



Study of Island Formation in Epitaxially Strained Films on Unbounded Domains

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Abstract

We consider a variational model related to the formation of islands in heteroepitaxial growth on unbounded domains. We first derive the scaling regimes of the minimal energy in terms of the volume of the film and the amplitude of the crystallographic misfit. For small volumes, non-existence of minimizers is then proven. This corresponds to the experimentally observed wetting effect. On the other hand, we show the existence of minimizers for large volumes. We finally study the asymptotic behavior of the optimal shapes.

1. Introduction

We consider the epitaxial deposition of a thin crystalline film on a relatively thick rigid substrate with a misfit between the lattice parameters of the film and those of the substrate. Experimental and numerical observations suggest that the shape of the film changes with increasing volume (see [3, 15, 19, 30, 32]). At small volumes, one typically observes a very thin flat layer (“wetting”), while at larger volumes, compact islands form. This transition is often explained as the result of a competition between two opposing types of energies, namely, the stored strain energy due to the crystallographic misfit, and the surface energy of the film’s free surface. Heuristically, at small volumes, the surface energy dominates, and complex structures are avoided, while at larger volumes, the film forms patterns to release elastic energy at the price of an additional surface energy.

We study, analytically, a two-dimensional variational model introduced in [31] (see also [3, 10, 14]), to describe the surface morphologies of the epitaxially strained film.

The main difference to the previous analytical works (see [10, 12, 14, 18]) is that the model explicitly allows for wetting, which corresponds to film profiles with unbounded support. We assume that the film occupies a domain Ω_h which

can be described as a subgraph of a height profile function $h: \mathbb{R} \rightarrow [0, \infty)$, i.e., $\Omega_h := \{\mathbf{x} := (x, y) \in \mathbb{R}^2: 0 < y < h(x)\}$. The energy functional is then given by

$$F(u, h) := \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} + \int_{\mathbb{R}} \left(\sqrt{1 + h'^2} - 1 \right) \, dx, \quad (1.1)$$

where $u: \Omega_h \rightarrow \mathbb{R}$. For fixed volume $d > 0$ of the film, we look for profile functions h and associated displacement functions $u: \Omega_h \rightarrow \mathbb{R}$ that minimize the total energy (1.1) subject to the constraints $\int_{\mathbb{R}} h \, dx = d$ and $u(x, 0) = e_0 x$ for all $x \in \mathbb{R}$. The latter condition describes the crystallographic misfit between the substrate and the film, where $e_0 > 0$ measures its amplitude.

The first term in (1.1) models the strain elastic energy in the film. Recall that we assume that there is a mismatch between the two crystal lattices, i.e., there is no stress-free configuration possible, and consequently, a strain is induced in the film during deposition. The second term in (1.1) models the extra surface energy due to the rearrangement of the atoms in the film. All typical surface energy constants per unit length are normalized to one. Let us notice that the functional F bears many similarities with models for capillary surfaces [20, 27].

We point out that, as noted before, in contrast to many previous works (see [10, 14, 18]), we do not assume a periodic pattern of islands and do not restrict matters to a single island on a compact domain. The main difference is that in (1.1) the support of the height profile function h may be unbounded, which can lead to a loss of compactness for low energy sequences. A short comparison to the compact setting is given in Proposition 4.6. Many of our results, however, build on techniques developed in the works on compactly supported islands.

Let us make some comments on several simplifications built into the model. First, the displacement function $u \in H_{\text{loc}}^1(\Omega_h)$ and the elastic energy term $|\nabla u|^2$ are scalar valued simplifications of a typical geometrically linear elastic energy density $W(U) = \mu |\frac{1}{2}(\nabla U + \nabla^T U)|^2$ for a displacement $U: \Omega_h \rightarrow \mathbb{R}^2$, μ being a typical elastic modulus. Based on the analysis in [18], we expect that the simplified energy contains, at least qualitatively, all relevant information. We note that the proofs of the scaling laws can be carried over to the elasticity setting, and are generalized to the three-dimensional setting in Section 5. Second, we assume that the domain occupied by the film can be described as a subgraph of the profile function h , which has the effect of preventing the formation of droplets or nanorings (see, for example [34]). Third, we do not take into account any plastic effects, such as misfit dislocations (see, for example [28]). Finally, we consider only the stationary setting, and refer to [8, 11, 29] for some recent results on the time evolution problem for the compact setting.

We consider two different types of approximations of the surface energy, namely for small and large slopes $|h'|$. Many physical models are based on the assumption that for small volumes of the deposited film one expects small slopes of the film's profile function (see [32, 33] or [20] where a similar simplification is used in the study of sliding liquid drops). This corresponds to the approximation (we ignore the factor $1/2$)

$$\int_{\mathbb{R}} \left(\sqrt{1 + h'^2} - 1 \right) \, dx \sim \int_{\mathbb{R}} h'^2 \, dx =: S_s(h). \quad (1.2)$$

If one expects, however, the formation of an island, the small slope approximation might not be appropriate anymore, and we compare it to the large slope approximation

$$\int_{\mathbb{R}} \left(\sqrt{1+h'^2} - 1 \right) dx \sim \int_{\mathbb{R}} |h'| dx =: S_{\ell}(h). \quad (1.3)$$

If we insert either of the approximations (1.2) or (1.3) into (1.1), then, due to the specific structure of the elastic energy term, we can rescale the problem to set $e_0 = 1$, i.e., we consider (see Section 3 for a detailed derivation)

$$F_{s/\ell}(V) := \inf \left\{ \int_{\Omega_h} |\nabla u|^2 dx + S_{s/\ell}(h) : h \in H^1(\mathbb{R}), h \geq 0, \int_{\mathbb{R}} h dx = V, \right. \\ \left. u(x, 0) = x \right\}. \quad (1.4)$$

It turns out that in both cases there are two scaling regimes of the energy, namely (see Propositions 3.9 and 4.3)

$$F_s(V) \sim \min\{V, V^{4/5}\} \quad \text{and} \quad F_{\ell}(V) \sim \min\{V, V^{2/3}\}.$$

Heuristically, these scaling laws reflect the transition from a wetting regime in which the surface energy dominates and, consequently, the film forms a thin flat layer, to a regime in which a compactly supported island forms, in which case the optimal energy comes from the competition between elastic and surface energy. We note that in this model, the surface energy prefers a flat layer, while the elastic energy favors oscillations. This is in contrast to many other physical situations where the surface energy typically favors compact shapes, and consequently minimizers exist for small volumes, but not for large volumes (see, for example [17,21,22,26] for some recent works). The situation is the opposite here: we prove that for large volumes, there always exists a minimizer, while for small volumes we prove the non-existence of minimizers in the case of the small slope approximation (see Proposition 3.13). This is due to a loss of compactness of low-energy sequences, which corresponds to the wetting effect. Note that this phenomenon does not occur in case of a non-vanishing miscut angle (see [12]).

To study more quantitatively the optimal shape of an island once it is formed, the limit $V \rightarrow \infty$ is considered. If properly rescaled, the asymptotic shape turns out to be a parabola (in the case of the small slope approximation) and a rectangle (in the case of the large slope approximation). We should stress the fact that even though it sheds some light on what can be (mathematically) predicted by our models, this asymptotic analysis does not tell much about the physics. In fact, it corresponds to very large mismatch e_0 and/or large volume d , for which the model is not expected to be relevant anymore. Moreover, since in the original variables, minimizers tend to have large slopes, the small slope approximation is also questionable in this regime. We believe that adapting our analysis to the original functional (1.1) would lead to results similar to the one obtained in the large slope approximation.

The remaining part of the text is organized as follows. After setting the notation in Section 2, we first consider the small slope approximation. Some qualitative

properties are derived in Section 3.1, and the scaling law for the minimal energy is proven in Proposition 3.9. The scaling law is then refined to show more quantitative results. More precisely, we show that there is a range of volumes $0 < V < \bar{V}$, for which the minimal energy is exactly equal to V , and consequently, there does not exist a minimizer for $0 < V < \bar{V}$ (see Proposition 3.13). On the other hand, for volumes such that $F_s(V) < V$ (i.e., if $V > \bar{V}$), there always exists a smooth minimizer, which has compact and connected support and meets the substrate at zero angle (see Proposition 3.16 and Theorem 3.18). Regularity properties and estimates on the support and its maximal height are provided (see Sections 3.4 and 3.5). The regularity of a minimizer (see Section 3.4) is shown using arguments from [10] with minor changes required due to a different form of the surface energy. Though the proof follows a standard approach, for the sake of completeness we include it in the paper. Finally, the asymptotic behavior for large $V \rightarrow \infty$ is studied. It is shown, that, when properly rescaled, minimizers converge to a parabola, and away from a boundary layer, this convergence occurs at an exponential rate in the L^2 -topology.

Subsequently, in Section 4, the large slope approximation (1.3) is considered. Note that this approximation comes along with a loss of regularity of admissible profile functions h , and we consider the relaxation following [10]. We prove the scaling law of the minimal energy F_ℓ (see Proposition 4.3), and show that in the regime $F_\ell(V) < V$ there always exists a minimizer with connected support (see Proposition 4.10). If properly rescaled, a sequence of minimizing profiles converges, away from a boundary layer, to a rectangular shape for large volumes at an exponential rate in the L^1 -topology (see Proposition 3.39). Recall that for the small slope approximation, non-existence of minimizers at small volumes is due to the fact that $F_s(V) = V$. For the large slope approximation, we only get the weaker result $F_\ell(V)/V \rightarrow 1$ as $V \rightarrow 0$ (see Proposition 4.7).

Finally, in Section 5, the three-dimensional setting is considered, and the scaling laws for both types of approximations are discussed.

2. Notation and Preliminary Results

In this section, we set the notation and collect some results that will be used later. Throughout the text we denote by C and c constants that may vary from expression to expression. The symbols \sim , \gtrsim , \lesssim indicate estimates that hold up to a constant. For instance, $f \lesssim g$ denotes the existence of a constant $C > 0$ such that $f \leq Cg$. For $\Omega \subset \mathbb{R}^2$, we denote by $\mathcal{H}^1(\Omega)$ its one-dimensional Hausdorff measure, and by $|\Omega|$ its two-dimensional Lebesgue measure. When it exists, we will denote by ν its inward normal. Given two sets $A, B \subset \mathbb{R}^2$, we define their Hausdorff distance as $d_{\mathcal{H}}(A, B) := \inf\{r > 0 : A \subset N(B, r) \text{ and } B \subset N(A, r)\}$, where $N(A, r) := \{x \in \mathbb{R}^2 : d(x, A) < r\}$, and $d(x, A)$ denotes the distance from x to A . Given a vector $\mathbf{x} := (x, y) \in \mathbb{R}^2$, we denote by $|\mathbf{x}| := (x^2 + y^2)^{1/2}$ its Euclidean norm.

We will use the following rescaling property for functions on rectangles.

Lemma 2.1. *If $u \in H^1([0, \ell] \times [0, L]; \mathbb{R})$, with $u(x, 0) = x$, then letting $v(x, y) := \frac{1}{\ell}u(\ell x, \ell y)$, there holds*

$$\int_{[0,\ell] \times [0,L]} |\nabla u|^2 \, d\mathbf{x} = \ell^2 \int_{[0,1] \times [0,L/\ell]} |\nabla v|^2 \, d\mathbf{x}. \quad (2.1)$$

As a consequence, for every $\lambda > 0$, there exists $C(\lambda) > 0$ such that

$$\int_{[0,\ell] \times [0,\lambda\ell]} |\nabla u|^2 \, d\mathbf{x} \geq C(\lambda)\ell^2. \quad (2.2)$$

The following lemma describes the behavior of the elastic energy for small thickness of the film. It can be seen as a simple special case of a dimension reduction argument (see [2,24]). A proof for the more complicated case of vector-valued functions can be found in [18].

Lemma 2.2. *There holds*

$$\lim_{\varepsilon \rightarrow 0^+} \min_{u(x,0)=x} \frac{1}{\varepsilon} \int_{[0,1] \times [0,\varepsilon]} |\nabla u|^2 \, d\mathbf{x} = 1. \quad (2.3)$$

Remark 2.3. The analogous statements hold for typical elastic energy functionals for deformations $U: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, i.e.,

$$\lim_{\varepsilon \rightarrow 0^+} \min_{U(x,0)=(x,0)} \frac{1}{\varepsilon} \int_{[0,1] \times [0,\varepsilon]} W(\nabla U) \, d\mathbf{x} = 1 \quad (2.4)$$

if $W(\nabla U) = |\nabla U|^2$ or $W(\nabla U) = \frac{1}{2}(\nabla U + \nabla^T U)^2$ (see [18] and the references given there). This allows us to carry over the qualitative results to the linear elasticity setting.

3. Small Slope Approximation

In this section we consider the small slope approximation $\sqrt{1+h'^2} - 1 \sim h'^2$ and study

$$F_s(e_0, d) := \inf \left\{ \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} + \int_{\mathbb{R}} h'^2 \, dx : h \in H^1(\mathbb{R}), h \geq 0, \right. \\ \left. u \in H_{loc}^1(\Omega_h), \int_{\mathbb{R}} h \, dx = d, u(x, 0) = e_0 x \text{ if } x \in \text{supp } h \right\}. \quad (3.1)$$

Note that by Gagliardo–Nirenberg inequality we have $h \in H^1(\mathbb{R})$ if $h \in L^1(\mathbb{R})$ and $h' \in L^2(\mathbb{R})$.

One main difference between the model considered here and the related models on compact domains (see [10,14,18]) is that it behaves well under rescaling as shown by the following lemma (compare also [12, Section 5]):

Lemma 3.1. *For $h \in H^1(\mathbb{R})$, $h \geq 0$, $u \in H_{loc}^1(\Omega_h)$ with $\nabla u \in L^2(\Omega_h)$, and $\lambda > 0$, letting $u_\lambda(x, y) = \frac{1}{\lambda}u(\lambda x, \lambda y)$ and $h_\lambda(x) = \frac{1}{\lambda}h(\lambda x)$, we have*

$$\int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} + \int_{\mathbb{R}} h'^2 \, dx = \lambda^2 \int_{\Omega_{h_\lambda}} |\nabla u_\lambda|^2 \, d\mathbf{x} + \lambda \int_{\mathbb{R}} h_\lambda'^2 \, dx$$

and $\int_{\mathbb{R}} h \, dx = \lambda^2 \int_{\mathbb{R}} h_\lambda \, dx$.

By rescaling, we can eliminate one of the two parameters d or e_0 . We will renormalize such that $e_0 = 1$:

Proposition 3.2. *Let $e_0 > 0$ and $d > 0$. Set $V := e_0^4 d$ and*

$$F_s(V) := \inf \left\{ \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} + \int_{\mathbb{R}} h^2 \, dx : (u, h) \in \mathcal{A}_V \right\}, \tag{3.2}$$

where the set of admissible pairs is given by

$$\mathcal{A}_V := \left\{ (u, h) : h \in H^1(\mathbb{R}), h \geq 0, u \in H_{loc}^1(\Omega_h), \int_{\mathbb{R}} h \, dx = V, \right. \\ \left. u(x, 0) = x \text{ if } x \in \text{supp } h \right\}. \tag{3.3}$$

Then

$$F_s(e_0, d) = \frac{1}{e_0^2} F_s(V).$$

Proof. Let (u, h) be admissible for $F_s(e_0, d)$, and let $\lambda := \frac{1}{e_0^2}$. Using the notation of Lemma 3.1, set

$$\tilde{h}(x) := h_\lambda(x), \quad \text{and} \quad \tilde{u}(x, y) := \frac{1}{e_0} u_\lambda(x, y). \tag{3.4}$$

Then $(\tilde{u}, \tilde{h}) \in \mathcal{A}_V$ and, by Lemma 3.1,

$$\int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} + \int_{\mathbb{R}} h^2 \, dx = \frac{1}{e_0^2} \left(\int_{\Omega_{\tilde{h}}} |\nabla \tilde{u}|^2 \, d\mathbf{x} + \int_{\mathbb{R}} \tilde{h}^2 \, dx \right).$$

Since (3.4) induces a bijective correspondence between the admissible pairs for $F_s(e_0, d)$ and \mathcal{A}_V , this proves the assertion. \square

In this section we study the problem (3.2). We first prove the scaling law of the optimal energy. It is shown that there exists a critical volume $\bar{V} > 0$ such that for volumes $0 < V < \bar{V}$, we have $F_s(V) = V$, which leads to the non-existence of minimizers. We also prove that for $V > \bar{V}$, we have $F_s(V) < V$ and that there exists a compact connected smooth minimizer of (3.2), which has zero contact angle with the substrate. Moreover, we provide estimates on the size of the support of this island together with estimates on its maximal height. We finally investigate the large volume limit and prove that, when suitably rescaled, the minimizers converge to a parabola and that away from a boundary layer this convergence is exponentially fast in the (strong) L^2 -topology. We point out that we investigate the asymptotic behavior mostly for mathematical reasons to better understand the shapes of minimizers. The physical model, and in particular the small slope approximation implemented here, is expected to give more reliable results in the case of small volumes.

3.1. First Properties of the Minimization Problem

For every $V > 0$ and every $(u, h) \in \mathcal{A}_V$, we set

$$E_V(u, h) := \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} \quad \text{and} \quad S_V(h) := \int_{\mathbb{R}} h^2 \, dx.$$

When it is clear from the context, we will often drop the explicit dependence on (u, h) . As a simple consequence of Lemma 3.1, we have the following important property of F_s .

Proposition 3.3. $F_s: (0, \infty) \rightarrow \mathbb{R}$ is concave and thus locally Lipschitz continuous.

Proof. Let $V > 0$ be fixed. Then, for every $V_0 > 0$ and for every admissible competitor $(u, h) \in \mathcal{A}_{V_0}$, by Lemma 3.1, $F_s(V) \leq \frac{V}{V_0} E_{V_0}(u, h) + \left(\frac{V}{V_0}\right)^{1/2} S_{V_0}(h)$. Hence

$$F_s(V) = \inf_{V_0 > 0} \inf_{(u, h) \in \mathcal{A}_{V_0}} \left(\frac{V}{V_0} E_{V_0}(u, h) + \left(\frac{V}{V_0}\right)^{1/2} S_{V_0}(h) \right), \quad (3.5)$$

and since for every $V_0 > 0$ and $(u, h) \in \mathcal{A}_{V_0}$, the function $V \mapsto \frac{V}{V_0} E_{V_0}(u, h) + \left(\frac{V}{V_0}\right)^{1/2} S_{V_0}(h)$ is concave, F_s is the infimum of concave functions and therefore also concave. \square

The scaling behavior from Lemma 3.1 is typical for various discrete and continuous models for epitaxial growth, and thus, similar properties hold for a large class of models (see also [12, 32]). Using the same rescaling we obtain the following result:

Proposition 3.4. For every $V > 0$, if

$$F_s(V) = \min \left\{ \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} + \int_{\mathbb{R}} h^2 \, dx : (u, h) \in \mathcal{A}_V \right\} = E_V(u^*, h^*) + S_V(h^*),$$

i.e., if the minimum is attained, then $\{h^* > 0\}$ is connected.

