Arch. Rational Mech. Anal. 218 (2015) 1239–1262
Digital Object Identifier (DOI) 10.1007/s00205-015-0879-5

Macroscopic Description of Microscopically Strongly Inhomogenous Systems: A Mathematical Basis for the Synthesis of Higher Gradients Metamaterials

A. Carcaterra, F. dell'Isola, R. Esposito & M. Pulvirenti

Communicated by C. Dafermos

Abstract

We consider the time evolution of a one dimensional n-gradient continuum. Our aim is to construct and analyze discrete approximations in terms of physically realizable mechanical systems, referred to as microscopic because they are living on a smaller space scale. We validate our construction by proving a convergence theorem of the microscopic system to the given continuum, as the scale parameter goes to zero.

1. Introduction

Continua with exotic behaviors are attracting increasing attention because of their technological applications (see for example [1,11,19,24,26,29] and references therein). In this paper we address what, in a sense, is an inverse problem: given a continuum model we seek for those mechanical systems which, at a certain length scale, behave as specified by the chosen continuum model. The aim is to understand the microscopic properties of such systems to obtain information on how to realize (synthesize) them, at least in principle.

To be more precise, we are interested in a metamaterial which, roughly speaking, is an array of elementary individuals, much smaller than the typical macroscopic size, arranged in periodic structures and exhibiting unusual macroscopic behavior.

In our mathematical analysis we want to consider such a continuous system as described by a partial differential equation generated by a Lagrangian which summarizes all the macroscopic properties we may desire. Then we discretize this system and manage to identify such a discretization as a real conservative mechanical model. In other words we start with macroscopic behavior and describe one possible microscopic interaction which realizes it at a macroscopic level. Finally we give a mathematical foundation to this procedure by proving a convergence result.

From a mathematical point of view, we underline once more that this is an inverse problem, compared to the one (largely unsolved) formulated by D. Hilbert in his famous speach in 1900 at ICM in Paris (see [15]) in which he encouraged attempts to prove rigorously the transition from particle systems to fluid dynamics (Hilbert's 6-th problem). However it is worth stressing that we are working in the framework of continuum mechanics, but our microscopic elements, even if small in macro units, are large compared with molecular scales.

We conclude this introduction by spending some more time discussing metamaterials, collocating them in the framework of generalized continua, with a particular emphasis on the pioneering work of G. Piola (see [2,9,25]).

The rest of the paper is organized as follows. In Section 2 we introduce continuous and discrete Lagrangians and discuss the identification problem, namely we specify the mechanical systems outlined by the discretization procedure. In Section 3 we formulate and solve the associated convergence problem.

We remark that our work concerns one-dimensional systems only. This is of course a severe limitation, but, on the other hand, it is a natural setting to start with.

1.1. Mechanical Metamaterials

By suitably rephrasing ENGHETA and ZIOLKOWSKI [11] and ZOUHDI ET AL. [29], metamaterials are materials which are first theoretically conceived and then engineered to have properties very unlikely to be found in nature.

They are obtained by suitably assembling multiple individual elements constructed with already available microscopic materials, but usually arranged in (quasi-)periodic sub-structures. Indeed the properties of metamaterials do not depend only on those of their component materials, but also on the topology of their connections and the nature of their mutual interaction forces. In literature there is currently specified a particular class of metamaterials, so called mechanical metamaterials, those in which the particular properties which are "designed" for the newly synthesized material are purely mechanical. The present paper deals exactly with such a class.

We explicitly remark here that in the present paper we use the adjective "microscopic" or "micro-" meaning all those length scales which are (much) smaller than the scale at which continuum mechanics is applicable. In particular we do not attach any value in SI units to each considered length scale.

The particular shape, geometry, size, orientation and arrangement of the elementary individual elements can affect, for instance, the propagation of waves of light or sound in a not-already-observed manner. In this way one can create material properties which cannot be found in conventional materials.

Particularly promising are those micro-structures which present high-contrast in microscopic properties. These structures, once homogenized, have been shown to produce generalized continua (see for example [1,5,26]). These micro-structures, although remaining quasi-periodical, are conceived so that some of the physical properties which are characterizing their behavior are diverging when the size of the representative elementary volume tends to zero, while simultaneously some others are vanishing in this limit.

To give a hint of the possible applications of newly designed metamaterials we list here some among the papers which are more relevant to our results, especially in the perspective of their extension to two dimensional and three dimensional systems. In [18] it is shown how to synthesize a composite medium exhibiting negative effective bulk modulus, negative effective mass density (see also [5]), or both properties. In [16] materials with negative Poisson's ratio (auxetics) were designed, and they were fabricated in 1999 (see Xu et al. [28]). One of the most famous examples of such materials is the Goretex whose negative Poisson ratio opened unexpected possibilities regarding things like vascular surgery.

The damping effects can be also suitably designed using special selection of the material microstructure as reported in [3,4], or acoustic and optical effects such as negative refraction, lensing and cloaking [6,21].

All described materials can be modeled at a micro-level as finite dimensional Lagrangian systems and their effective properties are all obtained via a kind of homogenization procedure.

1.2. Generalized Continua

In the first half of the nineteenth century the design of structures became an intellectual activity based on the rigorous application of predictive mathematical models. These models were formulated by means of a precise postulation process and originated a series of problems or exercises directly motivated by the engineering applications, which were solved by means of the use of the then newly developed techniques of mathematical analysis.

The model describing the mechanical behaviour of materials introduced by Cauchy—although very accurate for a large class of phenomena—cannot be applied to all materials in every physical condition.

More general models were formulated by Gabrio Piola in the same years, but only recently they were considered in engineering for applications.

In some formulations of continuum mechanics, the possibility of the dependence of deformation energy on higher gradients of displacement is rejected, due to an apparent (see [7]) incompatibility with the second principle of thermodynamics ([10,14]). On the other hand, physicists, for instance Landau and Lifshitz [17], always considered this dependence as admissible, as they are accustomed to basing the postulation of physical theories on the principle of least action or on the principle of virtual works, which is exactly the same starting point as that of Piola [25].

Actually, when introducing Piola continua, the true conceptual frame settled by Cauchy, Navier and Poisson is to be drastically modified. The concept of stress becomes secondary and the main role is played by deformation measures together with action and dissipation functionals. The Euler–Lagrange equation obtained in this more encompassing modeling process cannot be regarded anymore as coinciding with the balance of force unless one generalizes the concept of force. This can be done by introducing generalized actions as the dual quantities in the work of the gradients of displacements (see for example [8,12,13,20,22,23,27]).

Actually the same concept of contact interaction has to be completely modified, and the crucial point of determining the correct boundary conditions which can be assigned in generalized continua theory has been addressed only very recently (see for example [8]), following the original ideas by PIOLA [25].

2. Microscopic and Macroscopic Descriptions

In what follows we will consider two length scales, l and L, with $l \ll L$. We will call microscopic or micro the description at the length scale l, while macroscopic or macro will be the attribute relative to the description which is suitable at the length scale L.

We assume that the most suitable micro-description at micro-scale is "discrete", that is based on the model "material particle" (as done by Poisson, Navier and in some works by Piola), while the description which has to be used at the macro-level is that of a continuum, as introduced, for example, by Lagrange, Cauchy or again Piola.

We remark however that we will not limit our attention to systems which verify the assumptions put forward by Cauchy and Navier. We will consider, actually, those continua which have been considered by Piola (and then by many others, including Toupin, Green, Rivlin and Mindlin), that is, the so called higher gradient continua.

To quantify the above considerations we will introduce, in the sequel, a small parameter $\varepsilon > 0$ indicating the ratio between typical micro and macro scales, possibly to be sent to zero to outline a suitable asymptotic behavior.

2.1. The Basic Macroscopic Continuous Model

Let $I = (0, L) \subset \mathbb{R}$ be a finite interval assumed as a reference configuration of the considered one-dimensional continuum. We label each element of the continuum with the coordinate $x \in I$ of its placement in the reference configuration. The actual configuration of the continuum is described by the displacement field $u = u(x, t)$ which represents the horizontal displacement at time t of the element x from its position in the reference configuration.

Fixing an integer $n \geq 1$, for such a system we introduce the Lagrangian

$$\mathcal{L}(u, \dot{u}) = \frac{1}{2} \int_I |\dot{u}(x)|^2 - \int_I \Phi \left(u(x), Du(x), \ldots, D^n u(x) \right). \tag{2.1}$$

Here, $D^k u$ is the k-th x-derivative of u and

$$\mathbb{R}^{n+1} \ni \underline{\xi} = (\xi_0, \ldots, \xi_n) \mapsto \Phi(\underline{\xi}) \in \mathbb{R} \tag{2.2}$$

is a function whose properties will be specified later on.

Note that $\Phi(u, Du, \ldots, D^n u)$ is the potential energy density corresponding to the displacement u and describes the constitutive properties of the medium under investigation.

The action on the time interval $(0, T)$ is consequently defined as

$$\mathcal{A} = \int_0^T \mathcal{L}\left(u(\,\cdot\,, t), \dot{u}(\,\cdot\,, t)\right), \tag{2.3}$$

where $\dot{u}(x, t) = \partial_t u(x, t)$ is the time derivative.

To deduce the Euler–Lagrange equations from the stationary action principle, we have first to specify the kinematic boundary condition for our problem. In the sequel we shall assume either

- periodic boundary conditions. Namely the reference configuration is $\mathcal{C}$, a circle of radius $\frac{L}{2\pi}$ (the points 0 and L are identified), or
- Dirichlet boundary conditions. Namely u and its first $n-1$ derivatives vanish at 0 and L.

With the above boundary conditions, no boundary terms appear when performing the integrations by parts needed to obtain the equation of the motion (2.4) below.

Note also that the maximal order of the spatial derivatives appearing in the equation of motion (2.4) is $2n$.

The equation of motion, as a consequence of the stationary action principle and the boundary conditions, is (with $D^0u = u$)

$$\ddot{u} = -\sum_{\alpha=0}^{n} (-1)^{\alpha} D^{\alpha} \partial_{\xi_\alpha} \Phi \left(u, Du, \dots, D^n u\right). \tag{2.4}$$

We could also include, in the present context, a given external potential with a very minor effort. We avoid doing so for notational simplicity.

