0,\infty))$ for some $p(t) \in \mathscr{P}_X$ for all $t > 0$ and for sufficiently regular initial data. The space $\mathscr{P}_Y$ is used to describe analogously the

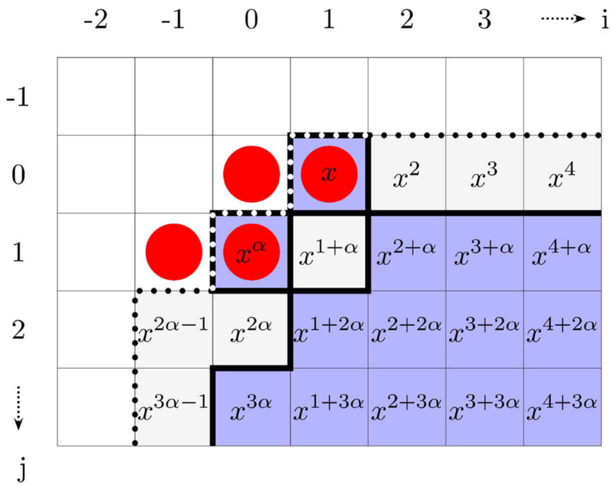

Fig. 2. The two sets illustrate the monomials $x^{i+j\alpha}$ included into the spaces $\mathscr{P}_X$ (*solid boundary*) and $\mathscr{P}_Y$ (*dotted boundary*). The four elements in the kernel of A are marked by *solid circles*

expansion of $N(f)$. Some facts about the relation of $\mathscr{P}_X$, $\mathscr{P}_Y$ and the operator (15) are given in the appendix.
From (32) it follows that for any $f \in H_k$, the expansion of f at the contact point is defined up to an order of x^{δ_k}. The corresponding subspaces of $\mathscr{P}_X$ and $\mathscr{P}_Y$ are denoted by $\mathscr{P}_{X,\delta_k}$, respectively $\mathscr{P}_{Y,\delta_k}$. We call the orthogonal projection p_{δ_k} of p on the space $\mathscr{P}_{X,\delta_k}$ the *expansion of f of order* δ_k. Note that by (32), p_{δ_k} is well-defined for $f \in X_k$.
For $k \geqq 0$, we define the space $\mathscr{D}_X$ as the set

$$\mathscr{D}_X = \Big\{ f \in C^\infty((0,\infty)) \ : \ \exists p \in \mathscr{P}_X \text{ s.t. } \lim_{x\to 0} \partial^k (f-p)(x) = 0 \ \forall k \in \mathbb{N} \\ \text{and } \exists R > 0 \text{ s.t. } \operatorname{supp} f \in [0,R)] \Big\}. \tag{37}$$

For $f \in \mathscr{D}_X$, $k \in \mathbb{N}_0$, we define the norm $\|f\|_{X_k}$ by

$$\|f\|_{X_k}^2 := \sum_{j=0}^{k} [f]_{X_j}^2, \quad \text{where } [f]_{X_k} := \left[f - p_{\delta_j}\right]_{H_j} \tag{38}$$

and where $p_{\delta_j} \in \mathscr{P}_{X,\delta_j}$ is the expansion of f of order δ_j. The corresponding space X_k, is the completion of $\mathscr{D}_X$ with respect to (38). For $g \in \mathscr{D}_Y$, $k \in \mathbb{N}_0$, we correspondingly define

$$\|g\|_{Y_k}^2 := \sum_{j=0}^{k} [g]_{Y_j}^2, \quad \text{where } [g]_{Y_k} := \left[g - p_{\delta_j}\right]_{H_j}, \tag{39}$$

where $p_{\delta_j} \in \mathscr{P}_{Y,\delta_j}$ is the expansion of g of order δ_j. The spaces $\mathscr{D}_Y$, Y_k are defined analogously. Note that X_k and Y_k are Hilbert spaces. Related norms have been used for example in [21,31]. Let $p_{\delta_k} \in \mathscr{P}_X$ be the expansion of f of order δ_k. In Proposition 5.5, we show that all coefficients of p_{δ_k} are estimated by $\|f\|_{X_k}$, that is

$$\|p_{\delta_k}\|_{\mathscr{P}} \leqq C_n \|f\|_{X_k}. \tag{40}$$

A short calculation yields that the norm (38) can be estimated by

$$c_{nk}(\|f - \zeta p_{\delta_k}\|_{H_k} + \|p_{\delta_k}\|_{\mathscr{P}}) \leqq \|f\|_{X_k} \leqq C_{nk}(\|f - \zeta p_{\delta_k}\|_{H_k} + \|p_{\delta_k}\|_{\mathscr{P}}), \tag{41}$$

where $\zeta \in C_c^\infty([0,\infty))$ is any fixed universal cut-off function with

$$\zeta(x) = 1 \text{ for } x \leqq \tfrac{1}{4}, \quad \zeta(x) = 0 \text{ for } x \geqq \tfrac{1}{2}, \quad \zeta_x(x) \leqq 0 \text{ for } x \geqq 0. \tag{42}$$

The estimate (41) shows that the expression

$$\|f - \zeta p_{\delta_k}\|_{H_k} + \|p_{\delta_k}\|_{\mathscr{P}} \tag{43}$$

provides an equivalent way to define the norm $\|\cdot\|_{X_k}$. The form (43) has been used in [31], the version (38) of the norm is used in [21]. The subtraction of regular polynomial expansions has been used in the theory of elliptic equations on non smooth domains, for some introduction and references, see for example [35]. In the following we will use both representations of the norm. The corresponding statements also hold for the space Y_k and its norm.

3.3. Conditions on ε

For given $k \in \mathbb{N}_0$, we will assume that ε satisfies the following four conditions. The first condition is

$$\delta_0, \ldots, \delta_k \notin \mathbb{N}_0 \cup \{\alpha - 1, \alpha\}, \tag{44}$$

where the (ε-dependent) numbers δ_ℓ are defined in (30). This condition avoids critical scalings in the application of Hardy's inequality (Lemma 5.1) and the elliptic estimates for the linear part of the operator (Lemma 6.3). The second condition ensures coercivity of the linearized parabolic evolution operator (Lemma 6.2):

$$\left(n \in \left(0, \frac{5}{2}\right) \text{ and } \varepsilon \in [0, \varepsilon_0)\right) \text{ or } \left(n \in \left[\frac{5}{2}, 3\right) \text{ and } \varepsilon \in \left[0, \frac{3-n}{2}\right)\right), \tag{45}$$

where $\varepsilon_0 \in (0, \frac{1}{4})$ is the constant from Lemma 6.2. The next condition ensures that the $\|\cdot\|_{H_2}$-norm is strong enough to formulate the condition $f(0) = 0$. This is used in the proof of Lemma 7.1. The condition is:

$$\varepsilon < 3 - n. \tag{46}$$

The last condition is related to the nonlinearity of the model:

$$\varepsilon > 4n - 11. \tag{47}$$

This condition ensures that $N(f)$ is locally L^2 integrable in terms of the measure $x^\varepsilon \mathrm{d}x$, related to our base norm $[\cdot]_{H_0}$. We need this property to test the equation. It can be easily checked that for $n \in (0, \frac{14}{5})$ there is always a non-empty interval of parameters ε which satisfy the conditions (44)–(47). Indeed, for $n \in (0, \frac{5}{2})$, conditions (45)–(47) are satisfied for all $\varepsilon \in [0, \min\{\varepsilon_0, 3-n\})$. For $n \in [\frac{5}{2}, \frac{14}{5})$, conditions (45)–(47) are satisfied for all $\varepsilon \in (\max\{0, \frac{4n-11}{2}\}, \frac{3-n}{2})$. Furthermore, (44) holds for all but a finite number of values for ε for any given $n \in (0, 3)$ and $k \in \mathbb{N}$. Also, the two conditions (46) and (47) cannot be both satisfied if $n \geqq \frac{14}{5}$.

3.4. Discussion on the Spaces and Their Relation to the Model

We shortly discuss the differences of the spaces H_k and X_k and their relation to the model (2)–(3). Let $n \in (0, 3)\backslash\{1, 2\}$, $k \in \mathbb{N}$, and suppose that ε satisfies (44)–(47). By (74) and by (44), it follows that $x^\beta \zeta \in H_k$ for $\beta \notin \mathbb{Z}$ is equivalent to $\beta > \delta_k$ where ζ is the cut-off function from (42). By (30),

$$\delta_0 = -\frac{1}{2}(1+\varepsilon), \quad \delta_1 = \frac{1}{4}(2-n-2\varepsilon), \quad \delta_2 = \frac{1}{2}(3-n-\varepsilon). \tag{48}$$

By (45)–(46), we have $\delta_0 < 0$, $0 < \delta_2 < 3-n$ and therefore

$$X_2 = \{f \in H_2 : f(0) = 0\}. \tag{49}$$

In particular, the trace of f at $x = 0$ is well-defined in H_k, $\mathring{H}_k$ and X_k for $k \geqq 2$. By the definition of the spaces, for $k \geqq 2$, we have $f(0) = 0$ for all $f \in X_k$ or $f \in \mathring{H}_k$, but generally not for functions $f \in H_k$.

We next discuss the relation of the norms to the nonlinear operator: the nonlinearity of (15) is characterized by its scaling invariance and by the expansion of the nonlinearity $N(f)$. Observe that equation (15) is invariant with respect to the scaling

$$(x, t, f) \mapsto \left(\lambda x, \lambda^{4-n} t, f\right). \tag{50}$$

By (31), the only homogeneous norm $[f]_{H_k}$, $k \in \mathbb{R}$, which is formally scaling invariant under the rescaling (50) of x and f is $[\cdot]_{H_{k_{\text{crit}}}}$, where

$$k_{\text{crit}} = \frac{2+2\varepsilon}{4-n} \in \left(\frac{1}{4}, 2\right). \tag{51}$$

Note that it is possible to define homogeneous norms of type $[\cdot]_{H_k}$ for general $k \in \mathbb{R}$ in terms of the Mellin transform, see Section 6. In this paper, we will, however, only use norms with integer indices. Note that $x^{2\alpha-1}$ is the most singular expansion of Y_0 for large n (and also the most singular expansion which generically appears in $N(f)$). Indeed, (47) is equivalent to $2\alpha - 1 > \delta_0$. It follows that

$$Y_0 = H_0. \tag{52}$$

3.5. Norms in Space and Time

Since the considered model is a boundary value problem, we need to consider parabolic norms where the maximal number of space and time derivatives are coupled. For $f \in C^\infty([0,\infty), \mathscr{D}_X)$ and for $k \geqq 0$, we define

$$\|f\|^2_{TX_k} := \sum_{2 \leqq 4i+j \leqq k} \left[\partial_t^i (f - p_{\delta_j})\right]^2_{L^2(H_j)} + \sum_{0 \leqq 4i+j \leqq k-2} \left[\partial_t^i (f - p_{\delta_j})\right]^2_{C^0(H_j)}, \tag{53}$$

where $i, j \in \mathbb{N}_0$ and where $p_{\delta_j} \in \mathscr{P}_X$ is the expansion of f of order δ_j. For $g \in C^\infty([0,\infty), \mathscr{D}_Y)$, $k \geqq 0$ and $n \in (0, \frac{5}{2}]$, we correspondingly define

$$\|g\|^2_{TY_k} := \sum_{0 \leqq 4i+j \leqq k} \left[\partial_t^i (g - q_{\delta_j})\right]^2_{L^2(H_j)} + \sum_{0 \leqq 4i+j \leqq k-2} \left[\partial_t^i (g - q_{\delta_j})\right]^2_{C^0(H_j)}, \tag{54}$$

where $i, j \in \mathbb{N}_0$ and where q_{δ_j} is the expansion of g of order δ_j. For $n \in (\frac{5}{2}, 3)$, we also include the homogeneous norm (27) with negative index, that is we set

$$\|g\|^2_{TY_k} := \text{right hand side of (54)} + \sum_{0 \leqq 4i-2 \leqq k} \left[\partial_t^i g\right]^2_{L^2(H_{-2})}. \tag{55}$$

The reason for including the norms with negative indices into the definition of the norm for large values of n is the following: if $f \in TX_{k+2}$ is the solution of (2), then in general the power $x^{2\alpha-1}$ is included in the expansion of $N(f)$, cf. Lemma 10.1. We hence need to control the coefficient in front of the expansion $x^{2\alpha-1}$ by the norm $\|g\|_{TY_k}$. However, for $n \in (\frac{5}{2}, 3)$, we have $2\alpha - 1 = 5 - 2n < \delta_0$. This is the reason why we need to include the homogeneous norm $[\cdot]_{H_{-2}}$ into the norm $\|\cdot\|_{TY_k}$. On the other hand, in general, the only control, we have on f for $x \to \infty$ is $f(x_k) \leqq C x_k^{-\frac{1}{2}}$ for some sequence $x_k \to \infty$ since $X_2 \subset L^2(0,\infty)$. In particular, we might have $\|N(f)\|_{H_{-2}} = \infty$ for $n \in (0, \frac{5}{2})$. This is the reason, why we cannot include the norms with negative index for small values of n, see also Lemma 8.3 for more details. Note also that we include the supremum norms into the definition of $\|\cdot\|_{X_k}$ because we do not have a trace estimate in these norms. Note also that the second sum is empty for both (53) and (54) for $k < 2$.

The spaces TX_{k+2} are defined by completion of $C_c^\infty([0,\infty), \mathscr{D}_X)$ with respect to the norms $\|\cdot\|_{TX_{k+2}}$, correspondingly the spaces TY_{k-2} are defined by completion of $C_c^\infty([0,\infty), \mathscr{D}_Y)$ with respect to the norm $\|\cdot\|_{TY_{k-2}}$. Let $T > 0$, $\Omega \subset \mathbb{R}$ and let $Q_T = (0,T) \times \Omega$. The corresponding spaces $TX_{k+2}(Q_T)$ and norms $\|\cdot\|_{TX_{k+2}(Q_T)}$ (respectively $TY_{k-2}(Q_T)$ and norms $\|\cdot\|_{TY_{k-2}(Q_T)}$) are defined analogously where the interval of integration in time is $(0,T)$ and where we use (28) instead of (25).

3.6. Compatibility Conditions

In order to have higher regularity of the solution, we need to impose compatibility conditions for the initial data. For $f_{\text{in}} \in X_k$, $g \in TX_{k-2}$, consider first a solution $f \in TX_{k+2}$ of the linear equation

$$\begin{cases} f_t + Af \ = \ g & \text{for } (x,t) \in (0,\infty)^2, \\ f \ = \ 0 & \text{for } x = 0, \\ f \ = \ f_{\text{in}} & \text{for } t = 0. \end{cases} \tag{56}$$

Recall that the condition $f = 0$ for $x = 0$ is included in the definition of the space TX_{k+2}. Derivating (56) in time and inductively using (56), we get

$$\partial_t^j f \ = \ (-A)^j f + \sum_{i=0}^{j-1} (-A)^i \partial_t^{j-i-1} g. \tag{57}$$

Since $f = 0$ at $x = 0$, we also have $\partial_t^j f_{|x=0} = 0$, for all $j \in \mathbb{N}_0$ such that this expression is well-defined. By evaluating (57) at $t = x = 0$, it hence follows that any solution $f \in TX_{k+2}$ of the parabolic equation (56), the initial data and right hand side necessarily satisfies $(-A)^j f_{\text{in}} + \sum_{i=0}^{j-1} (\partial_t^i (-A)^{j-1-i} g)_{|t=0} = 0$ at $x = 0$, for all $j \in \mathbb{N}_0$ such that the identity is well-defined. More generally, since $f \in TX_k$ we have $\partial_j^t f \in C^0(X_{k-4j})$ and hence

$$(-A)^j f_{\text{in}} + \left(\sum_{i=0}^{j-1} (-A)^i \partial_t^{j-i-1} g \right)_{|t=0} \in \ X_{k-4j} \quad \text{for } 0 \leqq 2 + 4j \leqq k. \tag{58}$$

We call (58) the compatibility conditions up to order k for the linear problem with initial data f_{in} and right hand side g. With the identification $g := N(f)$, the conditions (58) can also be interpreted as compatibility conditions for the nonlinear equation (15). Indeed, by repeated use of (15), all time derivatives of g which appear in (58) can be replaced by space derivatives. The $(j = 1, k = 6)$-compatibility condition for the nonlinear equation is for example $N(f_{\text{in}}) - Af_{\text{in}} \in X_2$, the $(j = 2, k = 10)$-compatibility condition is $N'(f)(N(f_{\text{in}}) - Af_{\text{in}}) - AN(f_{\text{in}}) + A^2 f_{\text{in}} \in X_2$. We formalize this observation as follows: For any smooth $Q : \mathbb{R} \to \mathbb{R}$, we define $\mathscr{D}_{A-N} Q(f)$ by $Q'(f)(N(f) - Af)$. This inductively defines $\mathscr{D}_{A-N}^i N(f_{\text{in}})$ for $i \in \mathbb{N}$. The compatibility condition for the nonlinear operator is then given by

$$(-A)^j f_{\text{in}} + \sum_{i=0}^{j-1} \left((-A)^i \mathscr{D}_{A-N}^{j-i-1} N(f_{\text{in}}) \right)_{|t=0} \in \ X_{k-4j} \quad \text{for } 0 \leqq 2 + 4j \leqq k. \tag{59}$$

With this understanding, we will refer to (59) as the compatibility condition of order k for the nonlinear problem (15) with initial data f_{in}.

4. Main Results

We establish well-posedness for the model (2) in certain classes of weighted Sobolev spaces. In particular, we show long-time existence for initial data which are a perturbation of the stationary solution. We also show short-time existence for compactly supported initial data.

Our first result establishes the existence and uniqueness of classical solutions for $n \in (0, \frac{14}{5})\backslash\{1, 2\}$ for initial data in H_2. The result holds in a scale of weighted Sobolev spaces, represented by the parameter ε.