Proof. Fix $V > 0$, and suppose that there exists a minimizer. It suffices to show that for every $(u, h) \in \mathcal{A}_V$, for which $\{h > 0\}$ is not connected, there exists $(\bar{u}, \bar{h}) \in \mathcal{A}_V$ with lower total energy. Assume that $h = h^{(1)} + h^{(2)}$ with $h^{(1)} \geq 0$, $h^{(2)} \geq 0$, $\{h^{(1)} > 0\} \cap \{h^{(2)} > 0\} = \emptyset$, $0 < \int_{\mathbb{R}} h^{(1)} \, dx = V_1 < V$, $\int_{\mathbb{R}} h^{(2)} \, dx = V - V_1$, and $h^{(1)}, h^{(2)} \in H^1(\mathbb{R})$. Up to translation, we can further assume that $h^{(1)}$ and $h^{(2)}$ were chosen such that

$$\{h^{(1)} > 0\} \subset \mathbb{R}^- \quad \text{and} \quad \{h^{(2)} > 0\} \subset \mathbb{R}^+. \quad (3.6)$$

We set $u^{(1)} := u|_{\Omega_{h^{(1)}}}$ and $u^{(2)} := u|_{\Omega_{h^{(2)}}}$. Then $(u^{(i)}, h^{(i)}) \in \mathcal{A}_{V_i}$ for $i = 1, 2$. We note that

$$E_V(u, h) + S_V(h) = E_{V_1}(u^{(1)}, h^{(1)}) + S_{V_1}(h^{(1)}) + E_{V_2}(u^{(2)}, h^{(2)}) + S_{V_2}(h^{(2)}).$$

We now build a competitor as follows. Consider the two components separately and rescale them. Precisely, for $0 \leq \mu \leq V$, set $\lambda_1 := \sqrt{\frac{V_1}{\mu}}$ and $\lambda_2 := \sqrt{\frac{V_2}{V-\mu}}$, and, using the notation of Lemma 3.1 consider

$$h_\mu := h_{\lambda_1}^{(1)} + h_{\lambda_2}^{(2)}(\cdot - \tau_\mu),$$

where τ_μ is such that $\{h_{\lambda_1}^{(1)} > 0\} \cap \{h_{\lambda_2}^{(2)}(\cdot - \tau_\mu) > 0\} = \emptyset$ [which exists thanks to (3.6)], and the associated u_μ . Note that $h = h_{V_1}$, and, by Lemma 3.1, $\int_{\mathbb{R}} h_\mu \, dx = \frac{\mu}{V_1} V_1 + \frac{V-\mu}{V_2} V_2 = V$ for every $\mu \in [0, V]$. Hence $(u_\mu, h_\mu) \in \mathcal{A}_V$ for every $\mu \in [0, V]$. Let

$$f(\mu) := E_V(u_\mu, h_\mu) + S_V(h_\mu),$$

so that by Lemma 3.1,

$$\begin{aligned} f(\mu) &= \frac{\mu}{V_1} E_{V_1}(u^{(1)}, h^{(1)}) + \sqrt{\frac{\mu}{V_1}} S_{V_1}(h^{(1)}) + \frac{V-\mu}{V_2} E_{V_2}(u^{(2)}, h^{(2)}) \\ &\quad + \sqrt{\frac{V-\mu}{V_2}} S_{V_2}(h^{(2)}). \end{aligned}$$

Since $S_{V_1}(h^{(1)}) > 0$ and $S_{V_2}(h^{(2)}) > 0$, the function f is strictly concave. Therefore, it attains its minimum at the boundary, that is, at $\mu = 0$ or at $\mu = V$. This shows that there is a configuration with strictly lower energy than (u, h) and proves that the minimizer must be connected. \square

Using a different rescaling we obtain the following:

Lemma 3.5. *If $V > 0$, then*

$$\begin{aligned} F_S(V) &= \inf \left\{ \int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}} h'^2 \, dx : (u, h) \in \mathcal{A}_V; \right. \\ &\quad \left. \partial_y u \equiv 0 \text{ or } \int_{\Omega_h} (\partial_y u)^2 \, dx = \frac{3}{4} \int_{\mathbb{R}} h'^2 \, dx \right\}. \end{aligned}$$

Proof. For $(u, h) \in \mathcal{A}_V$ consider the equivalence class $\{(u_\lambda, h_\lambda) : \lambda > 0\} \subset \mathcal{A}_V$ given by the anisotropic rescaling $u_\lambda(x, y) = \frac{1}{\lambda} u(\lambda x, \frac{1}{\lambda} y)$ and $h_\lambda(x) = \lambda h(\lambda x)$. Then

$$\int_{\Omega_{h_\lambda}} |\nabla u_\lambda|^2 \, dx + \int_{\mathbb{R}} h_\lambda'^2 \, dx = \int_{\Omega_h} \left((\partial_x u)^2 + \frac{1}{\lambda^4} (\partial_y u)^2 \right) \, dx + \lambda^3 \int_{\mathbb{R}} h'^2 \, dx.$$

Suppose that $\partial_y u \not\equiv 0$. Since for every $(u, h) \in \mathcal{A}_V$ we have $\int_{\mathbb{R}} h'^2 \, dx > 0$, within one equivalence class, the energy is minimized for $\lambda = \frac{4}{3} \frac{\int_{\Omega_h} (\partial_y u)^2 \, dx}{\int_{\mathbb{R}} h'^2 \, dx} > 0$. Therefore, from each such equivalence class, only the element with $\int_{\Omega_h} (\partial_y u)^2 \, dx = \frac{3}{4} \int_{\mathbb{R}} h'^2 \, dx$ is relevant for the infimum. \square

Following [3, 10, 18], we prove the lower semicontinuity of the energy and density of Lipschitz configurations.

Proposition 3.6. *For every sequence $(u_n, h_n) \in \mathcal{A}_V$ with $\sup_n E_V(u_n, h_n) + S_V(h_n) \leq C$, there exists (u, h) with $h \in H^1(\mathbb{R})$, $h \geq 0$ and $u \in H^1_{loc}(\Omega_h)$ such that up to a subsequence, h_n converges to h in $L^\infty_{loc}(\mathbb{R})$, u_n converges weakly in $H^1_{loc}(\Omega_h)$ to u , $u(x, 0) = x$, and*

$$\liminf_{n \rightarrow +\infty} (E_V(u_n, h_n) + S_V(h_n)) \geq \int_{\Omega_h} |\nabla u|^2 dx + \int_{\mathbb{R}} h^2 dx.$$

Moreover, if $\{h_n\}$ is tight, then $h_n \rightarrow h$ in $L^1(\mathbb{R})$, and $\int_{\mathbb{R}} h dx = V$.

Conversely, for every $(u, h) \in \mathcal{A}_V$ with $E_V(u, h) + S_V(h) < \infty$, there exists a sequence $(u_n, h_n) \in \mathcal{A}_V$ such that h_n is Lipschitz continuous with bounded support and such that h_n converges to h in $L^\infty(\mathbb{R})$, u_n converges weakly in $H^1_{loc}(\Omega_h)$ to u , and

$$\liminf_{n \rightarrow +\infty} (E_V(u_n, h_n) + S_V(h_n)) \leq E_V(u, h) + S_V(h).$$

Proof. Let $(u_n, h_n) \in \mathcal{A}_V$ be a sequence such that $\sup_n E_V(u_n, h_n) + S_V(h_n) \leq C$. From the compact embedding of $H^1_{loc}(\mathbb{R})$ in $L^\infty_{loc}(\mathbb{R})$, we get that (up to a subsequence), h_n converges in $L^\infty_{loc}(\mathbb{R})$ to some (continuous) function h . This implies the local Hausdorff convergence of $\mathbb{R}^2 \setminus \Omega_{h_n}$ to $\mathbb{R}^2 \setminus \Omega_h$, from which as in [3, 10] we infer the existence of a function $u \in H^1(\Omega_h)$ with $u(x, 0) = x$ such that u_n converges weakly in $H^1_{loc}(\Omega_h)$, and as in [3, 10] we obtain the lower semicontinuity

$$\liminf_{n \rightarrow +\infty} (E_V(u_n, h_n) + S_V(h_n)) + \int_{\Omega_h} |\nabla u|^2 dx + \int_{\mathbb{R}} h^2 dx.$$

If $\{h_n\}$ is tight, then $h_n \rightarrow h$ in $L^1(\mathbb{R})$, which implies that $\int_{\mathbb{R}} h dx = V$.

Let now $(u, h) \in \mathcal{A}_V$ be such that $E_V(u, h) + S_V(h) < \infty$. It is readily seen that we can approximate h from below by compactly supported height profiles so that we will assume from now on that h itself is compactly supported. It suffices to approximate h by a sequence of Lipschitz functions h_n with $0 \leq h_n \leq h$ and $S_V(h_n) \leq S_V(h)$. We refer the reader to [3, 10, 18] for the treatment of the volume constraint. Following [3], for $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we define

$$h_n(x) := \inf_{x' \in \mathbb{R}} (h(x') + n|x - x'|)$$

to be the Yosida transform of h . The latter is an n -Lipschitz function which satisfies $0 \leq h_n \leq h$. As proven in [3], Ω_{h_n} converges to Ω_h in the Hausdorff topology. Since h_n and h are continuous functions, the set $\{h_n < h\}$ is open and thus made of a countable union of disjoint intervals (a_k, b_k) , $k \in \mathbb{N}$. On each of these intervals (see [3]),

$$h_n(x) = \min\{h(a_k) + n|x - a_k|, h(b_k) + n|x - b_k|\}$$

so that $\int_{a_k}^{b_k} h_n^2 dx \leq \int_{a_k}^{b_k} h^2 dx$. From $h' = h'_n$ almost everywhere on $\{h_n = h\}$, we obtain that $S_V(h_n) \leq S_V(h)$ and that h_n converges in L^∞ to h . \square

We now prove an interpolation inequality which will be useful later.

Proposition 3.7. *For every $h \in H^1(\mathbb{R})$, we have*

$$\|h\|_{L^\infty(\mathbb{R})} \leq \left(\frac{9}{16}\right)^{1/3} \left(\int_{\mathbb{R}} |h(x)| \, dx\right)^{1/3} \left(\int_{\mathbb{R}} h'(x)^2 \, dx\right)^{1/3}. \quad (3.7)$$

Proof. Without loss of generality, we may assume that $h \geq 0$ (otherwise consider $|h|$). By rescaling the dependent and the independent variables, we may assume that $\|h\|_{L^\infty} = 1$ and $\int_{\mathbb{R}} h \, dx = 1$. Indeed, suppose that the inequality holds for some $h \in H^1$. For $M > 0$ and $\lambda > 0$ consider the rescaled function $\tilde{h}(x) = Mh(\lambda x)$. Then, by changing variables, $\tilde{x} = \frac{x}{\lambda}$, we get that (3.7) holds for \tilde{h} since

$$\begin{aligned} \|\tilde{h}\|_{L^\infty} &= M\|h\|_{L^\infty} \leq M \left(\frac{9}{16}\right)^{1/3} \left(\int_{\mathbb{R}} |h(x)| \, dx\right)^{1/3} \left(\int_{\mathbb{R}} h'(x)^2 \, dx\right)^{1/3} \\ &= M \left(\frac{9}{16}\right)^{1/3} \left(\frac{1}{M} \int_{\mathbb{R}} |(Mh)(x)| \, dx\right)^{1/3} \left(\frac{1}{M^2} \int_{\mathbb{R}} (Mh)'(x)^2 \, dx\right)^{1/3} \\ &= \left(\frac{9}{16}\right)^{1/3} \left(\lambda \int_{\mathbb{R}} |(Mh)(\lambda \tilde{x})| \, d\tilde{x}\right)^{1/3} \left(\lambda \int_{\mathbb{R}} \frac{1}{\lambda^2} (Mh)'(\lambda \tilde{x})^2 \, d\tilde{x}\right)^{1/3} \\ &= \left(\frac{9}{16}\right)^{1/3} \left(\int_{\mathbb{R}} |\tilde{h}(x)| \, dx\right)^{1/3} \left(\int_{\mathbb{R}} \tilde{h}'(x)^2 \, dx\right)^{1/3}. \end{aligned}$$

By translation and symmetric decreasing rearrangement (see [25]), we can further restrict ourselves to functions h with $h(0) = \sup h = 1$, which are even and non-increasing on $[0, +\infty)$. Let

$$g(x) := \begin{cases} \left(1 - \frac{|x|}{x_0}\right)^2, & \text{if } x \in (-x_0, x_0), \\ 0, & \text{else,} \end{cases}$$

where $x_0 > 0$ is chosen so that the volume constraint is satisfied, i.e., $x_0 = 3/2$. Let us prove that g is the minimizer of $\int_{\mathbb{R}} h'^2 \, dx$ in the class

$$\mathcal{M} := \left\{ h \in H^1(\mathbb{R}) : h \geq 0, \|h\|_{L^\infty} = h(0) = 1, \int_{\mathbb{R}} h \, dx = 1, \right. \\ \left. h(x) = h(-x), h \text{ non-increasing on } [0, \infty) \right\}.$$

Let $h \in \mathcal{M}$. Then

$$\begin{aligned} \int_0^{+\infty} h'^2 \, dx &= \int_0^{+\infty} g'^2 \, dx + 2 \int_0^{+\infty} g'(h-g)' \, dx + \int_0^{+\infty} |(h-g)'|^2 \, dx \\ &= \int_0^{+\infty} g'^2 \, dx - 2 \int_0^{x_0} g''(h-g) \, dx + \int_0^{+\infty} |(h-g)'|^2 \, dx \end{aligned}$$

where we used integration by parts and the fact that $g'(x_0) = 0, h(0) = g(0) = 1$. Since on $[0, x_0]$, $g'' = \frac{2}{x_0^2}$ and since $\int_0^{+\infty} (h-g) \, dx = 0$, we further obtain that

$$\int_0^{x_0} g''(h-g) \, dx = \frac{2}{x_0^2} \int_0^{x_0} (h-g) \, dx = -\frac{2}{x_0^2} \int_{x_0}^{+\infty} h \, dx$$

so that

$$\begin{aligned} \int_0^{+\infty} h^2 dx &= \int_0^{+\infty} g^2 dx + \frac{4}{x_0^2} \int_{x_0}^{+\infty} h dx \\ &+ \int_0^{+\infty} |(h-g)'|^2 dx \geq \int_0^{+\infty} g^2 dx. \end{aligned}$$

By symmetry, this shows that g minimizes $\int_{\mathbb{R}} h^2 dx$ in \mathcal{M} . Using that $\int_0^{x_0} g^2 dx = 8/9$, we obtain (3.7). \square

Remark 3.8. In terms of the energy, (3.7) can be rephrased as

$$\|h\|_{L^\infty(\mathbb{R})} \leq \left(\frac{9}{16}\right)^{1/3} V^{1/3} S_V(h)^{1/3}.$$

3.2. Scaling Law

In this section, we prove the following scaling law for the energy.

Proposition 3.9. *There exists a constant $c_0 > 0$ such that for every $V > 0$,*

$$c_0 \min\{V, V^{4/5}\} \leq F_s(V) \leq c_0^{-1} \min\{V, V^{4/5}\}.$$

Moreover, there exists $C > 0$ with the following property: If V is large enough, and $(u, h) \in \mathcal{A}_V$ with $E_V(u, h) + S_V(h) \leq \frac{1}{c_0} V^{4/5}$, then $\max h \leq C V^{3/5}$.

Proof. We prove the upper bound first. For that, we have to construct two elements from \mathcal{A}_V for $V > 0$. First, for $N \in \mathbb{N}$, set

$$h_N(x) := \begin{cases} \frac{V}{N^2}x + \frac{V}{N}, & \text{if } -N \leq x \leq 0, \\ -\frac{V}{N^2}x + \frac{V}{N}, & \text{if } 0 \leq x \leq N, \\ 0, & \text{if } |x| \geq N, \end{cases}$$

and set $u_N(x, y) := x$ if $(x, y) \in \Omega_{h_N}$. Then $(u_N, h_N) \in \mathcal{A}_V$, and $E_V(u_N, h_N) + S_V(h_N) = V + \frac{2V^2}{N^3} \leq 2V$ for N large enough. Note that we even have

$$E_V(u_N, h_N) + S_V(h_N) \rightarrow V \quad \text{as } N \rightarrow \infty. \quad (3.8)$$

For the other regime, set

$$\tilde{h}(x) := \begin{cases} V^{1/5}x + V^{3/5}, & \text{if } -V^{2/5} \leq x \leq 0, \\ -V^{1/5}x + V^{3/5}, & \text{if } 0 \leq x \leq V^{2/5}, \\ 0, & \text{if } |x| \geq V^{2/5}, \end{cases}$$

and let \tilde{u} be the restriction to $\Omega_{\tilde{h}}$ of

$$\tilde{u}(x, y) := \begin{cases} x(1 - V^{-2/5}y), & \text{if } y \leq V^{2/5}, \\ 0, & \text{if } y \geq V^{2/5}. \end{cases}$$

Then $(\tilde{u}, \tilde{h}) \in \mathcal{A}_V$, and

$$\begin{aligned} E_V(\tilde{u}, \tilde{h}) + S_V(\tilde{h}) &\leq \int_{-V^{2/5}}^{V^{2/5}} \int_0^{V^{2/5}} (1 - V^{-2/5}y)^2 + V^{-4/5}x^2 \, d\mathbf{x} + 2 \int_0^{V^{2/5}} V^{2/5} \, dx \\ &= \left(\frac{10}{3} + 2\right) V^{4/5}. \end{aligned}$$

For the lower bound, thanks to Proposition 3.6, we can assume that h is a Lipschitz function with $\int_{\mathbb{R}} h \, dx = V$ and $\{h > 0\}$ compact and connected (otherwise consider each of the (possibly infinitely many) connected components separately). Let then $u \in H^1(\Omega_h)$ be the minimizer of the Dirichlet energy in Ω_h with $u(x, 0) = x$. Let $x_1 \in \mathbb{R}$ be such that $h(x_1) > 0$, and let $t_1 > 0$ be the maximal $t > 0$ such that the square $[x_1, x_1 + t] \times [0, t]$ is below the graph of h , i.e., $[x_1, x_1 + t] \times [0, t] \subset \overline{\Omega_h}$. Observe that the maximality of t_1 implies the existence of a point $\bar{x}_1 \in [x_1, x_1 + t_1]$ such that $h(\bar{x}_1) = t_1$. Now let

$$V_1 := \int_{x_1}^{x_1+t_1} h \, dx, \quad E_1 := \int_{x_1}^{x_1+t_1} \int_0^{h(x)} |\nabla u|^2 \, d\mathbf{x}, \quad \text{and} \quad S_1 := \int_{x_1}^{x_1+t_1} h^2 \, dx.$$

We want to show that $E_1 + S_1 \gtrsim \min\{V_1, V_1^{4/5}\}$. By Lemma 2.1, $E_1 \geq \int_{[x_1, x_1+t_1] \times [0, t_1]} |\nabla u|^2 \, dx \geq Ct_1^2$, hence $E_1 \geq \frac{C}{2} V_1$ provided that $V_1 \leq 2t_1^2$.

Let us now assume that $V_1 > 2t_1^2$. Since $\max_{[x_1, x_1+t_1]} h \geq \frac{V_1}{t_1}$, we have

$$\begin{aligned} S_1 &\geq \frac{1}{t_1} \left(\max_{[x_1, x_1+t_1]} h - \min_{[x_1, x_1+t_1]} h \right)^2 \geq \frac{1}{t_1} \left(\frac{V_1}{t_1} - t_1 \right)^2 \\ &= \frac{1}{t_1} \left(\frac{V_1}{2t_1} + \frac{V_1}{2t_1} - t_1 \right)^2 \geq \frac{V_1^2}{4t_1^3}. \end{aligned}$$

Since $E_1 \geq Ct_1^2$, we obtain

$$E_1 + S_1 \geq Ct_1^2 + \frac{V_1^2}{4t_1^3} \geq CV_1^{4/5},$$

where the last inequality follows from Young’s inequality.

We have thus shown that the energy in $[x_1, x_1 + t_1]$ is bounded from below by $c_0 \min\{V_1, V_1^{4/5}\}$. We define iteratively $x_2 := x_1 + t_1$, $x_{i+1} := x_i + \sum_{k=1}^i t_k$, and repeat the process in each interval $[x_i, x_i + t_i]$, and similarly in the opposite direction (i.e., going to the left) starting at x_1 . Since h is Lipschitz we cover with this procedure the whole set $\{h > 0\}$, and the lower bound follows.

It remains to show that if V is large enough, and $(u, h) \in \mathcal{A}_V$ with $E_V(u, h) + S_V(h) \leq \frac{1}{c_0} V^{4/5}$, then $\max h \geq CV^{3/5}$ for some $C > 0$ independent of V . Having t_i and V_i constructed in the previous part of the proof, let us assume that V_1 is the largest among all $\{V_i\}$. Then we have

$$\begin{aligned} \frac{V^{4/5}}{c_0} &\geq E_V(u, h) + S_V(h) \geq \sum_i c_0 \min \left\{ V_i, V_i^{4/5} \right\} \\ &\geq c_0 \min\{1, V_1^{-1/5}\} \sum_i V_i = c_0 \min \left\{ 1, V_1^{-1/5} \right\} V, \end{aligned}$$

which gives $c_0^{10} V \leq \max\{1, V_1\}$. Assume now that $V \geq c_0^{-10}$ so that $V_1 \geq c_0^{10} V$. If we set $M := \max_{[x_1, x_1+t_1]} h$, then as before $M \geq V_1/t_1$, which implies $t_1 \geq V_1/M$. Since $E_1 \geq Ct_1^2$, we have

$$V^{4/5}/c_0 \geq E_1 \gtrsim t_1^2 \geq (V_1/M)^2 \geq c_0^{20} V^2 M^{-2},$$

which implies $\max h \geq M \gtrsim V^{3/5}$. \square

Remark 3.10. Using the scaling law (see Proposition 3.9) and (3.7), we find that $\max h \lesssim V^{3/5}$, and so the previous proposition implies that $\max h \sim V^{3/5}$ for sufficiently large V . Using this we see that the size of the support of h is at least of order $\max\{1, V^{2/5}\}$.

In terms of the original parameters e_0 and d , the scaling law reads as follows.

Proposition 3.11. *There exists a positive constant c such that for every $V > 0$,*

$$c \min \left\{ e_0^2 d, d^{4/5} e_0^{6/5} \right\} \leq F(V) \leq \frac{1}{c} \min \left\{ e_0^2 d, d^{4/5} e_0^{6/5} \right\}.$$

Remark 3.12. In the original coordinates, for $e_0^4 d \gtrsim 1$, the typical island is of height $e_0^{2/5} d^{3/5}$ and of width $\left(\frac{d}{e_0}\right)^{2/5}$.

3.3. Existence and Non-Existence of Minimizers

Let us start by studying the non-existence case for small V , i.e., the wetting regime. For this, we prove that for a non-trivial range of volumes $0 < V \leq \bar{V}$, the infimum of the energy $F_s(V)$ is equal to V .

Proposition 3.13. *There exists $\bar{V} > 0$ such that for every $0 < V \leq \bar{V}$ we have $F_s(V) = V$. As a consequence, there exists no minimizer of (3.2) for $0 < V < \bar{V}$.*

Proof. Recall that by the construction of flat layers in the proof of the scaling law [see (3.8)], we have $F_s(V) \leq V$. It remains to show the reverse inequality, i.e., that for every $(u, h) \in \mathcal{A}_V$, we have $E_V(u, h) + S_V(h) \geq V$ if V is small enough. By density, it suffices to consider $(u, h) \in \mathcal{A}_V$ for which h is Lipschitz continuous (see Proposition 3.6), and u being the minimizer of the Dirichlet energy in Ω_h subject to the boundary condition $u(x, 0) = x$ on $\{h > 0\}$. In particular, $-\Delta u = 0$ in Ω_h . Testing the Laplace equation with $u(x, y) - x$, we obtain

$$0 = \int_{\Omega_h} \operatorname{div}(\nabla u) \cdot (u - x) \, d\mathbf{x} = - \int_{\Omega_h} \left(|\nabla u|^2 - \partial_x u \right) \, d\mathbf{x} + \int_{\partial\Omega_h} \partial_\nu u \cdot (u - x) \, d\mathcal{H}^1.$$

The boundary integral vanishes, because $\partial_\nu u = 0$ if $y = h(x)$, and $u(x, 0) = x$. Hence

$$\int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} = \int_{\Omega_h} \partial_x u \, d\mathbf{x},$$

and integration by parts yields

$$\begin{aligned} \int_{\Omega_h} \partial_x u \, d\mathbf{x} &= \int_{\Omega_h} \operatorname{div} \begin{pmatrix} u \\ 0 \end{pmatrix} \, d\mathbf{x} = \int_{\partial\Omega_h} \begin{pmatrix} u \\ 0 \end{pmatrix} \cdot \nu \, d\mathcal{H}^1 \\ &= \int_{\{y=h(x)\}} u \nu_x \, d\mathcal{H}^1 = - \int_{\mathbb{R}} u(x, h(x)) h'(x) \, dx. \end{aligned}$$

Here we used that $\nu = \frac{(-h'(x), 1)}{\sqrt{1+h'^2}}$, and so $\nu_x \, d\mathcal{H}^1 = -h'(x) \, dx$.