Now we specify Φ by assuming that

$$\Phi(\underline{\xi}) = \frac{1}{2}(\underline{\xi}, Q\underline{\xi}) + R(\underline{\xi}), \tag{2.5}$$

that is, the quadratic part of Φ is a quadratic form in terms of the displacement and its derivatives, contained in the vector ξ. $Q = \{Q_{\alpha,\beta}\}_{\alpha,\beta=0}^{n}$ is a symmetric (without loss of generality) constant matrix with $Q_{n,n} \neq 0$.

On the non-linear part R we shall do suitable assumptions later on. We start by requiring that

$$R(\underline{0}) = 0, \quad R(\underline{\xi}) = O\left(|\xi|^3\right); \tag{2.6}$$

namely, the quadratic part of the interaction is fully expressed by the matrix Q.

The fact that Φ is not dependant explicitly on x is a consequence of the macroscopic homogeneity of the continuum (although it may be strongly inhomogeneous at microscopic scales). This implies that Q is constant.

As a first step we show that, in contrast with the fairly general nature of the model, the quadratic part can be considerably simplified. Indeed, symmetrizing, integrating by parts and using the periodic or Dirichlet boundary conditions, we get:

$$\begin{aligned}
\mathcal{U} &\doteq \frac{1}{2} \sum_{\alpha,\beta=0}^{n} Q_{\alpha,\beta} \int_I D^{\alpha} u D^{\beta} u \\
&= \frac{1}{4} \sum_{\gamma=0}^{2n} \sum_{\substack{\alpha,\beta\geq 0: \\ \alpha+\beta=\gamma}} Q_{\alpha,\beta} \int_I u D^{\gamma} u \big[(-1)^{\alpha} + (-1)^{\beta}\big]
\end{aligned}$$

$$= \frac{1}{4} \sum_{\gamma=0}^{n} \sum_{\substack{\alpha,\beta \geq 0: \\ \alpha+\beta=2\gamma}} Q_{\alpha,\beta} \big[(-1)^\alpha + (-1)^\beta\big](-1)^\gamma \int_I |D^\gamma u|^2$$

$$= \frac{1}{2} \sum_{\gamma=0}^{n} A_\gamma \int_I |D^\gamma u|^2, \tag{2.7}$$

where

$$A_\gamma = \frac{1}{2} \sum_{\substack{\alpha,\beta \geq 0: \\ \alpha+\beta=2\gamma}} Q_{\alpha,\beta} \big[(-1)^\alpha + (-1)^\beta\big](-1)^\gamma. \tag{2.8}$$

Note that in the first step in (2.7) we have used the symmetry of $Q_{\alpha,\beta}$ and in the second step we used that $(-1)^\alpha + (-1)^\beta = 0$ if $\alpha + \beta$ is odd. In the third step we have again integrated by parts.

As a consequence of this analysis, without loss of generality, we can assume Φ of the form

$$\Phi = \frac{1}{2} \sum_{\alpha=0}^{n} A_\alpha |\xi_\alpha|^2 + R(\underline{\xi}), \tag{2.9}$$

with $A_n \neq 0$ and the equations of motion are

$$\ddot{u} = -\sum_{\alpha=0}^{n} (-1)^\alpha A_\alpha \Delta^\alpha u - \sum_{\alpha=0}^{n} (-1)^\alpha D^\alpha \partial_{\xi_\alpha} \mathbb{R}\left(u, Du, \ldots, D^n u\right), \tag{2.10}$$

where $\Delta = D^2$ denotes the Laplacian. Note that in the linear part only even derivatives are allowed.

2.2. Formal Discretization

In view of the construction of the mechanical (microscopic) system with a finite number of degrees of freedom, we introduce a finite lattice of mesh ε in I. The lattice points are $\{0, \varepsilon, 2\varepsilon \ldots, k\varepsilon, \ldots N\varepsilon\}$ with the obvious condition $N\varepsilon = L$. When considering periodic boundary conditions we clearly identify 0 with εN.

We associate to each lattice point a *microscopic* particle of unitary mass labelled by the index $i \in \{0, \ldots, N\}$ and denote by u_i the displacement of the particle i from the reference position $i\varepsilon$. The array $u_\varepsilon = \{u_i\}_{i=0}^N$ is the discretized displacement field.

The discretized Lagrangian takes the form

$$\mathcal{L}_\varepsilon(u_\varepsilon, \dot{u}_\varepsilon) = \frac{1}{2} \sum_{i=0}^{N} \varepsilon \dot{u}_i^2 - U(u_\varepsilon), \tag{2.11}$$

where

$$U(u_\varepsilon) = \sum_{i=0}^{N} \varepsilon \left[\frac{1}{2} \sum_{\alpha=0}^{n} A_\alpha \left|(D_\varepsilon^\alpha u_\varepsilon)_i\right|^2 + R((\underline{D_\varepsilon u_\varepsilon})_i) \right], \tag{2.12}$$

where $\underline{(D_\varepsilon u_\varepsilon)}_i = \{(D_\varepsilon^\alpha u_\varepsilon)_i\}_{\alpha=0}^n$,

$$D_\varepsilon^\alpha u = \begin{cases} \Delta_\varepsilon^{\frac{\alpha}{2}} u_\varepsilon, & \alpha \text{ even}, \\ D_\varepsilon^+ \Delta_\varepsilon^{\frac{\alpha-1}{2}} u_\varepsilon, & \alpha \text{ odd}. \end{cases} \tag{2.13}$$

Here D_ε^+ and D_ε^-, defined as

$$\left(D^+ u_\varepsilon\right)_i = \frac{u_{i+1} - u_i}{\varepsilon}, \quad (D^- u_\varepsilon)_i = \frac{u_i - u_{i-1}}{\varepsilon}, \tag{2.14}$$

are the right and left discrete derivatives respectively and Δ_ε, defined by

$$(\Delta_\varepsilon u_\varepsilon)_i = \left(D_\varepsilon^+ D_\varepsilon^- u_\varepsilon\right)_i = \left(D_\varepsilon^- D_\varepsilon^+ u_\varepsilon\right)_i = \frac{1}{\varepsilon^2}\left(u_{i+1} + u_{i-1} - 2u_i\right), \tag{2.15}$$

is the discrete Laplacian.

To complete the above definitions we need to define the discrete derivatives at the boundary. For periodic boundary conditions it is enough to use the following convention: for any $k \in \mathbb{Z}$,

$$u_{N+k} = u_k. \tag{2.16}$$

For the Dirichlet boundary condition, we have to think of the first and last n particles frozen in their reference position. Hence we assume the constraints

$$u_i = 0, \quad i \in \{0, \ldots, n-1\} \cup \{N-n+1, \ldots, N\}. \tag{2.17}$$

The equations of motion are

$$\ddot{u}_i = F_i, \quad F_i = -\frac{\partial U}{\partial u_i}, \tag{2.18}$$

with the index i running from 1 to N in the periodic case and on the set of i's for which u_i is not constrained in the Dirichlet case. We notice that the choice of the right derivative (as well as any other possible discretization) is arbitrary. The only restriction that we have is the mechanical realizability (in principle) of this system. We are going to discuss this point in the next subsection.

We finally remark that F_i depends on u_j, with $|i-j| \le n$. However this is an almost local contribution because n is fixed and those u_j's influencing u_i are at macroscopic distance $O(\varepsilon)$.

2.3. Realizable Syntheses

The aim of this subsection is to show that, at least in the simplest case of linear forces, the above introduced discrete system corresponds to a system of particles interacting via two-body forces of range not larger than n. Therefore, it can be realized by suitably assembling mechanical elements.

Let us consider the linear system introduced in (2.4) with $R = 0$ and its discrete counterpart (2.18). It can be checked that

$$F_i = -\sum_{k=0}^{n} (-1)^k A_k \Delta_\varepsilon^k u_\varepsilon(x_i). \tag{2.19}$$

Therefore, the force acting on the particle i is expressed as a linear combination of discrete derivatives up to the order $2n$.

We want to show that F_i can be interpreted as the result of the action of a system of linear pairwise forces with suitable range. More precisely, we want to find ε-dependent coefficients $k_{i,j}$ such that

$$F_i = \sum_j k_{i,j}(u_j - u_i) \tag{2.20}$$

and hence

$$U(u_1, \dots, u_N) = \frac{1}{2} \sum_{i,j=1}^{N} k_{i,j}(u_i - u_j)^2. \tag{2.21}$$

We prove below that for any p,

$$(\Delta^p u)_i = \sum_j K_{i,j}^p (u_j - u_i), \tag{2.22}$$

with $K_{i,j}^p$ other suitable constants. Once (2.22) is proved, we can conclude that (2.20) holds with

$$k_{i,j} = \sum_{p=0}^{n} (-1)^p A_p K_{i,j}^p. \tag{2.23}$$

Note that the constants $k_{i,j}$ are not necessarily all positive even if the A_α are all positive.

The constants $K_{i,j}^p$ are given by the recursive Equation (2.28) below. It implies that, for any p, $K_{i,j}^p$ vanishes for $|i - j| > p$, thus $k_{i,j} = 0$ if $|i - j| > n$. Moreover, in the periodic case, $K_{i,j}^p$ depends only on the difference $i - j$ and is symmetric in the exchange $i \leftrightarrow j$ and hence the *action–reaction* principle is satisfied.

We prove (2.22) by recurrence.