Theorem 4.1. (Well-posedness) *Let $n \in (0, \frac{14}{5})\backslash\{1, 2\}$ and suppose that ε satisfies* (44)–(47) *for $k = 4$. Then there is $\delta > 0$ (depending on n, ε) such that for any $f_{\text{in}} \in H_2$ with $f_{\text{in}}(0) = 0$ and $\| f_{\text{in}}\|_{H_2} \leqq \delta$, there is a unique solution $f \in TX_4$ of* (15)*. Furthermore, there is $C < \infty$ (depending on n, ε) such that*

$$\| f \|_{TX_4} \leqq C\| f_{\text{in}}\|_{H_2}. \tag{60}$$

Note that no compatibility condition needs to be assumed for Theorem 4.1. Let us also remark that well-posedness in the partial wetting case for $n = 1$ has been established in [32]. The solutions in this case are smooth and in particular, well-posedness in this case holds for regular Sobolev spaces. Well-posedness for $n = 2$ has been shown in [31]. In this case the solutions have a logarithmic expansion at the moving contact point. Together with these results, Theorems (4.1) hence establishes well-posedness for (2) in the range of parameters $n \in (0, \frac{14}{5})$.

The second result addresses the case of initial data with higher regularity:

Theorem 4.2. (Higher regularity) *Let $n \in (0, \frac{14}{5})\backslash\{1, 2\}$, $k \geqq 2$ and suppose that ε satisfies* (44)–(47)*. There is $\delta > 0$ (depending on n, ε, k) such that for any $f_{\text{in}} \in X_k$ with $\| f_{\text{in}}\|_{X_k} \leqq \delta$ and such that the compatibility conditions* (59) *are satisfied up to order k, there is a unique solution $f \in TX_{k+2}$ of* (15)*. Furthermore, there is $C < \infty$ (depending on n, ε, k) such that*

$$\| f \|_{TX_{k+2}} \leqq C\| f_{\text{in}}\|_{X_k}. \tag{61}$$

Indeed, one would expect that the solution becomes analytic as a function of the two variables x and x^{α} for positive times. Our next result shows that the solution indeed gains regularity (viewed as a function of two variables):

Corollary 4.3. (Smoothing) *Suppose that the assumptions of Theorem 4.1 hold. Then for all $k \in \mathbb{N}$, there is $T < \infty$ and $C < \infty$ (both depending on n, ε, k) such that the solution f of Theorem 4.1 satisfies*

$$\| f \|_{TX_{k+2}([T,\infty)\times(0,\infty))} \leqq C\| f_{\text{in}}\|_{X_2}. \tag{62}$$

The results of Theorem 4.2 can be rephrased in terms of the profile h:

Corollary 4.4. *Suppose that the assumptions of Theorem 4.2 hold. Let f be the solution from Theorem 4.2 and let $h = x + \int_0^x f$.*

1. *The function h solves the thin-film equation (2). The profile h has an expansion up to order $\delta_k + 1$. It has the form*
$$h(x - s(t), t) = x + \sum_{i+j\alpha-1\in(\delta_2,\delta_k)\cap I_X} c_{i+j\alpha}(t)x^{i+j\alpha} + o(x^{\delta_k}) \tag{63}$$
$$= x + c_{1+\alpha}(t)x^{1+\alpha} + c_{1+2\alpha}(t)x^{1+2\alpha} + \cdots + c_{2+\alpha}(t)x^{2+\alpha} + \cdots$$
for some time-dependent coefficients $c_{ij}(t)$ and where $\alpha := 3-n$. The position of the contact point $s(t)$ is given as the solution of the ordinary differential equation
$$\dot{s}(t) = \alpha(\alpha - 1)c_\alpha(t) \tag{64}$$
with $s(0) = 0$. In particular, the speed of propagation is finite.
2. *The function h can be written in the form $h(t, x) = H(t, x, x^\alpha)$ for some function $H(t, x, y)$. For any $m > 0$ there exists $T > 0$ such that $H(t, \cdot, \cdot) \in C^m([0, \infty)^2)$ for all $t > T$.*

We also have short-time existence, uniqueness and regularity for (2) in the case of initial data in the form of a single droplet:

Theorem 4.5. (Droplet case) *Let $n \in (0, \frac{14}{5})\backslash\{1, 2\}$, $k \geqq 2$, $\Omega = (0, 1)$ and suppose that ε satisfies (44)–(47). Suppose that $h_{\text{in}} \in H^1(\Omega)$ with $h_{\text{in}} > 0$ in $(0, 1)$ and with $h_{\text{in}} = 0$, $|h_{\text{in,x}}| = 1$ on $\partial\Omega$. Suppose that $f_0 := [h_{\text{in}} - \frac{1}{2}x(1 - x)]_x \in X_k(\Omega)$ satisfies the corresponding compatibility condition to (59). Then there exists $\tau > 0$ and a unique short–time solution h of (21)–(22) with initial data h_{in} in the time interval $[0, \tau]$ and such that $f := [h - \frac{1}{2}(1 - x^2)]_x \in TX_{k+2}([0, \tau] \times \Omega)$. Furthermore, we have*
$$\|f\|_{TX_{k+2}([0,\tau]\times\Omega)} \leqq C\|f_{\text{in}}\|_{X_k(\Omega)},$$
where $C < \infty$ depends on n, ε, k. The solution depends continuously on the initial data.

Notation In the following, we do not explicitly differentiate in the notation if a constant depends on the parameters k, n, ε, that is we write $C = C_{nk\varepsilon}$. Throughout the paper, we also use the notation $\alpha = 3 - n$.

5. Some Interpolation Inequalities

In this section, we state and prove some basic estimates which are useful when working with the weighted spaces H_k and X_k.
We first recall Hardy's inequality for L^2-norms and L^∞-norms:

Lemma 5.1. (Hardy's inequalities) *Let $\gamma \neq 0$ and suppose that $x^{\gamma+1}f_x \in L^2((0, \infty), \frac{\mathrm{d}x}{x})$. If $\gamma < 0$, then $c := \lim_{x\to 0} f(x)$ is well-defined. If $\gamma > 0$, then $c := \lim_{x\to\infty} f(x)$ is well-defined. In both cases, we have*
$$\int_0^\infty |x^\gamma(f - c)|^2 \frac{\mathrm{d}x}{x} \leqq \frac{1}{\gamma^2}\int_0^\infty |x^{\gamma+1}f_x|^2 \frac{\mathrm{d}x}{x}, \tag{65}$$
$$\sup_{x\in(0,\infty)} |x^\gamma(f - c)|^2 \leqq \frac{4}{\gamma}\int_0^\infty |x^{\gamma+1}f_x|^2 \frac{\mathrm{d}x}{x}. \tag{66}$$

Proof. We sketch the proof for the case $\gamma < 0$, the argument for $\gamma > 0$ proceeds analogously, see for example the related proof in the appendix of [24]. In order to see that the limit $\lim_{x\to 0} f(x)$ exists, we note that for any $0 < y < z < 1$, we have

$$|f(z) - f(y)| \leqq \int_y^z |x f_x| \frac{\mathrm{d}x}{x} \leqq \left(\int_y^z |x^\gamma f_x|^2 \frac{\mathrm{d}x}{x}\right)^{1/2} \left(\int_0^z |x^{-\gamma}|^2 \frac{\mathrm{d}x}{x}\right)^{1/2},$$

which implies that $f(x_k)$ is a Cauchy sequence for every sequence $x_k \to 0$. By replacing f by $f - c$, we may hence assume without loss of generality that $f(0) = 0$. Estimates (65)–(66) then follow by integration by parts and application of Cauchy–Schwarz. □

The next two lemmas yield control of the asymptotic expansion of type $\mathscr{P}_X$ in terms of weighted L^2-estimates. The first lemma is a multiplicative variant of a corresponding lemma in [21]:

Lemma 5.2. *Let $\gamma_1 < \gamma_2$. Then for any $\beta \in [\gamma_1, \gamma_2]$, there is $C_\beta < \infty$ (depending on β), such that for any $\lambda \in \mathbb{R}$ we have*

$$\lambda^2 \leqq C_\beta \left(\int_0^\infty \left|\frac{1}{x^{\gamma_1}} f\right|^2 \frac{\mathrm{d}x}{x}\right)^{\frac{\gamma_2-\beta}{\gamma_2-\gamma_1}} \left(\int_0^\infty \left|\frac{1}{x^{\gamma_2}}(f - \lambda x^\beta)\right|^2 \frac{\mathrm{d}x}{x}\right)^{\frac{\beta-\gamma_1}{\gamma_2-\gamma_1}} \tag{67}$$

for any $f : (0, \infty) \to \mathbb{R}$ such that the right hand side of (67) is well-defined and finite.

Proof. For all $R > 0$, we have

$$\begin{aligned} \lambda^2 &\leqq C_\beta R^{-2\beta} \int_R^{2R} |\lambda x^\beta|^2 \frac{\mathrm{d}x}{x} \\ &\leqq C_\beta R^{-2\beta} \int_R^{2R} |f|^2 \frac{\mathrm{d}x}{x} + C_\beta R^{-2\beta} \int_R^{2R} |\lambda x^\beta - f|^2 \frac{\mathrm{d}x}{x} \\ &\leqq C_\beta R^{2(\gamma_1-\beta)} \int_0^\infty \left|\frac{1}{x^{\gamma_1}} f\right|^2 \frac{\mathrm{d}x}{x} + C_\beta R^{2(\gamma_2-\beta)} \int_0^\infty \left|\frac{1}{x^{\gamma_2}}(f - \lambda x^\beta)\right|^2 \frac{\mathrm{d}x}{x}. \end{aligned} \tag{68}$$

Estimate (67) follows by minimizing the right hand side of (68) in R, that is with

$$R := \left(\int_0^\infty \left|\frac{1}{x^{\gamma_1}} f\right|^2 \frac{\mathrm{d}x}{x}\right)^{\frac{1}{2(\gamma_2-\gamma_1)}} \left(\int_0^\infty \left|\frac{1}{x^{\gamma_2}}(f - \lambda x^\beta)\right|^2 \frac{\mathrm{d}x}{x}\right)^{\frac{-1}{2(\gamma_2-\gamma_1)}}.$$

□

The next lemma provides a general interpolation result of polynomially weighted L^2-norms:

Lemma 5.3. *Let $\gamma_1 < \gamma_2 < \gamma_3$. Then for any $\beta_1 \in (\gamma_1, \gamma_2)$ and $\beta_2 \in (\gamma_2, \gamma_2)$, there exists a constant $C < \infty$, which only depends on $\beta_2 - \gamma_2$ and $\gamma_3 - \beta_2$ such that for any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have*

$$\int_0^\infty \left| \frac{1}{x^{\gamma_2}} (f - \lambda_1 x^{\beta_1}) \right|^2 \frac{\mathrm{d}x}{x} \tag{69}$$
$$\leqq C \left(\int_0^\infty \left| \frac{1}{x^{\gamma_1}} f \right|^2 \frac{\mathrm{d}x}{x} \right)^{\frac{\gamma_3 - \gamma_2}{\gamma_3 - \gamma_1}} \left(\int_0^\infty \left| \frac{1}{x^{\gamma_3}} \left(f - \lambda_1 x^{\beta_1} - \lambda_2 x^{\beta_2} \right) \right|^2 \frac{\mathrm{d}x}{x} \right)^{\frac{\gamma_2 - \gamma_1}{\gamma_3 - \gamma_1}}$$

for any $f : (0, \infty) \to \mathbb{R}$ such that the right hand side of (67) is well-defined and finite.

Proof. For $R > 0$ to be fixed later, we apply the decomposition

$$\int_0^\infty \left| \frac{1}{x^{\gamma_2}} (f - \lambda_1 x^{\beta_1}) \right|^2 \frac{\mathrm{d}x}{x} = \int_0^R \left| \frac{1}{x^{\gamma_2}} (f - \lambda_1 x^{\beta_1}) \right|^2 \frac{\mathrm{d}x}{x} + \int_R^\infty \left| \frac{1}{x^{\gamma_2}} (f - \lambda_1 x^{\beta_1}) \right|^2 \frac{\mathrm{d}x}{x}. \tag{70}$$

The first term on the right hand side of (70) is estimated by

$$\begin{aligned} \int_0^R \left| \frac{1}{x^{\gamma_2}} \left(f - \lambda_1 x^{\beta_1} \right) \right|^2 \frac{\mathrm{d}x}{x} &\leqq 2 \int_0^R \left| \frac{1}{x^{\gamma_2}} \left(f - \lambda_1 x^{\beta_1} - \lambda_2 x^{\beta_2} \right) \right|^2 \frac{\mathrm{d}x}{x} \\ &\quad + 2 \int_0^R \left| \lambda_2 x^{\beta_2 - \gamma_2} \right|^2 \frac{\mathrm{d}x}{x} \\ &\leqq 2R^{2(\gamma_3 - \gamma_2)} \int_0^R \left| \frac{1}{x^{\gamma_3}} \left(f - \lambda_1 x^{\beta_1} - \lambda_2 x^{\beta_2} \right) \right|^2 \frac{\mathrm{d}x}{x} + \frac{\lambda_2^2}{\beta_2 - \gamma_2} R^{2(\beta_2 - \gamma_2)}, \end{aligned} \tag{71}$$

where in order to get the last estimate we used that $x^{\gamma_3 - \gamma_2} < R^{\gamma_3 - \gamma_2}$ for $x < R$ and $\gamma_3 > \gamma_2$. The second term on the right hand side of (70) is similarly estimated by

$$\begin{aligned} \int_R^\infty \left| \frac{1}{x^{\gamma_2}} \left(f - \lambda_1 x^{\beta_1} \right) \right|^2 \frac{\mathrm{d}x}{x} &\leqq 2 \int_R^\infty \left| \frac{1}{x^{\gamma_2}} f \right|^2 \frac{\mathrm{d}x}{x} + 2 \int_R^\infty \left| \lambda_1 x^{\beta_1 - \gamma_2} \right|^2 \frac{\mathrm{d}x}{x} \\ &\leqq 2R^{2(\gamma_1 - \gamma_2)} \int_R^\infty \left| \frac{1}{x^{\gamma_1}} f \right|^2 \frac{\mathrm{d}x}{x} + \frac{\lambda_1^2}{\gamma_2 - \beta_1} R^{2(\beta_1 - \gamma_2)}, \end{aligned} \tag{72}$$

where in order to get the last estimate we used that $x^{\gamma_1 - \gamma_2} < R^{\gamma_1 - \gamma_2}$ for $R < x$ and $\gamma_1 < \gamma_2$. We introduce the notation $[f]_\gamma^2 = \int_0^\infty |\frac{1}{\gamma} f|^2 \frac{\mathrm{d}x}{x}$. Furthermore, we set $f_0 = f - \lambda_1 x^{\beta_1}$ and $f_{00} = f - \lambda_1 x^{\beta_1} - \lambda_2 x^{\beta_2}$. With this notation, (70), (71) and (72) imply

$$[f_0]_{\gamma_2} \leqq C \left(R^{\gamma_1 - \gamma_2} [f]_{\gamma_1} + \lambda_1 R^{\beta_1 - \gamma_2} + R^{\gamma_3 - \gamma_2} [f_{00}]_{\gamma_3} + \lambda_2 R^{\beta_2 - \gamma_2} \right).$$

We apply Lemma 5.2 twice to obtain

$$[f_0]_{\gamma_2} \leqq C\left(R^{\gamma_1-\gamma_2}[f]_{\gamma_1} + R^{\beta_1-\gamma_2}[f]_{\gamma_1}^{\frac{\gamma_2-\beta_1}{\gamma_2-\gamma_1}}[f_0]_{\gamma_2}^{\frac{\beta_1-\gamma_1}{\gamma_2-\gamma_1}} \right.$$
$$\left. +R^{\gamma_3-\gamma_2}[f_{00}]_{\gamma_3} + R^{\beta_2-\gamma_2}[f_0]_{\gamma_2}^{\frac{\gamma_3-\beta_2}{\gamma_3-\gamma_2}}[f_{00}]_{\gamma_3}^{\frac{\beta_2-\gamma_2}{\gamma_3-\gamma_2}}\right).$$

Two times application of Young's inequality then yields

$$[f_0]_{\gamma_2} \leqq C\left(R^{\gamma_1-\gamma_2}[f]_{\gamma_1} + R^{\gamma_3-\gamma_2}[f_{00}]_{\gamma_3}\right). \tag{73}$$

Estimate (69) follows by minimizing the right hand side of (73) in R. □

We next apply the above inequalities to the weighted spaces, used in this paper. We will frequently use our assumption (44) on ε which excludes the critical case $\gamma = 0$ in the Hardy inequality (Lemma 5.1). We first consider the space $\mathring{H}_k$. Multiple application of Hardy's inequality yields:

Lemma 5.4. *Let $n \in (0, 3)$, $k \in \mathbb{N}_0$, and suppose that $\varepsilon > 0$ satisfies (44). Then we have for any $f \in \mathring{H}_k$,*

$$\int_0^\infty |x^{-\gamma+j}\partial_x^j f|^2 \frac{\mathrm{d}x}{x} \leqq C\|f\|_{H_k}^2, \quad \forall 0 \leqq j \leqq k \text{ and } \gamma \in [\delta_j, \delta_k], \tag{74}$$

$$\sup_{x\in(0,\infty)} |x^{-\gamma+j}\partial_x^j f|^2 \leqq C\|f\|_{H_k}^2, \quad \forall 0 \leqq j \leqq k-1 \text{ and } \gamma \in [\delta_j, \delta_k]. \tag{75}$$

The constants $C < \infty$ in the estimates depend on k, j, ε, γ.

Proof. The assertion follows by repeated application of Hardy's inequalities (65) and (66). Note that the assumption (44) excludes the critical case for Hardy's inequality. □

The previous estimates can be applied to obtain information on the coefficients in the expansion for functions in X_k:

Proposition 5.5. *Let $n \in (0, 3)$, $k \in \mathbb{N}_0$ and suppose that ε satisfies (44). Let $f \in X_k$ and for $\gamma \in (0, \infty)$, let $p_\gamma = \sum_{\beta\in[\delta_2,\gamma]\cap I_X} c_\beta x^\beta$ be the expansion of f of order γ. Then for all $0 \leqq j \leqq i \leqq k$ and for all $\beta \in [\delta_i, \delta_k]$, we have*

$$\left(\int_0^\infty \left|\frac{1}{x^\beta}\left(x^j\partial_x^j(f-p_\beta)\right)\right|^2 \frac{\mathrm{d}x}{x}\right)^{1/2} \leqq C\,[f-p_{\delta_i}]_{H_i}^{\frac{\delta_k-\beta}{\delta_k-\delta_i}}\,[f-p_{\delta_k}]_{H_k}^{\frac{\beta-\delta_i}{\delta_k-\delta_i}}. \tag{76}$$

Furthermore, if $j \leqq i-1$, then we have

$$\sup_{x\in(0,\infty)}\left|\frac{1}{x^\beta}\left(x^j\partial_x^j(f-p_\beta)\right)\right| \leqq C\,[f-p_{\delta_i}]_{H_i}^{\frac{\delta_k-\beta}{\delta_k-\delta_i}}\,[f-p_{\delta_k}]_{H_k}^{\frac{\beta-\delta_i}{\delta_k-\delta_i}}. \tag{77}$$

Moreover, for all $\beta \in [\delta_i, \delta_k] \cap I_X$ *we have*

$$|c_\beta| \leqq C \left[f - p_{\delta_i}\right]_{H_i}^{\frac{\delta_k-\beta}{\delta_k-\delta_i}} \left[f - p_{\delta_k}\right]_{H_k}^{\frac{\beta-\delta_i}{\delta_k-\delta_i}}. \tag{78}$$

The constants $C < \infty$ *in the estimates depend on* j, i, k, β. *Note that the right hand sides in* (76)–(78) *are estimated by above by* $C\|f\|_{X_k}$.