By the fundamental theorem of calculus, $u(x, h(x)) = x + \int_0^{h(x)} \partial_y u \, dy$, and we deduce

$$\int_{\mathbb{R}} u(x, h(x)) h'(x) \, dx = \int_{\mathbb{R}} x h'(x) \, dx + \int_{\Omega_h} \partial_y u(x, y) h'(x) \, d\mathbf{x}.$$

For the first integral on the right-hand side, integration by parts implies

$$\int_{\mathbb{R}} x h'(x) \, dx = - \int_{\mathbb{R}} h(x) \, dx = -V,$$

and so altogether,

$$\int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} = V - \int_{\Omega_h} \partial_y u(x, y) h'(x) \, d\mathbf{x}.$$

We add $S_V = \int_{\mathbb{R}} h^2 \, dx$ to both sides of the equation to obtain

$$E_V + S_V = \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} + S_V = V + S_V - \int_{\Omega_h} \partial_y u(x, y) h'(x) \, d\mathbf{x}. \tag{3.9}$$

To show that the energy of (u, h) is not smaller than V , it suffices to prove that

$$\int_{\Omega_h} \partial_y u(x, y) h'(x) \, d\mathbf{x} \leq S_V.$$

By Lemma 3.5, we have either $\partial_y u \equiv 0$, in which case the inequality holds trivially true, or, using Hölder's inequality,

$$\begin{aligned} \int_{\Omega_h} \partial_y u(x, y) h'(x) \, d\mathbf{x} &\leq \left(\int_{\Omega_h} (\partial_y u)^2(x, y) \, d\mathbf{x} \right)^{1/2} \left(\int_{\Omega_h} h'^2(x) \, d\mathbf{x} \right)^{1/2} \\ &\leq \sqrt{\frac{3}{4}} S_V \sqrt{\sup h}, \end{aligned} \tag{3.10}$$

which implies that $E_V(u, h) + S_V(h) \geq V$ for any admissible $(u, h) \in \mathcal{A}_V$ which satisfies $\sup h \leq 4/3$. Note that by (3.7),

$$\sup h \leq \left(\frac{9}{16} \right)^{1/3} V^{1/3} S_V^{1/3} \leq \left(\frac{9}{16} \right)^{1/3} V^{2/3}.$$

Hence, the infimum of the energy is exactly V for $0 < V \leq \frac{2^5}{9\sqrt{3}}$, and so there exists a maximal $\bar{V} > 0$ such that $F_S(V) = V$ for $V \in (0, \bar{V}]$.

To show non-existence of a minimizer, let us argue by contradiction and assume that there exists a minimizer $(u, h) \in \mathcal{A}_V$ for some V with $0 < V < \bar{V}$. Choose V_0 with $V < V_0 \leq \bar{V}$, and set $\lambda := \sqrt{\frac{V}{V_0}}$. Then by Lemma 3.1, since $(u_\lambda, h_\lambda) \in \mathcal{A}_{V_0}$ and $S_V(h) > 0$, we obtain a contradiction:

$$\begin{aligned} V_0 = F_s(V_0) &\leq E_{V_0}(u_\lambda, h_\lambda) + S_{V_0}(h_\lambda) = \frac{V_0}{V} E_V(u, h) + \left(\frac{V_0}{V}\right)^{1/2} S_V(h) \\ &< \frac{V_0}{V} (E_V(u, h) + S_V(h)) = V_0. \end{aligned}$$

□

We next consider the regime $\{V > \bar{V}\}$ and aim to prove that minimizers exist for every $V > \bar{V}$. For that, we need some auxiliary properties. For $V > 0$, we set

$$\beta(V) := \frac{F_s(V)}{V}. \quad (3.11)$$

By (3.5) and $F_s(V) \leq V$, we see that $\beta(V) \leq 1$ for every V , and that β is a non-increasing function of V . Using that $F_s(V) = V$ for $V \in (0, \bar{V}]$, we can say more about β :

Lemma 3.14. *The function β is strictly decreasing in the region $\{F_s(V) < V\} = \{\beta < 1\}$.*

Proof. We assume the contrary, i.e., that there exist $\bar{V} < V_0 < V_1 < \infty$ and $\beta_0 < 1$ such that $F_s(V) = \beta_0 V < V$ for all $V \in [V_0, V_1] \subset (\bar{V}, \infty)$. We use the concavity of F_s , $F_s(0) = 0$, and the previous assumption, to get for every $V \in (0, \bar{V})$

$$\begin{aligned} F_s(V) &\geq \left(1 - \frac{V}{V_0}\right) F_s(0) + \frac{V}{V_0} F_s(V_0) = \frac{V}{V_0} F_s(V_0) = \beta_0 V, \quad \text{and} \\ \beta_0 V_0 = F_s(V_0) &\geq \frac{V_1 - V_0}{V_1 - V} F_s(V) + \frac{V_0 - V}{V_1 - V} F_s(V_1). \end{aligned}$$

Since $F_s(V_1) = \beta_0 V_1$, the second relation simplifies to $F_s(V) \leq \beta_0 V$, which together with the first relation implies $F_s(V) = \beta_0 V$. This is a contradiction, since $\beta_0 < 1$ and by assumption $F_s(V) = V$. □

The essential step to prove the existence of minimizers is to derive the compactness of minimizing sequences. It is based on the following auxiliary lemma:

Lemma 3.15. *Let $V > \bar{V}$ and $\delta > 0$. Then there exist a length $l = l(V, \delta) > 0$ and $C(V, \delta) > 0$ with the following property: for every $(u, h) \in \mathcal{A}_V$ with $\varepsilon := E_V(u, h) + S_V(u) - F_s(V) \leq C(V, \delta)$, and every $x_0 < x_1$, with $x_1 - x_0 = l$, we have*

$$\int_{-\infty}^{x_0} h \, dx \leq \delta \quad \text{or} \quad \int_{x_1}^{\infty} h \, dx \leq \delta. \quad (3.12)$$

Proof. Let $V > \bar{V}$ and $\delta > 0$. Note that by the strict monotonicity of β in $[\bar{V}, \infty)$, we have $\beta(V) < \beta\left(\frac{V}{2}\right)$. Choose $C(V, \delta) > 0$ and let $0 < \alpha < 1$ be such that

$$C(V, \delta) < \frac{\delta}{2} \left(\beta\left(\frac{V}{2}\right) - \beta(V) \right), \quad \text{and} \quad \alpha < \left(\frac{\delta}{156} \left[\beta\left(\frac{V}{2}\right) - \beta(V) \right] \right)^2. \tag{3.13}$$

Assume further that $\alpha > 0$ is such that $\frac{V}{3\alpha} = n \in \mathbb{N}$. Set $l := \frac{V}{\alpha}$, and let $[x_0, x_1]$ be an arbitrary interval of length $x_1 - x_0 = l$. Since $\alpha^{-1}V = 3n$ with $n \in \mathbb{N}$, we can decompose $[x_0, x_1]$ as an essentially disjoint union of $3n$ intervals of length 1, i.e.,

$$[x_0, x_1] = \bigcup_{k=0}^{3n-1} [x_0 + k, x_0 + (k + 1)].$$

Since $\int_{\mathbb{R}} h \, dx = V = n \cdot 3\alpha$, there are at most n intervals which satisfy $\int_{I_k} h \, dx \geq 3\alpha$. Similarly, since $S_V(h) < V$, we find at most n intervals such that $\int_{I_k} h^2 \, dx \geq 3\alpha$. Hence, among I_k there is an interval I such that

$$\int_I h \, dx \leq 3\alpha \quad \text{and} \quad \int_I h^2 \, dx \leq 3\alpha, \tag{3.14}$$

in particular

$$3\alpha \geq \int_I h^2 \, dx \geq \left(\sup_I h - \inf_I h \right)^2.$$

Since we also have $3\alpha \geq \int_I h \, dx \geq \inf_I h$, we get that

$$\sup_I h = \left(\sup_I h - \inf_I h \right) + \inf_I h \leq \sqrt{3\alpha} + 3\alpha \leq 3(\alpha + \alpha^{1/2}). \tag{3.15}$$

Without loss of generality (translating h), we may assume that $I = [0, 1]$. Now we “cut” the profile into three parts. Precisely, we set

$$h_0(x) := \begin{cases} h(x), & \text{if } x \leq 0, \\ \min(-2h(0)x + h(0), h(x)), & \text{if } 0 \leq x \leq \frac{1}{2}, \text{ and } u_0 := u|_{\Omega_{h_0}}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h_1(x) := \begin{cases} \min(2h(1)x - h(1), h(x)), & \text{if } \frac{1}{2} \leq x \leq 1, \\ h(x), & \text{if } x \geq 1, \quad \text{and } u_1 := u|_{\Omega_{h_1}}. \\ 0 & \text{otherwise,} \end{cases}$$

We use the notation

$$V_0 := \int_{-\infty}^{1/2} h_0 \, dx, \quad V_1 := \int_{1/2}^{\infty} h_1 \, dx, \quad V_{lo} := V - V_0 - V_1.$$

We may assume without loss of generality that $V_0 \leq V_1$, which implies that $V_0 \leq \frac{V}{2}$. Note that by construction $(u_0, h_0) \in \mathcal{A}_{V_0}$, $(u_1, h_1) \in \mathcal{A}_{V_1}$,

$$V_{lo} \leq \sup_I h \leq 3(\alpha + \alpha^{1/2}), \tag{3.16}$$

and

$$E_V(u, h) \geq E_{V_0}(u_0, h_0) + E_{V_1}(u_1, h_1). \tag{3.17}$$

Further, again by construction and (3.15),

$$\begin{aligned} S_V(u, h) &\geq S_{V_0}(h_0) + S_{V_1}(h_1) - \left[\int_0^{1/2} h_0^2 dx + \int_{1/2}^1 h_1^2 dx \right] \\ &\geq S_{V_0}(h_0) + S_{V_1}(h_1) - \left(2 \sup_I h \right)^2 \geq S_{V_0}(h_0) + S_{V_1}(h_1) \\ &\quad - 36(\alpha + \alpha^{1/2})^2. \end{aligned} \tag{3.18}$$

Since $F_s(V) = \beta(V)V$ with $\beta < 1$ we have by (3.17) and (3.18),

$$\begin{aligned} \beta(V)V + \varepsilon &= F_s(V) + \varepsilon = E_V(u, h) + S_V(h) \geq F_s(V_0) + F_s(V_1) - 36(\alpha + \alpha^{1/2})^2 \\ &= \beta(V_0)V_0 + \beta(V_1)V_1 - 36(\alpha + \alpha^{1/2})^2. \end{aligned}$$

Moreover, since β is non-increasing and since $V = V_0 + V_1 + V_{lo}$, and $V_0 \leq \frac{V}{2}$,

$$\begin{aligned} \beta(V_0)V_0 + \beta(V_1)V_1 &= \beta(V)V + V_0(\beta(V_0) - \beta(V)) + V_1(\beta(V_1) - \beta(V)) - \beta(V)V_{lo} \\ &\geq \beta(V)V + V_0(\beta(V/2) - \beta(V)) - V_{lo} \end{aligned}$$

which together implies $V_0(\beta(V/2) - \beta(V)) \leq \varepsilon + V_{lo} + 36(\alpha + \alpha^{1/2})^2$. Since $V > \bar{V}$, the strict monotonicity of β implies $\beta(V/2) - \beta(V) > 0$, and so, since $\alpha^2 < \alpha < \alpha^{1/2}$,

$$V_0 \leq \frac{\varepsilon + V_{lo} + 36(\alpha + \alpha^{1/2})^2}{\beta(V/2) - \beta(V)} \leq \frac{\varepsilon + 78\alpha^{1/2}}{\beta(V/2) - \beta(V)}. \tag{3.19}$$

By (3.13), we obtain that $V_0 < \delta$, and thus

$$\int_{-\infty}^{x_0} h dx = \int_{-\infty}^{x_0} h_0 dx \leq \int_{\mathbb{R}} h_0 dx = V_0 \leq \delta.$$

□

Proposition 3.16. *For any $V > \bar{V}$, a minimizer of (3.2) exists.*

Proof. Let V be such that $F_s(V) < V$, and let (u_n, h_n) be a minimizing sequence. First we claim that Lemma 3.15 implies tightness of (u_n, h_n) .

Indeed, let $0 < \delta < V/2$ be fixed and let $l = l(V, \delta)$ be obtained from Lemma 3.15. Then for n large enough the energy of (u_n, h_n) is close enough to

$F_s(V)$ [i.e., $\varepsilon := E_V(u_n, h_n) + S_V(h_n) - F_s(V)$ from Lemma 3.15 is small enough]. Then we choose x_n such that $\int_{-\infty}^{x_n} h_n \, dx = 2\delta$ and observe that by Lemma 3.15

$$\int_{x_n+l}^{\infty} h_n \, dx \leq \delta.$$

Hence, for n large enough we have $\int_{-\infty}^{x_n} h_n \leq 2\delta$ and $\int_{x_n+l}^{\infty} h_n \leq \delta$, which implies tightness of the minimizing sequence (up to translations). The existence of a minimizer then follows from the lower semicontinuity of the energy (see Proposition 3.6). \square

We will prove in Corollary 3.34 that also for $V = \bar{V}$ a minimizer exists.

3.4. Regularity of Minimizers

Notation In this section, for given $x \in \mathbb{R}$ ($x_i \in \mathbb{R}$, etc.), we will denote by z (z_i) a point in \mathbb{R}^2 defined by $\mathbf{z} := (x, y) = (x, h(x))$ ($\mathbf{z}_i = (x_i, y_i) = (x_i, h(x_i))$).

In this subsection we prove the regularity of minimizers of (3.2) if they exist. For this we follow the strategy of [10] (see also [9]) which in turn was inspired by [5]. Let us first notice that as in [10], the volume constraint can be relaxed. For $\mu > 0$ and $V > 0$ set

$$F_s^\mu(V) := \inf \left\{ \int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}} h^2 \, dx + \mu \left| \int_{\mathbb{R}} h \, dx - V \right| : (u, h) \in \mathcal{A} \right\}$$

with

$$\mathcal{A} := \left\{ (u, h) : h \in H^1(\mathbb{R}), h \geq 0, u \in H^1(\Omega_h), u(x, 0) = x \text{ if } x \in \text{supp } h \right\}.$$

Note that $\mathcal{A}_V = \{(u, h) \in \mathcal{A} : \int_{\mathbb{R}} h \, dx = V\}$. We have the following relation between F_s^μ and F_s :

Proposition 3.17. *Let $C \geq \frac{2}{c_0}$, where c_0 is the constant from Proposition 3.9. Then, for $\mu := C \min \{1, V^{-1/5}\}$, we have $F_s^\mu(V) = F_s(V)$.*

Proof. We closely follow the proof of [16, Theorem 2.8]. Let $V > 0$. First, since $\mathcal{A}_V \subset \mathcal{A}$, and

$$\int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}} h^2 \, dx + \mu \left| \int_{\mathbb{R}} h \, dx - V \right| = \int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}} h^2 \, dx$$

for every $(u, h) \in \mathcal{A}_V$, we have $F_s^\mu(V) \leq F_s(V)$. Assume now for the sake of contradiction that $F_s^\mu(V) < F_s(V)$, i.e., there exists $(u, h) \in \mathcal{A}$ such that¹

$$E(u, h) + S(h) + \mu \left| \int_{\mathbb{R}} h \, dx - V \right| < F_s(V). \tag{3.20}$$

¹ We denote here, for simplicity, $E(u, h) := \int_{\Omega_h} |\nabla u|^2 \, dx$ and $S(h) := \int_{\mathbb{R}} h^2 \, dx$.

We note first that by rescaling we can assume that $\int_{\mathbb{R}} h \, dx \leq V$. Indeed, assume that there exists $(u, h) \in \mathcal{A}$ with (3.20) and $\int_{\mathbb{R}} h \, dx =: W > V$. Then for the rescaled pair (see Lemma 3.1) $(u_\lambda, h_\lambda) \in \mathcal{A}_V$ with $\lambda := \sqrt{\frac{W}{V}} > 1$, we also have by Lemma 3.1 and (3.20),

$$E(u_\lambda, h_\lambda) + S(h_\lambda) < E(u, h) + S(h) < F_s(V).$$

On the other hand, if $\int_{\mathbb{R}} h \, dx \leq \frac{V}{2}$, then $\mu \left| \int_{\mathbb{R}} h \, dx - V \right| \geq \mu \frac{V}{2} = \frac{C}{2} \min\{1, V^{4/5}\} \geq F_s(V)$ by the choice of C and Proposition 3.9. Hence we can assume that $\int_{\mathbb{R}} h \, dx \geq \frac{V}{2}$. Now we construct a competitor from \mathcal{A}_V for $(u, h) \in \mathcal{A} \setminus \mathcal{A}_V$ by rescaling. Denoting $\alpha := \int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}} h^2 \, dx$ and $W := \int_{\mathbb{R}} h \, dx$, we find by (3.20) that

$$\alpha + \mu(V - W) = E(u, h) + S(h) + \mu \left| \int_{\mathbb{R}} h \, dx - V \right| < F_s(V) \leq \alpha \frac{V}{W}.$$

From this we get $\mu < \frac{\alpha}{W}$ [since otherwise $\alpha + \mu(V - W) \geq \frac{\alpha V}{W}$], and from the scaling law (Proposition 3.9) finally (recall that $W \geq V/2$)

$$\mu < \frac{\alpha}{W} \leq \frac{F_s(V)}{W} \leq \frac{1}{W} \frac{1}{c_0} \min\{V, V^{4/5}\} \leq \frac{2}{c_0} \min\{1, V^{-1/5}\},$$

from which we get a contradiction by the choice of C . \square

In order to prove more properties of the minimizers, we will need to use the Euler–Lagrange equation. For this, we will need to show first that minimizers of (3.2) are smooth.

Theorem 3.18. *If $(u, h) \in \mathcal{A}$ is a minimizer of (3.2), then h is analytic in $\{h > 0\}$ and satisfies the zero contact angle condition.*

Following [5, 9, 10], we divide the proof of the regularity of (u, h) into several lemmas. We first prove that h is locally Lipschitz continuous. Then, in the spirit of [5], we prove a uniform sphere condition. From this, we derive decay estimates for $|\nabla u|$ which as in [10] leads to the regularity of h .

Lemma 3.19. *Let the pair (u, h) be a minimizer of (3.2). Then in the set $\{h > 0\}$ the height profile h is a locally Lipschitz function.*

Proof. Let $m > 0$, and $x_0, x_1 \in \text{supp } h$ be such that $x_0 < x_1$, $h(x_0) < h(x_1)$ and $m \leq h(x)$ for $x \in [x_0, x_1]$. Since h is continuous, there exist

$$\begin{aligned} \bar{x}_0 &:= \max\{x \in [x_0, x_1] : h(x) = h(x_0)\}, \\ x_3 &:= \min\{x > x_1 : h(x) = h(x_0)\}, \quad \text{and} \\ x_2 &:= \max\{x < x_3 : h(x) = h(x_1)\}. \end{aligned}$$

Note that $h \geq h(x_0)$ in $[\bar{x}_0, x_3]$ and $h > h(x_0)$ in (\bar{x}_0, x_1) . We denote $\delta := h(x_1) - h(x_0)$.

Let us now define a competitor (\tilde{u}, \tilde{h}) for (u, h) . In $\mathbb{R} \setminus [\bar{x}_0, x_3]$ we set $\tilde{h} := h$, and in $[\bar{x}_0, x_3]$ we set

$$\tilde{h}(x) := \begin{cases} h(x_0) & \text{if } h(x_0) \leq h(x) < h(x_1) \\ h(x) - \delta & \text{if } h(x) \geq h(x_1). \end{cases}$$

Since $\tilde{h} \leq h$, we have $\Omega_{\tilde{h}} \subset \Omega_h$ and the following definition makes sense

$$\tilde{u}(\mathbf{x}) := u(\mathbf{x}), \quad \mathbf{x} \in \Omega_{\tilde{h}}. \tag{3.21}$$

We denote $M := \{x \in [\bar{x}_0, x_3] : h(x_0) \leq h(x) < h(x_1)\}$. Then $h'(x) = \tilde{h}'(x)$ for almost every $x \notin M$ and $\tilde{h}'(x) = 0$ for almost every $x \in M$, and so

$$\begin{aligned} \int_{\mathbb{R}} \tilde{h}'^2 dx &= \int_{\mathbb{R}} h'^2 dx - \int_M h'^2 dx \leq \int_{\mathbb{R}} h'^2 dx - \int_{\bar{x}_0}^{x_1} h'^2 dx \\ &\leq \int_{\mathbb{R}} h'^2 dx - \frac{\delta^2}{x_1 - \bar{x}_0}, \end{aligned}$$

where the last inequality follows from $\delta^2 = (h(x_1) - h(\bar{x}_0))^2 = \left(\int_{\bar{x}_0}^{x_1} h' dx\right)^2 \leq (x_1 - \bar{x}_0) \int_{\bar{x}_0}^{x_1} h'^2 dx$.