For $p = 1$, we have

$$(\Delta_\varepsilon u)_i = \varepsilon^{-2}(u_{i+1} + u_{i-1} - 2u_i) = \varepsilon^{-2}(u_{i+1} - u_i) + \varepsilon^{-2}(u_{i-1} - u_i). \tag{2.24}$$

Thus (2.22) is verified with

$$K_{i,i+1}^1 = K_{i,i-1}^1 = \varepsilon^{-2} \quad \text{and} \quad K_{i,j}^1 = 0 \quad \text{otherwise}. \tag{2.25}$$

Suppose now that (2.22) is true for $p = \ell - 1$:

$$\left(\Delta_\varepsilon^{\ell-1} u\right)_i = \sum_j K_{i,j}^{\ell-1}[u_j - u_i].$$

Then,

$$\begin{aligned}\left(\Delta_\varepsilon^\ell u\right)_i &= \left(\Delta_\varepsilon^{\ell-1}\Delta_\varepsilon u\right)_i = \sum_j K_{i,j}^{\ell-1}\left[(\Delta_\varepsilon u)_j - (\Delta_\varepsilon u)_i\right]\\ &= \sum_j K_{i,j}^{\ell-1}\left[\varepsilon^{-2}(u_{j+1}-u_j) + \varepsilon^{-2}(u_{j-1}-u_j) - \varepsilon^{-2}(u_{i+1}-u_i) - \varepsilon^{-2}(u_{i-1}-u_i)\right]\\ &= \sum_j K_{i,j}^{\ell-1}\Big[\varepsilon^{-2}(u_{j+1}-u_i) - \varepsilon^{-2}(u_j-u_i) + \varepsilon^{-2}(u_{j-1}-u_i) - \varepsilon^{-2}(u_j-u_i)\\ &\qquad -\varepsilon^{-2}(u_{i+1}-u_i) - \varepsilon^{-2}(u_{i-1}-u_i)\Big]. \end{aligned} \tag{2.26}$$

Using the change of index $j+1 \to j$ in the first term and $j-1 \to j$ in the second, we have

$$\begin{aligned}\left(\Delta_\varepsilon^\ell u\right)_i &= \sum_j K_{i,j-1}^{\ell-1}\varepsilon^{-2}(u_j-u_i) - K_{i,j}^{\ell-1}\varepsilon^{-2}(u_j-u_i) + K_{i,j+1}^{\ell-1}\varepsilon^{-2}(u_j-u_i)\\ &\quad -\varepsilon^{-2}K_{i,j}^{\ell-1}(u_j-u_i) - \varepsilon^{-2}K_{i,j}^{\ell-1}(u_{i+1}-u_i) - \varepsilon^{-2}K_{i,j}^{\ell-1}(u_{i-1}-u_i)]. \end{aligned} \tag{2.27}$$

Thus, (2.22) is verified with the following recursive definition of $K_{i,j}^\ell$:

$$K_{i,j}^\ell = \varepsilon^{-2}\Big[K_{i,j-1}^{\ell-1} + K_{i,j+1}^{\ell-1} - 2K_{i,j}^{\ell-1} - (\delta_{i+1,j} + \delta_{i-1,j})\sum_{j'} K_{i,j'}^{\ell-1}\Big], \tag{2.28}$$

for $\ell > 1$ and $K_{i,j}^1$ given by (2.25).

Equations (2.28) and (2.23) definitely solve the posed problem of identifying the topology of the microstructure connections, since they provide the coefficients $k_{i,j}$ only in terms of the coefficients A_p that characterize the continuous formulation of the macroscopic description of the elastic problem.

3. A Rigorous Result of Convergence

In this section we prove a convergence result of the discrete model introduced in the previous section to the prescribed continuous systems in the limit as the scale parameter goes to 0. We show the convergence of the solution of the discrete system to the continuous one in the energy norm of the system. To clarify the argument without the use of cumbersome notation, we present first a paradigmatic case for which we discuss both periodic and Dirichlet boundary conditions. The more general case is considered in Section 3.2 where we give the convergence proof only in the periodic case although the argument can be straightforwardly extended to the Dirichlet boundary conditions as well.

For the reader's convenience we rewrite the Lagrangian we are going to consider in this Section, namely:

$$\begin{aligned}\mathcal{L}(u,\dot u) &= \frac{1}{2}\int_I \mathrm{d}x|\dot u(x,t)|^2\\ &\quad -\frac{1}{2}\sum_{\alpha=1}^n A_\alpha \int_I \mathrm{d}x|D^\alpha u|^2(x,t))^2 - \int_I R\left(u, Du, D^2u\ldots\right). \end{aligned} \tag{3.1}$$

As we shall see later on, we will consider only nonlinear terms R depending on u and the first derivative only.

3.1. The Δ^2 Case-Dynamic Euler–Bernoulli Beam: "Elastica"

3.1.1. Periodic Boundary Conditions We consider the Lagrangian (3.1) with $A_0 = A_1 = 0$ and $A_2 = 1$. Moreover we focus on the linear case $R = 0$. Thus we have the following linear initial value problem in the circle, $\mathcal{C}$:

$$\ddot{u} = -\frac{\partial^4 u}{\partial x^4} := -\Delta^2 u, \tag{3.2}$$

$$u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x). \tag{3.3}$$

It is well known that there exists a unique classical solution as the initial data are assumed to be sufficiently smooth.

More precisely, we assume that

$$u_0 \in H^s, \quad v_0 \in H^r \quad \text{with} \quad s \geq 6, \quad r \geq 4, \tag{3.4}$$

where H^s denotes the Sobolev space endowed with norm

$$\|u\|_{H^s} = \sum_{\ell=0}^{s} \left\| D^\ell u \right\|_2^2,$$

and $\| \cdot \|_p$ is the $L^p(\mathcal{C})$-norm.

In this way, by using the well known energy method, we can prove the propagation (in time) of the H^s regularity for u and $\dot{u}$, yielding, in particular, $u \in C^5(\mathcal{C})$ (as consequence of the obvious inequality $\|u\|_\infty \leq C\|u\|_{H^1}$).

Next we consider the mechanical system of N particles, with coordinates u_i, $i = 1, \ldots N$, whose Lagrangian is given by (2.11) again with $A_0 = A_1 = 0$, $A_2 = 1$ and $R = 0$. The equations of motion are explicitly

$$\ddot{u}_i = \frac{1}{\varepsilon^4}(-u_{i+2} + 4u_{i+1} - 6u_i - u_{i-2} + 4u_{i-1}) \quad i = 1 \ldots N, \tag{3.5}$$

with the convention $u_{N+k} = u_k$ for any $k \in \mathbb{Z}$.

We want to compare the solutions of (3.2) with the corresponding ones of (3.5). To do this we first set

$$u_\varepsilon(x, t) = u_i(t) \quad \text{if} \quad x \in [i\varepsilon, (i+1)\varepsilon), \quad i \in \{1, \ldots, N\}. \tag{3.6}$$

In other words we introduce a function u_ε which is the step, left continuous, function (constant in the lattice interval) taking the value of the nearest left point of the lattice. Problem (3.5) is rephrased accordingly:

$$\ddot{u}_\varepsilon(x, t) = -\Delta_\varepsilon^2 u_\varepsilon(x, t) \quad x \in \mathcal{C}, \tag{3.7}$$

where

$$\Delta_\varepsilon u(x) = D_\varepsilon^+ D_\varepsilon^- u(x) \tag{3.8}$$

$$D^{\pm}u(x) = \pm\frac{1}{\varepsilon}(u(x \pm \varepsilon) - u(x)). \tag{3.9}$$

Notice that the Lagrangian (2.11), with $A_0 = A_1 = 0$, $A_2 = 1$ and $R = 0$, has the following continuous representation:

$$\mathcal{L}(u_\varepsilon, \dot{u}_\varepsilon) = \int_{\mathcal{C}} \mathrm{d}x \Big[\frac{1}{2}\dot{u}_\varepsilon(x,t)^2 - \frac{1}{2}(\Delta_\varepsilon u_\varepsilon(x,t))^2\Big]. \tag{3.10}$$

We suppose that, at the initial time, $u_\varepsilon, \dot{u}_\varepsilon$ are approximating $u, \dot{u}$ in the sense that

$$u_\varepsilon(x,0) = u_0(i\varepsilon), \quad \dot{u}_\varepsilon(x,0) = v_0(i\varepsilon) \quad \text{if} \quad x \in [i\varepsilon, (i+1)\varepsilon). \tag{3.11}$$

Note that, by the conservation of the energy, we have

$$\mathcal{E}[u(t)] := \frac{1}{2}\int_{\mathcal{C}} \mathrm{d}x \Big[|\dot{u}(t)|^2 + |\Delta u(t)|^2\Big] = \mathcal{E}[u(0)], \tag{3.12}$$

as well as

$$\mathcal{E}_\varepsilon[u_\varepsilon(t)] := \frac{1}{2}\int_{\mathcal{C}} \mathrm{d}x \Big[|\dot{u}_\varepsilon(t)|^2 + |\Delta_\varepsilon u_\varepsilon(t)|^2\Big] = \mathcal{E}_\varepsilon[u_\varepsilon(0)]. \tag{3.13}$$

Next we introduce the following function which controls the deviation of u_ε from u:

$$\begin{aligned} W_\varepsilon(t) = \frac{1}{2}\int_{\mathcal{C}} \mathrm{d}x \Big[&(u_\varepsilon(x,t) - u(x,t))^2 + (\dot{u}_\varepsilon(x,t) - \dot{u}(x,t))^2 \\ &+ [\Delta_\varepsilon(u_\varepsilon(x,t) - u(x,t))]^2\Big]. \end{aligned} \tag{3.14}$$

Computing the time derivative and using the equation of motion we get

$$\begin{aligned} \dot{W}_\varepsilon(t) &= \int_{\mathcal{C}} \mathrm{d}x\, (\dot{u}_\varepsilon(x,t) - \dot{u}(x,t))(u_\varepsilon(x,t) - u(x,t) + \ddot{u}_\varepsilon(x,t) - \ddot{u}(x,t)) \\ &\quad + \int_{\mathcal{C}} \mathrm{d}x\, \Delta_\varepsilon(u_\varepsilon(x,t) - u(x,t))\Delta_\varepsilon(\dot{u}_\varepsilon(x,t) - \dot{u}(x,t)) \\ &= \int_{\mathcal{C}} \mathrm{d}x\, (\dot{u}_\varepsilon(x,t) - \dot{u}(x,t))(u_\varepsilon(x,t) - u(x,t)) \\ &\quad - \int_{\mathcal{C}} \mathrm{d}x\, (\dot{u}_\varepsilon(x,t) - \dot{u}(x,t))\Delta_\varepsilon^2(u_\varepsilon(x,t) - u(x,t))] \\ &\quad + \int_{\mathcal{C}} \mathrm{d}x\, (\dot{u}_\varepsilon(x,t) - \dot{u}(x,t))(\Delta^2 u(x,t) - \Delta_\varepsilon^2 u(x,t)) \\ &\quad + \int_{\mathcal{C}} \mathrm{d}x\, \Delta_\varepsilon(u_\varepsilon(x,t) - u(x,t))\Delta_\varepsilon(\dot{u}_\varepsilon(x,t) - \dot{u}(x,t)). \end{aligned} \tag{3.15}$$

Now consider the following discrete integration by parts formula, namely

$$\int_{\mathcal{C}} f(x) D_\varepsilon^{\pm} g(x) = -\int_{\mathcal{C}} D_\varepsilon^{\mp} f(x) g(x), \tag{3.16}$$

valid for any couple of bounded functions f and g.