Proof. Note that the critical scaling of the Hardy inequalities is excluded by the condition (44). An application of Hardy's inequalities hence yields

$$\int_0^\infty \left|\frac{1}{\delta_k}\left(x^j \partial_x^j (f - p_{\delta_k})\right)\right|^2 \frac{\mathrm{d}x}{x} \leqq C\left[f - p_{d_k}\right]_{H_k}^2 \quad \forall 0 \leqq j \leqq k, \tag{79}$$

$$\sup_{x\in(0,\infty)} \left|\frac{1}{\delta_k}\left(x^j \partial_x^j (f - p_{\delta_k})\right)\right|^2 \leqq C\left[f - p_{d_k}\right]_{H_k}^2 \quad \forall 0 \leqq j \leqq k-1. \tag{80}$$

Indeed, consider $F := f - p_{d_k}$. To see (80), we then have to show for all $0 \leqq j \leqq k$,

$$\int_0^\infty \left|\frac{1}{\delta_k}\left(x^j \partial_x^j F\right)\right|^2 \frac{\mathrm{d}x}{x} \leqq C \int_0^\infty \left|\frac{1}{\delta_k}\left(x^k \partial_x^k F\right)\right|^2 \frac{\mathrm{d}x}{x}. \tag{81}$$

This estimate follows from Hardy's inequality (65), since $\lim_{x\to 0} \partial_x^i F = 0$ for $i = 1, \ldots, k$. The estimate (80) follows similarly from (66). By repeated application of Hardy's inequality (79), the right hand sides of (76)–(77) are estimated by below by

$$\begin{aligned}
&\left[f - p_{\delta_i}\right]_{H_i}^{\beta-\delta_i} \left[f - p_{\delta_k}\right]_{H_k}^{\delta_k-\beta} \\
&\quad \geqq c\left(\int_0^\infty \left|\frac{1}{x^{\delta_i}}\left(x^j \partial_x^j (f - p_{\delta_i})\right)\right|^2 \frac{\mathrm{d}x}{x}\right)^{\frac{\beta-\delta_i}{2}} \\
&\quad \left(\int_0^\infty \left|\frac{1}{x^{\delta_k}}\left(x^j \partial_x^j (f - p_{\delta_k})\right)\right|^2 \frac{\mathrm{d}x}{x}\right)^{\frac{\delta_k-\beta}{2}}
\end{aligned}$$

for some constant $c > 0$ (depending k, i, δ_k, δ_i). The L^2 estimates in (76) then follows by application of Lemma 5.2 on $F := \partial_x^j f$. Estimate (77) follows analogously by application of (66). Estimate (78) follows from (76) with $j = 0$ and Lemma 5.2. □

The next result provides point-wise bounds of f up to order $k - 1$ for any $f \in X_k$:

Lemma 5.6. *Let* $n \in (0, 3)$, $k \in \mathbb{N}_0$ *and suppose that* (44) *holds. Let* $f \in X_k$ *and let* p_{δ_k} *be the expansion of order* δ_k *of* f. *Then we have for all* $0 \leqq j \leqq k-1$, *there is* $C < \infty$ *(depending on n,ε,k) such that*

$$\left|\partial_x^j (f(x) - p_{\delta_k}(x))\right| \leqq C x^{\delta_k - j} \|f\|_{X_k} \quad \textit{for } x \in (0, \infty). \tag{82}$$

Furthermore, for any $R > 0$, *there is* $C_R < \infty$ *(depending on n,ε,k,R) such that for all* $0 \leqq j \leqq k-1$, *we have*

$$\left|\partial_x^j f(x)\right| \leqq C_R x^{\delta_j - j} \|f\|_{X_k} \quad \textit{for } x \in (R, \infty). \tag{83}$$

Proof. Estimate (82) follows directly (77) with $\beta = \delta_k$ where we note that the right hand side of (77) is estimated by $C\|f\|_{X_k}$. By (77), for every $R \in (0, \infty)$ there is $C_R < \infty$ such that

$$|\partial_x^j p_{\delta_j}(x)| \leqq C_R x^{\delta_j - j} \|f\|_{X_k} \quad \text{for } x \in (R, \infty). \tag{84}$$

Estimate (83) follows from (82) with $k = j$, (84) and the triangle inequality. □

The next lemma will be used for the estimate of the nonlinear operator. In contrary to Lemma 5.6, the estimates in Lemma 5.7 hold for derivatives up to order $j = k$.

Lemma 5.7. *Let $n \in (0, 3)$, $k \in \mathbb{N}_0$ and suppose that ε satisfies (44). Suppose that $f \in X_k$ and let $F : (0, \infty) \to \mathbb{R}$ be given by*

$$F(x) := \frac{1}{x} \int_0^x f(\tilde{x})\, \mathrm{d}\tilde{x}. \tag{85}$$

Then $F \in X_k$ and

$$\|F\|_{X_k} + \|x F_x\|_{X_k} \leqq C\|f\|_{X_k}. \tag{86}$$

Moreover, for all $0 \leqq j \leqq k$, we have

$$\left|\partial_x^j (F - P_{\delta_k})(x)\right| \leqq C x^{\delta_k - j} \|f\|_{X_k} \quad \textit{for } x \in (0, \infty), \tag{87}$$

where $P_{\delta_k} \in \mathscr{P}$ is the expansion of F of order δ_k and where the constants $C < \infty$ depend on n, ε, k. Furthermore, for $C_R < \infty$, depending on n, ε, k, R, we have

$$\left|\partial_x^j F(x)\right| \leqq C_R x^{\delta_j - j} \|f\|_{X_k} \quad \textit{for } x \in (R, \infty). \tag{88}$$

Proof. The expansion P_{δ_k} of F of order δ_k is given by

$$P_{\delta_k}(x) = \frac{1}{x} \int_0^x p_{\delta_k}(\tilde{x})\, \mathrm{d}\tilde{x},$$

where p_{δ_k} is the expansion of f of order δ_k. In particular, the coefficients of P_{δ_k} are controlled by the corresponding coefficients of p_{δ_k} and $\|P_{\delta_k}\|_{\mathscr{P}} \leqq C\|p_{\delta_k}\|_{\mathscr{P}} \leqq C\|f\|_k$. For $0 \leqq j \leqq k$, we decompose $f = f_0 + p_{\delta_j}$ and $F = F_0 + P_{\delta_j}$, in particular $F_0 = \frac{1}{x}\int_0^x f_0$ and hence $x F_{0x} = f_0 - F_0$. By Hardy's inequality, we thus have

$$\begin{aligned}
\int_0^\infty \left|x^{-\delta_j + j} \partial_x^j (x F_{0x})\right|^2 \frac{\mathrm{d}x}{x} &\leqq 2 \int_0^\infty \left|x^{-\delta_j + j} \partial_x^j f_0\right|^2 \frac{\mathrm{d}x}{x} \\
&\quad + 2 \int_0^\infty \left|x^{-\delta_j + j} \partial_x^j F_0\right|^2 \frac{\mathrm{d}x}{x} \\
&\overset{(65)}{\leqq} C \int_0^\infty \left|x^{-\delta_j + j} \partial_x^j f_0\right|^2 \frac{\mathrm{d}x}{x}.
\end{aligned}$$

This shows that (86) holds. The estimates (87) and (88) then follow similarly from the corresponding estimates (82) and (83) using Hardy's inequality. □

Proposition 5.5 allows to control the speed of the contact point:

Corollary 5.8. *Let* $n \in (0, 3)$ *and suppose that* ε *satisfies* (44)–(46) *for* $k = 4$. *Then*

$$\left|(x^{n-1} f_{xx})_{|x=0}\right| \leqq C\, [f]_{H_2}^{\frac{2}{5-n}} [f - p_4]_{H_4}^{\frac{3-n}{5-n}} \quad \textit{for all } f \in X_4. \tag{89}$$

Proof. Indeed, $(x^{n-1} f_{xx})_{|x=0} = (x^{2-\alpha} f_{xx})_{|x=0} = c_\alpha$ where c_α is the coefficient in the expansion of f corresponding to the power x^α with $\alpha = 3-n$. By (45)–(46), we have in particular $\alpha > -\varepsilon$ and $\varepsilon < 1$ and hence $\alpha \in (\delta_2, \delta_4)$. The estimate (89) then follows from (78). □

6. Coercivity and Elliptic Estimates

In this section, we investigate the elliptic operator A,

$$Af = (x^n f_{xx})_{xx},$$

viewed as an operator $A : X_4 \to X_0$. Recall that $H_0 = L^2((0, \infty))$, if $\varepsilon = 0$. We first observe that—for $\varepsilon = 0$—the operator A is symmetric and coercive with respect to L^2:

Lemma 6.1. (Symmetry for $\varepsilon = 0$) *Let* $n \in (0, 3)$ *and let* $\varepsilon = 0$. *Then*

$$\langle Af, g\rangle_{H_0} = \langle f, g\rangle_{H_2} = \langle f, Ag\rangle_{H_0} \qquad \forall f, g \in X_4. \tag{90}$$

Proof. Integrating by parts yields

$$\langle Af, g\rangle_{H_0} = \int_0^\infty (x^n f_{xx})_{xx} g = \int_0^\infty x^n f_{xx} g_{xx} = \langle f, g\rangle_{H_2}, \tag{91}$$

which yields the first identity in (90). The second identity follows by symmetry. The boundary terms in the above integration by parts vanish for all $f \in X_4$ as can be checked easily. □

The next lemma addresses coercivity of A:

Lemma 6.2. (Coercivity) *Let* $n \in (0, 3)$. *Then there is* $0 < \varepsilon_0 < \frac{1}{4}$ *such that for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *satisfying* (44) *for* $k = 2$, *we have*

$$c\, [f]_{H_2}^2 \leqq \langle Af, f\rangle_{H_0} \leqq C\, [f]_{H_2}^2 \quad \forall f \in X_4. \tag{92}$$

If $n \in [\frac{5}{2}, 3)$, *then* (92) *also holds for all* $\varepsilon \in [0, 3-n)$ *satisfying* (44) *for* $k = 2$.

Proof. Integrating by parts, we get

$$\begin{aligned}\langle Af, f\rangle_{H_0} &= \int_0^\infty x^\varepsilon (x^n f_{xx})_{xx} f \,\mathrm{d}x = \int_0^\infty x^n f_{xx} (x^\varepsilon f)_{xx} \,\mathrm{d}x \\ &= \int_0^\infty x^{n+\varepsilon} f_{xx}^2 + 2\varepsilon x^{n+\varepsilon-1} f_x f_{xx} + \varepsilon(\varepsilon-1) x^{n+\varepsilon-2} f f_{xx} \,\mathrm{d}x.\end{aligned}$$

Again integrating by parts, we have

$$2\varepsilon \int_0^\infty x^{n+\varepsilon-1} f_x f_{xx}\,\mathrm{d}x \;=\; -\varepsilon(n+\varepsilon-1)\int_0^\infty x^{n+\varepsilon-2} f_x^2\,\mathrm{d}x$$

$$\begin{aligned}\int_0^\infty x^{n+\varepsilon-2} f f_{xx}\,\mathrm{d}x &= -\int_0^\infty x^{n+\varepsilon-2} f_x^2\,\mathrm{d}x - |!(n+\varepsilon-2)\int_0^\infty x^{n+\varepsilon-3} f f_x\,\mathrm{d}x\\ &= -\int_0^\infty x^{n+\varepsilon-2} f_x^2\,\mathrm{d}x\\ &\quad + \frac{1}{2}(n+\varepsilon-2)(n+\varepsilon-3)\int_0^\infty x^{n+\varepsilon-4} f^2\,\mathrm{d}x.\end{aligned}$$

This implies

$$\langle Af, f\rangle_{H_0} \;=\; \int_0^\infty x^{n+\varepsilon} f_{xx}^2\,\mathrm{d}x + \varepsilon(2-n-2\varepsilon)\int_0^\infty x^{n+\varepsilon-2} f_x^2\,\mathrm{d}x \tag{93}$$

$$+\frac{1}{2}\varepsilon(\varepsilon-1)(n+\varepsilon-2)(n+\varepsilon-3)\int_0^\infty x^{n+\varepsilon-4} f^2\,\mathrm{d}x. \tag{94}$$

The upper bound follows by Hardy's inequality and the Cauchy–Schwarz inequality. In order to obtain the corresponding lower bound, we note that by Hardy's inequality (65) and for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small, the second integral in line (93) and the integral in line (94) can be estimated by a fraction of the integral on the right hand side of (93).
We next consider the case when $n \in [\frac{5}{2}, 3)$ and $\varepsilon \in (0, \frac{3-n}{2})$. In this case, the term in line (94) is positive, the second term on the right hand side of line (93) is negative. By Hardy's inequality, we have

$$\varepsilon(2-n-2\varepsilon)\int_0^\infty x^{n+\varepsilon-2} f_x^2\,\mathrm{d}x \overset{(65)}{\leqq} \frac{4\varepsilon(2-n-2\varepsilon)}{(n+\varepsilon-1)^2}\int_0^\infty x^{n+\varepsilon} f_{xx}^2\,\mathrm{d}x.$$

By (93)–(94) and by application of Hardy's inequality, it is hence enough to show

$$4\varepsilon(n+2\varepsilon-2) \;<\; (n+\varepsilon-1)^2 \quad \text{for } n \in \left[\frac{5}{2}, 3\right), \varepsilon \in \left[0, \frac{3-n}{2}\right). \tag{95}$$

Indeed, (95) is equivalent to $7\varepsilon^2 + 2\varepsilon(n-3) < (n-1)^2$. It is easy to check that this inequality is satisfied for all $\varepsilon \in [0, 3-n)$. This estimate and the above estimates are not optimal, but sufficient for our proof. This concludes the proof of the lower bound in (92). □

Recall that the Mellin transform of a function $f \in C_c^\infty((0,\infty))$ is defined by

$$\hat{f}(\lambda) \;=\; \int_0^\infty x^{-\lambda} f(x)\,\frac{\mathrm{d}x}{x}. \tag{96}$$

Notice that the Mellin transform of f equals the Laplace transform of $f \circ \Phi : \mathbb{R} \to \mathbb{R}$, where $\Phi(u) = e^u$. The properties of the Laplace transform hence transfer to the

Mellin transform via the change of variables $x = e^u$, $\partial_x = e^{-u}\partial_u$. For $\beta \in \mathbb{R}$ we, for example, have

$$\widehat{x^\beta f}(\lambda) = \hat{f}(\lambda - \beta), \quad \widehat{\partial_x f}(\lambda) = (\lambda + 1)\hat{f}(\lambda + 1), \tag{97}$$

as long as one of the two sides of the identity (97) is well-defined. For $f \in L^1(\mathbb{R}^n)$, the *strip of convergence* of the Mellin transform is defined as the (maximal) set of form $\mathscr{S} := (\gamma_1, \gamma_2) \times \mathbb{R} \subset \mathbb{C}$, $\gamma_1, \gamma_2 \in \mathbb{R}$ such that the integral (96) converges absolutely for all $\lambda \in \mathscr{S}$. In particular, $\hat{f}$ is analytic in $\mathscr{S}$ as a function of λ. Suppose that f has the strip of convergence $\mathscr{S} = (\gamma_1, \gamma_2) \times \mathbb{R}$. Then for any $\beta \in (\gamma_1, \gamma_2)$ and with the notation $x = e^u$, the function f can be recovered from $\hat{f}$ by application of the inverse Mellin transform:

$$f(x) = \int_{\Re\lambda = \beta} x^\lambda \hat{f}(\lambda)\, d\Im(\lambda). \tag{98}$$

Note that the right hand side of (98) does not depend on the choice of $\beta \in (\beta_1, \beta_2)$ since $e^{\lambda u}\hat{f}(\lambda)$ is analytic in $\mathscr{S}$ as a function of λ.
Plancherel's identity for the Mellin states that for all $\beta \in \mathbb{R}$, we have

$$\int_0^\infty |x^\beta f|^2 \frac{\mathrm{d}x}{x} = \int_{\Re\lambda = -\beta} |\hat{f}|^2\, d\Im(\lambda). \tag{99}$$

In view of Plancherel's identity, the Mellin transform is well-defined (by continuous extension of the linear operator defined in (96)) for all $\lambda \in \mathbb{C}$ such that the left hand side of (99) is finite for $\beta := \Re\lambda$. In this case, we say that the Mellin transform is well defined on the line $\Re\lambda = \beta$.
By Hardy's inequality (65), the Mellin transform of any f with $[f]_{H_k} < \infty$ is hence well-defined and $L^2(\frac{\mathrm{d}x}{x})$-integrable on the line $\Re\lambda = \delta_k$ and

$$[f]_{H_k}^2 = \int_{\Re\lambda = \delta_k} |(\lambda - (k-1)) \cdot \ldots \cdot (\lambda - 1)\lambda \hat{f}|^2\, d\Im(\lambda).$$

In view of the assumption (44), we have in particular

$$c \int_{\Re\lambda = \delta_k} |\lambda^k \hat{f}|^2\, d\Im(\lambda) \leqq [f]_{H_k}^2 \leqq C \int_{\Re\lambda = \delta_k} |\lambda^k \hat{f}|^2\, d\Im(\lambda) \tag{100}$$

for some constants $c > 0$, $C < \infty$, which only depend on n, k, ε. If $f \in H_k$, then $(\delta_0, \delta_k) \times \mathbb{R}$ is contained in the strip of absolute convergence. Indeed, for $\beta \in (\delta_0, \delta_k)$, we have

$$\int_0^\infty x^{-\beta}|f| \frac{\mathrm{d}x}{x} \leqq \left(\int_0^\infty |(x^{-\delta_0} + x^{-\delta_k}) f|^2 \frac{\mathrm{d}x}{x}\right)^{1/2} \times \left(\int_0^\infty \left|\frac{\min\{x^{\delta_0}, x^{\delta_k}\}}{x^\beta}\right|^2 \frac{\mathrm{d}x}{x}\right)^{1/2} \overset{(65)}{\leqq} C\|f\|_{H_k}. \tag{101}$$

By (97), for any $f \in C_c^\infty((0,\infty))$, we have

$$\widehat{Af}(\lambda - 1 - \alpha) \;=:\; p_A(\lambda)\hat{f}(\lambda), \tag{102}$$

where $\alpha = 3 - n$ and where the Mellin multiplier $p_A(\lambda)$ is given by

$$p_A(\lambda) \;:=\; \lambda(\lambda-1)(\lambda-\alpha)(\lambda-(\alpha-1)). \tag{103}$$

Elliptic regularity of the operator A in the spaces X_k is stated in the next lemma. We use the Mellin transform to obtain elliptic regularity. Alternatively, it is also possible to obtain elliptic estimates for polynomially weighted operators using real space methods. Such an approach is used for example in [21,22].