We now estimate $\int_{\mathbb{R}} \tilde{h} dx$:

$$\int_{\mathbb{R}} \tilde{h} dx = \int_{\mathbb{R}} h dx - \int_{\{\bar{x}_0 < x < x_3\}} \min\{h(x) - h(x_0), \delta\} dx \geq V - (x_3 - \bar{x}_0)\delta.$$

Since $h(x) \geq m$ in $[\bar{x}_0, x_3]$, we have that $V \geq \int_{\bar{x}_0}^{x_3} h(x) dx \geq (x_3 - \bar{x}_0)m$, and so

$$V \geq \int_{\mathbb{R}} \tilde{h} dx \geq V - \frac{\delta V}{m}. \tag{3.22}$$

Finally, by (3.21) we see that $\int_{\Omega_h} |\nabla u|^2 d\mathbf{x} \leq \int_{\Omega_{\tilde{h}}} |\nabla \tilde{u}|^2 d\mathbf{x}$.

Since (u, h) is a minimizer, the previous estimates and Proposition 3.17 imply

$$\begin{aligned} \int_{\Omega_h} |\nabla u|^2 d\mathbf{x} + \int_{\mathbb{R}} h^2 dx &\leq \int_{\Omega_{\tilde{h}}} |\nabla \tilde{u}|^2 d\mathbf{x} + \int_{\mathbb{R}} \tilde{h}'^2 dx + \mu \left| \int_{\mathbb{R}} \tilde{h} dx - V \right| \\ &\leq \int_{\Omega_h} |\nabla u|^2 d\mathbf{x} + \int_{\mathbb{R}} h^2 dx - \left(\frac{\delta^2}{x_1 - \bar{x}_0} - \mu \frac{\delta V}{m} \right), \end{aligned}$$

where $\mu = CV^{-1/5}$. Hence $\frac{\delta}{x_1 - \bar{x}_0} \leq \mu \frac{V}{m}$, which implies

$$\frac{h(x_1) - h(x_0)}{x_1 - x_0} \leq \frac{h(x_1) - h(x_0)}{x_1 - \bar{x}_0} = \frac{\delta}{x_1 - \bar{x}_0} \leq \frac{\mu V}{m} \lesssim \frac{V^{4/5}}{m}.$$

In the case $h(x_0) > h(x_1) \geq m$ we proceed analogously. Altogether we get that if both x_0 and x_1 belong to the set $\{x : h(x) \geq m\}$, then

$$|h(x_0) - h(x_1)| \lesssim |x_0 - x_1| \frac{V^{4/5}}{m},$$

in particular h is locally Lipschitz in the set $\{h > 0\}$. \square

We now prove that the graph of h satisfies a uniform sphere condition. The proof is inspired by, but slightly different from, the proof of a similar statement in [5] (see also [9, 10]). The main difference to the aforementioned papers is that in our setting, the surface energy is not invariant under rotation of the axis.

Lemma 3.20. *Let $V > 0$, and let the pair (u, h) be a minimizer of (3.2). Then, there exists a radius $r_0 = r_0(V) > 0$ with the following property: for every circle $S_r(\mathbf{x}_0)$ ($\mathbf{x}_0 := (x_0, y_0)$) and every interval $(a, b) \subset \mathbb{R}$ such that $(a, h(a)) \in S_r(\mathbf{x}_0)$, $h(a) > y_0$, $(b, h(b)) \in S_r(\mathbf{x}_0)$, $h(b) > y_0$, and such that the graph of h is above $S_r(\mathbf{x}_0)$ in (a, b) , we have that $r > r_0$.*

Proof. Let $S_r(\mathbf{x}_0)$ and $(a, b) \subset \mathbb{R}$ be as in the lemma. We define (\tilde{h}, \tilde{u}) , a competitor for (u, h) , by

$$\tilde{h}(x) := \begin{cases} h(x) & \text{if } x \notin (a, b) \\ h(a) + \frac{h(b)-h(a)}{b-a}(x-a) & \text{if } x \in (a, b), \end{cases}$$

and $\tilde{u}(\mathbf{x}) := u(\mathbf{x})$ for $\mathbf{x} \in \Omega_{\tilde{h}} \subset \Omega_h$. Since (u, h) is a minimizer, Proposition 3.17 implies

$$\int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} + \int_{\mathbb{R}} h'^2 \, dx \leq \int_{\Omega_{\tilde{h}}} |\nabla \tilde{u}|^2 \, d\mathbf{x} + \int_{\mathbb{R}} \tilde{h}'^2 \, dx + \mu \left| \int_{\mathbb{R}} \tilde{h} \, dx - V \right|.$$

Since $u = \tilde{u}$ in $\Omega_{\tilde{h}}$ and $\Omega_{\tilde{h}} \subset \Omega_h$, we see that $\int_{\Omega_{\tilde{h}}} |\nabla u|^2 \, d\mathbf{x} \leq \int_{\Omega} |\nabla \tilde{u}|^2 \, d\mathbf{x}$. We use this together with the fact that $h = \tilde{h}$ outside of (a, b) to derive

$$\int_a^b (h'^2 - \tilde{h}'^2) \, dx \leq \mu \left| \int_{\mathbb{R}} \tilde{h} \, dx - \int_{\mathbb{R}} h \, dx \right| = \mu \left| \int_a^b (h - \tilde{h}) \, dx \right|. \quad (3.23)$$

Using the definition of \tilde{h} and Hölder's inequality, we obtain for every $x \in (a, b)$

$$\begin{aligned} |h(x) - \tilde{h}(x)| &\leq \int_a^x \left| h'(x') - \frac{h(b) - h(a)}{b-a} \right| \, dx' \\ &\leq (b-a)^{1/2} \left(\int_a^x \left(h'(x') - \frac{h(b) - h(a)}{b-a} \right)^2 \, dx' \right)^{1/2}. \end{aligned} \quad (3.24)$$

We observe that $\int_a^b h' \, dx = h(b) - h(a)$ implies that

$$\int_a^b \left(h'(x') - \frac{h(b) - h(a)}{b-a} \right)^2 \, dx' = \int_a^b (h'^2(x') - \tilde{h}'^2(x')) \, dx'.$$

We plug this relation into (3.24) to show

$$\int_a^b (h'^2 - \tilde{h}'^2) \, dx \stackrel{(3.23)}{\leq} \mu \int_a^b |h - \tilde{h}| \, dx \leq \mu (b-a)^{3/2} \left(\int_a^b (h'^2 - \tilde{h}'^2) \, dx \right)^{1/2}.$$

Hence

$$\int_a^b (h'^2 - \tilde{h}'^2) \, dx \leq \mu^2(b-a)^3,$$

and subsequently

$$\int_a^b (h - \tilde{h}) \, dx \leq \mu(b-a)^3. \tag{3.25}$$

Let us now estimate $\int_a^b (h - \tilde{h}) \, dx$ using the fact that in the interval (a, b) the height function h is above the circle $S_r(\mathbf{x}_0)$, i.e., that $h(x) \geq y_0 + \sqrt{r^2 - (x - x_0)^2} =: f(x)$ for $x \in (a, b)$. The trapezoidal rule implies that

$$\int_a^b f(x) \, dx - (b-a) \frac{f(a) + f(b)}{2} = -\frac{(b-a)^3}{12} f''(\xi)$$

for some $\xi \in (a, b)$. Using that $\int_a^b \tilde{h} \, dx = (b-a) \frac{h(a)+h(b)}{2} = (b-a) \frac{f(a)+f(b)}{2}$, we get that

$$\int_a^b (h - \tilde{h}) \, dx \geq \int_a^b f \, dx - (b-a) \frac{f(a) + f(b)}{2} = -\frac{(b-a)^3}{12} f''(\xi).$$

Finally, we compute $f''(\xi) = -\frac{r^2}{(r^2 - (\xi - x_0)^2)^{3/2}}$ to show

$$\begin{aligned} \int_a^b (h - \tilde{h}) \, dx &\geq -\frac{(b-a)^3}{12} f''(\xi) = \frac{(b-a)^3}{12} \frac{r^2}{(r^2 - (\xi - x_0)^2)^{3/2}} \\ &\geq \frac{1}{12r} (b-a)^3. \end{aligned}$$

We conclude by combining the previous estimate with (3.25) to get $r \geq \mu/12$. \square

Arguing as in [9, 10], we obtain the following result:

Lemma 3.21. *Let the pair (u, h) be a minimizer of (3.2). Then for every point $x \in \text{supp } h$ there exists a ball $B_{r_0}(x_0, y_0) \subset \Omega_h \cup \{y \leq 0\}$ (with r_0 defined in Lemma 3.20) such that $\partial B_{r_0}(x_0, y_0) \cap \text{graph } h = (x, h(x))$.*

From this and [5, Lemma 3], we obtain:

Corollary 3.22. *Let the pair (u, h) be a minimizer of (3.2). Let $x_0 \in \mathbb{R}$ be such that $h(x_0) > 0$. Then there exists a neighborhood $U \subset \mathbb{R}$ of x_0 such that $h|_U$ is Lipschitz and admits left and right derivatives at every point of U , that are respectively left and right continuous.*

The previous corollary implies that to prove that h is a C^1 function in the set $\{h > 0\}$, it suffices to consider points $x_0 \in \mathbb{R}$, $h(x_0) > 0$, for which $h'_+(x_0) \neq h'_-(x_0)$, where h'_+ and h'_- denote the right and left derivatives. Following [9, 10] we call such points *corner points*. Our aim is to prove that if (u, h) is a minimizer of (3.2), then there are no corner points. In order to show this, we first obtain the following estimate on $|\nabla u|$ (which is also an important ingredient in the proof of [9, 10]):

Lemma 3.23. *Let the pair (u, h) be a minimizer of (3.2), and let x_0 be a corner point. Then there exist $\alpha > 1$ and $r_1 > 0$ such that $\int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} \leq \bar{C} \rho^\alpha$ for all $\rho \in (0, r_1)$, where $\bar{C} := r_1^{-\alpha} \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x}$.*

Proof. Since $h(x_0) > 0$, h is locally Lipschitz in the neighborhood of x_0 and we can find $\delta, L > 0$ such that

$$|h(x) - h(x')| \leq L|x - x'| \quad \forall x, x' \in (x_0 - \delta, x_0 + \delta). \quad (3.26)$$

Since by assumption, x_0 is a corner point, we can choose $r_1, 0 < r_1 < \min\{\delta, h(x_0)\}$, small enough such that for every $\rho \in (0, r_1)$ both the graph of $h|_{(x_0, +\infty)}$ and of $h|_{(-\infty, x_0)}$ intersect $S_\rho(\mathbf{z}_0)$ exactly once. For $\rho \in (0, r_1)$, let us denote the arc of $S_\rho(\mathbf{z}_0)$, which connects two intersections of the graph of h with $S_\rho(\mathbf{z}_0)$, and which belongs to Ω_h (i.e., the bottom arc), by A_ρ . By virtue of (3.26) this arc has length at most

$$\mathcal{H}^1(A_\rho) \leq 2\pi\rho(1 - \arctan(1/L)). \quad (3.27)$$

For any $a \in \mathbb{R}$ and any $\rho \in (0, r_1)$, since $\Delta(u - a) = 0$ in $B_\rho(\mathbf{z}_0) \cap \Omega_h$ and $\partial_\nu(u - a) = 0$ on $\partial\Omega_h \cap B_\rho(\mathbf{z}_0)$, we get

$$\int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} = \int_{A_\rho} (u - a) \partial_\nu u \, d\mathcal{H}^1. \quad (3.28)$$

Then Poincaré's inequality with the optimal constant implies

$$\begin{aligned} \int_{A_\rho} (u - \bar{u})^2 \, d\mathcal{H}^1 &\leq \left(\frac{\mathcal{H}^1(A_\rho)}{\pi} \right)^2 \int_{A_\rho} |\partial_\tau u|^2 \, d\mathcal{H}^1 \\ &\stackrel{(3.27)}{\leq} (2\rho(1 - \arctan(1/L)))^2 \int_{A_\rho} |\partial_\tau u|^2 \, d\mathcal{H}^1, \end{aligned} \quad (3.29)$$

where \bar{u} denotes the average of u on A_ρ and $\partial_\tau u$ denotes the derivative of u in the tangential direction. Hence, by Hölder's and Young's inequality we get from (3.28) with $a := \bar{u}$ and (3.29)

$$\begin{aligned} &\int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} \\ &= \int_{A_\rho} (u - \bar{u}) \partial_\nu u \, d\mathcal{H}^1 \\ &\leq \left(\int_{A_\rho} (u - \bar{u})^2 \, d\mathcal{H}^1 \right)^{1/2} \left(\int_{A_\rho} |\partial_\nu u|^2 \, d\mathcal{H}^1 \right)^{1/2} \\ &\leq 2\rho(1 - \arctan(1/L)) \left(\int_{A_\rho} |\partial_\tau u|^2 \, d\mathcal{H}^1 \right)^{1/2} \left(\int_{A_\rho} |\partial_\nu u|^2 \, d\mathcal{H}^1 \right)^{1/2} \\ &\leq 2\rho(1 - \arctan(1/L)) \frac{1}{2} \int_{A_\rho} (|\partial_\tau u|^2 + |\partial_\nu u|^2) \, d\mathcal{H}^1 \\ &= \rho(1 - \arctan(1/L)) \int_{A_\rho} |\nabla u|^2 \, d\mathcal{H}^1. \end{aligned}$$

We let $F(\rho) := \int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x}$ and observe that the last estimate can be rewritten as $F(\rho) \leq \rho(1 - \arctan(1/L))F'(\rho)$. By integrating this inequality, we obtain for any $\rho \in (0, r_1)$,

$$F(\rho) \leq F(r_1) \left(\frac{\rho}{r_1} \right)^\alpha,$$

where $1/\alpha = (1 - \arctan(1/L)) < 1$. To conclude we observe that $F(r_1) \leq \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x}$ implies

$$\int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} = F(\rho) \leq \rho^\alpha F(r_1) r_1^{-\alpha} \leq \bar{C} \rho^\alpha.$$

□

Following [10, Th. 3.13], we can now prove that in the set $\{h > 0\}$ there are no corner points, and so $h \in C^1(\{h > 0\})$:

Lemma 3.24. *Let the pair (u, h) be a minimizer of (3.2). Then in the set $\{h > 0\}$ the height profile h is a C^1 function.*

Proof. To prove the lemma it is enough to show that there are no corner points. Let us argue by contradiction and assume that x_0 is a corner point. Then, by Corollary 3.22 and Lemma 3.23 there exist $r_1 > 0$ and $\alpha > 1$ such that for $\rho \in (0, r_1)$

$$\int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} \leq \bar{C} \rho^\alpha, \tag{3.30}$$

and $h|_{(x_0-r_1, x_0+r_1)}$ is a Lipschitz function with right and left derivatives, which are respectively right and left continuous. Moreover, x_0 being a corner point implies $h'_+(x_0) \neq h'_-(x_0)$.

First, we observe that by Lemma 3.21, $h'_-(x_0) < h'_+(x_0)$, and so $\varepsilon := (h'_+(x_0) - h'_-(x_0))/4 > 0$. Then (possibly by diminishing r_1), we can assume that for every $x \in (x_0, x_0 + r_1)$

$$\left| \frac{h(x) - h(x_0)}{x - x_0} - h'_+(x_0) \right| \leq \varepsilon, \tag{3.31}$$

and similarly for $x \in (x_0 - r_1, x_0)$

$$\left| \frac{h(x) - h(x_0)}{x - x_0} - h'_-(x_0) \right| \leq \varepsilon. \tag{3.32}$$

By Lemma 3.20, for any $\rho \in (0, r_1)$ there exist unique points $x_l, x_r \in S_\rho(\mathbf{z}_0) \cap \partial\Omega_h$ with $x_l < x_0$ and $x_r > x_0$. Using x_l and x_r we define \tilde{h} , a competitor for h by

$$\tilde{h} := \begin{cases} h(x) & \text{if } x \notin [x_l, x_r] \\ a(x) & \text{if } x \in [x_l, x_r], \end{cases}$$

where $a(x) := h(x_l) + \frac{x-x_l}{x_r-x_l}(h(x_r) - h(x_l))$ is an affine function which connects $(x_l, h(x_l))$ and $(x_r, h(x_r))$. Using (3.30) and the fact that $\partial\Omega_{\tilde{h}} \cap B_\rho(\mathbf{z}_0)$ is Lipschitz, we can extend u to $\Omega_{\tilde{h}}$ (still denoted by u) such that

$$\int_{B_\rho(\mathbf{z}_0) \cap \Omega_{\tilde{h}}} |\nabla u|^2 \, d\mathbf{x} \leq \bar{C}_1 \rho^\alpha, \quad (3.33)$$

where \bar{C}_1 depends only on \bar{C} and the Lipschitz constant of $h|_{(x_0-r_1, x_0+r_1)}$. Since (u, h) is a minimizer of (3.2), Proposition 3.17 implies

$$\int_{\mathbb{R}} (h'^2 - \tilde{h}'^2) \, dx \leq \int_{\Omega_{\tilde{h}} \setminus \Omega_h} |\nabla u|^2 \, d\mathbf{x} + \mu \int_{\mathbb{R}} (\tilde{h} - h) \, dx. \quad (3.34)$$

First we will estimate the left-hand side of (3.34) from below. The definition of \tilde{h} implies

$$\int_{\mathbb{R}} (h'^2 - \tilde{h}'^2) \, dx = \int_{x_l}^{x_0} h'^2 \, dx + \int_{x_0}^{x_r} h'^2 \, dx - \int_{x_l}^{x_r} \tilde{h}'^2 \, dx. \quad (3.35)$$

We set $d_l := \frac{h(x_0)-h(x_l)}{x_0-x_l}$ and $d_r := \frac{h(x_r)-h(x_0)}{x_r-x_0}$. Then

$$\int_{x_l}^{x_0} h'^2 \, dx + \int_{x_0}^{x_r} h'^2 \, dx \geq d_l^2(x_0 - x_l) + d_r^2(x_r - x_0), \quad (3.36)$$

and

$$\int_{x_l}^{x_r} \tilde{h}'^2 \, dx = \frac{(d_l(x_0 - x_l) + d_r(x_r - x_0))^2}{x_r - x_l}. \quad (3.37)$$

We plug (3.36) and (3.37) into (3.35) to get

$$\int_{x_l}^{x_r} (h'^2 - \tilde{h}'^2) \, dx \geq d_l^2(x_0 - x_l) + d_r^2(x_r - x_0) - \frac{(d_l(x_0 - x_l) + d_r(x_r - x_0))^2}{x_r - x_l}.$$

A simple algebraic manipulation shows that

$$\begin{aligned} & d_l^2(x_0 - x_l) + d_r^2(x_r - x_0) - \frac{(d_l(x_0 - x_l) + d_r(x_r - x_0))^2}{x_r - x_l} \\ &= \frac{(x_0 - x_l)(x_r - x_0)}{x_r - x_l} (d_r - d_l)^2, \end{aligned}$$

and so the estimates on d_l and d_r [see (3.31) and (3.32)] imply that

$$\begin{aligned} \int_{x_l}^{x_r} (h'^2 - \tilde{h}'^2) \, dx &\geq \frac{(x_0 - x_l)(x_r - x_0)}{x_r - x_l} (h'_+(x_0) - h'_-(x_0) - 2\varepsilon)^2 \\ &\geq 2 \min\{x_0 - x_l, x_r - x_0\} \varepsilon^2, \end{aligned} \quad (3.38)$$

where in the last step we used that $2ab/(a+b) \geq \min\{a, b\}$ and the definition of ε . Let us now observe that since h is Lipschitz, we have that $x_0 - x_l \geq C\rho$ and

$x_r - x_0 \geq C\rho$, where C depends only on the Lipschitz constant of $h|_{(x_0-r_1, x_0+r_1)}$. Therefore, for any $\rho \in (0, r_1)$ and the corresponding x_l, x_r we obtain that

$$\int_{x_l}^{x_r} (h'^2 - \tilde{h}'^2) \, dx \geq C\rho, \tag{3.39}$$

where $C > 0$ depends on $h|_{(x_0-r_1, x_0+r_1)}$, but not on ρ .

Finally, we observe that $|\int_{\mathbb{R}} (\tilde{h}-h) \, dx| = |\int_{x_l}^{x_r} (\tilde{h}-h) \, dx| \leq |B_\rho|$, and so (3.34), (3.33), and (3.39) imply that there exists $\alpha > 1$ and constants $C > 0, \bar{C}_1$ such that for every $\rho \in (0, r_1)$ we have $C\rho \leq \bar{C}_1\rho^\alpha + \mu\pi\rho^2$, which yields a contradiction for sufficiently small ρ . This concludes the proof. \square

We proved that in the set $\{h > 0\}$ the height profile h is a C^1 function. It then follows that in fact h is more regular. Indeed, for any x_0 such that $h(x_0) > 0$ we observe that given $\varepsilon > 0$ there exists $r_1 > 0$ such that for any $\rho \in (0, r_1)$

$$\mathcal{H}^1(S_\rho(\mathbf{z}_0) \cap \Omega_h) \leq (1 + \varepsilon)\pi\rho. \tag{3.40}$$

Then we can repeat the proof of Lemma 3.23 while replacing (3.27) by (3.40) to show the following result.

Lemma 3.25. *Let the pair (u, h) be a minimizer of (3.2), and let $x_0 \in \mathbb{R}$ be such that $h(x_0) > 0$. Then for any $0 < \alpha < 2$ there exists $r_1 > 0$ such that for any $\rho \in (0, r_1)$*

$$\int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} \leq \bar{C}\rho^\alpha,$$

where $\bar{C} = r_1^{-\alpha} \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x}$. Moreover, given $\alpha < 2$, the corresponding r_1 depends [through relation (3.40)] only on the modulus of continuity of h' in the neighborhood of x_0 .