If we apply the above formula twice we conclude that the second and fourth terms in (3.15) cancel each other. On the other hand the first term is bounded by

$$\frac{1}{2}\int_{\mathcal{C}} \mathrm{d}x|\dot{u}-\dot{u}_\varepsilon|^2+|u-u_\varepsilon|^2 \le W.$$

The third term is bounded by

$$\frac{1}{2}\int_{\mathcal{C}} \mathrm{d}x(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))^2+\frac{1}{2}\int_{\mathcal{C}} |(\Delta^2-\Delta_\varepsilon^2)u(x,t)|^2.$$

Now the first term of the expression above is bounded by W. The second one, by the regularity of u and its derivatives up to the fifth order, is bounded, uniformly in $x \in \mathcal{C}$ and in t in any bounded interval, by a constant ω_ε vanishing as $\varepsilon \to 0$. Here and in the rest of the paper $\omega_\varepsilon \in \mathbb{R}$ denotes such a generic infinitesimal constant. In conclusion, by the Gronwall lemma,

$$W_\varepsilon(t) \le W_\varepsilon(0)\mathrm{e}^{2t}+\omega_\varepsilon t\mathrm{e}^{2t} \tag{3.17}$$

so that $W_\varepsilon(t)$ is vanishing, because $W_\varepsilon(0) \to 0$ by the regularity of u and the assumptions on initial data.

We summarize the above discussion in the following:

Theorem 3.1. *Suppose that u_0 and v_0 satisfy (3.4). Let $u(t)$ be the solution to (3.2) and $u_\varepsilon(t)$ be the step function defined by (3.6) with $u_i(t)$, $i=1,\dots,N$, solutions to (3.5) with initial data $u_i(0)=u_0(i\varepsilon)$ and $\dot{u}_i(0)=v_0(i\varepsilon)$. Then, for any $t \in \mathbb{R}$,*

$$\lim_{\varepsilon\to 0} W_\varepsilon(t)=0.$$

3.1.2. Dirichlet Boundary Conditions For the Dirichlet boundary conditions we replace the circle $\mathcal{C}$ with the interval $I=[0,L]$. The Equation (3.2) is well posed with the boundary conditions

$$u(0,t)=u'(0,t)=u(L,t)=u'(L,t)=0. \tag{3.18}$$

Remark. Several other boundary conditions may have an interest in engineering applications and a physical meaning. For instance the conditions $u(0)=u''(0)=0$, $u(L)=u''(L)=0$, characterize a beam with pivots applied at its endpoints, while the conditions which we consider here are relative to clamped-clamped beams. We do not consider in this paper the other possible boundary conditions, as the focus of this paper is different.

Again, by using the energy method, we can construct a solution with H^s regularity, by assuming

$$u_0 \in H_0^2 \cap H^s, \quad v_0 \in H_0^2 \cap H^r \quad \text{with} \quad s \ge 6, \quad r \ge 4. \tag{3.19}$$

Here H_0^2 (introduced to take into account the boundary conditions) is defined as the space of the H^2 functions vanishing in 0 and L, together with their first derivative.

The corresponding discrete system is constituted by $N-3$ particles with coordinates u_i, $i=2,\dots,N-2$ and

$$u_0 = u_1 = u_{N-1} = u_N = 0 \tag{3.20}$$

are the constraints corresponding to the Dirichlet boundary conditions.

With this position, the explicit equations of motion are

$$\ddot{u}_i = \frac{1}{\varepsilon^4}(-u_{i+2} + 4u_{i+1} - 6u_i - u_{i-2} + 4u_{i-1}) \qquad i = 2\dots N-2. \tag{3.21}$$

As before, we introduce the left continuous step function

$$u_\varepsilon(x,t) = u_i(t) \quad \text{if} \quad x \in [i\varepsilon, (i+1)\varepsilon), \quad i \in \{0,\dots,N-1\}, \tag{3.22}$$

but we find it convenient to think of it as a function on $\mathbb{R}$ extended with value 0 outside I. Then (3.21) can be rewritten similarly to (3.7) as

$$\ddot{u}_\varepsilon(x,t) = -\Delta_\varepsilon^2 u_\varepsilon(x,t) \quad x \in I_\varepsilon = (2\varepsilon, L-\varepsilon). \tag{3.23}$$

Note that the values of u_i are frozen for $i = 0, 1, N-1, N$, so that $u_\varepsilon = 0$ in $I_\varepsilon^c = I - I_\varepsilon$. We also think of the solution u of the continuous equation as extended with value 0 outside of I.

Next we introduce the function $W_\varepsilon(t)$ as

$$W_\varepsilon(t) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))^2 + \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x[\Delta_\varepsilon(u_\varepsilon(x,t)-u(x,t))]^2. \tag{3.24}$$

Note that this function differs from the one defined by integrating on I instead of $\mathbb{R}$ because Δ_ε is non-local. It is actually larger and hence provides a stronger control of the convergence. Now we compute again the time derivative of W, as before, and we get

$$\dot{W}_\varepsilon(t) = \int_{\mathbb{R}} \mathrm{d}x(\dot{u}_\varepsilon(x,t) - \dot{u}(x,t))(\ddot{u}_\varepsilon(x,t) - \ddot{u}(x,t)) \\ + \int_{\mathbb{R}} \mathrm{d}x\,\Delta_\varepsilon(u_\varepsilon(x,t) - u(x,t))\Delta_\varepsilon(\dot{u}_\varepsilon(x,t) - \dot{u}(x,t)). \tag{3.25}$$

By twice using the discrete integration by parts formula

$$\int_{\mathbb{R}} \mathrm{d}x f D_\varepsilon^{\pm} g = -\int_{\mathbb{R}} g D_\varepsilon^{\mp} f,$$

valid for any couple of bounded compactly supported functions f and g, the second term becomes, as before

$$\int_{\mathbb{R}} \mathrm{d}x(\dot{u}_\varepsilon(x,t) - \dot{u}(x,t))\Delta_\varepsilon^2(u_\varepsilon(x,t) - u(x,t)).$$

As for the the first term, we need to use the equations of motion (3.2) for u and (3.23) for u_ε. Note that the last ones hold only in I_ε. Thus, using that $\ddot{u}_\varepsilon = 0$ in $\mathbb{R} - I_\varepsilon$, the first term becomes

$$
\begin{aligned}
&-\int_{I_\varepsilon} \mathrm{d}x(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))(\Delta_\varepsilon^2 u_\varepsilon(x,t)\\
&-\Delta^2 u(x,t)) - \int_{\mathbb{R}-I_\varepsilon}(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))(-\Delta^2 u(x,t))\\
&= -\int_{\mathbb{R}} \mathrm{d}x(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))(\Delta_\varepsilon^2 u_\varepsilon(x,t)-\Delta^2 u(x,t))\\
&\quad + \int_{\mathbb{R}-I_\varepsilon}(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))\Delta_\varepsilon^2 u_\varepsilon(x,t)). \qquad (3.26)
\end{aligned}
$$

By adding and subtracting the term $\int_{\mathbb{R}} \mathrm{d}x(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))(\Delta_\varepsilon^2 u_\varepsilon(x,t)-\Delta_\varepsilon^2 u(x,t))$ the above term becomes

$$
\begin{aligned}
&-\int_{\mathbb{R}} \mathrm{d}x(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))(\Delta_\varepsilon^2 u_\varepsilon(x,t)-\Delta_\varepsilon^2 u(x,t))\\
&-\int_{\mathbb{R}} \mathrm{d}x(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))(\Delta_\varepsilon^2 u(x,t)-\Delta^2 u(x,t))\\
&+\int_{\mathbb{R}-I_\varepsilon}(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))\Delta_\varepsilon^2 u_\varepsilon(x,t)). \qquad (3.27)
\end{aligned}
$$

Putting together all these terms we conclude that

$$
\begin{aligned}
\dot{W} = &-\int_{\mathbb{R}} \mathrm{d}x(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))\left(\Delta_\varepsilon^2 u(x,t)-\Delta^2 u(x,t)\right)\\
&+\int_{\mathbb{R}-I_\varepsilon}(\dot{u}_\varepsilon(x,t)-\dot{u}(x,t))\Delta_\varepsilon^2 u_\varepsilon(x,t)). \qquad (3.28)
\end{aligned}
$$

The first term in the right hand side of (3.28) goes to 0 as in the periodic case, by the regularity u. The second term is the novelty of the Dirichlet case. In order to estimate it, note that $\dot{u}_\varepsilon = 0$ outside of I_ε, hence we need to estimate $\int_{\mathbb{R}-I_\varepsilon} \dot{u}(x,t)\Delta_\varepsilon^2 u_\varepsilon(x,t))$.

By the boundary conditions on u ($u = 0$ and $u' = 0$ in 0 and L for any t) we have the result that (by our assumptions $|\Delta\dot{u}(x,t)|$ is bounded)

$$
|\dot{u}(x,t)| \leq \frac{1}{2}\sup_{x\in\mathcal{I}}|\Delta\dot{u}(x,t)|\varepsilon^2 \quad x \in I_\varepsilon^c. \qquad (3.29)
$$

Furthermore,

$$
\Delta_\varepsilon^2 u_\varepsilon(x) = \varepsilon^{-2}(\Delta_\varepsilon u_\varepsilon(x+\varepsilon)+\Delta_\varepsilon u_\varepsilon(x-\varepsilon)-2\Delta_\varepsilon u_\varepsilon(x)).
$$

Rewriting the total energy (3.13) in a more explicit form,

$$
\mathcal{E}[u_\varepsilon] = \frac{1}{2}\sum_{i=2}^{N-2}\varepsilon|\dot{u}_\varepsilon(\varepsilon i)|^2 + \frac{1}{2}\sum_{i=1}^{N}\varepsilon|\Delta_\varepsilon u_\varepsilon(i\varepsilon)|^2, \qquad (3.30)
$$

and we obtain, at any time and for any x in $\mathcal{I}_\varepsilon$,

$$
|\Delta_\varepsilon u_\varepsilon(x)| \leq \frac{\sqrt{2E_0}}{\sqrt{\varepsilon}}, \qquad (3.31)
$$

where $E_0 = \mathcal{E}(u(0))$ is the energy of the initial data. Hence,

$$\sup_{x\in I}\left|\Delta_\varepsilon^2 u_\varepsilon(x)\right| \le 4\sqrt{2E_0}\varepsilon^{-\frac{5}{2}}. \tag{3.32}$$

Combining (3.29) and (3.32) and using the fact that the integration is restricted to the set $I - I_\varepsilon$, whose measure is 4ε (reminding the reader that $\dot{u} = 0$ outside $\mathcal{I}$), we conclude that

$$\left|\int_{\mathbb{R}-I_\varepsilon} \dot{u}(x,t)\Delta_\varepsilon^2 u_\varepsilon(x,t))\right| \le C\sqrt{\varepsilon}.$$

The rest of the argument proceeds as before and we conclude that $W_\varepsilon(t) \to 0$.