Lemma 6.3. (Elliptic regularity) *Let* $n \in (0,3)\backslash\{1,2\}$, $k \in \mathbb{N}_0$, *and suppose that* ε *satisfies* (44) *for* $k+4$. *Suppose that* $g \in Y_k$ *and let* q *be the expansion of* g *of order* δ_k. *Then the general solution of* $Af = g$ *is given by* $f = f_0 + p + w$, *where* $f_0 \in \mathring{H}_k$, *where* $p \in \mathscr{P}_X \cap (\mathrm{kern}\, A)^\perp$ *solves* $Ap = q$ *and where* $w \in \mathrm{kern}\, A$. *Furthermore,*

$$c\,[g-q]_{H_k} \leqq [f-p]_{H_{k+4}} \leqq C\,[g-q]_{H_k}\,, \tag{104}$$

where $c > 0$, $C < \infty$ *depend on* k.

Proof. We set $g_0 = g - q$. If $q^* = x^\beta$ with $\beta \in I_Y$, then $p^* = C_\beta x^{\beta+1+\alpha}$ solves $Ap^* = q^*$ for some explicitly given constant $C_\beta \neq 0$. Therefore, for $q \in \mathscr{P}_Y$, there is precisely one $p_0 \in \mathscr{P}_X$, orthogonal to kern A, such that $Ap = q$. If q is an expansion of order δ_k, then p is an expansion of order δ_{k+4}. It hence remains to find a special solution of the equation $Af_0 = g_0$. Since $[g_0]_{H_k} < \infty$, the Mellin transform $\widehat{g_0}$ is well-defined on the line $\Re\lambda = \delta_k$. Hence, $\widehat{g_0}(\cdot - 1 - \alpha)$ is well-defined on the line $\Re\lambda = \delta_{k+4}$, since $\delta_{k+4} - \delta_k = 1 + \alpha$. The application of the Mellin transform on $Af_0 = g_0$ yields

$$\widehat{f_0}(\lambda) \overset{(102)}{=} \frac{\widehat{g_0}(\lambda-1-\alpha)}{\lambda(\lambda-1)(\lambda-\alpha)(\lambda-(\alpha-1))} \quad \text{for } \Re\lambda = \delta_{k+4}, \tag{105}$$

which defines $\widehat{f_0}$ on the line $\Re\lambda = \delta_{k+4}$. The function f_0 can be recovered by the inverse Mellin transform:

$$f_0(x) \;:=\; \int_{\Re\lambda=\delta_{k+4}} x^\lambda \widehat{f_0}(\lambda)\, d\Im(\lambda). \tag{106}$$

Indeed, the such defined function f_0 satisfies $Af_0 = g_0$. Furthermore, $f := f_0 + p$ satisfies $Af = g$. By (105) and in view of (44), this implies

$$c|\widehat{g_0}(\lambda-1-\alpha)| \;\leqq\; |\lambda^4 \widehat{f_0}(\lambda)| \leqq C|\widehat{g_0}(\lambda-1-\alpha)| \quad \text{for } \Re\lambda = \delta_{k+4} \tag{107}$$

for constants $c > 0$, $C < \infty$, which depend on n, k, ε. By Plancherel's identity (100) and by (44), we have

$$\begin{aligned} [f_0]^2_{H_{k+4}} &\overset{(100)}{\leqq} C \int_{\Re\lambda=\delta_{k+4}} \left|\lambda^{k+4}\widehat{f_0}\right|^2 d\Im(\lambda) \\ &\overset{(107)}{\leqq} C \int_{\Re\lambda=\delta_k} |\lambda^k \widehat{g_0}|^2\, d\Im(\lambda) \overset{(100)}{\leqq} C\,[g_0]^2_{H_k}\,. \end{aligned} \tag{108}$$

The argument for the reverse estimate follows analogously using the first estimate in (107). This yields (104) and hence concludes the proof. Since the equation is a fourth order ODE, the solution space is four-dimensional, the above method provides all solutions. □

7. The Degenerate Parabolic Operator

Before we address the degenerate parabolic equation, we first consider the corresponding resolvent equation. We hence consider for given $g \in X_{k-2}$, $f_{\rm in} \in X_k$, $k \in \mathbb{N}_0$, the resolvent equation

$$\begin{cases} f + Af \;=\; f_{\rm in} + g & \text{for } x \in (0,\infty), \\ f(0) = 0, \\ \lim_{x\to 0}(x^n f_{xx}) = 0. \end{cases} \tag{109}$$

We will derive the existence, uniqueness and regularity of solutions for (109). With this result, we then establish the corresponding statements for the parabolic equation. Note that both boundary conditions in (109) are also included in the definition of the spaces X_k for sufficiently large $k \in \mathbb{N}_0$.
The next lemma is concerned with the existence, uniqueness and regularity of solutions for the resolvent equation (109):

Lemma 7.1. (Resolvent equation) *Let $n \in (0,3)\backslash\{1,2\}$ and suppose that ε satisfies* (44)–(46) *for $k = 4$. Let $f_{\rm in} \in H_2$ with $f_{\rm in}(0) = 0$ and let $g \in H_0$. Then there is a unique solution $f \in X_4$ of* (109). *It satisfies*

$$c\,[f - f_{\rm in}]^2_{H_0} + [f]^2_{H_2} + c\,[f - p_4]^2_{H_4} \;\leqq\; [f_{\rm in}]^2_{H_2} + C\,[g]^2_{H_0}\,, \tag{110}$$

where $p_4 = \sum_{\beta\in[\delta_2,\delta_4]\cap I_X} c_\beta x^\beta$ is the generalized expansion of f of order δ_4.

Proof. We say that $f \in X_2$ is a weak solution of (109) if

$$\int_0^\infty x^\varepsilon f\varphi\,\mathrm{d}x + \int_0^\infty x^n f_{xx}(x^\varepsilon\varphi)_{xx}\,\mathrm{d}x \;=\; \int_0^\infty x^\varepsilon(f_{\rm in} + g)\varphi\,\mathrm{d}x \tag{111}$$

for all $\varphi \in X_2$, where we recall (49), that is $X_2 = \{f \in H_2 : f(0) = 0\}$. Existence of a weak solution follows by the Lemma of Lax-Milgram: Indeed, by Lemma 6.2, the bilinear form $b : X_2 \times X_2 \to \mathbb{R}$, defined by the left hand side of (111), is continuous and coercive with respect to the norm $\|\cdot\|_{H_2}$. Furthermore, $X_2 \subset H_2$ is a Hilbert space with respect to the norm $\|\cdot\|_{H_2}$ and H_0 is a subspace of the dual space of X_2. Application of Lax–Milgram's Lemma, then yields a unique weak solution $f \in X_2$ of (111). By ODE theory, this solution is also a classical solution of (109).
We next show that f satisfies the second boundary condition in (109). Since $Af = f - g \in H_0$ and by Lemma 6.3, we have $(f - c_* x^{\alpha-1}\zeta) \in X_4$ for some $c_* \in \mathbb{R}$, where ζ is the cut-off function from (42) and where $\alpha = 3 - n$. By the definition of X_4, we have

$$\lim_{x\to 0}(x^n f_{xx}) = c_*(\alpha - 1)(\alpha - 2), \tag{112}$$

where the assumption (44) ensures that $(\alpha-1)(\alpha-2) \neq 0$. Hence, $f \in X_4$ and the second boundary condition in (109) follow, if $c^* = 0$. If $n > 1+\varepsilon$, or equivalently $\alpha - 1 < \delta_2$, then $\zeta x^{\alpha-1} \notin H_2$ for any cut-off function ζ satisfying (42). On the other hand, we have $f \in X_2 \subset H_2$. Therefore, it follows that $c_* = 0$ if $n > 1+\varepsilon$. In the following, we assume that $n \leqq 1+\varepsilon$. We need to show that $c^* = 0$. For this, we will use that the second boundary condition in (109) holds as a natural boundary condition for (111). Indeed, let $\tilde{f}$ be the weak solution (111) for $\varepsilon = 0$. By Lemma 6.3, we have $\tilde{f}_0 := \tilde{f} - c_* x^{\alpha-1}$ with $\tilde{f}_0 \zeta \in X_{4,\varepsilon=0}$ for some $\tilde{c}_* \in \mathbb{R}$. We choose a test function $\varphi \in C_c^\infty([0,\infty))$. Since $\tilde{f}$ is a classical solution of the resolvent equation in $(0,\infty)$, integration by parts yields

$$0 = \int_0^\infty \tilde{f} g \, \mathrm{d}x + \int_0^\infty x^n \tilde{f}_{xx} \varphi_{xx} \, \mathrm{d}x - \int_0^\infty (f_{\text{in}} + g)\varphi \, \mathrm{d}x \tag{113}$$

$$= \lim_{x\to 0} \left((x^n \tilde{f}_{xx})_x \varphi \right) - \lim_{x\to 0} \left(x^n \tilde{f}_{xx} \varphi_x \right) = \tilde{c}_* (\alpha-1)(\alpha-2)(\varphi(0) - \varphi_x(0)), \tag{114}$$

and hence $\tilde{c}_* = 0$ with the choice $\varphi(0) = 0$, $\varphi_x(0) = 1$. In particular, $\tilde{f}$ satisfies the natural boundary condition $\lim_{x\to 0} x^n \tilde{f}_{xx} = 0$ and we have $\tilde{f} \in X_{4,\varepsilon=0}$. Since $X_{4,\varepsilon=0} \subset H_2$, it follows that $\tilde{f} \in H_2$. By the uniqueness of weak solutions in H_2, we obtain $\tilde{f} = f$ and $c_* = \tilde{c}_* = 0$. In particular, $f \in X_4$ satisfies (109).
It remains to show that (110) holds. Indeed, since $f \in X_4$, we have, in particular, $Af \in H_0$. In order to get (110), we use Af as a test function in (109), which yields

$$\langle f, Af \rangle_{H_0} + \langle Af, Af \rangle_{H_0} = \langle g, Af \rangle_{H_0} + \langle f_{\text{in}}, Af \rangle_{H_0}. \tag{115}$$

By Lemmas 6.2 and 6.3, this implies

$$[f]_{H_2}^2 + c\,[f - p_4]_{H_4}^2 \leqq [g]_{H_0} [f - p_4]_{H_4} + [f_{\text{in}}]_{H_2} [f]_{H_2}.$$

Young's inequality yields

$$[f]_{H_2}^2 + c\,[f - p_4]_{H_4}^2 \leqq [f_{\text{in}}]_{H_2}^2 + C\,[g]_{H_0}^2. \tag{116}$$

The term $[f - f_{\text{in}}]_{H_0}$ is estimated using the equation together with (116). □

Remark 7.2. (Self-adjointness for $\varepsilon = 0$) If $\varepsilon = 0$ satisfies (44)–(46) for $k = 4$, then the above results show that $A : X_4 \to L^2$ is self-adjoint for $\alpha \in (0, \frac{7}{4}) \backslash \{1, 2\}$ and $\varepsilon = 0$. Indeed, A is symmetric (Lemma 6.1) and closed (which follows from Lemma 6.3). Lemma 7.1 implies that -1 is in the resolvent set of A. Self-adjointness of A then follows [38, p. 137]. The self-adjointness of the operator for $\varepsilon = 0$ will not be used in the sequel.

We next show well-posedness for the linear parabolic equation:

Proposition 7.3. (Parabolic equation) *Let $n \in (0,3)\backslash\{1,2\}$ and suppose that ε satisfies (44)–(46) for $k = 4$. Suppose that $f_{\text{in}} \in H_2$ with $f_{\text{in}}(0) = 0$ and let $g \in TY_0$. Then there is a unique solution $f \in TX_4$ of*

$$\begin{cases} f_t + Af = g & \text{for } (t,x) \in (0,\infty)^2, \\ f = 0 & \text{for } x = 0, \\ f = f_{\text{in}} & \text{for } t = 0. \end{cases} \tag{117}$$

It satisfies

$$\|f\|_{TX_4} \leqq C\left(\|f_{\mathrm{in}}\|_{H_2} + \|g\|_{L^2(H_0)}\right). \tag{118}$$

Proof. Our argument is based on Lemma 7.1 together with a time-discretization argument. We set $f^{(0)} = f_{\mathrm{in}} \in H_2$ and $g^{(j)} := \frac{1}{\tau}\int_{j\tau}^{(j+1)\tau} g \,\mathrm{d}t$ for $j \in \mathbb{N}_0$, where $\tau > 0$ represents the time step size. We then define $f^{(j+1)} \in X_4$ recursively for any $j \in \mathbb{N}_0$ as solution of the resolvent equation

$$f^{(j+1)} + \tau A f^{(j+1)} = f^{(j)} + \tau g^{(j)} \quad \text{for } x \in (0,\infty) \tag{119}$$

with $f^{(j+1)}(0) = 0$. Equation (119) is equivalent to (109) by the change of variables $x \mapsto \tau^{\frac{1}{1+\alpha}} x$ and $g \mapsto \tau g$. Correspondingly, by Lemma 7.1 we get a solution of (119) which satisfies the following rescaled version of estimate (110):

$$\begin{aligned} &c\tau\left[\frac{1}{\tau}(f^{(j+1)} - f^{(j)})\right]_{H_0}^2 + \left[f^{(j+1)}\right]_{H_2}^2 + c\tau\left[f^{(j+1)} - p_4^{(j+1)}\right]_{H_4}^2 \\ &\quad\leqq \left[f^{(j)}\right]_{H_2}^2 + C\tau\left[g^{(j+1)}\right]_{H_0}^2, \end{aligned} \tag{120}$$

where $p_4^{(j+1)}$ is the generalized expansion of $f^{(j+1)}$ of order δ_4. We define $f^h \in TX_4$ as follows. We first set $(f^h)_{|t=t_j} := f^j$ where $t_j = jh$ and for any $j \in \mathbb{N}_0$. For $t \in (t_j, t_{j+1})$, we define f^h as the linear interpolation between $(f^h)_{|t=t_j}$ and $(f^h)_{|t=t_{j+1}}$. Taking the sum over (120), f^h satisfies the estimate

$$\begin{aligned} &c\int_0^T \left[\partial_t f^h\right]_{H_0}^2 \,\mathrm{d}t + \sup_t \left[f^h\right]_{H_2}^2 + c\int_0^T \left[f^h - p_4^h\right]_{X_4}^2 \,\mathrm{d}t \\ &\quad\leqq [f_{\mathrm{in}}]_{H_2}^2 + C\int_0^T \left[g^h\right]_{H_0}^2 \,\mathrm{d}t. \end{aligned} \tag{121}$$

In the limit $\tau \to 0$, we have uniform convergence of f^h to a solution $f \in TX_4$ of (117) which satisfies the corresponding estimate to (121), that is

$$c\,[\partial_t f]_{L^2(H_0)}^2 + [f]_{C^0(H_2)} + c[f - p_4]_{L^2(X_4)}^2 \leqq [f_{\mathrm{in}}]_{H_2}^2 + C[g]_{L^2(Y_0)}^2. \tag{122}$$

The estimate (118) follows, thus concluding the proof of Proposition 7.3. We refer to for example [24] for a more detailed version of a similar argument. □

Higher regularity in space and time is achieved by considering the corresponding equation to (117) which is derivated k-times in *time*. For a Cauchy problem (and some boundary value problems), the usual way to obtain higher regularity would be to apply the operator A on both sides of the equation (117). However, our solution space X_k is not closed under application of the operator A, that is $AX_{k+4} \not\subseteq X_k$ in general. In fact, we only have $AX_{k+4} \subseteq Y_k$. Note that the choice of spaces is dictated by the nonlinearity of the model and not well-adapted for the linear operator.