Similar ideas as in the proof of Lemma 3.24 then give the following result.

Proposition 3.26. *Let the pair (u, h) be a minimizer of (3.2). Then for every $\beta \in (0, 1/2)$ the height profile $h|_{\{h>0\}}$ is a $C_{loc}^{1,\beta}$ function.*

Proof. Let $\bar{x} \in \mathbb{R}, h(\bar{x}) > 0$, and $\alpha \in (0, 2)$ be fixed. Then by Lemma 3.25 and the fact that $h \in C^1(\{h > 0\})$ there exist \mathcal{U} , a neighborhood of \bar{x} , a radius $r_1 > 0$, and a constant C_1 such that for any point $x \in \mathcal{U}$ and any $\rho \in (0, r_1)$ we have

$$\int_{B_\rho(\mathbf{z}) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} \leq C_1\rho^\alpha. \tag{3.41}$$

Moreover, we can assume that h' is bounded in \mathcal{U} .

Let $x_0 \in \mathcal{U}$ be fixed. Using standard extension argument and estimate (3.41) we can extend u to $B_\rho(\mathbf{z}_0) \setminus \Omega_h$ (the extension still denoted by u) such that

$\int_{B_\rho(\mathbf{z}_0)} |\nabla u|^2 \, d\mathbf{x} \leq \bar{C}_1 \rho^\alpha$, where \bar{C}_1 is independent of ρ and the choice of x_0 . Now let $x_1 > x_0$ be such that $|\mathbf{z}_0 - \mathbf{z}_1| < r_1$. We set $\rho := |\mathbf{z}_0 - \mathbf{z}_1|$ and define \tilde{h} by

$$\tilde{h}(x) := \begin{cases} h(x) & \text{if } x \notin [x_0, x_1] \\ a(x) & \text{if } x \in [x_0, x_1], \end{cases}$$

where $a(x) := h(x_0) + \frac{x-x_0}{x_1-x_0}(h(x_1) - h(x_0))$ is an affine function connecting \mathbf{z}_0 and \mathbf{z}_1 . We observe that $\Omega_{\tilde{h}} \subset \Omega_h \cup B_\rho(\mathbf{z}_0)$, and so $(\tilde{h}, u|_{\Omega_h \cup B_\rho})$ is a well defined competitor for (u, h) . Since (u, h) is a minimizer of (3.2), Proposition 3.17 implies

$$\int_{x_0}^{x_1} (h'^2 - \tilde{h}'^2) \, dx \leq \int_{B_\rho(\mathbf{z}_0) \setminus \Omega_h} |\nabla u|^2 \, d\mathbf{x} + \mu |B_\rho(\mathbf{z}_0)| \leq \bar{C}_1 \rho^\alpha + \mu \pi \rho^2 \leq \bar{C} \rho^\alpha.$$

Since h' is bounded in \mathcal{U} , we have that $x_1 - x_0 \geq C^{-1} \rho$, where C does not depend on ρ or x_1 . Then

$$\int_{x_0}^{x_1} \left(h'(x) - \int_{x_0}^{x_1} h' \right)^2 \, dx = \int_{x_0}^{x_1} (h'^2 - \tilde{h}'^2) \, dx \leq C \rho^{-1} \int_{x_0}^{x_1} (h'^2 - \tilde{h}'^2) \, dx,$$

and so

$$\int_{x_0}^{x_1} \left(h'(x) - \int_{x_0}^{x_1} h' \right)^2 \, dx \leq C \rho^{\alpha-1}. \tag{3.42}$$

A relation similar to (3.42) holds also for the choice $x_1 < x_0$, and we can use [1, Th. 7.51] to conclude that $h \in C^{1,(\alpha-1)/2}(\bar{\mathcal{U}})$. \square

We showed that $h \in C_{\text{loc}}^{1,\beta}(\{h > 0\})$ for any $\beta \in (0, 1/2)$, and so (u, h) satisfies all the assumptions of the following theorem (see [1, Th. 7.49]).

Theorem 3.27. *Let Ω be an open set in \mathbb{R}^2 , $g \in L^\infty(\Omega)$ and $u \in H^1(\Omega)$ be a solution of the Neumann problem*

$$\begin{aligned} -\Delta u &= g \quad \text{in } \Omega \\ \partial_\nu u &= 0 \quad \text{on } S. \end{aligned}$$

If $S \subset \partial\Omega$ is a $C^{1,\beta}$ curve relatively open in $\partial\Omega$, $\beta < 1$, then ∇u has a $C^{0,\beta}$ extension up to S .

We apply this theorem to show that for every $s > 0$, $|\nabla u|$ is $C^{0,\beta}$ in the neighborhood of $\partial\Omega_h \cap \{y > s\}$. As a consequence we obtain the following:

Proposition 3.28. *A minimizer (u, h) of (3.2) satisfies the Euler–Lagrange equation*

$$|\nabla u|^2(x, h(x)) - 2h''(x) = \lambda_V \quad \text{for almost every } x \in \{h > 0\}, \tag{3.43}$$

where $\lambda_V = \lambda(V)$ is a constant that depends on V , namely the Lagrange multiplier associated to the volume constraint.

Having (3.43), a simple bootstrap argument implies that in fact $h \in C^\infty(\{h > 0\})$ and $u \in C^\infty(\Omega_h)$ (see [10] for more details). If κ denotes the mean curvature of $\partial\Omega_h$, observing that (3.43) can be rewritten as

$$\kappa = \frac{|\nabla u|^2 - \lambda_V}{2} v_y^3,$$

we see that [23, Th. 3.1] implies that h is analytic in $\{h > 0\}$. To prove Theorem 3.18 it remains to show that h satisfies the zero contact angle condition. To do so we first derive a lemma analogous to Lemma 3.23 which applies to the points of contact with the substrate:

Lemma 3.29. *Let the pair (u, h) be a minimizer of (3.2), and let $x_0 \in \text{supp } h$ be such that $h(x_0) = 0$. Then $x_0 \in \partial \text{supp } h$ and there exist $r_0 > 0$ and \bar{C} such that for every $\rho \in (0, r_0)$*

$$\int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} \leq \bar{C} \rho^{4/3}. \tag{3.44}$$

Proof. First note that by Proposition 3.4, the set $\{h > 0\}$ is connected if (u, h) is a minimizer of (3.2). Therefore, if $h(x_0) = 0$ and $x_0 \in \text{supp } h$, then $x_0 \in \partial \text{supp } h$. Consequently, h vanishes in $(-\infty, x_0)$ or in (x_0, ∞) . Let us now assume that h vanishes in $(-\infty, x_0)$, the other case being symmetric. Then by Lemma 3.21 and [5, Lemma 3] there exists a radius $r_0 > 0$ such that $h|_{(x_0-r_0, x_0+r_0)}$ has left and right derivatives at every point, that are respectively left and right continuous (but could possibly attain infinite values). Moreover, we can assume (by possibly diminishing $r_0 > 0$) that for every $\rho \in (0, r_0)$, the graph of h intersects $S_\rho((x_0, 0))$ in exactly two points $(x_0 - \rho, 0)$ and $(x_r, h(x_r))$.

Let us fix $\rho \in (0, r_0)$ and the corresponding x_r . Since u minimizes the Dirichlet integral in Ω_h subject to boundary conditions $u(x, 0) = x$, we get that

$$\int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} \nabla u \cdot \nabla(u - x) \, d\mathbf{x} = \int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} (u - x) \partial_\nu u \, d\mathcal{H}^1.$$

Let $\delta := 1/5$. By Hölder’s and Young’s inequality the previous relation gives

$$\begin{aligned} \int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} &= \int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} (u - x) \partial_\nu u \, d\mathcal{H}^1 + \int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} \nabla u \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, d\mathbf{x} \\ &\leq \left(\int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} (\partial_\nu u)^2 \, d\mathcal{H}^1 \right)^{1/2} \left(\int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} (u - x)^2 \, d\mathcal{H}^1 \right)^{1/2} \\ &\quad + \delta \int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} + \frac{C}{\delta} |B_\rho(\mathbf{z}_0)|. \end{aligned} \tag{3.45}$$

We estimate the second integral on the right-hand side using Wirtinger’s inequality, which states that if $f(0) = 0$, then

$$\int_0^l f(t)^2 \, dt \leq \left(\frac{2l}{\pi} \right)^2 \int_0^l f'(t)^2 \, dt.$$

More precisely, we apply Wirtinger's inequality to $u(x, y) - x$ on $S_\rho(\mathbf{z}_0) \cap \Omega_h$ (observe that $u(x, 0) - x = 0$ if $x = 0$), which, together with Young's inequality, yields

$$\begin{aligned} \int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} (u - x)^2 \, d\mathcal{H}^1 &\leq \rho^2 \int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} |\partial_\tau(u - x)|^2 \, d\mathcal{H}^1 \\ &\leq \rho^2 \int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} (1 + \delta)(\partial_\tau u)^2 + (1 + \delta^{-1}) \, d\mathcal{H}^1, \end{aligned}$$

where τ denotes the tangent vector to S_ρ , and we used that $|\partial_\tau x| \leq 1$. Hence, using Young's inequality again and $|S_\rho(\mathbf{z}_0) \cap \Omega_h| \leq \rho\pi/2$, it follows from (3.45):

$$\begin{aligned} (1 - \delta) \int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} &\leq \frac{C}{\delta} \rho^2 + \frac{\rho}{2} \int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} \left[(\partial_\nu u)^2 \right. \\ &\quad \left. + (1 + \delta)(\partial_\tau u)^2 + (1 + \delta^{-1}) \right] \, d\mathcal{H}^1 \\ &\leq C\rho^2\delta^{-1} + \frac{\rho(1 + \delta)}{2} \int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathcal{H}^1. \end{aligned}$$

Since $\delta = 1/5$, one has $(1 - \delta)/(1 + \delta) = 2/3$, and so

$$\int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} \leq C_1\rho^2 + \frac{3}{4}\rho \int_{S_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathcal{H}^1.$$

If we denote $G(\rho) := \int_{B_\rho(\mathbf{z}_0) \cap \Omega_h} |\nabla u|^2 \, d\mathbf{x} + 2C_1\rho^2$, then the last relation is equivalent to

$$G(\rho) \leq \frac{3}{4}\rho G'(\rho).$$

Integrating this relation (the same way as we did in the end of the proof of Lemma 3.23) we obtain (3.44). \square

Proceeding along the lines of the proof of Lemma 3.24, we derive the following result from Lemma 3.29.

Lemma 3.30. *Let the pair (u, h) be a minimizer of (3.2). Let $x_0 \in \text{supp } h$ be such that $h(x_0) = 0$. Then $h'(x_0) = 0$, and so $h \in C^1(\mathbb{R})$.*

3.5. More Qualitative Results

It has been observed for various variational models for the epitaxial growth that minimizers are often not unique. For instance, for a model for periodic island formation, there is a regime of volumes and periods in which the flat configuration is not the only minimizing configuration (see [14, Theorem 2.13]). We do not obtain uniqueness of minimizers here, but part of the following result is the weaker statement that for almost every V , any two minimizers have the same surface and the same elastic energy. For a similar result for faceted islands see [12].

Proposition 3.31. *The function $F_s: V \mapsto F_s(V)$ is Lipschitz continuous with Lipschitz constant less than $C \min \{1, V^{-1/5}\}$. Moreover, if F_s is differentiable at V , then*

$$F'_s(V) = \lambda_V = \frac{1}{V} \left(E_V(u, h) + \frac{1}{2} S_V(h) \right), \tag{3.46}$$

for every minimizing pair (u, h) .

At such points of differentiability, if (u, h) and (\tilde{u}, \tilde{h}) are two different minimizers of (3.2), then $E_V(u, h) = E_V(\tilde{u}, \tilde{h})$ and $S_V(h) = S_V(\tilde{h}) =: S_V$. Finally, there holds,

$$\overline{\lim}_{\varepsilon \rightarrow 0^+} \frac{F_s(V + \varepsilon) + F_s(V - \varepsilon) - 2F_s(V)}{\varepsilon^2} \leq -\frac{S_V}{4V^2}.$$

Proof. We already know from Proposition 3.3 that F_s is locally Lipschitz continuous. The estimate on the Lipschitz constant can be obtained by two different ways. The first approach is to use Proposition 3.17, and test for two volumes V and W the minimization problem F_s^μ with the minimizers for each of these volumes. Another approach is to compute the derivative of F_s directly. For this, we see that for $\varepsilon > 0$, using the rescaling argument (see Lemma 2.1),

$$F_s(V + \varepsilon) - F_s(V) \leq (1 + \varepsilon/V)E_V + (1 + \varepsilon/V)^{1/2}S_V - E_V - S_V,$$

where E_V and S_V are the elastic and surface energy of the minimizer, respectively. If F_s is differentiable at V , this implies by Proposition 3.9 that $F'_s(V) \leq \frac{1}{V}(E_V + \frac{1}{2}S_V) \leq F_s(V)/V \leq c_0^{-1} \min \{1, V^{-1/5}\}$ with $c_0 > 0$ from Proposition 3.9. Similarly, by rescaling from V to $V - \varepsilon$, we find by Lemma 3.1 that

$$\frac{F_s(V - \varepsilon) - F_s(V)}{-\varepsilon} \geq \frac{1}{\varepsilon}(E_V + S_V - (1 - \varepsilon/V)E_V - (1 - \varepsilon/V)^{1/2}S_V),$$

and thus $F'_s(V) \geq \frac{E_V + \frac{1}{2}S_V}{V} \geq \frac{c_0}{2} \min \{1, V^{-1/5}\}$, which implies $F'_s(V) = \frac{1}{V}(E_V + \frac{1}{2}S_V)$. Moreover, this also implies that two minimizers for volume V have the same elastic and the same surface energy. The same rescaling argument also gives the bound

$$\overline{\lim}_{\varepsilon \rightarrow 0^+} \frac{F_s(V + \varepsilon) + F_s(V - \varepsilon) - 2F_s(V)}{\varepsilon^2} \leq -\frac{1}{4V^2}.$$

It remains to show that

$$\lambda_V V = F_s(V) - \frac{1}{2} S_V \tag{3.47}$$

(where λ_V is the Lagrange multiplier) for $V > \bar{V}$. We test the Laplace equation for u in Ω_h with the function $y\partial_y u$ to find

$$0 = \int_{\Omega_h} -\operatorname{div} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} (y\partial_y u) \, d\mathbf{x} = \int_{\Omega_h} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} \cdot \begin{pmatrix} y\partial_{xy} u \\ y\partial_{yy} u + \partial_y u \end{pmatrix} \, d\mathbf{x}.$$

Using integration by parts we obtain

$$\begin{aligned} 2 \int_{\Omega_h} \partial_x u (y \partial_{xy} u) \, d\mathbf{x} &= - \int_{\Omega_h} (\partial_x u)^2 \, d\mathbf{x} + \int_{\partial\Omega_h} (\partial_x u)^2 y \nu_y \, d\mathbf{x}, \quad \text{and} \\ 2 \int_{\Omega_h} \partial_y u (y \partial_{yy} u) \, d\mathbf{x} &= - \int_{\Omega_h} (\partial_y u)^2 \, d\mathbf{x} + \int_{\partial\Omega_h} (\partial_y u)^2 y \nu_y \, d\mathbf{x}, \end{aligned}$$

which together with the previous relation imply

$$\int_{\Omega_h} \left[(\partial_x u)^2 - (\partial_y u)^2 \right] \, d\mathbf{x} = \int_{\partial\Omega_h} |\nabla u|^2 y \nu_y \, d\mathcal{H}^1.$$

Using the Euler–Lagrange equation, the right-hand side can be written as $-2 \int_{\mathbb{R}} h'^2 \, dx + \lambda_V \int_{\mathbb{R}} h \, dx$, which implies

$$\int_{\Omega} \left[(\partial_x u)^2 - (\partial_y u)^2 \right] \, d\mathbf{x} = -2S_V + \lambda_V V.$$

From this and Lemma 3.5, we finally obtain (3.47). From (3.47) we get that if F is differentiable, then $F'(V) = \lambda_V$. \square

We can now use this information to study the compactness properties of minimizers.

Proposition 3.32. *Let $V > \bar{V}$. Then for every minimizer (u, h) of (3.2), the height function h has bounded support, and $\mathcal{H}^1(\text{supp } h) \leq \frac{\lambda_V S_V}{1 - \lambda_V} \lesssim V^{3/5}$.*

Proof. Let us first prove that any minimizer of (3.2) is compactly supported. For the sake of contradiction, assume it is not. Since

$$\int_{\mathbb{R}} h'^2 \, dx + \int_{\mathbb{R}} \left(\int_0^{h(x)} |\nabla u|^2(x, y) \, dy \right) \, dx < +\infty,$$

for any $\varepsilon > 0$ and $K > 0$ there exist $x_1 < x_2$ such that $x_2 - x_1 \geq K$ and

$$|h'(x_i)| + \int_0^{h(x_i)} |\nabla u|^2(x_i, y) \, dy < \varepsilon, \quad i = 1, 2. \quad (3.48)$$

Using

$$u(x_2, h(x_2)) - u(x_1, h(x_1)) = \int_{x_1}^{x_2} \nabla u(x, h(x)) \cdot \begin{pmatrix} 1 \\ h'(x) \end{pmatrix} \, dx$$

we find

$$\begin{aligned} |u(x_1, h(x_1)) - u(x_2, h(x_2))| &\leq \int_{x_1}^{x_2} |\nabla u(x, h(x))| \sqrt{1 + h'^2(x)} \, dx \\ &\stackrel{\text{H\"older}}{\leq} \left(\int_{x_1}^{x_2} |\nabla u|^2(x, h(x)) \, dx \right)^{1/2} \left(\int_{x_1}^{x_2} (1 + h'^2(x)) \, dx \right)^{1/2}. \end{aligned}$$

We use the Euler–Lagrange equation (3.43) to replace the first term on the right-hand side of the previous relation, and get

$$\begin{aligned}
 & |u(x_1, h(x_1)) - u(x_2, h(x_2))| \\
 & \leq \left(\int_{x_1}^{x_2} (\lambda_V + 2h''(x)) \, dx \right)^{1/2} \left(\int_{x_1}^{x_2} (1 + h'^2(x)) \, dx \right)^{1/2} \\
 & \leq (\lambda_V(x_2 - x_1) + 2|h'(x_1) - h'(x_2)|)^{1/2} (x_2 - x_1 + S_V)^{1/2} \\
 & \stackrel{(3.48)}{\leq} (\lambda_V(x_2 - x_1) + 2\varepsilon)^{1/2} (x_2 - x_1 + S_V)^{1/2}.
 \end{aligned} \tag{3.49}$$

Finally, since $u(x, h(x)) = x + \int_0^{h(x)} \partial_y u(x, y) dy$, we have

$$\begin{aligned}
 |u(x_1, h(x_1)) - u(x_2, h(x_2))| & \geq |x_1 - x_2| - \int_0^{h(x_1)} |\partial_y u(x_1, y)| dy \\
 & \quad - \int_0^{h(x_2)} |\partial_y u(x_2, y)| dy \\
 & \stackrel{(3.48)}{\geq} |x_1 - x_2| - 2\varepsilon^{1/2} (\sup h)^{1/2},
 \end{aligned}$$

so that

$$x_2 - x_1 \leq (\lambda_V(x_2 - x_1) + 2\varepsilon)^{1/2} (x_1 - x_2 + S_V)^{1/2} + 2\varepsilon^{1/2} \sup h^{1/2}.$$

Since $\lambda_V < 1$ and $x_2 - x_1 \geq K$, we get a contradiction for K large enough. We thus see that the support of h must be bounded.

To prove the estimate on the size of the support of h , take $x_1 < x_2$ on the boundary of the support. From the zero contact angle condition (see Lemma 3.30) we get $h'(x_i) = 0$. Since $u(x_i, h(x_i)) = x_i$, (3.49) with $\varepsilon = 0$ implies $x_2 - x_1 \leq \lambda_V(x_2 - x_1 + S_V)$. Note that by Proposition 3.31 and the scaling law, $\lambda_V \leq \frac{V^{4/5}}{c_0 V}$. Putting things together, we get

$$x_2 - x_1 \leq \frac{\lambda_V S_V}{1 - \lambda_V} \lesssim V^{3/5}.$$

□

Remark 3.33. The bound on the size of support of h derived in Proposition 3.32 is slightly suboptimal since we expect from the proof of the scaling law (see Proposition 3.9) that $\mathcal{H}^1(\text{supp } h) \sim V^{2/5}$.

Using Proposition 3.32, we can prove the existence of a minimizer at the critical volume.

Corollary 3.34. *There exists a minimizer of (3.2) for $V = \bar{V}$.*

Proof. The existence of a minimizer will follow from a general estimate on λ_V . From (3.47) we know that for $V > \bar{V}$ we have $V > F_s(V) = \lambda_V V + S_V/2$, from which we get

$$\lambda_V < 1 - \frac{1}{2} \frac{S_V}{V}. \tag{3.50}$$

To show that λ_V is bounded away from 1 as $V \rightarrow \bar{V}$, it is enough to show that S_V/V does not tend to zero as $V \rightarrow \bar{V}$. Let us fix $V > \bar{V}$. Then, from (3.9), (3.10) and $F_s(V) \leq V$ we get that $V \geq V + S_V - \sqrt{\frac{3}{4}} S_V \sqrt{\sup h}$, which implies that $\sup h > \frac{4}{3}$. By Remark 3.8 we have $\sup h \leq (9/16)^{1/3} V^{1/3} S_V^{1/3}$, and thus

$$\frac{S_V}{V} > \left(\frac{4}{3}\right)^3 \frac{16}{9} V^{-2}.$$

We thus get with (3.50)

$$\lambda_V \leq \frac{16 (\sup h)^2}{9 V^2} \leq 1 - \frac{2^9}{3^5} V^{-2}$$

for any $V > \bar{V}$. By Proposition 3.32, the size of the support of h for any minimizer (u, h) (for any $V > \bar{V}$) is bounded by $\frac{\lambda_V}{1-\lambda_V} S_V$. Hence, the limit of minimizers as $V \rightarrow \bar{V}$ exists. This limit is a minimizer of $F_s(\bar{V})$ (as a consequence of a simple Γ -limit type argument along the lines of Proposition 3.6). \square

3.6. Asymptotic Analysis

For every $V > \bar{V}$ and every $(u, h) \in \mathcal{A}_V$ with u the minimizer of the Dirichlet energy in Ω_h subject to the boundary condition, we define an (anisotropically) rescaled height profile $\tilde{h}(x) := V^{-3/5} h(V^{2/5}x)$ and a rescaled energy

$$G_V(\tilde{h}) := V^{-4/5} (S_V(h) + E_V(u, h)) = \int_{\mathbb{R}} \tilde{h}'^2 dx + V^{-4/5} \int_{\Omega_h} |\nabla u|^2 dx.$$

Observe that the rescaled \tilde{h} satisfies $\int_{\mathbb{R}} \tilde{h} dx = 1$.