We summarize the above discussion in the following:

Theorem 3.2. *Suppose that u_0 and v_0 satisfy (3.19). Let $u(t)$ be the classical solution to (3.2) with boundary conditions (3.18) and initial values (3.3) and $u_\varepsilon(t)$ be the step function defined by (3.22) with $u_i(t)$, $i = 2, \ldots, N-2$, solutions to (3.21) with initial data $u_i(0) = u_0(i\varepsilon)$ and $\dot{u}_i(0) = v_0(i\varepsilon)$. Then, for any $t \in \mathbb{R}$,*

$$\lim_{\varepsilon\to 0} W_\varepsilon(t) = 0.$$

3.2. A n-th Gradient Case

Now we extend the previous argument to the more general setup corresponding to the Lagrangian (3.1), restricting the discussion to the simpler case of periodic boundary conditions. The Dirichlet boundary conditions can be handled as in the previous subsection but we avoid here unnecessary complications.

We assume the following conditions:

1.
$$A_0 > 0, \quad A_n > 0, \quad A_\alpha \ge 0, \quad \alpha = 1, \ldots, n-1 \tag{3.33}$$

2. We have already supposed that $R(\underline{0}) = 0$ and $R(\underline{\xi}) = O(|\underline{\xi}|^3)$. In addition we assume that, for $n = 1$, R depends only on u and, for $n \ge 2$, R depends only on u and Du. Moreover we assume $R \in C^{2n+2}(\mathbb{R}^2)$.

Remark 3.1. The positivity assumptions on the A_α's with $\alpha = 1, \ldots, n-1$, can be relaxed. In fact, let us define, for some $\varepsilon_0 > 0$,

$$\kappa = \sup_{\varepsilon\in(0,\varepsilon_0)} \sup_{u:\|D_\varepsilon u\|_2 \le 1} \frac{\|u\|_2^2}{\|D_\varepsilon u\|_2^2}, \tag{3.34}$$

with the supremum on u taken on all u with 0 average. Then it is enough to assume

$$\sum_{\alpha=1:A_\alpha<0}^{n-1} |A_\alpha|\kappa^{n-\alpha} \le \frac{1}{2}A_n \tag{3.35}$$

to make the argument of the proof still work. This remark allows us to consider, for instance, the case $\ddot{u} = (-\Delta^2 - \gamma\Delta)u$, with γ sufficiently small, excluded by (3.33).

Remark 3.2. The assumption on R concerning its dependence on u and Du only, is restrictive. We do not expect any surprise in assuming an explicit dependence on some higher derivatives. However, as we shall see in the course of the proof, more general assumptions would complicate the algebraic manipulations in dealing with the discrete derivatives in a consistent way.

As regards the initial data we assume

$$u_0 \in H^{2n+2}, \quad v_0 \in H^{n+2}, \tag{3.36}$$

and, as before, the H^s regularity is propagated. Clearly $u \in C^{2n+1}(\mathcal{C})$.

The explicit equation is

$$\ddot{u} + \sum_{\alpha=0}^{n} (-1)^\alpha A_\alpha \Delta^\alpha u + \partial_{\xi_0} R(u, Du) - D[\partial_{\xi_1} R(u, Du)] = 0. \tag{3.37}$$

Note that, thanks to the energy conservation,

$$\mathcal{E}[u] = \int_{\mathcal{C}} \mathrm{d}x \left[\frac{1}{2} \left\{ \dot{u}^2 + \sum_{\alpha=0}^{n} A_\alpha |D^\alpha u|^2 \right\} + R(u, Du) \right] = \mathcal{E}[u(0)], \tag{3.38}$$

and we get immediately an a priori bound on the L^2 norm of u, $\dot{u}$ and $D^n u$:

$$\frac{1}{2} \int_{\mathcal{C}} \mathrm{d}x \left[|\dot{u}|^2 + A_0 |u|^2 + A_n |D^n u|^2 \right] \leq \mathcal{E}[u(0)]. \tag{3.39}$$

Now we remind the reader of the discrete counterpart of the above setup, which corresponds to the discrete Lagrangian (2.11). Using the discontinuous function

$$u_\varepsilon(x, t) = u_i(t) \quad \text{if} \quad x \in [i\varepsilon, (i+1)\varepsilon), \tag{3.40}$$

as in the previous section, the discrete Lagrangian can be written as

$$\mathcal{L}_\varepsilon = \int_{\mathcal{C}} \mathrm{d}x \left[\frac{1}{2} |\dot{u}_\varepsilon(x,t)|^2 - \frac{1}{2} \sum_{\alpha=0}^{n} A_\alpha \left| D_\varepsilon^\alpha u_\varepsilon(x,t) \right|^2 - R\left(u_\varepsilon(x,t), D_\varepsilon^+ u_\varepsilon(x,t) \right) \right]. \tag{3.41}$$

We can write the associated equations of motion in terms of u_ε as

$$\ddot{u}_\varepsilon + \sum_{\alpha=0}^{n} (-1)^\alpha A_\alpha \Delta_\varepsilon^\alpha u_\varepsilon + \partial_{\xi_0} R\left(u_\varepsilon, D_\varepsilon^+ u \right) - D_\varepsilon^- \left[\partial_{\xi_1} R(u_\varepsilon, D_\varepsilon^+ u_\varepsilon) \right] = 0. \tag{3.42}$$

Also for the discrete system the energy conservation holds. Thus we have that

$$\mathcal{E}_\varepsilon[u_\varepsilon] = \int_{\mathcal{C}} \mathrm{d}x \left[\frac{1}{2} \left\{ \dot{u}_\varepsilon^2 + \sum_{\alpha=0}^{n} A_\alpha \left| D_\varepsilon^\alpha u \right|^2 \right\} + R\left(u_\varepsilon, D_\varepsilon^+ u_\varepsilon \right), \right] \tag{3.43}$$

is conserved, and hence, using that $R \geq 0$, we have the inequality

$$\frac{1}{2}\int_{\mathcal{C}} \mathrm{d}x \left[|\dot{u}_\varepsilon|^2 + A_0|u_\varepsilon|^2 + A_n \left|D^n_\varepsilon u_\varepsilon\right|^2\right] \leq \mathcal{E}_\varepsilon[u_\varepsilon(0)]. \tag{3.44}$$

Since $A_0 > 0$ and $A_n > 0$, the existence, globally in time, for the solution to the discrete system follows from this bound.

We start by proving the convergence of the discrere system to the continuous one in the linear case, namely when $R = 0$,

$$\ddot{u} + \sum_{\alpha=0}^{n}(-1)^\alpha A_\alpha \Delta^\alpha u = 0. \tag{3.45}$$

Similarly, the discrete system becomes

$$\ddot{u}_\varepsilon + \sum_{\alpha=0}^{n}(-1)^\alpha A_\alpha \Delta^\alpha_\varepsilon u_\varepsilon = 0. \tag{3.46}$$

We introduce

$$\begin{aligned} W_\varepsilon(t) = \frac{1}{2}\int_{\mathcal{C}} \mathrm{d}x \left\{|u(x,t) - u_\varepsilon(x,t)|^2 + |\dot{u}(x,t) - \dot{u}_\varepsilon(x,t)|^2\right\} \\ + \int_{\mathcal{C}} \mathrm{d}x \sum_{\alpha=0}^{n} A_\alpha \left|D^\alpha_\varepsilon[u(x,t) - u_\varepsilon(x,t)]\right|^2. \end{aligned} \tag{3.47}$$

The time derivative of W is:

$$\begin{aligned} &\frac{d}{\mathrm{d}t}W_\varepsilon \\ &= \int_{\mathcal{C}} \mathrm{d}x \left\{(\dot{u} - \dot{u}_\varepsilon)(u - u_\varepsilon + \ddot{u} - \ddot{u}_\varepsilon) + \sum_{\alpha=0}^{n} A_\alpha D^\alpha_\varepsilon(\dot{u} - \dot{u}_\varepsilon) D^\alpha_\varepsilon(u - u_\varepsilon)\right\} \\ &= \int_{\mathcal{C}} \mathrm{d}x (\dot{u} - \dot{u}_\varepsilon)\left\{(u - u_\varepsilon + \ddot{u} - \ddot{u}_\varepsilon) + \sum_{\alpha=0}^{n}(-1)^\alpha A_\alpha \Delta^\alpha_\varepsilon(u - u_\varepsilon)\right\} \\ &= \int_{\mathcal{C}} \mathrm{d}x (\dot{u} - \dot{u}_\varepsilon)\left\{(u - u_\varepsilon) + \sum_{\alpha=0}^{n}(-1)^\alpha A_\alpha \left[\Delta^\alpha_\varepsilon u_\varepsilon - \Delta^\alpha u - \Delta^\alpha_\varepsilon u_\varepsilon + \Delta^\alpha_\varepsilon u\right]\right\} \\ &= \int_{\mathcal{C}} \mathrm{d}x (\dot{u} - \dot{u}_\varepsilon)\left\{u - u_\varepsilon + \sum_{\alpha=0}^{n}(-1)^\alpha A_\alpha \left(\Delta^\alpha_\varepsilon u - \Delta^\alpha u\right)\right\}. \end{aligned}$$

In the second step we have integrated by parts α times, in the third we have used the equations of motion. The last step follows by canceling two equal terms with opposite sign.