Proposition 7.4. (Parabolic equation—higher regularity) *Let* $n \in (0,3)\setminus\{1,2\}$, $k \geqq 2$ *and suppose that* ε *satisfies* (44)–(46). *Suppose that* $f_{\mathrm{in}} \in X_k$ *and* $g \in TY_{k-2}$ *satisfy the compatibility condition* (58). *Then there is a unique solution* $f \in TX_{k+2}$ *of* (117). *It satisfies*

$$\|f\|_{TX_{k+2}} \leqq \|f_{\mathrm{in}}\|_{X_k} + C\|g\|_{TY_{k-2}} + \|f\|_{L^2_t L^2_x(Q_T)}. \tag{123}$$

Proof. We first show the assertion for the case $k = 6$. In particular, we have $f_{\mathrm{in}} \in X_6$, $g \in L^2(Y_4) \cap C^0(Y_2)$ and $g_t \in L^2(Y_0)$. Recall that $Y_0 = H_0$, (52). Let $f^{(1)}$ be the solution of the formally in time derivated equation

$$\begin{cases} \partial_t f^{(1)} + Af^{(1)} = \partial_t g & \text{for } (x,t) \in (0,\infty)^2, \\ f^{(1)} = 0 & \text{for } x = 0, \\ f^{(1)}_{|t=0} = f^{(1)}_{\mathrm{in}} := g_{|t=0} - Af_{\mathrm{in}} & \text{for } t = 0. \end{cases} \tag{124}$$

By the compatibility condition (58), we have $f^{(1)}_{\mathrm{in}} = g_{|t=0} - Af_{\mathrm{in}} = 0$ at $x = 0$ and $f^{(1)}_{\mathrm{in}} \in H_2$. Also note that $\partial_t g \in TY_0$. By Proposition 7.3, we hence get a solution $f^{(1)} \in TX_4$ of (124) which satisfies

$$\begin{aligned} \|f^{(1)}_t\|_{L^2(H_0)} + \|f^{(1)}\|_{C^0(H_2)} + \|f^{(1)}\|_{L^2(X_4)} &\leqq C\left(\|f^{(1)}_{\mathrm{in}}\|_{H_2} + \|g_t\|_{L^2(Y_0)}\right) \\ &\leqq C\left(\|f_{\mathrm{in}}\|_{X_6} + \|g\|_{TY_4}\right). \end{aligned} \tag{125}$$

By a straightforward calculation, it follows that the time–integrated function $f_{\mathrm{in}} + \int_0^t f^{(1)} \mathrm{d}\tilde{t} \in L^2(X_4)$ is a solution of (117). By uniqueness, we hence get $f^{(1)} = f_t$ and in particular $f_t \in L^2(X_4)$. Hence

$$\|f_{tt}\|_{L^2(H_0)} + \|f_t\|_{C^0(H_2)} + \|f_t\|_{L^2(X_4)} \overset{(125)}{\leqq} C\left(\|f_{\mathrm{in}}\|_{X_6} + \|g\|_{TY_4}\right). \tag{126}$$

By (117), we have $Af = g - f_t \in L^2(Y_4) \cap C^0(Y_2)$. By Lemma 6.3, this yields $f \in L^2(X_8) \cap C^0(X_6)$ and the corresponding estimates. Together with (126), this implies

$$\|f\|_{TX_8} \leqq C\left(\|f_0\|_{X_6} + \|g\|_{TY_4}\right).$$

Bootstrapping this argument eventually yields the assertion of the lemma for all $k \in 2+4\mathbb{N}_0$. Now suppose that $k \in \mathbb{N}$ with $k \geqq 2$ and f_{in}, g satisfy the compatibility conditions (58) of order k. Let K be the smallest integer $K \geqq k$ with $K \in 2+4\mathbb{N}$. We approximate the initial data f_{in} and the right hand side g in terms of $\|\cdot\|_{X_K}$ (respectively $\|\cdot\|_{TX_{K-2}}$) by functions in X_K (respectively in TX_{K-2}) which satisfy the compatibility conditions up to order $k+2$. The existence of a solution for the approximated data together with the corresponding estimates then follows from the arguments above. In the limit, we obtain that the original problem has a solution in the space $f \in X_{k+2}$ together with the corresponding estimates. □

As a consequence of the linear theory, we obtain the following extension lemma for functions in X_k:

Lemma 7.5. (Extension Lemma) *Let* $n \in (0,3)\setminus\{1,2\}$, $j_0 \in \mathbb{N}_0$, *let* $k = 2 + 4j_0$ *and suppose that* ε *satisfies* (44)–(46). *Suppose that* $\varphi_j \in X_{k-4j}$ *for* $j = 0, \ldots, j_0$. *Then there is* $w \in TX_{k+2}$ *such that*

$$\partial_t^j w_{|t=0} = \varphi_j \quad \textit{and} \quad \|w\|_{TX_{k+2}} \leqq C \sum_{j=0}^{j_0} \|\varphi_j\|_{X_{k-4j}}. \tag{127}$$

Proof. Let $p_{\varphi_j,\delta_{k-4j}} \in \mathscr{P}_X$ be the expansion of the function φ_j up to order δ_{k-4j} and let $\varphi_j^{(1)} := p_{\varphi_j,\delta_{k-4j}}\zeta$ for some smooth cut-off function ζ as in (42). We define

$$w^{(1)}(t,x) = \zeta(t) \sum_{j=0}^{j_0} \frac{t^j}{j!} \varphi_j^{(1)}(x). \tag{128}$$

Then $w^{(1)}$ satisfies $\partial_t^j w^{(1)}_{|t=0} = \varphi_j^{(1)}$ and the estimate in (127) with w replaced by $w^{(1)}$. It hence remains to prove the assertion of the lemma with φ_j replaced by $\varphi_j^{(0)} := \varphi_j - \varphi_j^{(1)} \in \mathring{H}_{k-4j}$.
We proceed by induction in j_0. If $j_0 = 0$ (and $k = 2$), then $w \in TX_4$ is given as the solution of (117) with initial data $\varphi_0^{(0)} \in X_2$ and right hand side $g = 0$. Now, suppose that $j_0 \geqq 1$ and that the assertion of the lemma holds for $j_0 - 1$. Since $\varphi_j^{(0)} \in \mathring{H}_{k-4j}$, we have $\varphi_{j+1}^{(0)} - A\varphi_j^{(0)} \in \mathring{H}_{k-4j-4} \subset X_{(k-4)-4j}$ for $j = 0, \ldots, j_0 - 1$. Hence, by the induction assumption there is $g \in TX_{k-2}$ such that for $j = 0, \ldots, j_0 - 1$, we have

$$\partial_t^j g_{|t=0} = \varphi_{j+1}^{(0)} + A\varphi_j^{(0)} \quad \text{and} \ \|g\|_{TX_{k-2}} \leqq C \sum_{j=0}^{j_0-1} \|\varphi_{j+1}^{(0)} - A\varphi_j^{(0)}\|_{X_{k-4j}}$$
$$\leqq C \sum_{j=1}^{j_0} \|\varphi_j\|_{X_{k-4j}}.$$

Then we define $w \in TX_{k+2}$ as the solution of

$$\begin{cases} \partial_t w + Aw = g & \text{for } (x,t) \in (0,\infty)^2, \\ w = 0 & \text{for } x = 0, \\ w = \varphi_0^{(0)} & \text{for } t = 0. \end{cases} \tag{129}$$

The choice of g is such that the compatibility conditions are satisfied for this evolution problem and such that the solution w of (129) satisfies the initial conditions in (127). By the parabolic estimate in Proposition 7.4, f also satisfies the estimates in (127). $\square$

Remark 7.6. Using the parabolic estimates in [24] and [31] one easily gets a corresponding extension lemma for the case $n = 1$, $n = 2$ in the spaces used in these papers.

8. Nonlinear Estimates

In this section, we estimate the nonlinear operator $N(f)$. We recall that

$$N(f, \tilde{f}) \overset{(17)}{=} ((x^n - h^n)\tilde{f}_{xx})_{xx} + f_x(x^{n-1}\tilde{f}_{xx})_{|x=0}, \tag{130}$$

where $h = x + \int_0^x f \,\mathrm{d}\tilde{x}$ and $N(f) = N(f, f)$. With the definition

$$F(x) := \frac{1}{x}\int_0^x f \,\mathrm{d}\tilde{x} = \frac{h-x}{x}, \tag{131}$$

that is $\frac{h}{x} = 1 + F$, the operator N can be written in the form

$$\begin{aligned} N(f, \tilde{f}) &\overset{(130)}{=} \left(x^n\left(1 - \left(\frac{h}{x}\right)^n\right)\tilde{f}_{xx} + (h-x)(x^{n-1}\tilde{f}_{xx})_{|x=0}\right)_{xx} \\ &= \left(x^n(1-(1+F)^n)\tilde{f}_{xx} + xF(x^{n-1}\tilde{f}_{xx})_{|x=0}\right)_{xx}. \end{aligned} \tag{132}$$

By the generalized binomial theorem [1, 3.6.9] for every $n > 0$ and $X \in (-1, 1)$, we have

$$(1+X)^n = \sum_{r=0}^{\infty}\binom{n}{r}X^r, \quad \text{where } c_{nr} := \prod_{j=1}^{r}\frac{n-j+1}{j}; \tag{133}$$

the series (133) converges absolutely and uniformly for $X \in (-\lambda, \lambda)$ with $\lambda \in (-1, 1)$. By Lemma 5.4 and Lemma 5.7, we have

$$\|F\|_{L^\infty((0,\infty))} < C\|f\|_{H_2} \leqq C\|f\|_{X_k} < \frac{1}{2}, \tag{134}$$

for $k \geqq 2$ and if $\|f\|_{X_k}$ is sufficiently small. In this case, the binomial formula (133) can be point-wise applied to (132) and we obtain,

$$N(f,\tilde f) \overset{(132)}{=} \left(x^n \tilde f_{xx} \sum_{r=1}^{\infty} c_{nr} F^r \; - (x^{n-1}\tilde f_{xx})_{|x=0} x F\right)_{xx} \tag{135}$$
$$= \left(x^n \tilde f_{xx} \sum_{r=2}^{\infty} c_{nr} F^r \; + x^n F \left(n \tilde f_{xx} - (x^{n-1}\tilde f_{xx})_{|x=0} x^{1-n}\right)\right)_{xx},$$

since $c_{n1} = n$. Correspondingly, we use the decomposition

$$N(f,\tilde f) \;=\; N_1(f,\tilde f) + N_2(f,\tilde f), \tag{136}$$

where

$$N_1(f,\tilde f) \;:= \left(x^n \sum_{r=2}^{\infty} c_{nr} F^r \tilde f_{xx}\right)_{xx}, \tag{137}$$
$$N_2(f,\tilde f) \;:= \left(x^n F \left(\tilde f_{xx} - \left(x^{n-1}\tilde f_{xx}\right)_{|x=0} x^{1-n}\right)\right)_{xx}. \tag{138}$$

Note that N_1 is a highly nonlinear, local operator, while N_2 is bilinear and nonlocal. If $\|F\|_{L^\infty((0,\infty))} < 1$ and $F \in C^k((0,\infty))$, then also $\sum_r \partial_x^k(F^r)$ converges uniformly for all $x \in (0,\infty)$. Therefore, summation and differentiation can be exchanged if (134) holds and if F is sufficiently regular.
We will need the following auxiliary estimate:

Lemma 8.1. *Let $n \in (0,3)$, $k \in \mathbb{N}_0$ and suppose that ε satisfies (44)–(46). Let $f \in X_k$ and let F be defined by (131). Let $p_{\delta_k} \in \mathscr{P}_X$ be the expansion of f of order δ_k and let $P_{\delta_k} = \frac{1}{x}\int_0^x p_{\delta_k}$ be the corresponding expansion for F. Then there is a universal constant $c_0 > 0$ such that if $\|f\|_{X_k} \leqq c_0$, the following holds: for all $0 \leqq j \leqq k$, we have*

$$\left|\partial_x^j\left(\sum_{r=2}^{\infty} F^r - P_{\delta_k}^r\right)(x)\right| \;\leqq\; C x^{\delta_k - j}\|f\|_{X_k} \quad \textit{for } x \in (0,\infty). \tag{139}$$

Furthermore, for all $R \in (0,\infty)$ there is a constant $C_R < \infty$, depending on k and R, such that

$$\left|\partial_x^j\left(\sum_{r=2}^{\infty} F^r\right)(x)\right| \;\leqq\; C_R x^{\delta_j - j}\|f\|_{X_k} \quad \textit{for } x \in (R,\infty). \tag{140}$$

Proof. By definition, all coefficients of P_{δ_k} are controlled by $\|F\|_{X_k}$. By (134), we have $\|F\|_{L^\infty} < \frac{1}{2}$ if we choose c_0 sufficiently small. Then the series $\sum_{r=2}^{\infty} F^r(x)$ is well-defined for all $x \in (0,\infty)$. We first give the argument for (140). By (88), we get for $j = 0$

$$\sum_{r=2}^{\infty} |F(x)|^r \;\leqq\; C|F(x)| \overset{(88)}{\leqq} C_R x^{\delta_0}\|f\|_{X_k} \quad \text{for } x \in (R,\infty). \tag{141}$$

This yields (88) for $j = 0$. For $1 \leqq j \leqq k$, we calculate

$$\left|\partial_x^j (F^r)(x)\right| \leqq C|F(x)|^{\max\{0,r-j\}} \sum_{m=1}^{j} \sum_{(i_1,\ldots,i_m)} \left|(\partial_x^{i_1} F)\ldots(\partial_x^{i_m} F)(x)\right|, \tag{142}$$

with the second sum being taken over all multiindices $(i_1, \ldots, i_m) \in \mathbb{N}_0^m$ with $i_1 + \cdots + i_m = j$. In view of (88), we get for $j \geqq 1$

$$\begin{aligned}\sum_{i_1+\cdots+i_m=j} \left|(\partial_x^{i_1} F\ldots\partial_x^{i_m} F)(x)\right| &\overset{(88)}{\leqq} C_R \sum_{m=1}^{j} x^{(\delta_{i_1}-i_1)+\cdots+(\delta_{i_m}-i_m)} \|f\|_{X_k} \\ &\overset{(30)}{=} C_R\, x^{\delta_j - j - \frac{1}{2}(m-1)(\varepsilon+1)} \|f\|_{X_k} \quad \text{for } x \in (R,\infty),\end{aligned}$$

where we have used that $\|f\|_{X_k} \leqq c_0 < 1$ for c_0 sufficiently small. By (45), we have in particular $\varepsilon \geqq -1$ and hence $x^{\delta_j - j - \frac{1}{2}(m-1)(\varepsilon+1)} \leqq C_R x^{\delta_j - j}$ for $x \in (R, \infty)$. Taking the sum over $r = 2$ to infinity of (142), we hence obtain

$$\partial_x^j \left(\sum_{r=2}^{\infty} |F(x)|^r\right) \leqq C_R x^{\delta_j - j} \|f\|_{X_k} \sum_{r=2}^{\infty} |F(x)|^{\max\{0,r-j\}} \leqq C_R x^{\delta_j - j} \|f\|_{X_k} \quad \text{for } x \in (R,\infty).$$

This concludes the proof of (140). For the estimate of (139), we note that $|F(x)|$, $|p_F(x)|$, $|(F - p_F)(x)| \leqq C x^{\delta_k}$ for all $x \in (0, \infty)$ and hence

$$\left|\sum_{r=2}^{\infty} \left(F^r - p_F^r\right)(x)\right| \overset{(133)}{\leqq} C x^{\delta_k} \|f\|_{X_k}, \tag{143}$$

The corresponding estimate for higher derivatives follows by using an analogous calculation as in (142). □

For notational convenience, we use the notation

$$\|f\|_{X_{k,2}} := \left(\sum_{j=2}^{k} [f - p_{\delta_j}]_{H_j}^2\right)^{\frac{1}{2}}. \tag{144}$$

for the semi-norm, where the homogeneous norms are included for indices from $j = 2$: We next collect some estimates of the nonlinear operator $N(f, \tilde{f})$:

Lemma 8.2. (Basic nonlinear estimate) *Let $n \in (0,3)\backslash\{1,2\}$, $k \geqq 2$ and suppose that ε satisfies (44)–(47). Then there is a constant $c_0 > 0$ such that for any $f \in X_k$, $\tilde{f} \in X_{k+2}$ with $\|f\|_{X_k} \leqq c_0$, we have $N(f, \tilde{f}) \in Y_{k-2}$ and*

$$\left\|N(f,\tilde{f})\right\|_{Y_{k-2}} \leqq C\left(\|f\|_{X_k}\|\tilde{f}\|_{X_{k+2,2}} + \|\tilde{f}\|_{X_k}\|f\|_{X_{k+2,2}}\right). \tag{145}$$

Proof. It suffices to show for any $j \in \mathbb{N}$ with $2 \leqq j \leqq k$,

$$\left[N(f,\tilde f) - q_{\delta_{j-2}}\right]_{H_{j-2}} \leqq C\left(\|f\|_{X_j}\|\tilde f\|_{X_{j+2,2}} + \|\tilde f\|_{X_j}\|f\|_{X_{j+2,2}}\right) =: \mathscr{R}, \tag{146}$$

where $q_{\delta_{j-2}}$ is the generalized expansion of $N(f,\tilde f)$ of order δ_{j-2}. Here, and in the remainder of the proof, we denote by $\mathscr{R}$ any multiple by a constant depending only on n, ε, k of the right hand side of (146). By Lemma 10.1, we have $q_{\delta_{j-2}} \in \mathscr{P}_Y$ and $\|q_{\delta_{j-2}}\|_{\mathscr{P}} \leqq \mathscr{R}$. Let ζ be the cut-off function defined in (42). We have

$$\begin{aligned}
&\left[N(f,\tilde f) - q_{\delta_{j-2}}\right]_{H_{j-2}} \\
&\quad\leqq \left[\zeta(N(f,\tilde f) - q_{\delta_{j-2}})\right]_{H_{j-2}} + \left[(1-\zeta)N(f,\tilde f)\right]_{H_{j-2}} + \left[(1-\zeta)q_{\delta_{j-2}}\right]_{H_{j-2}} \\
&\quad\leqq \left[\zeta(N(f,\tilde f) - q_{\delta_{j-2}})\right]_{H_{j-2}} + \left[(1-\zeta)N(f,\tilde f)\right]_{H_{j-2}} + \mathscr{R}.
\end{aligned} \tag{147}$$

We use Lemma 10.1 to replace the expansion $q_{\delta_{j-2}}$ in (147) by another expansion which better reflects the nonlinear structure of the operator and at the same time agrees with $q_{\delta_{j-2}}$ up to an order of δ_{j-2}. Let $\zeta_2(x) := \zeta(\frac{x}{2})$; in particular, $\zeta_2\zeta = \zeta$. We decompose

$$\tilde f = \tilde f_0 + \zeta_2 \sum_{\beta \in [\delta_2, \delta_{j+2}] \cap \mathscr{I}_X} \tilde c_{\tilde\beta} x^{\tilde\beta}, \tag{148}$$

where $\tilde c_{\tilde\beta}$ are the coefficients of the expansion of $\tilde f$. In particular, we have $\tilde f_0 = o(x^{\delta_{j+2}})$ for small x. For any $\tilde\beta \in (\delta_2, \delta_{j+2}) \cap I_X$, let p_β be the expansion of f of order β, where $\beta \in (\delta_2, \delta_{j+2})$ is the "dual exponent" of $\tilde\beta$, given by

$$\beta := \delta_2 + \delta_{j+2} - \tilde\beta \in (\delta_2, \delta_{j+2}). \tag{149}$$