Theorem 3.35. *For every sequence $V_n \rightarrow +\infty$ and every minimizer (u_{V_n}, h_{V_n}) of (3.2), the corresponding \tilde{h}_{V_n} (possibly translated) converge, up to a subsequence, in $L^\infty(\mathbb{R})$ to some function h , which minimizes the functional*

$$G(h) := \left(\inf_{u(x,0)=x} \int_{\{h>0\} \times [0,+\infty)} |\nabla u|^2 dx \right) + \int_{\mathbb{R}} h'^2 dx \tag{3.51}$$

under the constraint $\int_{\mathbb{R}} h dx = 1$.

Proof. We start by noticing that Proposition 3.9 and (3.7) give $V^{3/5} \lesssim \max h \lesssim V^{1/3} S_V^{1/3}$, which implies $S_V \gtrsim V^{4/5}$. If $F_s(V)$ is differentiable at V , then Proposition 3.31 gives $\beta'(V) = \left(\frac{F_s(V)}{V}\right)' = -\frac{S_V}{2V^2}$. This together with $S_V \gtrsim V^{4/5}$ implies $\beta'(V) \leq -CV^{-6/5}$, and so

$$\beta(V) \leq \beta(V/2) - C \int_{V/2}^V t^{-6/5} dt = \beta(V/2) - CV^{-1/5}(2^{1/5} - 1). \tag{3.52}$$

Since $G_{V_n}(\tilde{h}_{V_n}) = V_n^{-4/5}(S_{V_n}(h_{V_n}) + E_{V_n}(u_{V_n}, h_{V_n}))$ and by the scaling law (see Proposition 3.9) $F_s(V_n) = S_{V_n}(h_{V_n}) + E_{V_n}(u_{V_n}, h_{V_n}) \lesssim V_n^{4/5}$, we see that $G_{V_n}(\tilde{h}_{V_n}) \leq C$.

Let us prove that (after possible translation) the sequence \tilde{h}_{V_n} is tight. For this we follow the argument from the proof of Lemma 3.15. Since $\int_{\mathbb{R}} \tilde{h}_{V_n} dx = 1$ and $\int_{\mathbb{R}} \tilde{h}_{V_n}^2 dx \leq C$ (independently of V_n), for every $\varepsilon > 0$ there exists $l = l(\varepsilon)$ such that in every interval of length at least l we can cut the profile \tilde{h}_V in two parts of volume α_1 and α_2 , respectively, with $1 - (\alpha_1 + \alpha_2) \leq \varepsilon$ (for the precise construction we refer to the proof of Lemma 3.15). Moreover, we can assume that the cost (surface energy) of the cut is bounded by a multiple of ε . Then we get

$$\beta(V_n)V_n^{1/5} + C\varepsilon = G_{V_n}(\tilde{h}_{V_n}) + C\varepsilon \geq V_n^{1/5}(\alpha_1\beta(V_n\alpha_1) + \alpha_2\beta(V_n\alpha_2)),$$

and thus

$$\begin{aligned} \varepsilon(C + V_n^{1/5}\beta(V_n)) &\geq V_n^{1/5}(\alpha_1(\beta(V_n\alpha_1) - \beta(V_n)) + \alpha_2(\beta(V_n\alpha_2) - \beta(V_n))) \\ &\geq V_n^{1/5}(\alpha_1(\beta(V_n/2) - \beta(V_n))), \end{aligned}$$

where we used that β is non-increasing and $\alpha_1 < 1/2$. Using that $V_n^{1/5}\beta(V_n) \leq C$ and (3.52) we find

$$\alpha_1 \leq C\varepsilon,$$

which by the same argument as in the proof Proposition 3.16 implies tightness of (possibly shifted) \tilde{h}_{V_n} . As a consequence of tightness we get $L^1(\mathbb{R})$ convergence of (possibly a subsequence of) \tilde{h}_{V_n} to h . Using the compact embedding (on bounded domains) of H^1 into L^∞ we get locally uniform convergence of \tilde{h}_{V_n} to h . Moreover, tightness of the sequence \tilde{h}_{V_n} implies that outside of a compact set we can use a half-line version of (3.7) to show that \tilde{h}_{V_n} is uniformly small there. These two facts together yield uniform convergence.

Let us now show that

$$\lim_{V_n \rightarrow +\infty} G_{V_n}(\tilde{h}_{V_n}) \geq G(h).$$

Since the surface energy is lower semicontinuous, to prove the previous relation it is enough to prove the inequality for the elastic part of the energy. For every $\varepsilon > 0$ and for V_n large enough we can assume that $\{h > \varepsilon\} \subset \{\tilde{h}_{V_n} > \varepsilon/2\}$, and so

$$\begin{aligned} \int_{\Omega_{hV_n}} |\nabla u_{V_n}|^2 \, d\mathbf{x} &\geq \int_{\Omega_{hV_n} \cap [\{hV_n > \varepsilon V_n^{3/5}/2\} \times [0, \infty)]} |\nabla u_{V_n}|^2 \, d\mathbf{x} \\ &\geq \int_{\{h > \varepsilon V_n^{3/5}\} \times [0, \varepsilon V_n^{3/5}]} |\nabla u_{V_n}|^2 \, d\mathbf{x}. \end{aligned}$$

Using the change of variables $x = V_n^{2/5} \hat{x}$, $y = V_n^{2/5} \hat{y}$, and $u_{V_n} = V_n^{-2/5} \hat{u}_{V_n}$ we find

$$V_n^{-4/5} \int_{\Omega_{hV_n}} |\nabla u_{V_n}|^2 \, d\mathbf{x} \geq \int_{\{h > \varepsilon/2\} \times [0, \varepsilon V_n^{1/5}]} |\nabla \hat{u}_{V_n}|^2 \, d\mathbf{x}.$$

Since for any interval $I \subset \mathbb{R}$

$$\lim_{L \rightarrow \infty} \min_{u(x,0)=x} \int_{I \times [0, L]} |\nabla u|^2 \, d\mathbf{x} = \min_{u(x,0)=x} \int_{I \times [0, \infty)} |\nabla u|^2 \, d\mathbf{x},$$

we obtain

$$\varliminf_{n \rightarrow +\infty} V_n^{-4/5} \int_{\Omega_{hV_n}} |\nabla u_{V_n}|^2 \, d\mathbf{x} \geq \min_{u(x,0)=x} \int_{\{h > \varepsilon/2\} \times [0, +\infty)} |\nabla u|^2 \, d\mathbf{x}.$$

By letting $\varepsilon \rightarrow 0$ we obtain the desired lower bound.

For any other admissible function g , it is easily seen that

$$\overline{\lim}_{V_n \rightarrow +\infty} G_{V_n}(g) \leq G(g).$$

By a classical argument of Γ -convergence (see [4]), we deduce that h is a minimizer of G . \square

Remark 3.36. Notice that

$$G(h) := C_W \sum_{i \in \mathbb{N}} (b_i - a_i)^2 + \int_{\mathbb{R}} h^2 \, dx,$$

where the intervals (a_i, b_i) are the connected components of $\{h > 0\}$, and C_W is a constant defined by

$$C_W := \inf \left\{ \int_{[0,1] \times [0,+\infty)} |\nabla u|^2 \, d\mathbf{x} : u \in H_{\text{loc}}^1([0,1] \times [0, \infty)), u(x,0) = x \right\}. \quad (3.53)$$

We now study the limiting problem.

Proposition 3.37. *The minimization problem (3.51) admits (up to translations) a unique minimizer \bar{h} given by*

$$\bar{h}(x) := \begin{cases} \frac{3}{2} \ell^{-3} (\ell^2 - x^2) & \text{if } x \in [-\ell, \ell] \\ 0 & \text{if } x \notin [-\ell, \ell], \end{cases}$$

where $\ell := \left(\frac{9}{16C_W} \right)^{1/5}$.

Proof. The existence of a minimizer follows either by Theorem 3.35 or by the following direct argument, which is similar in spirit to [18, Proposition 4.5]. Let h be an admissible function. If S_i denotes the surface energy of the i -th connected component of h , ℓ_i its length, and V_i its volume (assuming V_i are in a non-increasing order), we observe that $h(x) = \int_{a_i}^x h' dx$ for $x \in [a_i, b_i]$, and so $V_i \leq \ell_i^{3/2} S_i^{1/2}$. We sum this inequality for $i \geq n$ and apply Hölder's inequality to get

$$\sum_{i \geq n} V_i \leq \left(\sum_{i \geq n} \ell_i^3 \right)^{1/2} \left(\sum_{i \geq n} S_i \right)^{1/2}.$$

Since $\sum_{i \in \mathbb{N}} C_W \ell_i^2 + S_i = G(h) \leq C$, we get that $n \ell_n^2 \leq C$, that is, $\ell_n \leq C n^{-1/2}$.

We deduce $\sum_{i \geq n} V_i \leq C n^{-1/2}$, which shows tightness of a minimizing sequence.

Let now \bar{h} be a minimizer. Then in each of its connected components $[c_i - \ell_i, c_i + \ell_i]$, \bar{h} satisfies $\bar{h}'' = -\lambda_i$, and so $\bar{h} = -\frac{\lambda_i}{2}((x - c_i)^2 - \ell_i^2)$. Since $\int_{c_i - \ell_i}^{c_i + \ell_i} \bar{h} dx = V_i$, the form of \bar{h} implies $\lambda_i = \frac{3V_i}{2\ell_i^3}$. Then by direct computation the energy inside $[c_i - \ell_i, c_i + \ell_i]$ equals

$$4C_W \ell_i^2 + \frac{2}{3} \lambda_i^2 \ell_i^3 = 4C_W \ell_i^2 + \frac{3V_i^2}{2\ell_i^3}.$$

This expression is minimized (under the constraint of volume V_i) by $\ell_i = \left(\frac{9}{16C_W}\right)^{1/5} V_i^{2/5}$. Then the total energy $G(\bar{h})$ is

$$\sum_i \left(324 \cdot C_W^3\right)^{1/5} V_i^{4/5}$$

with the constraint $\sum V_i = 1$. Thus, this energy is minimized by a single island. □

Remark 3.38. By the uniqueness of the minimizer of G , we see that the whole sequence \tilde{h}_V (possibly translated) converges in L^∞ to \bar{h} .

We will finally prove the exponential rate of convergence of optimal profiles. For that, we need the following quantitative inequality, which can be considered as a very simple quantitative isoperimetric inequality (see [6, 13]).

Proposition 3.39. *Let $L > 0$, $V > 0$, and let $h_{\min} \in H^1(\mathbb{R})$ be the minimizer of $\int_{-L}^L h^2 dx$ under the constraints $h(-L) = h(L) = 0$ and $\int_{-L}^L h dx = V$. Then for every other $h \in H^1(\mathbb{R})$ satisfying the same constraints,*

$$\int_{-L}^L h^2 dx - \int_{-L}^L h_{\min}^2 dx \geq \frac{1}{4L^2} \int_{-L}^L |h - h_{\min}|^2 dx. \tag{3.54}$$

Proof. Let L , V , and h_{\min} be as in the statement. Then, for every competitor h we write $h = (h - h_{\min}) + h_{\min}$, and so

$$\int_{-L}^L h'^2 dx = \int_{-L}^L (h - h_{\min})'^2 dx + \int_{-L}^L h_{\min}'^2 dx + 2 \int_{-L}^L h'_{\min} (h - h_{\min})' dx.$$

Since h''_{\min} is constant and $\int_{-L}^L h dx = \int_{-L}^L h_{\min} dx$, integration by parts implies

$$\int_{-L}^L h'^2 dx = \int_{-L}^L (h - h_{\min})'^2 dx + \int_{-L}^L h_{\min}'^2 dx.$$

For $x \in [-L, L]$ we have

$$|h(x) - h_{\min}(x)|^2 \leq \left(\int_{-L}^x |h' - h'_{\min}| dx \right)^2 \leq 2L \int_{-L}^L (h - h_{\min})'^2 dx,$$

and so by integration we obtain

$$\begin{aligned} \int_{-L}^L |h - h_{\min}|^2 dx &\leq 4L^2 \int_{-L}^L (h - h_{\min})'^2 dx \\ &= 4L^2 \left(\int_{-L}^L h'^2 dx - \int_{-L}^L h_{\min}'^2 dx \right), \end{aligned}$$

which shows the claim. \square

We now prove the exponential convergence of \tilde{h}_V to a truncated parabola. To state our result, we will need the following notation. Let $V > \bar{V}$ and \tilde{h}_V , a minimizer of G_V , be fixed. Then for $s > 0$, we let \tilde{I}_s be the largest connected component of $\{\tilde{h}_V > s\}$ and \bar{h}_s be the minimizer of $\int_{\tilde{I}_s} h'^2 dx$ with the constraint $\int_{\tilde{I}_s} h dx = \int_{\tilde{I}_s} \tilde{h}_V dx$ and $h = \tilde{h}_V$ on the boundary of \tilde{I}_s .

Proposition 3.40. *Let $\varepsilon > 0$. Then there exist constants $C_0 = C_0(\varepsilon)$ and $C_1 = C_1(\varepsilon)$ such that for every $V > \bar{V}$ and for every minimizer \tilde{h}_V of G_V ,*

$$\|\tilde{h}_V - \bar{h}_s\|_{L^2(\tilde{I}_s)} \leq C_0 \exp(-C_1 V^{1/5}) \quad \forall s \geq \varepsilon.$$

Proof. Let $\varepsilon > 0$, $V > \bar{V}$, and \tilde{h}_V be as in the statement. If V is large enough, then

$$\|\tilde{h}_V - \bar{h}\|_{L^\infty} \leq \varepsilon/2.$$

We observe that for any $s \geq \varepsilon$, this implies $\{\tilde{h}_V > s\} \subset \{\bar{h} > \varepsilon/2\}$, and so $\mathcal{H}^1(\tilde{I}_s) \leq \mathcal{H}^1(\{\bar{h} > \varepsilon/2\}) \leq \mathcal{H}^1(\text{supp } \bar{h}) = C$ for any $s \geq \varepsilon$. From this follows that if I_s denotes the largest connected component of $\{h_V > s\}$, where h_V is obtained by the inverse rescaling of \tilde{h}_V , then

$$\mathcal{H}^1(I_s) \leq CV^{2/5} \quad \forall s \geq \varepsilon V^{3/5}. \quad (3.55)$$

Now we claim that for some $t \in [2\varepsilon V^{3/5}, 3\varepsilon V^{3/5}]$, we have

$$\begin{aligned} & \|u_V(\cdot, t)\|_{\dot{H}^{1/2}(I_t \times \{t\})}^2 \\ & := \min_{v(\cdot, t) = u_V(\cdot, t)} \int_{I_t \times [t, +\infty)} |\nabla v|^2 \, d\mathbf{x} \lesssim V^{4/5} \varepsilon^{-1} \exp(-C\varepsilon V^{1/5}). \end{aligned} \quad (3.56)$$

Indeed, fix $s \geq \varepsilon V^{3/5}$. Then since u_V is the minimizer of the Dirichlet energy, it satisfies the Laplace equation with Neumann boundary conditions at the upper part of the boundary. Denoting by \bar{u}_V the average value of u_V on $I_s \times \{s\}$, $\Omega_V^s := \Omega_{\tilde{h}_V} \cap (I_s \times [s, +\infty))$, and using Hölder's and Poincaré's inequalities, we get

$$\begin{aligned} \int_{\Omega_V^s} |\nabla u_V|^2 \, d\mathbf{x} &= \int_{I_s \times \{s\}} (u_V - \bar{u}_V) \partial_y u_V \, dx \\ &\leq \left(\int_{I_s \times \{s\}} |u_V - \bar{u}_V|^2 \, dx \right)^{1/2} \left(\int_{I_s \times \{s\}} (\partial_y u_V)^2 \, dx \right)^{1/2} \\ &\leq \frac{\mathcal{H}^1(I_s)}{\pi} \left(\int_{I_s \times \{s\}} (\partial_x u_V)^2 \, dx \right)^{1/2} \left(\int_{I_s \times \{s\}} (\partial_y u_V)^2 \, dx \right)^{1/2} \\ &\leq \frac{\mathcal{H}^1(I_s)}{2\pi} \int_{I_s \times \{s\}} |\nabla u_V|^2 \, dx \stackrel{(3.55)}{\leq} C V^{2/5} \int_{I_s \times \{s\}} |\nabla u_V|^2 \, dx. \end{aligned}$$

Since for $F(s) := \int_{\Omega_V^s} |\nabla u_V|^2 \, d\mathbf{x}$, $s \geq \varepsilon V^{3/5}$, this is equivalent to $F(s) \leq -C V^{2/5} F'(s)$, an integration in s yields that for $s \geq \varepsilon V^{3/5}$,

$$\begin{aligned} \int_{\Omega_V^s} |\nabla u_V|^2 \, d\mathbf{x} &\leq \int_{\Omega_h} |\nabla u_V|^2 \, d\mathbf{x} \cdot \exp\left(-C \frac{s - \varepsilon V^{3/5}}{V^{2/5}}\right) \\ &\leq V \exp\left(-C \frac{s - \varepsilon V^{3/5}}{V^{2/5}}\right). \end{aligned}$$

In particular, the previous relation with $s = 2\varepsilon V^{3/5}$ implies

$$\int_{2\varepsilon V^{3/5}}^{3\varepsilon V^{3/5}} \int_{I_s} |\nabla u_V(x, s)|^2 \, dx \, ds \leq V \exp(-C\varepsilon V^{1/5}),$$

and so there exists $t \in [2\varepsilon V^{3/5}, 3\varepsilon V^{3/5}]$ such that

$$\int_{I_t \times \{t\}} |\nabla u_V|^2 \, d\mathbf{x} \leq V^{2/5} \varepsilon^{-1} \exp(-C\varepsilon V^{1/5}).$$

Finally, we use Wirtinger's inequality

$$\|u_V - \bar{u}_V\|_{L^2(I_t \times \{t\})} \lesssim \mathcal{H}^1(I_t) \cdot \|u_V\|_{\dot{H}^1(I_t \times \{t\})},$$

where \bar{u}_V denotes the average of u_V on $(I_t \times \{t\})$, together with

$$\|u_V\|_{\dot{H}^{1/2}(I_t \times \{t\})}^2 \leq \|u_V - \bar{u}_V\|_{L^2(I_t \times \{t\})} \cdot \|u_V\|_{\dot{H}^1(I_t \times \{t\})}$$

(see, for example [7, Eq. (9)]), to get $\|u_V\|_{\dot{H}^{1/2}(I_t \times \{t\})} \lesssim \mathcal{H}^1(I_t)^{1/2} \cdot \|u_V\|_{\dot{H}^1(I_t \times \{t\})}$ and thus (3.56).