Hence the linear case goes exactly as in previous subsection, because $|\Delta^\alpha_\varepsilon u - \Delta^\alpha u| \leq \omega_\varepsilon$ for $\alpha \leq n$ and $u \in C^{2n+1}(\mathcal{C})$.

We summarize the results for the linear case in the following:

Theorem 3.3. *Assume that* $R = 0$ *and suppose that* u_0 *and* v_0 *satisfy* (3.36). *Let* $u(t)$ *be the classical solution to* (3.37) *and* $u_\varepsilon(t)$ *be the step function defined by* (3.6) *with* $u_i(t)$, $i = 1, \ldots, N$, *solutions to* (3.45) *with initial data* $u_i(0) = u_0(i\varepsilon)$ *and* $\dot{u}_i(0) = v_0(i\varepsilon)$.

Then, for any $t \in [0, T]$,

$$\|u(t) - u_\varepsilon(t)\|_\varepsilon \to 0, \quad as\ \varepsilon \to 0,$$

where $\|\cdot\|_\varepsilon$ *is the* ε*-dependent norm defined by*

$$\|u\|_\varepsilon^2 = \int_{\mathcal{C}} \mathrm{d}x \left\{ |\dot{u}|^2 + |u|^2 + \sum_{k=1}^{n} \left| D_\varepsilon^k u \right|^2 \right\}. \tag{3.48}$$

Next we consider the nonlinear case. Now the equations of motion are (3.37) and (3.42) for the continuous and discrete systems respectively. Defining W_ε by (3.47), by the same computation, we have, again using the summation by parts formula,

$$\begin{aligned}
\frac{d}{\mathrm{d}t} W_\varepsilon &= \int_{\mathcal{C}} \mathrm{d}x (\dot{u} - \dot{u}_\varepsilon) \Big\{ (u - u_\varepsilon + \ddot{u} - \ddot{u}_\varepsilon \Big\} + \sum_{\alpha=0}^{n} A_\alpha D_\varepsilon^\alpha (\dot{u} - \dot{u}_\varepsilon)(D_\varepsilon^\alpha (u - u_\varepsilon) \\
&= \int_{\mathcal{C}} \mathrm{d}x (\dot{u} - \dot{u}_\varepsilon) \left\{ (u - u_\varepsilon + \ddot{u} - \ddot{u}_\varepsilon) + \sum_{\alpha=0}^{n} (-1)^\alpha A_\alpha \Delta_\varepsilon^\alpha (u - u_\varepsilon) \right\} \\
&= \int_{\mathcal{C}} \mathrm{d}x (\dot{u} - \dot{u}_\varepsilon) \Big\{ (u - u_\varepsilon) + \sum_{\alpha=0}^{n} (-1)^\alpha A_\alpha \Big[\Delta_\varepsilon^\alpha u_\varepsilon - \Delta^\alpha u - \Delta_\varepsilon^\alpha u_\varepsilon + \Delta_\varepsilon^\alpha u \Big] \\
&\quad + \Big[-\partial_{\xi_0} R(u, Du) + D\partial_{\xi_1} R(u, Du) + \partial_{\xi_0} R\left(u_\varepsilon, D_\varepsilon^+ u_\varepsilon\right) - D_\varepsilon^- \partial_{\xi_1} R\left(u_\varepsilon, D_\varepsilon^+ u_\varepsilon\right) \Big] \Big\} \\
&= \int_{\mathcal{C}} \mathrm{d}x (\dot{u} - \dot{u}_\varepsilon) \Big\{ u - u_\varepsilon + \sum_{\alpha=0}^{n} (-1)^\alpha A_\alpha \left(\Delta_\varepsilon^\alpha u - \Delta^\alpha u \right) \\
&\quad + \Big[-\partial_{\xi_0} R(u, Du) + D\partial_{\xi_1} R(u, Du) + \partial_{\xi_0} R\left(u_\varepsilon, D_\varepsilon^+ u_\varepsilon\right) - D_\varepsilon^- \partial_{\xi_1} R\left(u_\varepsilon, D_\varepsilon^+ u_\varepsilon\right) \Big] \Big\}.
\end{aligned} \tag{3.49}$$

To control the non-linear terms we proceed by estimating:

$$T_1 = \partial_{\xi_0} R(u, Du) - \partial_{\xi_0} R\left(u_\varepsilon, D_\varepsilon^+ u_\varepsilon\right) = T_1^1 + T_1^2 \tag{3.50}$$

and

$$T_2 = D_\varepsilon^- \left[\partial_{\xi_1} R\left(u_\varepsilon, D_\varepsilon^+ u_\varepsilon\right)\right] - D\left[\partial_{\xi_1} R(u, Du)\right] = T_2^1 + T_2^2 \tag{3.51}$$

where

$$T_1^1 = \partial_{\xi_0} R(u, Du) - \partial_{\xi_0} R\left(u, D_\varepsilon^+ u\right), \tag{3.52}$$

$$T_1^2 = \partial_{\xi_0} R(u, D_\varepsilon^+ u) - \partial_{\xi_0} R\left(u_\varepsilon, D_\varepsilon^+ u_\varepsilon\right), \tag{3.53}$$

$$T_2^1 = D_\varepsilon^- \left[\partial_{\xi_1} R\left(u, D_\varepsilon^+ u\right)\right] - D\left[\partial_{\xi_1} R(u, Du)\right], \tag{3.54}$$

and

$$T_2^2 = D_\varepsilon^- \left[\partial_{\xi_1} R(u_\varepsilon, D_\varepsilon^+ u_\varepsilon)\right] - D_\varepsilon^- \left[\partial_{\xi_1} R\left(u, D_\varepsilon^+ u\right)\right]. \tag{3.55}$$

The bound (3.39) and Poincaré inequality imply that the L^∞ norms of u, Du and $D_\varepsilon^\pm u$ are bounded uniformly in ε. Thus, by the local Lipschitz continuity of $\partial_{\xi_0} R$, we have

$$\left|T_1^1\right| \le C\left|Du - D_\varepsilon^+ u\right| \le \omega_\varepsilon,$$

by the regularity of u. Thus, by the energy bounds (3.39) and (3.44) we get

$$\left|\int_{\mathcal{C}} \mathrm{d}x(\dot{u} - \dot{u}_\varepsilon)T_1^1\right| \le C\left[\mathcal{E}(u(0)) + \mathcal{E}_\varepsilon(u_\varepsilon(0))\right]^{\frac{1}{2}} \omega_\varepsilon.$$

To control T_1^2 we need L^∞ bounds for u_ε and $D_\varepsilon u_\varepsilon$. They follow from the conservation of the energy for the discrete system by means of the following:

Lemma 1. *Let f be a step function on $\mathcal{C}$ left continuous in the points $i\varepsilon$. Suppose that*

$$\|f\|_{H_\varepsilon^1}^2 = \int_{\mathcal{C}} \mathrm{d}x\left(|f|^2 + \left|D_\varepsilon^+ f\right|^2\right)$$

is bounded. Then

$$\|f\|_\infty \le C\|f\|_{H_\varepsilon^1}.$$

Proof. Let $x_0 = i_0\varepsilon$ be any point such that $|f(x_0)|^2 \le \frac{1}{|\mathcal{C}|}\int_{\mathcal{C}} \mathrm{d}x|f|^2$. Note that such a point does exist otherwise we would obtain a contradiction ($\int_{\mathcal{C}} \mathrm{d}x|f|^2 > \int_{\mathcal{C}} \mathrm{d}x|f|^2$). For $x = (i_0 + k)\varepsilon$ we have

$$f^2(x) = f^2(x_0) + \sum_{h=0}^{k-1}\left[f^2(x_0 + (h+1)\varepsilon) - f^2(x_0 + h\varepsilon)\right].$$

Since

$$\left|f^2(x+\varepsilon) - f^2(x)\right| = |[f(x+\varepsilon) + f(x)][f(x+\varepsilon) - f(x)]$$
$$= \varepsilon[f(x+\varepsilon) + f(x)]D_\varepsilon^+ f| \le \frac{1}{2}\varepsilon[f(x+\varepsilon) + f(x)]^2 + \frac{1}{2}\varepsilon\left|D_\varepsilon^+ f\right|^2,$$

we conclude that

$$\left|f^2(x)\right| \le \left(\frac{1}{|\mathcal{C}|} + 1\right)\int_{\mathcal{C}} \mathrm{d}x|f|^2 + \frac{1}{2}\int_{\mathcal{C}} \mathrm{d}x\left|D_\varepsilon^+ f\right|^2 \le \left(\frac{1}{|\mathcal{C}|} + 1\right)\|f\|_{H_\varepsilon^1}^2.$$

□

Lemma 1 and the energy bound (3.44) imply that the L^∞ norms of u_ε and $D_\varepsilon^+ u_\varepsilon$ are bounded uniformly in ε. Thus we can use the Lipschitz continuity of $\partial_{\xi_0} R$ to get:

$$\left|\int_{\mathcal{C}} \mathrm{d}x(\dot{u} - \dot{u}_\varepsilon)T_1^2\right| \le KW_\varepsilon,$$

with K the Lipschitz constant of $\partial_{\xi_0} R$ in the ball of radius $\max\{\|u\|_\infty, \|Du\|_\infty, \|u_\varepsilon\|_\infty, \|D_\varepsilon u_\varepsilon\|_\infty\}$.