By Lemma 10.1, we then have

$$\left\| N\left(p_\beta, \tilde c_{\tilde\beta} x^{\tilde\beta}\right)\right\|_{\mathscr{P}} \leqq \mathscr{R}$$

and furthermore

$$q_{\delta_{j-2}} = \sum_{\tilde\beta \in [\delta_2, \delta_{j+2}] \cap I_X} N\left(p_\beta, \tilde c_{\tilde\beta} x^{\tilde\beta}\right) + o(x^{\delta_{j-2}}). \tag{150}$$

In view of (147) and since N is linear in the second argument, we thus obtain

$$\begin{aligned}
\left[\zeta\left(N(f,\tilde f) - q_{\delta_{j-2}}\right)\right]_{H_{j-2}} &\overset{(150)}{\leqq} \left[\zeta N(f,\tilde f_0)\right]_{H_{j-2}} \\
&+ \sum_{\tilde\beta \in [\delta_2, \delta_{j+2}] \cap I_X} \left[\zeta N\left(f, \tilde c_{\tilde\beta} x^{\tilde\beta}\right) - N\left(p_\beta, \tilde c_{\tilde\beta} x^{\tilde\beta}\right)\right]_{H_{j-2}} + \mathscr{R}.
\end{aligned} \tag{151}$$

By (147) and (151), we hence get

$$\left[N(f,\tilde f)-q_{\delta_{j-2}}\right]_{H_{j-2}} \leqq I_1+I_2+I_3+\mathscr{R}, \tag{152}$$

where

$$I_1 := \left[(1-\zeta)N(f,\tilde f)\right]_{H_{j-2}}, \qquad I_2 := \left[\zeta N(f,\tilde f_0)\right]_{H_{j-2}}, \tag{153}$$
$$I_3 := \sum_{\tilde\beta\in[\delta_2,\delta_{j+2}]\cap I_X} \left[\zeta N(f,\tilde c_{\tilde\beta}x^{\tilde\beta}) - N(p_\beta,\tilde c_{\tilde\beta}x^{\tilde\beta})\right]_{H_{j-2}}.$$

It remains to estimate the terms I_1, I_2, and I_3.
Estimate of I_1*:* In order to estimate the homogeneous norm $[\cdot]_{H_{j-2}}$ in I_1, $j-2$ derivatives need to be applied. By application of the Leibniz rule, some of these derivatives are applied on the cut-off function ζ. It is easy to see that in this case, the resulting terms is of lower order an can be estimates easily. Indeed, since ζ_x is supported in $(1,2)$, we have $1\leqq x\leqq 2$ in this interval and in particular hence all weights are equivalent in the sense that $x^\alpha \leqq C_{\alpha,\beta}x^\beta$ for all $x\in(1,2)$ and all $\alpha,\beta\in\mathbb{R}$. Also recall that, $1-\zeta$ is supported in $(1,\infty)$. It is therefore enough to show the estimate

$$\left[N(f,\tilde f)\right]_{H_{j-2}} \leqq \mathscr{R}. \tag{154}$$

with the additional assumptions $f,\tilde f\in H_{j+2}$ and supp $f,\tilde f\in[1,\infty)$. The estimate for I_1 follows easily.
We use the decomposition (137)–(138). We get

$$\left[N_1(f,\tilde f)\right]^2_{H_{j-2}} \overset{(137)}{=} \int_1^\infty \left| x^{-\delta_{j-2}+j-2}\partial_x^j\left(x^n\sum_{r=2}^\infty F^r\tilde f_{xx}\right)\right|^2 \frac{\mathrm{d}x}{x} \tag{155}$$
$$\leqq C\int_1^\infty \left|x^{(\delta_i-\delta_{j-2}+n-4)+(j-i-n+2)}\partial_x^{j-i}(x^n\tilde f_{xx})\right|^2\frac{\mathrm{d}x}{x}$$
$$\times\sum_{i=0}^j \sup_{x\in(1,\infty)}\left|x^{-\delta_i+i}\partial_x^i\left(\sum_{r=2}^\infty F^r\right)\right|^2.$$

We calculate

$$\delta_i-\delta_{j-2}+n-4=\delta_j-\delta_{j+2} = -\tfrac14(j+2-i)(4-n)$$
$$= -\delta_{j+2-i}-\tfrac12(\varepsilon+1) < -\delta_{j+2-i}.$$

Since supp f, supp $\tilde f\subset(1,\infty)$, by Hardy's inequality (65) and by (140), we thus get

$$\left[N_1(f,\tilde f)\right]^2_{H_{j-2}} \leqq C\left[\tilde f\right]^2_{H_{j+2-i}}\sum_{i=0}^j\sup_{x\in(1,\infty)}|x^{-\delta_i+i}\partial_x^i\left(\sum_{r=2}^\infty F^r\right)|^2 \overset{(140)}{\leqq} \mathscr{R}.$$

For the estimate of $N_2(f, \tilde{f})$, we decompose

$$\left[N_2(f,\tilde{f})\right]^2_{H_{j-2}} \overset{(138)}{\leqq} \int_1^\infty \left|x^{-\delta_{j-2}+(j-2)}\partial_x^j\left(x^nF\tilde{f}_{xx}\right)\right|^2 \frac{\mathrm{d}x}{x} \tag{156}$$

$$+\left(x^{n-1}\tilde{f}_{xx}\right)^2_{|x=0}\int_1^\infty \left|x^{-\delta_{j-2}+(j-2)}\partial_x^j(xF)\right|^2 \frac{\mathrm{d}x}{x}. \tag{157}$$

The term on the right hand side of line (156) is estimated analogously as before using (88) instead of (140). By (89) and by application of Hardy's inequality, we furthermore get

$$\begin{aligned}
&(x^{n-1}\tilde{f}_{xx})^2_{|x=0}\int_1^\infty \left|x^{-\delta_{j-2}+(j-2)}\partial_x^j(xF)\right|^2 \frac{\mathrm{d}x}{x}\\
&\overset{(65)}{\leqq} C\|\tilde{f}\|^2_{X_{j+2,2}}\int_1^\infty \left|x^{-\delta_{j-2}+(j-2)}\partial_x^{j-1}f\right|^2 \frac{\mathrm{d}x}{x}\\
&\overset{(78)}{\leqq} C\|\tilde{f}\|^2_{X_{j+2,2}}\|f\|^2_{X_{j-1}} \leqq \mathscr{R},
\end{aligned}$$

since $1 \leqq x^\alpha$ for $x \in (1,\infty)$ and $\alpha > 0$. This concludes the estimate of I_1.
Estimate of I_2: As in step 1, we can neglect terms where the derivative is applied to the cut-off functions, since this yields lower order terms. Hence, we will show the estimate

$$\left[N(f,\tilde{f}_0)\right]_{H_{j-2}} \leqq \mathscr{R}. \tag{158}$$

with the additional assumption that supp $f_0 \subset [0,2]$; the asserted estimate for I_2 then follows easily. We use the decomposition (137)–(138). Note that $N_2(f,\tilde{f}_0) = 0$ so that it remains to show the estimate for N_1. We have

$$\left[N_1(f,\tilde{f}_0)\right]^2_{H_{j-2}} = \int_0^\infty \left|x^{-\delta_{j-2}+(j-2)}\partial_x^j\left(x^n\sum_{r=2}^\infty F^r\tilde{f}_{0xx}\right)\right|^2 \frac{\mathrm{d}x}{x} \tag{159}$$

$$\leqq C\sum_{i=0}^j\int_0^\infty \left|x^{-\delta_{j-2}+(j-2)}\partial_x^i\left(\sum_{r=2}^\infty F^r\right)\partial_x^{j-i}(x^n\tilde{f}_{0xx})\right|^2 \frac{\mathrm{d}x}{x}. \tag{160}$$

By Lemma 8.1, by Hardy's inequality and since $\delta_{j+2}-\delta_{j-2} = 4-n$, we get

$$\begin{aligned}
[N_1(f,f_0)]^2_{H_{j-2}} &\overset{(87)}{\leqq} C\|f\|^2_{H_j}\sum_{i=0}^j\int_0^\infty \left|x^{-\delta_{j-2}+(j-2)-i}\partial_x^{j-i}(x^n\tilde{f}_{0xx})\right|^2 \frac{\mathrm{d}x}{x}\\
&\overset{(65)}{\leqq} C\|f\|^2_{H_j}\int_0^\infty \left|x^{-\delta_{j+2}+(j+2)}\partial_x^{j+2}\tilde{f}_0\right|^2 \frac{\mathrm{d}x}{x}\\
&\overset{(65)}{\leqq} C\|f\|^2_{H_j}\|f\|^2_{X_{j+2,2}} \leqq \mathscr{R}.
\end{aligned}$$

This concludes the proof of (158).

Estimate of I_3*:* As in the previous steps, we will show the corresponding estimate without the cut-off function, using an additional assumption on the support of the considered functions. Under the assumption that $\mathrm{supp}(F - P_{\delta_j}) \subset [0, 2)$, we claim that for every $\tilde{\beta} \in [\delta_2, \delta_{j+2}] \cap I_X$,

$$\left[N\left(f, \tilde{c}_{\tilde{\beta}} x^{\tilde{\beta}}\right) - N\left(p_\beta, \tilde{c}_{\tilde{\beta}} x^{\tilde{\beta}}\right)\right]_{H_{j-2}} \leqq \mathscr{R}. \tag{161}$$

We present the estimate for N_1, the estimate for N_2 proceeds similarly. Let $P_\beta = \frac{1}{x}\int_0^x p_\beta$ be the expansion of F of order β. We have

$$\begin{aligned}
&\left[N_1\left(f, \tilde{c}_{\tilde{\beta}} x^{\tilde{\beta}}\right) - N_1\left(p_\beta, \tilde{c}_{\tilde{\beta}} x^{\tilde{\beta}}\right)\right]^2_{H_{j-2}} \\
&\leqq C \sum_{i=0}^{j} \int_0^2 \left| x^{i-\beta} \partial_x^i \sum_{r=0}^{\infty} \left(F^r - P_\beta^r\right)\right|^2 \\
&\quad \times \left| x^{-\delta_{j-2}+j-2+\beta-i} \partial_x^{j-i} \left(x^n \left(\tilde{c}_{\tilde{\beta}} x^{\tilde{\beta}}\right)_{xx}\right)\right|^2 \frac{\mathrm{d}x}{x}.
\end{aligned}$$

By application of Lemma 8.1, we obtain

$$\begin{aligned}
&\left[N_1\left(f, \tilde{c}_{\tilde{\beta}} x^{\tilde{\beta}}\right) - N_1\left(p_\beta, \tilde{c}_{\tilde{\beta}} x^{\tilde{\beta}}\right)\right]^2_{H_{j-2}} \overset{(87)}{\leqq} \mathscr{R} \int_0^2 \left| x^{-\delta_{j-2}+(j-2)+\beta+n+\tilde{\beta}-j-2}\right|^2 \frac{\mathrm{d}x}{x} \\
&\overset{(149)}{=} \mathscr{R} \int_0^2 \left| x^{\delta_{j+2}-\delta_{j-2}+\delta_2+n-4}\right|^2 \frac{\mathrm{d}x}{x} = \mathscr{R} \int_0^2 |x^{\delta_2}|^2 \frac{\mathrm{d}x}{x} \overset{(66)}{\leqq} \mathscr{R},
\end{aligned}$$

since $\delta_{j+2} - \delta_{j-2} = 4 - n$ and $\delta_2 > 0$. This concludes the estimate of I_3. □

We next give a corresponding estimate for the nonlinear operator in terms of the H^{-2}-norm. This estimate is needed if $n \in (\frac{5}{2}, 3)$, see also (55).

Lemma 8.3. (Nonlinear estimate by $[\cdot]_{H_{-2}}$-norm) *Let* $n \in (\frac{5}{2}, \frac{14}{5})$ *and suppose that* ε *satisfies the assumptions* (44)–(47) *for* $k = 4$. *Then there is a constant* $c > 0$ *such that if* $\|f\|_{H_2} \leqq c$, *then*

$$\left[N(f, \tilde{f})\right]_{H_{-2}} \leqq C\|f\|_{H_2} \|\tilde{f}\|_{X_{4,2}} \quad \text{for all } f \in H_2,\ \tilde{f} \in X_4. \tag{162}$$

Proof. We define

$$G(x) := (x^n - h^n)\tilde{f}_{xx} + xF\left(x^{n-1}\tilde{f}_{xx}\right)_{|x=0}, \tag{163}$$

where F is given by (131). In particular, we have $G_{xx} = N(f, \tilde{f})$. By Lemma 5.4, we have $h \leqq x\|f\|_{H_2}$ and hence

$$\int_0^\infty \frac{1}{x^n} \left|(h^n - x^n)\tilde{f}_{xx}\right| \mathrm{d}x \leqq C\|f\|^2_{H_2} \int_0^\infty x^n \tilde{f}^2_{xx}\, \mathrm{d}x \leqq C\|f\|^2_{H_2} \left[\tilde{f}\right]^2_{H_2}.$$

Similarly, using (89), we get

$$\int_0^\infty \frac{1}{x^n}\left|xF(x^{n-1}\tilde{f}_{xx})_{|x=0}\right|^2\,\mathrm{d}x \overset{(65)}{\leqq} C\left(x^{n-1}\tilde{f}_{xx}\right)^2_{|x=0}\int_0^\infty x^{-n}f^2\,\mathrm{d}x$$
$$\overset{(65)}{\leqq} C\left(x^{n-1}\tilde{f}_{xx}\right)^2_{|x=0}\int_0^2\left|x^{\frac{-n+1}{2}+2}f_{xx}\right|^2\frac{\mathrm{d}x}{x} \overset{(89),(76)}{\leqq} C\|\tilde{f}\|^2_{X_{4,2}}\|f\|^2_{H_2},$$

since $-\frac{n-1}{2} > -\delta_2$, thus concluding the proof of (162). □

For the fix-point argument, we also need an estimate for the difference:

Lemma 8.4. (Nonlinear estimate for differences) *Suppose that the assumptions of Lemma 8.2 hold. Then there is a constant $c_0 > 0$ such that for any $f, \tilde{f} \in X_{k+2}$ with $\|f\|_{X_k}, \|\tilde{f}\|_{X_k} \leqq c_0$, we have $N(f,\tilde{f}) \in Y_{k-2}$ and*

$$\left\|N(f) - N(\tilde{f})\right\|_{Y_{k-2}} \tag{164}$$
$$\leqq C\left(\|f\|_{X_k} + \|\tilde{f}\|_{X_k}\right)\|f-\tilde{f}\|_{X_{k+2,2}}$$
$$+ C\|f-\tilde{f}\|_{X_k}\left(\|f\|_{X_{k+2,2}} + \|\tilde{f}\|_{X_{k+2,2}}\right).$$

Proof. Since the argument proceeds similarly as in the proof of Lemma 8.2, we only present the main ideas of the proof. Since N is linear in the second argument, we have

$$\left\|N(f) - N(\tilde{f})\right\|_{Y_{k-2}} \leqq \left\|N(f,f) - N(\tilde{f},f)\right\|_{Y_{k-2}} + \left\|N(\tilde{f}, f-\tilde{f})\right\|_{Y_{k-2}}. \tag{165}$$

The second term on the right hand side of (165) is estimated by Lemma 8.2. In order to estimate the first term, we use the decomposition (137)–(138). Since N_2 is bilinear, the corresponding estimate follows directly from the argument in Lemma 8.2. It hence remains to estimate

$$I := [N_1(f,f) - N_1(\tilde{f},f)]_{Y_{k-2}} \tag{166}$$

for any $j \in \mathbb{N}$ with $2 \leqq j \leqq k$. Let $q_{\delta_{j-2}}$ be the generalized expansion of $N(f,f) - N(\tilde{f},f)$ of order δ_{j-2}. Then

$$I = \left[N_1(f,f) - N_1(\tilde{f},f) - q_{\delta_{j-2}}\right]_{H_{j-2}}. \tag{167}$$

We decompose $\tilde{f}$ as in (148) and for given $\tilde{\beta}$, we define β by (149). Let p_β be the expansion of f of order β. Let F be defined by (131) and let $P_\beta = \frac{1}{x}\int_0^x p_\beta$ be the corresponding expansion of F. Analogously, for given $\tilde{f}$, we define $\tilde{p}_\beta$, $\tilde{F}$ and $\tilde{P}_\beta$. We then have

$$q_{\delta_{j-2}}(x) - \sum_{\tilde{\beta}\in[\delta_2,\delta_{j+2}]\cap I_X}\left(\sum_{r=2}^\infty\left(P^r_\beta(x) - \tilde{P}^r_\beta(x)\right)\left(\tilde{c}_{\tilde{\beta}}x^{\tilde{\beta}}\right)_{xx}\right)_{xx} = o(x^{\delta_{j-2}}), \tag{168}$$

for $x \leqq 2$ and c_0 sufficiently small, cf. (150) and the proof of Lemma 10.1. By (133), the above expansions have the difference $P_\beta - \tilde{P}_\beta$ in at least one factor. The estimate of (167) then proceeds analogously to the corresponding argument in the proof of Lemma 8.2. □

We also need the corresponding estimate in space-time norms:

Proposition 8.5. (Nonlinear estimate in space-time norms) *Suppose that the assumptions of Lemma 8.2 hold. Let* $0 < T \leqq \infty$ *and let* $Q_T = (0,T) \times (0,\infty)$. *Then there is a constant* $c_0 > 0$ *such that if* $f, \tilde{f} \in TX_{k+2}(Q_T)$ *satisfy* $\|f\|_{TX_{k+2}(Q_T)}, \|\tilde{f}\|_{TX_{k+2}(Q_T)} \leqq c_0$, *then* $N(f,\tilde{f}) \in TY_{k-2}(Q_T)$ *and*

$$\|N(f,f) - N(\tilde{f},\tilde{f})\|_{TY_{k-2}(Q_T)} \leqq C\|f - \tilde{f}\|_{TX_{k+2}(Q_T)} \times \left(\|\tilde{f}\|_{TX_{k+2}(Q_T)} + \|\tilde{f}\|_{TX_{k+2}(Q_T)}\right). \quad (169)$$

Proof. We will show the estimate

$$\|N(f,\tilde{f})\|_{TY_{k-2}} \leqq C\|f\|_{TX_{k+2}}\|\tilde{f}\|_{TX_{k+2}}. \quad (170)$$