As a final step of the proof, for $s \geq 3\varepsilon$ we want to construct (\tilde{u}, \tilde{h}) , a competitor for $(\tilde{u}_V, \tilde{h}_V)$. Outside of \tilde{I}_s let $\tilde{h} := \tilde{h}_V$, and in \tilde{I}_s let $\tilde{h} := \tilde{h}_s$. Take then \tilde{u} to be equal to u_V outside of Ω_V^t and to the restriction of a minimizer of

$$\min_{u(\cdot, t) = u_V(\cdot, t)} \int_{I_t \times [t, +\infty)} |\nabla u|^2 \, dx$$

elsewhere. By minimality of (\tilde{h}_V, u_V) we infer that

$$\int_{\tilde{I}_s} \tilde{h}_V^2 \, dx + V^{-4/5} \int_{\Omega_V^s} |\nabla u_V|^2 \, dx \leq \int_{\tilde{I}_s} \tilde{h}^2 \, dx + V^{-4/5} \|u_V(\cdot, t)\|_{\dot{H}^{1/2}(I_t \times \{t\})}^2,$$

hence

$$\int_{\tilde{I}_s} (\tilde{h}_V^2 - \tilde{h}^2) \, dx \leq V^{-4/5} \|u_V(\cdot, t)\|_{\dot{H}^{1/2}(I_t \times \{t\})}^2 \stackrel{(3.56)}{\lesssim} \varepsilon^{-1} \exp(-C\varepsilon V^{1/5}).$$

Using Proposition 3.39 we conclude the proof. \square

4. The Large Slope Approximation

If at large volume, a compact island forms, then $|h'|$ is expected to be large on its support. In this case, the small slope approximation $\sqrt{1 + h'^2} - 1 \sim h'^2$ from Section 3 might not be appropriate, and we rather work with the large slope approximation

$$\sqrt{1 + h'^2} - 1 \sim |h'|, \tag{4.1}$$

that is, we consider now for $h \in W^{1,\infty}(\mathbb{R})$ and $u \in H_{\text{loc}}^1(\Omega_h)$ the functional

$$\int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}} |h'|. \tag{4.2}$$

Note that this approximation comes along with a loss of regularity since for low-energy configurations, h is no longer bounded in $H^1(\mathbb{R})$ but only in $BV(\mathbb{R})$. Hence, we consider the relaxation of the energy as determined in the case of compact support in [10]. We follow the notation of [3, 10, 18]. If $h: \mathbb{R} \rightarrow [0, +\infty)$ is lower semicontinuous, then we denote the pointwise variation of h by

$$\text{Var } h := \sup \left\{ \sum_{i=1}^n |h(x_i) - h(x_{i-1})| : x_0 < \dots < x_n \right\},$$

and set

$$\int_{\mathbb{R}} |h'| := \inf \{ \text{Var } g : g = h \text{ almost everywhere} \}.$$

If $\int_{\mathbb{R}} |h'|$ is finite then h is said to be of bounded pointwise variation (see [1]). For a function h of bounded pointwise variation (BV), set

$$h^-(x) := \min \{h(x^+), h(x^-)\} = \liminf_{z \rightarrow x} h(z),$$

$$h^+(x) := \max \{h(x^+), h(x^-)\} = \limsup_{z \rightarrow x} h(z),$$

where $h(x^\pm) := \lim_{z \rightarrow x^\pm} h(z)$. We denote by Γ_{cuts} the at most countable collection of vertical cuts,

$$\Gamma_{\text{cuts}} := \{(x, y) : x \in S(h), h(x) \leq y \leq h^-(x)\}$$

where $S(h) := \{x : h(x) < h^-(x)\}$. For h lower semicontinuous and of bounded pointwise variation, we set

$$E_V(u, h) := \int_{\Omega_h} |\nabla u|^2 \, dx \quad \text{and} \quad S_V^{(\ell)}(h) := \int_{\mathbb{R}} |h'| + 2\mathcal{H}^1(\Gamma_{\text{cuts}}).$$

Note that $\mathcal{H}^1(\Gamma_{\text{cuts}}) = 0$ for locally Lipschitz functions h . If for a sequence of bounded energy we restrict ourselves to a compact set, we are in the situation of [10, 18], and we obtain a local compactness result by [10, Lemma 2.1, Proposition 2.2, and Theorem 2.8]. The result becomes a global result, if we have strong L^1 -convergence of $\{h_n\}$, which, in turn, follows from tightness.

Proposition 4.1. *Assume (u_n, h_n) is admissible for (4.2) with $\int_{\mathbb{R}} h_n(x) \, dx = V$. Then there exists a subsequence (u_n, h_n) (not relabeled) with the following properties: We have (up to translations) $h_n \rightarrow h$ in $L^1_{\text{loc}}(\mathbb{R})$ with*

$$h(x) := \inf \left\{ \varliminf_{n \rightarrow \infty} h_n(x_n) : x_n \rightarrow x \right\}.$$

It holds that $\int_{\mathbb{R}} h \, dx \leq V$. Further, $\mathbb{R}^2 \setminus \Omega_{h_n} \rightarrow \mathbb{R}^2 \setminus \Omega_h$ in the local Hausdorff topology, and $u_n \rightarrow u$ in $H^1_{\text{loc}}(\Omega_h)$. If $\{h_n\}$ is tight, then $h_n \rightarrow h$ in $L^1(\mathbb{R})$, $\int_{\mathbb{R}} h \, dx = V$, and

$$\varliminf_{n \rightarrow \infty} \left(S_V^{(\ell)}(h_n) + E_V(u_n, h_n) \right) \geq S_V^{(\ell)}(h) + E_V(u, h).$$

Conversely, if h is a lower semicontinuous, nonnegative function of bounded pointwise variation with $\int_{\mathbb{R}} h \, dx = V$ and $u \in H^1_{\text{loc}}(\Omega_h)$, then there exists a sequence of compactly supported Lipschitz functions h_n with $\int_{\mathbb{R}} h_n(x) \, dx = V$, and $u_n \in H^1(\Omega_{h_n})$ such that $h_n \rightarrow h$ in $L^1(\mathbb{R})$, $u_n \rightarrow u$ in $H^1_{\text{loc}}(\Omega_h)$, and

$$S_V^{(\ell)}(h) + E_V(u, h) \geq \overline{\varliminf}_{n \rightarrow \infty} \left(S_V^{(\ell)}(h_n) + E_V(u_n, h_n) \right).$$

For $V > 0$, we set (see [10, (2.8)])

$$\mathcal{A}_V^\ell := \left\{ (u, h) : h \in BV(\mathbb{R}), h \text{ l.s.c.}, h \geq 0, u \in H_{\text{loc}}^1(\Omega_h), \int_{\mathbb{R}} h \, dx = V, u(x, 0) = x \text{ if } x \in \text{supp } h \right\},$$

and

$$F_\ell(V) := \inf \left\{ \int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}} |h'| + 2\mathcal{H}^1(\Gamma_{\text{cuts}}) : (u, h) \in \mathcal{A}_V^\ell \right\}. \quad (4.3)$$

4.1. Basic Properties

Note that the rescaling property for the small slope approximation from Lemma 3.1 carries over to the case of the large slope approximation (4.1). Precisely, for a lower semicontinuous function $h \in BV(\mathbb{R})$, $u \in H_{\text{loc}}^1(\Omega_h)$ with $\nabla u \in L^2(\Omega_h)$ and $\lambda > 0$, consider the rescaled quantities $h_\lambda \in BV(\mathbb{R})$ and $u_\lambda \in H_{\text{loc}}^1(\Omega_h)$ given by $h_\lambda(x) := \frac{1}{\lambda}h(\lambda x)$ and $u_\lambda(x, y) := \frac{1}{\lambda}u(\lambda x, \lambda y)$. Then $\int_{\mathbb{R}} h \, dx = \lambda^2 \int_{\mathbb{R}} h_\lambda(x) \, dx$, and

$$\begin{aligned} & \int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}} |h'| + 2\mathcal{H}^1(\Gamma_{\text{cuts}}) \\ &= \lambda^2 \int_{\Omega_{h_\lambda}} |\nabla u_\lambda|^2 \, dx + \lambda \int_{\mathbb{R}} |h'_\lambda| + 2\lambda \mathcal{H}^1(\Gamma_{\text{cuts}}). \end{aligned}$$

Further, we again have $F_\ell(V) \leq V$ by considering a sequence $h_n(x) = \frac{V}{n} \chi_{(0,n)}$ and $u_n(x, y) = x$ in Ω_{h_n} . Hence, some results from the previous sections carry over essentially verbatim to the large slope approximation. We collect some properties in the following proposition.

Proposition 4.2. (i) F_ℓ is concave, and hence in particular locally Lipschitz continuous. At every point of differentiability,

$$F'_\ell(V) = \frac{1}{V} \left(E_V(u, h) + \frac{1}{2} S_V^{(\ell)}(h) \right),$$

for every minimizing pair (u, h) .

Minimizers for a fixed volume have the same surface and the same elastic energy. Further,

$$\overline{\lim}_{\varepsilon \rightarrow 0} \frac{F_\ell(V + \varepsilon) + F_\ell(V - \varepsilon) - 2F_\ell(V)}{\varepsilon^2} \leq -\frac{S_V^{(\ell)}}{4V^2}.$$

(ii) For every $V > 0$, if a minimizer exists, that is, if there is $(u^*, h^*) \in \mathcal{A}_V^\ell$ such that $F_\ell(V) = E_V(u^*, h^*) + S_V^{(\ell)}(h^*)$, then $\overline{\{h^* > 0\}}$ is connected.

(iii) For every $V > 0$,

$$F_\ell(V) = \inf \left\{ \int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}} |h'| : (u, h) \in \mathcal{A}_V^\ell, h \text{ Lipschitz}, \partial_y u \equiv 0 \text{ or} \right. \\ \left. \int_{\Omega_h} (\partial_y u)^2 \, dx = \frac{1}{4} \int_{\mathbb{R}} |h'| \right\}.$$

(iv) We have

$$\|h\|_{L^\infty(\mathbb{R})} \leq \int_{\mathbb{R}} |h'|.$$

Proof. (i) See proofs of Propositions 3.3 and 3.31.

(ii) See proof of Proposition 3.4.

(iii) Consider the anisotropic volume-preserving scaling from the proof of Lemma 3.5, that is, $h_\lambda(x) := \lambda h(\lambda x)$, and $u_\lambda(x) := \frac{1}{\lambda} u(\lambda x, \frac{1}{\lambda} y)$. Then

$$\int_{\Omega_h} |\nabla u_\lambda|^2 \, dx + \int_{\mathbb{R}} |h'_\lambda| = \int_{\Omega_h} \left((\partial_x u)^2 + \frac{1}{\lambda^4} (\partial_y u)^2 \right) \, dx + \lambda \int_{\mathbb{R}} |h'|,$$

and for $\partial_y u \not\equiv 0$, minimization in λ yields the claim. \square

4.2. Scaling Law

The scaling law for F_ℓ can be derived arguing along the lines of the proof of Proposition 3.9.

Proposition 4.3. *There is a constant $c_0 > 0$ such that for all $V > 0$, we have*

$$c_0 \min\{V, V^{2/3}\} \leq F_\ell(V) \leq \frac{1}{c_0} \min\{V, V^{2/3}\}.$$

Further, there is a constant $c > 0$ with the following property: If V is large enough, and (u, h) is admissible for (4.2) with $F(u, h) \leq \frac{1}{c_0} \min\{V, V^{2/3}\}$, then $\sup h \geq cV^{2/3}$.

Proof. We prove the upper bound first. If $V \leq 1$, for $L > 0$ we set $h := VL^{-1}\chi_{[0,L]}$ and $u(x, y) := x$ for all $(x, y) \in \Omega_h$. Then

$$F_\ell(V) \leq \int_0^L \int_0^{VL^{-1}} |\nabla u|^2 \, dx + \int_{\mathbb{R}} |h'| = V + 2VL^{-1}.$$

Since $L > 0$ can be chosen arbitrary large, we have that $F_\ell(V) \leq V$.

In the case $V \geq 1$, choose $L := V^{1/3}$, $h := \frac{V}{L}\chi_{[0,L]}$, and let u be a minimizer of the Dirichlet energy in $\Omega_h = [0, L] \times [0, h]$ subject to $u(x, 0) = x$. Since $L = V^{1/3} \ll V^{2/3} \sim \frac{V}{L}$, we have

$$\int_{\Omega_h} |\nabla u|^2 \, dx \sim L^2 \sim V^{2/3} \quad \text{and} \quad \int_{\mathbb{R}} |h'| = 2V^{2/3}.$$

The lower bound together with the estimate on $\sup h$ can be obtained by repeating the proof of Proposition 3.9 replacing the estimate $S_1 \gtrsim \frac{V^2}{t^3}$ by $S_1^{(\ell)} \gtrsim \frac{V_1}{t_1}$. \square

Remark 4.4. Just as for F_s , using (iv) of Proposition 4.2, we can obtain that for an almost minimizer, $\sup h \sim V^{2/3}$, and the size of the support of h is at least $\sim V^{1/3}$.

Remark 4.5. If we rescale $V := e_0^4 d$, and $F_\ell(e_0, d) = \frac{1}{e_0^2} F_\ell(V)$, we find

$$\min F_\ell(e_0, d) \sim \min \left\{ e_0^2 d, e_0^{2/3} d^{2/3} \right\}.$$

Over the last years, much work has been devoted to the analysis of island formation in a compact setting (see [10, 14, 18]). Precisely, assuming that $h : [0, 1] \rightarrow [0, \infty)$ is Lipschitz, set

$$\begin{aligned} \tilde{F}(d, e_0) := \inf \left\{ \int_{\Omega_h} |\nabla u|^2 \, dx + \int_0^1 (\sqrt{1+h^2} - 1) \, dx : \right. \\ \left. h \in W^{1,\infty}(\mathbb{R}), u \in H^1(\Omega_h), \int_0^1 h \, dx = d, \right. \\ \left. u(x, 0) = e_0 x, h(0) = h(1) = 0 \right\}. \end{aligned} \tag{4.4}$$

In this case, the surface energy is always bounded below by $\min\{d, d^2\}$ since by the compact support there is a point $x^* \in (0, 1)$ with $h(x^*) \geq d$, and thus

$$\begin{aligned} \int_0^1 (\sqrt{1+h^2} - 1) \, dx &= \int_0^{x^*} (\sqrt{1+h^2} - 1) \, dx + \int_{x^*}^1 (\sqrt{1+h^2} - 1) \, dx \\ &\geq \sqrt{1+(2d)^2} - 1 \geq \begin{cases} cd & \text{if } d \text{ is large} \\ cd^2 & \text{if } d \text{ is small.} \end{cases} \end{aligned} \tag{4.5}$$

Consequently, for compact support, the scaling law is the following.

Proposition 4.6. *The following holds: For all $d, e_0 > 0$*

$$\tilde{F}(d, e_0) \sim \max \left\{ \min\{d, d^2\}, \min \left\{ d e_0^2, d^{2/3} e_0^{2/3} \right\} \right\}. \tag{4.6}$$

Proof. We prove the upper bound first.

- (i) If $d \lesssim 1$, let Ω_h be a triangle of length 1 and height $2d$, and set $u(x, y) = e_0 x$ in Ω_h . Then, since $|h'|$ is small,

$$\tilde{F}(d, e_0) \lesssim e_0^2 d + d^2 \lesssim \begin{cases} d^2 & \text{if } e_0^2 \lesssim d \\ e_0^2 d & \text{if } e_0^2 \gtrsim d. \end{cases}$$

- (ii) If $1 \lesssim e_0^2 \lesssim d$, let Ω_h be a Lipschitz approximation of a rectangle of length 1 and height d , that is, $\text{supp } h' \subset (0, \varepsilon) \cup (1 - \varepsilon, 1)$ for $0 < \varepsilon \ll 1$ and let u be a minimizer of the Dirichlet energy in Ω_h subject to the boundary condition $u(x, 0) = e_0 x$. Then, since $|h'|$ is large on its support,

$$\tilde{F}(d, e_0) \lesssim e_0^2 + d \lesssim d.$$

(iii) If $\frac{1}{e_0^4} \lesssim d \lesssim e_0^2$, let Ω_h be a Lipschitz approximation of a rectangle of length $L \sim \frac{d^{1/3}}{e_0^{2/3}} \lesssim 1$, and height $h \sim d^{2/3} e_0^{2/3}$, and let u be a minimizer of the Dirichlet integral in Ω_h subject to the boundary condition $u(x, 0) = e_0 x$. Then, since $|h'|$ is large on its support,

$$\tilde{F}(d, e_0) \lesssim e_0^2 L^2 + d^{2/3} e_0^{2/3} \sim e_0^{2/3} d^{2/3}.$$

The lower bound follows from (4.5) and a proof similar to that of Proposition 3.9 since in case $V_1 > 2t_1^2$, the slope used to estimate the surface energy is large. \square

The scaling law (4.6) essentially resembles results from [18], where a model without normalization of the surface energy has been considered, that is,

$$\tilde{F}_2(d, e_0) := \inf \left\{ \int_{\Omega_h} |\nabla u|^2 \, dx + \int_0^1 \sqrt{1 + h'^2} \, dx : h \in W^{1,\infty}(\mathbb{R}), u \in H^1(\Omega_h), \int_0^1 h \, dx = d, u(x, 0) = e_0 x, h(0) = h(1) = 0 \right\}.$$

There, the scaling law turns out to be

$$\tilde{F}_2(d, e_0) \sim \max \left\{ 1, d, e_0^{2/3} d^{2/3} \right\}.$$

We note that the proof uses essentially the large slope approximation $\sqrt{1 + h'^2} \geq \max\{1, |h'|\}$.

4.3. Existence of Minimizers

In case of the small slope approximation, non-existence of minimizers follows from the fact that there is a regime of volumes for which $F_s(V) = V$ (see Proposition 3.13). In case of the large slope approximation, we can prove only a weaker statement in this direction.

Proposition 4.7. *For every $\delta > 0$ there is $\tilde{V}(\delta) > 0$ such that for every $0 < V < \tilde{V}$,*

$$(1 - \delta)V \leq F_\ell(V) \leq V.$$

In particular,

$$\lim_{V \rightarrow 0} \frac{1}{V} F_\ell(V) = 1.$$

Proof. First, the upper bound $F_\ell(V) \leq V$ is showed in the proof of Proposition 4.3. To prove the other estimate, we refine the argument from the proof of the lower bound for the minimal energy in Proposition 4.3.

Let h be any locally Lipschitz function with $\int_{\mathbb{R}} h \, dx = V$ and let $0 < \delta < 1$ be fixed. Let $\lambda > 0$ be a parameter which will be chosen below. Pick $x_1 \in \mathbb{R}$ with $h(x_1) > 0$ and define $t_1 := \max\{t > 0 : [x_1, x_1 + t] \times [0, \lambda t] \subset \Omega_h\}$. Set $V_1 := \int_{x_1}^{x_1+t_1} h \, dx > 0$.

Let us first prove that there exists $\tilde{V} := \tilde{V}(\delta)$ such that if $V_1 < \tilde{V}$, then

$$\int_{x_1}^{x_1+t_1} \int_0^{h(x)} |\nabla u|^2 dx dy + \int_{x_1}^{x_1+t_1} |h'| \geq (1-\delta)V_1. \quad (4.7)$$

Assume for the sake of contradiction that

$$\int_{x_1}^{x_1+t_1} \int_0^{h(x)} |\nabla u|^2 dx dy + \int_{x_1}^{x_1+t_1} |h'| < (1-\delta)V_1. \quad (4.8)$$

Since h is locally Lipschitz, we have that $\max_{[x_1, x_1+t_1]} h - \min_{[x_1, x_1+t_1]} h \leq \int_{x_1}^{x_1+t_1} |h'|$. Then $\max_{[x_1, x_1+t_1]} h \geq V_1/t_1$ and $\min_{[x_1, x_1+t_1]} h = \lambda t_1$, which implies that $V_1/t_1 - \lambda t_1 < (1-\delta)V_1$, and subsequently

$$V_1(1 - (1-\delta)t_1) < \lambda t_1^2. \quad (4.9)$$

Next, by Lemma 2.2, there exists $\psi = \psi(\lambda)$ such that

$$\int_{x_1}^{x_1+t_1} \int_0^{\lambda t_1} |\nabla u|^2 dx \geq \psi \lambda t_1^2, \quad \text{and } \psi \rightarrow 1 \text{ as } \lambda \rightarrow 0. \quad (4.10)$$

In particular, we can choose $\lambda = \lambda(\delta) > 0$ small enough such that $\psi(\lambda) > 1-\delta$. By (4.10), the assumption (4.8) implies $\psi \lambda t_1^2 < (1-\delta)V_1$. We combine this with (4.9) to obtain

$$t_1 > \frac{1 - \frac{1-\delta}{\psi}}{1-\delta} > 0. \quad (4.11)$$

Now we define $\tilde{V}(\delta) := \lambda \left(\frac{1-\frac{1-\delta}{\psi}}{1-\delta} \right)^2 > 0$ and observe that (4.11) implies $V_1 \geq \lambda t_1^2 \geq \tilde{V}$, a contradiction to the assumption $V_1 < \tilde{V}$.

If we choose \tilde{V} as above, we can continue the same way as in the proof of Proposition 3.9 and use that $V_i \leq V < \tilde{V}$ to get

$$F_\ell(V) \geq (1-\delta) \sum_i V_i = (1-\delta)V. \quad (4.12)$$

□

This slightly weaker statement still allows us to derive an analogue to Lemma 3.14. We define β_ℓ by $F_\ell(V) =: \beta_\ell(V)V$. Note that again $\beta_\ell(V) < 1$ for large V .

Lemma 4.8. *The function β_ℓ is strictly decreasing in $\{F_\ell(V) < V\} = \{\beta_\ell < 1\}$.*

Proof. Assume for a contradiction that there are $V_0 < V_1$ such that $F_\ell(V) = \beta_0 V < V$ for all $V \in [V_0, V_1]$ with some $0 < \beta_0 < 1$. By Proposition 4.7, there exists $\tilde{V} > 0$ such that $F_\ell(V) \geq \frac{1}{2}(1 + \beta_0)V$ for all $V < \tilde{V}$. As in the proof of Lemma 3.14, concavity of F_ℓ and $F_\ell(0) = 0$ imply that $F_\ell(V) = \beta_0 V$ for $V \in (0, \tilde{V})$, which yields a contradiction. □

Proceeding along the lines of Section 3, we show tightness of minimizing sequences for $V \in \{\beta_\ell < 1\}$.