The bound of T_2, involving discrete derivatives, requires the following chain rule formula for the discrete derivative of a composite function:

Lemma 2. *If f has continuous first derivative f', then for any function g and for any x there exist $\lambda_{\varepsilon,x} \in (0, 1)$ such that*

$$D_\varepsilon^{\pm} f(g(x)) = f'(\zeta_\varepsilon(x)) D_\varepsilon^{\pm} g(x), \quad \textit{with } \zeta_\varepsilon(x) = g(x) + \varepsilon\lambda_{\varepsilon,x} D_\varepsilon^{\pm} g(x).$$

Proof. By the mean value theorem, for D_ε^+ we have

$$\begin{aligned} D_\varepsilon^+ f(g(x)) &= \varepsilon^{-1}[f(g(x+\varepsilon) - f(g(x)] \\ &= \varepsilon^{-1} \int_{g(x)}^{g(x+\varepsilon)} \mathrm{d}z f'(z) = \varepsilon^{-1}[g(x+\varepsilon) - g(x)] f'(\zeta) \end{aligned}$$

for a suitable ζ in the interval with extremes $g(x)$ and $g(x+\varepsilon)$: $\zeta = g(x) + \lambda_{\varepsilon,x}[g(x+\varepsilon) - g(x)] = g(x) + \varepsilon\lambda_{\varepsilon,x} D_\varepsilon^+ g(x)$ for some $\lambda_{\varepsilon,x} \in (0, 1)$. In the same way the statement for D_ε^- follows. □

By the chain rule,

$$\begin{aligned} T_2^1 = {} & \partial^2_{\xi_0,\xi_1} R\left(\zeta_\varepsilon(x), D_\varepsilon^+ u\right) D_\varepsilon^- u - \partial^2_{\xi_0,\xi_1} R(u, Du) Du \\ & + \partial^2_{\xi_1^2} R(u, \eta_\varepsilon(x)) \Delta_\varepsilon u - \partial^2_{\xi_1} R(u, Du) \Delta u, \end{aligned}$$

where

$$\zeta_\varepsilon(x) = u(x) + \varepsilon\lambda_{\varepsilon,x} D_\varepsilon^- u(x)$$

and

$$\eta_\varepsilon(x) = D_\varepsilon u(x) + \varepsilon\mu_{\varepsilon,x} \Delta_\varepsilon u(x),$$

with $\lambda_{\varepsilon,x} \in (0, 1)$, $\mu_{\varepsilon,x} \in (0, 1)$. However,

$$\begin{aligned} & \partial^2_{\xi_0,\xi_1} R\left(\zeta_\varepsilon(x), D_\varepsilon^+ u\right) D_\varepsilon^- u - \partial^2_{\xi_0,\xi_1} R(u, Du) Du \\ & \quad = \partial^2_{\xi_0,\xi_1} R(u, Du) \left[D_\varepsilon^- u - Du\right] + D_\varepsilon^- u \left[\partial^2_{\xi_0,\xi_1} R(\zeta_\varepsilon(x), D_\varepsilon^+ u) - \partial^2_{\xi_0,\xi_1} R(u, Du)\right]. \end{aligned}$$

The smoothness of u and $D_\varepsilon u$ and the Lipschitz continuity of $\partial^2_{\xi_0,\xi_1} R$ yield

$$\left|\partial^2_{\xi_0,\xi_1} R(\zeta_\varepsilon(x), D_\varepsilon^+ u) - \partial^2_{\xi_0,\xi_1} R(u, Du)\right| \le C|D_\varepsilon^+ u - Du|) \le \omega_\varepsilon,$$

so this term also goes to 0 by the regularity of u.

Similarly,

$$\begin{aligned} & \partial^2_{\xi_1^2} R(u, \eta_\varepsilon(x)) \Delta_\varepsilon u - \partial^2_{\xi_1^2} R(u, Du) \Delta u \\ & \quad = \partial^2_{\xi_1^2} R(u, Du)[\Delta_\varepsilon u - \Delta u] + \left[\partial^2_{\xi_1^2} R(u, \eta_\varepsilon(x)) - \partial^2_{\xi_1^2} R(u, Du)\right] \Delta_\varepsilon u. \end{aligned}$$

The first part goes to 0 by the regularity of u. By the boundedness of u, Du, $D_\varepsilon^\pm u$ and $\Delta_\varepsilon u$, we can use the Lipschitz continuity of $\partial^2_{\xi_1^2} R$ to get the bound

$$\left|\partial^2_{\xi_1^2} R(u(x), \eta_\varepsilon(x),) - \partial^2_{\xi_1^2} R(u(x), Du(x))\right| \le K|\eta_\varepsilon(x) - Du(x)|.$$

Since $\eta_\varepsilon(x) - Du(x) = D_\varepsilon^+ u(x) - Du(x) + \varepsilon\mu_{\varepsilon,x}\Delta_\varepsilon u(x)$,

$$|\eta_\varepsilon(x) - Du(x)| \le \left|D_\varepsilon^+ u(x) - Du(x)\right| + \varepsilon\mu_{\varepsilon,x}|\Delta_\varepsilon u(x)|,$$

and hence

$$\begin{aligned}&\left|\partial^2_{\xi_1^2} R(u(x), \eta_\varepsilon(x)) - \partial^2_{\xi_1^2} R(u(x), Du(x))\right| |\Delta_\varepsilon u(x)|\\ &\le \left(\left|D_\varepsilon^+ u(x) - Du(x)\right| + \varepsilon\mu_{\varepsilon,x}|\Delta_\varepsilon u(x)|\right) |\Delta_\varepsilon u(x)|.\end{aligned}$$

However,

$$|\Delta_\varepsilon u(x)| \le |(\Delta_\varepsilon - \Delta)u(x)| + |\Delta u(x)|.$$

By the propagation of the initial regularity, $\|\Delta u(\cdot, t)\|_\infty$ is bounded for any $t \in (0, T)$. Therefore

$$\left|\partial^2_{\xi_1^2} R(u(x), \eta_\varepsilon(x))\Delta_\varepsilon u(x) - \partial^2_{\xi_1^2} R(u(x), Du(x))\right| |\Delta_\varepsilon u(x)| \le \omega_\varepsilon.$$

As for the term T_2^2, we use again the chain rule:

$$\begin{aligned}T_2^2 &= D_\varepsilon^-\left[\partial_{\xi_1} R(u_\varepsilon, D_\varepsilon^+ u_\varepsilon)\right] - D_\varepsilon^-\left[\partial_{\xi_1} R(u, D_\varepsilon^+ u)\right]\\ &= \partial^2_{\xi_0,\xi_1} R\left(\zeta_\varepsilon(x), D_\varepsilon^+ u_\varepsilon\right) D_\varepsilon^- u_\varepsilon - \partial^2_{\xi_0,\xi_1} R\left(\tilde\zeta_\varepsilon(x), D_\varepsilon^+ u\right) D_\varepsilon^- u +\\ &\quad + \partial^2_{\xi_1^2} R(u_\varepsilon, \eta_\varepsilon(x))\Delta_\varepsilon u_\varepsilon - \partial^2_{\xi_1^2} R(u, \tilde\eta_\varepsilon(x))\Delta_\varepsilon u,\end{aligned}$$

where

$$\begin{aligned}&\zeta_\varepsilon(x) = u_\varepsilon(x) + \varepsilon\lambda_{\varepsilon,x} D_\varepsilon^- u_\varepsilon(x), \quad \tilde\zeta_\varepsilon(x) = u(x) + \varepsilon\lambda_{\varepsilon,x} D_\varepsilon^- u(x)\\ &\eta_\varepsilon(x) = D_\varepsilon^+ u_\varepsilon(x) + \varepsilon\mu_{\varepsilon,x}\Delta_\varepsilon u_\varepsilon(x), \quad \tilde\eta_\varepsilon(x) = D_\varepsilon^+ u(x) + \varepsilon\mu_{\varepsilon,x}\Delta_\varepsilon u(x).\end{aligned}$$

We use the energy bound and Lemma 1 to get the boundedness of ζ_ε and $D_\varepsilon^\pm u_\varepsilon$ and thus the Lipschitz continuity of $\partial^2_{\xi_0,\xi_1} R$, so that

$$\begin{aligned}&\left|\partial^2_{\xi_0,\xi_1} R\left(\zeta_\varepsilon, D_\varepsilon^+ u_\varepsilon\right) D_\varepsilon^- u_\varepsilon - \partial^2_{\xi_0,\xi_1} R\left(\tilde\zeta_\varepsilon, D_\varepsilon^+ u\right) D_\varepsilon^- u\right|\\ &\le \left|\partial^2_{\xi_0,\xi_1} R\left(\tilde\zeta_\varepsilon, D_\varepsilon^+ u\right)\right| \left|D_\varepsilon^- u_\varepsilon - D_\varepsilon^- u\right| + K|D_\varepsilon^- u_\varepsilon||\zeta_\varepsilon(x) - \tilde\zeta_\varepsilon(x)|.\end{aligned}$$

However,

$$\left|\zeta_\varepsilon(x) - \tilde\zeta_\varepsilon(x)\right| = |u_\varepsilon(x) - u(x)| + \varepsilon\Big(\left|D_\varepsilon^- u_\varepsilon\right| + \left|D_\varepsilon^- u\right|\Big).$$

so that

$$\int_{\mathcal{C}} |\mathrm{d}x|\dot u - \dot u_\varepsilon| \left|D_\varepsilon^+\left[\partial_{\xi_1} R\left(u_\varepsilon, D_\varepsilon^+ u_\varepsilon\right)\right] - D_\varepsilon\left[\partial_{\xi_1} R\left(u, D_\varepsilon^+ u\right)\right]\right|$$

$$\le CW_\varepsilon + \frac{1}{2}\varepsilon^2(\mathcal{E}[u(0)] + \mathcal{E}_\varepsilon[u_\varepsilon(0)]).$$

The term $\partial^2_{\xi_1^2} R(u_\varepsilon, \eta_\varepsilon(x))\Delta_\varepsilon u_\varepsilon - \partial^2_{\xi_1^2} R(u, \tilde{\eta}_\varepsilon(x))\Delta_\varepsilon u$ is more delicate because, in order to use Lipschitz continuity, we need to bound $\eta_\varepsilon(x)$ and hence the supremum of $\Delta_\varepsilon u_\varepsilon$. However Lemma 1 and energy conservation are not enough when $n = 2$, but we need really to bound $\varepsilon\Delta_\varepsilon u_\varepsilon$ and we can take advantage of this extra ε. Indeed, by the energy conservation and the positivity assumptions on R and A,

$$\varepsilon A_{n,n}\left(D^n_\varepsilon u_\varepsilon(x)\right)^2 \le C,$$

and hence

$$\| D^n_\varepsilon u_\varepsilon| \le \frac{C}{\sqrt{\varepsilon}}, \tag{3.56}$$

implying that $|\eta_{\varepsilon,x}| \le C$ (if $n > 2$ we get a better estimate). Thus we have

$$\left|\partial_1^2 R(u_\varepsilon, \eta_\varepsilon)\Delta_\varepsilon u_\varepsilon - \partial_1^2 R(u, \tilde{\eta}_\varepsilon)\Delta_\varepsilon u\right| \le \left|\partial^2_{\xi_1^2} R\left(u, \tilde{\eta}_\varepsilon(x)\right)\right| |\Delta_\varepsilon u - \Delta_\varepsilon u_\varepsilon|$$
$$+ \left|\partial^2_{\xi_1^2} R(u, \tilde{\eta}_\varepsilon(x)) - \partial^2_{\xi_1^2} R(u, \eta_\varepsilon(x))\right| |\Delta_\varepsilon u_\varepsilon|$$
$$+ \left|\partial^2_{\xi_1^2} R(u_\varepsilon, \eta_\varepsilon(x))\right| - \partial^2_{\xi_1^2} R(u, \eta_\varepsilon(x))||\Delta_\varepsilon u_\varepsilon|.$$