The proof for (169) is a straightforward extension of this estimate using the arguments in the proof of Lemma 8.4. In order to prove (169), it is enough to show for all $\ell, i, j \in \mathbb{N}_0$ with $0 \leqq \ell \leqq k$ and $i + j \leqq \ell$,

$$\left\|\partial_t^i N(\tilde{f},f)\right\|_{Y_{j-2}} \leqq \sum_{i_1+i_2=i} \left(\|\partial_t^{i_1} f\|_{X_{\ell-i_1}}\|\partial_t^{i-i_1}\tilde{f}\|_{X_{\ell+2-(i-i_1),2}} + \|\partial_t^{i_1} f\|_{X_{\ell+2-i_1}}\|\partial_t^{i_2}\tilde{f}\|_{X_{\ell-i_2,2}}\right).$$

Indeed, estimate (170) follows by integrating the square of (171) in time, application of the Cauchy–Schwarz inequality (with L^2/L^∞) and taking the sum over ℓ, i, j. It remains to show the estimate (171). In the following, by $\mathscr{R}$ we denote any multiple (by a constant only depending on n, ε, k) of the right hand side of (171).
Let $q_{\delta_j} \in \mathscr{P}_Y$ be the expansion of $N(f,\tilde{f})$ of order δ_j and let $h := x + \int_0^x f\,\mathrm{d}\hat{x}$. We then have

$$[\partial_t^i N(f,\tilde{f})]^2_{Y_{j-2}} \overset{(39)}{=} \int_0^\infty x^{-\delta_j+j}\left|\partial_t^i\partial_x^{j-2}\left(\left((h^n - x^n)\tilde{f}_{xx}\right)_{xx} - q_{\delta_j}\right)\right|^2 \frac{\mathrm{d}x}{x} \leqq \mathscr{R}. \quad (171)$$

We will show the argument in the case $q_{\delta_j} = 0$. The extension to the general case follows by a straightforward adaption of the methods in the proof of Lemma 8.2. Inequality (171) for $i = 0$ follows from Lemma 8.4. It hence remains to prove the inequality for $0 < i \leqq \ell$ and $4j \leqq \ell - i$. If all the time derivatives in (171) are applied to $\tilde{f}$, then by Lemma 8.2 with $\tilde{f}$ replaced by $\partial_t^i\tilde{f}$, we get

$$\int_0^\infty x^{-\delta_j+j}\left|\partial_x^j\left((h^n - x^n)\partial_t^i\tilde{f}_{xx}\right)\right|^2 \frac{\mathrm{d}x}{x} \leqq C\left(\|f\|^2_{X_{j+2,2}}\|\partial_t^i f\|^2_{X_j} + \|f\|^2_{X_j}\|\partial_t^i f\|^2_{X_{j+2,2}}\right) \leqq \mathscr{R}.$$

We next consider the case when at least one time derivative falls on the factor $h^n - x^n$. We then have to estimate terms of the form

$$\int_0^\infty x^{-\delta_j+j} \left| h_0^{n-j-i+q} \Pi_{m=1}^q \partial_t^{i_m} \partial_x^{j_m} \varphi_m \right|^2 \frac{\mathrm{d}x}{x} \leqq \mathscr{R}, \tag{172}$$

where $\varphi_j \in \{f_0, \tilde{f}_0\}$ and where $\sum_{m=1}^q i_m = i$, $\sum_{m=1}^q j_m = j$ with $0 \leqq j_1 \leqq j_2 \leqq \cdots \leqq j_q$. The estimate of the terms in (172) follows by similar arguments as in the proof of Lemma 8.2. □

9. Localization

In this section, we collect some results which will are helpful for the proof of Theorem 4.5. Indeed, this Theorem can be obtained by a localization of the corresponding estimates in the proof of Theorem 4.1.
The first ingredient in the proof is the following extension lemma:

Lemma 9.1. (Extension lemma) *Let $n \in (0, 3)\backslash\{1, 2\}$, $k = 2 + 4j_0$, $j \in \mathbb{N}_0$ and suppose that ε satisfies (44)–(46). Suppose that $\varphi_j \in X_{2+k-4j}$ for $j = 2, \ldots, k$ and suppose that $h_{\text{in}}(x) := \frac{1}{2}(1 - x^2) + \int_{-1}^x \varphi_0 \, \mathrm{d}x$ satisfies $h > 0$ and (22). Then there is $w \in TX_{k+2}(Q)$ such that $h(t, x) := \frac{1}{2}(1 - x^2) + \int_{-1}^x w(t, \tilde{x}) \, \mathrm{d}\tilde{x}$ satisfies (22) and such that*

$$\partial_t^j w_{|t=0} = \varphi_j, \quad \text{and} \quad \|w\|_{TX_{k+2}} \leqq C \sum_{j=0}^{j_0} \|\varphi_j\|_{X_{2+k-4j}}. \tag{173}$$

If $\int_0^\infty \varphi_1 \, \mathrm{d}x = 0$, then $\int_0^\infty w \, \mathrm{d}x = \int_0^\infty \varphi_0 \, \mathrm{d}x$ for all $t > 0$.

Proof. Let ζ_1 be a smooth cut-off function with $\operatorname{supp} \zeta_1 \in [-1, \frac{1}{2}$ and let $\zeta_2 := 1 - \zeta_1$. By Lemma 7.5, there is an extension w_1 satisfying (173) where the functions φ_j are replaced by $\varphi_j^{(1)} \zeta_1$. Correspondingly, there is an extension w_2 satisfying (173) where the functions φ_j are replaced by $\varphi_j^{(1)} \zeta_2$. Note that $\|\varphi_j^{(i)}\|_{X_{2+k-4j}} \leqq C\|\varphi_j\|_{X_{2+k-4j}}$, $i = 1, 2$. It follows that $w := w_1 + w_2$ satisfies (173). Clearly, □

The idea is to apply the implicit function theorem on the linearization $\delta\mathscr{L}(w)$ of $\mathscr{L}(h)$ around w. In order to apply this theorem, we need to show maximal regularity of $\delta\mathscr{L}(w)$ and continuous differentiability of $\mathscr{L}$ in a neighborhood of w. This is the content of the next proposition:

Proposition 9.2. *Let $n \in (0, \frac{14}{5})\backslash\{1, 2\}$, let $k \geqq 2$ and suppose that ε satisfies (44)–(47). Suppose that $h_{\text{in}} : \Omega \to \mathbb{R}$ is positive in Ω and satisfies (22) with $\overline{h}_{\text{in}} \in X_k(\Omega)$ and let w be the extension of h_{in} from Lemma 9.1. Then for τ sufficiently small, we have*

1. *For any* $f_{\text{in}} \in X_k(\Omega)$ *and* $g \in TY_{k-2}(Q_t)$ *such that* (22) *and the corresponding compatibility condition to* (58) *are satisfied for* f_{in} *and* g *there exists a unique* $f \in TX_{k+2}(Q_t)$ *such that*

$$\delta\mathscr{L}(w)f = g \quad \text{in } Q_\tau \quad \text{and} \quad f|_{t=0} = f_{\text{in}} \quad \text{in } \Omega,$$

where $Q_\tau = [0, \tau] \times \Omega$. *Furthermore*

$$\|f\|_{TX_{k+2}(Q_\tau)} \leqq C \left(\|g\|_{TY_{k-2}(Q_\tau)} + \|f_{\text{in}}\|_{X_k(\Omega)}\right),$$

where C *depends on* k *and* $\|h_{\text{in}}\|_{X_k(\Omega)}$.

2. *There is an* $\delta > 0$, *such that* $\mathscr{L} : TX_{k+2}(Q_\tau) \to TY_{k-2}(Q_\tau)$ *is continuously differentiable for all* $\tilde{w}$ *in* X_τ *with* $\|w - \tilde{w}\|_{TX_{k+2}(Q_\tau)} < \delta$.

The proof of these statements follows by a localization of the global results on existence and uniqueness in Theorem 4.1. We refer the reader to the analogous localization arguments in [18,23,31] for further details. The general scheme applied in these papers also applies here.

10. Proof of the Theorems

In this section, we give the proof of the Theorems 4.1, 4.2 and 4.5 and of the Corollaries 4.3 and 4.4. We use fix-point arguments, similarly as in [31]. In order to apply the fixed point argument in the case of higher regularity, it is necessary to ensure that the compatibility conditions are satisfied in each step of the argument. In the fix-point in the proof of Theorem 2.2 in [31], we have missed this aspect. The argument in Theorem 4.2 closes this gap for the corresponding proof in [31].

Proof of Theorem 4.1. The proof of Theorem 4.1 follows by application of a contraction argument. Let $f_{\text{in}} \in X_k$ with $\|f_{\text{in}}\|_{X_k} \leqq \delta$, where δ will be fixed later. For $\eta > 0$ to be fixed later, we define the complete metric space

$$E := \{f \in TX_4 : f_{|t=0} = f_{\text{in}} \text{ and } \|f\|_{TX_4} \leqq \eta\}. \tag{174}$$

Indeed, for any $f \in TX_4$ with $k \geqq 2$, the trace of f at $t = 0$ is controlled as an L^2-function. Furthermore, E is not empty since the solution $f^{(0)}$ of the linear parabolic problem (117) with right hand side $g = 0$ and initial data f_{in}, with $\|f_{\text{in}}\|_{X_k} \leqq \delta$, is an element of E for $\delta = \delta(\eta)$ sufficiently small. For $f \in E$, we define $S(f)$ as the solution of

$$\begin{cases} \partial_t S(f) + AS(f) = N(f) & \text{for } (x, t) \in (0, \infty)^2, \\ S(f) = 0 & \text{for } x = 0, \\ S(f) = f_{\text{in}} & \text{for } t = 0. \end{cases} \tag{175}$$

Let $f_1, f_2 \in E$ and let $f := f_1 - f_2$. In particular, $S(f_1) - S(f_2)$ solves (117) with right hand side $N(f_1) - N(f_2)$ and vanishing initial data. By (118) it satisfies

$$\|S(f_1) - S(f_2)\|_{TX_4} \leqq C\|N(f_1) - N(f_2)\|_{TY_0}. \tag{176}$$

By Proposition 8.5, we hence get

$$\begin{aligned}\|S(f_1)-S(f_2)\|_{TX_4} &\leqq C\|N(f_1)-N(f_2)\|_{TY_0}\\ &\overset{(169)}{\leqq} C\|f_1-f_2\|_{TX_4}\left(\|f_1\|_{TX_4}+\|f_2\|_{TX_4}\right)\\ &\leqq C\eta\|f_1-f_2\|_{TX_4}.\end{aligned}$$

Hence, S is a contraction if $\eta>0$ is sufficiently small. Similarly, by (118) and (169), we get

$$\|S(f)\|_{TX_4} \leqq \|f_{\rm in}\|_{X_2}+C\|N(f)\|_{TY_0} \overset{(169)}{\leqq} C\left(\delta+\eta^2\right), \tag{177}$$

and hence $S(E)\subseteq E$ for $\delta:=\eta^2$ and for $\eta>0$ sufficiently small. The above estimates show that $S:E\to E$ is a contraction operator. By Banach's fixed point theorem, we thus obtain the existence of a unique $f\in E$ with $S(f)=f$. By (175), $S(f)$ solves the nonlinear equation. By (178) and for η sufficiently small, we get

$$\|f\|_{TX_4} \leqq \|f_{\rm in}\|_{X_2}+C\|f\|_{TX_4}^2 \leqq \|f_{\rm in}\|_{X_2}+\frac{1}{2}\|f\|_{TX_4} \tag{178}$$

which yields (60). This concludes the proof of Theorem 4.1. □

Proof of Theorem 4.2. Let $f_{\rm in}\in X_k$ with $\|f_{\rm in}\|_{X_k}\leqq\delta$, where $\delta>0$ will be fixed later and suppose that $f_{\rm in}$ satisfies the compatibility conditions. By Lemma 7.5 there is an extension $w\in TX_{k+2}$ (in particular $w_{|x=0}=0$) with $w_{|t=0}=f_{\rm in}$,

$$\partial_t^j w_{|t=0}=\left(\partial_t^{j-1}(N(w)-Aw)\right)_{|t=0} \qquad \text{for } 0\leqq 2+4j\leqq k \tag{179}$$

and $\|w\|_{TX_{k+2}}\leqq C\|f_{\rm in}\|_{X_k}$. For this, we choose w as in the extension Lemma 7.5, where the functions $\varphi^{(j)}$ are inductively defined by $\varphi^{(0)}=f_{\rm in}$ and by (179) with $\partial_t^{j-1}w_{|t=0}$ replaced by $\varphi^{(j-1)}$. By the compatibility conditions, we have $\varphi^{(j)}\in X_{k-4j}$ for $0\leqq 2+4j\leqq k$ so that the extension lemma is applicable. Similarly to (174), we define

$$E := \{f\in TX_{k+2}:\|f\|_{TX_{k+2}}\leqq\eta,\quad \partial_t^j f_{|t=0}=0 \quad \text{for } j=0,\dots,k\}. \tag{180}$$

For $f\in E$, we define $S(f)\in TX_{k+2}$ by $F=:w+S(f)$ where F is the solution of

$$\begin{cases} F_t+AF = N(w+f) & \text{for } x\in(0,\infty),\\ F = 0 & \text{for } x=0,\\ F = f_{\rm in} & \text{for } t=0.\end{cases} \tag{181}$$

The extension w is constructed such that the compatibility conditions (58) are satisfied up to order k for $f_{\rm in}$ and $N(w+f)$. Indeed by the definition of E, we have $(\partial_t^j N(w+f)-\partial_t^j N(w))_{|x=0}=0$ and $\partial_t^j N(w+f)-\partial_t^j N(w)\in H_{k-4j}$. The compatibility conditions then follow inductively from (179). By the parabolic estimate in Proposition 7.4, we hence get a unique solution $w + S(f)$ and the estimate

$$\|w + S(f)\|_{TX_{k+2}} \leqq C(\|f_{\text{in}}\|_{X_k} + \|N(w + f)\|_{TY_{k-2}}). \tag{182}$$

By Proposition 8.5, for all $f, \tilde{f} \in E$, we have

$$\begin{aligned} &\|N(w + S(f)) - N(w + S(\tilde{f}))\|_{TY_{k-2}(Q_T)} \\ &\quad \leqq C\|f - \tilde{f}\|_{TX_{k+2}(Q_T)}(\|w + \tilde{f}\|_{TX_{k+2}(Q_T)} + \|w + \tilde{f}\|_{TX_{k+2}(Q_T)}). \end{aligned} \tag{183}$$

Taking $j - 1$ derivatives of (181) in time and evaluating the result at $x = 0$, we get

$$\begin{aligned} \left(\partial_t^j S(f)\right)_{|x=0} &= \partial_t^{j-1} (N(w + f) - Aw - w_t)_{|x=0} - \left(\partial_t^{j-1} AS(f)\right)_{|x=0} \\ &\overset{(179)}{=} -A \left(\partial_t^{j-1} S(f)\right)_{|x=0}. \end{aligned} \tag{184}$$

From (184), we obtain inductively that $\partial_t^j S(f) = 0$ for $0 \leqq 2 + 4j \leqq k$. Analogously as in the proof of Theorem 4.1, it then follows that the operator S is a contraction operator in E if δ is chosen sufficiently small. By Banach's fixed point argument hence there is a unique $f \in E$ such that $F := w + f$ satisfies (181). In particular F solves (15) and satisfies (61). □

Proof of Corollary 4.3. By Theorem 4.1 there exists a solution satisfying (60). We argue by induction. Let $k \in \mathbb{N}$, $k \geqq 4$ and $\delta_k > 0$ be the constant from the assumptions of Theorem 4.2. For the induction argument, we assume there is $t_k > 0$ such that the compatibility conditions hold up to order k at $t = t_k$ and

$$f_{|t=t_k} \in X_k \quad \text{and} \quad \|f_{|t=t_k}\|_{X_k} \leqq \delta_k. \tag{185}$$

By Theorem 4.1, this assumption holds true for $k = 2$. By Theorem 4.2, we then have $f \in TX_{k+2}([t_k, \infty) \times \mathbb{R}_+)$ and $\|f\|_{TX_{k+2}([t_k,\infty)\times\mathbb{R}_+)} \leqq C\delta_k$. By Fubini's theorem it follows that there is a time $t_{k+2} > t_k$ which depends on t_k, δ_k, δ_{k+2} and C_{k+2} (the constants in the assumptions of Theorem 4.1 and in (60)) such that $f_{|t=t_{k+2}} \in X_{k+2}$ and $\|f_{|t=t_{k+2}}\|_{X_{k+2}} \leqq \delta_{k+2}$. In particular, the compatibility condition (59) up to order $k + 2$ hold at $t = t_{k+2}$, see also the discussion next to (59). The statement of Corollary 4.3 then follows by induction. □

Proof of Corollary 4.4. Clearly, h is a solution of (2) if f is a solution of (15). By Theorem 4.1, the solution f satisfies $f(t, \cdot) \in X_k$ for every $t > 0$. By Proposition 5.5 and by definition (38) of the norm, for every $t > 0$ there is an expansion $p \in \mathscr{P}_X$ of order δ_k such that

$$|f(x) - p(x)| \leqq Cx^{\delta_k}\|f\|_{C^0(X_k)} \leqq Cx^{\delta_k}\|f_{\text{in}}\|_{X_k} \tag{186}$$

for all $x \in (0, \infty)$. This implies that h has the expansion of type (63). The estimate (64), then follows from (11). The second statement of the corollary follows from Corollary 4.3. This concludes the proof of Corollary 4.4. □