However, by (3.56),

$$|\eta_\varepsilon(x) - \tilde{\eta}_\varepsilon(x)| = \left|D^-_\varepsilon(u_\varepsilon(x) - u(x))\right| + \varepsilon(|\Delta_\varepsilon u_\varepsilon| + |\Delta_\varepsilon u|) \le \omega_\varepsilon.$$

Therefore, by the Cauchy–Schwartz inequality and conservation of energy, we obtain that

$$\int_{\mathcal{C}} \mathrm{d}x|\dot{u} - \dot{u}_\varepsilon|\left|\partial^2_{\xi_1^2} R(u_\varepsilon, \eta_\varepsilon)\Delta_\varepsilon u_\varepsilon - \partial^2_{\xi_1^2} R(u, \tilde{\eta}_\varepsilon)\Delta_\varepsilon u\right| \le CW + \omega_\varepsilon.$$

Collecting all the terms, we conclude that there are constant $C > 0$ such that

$$\frac{d}{\mathrm{d}t} W_\varepsilon \le CW_\varepsilon + \omega_\varepsilon,$$

and hence, by the Gronwall lemma,

$$|W_\varepsilon(t)| \le |W_\varepsilon(0)|\mathrm{e}^{Ct} + t\mathrm{e}^{Ct}\omega_\varepsilon \quad \text{as } \varepsilon \to 0.$$

We summarize the results in the following:

Theorem 3.4. *Suppose that u_0 and v_0 satisfy (3.36). Let $u(t)$ be the solution to (3.37) and $u_\varepsilon(t)$ be the step function defined by (3.6) with $u_i(t)$, $i = 1, \ldots, N$, solutions to (3.45) with initial data $u_i(0) = u_0(i\varepsilon)$ and $\dot{u}_i(0) = v_0(i\varepsilon)$. Then for any $t > 0$,*

$$\|u(t) - u_\varepsilon(t)\| \to 0, \quad \textit{as } \varepsilon \to 0,$$

in the norm defined in (3.48).

Acknowledgments A.C. thanks CNIS-Roma, Italy; A.C., F.d'I. and M.P. thank M&MOCS, Cisterna di Latina, Italy for their support.

References

1. Alibert, J.J., Seppecher, P. dell'Isola, F.: Truss modular beams with deformation energy depending on higher displacement gradients. *Math. Mech. Solids* **8**, 51–73 (2003)
2. Auffray, N., dell'Isola, F., Eremeyev, V., Madeo, A., Rosi, G.: Analytical continuum mechanics a la Hamilton–Piola: least action principle for second gradient continua and capillary fluids. arXiv preprint arXiv:1305.6744 (2013) to appear in Mathematics and Mechanics of Solids (2014). doi:10.1177/1081286513497616
3. Carcaterra, A, Akay, A.: Dissipation in a finite-size bath. *Phys. Rev. E*, **84**, 011121.1–011121.4 (2011)
4. Carcaterra, A., Akay, A.: Theoretical foundation of apparent damping and irreversible energy exchange in linear conservative dynamical systems. *J. Acoust. Soc. Am.* **121**, 1971–1982 (2007)
5. Chesnais, C., Boutin, C., Stèphane, H.: Effects of the local resonance on the wave propagation in periodic frame structures: generalized Newtonian mechanics. *J. Acoust. Soc. Am.* **132**, 2873 (2012)
6. Craster, R.V., Guenneau, S. (eds.): Acoustic Metamaterials. *Springer Series in Material Science*, vol. 166, Springer, Berlin, 2013
7. dell'Isola, F., Seppecher, P.: The relationship between edge contact forces, double force and interstitial working allowed by the principle of virtual power. *Comptes rendus de l'Acadmie des Sciences Serie IIb* **321**, 303–308 (1995)
8. dell'Isola, F., Seppecher P., Madeo A.: How contact interactions may depend on the shape of Cauchy cuts in Nth gradient continua: approach à la D'Alembert. *Zeitschrift fü r angewandte Mathematik und Physik* **63.6**, 1119–1141 (2012)
9. dell'Isola F., Andreaus, U., Placidi L.: At the origins and in the vanguard of peridynamics, non-local and higher gradient continuum mechanics. An underestimated and still topical contribution of Gabrio Piola. arXiv preprint arXiv:1310.5599 (2013) to appear in Mathematics and Mechanics of Solids (2014) doi:10.1177/1081286513509811
10. Dunn, J.E., Serrin, J.: *On the thermomechanics of interstitial working in The Breadth and Depth of Continuum Mechanics*. Springer, Berlin Heidelberg, pp. 705–743 1986
11. Engheta, N., Ziolkowski, R.W.: *Metamaterials: Physics and Engineering, Explorations*. Wiley, New York, 2006
12. Germain, P.: The method of virtual power in continuum mechanics. Part 2: microstructure. *SIAM J. Appl. Math.* **25**, 556–575 (1973)
13. Green A.E., Rivlin R.S.: Multipolar continuum mechanics. *Arch. Ration. Mech. Anal.* **17**(2), 113–114 (1964)
14. Gurtin, M.E.: Thermodynamics and the possibility of spatial interaction in elastic materials. *Arch. Ration. Mech. Anal.* **19.5**, 339–352 (1965)
15. Hilbert, D.: Begründung der kinetischen Gastheorie, *Math. Ann.* **72** 331–407 (1916/17)
16. Kolpakovs, A.G.: Determination of the average characteristics of elastic frameworks. *J. Appl. Math. Mech.* **49**(6), 739–745 (1985)
17. Landau, L.D., Lifshitz, E.M.: *Quantum Mechanics: Non-Relativistic Theory*, 3 (3rd edn.). Pergamon Press, Oxford, 1977
18. Lee, S.H., Park, C.M., Seo, Y.M., Wang, Z.G., Kim, C.K.: Composite acoustic medium with simultaneously negative density and modulus. *Phys. Rev. Lett.* **104** (5), (2010). Bibcode:2010PhRvL.104e4301L. doi:10.1103/PhysRevLett.104.054301
19. Madeo, A., Placidi, L., Rosi, G.: Towards the design of meta-materials with enhanced damage sensitivity: second gradient porous materials. *Res. Nondestruct. Eval.* (2013). doi:10.1080/09349847.2013.853114
20. Milton, G.W., Cherkaev, A.V.: Which elasticity tensors are realizable? *J. Eng. Mater. Technol.* **117**(4), 483 (1995)
21. Milton, G.M., Briane, M., Willis, J.R.: On cloaking for elasticity and physical equations with a transformation invariant form. *New J. Phys.* **8**, 248 (2006)
22. Mindlin, R.D.: Second gradient of strain and surface tension in linear elasticity. *Int. J. Solids Struct.* **1**, 417–438 (1965)

23. NEFF, P., GHIBA, I.D., MADEO, A., PLACIDI, L., ROSI, G.: A unifying perspective: the relaxed linear micromorphic continuum. *Continuum Mech. Thermodyn.* 1–43 (2014). doi:10.1007/s00161-013-0322-9.
24. PICCARDO, G., RANZI, G., LUONGO, A.: A complete dynamic approach to the generalized beam theory cross-section analysis including extension and shear modes. *Math. Mech. Solids* **19**(8), 900–924 (2014)
25. PIOLA G.: *Memoria intorno alle equazioni fondamentali del movimento di corpi qualsivogliono considerati secondo la naturale loro forma e costituzione*, Modena, Tipi del R.D. Camera, 1846 translated by F.dell'Isola, U. Andreaus and L.Placidi in "The complete works of Gabrio Piola : Volume I" U. Andreaus, F.dell'Isola, R. Esposito, S. Forest, G.Maier, U. Perego, Editors Springer Verlag (see also www.fdellisola.it), vol. 38, 2014
26. SEPPECHER, P., JEAN-JACQUES ALIBERT, J.-J. DELL ISOLA, F.: Linear elastic trusses leading to continua with exotic mechanical interactions. *J. Phys.: Conf. Ser.* **319**(1). IOP Publishing (2011)
27. TOUPIN R.A.: Elastic materials with couple-stresses. *Arch. Ration. Mech. Anal.* **11**, 385–414 (1962)
28. XU, B., ARIAS, F., BRITTAIN, S.T., ZHAO, X.-M., GRZYBOWSKI, B., TORQUATO, S., WHITESIDES, G.M.: Making negative Poisson's ratio microstructures by soft lithography. *Adv. Mater.* **11**(14), 1186–1189 (1999)
29. ZOUHDI, S., ARI S., VINOGRADOV, A.P.: *Metamaterials and Plasmonics: Fundamentals, Modelling, Applications*. Springer-Verlag, New York, 2008

Dipartimento di Ingegneria Meccanica ed Aeronautica,
Università di Roma La Sapienza,
Via Eudossiana 18, 00184 Rome, Italy.

and

Dipartimento di Ingegneria Strutturale e Geotecnica,
Università di Roma La Sapienza,
Via Eudossiana 18, 00184 Rome, Italy.

and

International Research Center M&MOCS,
Università dell'Aquila, Palazzo Caetani,
04012 Cisterna di Latina (LT), Italy.

and

Diparimento di Matematica,
Università di Roma La Sapienza,
Piazzale Aldo Moro 5,
00185 Roma, Italy.
e-mail: pulviren@mat.uniroma1.it

(*Received February 25, 2014 / Accepted May 4, 2015*)
Published online May 21, 2015 –