Proof of Theorem 4.5. The proof of Theorem 4.5 is based on Proposition 9.2. Using this proposition, Theorem 4.5 follows by application of the inverse function theorem. We refer to similar arguments, as, for example, [23,32], and keep the details brief. We linearize $\mathscr{L}$ at an appropriately constructed extension w. We then show boundedness and differentiability for $\mathscr{L}$ and invertibility and maximal regularity for its linearization $\delta\mathscr{L}(w)$ at w. Suppose that $f_{\text{in}} \in X_{k+1/2}$ satisfies the compatibility conditions up to order k.
We first construct the extension $w \in TX_{k+1}(Q_\tau)$ such that with $w_{|t=0} = f_{\text{in}}$,

$$\partial_t^j \mathscr{L}(w)_{|t=0} = 0 \quad \text{for } 0 \leqq 2 + 4j \leqq k \tag{187}$$

and $\|w\|_{TX_{k+2}(Q_\tau)} \leqq C\|f_{\text{in}}\|_{X_k}$. The existence of such a function w follows from the extension Lemma 9.1 since the compatibility conditions hold, see also the argument in the proof of Theorem 4.2. Let $\delta\mathscr{L}(w)$ be the linearization of $\mathscr{L}$ around w. We define the operator

$$\mathscr{M} : TX_{k+2}(Q_\tau) \to X_k(\Omega) \times TY_{k-2}(Q_\tau) \quad \text{with} \quad \mathscr{M}(f) := (f_{|t=0}, \mathscr{L}f).$$

By Proposition 9.2, the operator $\mathscr{M}$ is bounded, continuously differentiable near w. Furthermore, $\delta\mathscr{M}(w)$ is invertible with bounded inverse for τ small enough. We define $\varphi := \mathscr{L}(w) \in TY_{k-2}(Q_\tau)$, that is $\mathscr{M}(w) = (f_{\text{in}}, \varphi)$. By the inverse mapping theorem there is a neighborhood $\mathscr{U}$ of w and a neighborhood $\mathscr{V}$ of (f_{in}, φ) such that $\mathscr{M} : \mathscr{U} \to \mathscr{V}$ is a diffeomorphism. By (187) we have $\partial_t^j \varphi_{|t=0} = 0$ for $0 \leqq 2 + 4j \leqq k$. It follows that $\|\varphi\|_{TX_k(Q_\tau)} \to 0$ for $\tau \to 0$. Hence, there is $\tilde{\tau} \in (0, \tau)$ and $\tilde{\varphi} \in TY_{k-2}(Q_\tau)$ with $\tilde{\varphi} = 0$ for $t \in (0, \tilde{\tau})$ and such that $(f_{\text{in}}, \tilde{\varphi}) \in \mathscr{V}$. Hence, there is $f \in \mathscr{U}$ with $\mathscr{M}(f) = (f_{\text{in}}, \tilde{\varphi})$. In particular, the function f solves $f = f_{\text{in}}$ at $t = 0$ and $\mathscr{L}(f) = 0$ for $t \in (0, \tilde{\tau})$. Correspondingly, $h(x) = x(1-x) + \int_0^x f(x')\mathrm{d}x'$ solves (21) for $t \in (0, \tilde{\tau})$. □

Appendix

In the appendix, we investigate how the operators $\partial_t + A$ and N act on expansions of type $\mathscr{P}_X$. We first consider the nonlinear operator N:

Lemma 10.1. *Let $n \in (0,3)\backslash\{1,2\}$. Let $f, \tilde{f} \in \mathscr{P}_X$, where*

$$f = \sum_{\beta \in I_X} f_\beta x^\beta \quad \textit{and} \quad \tilde{f} = \sum_{\tilde{\beta} \in I_X} \tilde{f}_{\tilde{\beta}} x^{\tilde{\beta}} \qquad \textit{for some } f_\beta, \tilde{f}_{\tilde{\beta}} \in \mathbb{R}. \tag{188}$$

Then there is a constant $c_0 > 0$ such that if $\|f\|_{\mathscr{P}} \leqq c_0$, then $g := N(f, \tilde{f}) \in \mathscr{P}_Y$ is well-defined for $x \in (0, 2)$. With the notation $g = \sum_{\beta \in I_Y} g_\beta x^\beta$, we furthermore have

$$|g_\gamma| \leqq C \sum_{\beta + \tilde{\beta} \leqq \delta_{k+4}} |f_\beta||\tilde{f}_{\tilde{\beta}}| \quad \forall \gamma \in \mathscr{P}_Y \textit{ with } \gamma \leqq \delta_k. \tag{189}$$

Proof of Lemma 10.1 We use the decomposition $N = N_1 + N_2$ given in (137), (138). By (134), N_2 is well-defined for $x \in (0, 2)$ if c_0 is sufficiently small. We first argue that $g \in \mathscr{P}_Y$. By Lemma 5.7, we have $F \in \mathscr{P}_X$, where F is defined in (131). We calculate,

$$\begin{aligned} F(x) &= \frac{f_{1+\alpha}}{2}x + \frac{f_\alpha}{\alpha+1}x^\alpha + \sum_{i+j\geqq 3, i\geqq 0, j\geqq 1} \frac{f_{i+j\alpha}}{i+1+j\alpha}x^{i+j\alpha}, \\ \tilde{f}_{xx}(x) &= \alpha(\alpha-1)\tilde{f}_\alpha x^{\alpha-2} + \sum_{i+j\geqq 1, i\geqq 0, \geqq 1} \tilde{f}_{i+j\alpha}x^{i+j\alpha}. \end{aligned} \tag{190}$$

In the sequel, by γ_{ij} we denote constants which depend only on the coefficients f_β with $\beta \leqq i + j\alpha$. By $\tilde{\gamma}_{ij}$ we denote constants which depend on the coefficients $\tilde{f}_{\tilde{\beta}}$ with $\tilde{\beta} \leqq i + j\alpha$. By $\bar{\gamma}_{ij}$ we denote constants which depend on sums of products of coefficients f_β and $\tilde{f}_\beta$ for $\beta + \tilde{\beta} \leqq i + j\alpha$. We have,

$$\sum_{r=2}^{\infty} c_{nr}F^r = \sum_{i+j\geqq 2, i\geqq 0, j\geqq 0} \gamma_{ij}x^{i+j\alpha}, \tag{191}$$

where c_{nr} are the binomial coefficients, cf. (137). The sum in (191) converges absolutely for c_0 sufficiently small and by (86). From the definition (137) of N_1, we hence get

$$\begin{aligned} N_1(f, \tilde{f}) &\overset{(137)}{=} \left(x^{3-\alpha}\left(\sum_{r=2}^{\infty} c_{nk}F^r\right)\tilde{f}_{xx}\right)_{xx} \\ &\overset{(190)}{=} \left(x^{3-\alpha}\sum_{i+j\geqq 2, i\geqq 0, j\geqq 0}\gamma_{ij}x^{i+j\alpha}\right)\left(\sum_{i+j\geqq -1, i\geqq -2, j\geqq 1}\tilde{\gamma}_{ij}x^{i+j\alpha}\right)_{xx} \\ &= \sum_{i+j\geqq 1, i\geqq -1, j\geqq 0}\bar{\gamma}_{ij}x^{i+j\alpha}. \end{aligned} \tag{192}$$

From the definition (138) of N_2, we get

$$\begin{aligned} N_2(f, \tilde{f}) &\overset{(138)}{=} \left(x^{3-\alpha}F\left(\tilde{f} - \tilde{f}_{01}x^\alpha\right)_{xx}\right)_{xx} \\ &\overset{(190)}{=} \left(x^{3-\alpha}\left(\sum_{i+j\geqq 1, i\geqq 0, j\geqq 0}\gamma_{ij}x^{i+j\alpha}\right)\left(\sum_{i+j\geqq 1, i\geqq 0, j\geqq 1}\tilde{\gamma}_{ij}x^{i+j\alpha}\right)\right)_{xx} \\ &= \sum_{i+j\geqq 2, i\geqq 1, j\geqq 0}\bar{\gamma}_{ij}x^{i+j\alpha}. \end{aligned} \tag{193}$$

Equations (192) and (193) yield $N(f, \tilde{f}) \in \mathscr{P}_Y$. The estimate (189) is a consequence of the scaling invariance of the operator and can be read off easily from the argument below. □

The following lemma demonstrates how solutions of the $\partial_t + A$ can be constructed in the space of generalized expansions. It is not used in the proof of the main theorems:

Lemma 10.2. *Let* $P_{f_{\rm in}} \in \mathscr{P}_{X,\delta_k}$, $g \in L^2(\mathscr{P}_{Y,\delta_{k-2}})$ *such that* (58) *holds for* $f_{in} = P_{f_{in}}$ *and* $g = P_g$. *Then there exists* $P_f \in L^2(\mathscr{P}_{X,\delta_{k+2}})$, *a solution of*

$$\partial_t P_f + A P_f \ = \ P_g, \quad \Pi_k P_{f\,|t=0} \ = \ P_{f_{\rm in}}, \tag{194}$$

where Π_k *is the orthogonal projection onto* P_{X,δ_k}. *Let* $n \in (0, \frac{14}{5})\backslash\{1, 2\}$. *Furthermore, for any given cut-off function* $\zeta \in C_c^\infty([0,\infty))$, *with* $\zeta = 1$ *in* $[0, 1]$ *and* $\zeta = 0$ *in* $[2,\infty)$, *we have*

$$\|\zeta P_f\|_{TX_{k+2}} \leqq C\left(\|\zeta P_{f_{\rm in}}\|_{X_{k+2}} + \|\zeta P_g\|_{TY_{k-2}}\right), \tag{195}$$

where C *only depends on* ζ.

Proof. Let us assume that g and f are given by

$$f(t) \ = \sum_{\beta\in[\delta_2,\delta_{k+2}]\cap\in I_X} c_\beta(t)x^{i+j\alpha}, \quad g(t) \ = \sum_{\beta\in[\delta_2,\delta_{k-2}]\cap I_Y} d_\beta(t)x^{i+j\alpha}. \tag{196}$$

The coefficients of $f_{\rm in}$ are correspondingly denoted by $c_\beta(0)$. Let $\Delta := 1+\alpha$. Equality (194) then takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}c_\beta(t) + M_{\beta+\Delta}c_{\beta+\Delta}(t) \ = \ d_\beta(t), \quad i \geqq 0, \tag{197}$$

for $\beta \in I_Y$ and with $M_{\beta+\Delta} \neq 0$. Let $\beta \in I_Y$ such that with $\beta - \Delta \notin I_Y$ and let $\tilde{c}_i := c_{\beta+i\Delta}$, $\tilde{d}_i := d_{\beta+i\Delta}$, $\tilde{M}_i := M_{\beta+i\Delta}$. Then we get the chain of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{c}_i(t) + \tilde{M}_{i+1}\tilde{c}_{i+1}(t) \ = \ \tilde{d}_i(t) \quad \text{for } i \in \mathbb{N}_0. \quad \tilde{c}_i(0) = \tilde{c}_{i,\rm in}. \tag{198}$$

It is enough to find the solution for a system of type (198). Since $f_\in \in \mathscr{P}_X$, we have $\tilde{c}_{0,\rm in} = 0$, cf. Fig. 2. We choose $\tilde{c}_0(t) = 0$. The solution can be inductively calculated by $\tilde{M}_{j+1}\tilde{c}_{j+1} = \tilde{d}_j - \frac{\mathrm{d}}{\mathrm{d}t}\tilde{c}_j$. The estimate (195) follows. Note that the compatibility condition (58) implies $c_{2i}(t) = c_{2i,\rm in}$ for all $i \geqq 0$. This solves (198) and hence (197). The estimate follows easily using the equation. □

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References

1. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions. National Bureau of Standards, United Statement Department of Commerce (1972)

2. Angenent, S.: Analyticity of the interface of the porous media equation after the waiting time. *Proc. Am. Math. Soc.* **102**(2), 329–336 (1988)
3. Angenent, S.: Local existence and regularity for a class of degenerate parabolic equations. *Math. Ann.* **280**(3), 465–482 (1988)
4. Angenent, S.: Solutions of the one-dimensional porous medium equation are determined by their free boundary. *J. Lond. Math. Soc. (2)* **42**(2), 339–353 (1990)
5. Barrett, J., Blowey, J., Garcke, H.: Finite element approximation of a fourth order nonlinear degenerate parabolic equation. *Numer. Math.* **80**(4), 525–556 (1998)
6. Beretta, E., Bertsch, M., Dal Passo, R.: Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. *Arch. Ration. Mech. Anal.* **129**(2), 175–200 (1995)
7. Bernis, F.: Finite speed of propagation and continuity of the interface for thin viscous flows. *Adv. Differ. Equ.* **1**(3), 337–368 (1996)
8. Bernis, F.: Finite speed of propagation for thin viscous flows when $2 \leqq n < 3$. *C. R. Acad. Sci. Paris Sér. I Math.* **322**(12), 1169–1174 (1996)
9. Bernis, F., Friedman, A.: Higher order nonlinear degenerate parabolic equations. *J. Differ. Equ.* **83**(1), 179–206 (1990)
10. Bertozzi, A.: The mathematics of moving contact lines in thin liquid films. *Notices Am. Math. Soc.* **45**(6), 689–697 (1998)
11. Bertozzi, A., Pugh, M.: The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. *Commun. Pure Appl. Math.* **49**(2), 85–123 (1996)
12. Bertsch, M., Dal Passo, R., Garcke, H., Grün, G.: The thin viscous flow equation in higher space dimensions. *Adv. Differ. Equ.* **3**(3), 417–440 (1998)
13. Bertsch, M., Giacomelli, L., Karali, G.: Thin-film equations with "partial wetting" energy: existence of weak solutions. *Phys. D* **209**(1–4), 17–27 (2005)
14. Blake, T.: The physics of moving wetting lines. *J. Coll. Interf. Sc.* **299**(1), 1–13 (2006)
15. Caffarelli, L., Friedman, A.: Regularity of the free boundary for the one-dimensional flow of gas in a porous medium. *Am. J. Math.* **101**(6), 1193–1218 (1979)
16. Dal Passo, R., Garcke, H., Grün, G.: On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions. *SIAM J. Math. Anal.* **29**(2), 321–342 (electronic) (1998)
17. Dal Passo, R., Giacomelli, L., Grün, G.: A waiting time phenomenon for thin film equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30**(2), 437–463 (2001)
18. Daskalopoulos, P., Hamilton, R.: Regularity of the free boundary for the porous medium equation. *J. Am. Math. Soc.* **11**(4), 899–965 (1998)
19. Eggers, J.: Drop formation—an overview. *ZAMM Z. Angew. Math. Mech.* **85**(6), 400–410 (2005)
20. Gennes, P.D., Brochard-Wyart, F., Quere, D.: Capillarity and Wetting Phenomena. Springer, Berlin, 2004
21. Giacomelli, L., Gnann, M.V., Knüpfer, H., Otto, F.: Well-posedness for the Navier-slip thin-film equation in the case of complete wetting. *J. Differ. Equ.* **257**(1), 15–81 (2014). doi:10.1016/j.jde.2014.03.010
22. Giacomelli, L., Gnann, M.V., Otto, F.: Regularity of source-type solutions to the thin-film equation with zero contact angle and mobility exponent between 3/2 and 3. *Eur. J. Appl. Math.* **24**(5), 735–760 (2013). doi:10.1017/S0956792513000156
23. Giacomelli, L., Knüpfer, H.: A free boundary problem of fourth order: classical solutions in weighted Hölder spaces. *Commun. Partial Differ. Equ.* **35**(11), 2059–2091 (2010). doi:10.1080/03605302.2010.494262
24. Giacomelli, L., Knüpfer, H., Otto, F.: Smooth zero-contact-angle solutions to a thin-film equation around the steady state. *J. Differ. Equ.* **245**(6), 1454–1506 (2008)
25. Giacomelli, L., Shishkov, A.: Propagation of support in one-dimensional convected thin-film flow. *Indiana Univ. Math. J.* **54**(4), 1181–1215 (2005)
26. Greenspan, H.: Motion of a small viscous droplet that wets a surface. *J. Fluid Mech.* **84**, 125–143 (1978)

27. Grün, G.: Droplet spreading under weak slippage: a basic result on finite speed of propagation. *SIAM J. Math. Anal.* **34**(4), 992–1006 (electronic) (2003). doi:10.1137/S0036141002403298
28. Grün, G.: Droplet spreading under weak slippage—existence for the Cauchy problem. *Commun. Partial Differ. Equ.* **29**(11–12), 1697–1744 (2004)
29. Huh, C., Scriven, L.: Hydrodynamic model of steady movement of a solid/liquid/fluid contact line. *J. Coll. Int. Sc.* **35**(1), 85–101 (1971)
30. Hulshof, J., Shishkov, A.E.: The thin film equation with $2 \leqq n < 3$: finite speed of propagation in terms of the L^1-norm. *Adv. Differ. Equ.* **3**(5), 625–642 (1998)
31. Knüpfer, H.: Navier slip thin-film equation for partial wetting. *Commun. Pure Appl. Math.* **64**(9), 1263–1296 (2011)
32. Knüpfer, H., Masmoudi, N.: Darcy's flow with prescribed contact angle: well-posedness and lubrication approximation. *Arch. Rational Mech. Anal.* (2015). doi:10.1007/s00205-015-0868-8
33. Knüpfer, H., Masmoudi, N.: Well-posedness and uniform bounds for a nonlocal third order evolution operator on an infinite wedge. *Commun. Math. Phys.* **320**(2), 395–424 (2013). doi:10.1007/s00220-013-1708-z
34. Koch, H.: Non-euclidean singular integrals and the porous medium equation. Habilitation Thesis, Dortmund (1999)
35. Kozlov, V., Mazya, V., Rossmann, J.: Elliptic boundary value problems in domains with point singularities, *Mathematical Surveys and Monographs*, vol. 52. Am. Math. Soc., Providence, 1997
36. Oron, A., Davis, S., Bankoff, S.: Long-scale evolution of thin liquid films. *Rev. Mod. Phys.* **69**(3), 931–980 (1997)
37. Otto, F.: Lubrication approximation with prescribed nonzero contact angle. *Commun. Partial Differ. Equ.* **23**(11–12), 2077–2164 (1998)
38. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. I, 2nd edn. Academic Press Inc., New York, 1980. Functional analysis
39. Ren, W., Hu, D., Weinan, E.: Continuum models for the contact line problem. *Phys. Fluids* **22**(10) (2010)
40. Reynolds, O.: On the theory of lubrication and its application to Mr. Beauchamp Tower's experiments, including an experimental determination of the viscosity of olive oil. *Proc. R. Soc. Lond.* **40**, 191–203 (1886)
41. Thompson, P., Troian, S.: A general boundary condition for liquid flow at solid surfaces. *Nature* **389**(6649), 360–362 (1997)
42. Wayner, P.: Spreading of a liquid film with a finite contact angle by the evaporation condensation process. *Langmuir* **9**(1), 294–299 (1993). doi:10.1021/la00025a056

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