Lemma 4.9. *Let $V \in \{\beta_\ell < 1\}$ and $\delta > 0$. Then there exist $\ell = \ell(V, \delta) > 0$ and $C(V, \delta) > 0$ with the following property: If $(u, h) \in \mathcal{A}_V^{(\ell)}$ with $\varepsilon := E_V(u, h) + S_V^{(\ell)}(h) - F_\ell(V) \leq C(V, \delta)$, and $x_0 < x_1$ with $x_1 - x_0 = \ell$, then*

$$\int_{-\infty}^{x_0} h \, dx \leq \delta \quad \text{or} \quad \int_{x_1}^{\infty} h \, dx \leq \delta.$$

Proof. We proceed along the lines of the proof of Lemma 3.15. We may assume that h is locally Lipschitz continuous. Fix $V \in \{\beta_\ell < 1\}$ and $\delta > 0$. Let

$$0 < C(V, \delta) \leq \frac{\delta}{2} \left(\beta_\ell \left(\frac{V}{2} \right) - \beta_\ell(V) \right), \quad \text{and} \quad 0 < \alpha < \frac{\delta}{36} \left(\beta_\ell \left(\frac{V}{2} \right) - \beta_\ell(V) \right)$$

be such that $V/(3\alpha) = n \in \mathbb{N}$. Define $\ell := \alpha^{-1}V$, and consider an interval $[x_0, x_1]$ of length ℓ , and write it as a disjoint union of $3n$ intervals of length 1. As in the proof of Lemma 3.15, taking into account the different surface energy term, there is an interval I such that

$$\int_I h \, dx \leq 3\alpha \quad \text{and} \quad \int_I |h'| \leq 3\alpha,$$

which implies

$$3\alpha \geq \int_I |h'| \geq \sup_I h - \inf_I h.$$

Since also $3\alpha \geq \int_I h \, dx \geq \inf_I h$, we have

$$\sup_I h = \left(\sup_I h - \inf_I h \right) + \inf_I h \leq 3\alpha + 3\alpha = 6\alpha.$$

Following the lines of the proof of Lemma 3.15, we make two cuts in I such that the profile is separated into two pieces of volumes $V_0 < V_1$, with $V = V_0 + V_1 + V_{lo}$, with $V_{lo} \leq 16\alpha$ such that the surface energy is increased by at most $S_{cut} \leq 2 \sup_I h \leq 12\alpha$. Then as in (3.19), $V_0 (\beta_\ell(V/2) - \beta(V)) \leq \varepsilon + V_{lo} + S_{cut}$, and thus,

$$V_0 \leq \frac{\varepsilon + V_{lo} + S_{cut}}{\beta_\ell(V/2) - \beta_\ell(V)} \leq \frac{\varepsilon + 6\alpha + 12\alpha}{\beta_\ell(V/2) - \beta_\ell(V)} < \delta.$$

The proof is concluded as in Lemma 3.15. \square

As worked out in Proposition 3.16, Lemma 4.9 implies tightness (up to translations) of minimizing sequences. Using lower semicontinuity of the energy we obtain the existence of minimizers:

Proposition 4.10. *Let V be such that $F_\ell(V) < V$. Then there exists a minimizer (u, h) of (4.3).*

4.4. Asymptotic Behavior

We proceed along the lines of Section 3.6. According to the slightly different scaling law (see Proposition 4.3), we define for admissible h with $\int_{\mathbb{R}} h \, dx = V$ the rescaled quantities by $\tilde{h}(x) := V^{-2/3}h(V^{1/3}x)$, and

$$G_V(\tilde{h}) := V^{-2/3} (E_V(u, h) + S_V(h)) = V^{-2/3} \int_{\Omega_{\tilde{h}}} |\nabla u|^2 \, dx + \int_{\mathbb{R}} |\tilde{h}'| + 2\mathcal{H}^1(\Gamma_{\text{cuts}}).$$

Note that $\int_{\mathbb{R}} \tilde{h} \, dx = 1$.

Theorem 4.11. *For every sequence $V_n \rightarrow \infty$, let (u_{V_n}, h_{V_n}) be a sequence of minimizers. Then there exists a subsequence such that the rescaled profile functions \tilde{h}_{V_n} (up to translations) converge to h in $L^1(\mathbb{R})$, which minimizes*

$$G(h) := \left(\inf_{u(x,0)=x} \int_{\{h>0\} \times [0,+\infty)} |\nabla u|^2 \, dx \right) + \int_{\mathbb{R}} |h'| + 2\mathcal{H}^1(\Gamma_{\text{cuts}}) \quad (4.13)$$

subject to the constraint $\int_{\mathbb{R}} h \, dx = 1$. Moreover, $\mathbb{R}^2 \setminus \Omega_{\tilde{h}_{V_n}}$ converges in the Hausdorff topology to $\mathbb{R}^2 \setminus \Omega_h$.

Proof. We first observe that $G_{V_n}(\tilde{h}_{V_n}) \leq C$, and that (up to translations) the sequence \tilde{h}_{V_n} is tight (see proofs of Theorem 3.35 and Lemma 4.9). Thus, a subsequence of \tilde{h}_{V_n} converges to h in $L^1(\mathbb{R})$. Thanks to the bound on the surface energy, we also have local Hausdorff convergence of $\mathbb{R}^2 \setminus \Omega_{\tilde{h}_{V_n}}$ to $\mathbb{R}^2 \setminus \Omega_h$ which then improves to Hausdorff convergence thanks to Proposition 4.2 (iv). By the lower semicontinuity, we obtain the lower bound for the surface energy, that is,

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}} |\tilde{h}'_{V_n}| + 2\mathcal{H}^1(\Gamma_{\text{cuts}}(\tilde{h}_{V_n})) \geq \int_{\mathbb{R}} |h'| + 2\mathcal{H}^1(\Gamma_{\text{cuts}}).$$

For the elastic energy, we proceed as in [18, Proposition 4.3]. By the Hausdorff convergence of $\Omega_{\tilde{h}_{V_n}}$ to Ω_h , we have convergence of the “boundary layers” $\{h_{V_n} > 0 \text{ and } h > 0\} \times [0, \infty)$ to $\{h > 0\} \times [0, \infty)$ in the local Hausdorff topology. Changing variables, $x = V_n^{1/3}\hat{x}$, $y = V_n^{1/3}\hat{y}$, $u_{V_n} = V_n^{-1/3}\hat{u}_{V_n}$, there is a subsequence such that $\hat{u}_{V_n} \rightharpoonup \hat{u}$ locally weakly in $H^1(\{h > 0\} \times [0, \infty))$. Thus as in [18, Proposition 4.3],

$$\begin{aligned} V_n^{-2/3} \int_{\Omega_{h_{V_n}}} |\nabla u_{V_n}|^2 \, dx &\geq \int_{\Omega_{\tilde{h}_{V_n}} \cap [\{h>0\} \times [0, \infty)]} |\nabla \hat{u}_{V_n}|^2 \, dx \\ &\geq \int_{\{h>0\} \times [0, \infty)} |\nabla \hat{u}_{V_n}|^2 \, dx. \end{aligned}$$

The lower bound follows. For the upper bound construction, it is enough considering $\tilde{h}_{V_n} = h$. We conclude by Γ -convergence, as in the proof of Theorem 3.35. \square

Remark 4.12. We note that

$$G(h) := C_W \sum_{i \in \mathbb{N}} (b_i - a_i)^2 + \int_{\mathbb{R}} |h'| + 2\mathcal{H}^1(\Gamma_{\text{cuts}}),$$

where the intervals (a_i, b_i) are the connected components of $\{h > 0\}$, and C_W is as in (3.53).

We next study the minimizer of the limiting functional.

Proposition 4.13. *The minimization problem (4.13) admits a unique minimizer up to translations, namely the rectangle with base $2^{1/3}C_W^{-1/3}$.*

Proof. The proof follows as in [18, Proposition 4.5], and we briefly sketch it only for the reader’s convenience. The optimal function h is of the form $h = \sum_{i=1}^N h_i \chi_{(a_i, b_i)}$ with $\sum_{i=1}^N h_i (b_i - a_i) = 1$, and $a_i < b_i < a_{i+1}$ since the rectangle minimizes $\int |h'|$ among profiles with given volume. We set $\ell_i := b_i - a_i$. Then by Remark 4.12, the energy is given by

$$C_W \sum_{i=1}^N \ell_i^2 + 2 \sum_{i=1}^N h_i.$$

Assume that there are two connected components, say, of lengths $\ell_1 \geq \ell_2 > 0$, then for $\eta \in [-h_1, \frac{\ell_2}{\ell_1} h_2]$, we consider the volume-preserving variation of h changing h_1 and h_2 to $h_1 + \eta$ and $h_2 - \eta \frac{\ell_1}{\ell_2}$, respectively. The minimality condition then gives $\ell_1 = \ell_2$, from which we deduce that $\ell_i = \ell \equiv \text{const}$ for every $i = 1, \dots, N$. The minimization problem then reduces to minimizing $C_W N \ell^2 + \frac{2}{\ell}$ subject to $N \in \mathbb{N}$, which yields $N = 1$ and $\ell = 2^{1/3}C_W^{-1/3}$. \square

To prove the exponential convergence, we have an analogue to Proposition 3.39.

Lemma 4.14. *For $L > 0$ consider $h_{\min} := \frac{V}{L} \chi_{[0, L]}$, which is the minimizer of $S_V^{(\ell)}$ among all $h \geq 0$ with $\int_{\mathbb{R}} h(x) dx = \int_0^L h(x) dx = V$. Then for all $h \in BV(\mathbb{R})$ with $\int_{\mathbb{R}} h(x) dx = \int_0^L h(x) dx = V$ we have*

$$\int_{\mathbb{R}} |h'| - \int_{\mathbb{R}} |h'_{\min}| \geq \frac{1}{L} \int_{\mathbb{R}} |h - h_{\min}| dx.$$

Proof. By rescaling, it suffices to consider $L = 1$ and $V = 1$, that is, $h_{\min} = \chi_{[0, 1]}$ and $\int_{\mathbb{R}} |h'_{\min}| = 2$. By approximation, we may assume that $h \in W^{1, \infty}(\mathbb{R})$. Let the function $h(x) - 1$ attain its non-negative maximum at $\bar{x} \in [0, 1]$. We have $\int_0^1 |h'| \geq 2h(\bar{x})$, and thus,

$$\int_0^1 |h'| - 2 \geq 2(h(\bar{x}) - 1) \geq 2 \int_0^1 (h - 1)_+ dx = \int_0^1 |h - h_{\min}| dx,$$

where we used $\int_0^1 h dx = 1$ to show that $2 \int_0^1 (h - 1)_+ dx = \int_0^1 |h - 1| dx$. \square

Finally, we prove exponential convergence of a sequence of minimizing profiles of G_V to the rectangle, which minimizes the limit functional (see Proposition 4.13). We denote by I_s the largest connected component of $\{h_V > s\}$, and by \tilde{I}_s the rescaled one. Given a function \tilde{h} , we denote by \bar{h}_s the function that agrees with \tilde{h} outside I_s , has the same volume as \tilde{h} , and is such that the surface energy term is minimized.

Proposition 4.15. *For every $\varepsilon > 0$ there exist constants C_0 and C_1 such that for every $V > \bar{V}$, and for every minimizer \tilde{h}_V of G_V , for all $s \geq \varepsilon$,*

$$\|\tilde{h}_V - \bar{h}_s\|_{L^1(\tilde{I}_s)} \leq C_0 \exp(-C_1 V^{1/3}).$$

Proof. Let $\varepsilon > 0$. Since (after possible translations), $\mathbb{R}^2 \setminus \Omega_{\tilde{h}_V}$ converges in Hausdorff topology to $\mathbb{R}^2 \setminus \Omega_{\tilde{h}}$ and since \bar{h} is a characteristic function,

$$\mathcal{H}^1(\tilde{h} > s) \leq C \quad \forall s \geq \varepsilon.$$

Hence, by rescaling, we obtain that for $s \geq \varepsilon V^{2/3}$, the largest connected component I_s of $\{h_V > s\}$ satisfies

$$\mathcal{H}^1(I_s) \leq C V^{1/3}. \quad (4.14)$$

By density, possibly slightly changing ε , we may instead of minimizers consider Lipschitz functions \tilde{H}_V with $\|\tilde{H}_V - \tilde{h}_V\|_{L^1} \leq \exp(-C_1 V^{1/3})$ and $E_V(u_V, H_V) + S_V^{(\ell)}(H_V) \leq E_V(u, h_V) + S_V^{(\ell)}(h_V) + \exp(-C_1 V^{1/3})$. Similarly to the derivation of (3.56) we then obtain that there is some $t \in [2\varepsilon V^{2/3}, 3\varepsilon V^{2/3}]$ with

$$\|u_V\|_{\tilde{H}^{1/2}(I_t \times \{t\})}^2 \leq V^{2/3} \varepsilon^{-1} \exp(-C\varepsilon V^{1/3}). \quad (4.15)$$

Indeed, since \tilde{H}_V is Lipschitz, and u_V is a minimizer of the Dirichlet energy, we have as in the proof of Proposition 3.40,

$$\int_{\Omega_V^s} |\nabla u_V|^2 \, d\mathbf{x} \leq C \mathcal{H}^1(I_s) \int_{I_s \times \{s\}} |\nabla u_V|^2 \, d\mathbf{x} \leq C V^{1/3} \int_{I_s \times \{s\}} |\nabla u_V|^2 \, d\mathbf{x},$$

that is, $f(s) := \int_{\Omega_V^s} |\nabla u_V|^2$ for $s \geq \varepsilon V^{2/3}$ satisfies $f(s) \leq -C V^{1/3} f'(s)$. Thus, for $s \geq \varepsilon V^{2/3}$, we have $f(s) \leq V \exp\left(-C \frac{s - \varepsilon V^{2/3}}{V^{1/3}}\right)$. In particular,

$$\int_{2\varepsilon V^{2/3}}^{3\varepsilon V^{2/3}} |\nabla u_V|^2 \, d\mathbf{x} \leq V \exp(-C\varepsilon V^{1/3}),$$

and it follows by Wirtinger's inequality that there is some $t \in [2\varepsilon V^{2/3}, 3\varepsilon V^{2/3}]$ such that (4.15) holds. Now we choose as a competitor the function \tilde{h}_t , which by definition agrees with \tilde{H} outside of I_t , and the corresponding optimal deformation u with boundary data $u(x, t) = u_V(x, t)$ for all $x \in I_t$. Using the almost optimality, we conclude as in Proposition 3.40, using Lemma 4.14. Note that the factor $V^{2/3}$ cancels with the rescaling factor of the elastic energy. \square

5. The Scaling Law in Three Space Dimensions

In this section we will identify the scaling law for the energy in the (2 + 1)-dimensional setting. More precisely, we consider a three dimensional analog of (3.2): given $V \in (0, \infty)$, we define

$$F_s^{3D}(V) := \inf_{(u,h)} \left\{ \int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} + \int_{\mathbb{R}^2} |\nabla h|^2 \, dx \, dy : h \in W^{1,\infty}(\mathbb{R}), u \in H_{\text{loc}}^1(\Omega_h), \int_{\mathbb{R}^2} h \, dx \, dy = V, u(x, y, 0) = (x, y) \text{ in } \text{supp } h \right\}, \quad (5.1)$$

where $\Omega_h := \{\mathbf{x} := (x, y, z) \in \mathbb{R}^3 : 0 < z < h(x, y)\}$.

Theorem 5.1. *There exists a positive constant c such that for every $V > 0$*

$$c \min\{V, V^{6/7}\} \leq F_s^{3D}(V) \leq c^{-1} \min\{V, V^{6/7}\}.$$

Proof. To prove the upper bound, we consider two different constructions—a thin layer and a pyramid.

Thin layer construction For $\varepsilon > 0$, let L be such that $\int_{\mathbb{R}^2} h \, dx \, dy = V$ for h defined by $h(x, y) := \min\{\varepsilon, [(L + \varepsilon) - \max(|x|, |y|)]_+\}$, where by $f_+ := \max\{0, f\}$ denotes the positive part of f . By setting $u(x, y, z) := (x, y)$ we get $\int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} = 2V$. We observe that $|\nabla h| = \chi_M$ with M the difference of two concentric squares with sidelengths $2(L + \varepsilon)$ and $2L$, and so

$$\int_{\mathbb{R}^2} |\nabla h|^2 \, dx \, dy = |M| = 4\varepsilon(2L + \varepsilon) \leq 4(\sqrt{V\varepsilon} + \varepsilon^2),$$

where we used that $(2L)^2\varepsilon \leq V$. Hence $F(V) \leq 2V + 4(\sqrt{V\varepsilon} + \varepsilon^2)$ for arbitrarily small $\varepsilon > 0$, which implies $F(V) \leq 2V$.

Pyramid construction Let $L := V^{2/7}$ and $H := 3V^{3/7}/4$. We define $h(x, y) := H(1 - \max(|x|, |y|)/L)_+$ and $u(x, y, z) := (x, y)(1 - z/L)_+$, and so $\int_{\mathbb{R}^2} h \, dx \, dy = V$. Then $|\nabla h| = \chi_{[-L, L]^2} H/L$, and $\int_{\mathbb{R}^2} |\nabla h|^2 \, dx \, dy = 4L^2(H/L)^2 = 4H^2$. To estimate the elastic energy we observe that for $(x, y, z) \in \Omega_h \cap \{z \in (0, L)\}$ we have $|\nabla u(x, y, z)|^2 = 2(1 - z/L)^2 + (x^2 + y^2)/L^2$, and $\nabla u = 0$ otherwise. Since $x^2 + y^2 \leq 2L^2$, we have that $\int_{\Omega_h} |\nabla u|^2 \, d\mathbf{x} \leq 4|\Omega_h \cap \{z < L\}| \leq 4(2L)^2L = 8L^3$, and finally $F(V) \leq 4H^2 + 8L^3 = (9/4 + 8)V^{6/7}$.

It remains to prove the lower bound. First we describe the notation. For $(x_0, y_0) \in \mathbb{R}^2$ and $l > 0$ we define the square $S_l(x_0, y_0) := [x_0, x_0 + l) \times [y_0, y_0 + l)$. Let Φ be a function, which for a given square $S_l(x_0, y_0)$ counts on what portion of slices the function h is larger than l :

$$\Phi(S_l(x_0, y_0)) := \left| \{x \in [x_0, x_0 + l) : h(x, y) \geq l \text{ for all } y \in [y_0, y_0 + l)\} \right| / l.$$

We observe that this definition makes sense since $h \in H^1(\mathbb{R}^2)$, and so there exists a representative which is defined everywhere on almost every slice. Let $\varepsilon > 0$ be such that $\int_{\{h \geq \varepsilon\}} h \, d\mathbf{x} \geq V/2$. Given ε , we assume that the following Calderón–Zygmund type lemma holds:

Lemma 5.2. *There exists a collection of disjoint squares $\{S_{l_n}(x_n, y_n)\}$ such that their union covers the set $\{h \geq \varepsilon\}$, and each square $S_{l_n}(x_n, y_n)$ from the collection satisfies:*

- $\Phi(S_{l_n}(x_n, y_n)) \leq 1/2$,
- *there exists a point (x'_n, y'_n) such that $S_{l_n/2}(x'_n, y'_n) \subset S_{l_n}(x_n, y_n)$ and $\Phi(S_{l_n/2}(x'_n, y'_n)) \geq 1/2$.*

We postpone the proof of the lemma and first show how the lemma implies the lower bound. Let $S_{l_n}(x_n, y_n)$ be one of the squares obtained in Lemma 5.2. We denote

$$S_n := S_{l_n}(x_n, y_n), \quad V_n := \int_{S_n} h \, dx \, dy, \quad S'_n := S_{l_n/2}(x'_n, y'_n),$$

$$X'_n := \{x \in [x'_n, x'_n + l_n/2) : h(x, y) \geq l_n/2 \text{ for all } y \in [y'_n, y'_n + l_n/2)\}.$$

Observe that

$$\begin{aligned} \int_{S_n \times (0, \infty) \cap \Omega_h} |\nabla u|^2 \, dx &\geq \int_{S'_n \times (0, \infty) \cap \Omega_h} |\nabla u|^2 \, dx \\ &\geq \int_{X'_n} \left(\int_{y'_n}^{y'_n + l_n/2} \int_0^{l_n/2} |\nabla u|^2 \, dy \, dz \right) dx \\ &\gtrsim |X'_n| l_n^2 \stackrel{\Phi(S'_n) \geq 1/2}{\gtrsim} l_n^3, \end{aligned} \quad (5.2)$$

where we used that for $x \in X'_n$ the whole square $\{x\} \times [y'_n, y'_n + l_n/2) \times [0, l_n/2) \subset \Omega_h$, and we used the one-dimensional argument to get $\int_{y'_n}^{y'_n + l_n/2} \int_0^{l_n/2} |\nabla u|^2 \, dy \, dz \gtrsim l_n^2$.

We consider two cases: $V_n < 2l_n^3$ and $V_n \geq 2l_n^3$. If $V_n < 2l_n^3$, then by (5.2) we have $V_n \lesssim \int_{S_n \times (0, \infty) \cap \Omega_h} |\nabla u|^2 \, dx$.

Now suppose that $V_n \geq 2l_n^3$. Since $\Phi(S_n) \leq 1/2$, we can find $x' \in [x_n, x_n + l_n)$ such that $h(x', y) \leq l_n$ for some $y' \in [y_n, y_n + l_n)$ and $\int_{y_n}^{y_n + l_n} |\partial_y h(x', y)|^2 \, dy \leq 2l_n^{-1} \int_{S_n} |\nabla h|^2 \, dx \, dy$. Then for any $y \in [y_n, y_n + l_n)$, Hölder's inequality implies

$$\begin{aligned} |h(x', y) - h(x', y')| &\leq \int_y^{y'} |\partial_y h(x', \hat{y})| \, d\hat{y} \\ &\leq l_n^{1/2} \left(2l_n^{-1} \int_{S_n} |\nabla h|^2 \, dx \, dy \right)^{1/2} \\ &= \left(2 \int_{S_n} |\nabla h|^2 \, dx \, dy \right)^{1/2}. \end{aligned} \quad (5.3)$$

Since $h(x', y') \leq l_n$, we get that

$$\max_{y \in [y_n, y_n + l_n)} h(x', y) \leq l_n + \left(2 \int_{S_n} |\nabla h|^2 \, dx \, dy \right)^{1/2}$$

and that $\int_{y_n}^{y_n+l_n} h(x', y) dy \leq l_n^2 + l_n \left(2 \int_{S_n} |\nabla h|^2 dx dy \right)^{1/2}$. Finally, another application of Hölder's inequality shows that for every $x \in [x_n, x_n + l_n]$:

$$\begin{aligned} & \int_{y_n}^{y_n+l_n} h(x, y) dy \\ & \leq \int_{y_n}^{y_n+l_n} h(x', y) dy + \left| \int_{y_n}^{y_n+l_n} h(x, y) dy - \int_{y_n}^{y_n+l_n} h(x', y) dy \right| \\ & \leq l_n^2 + l_n \left(2 \int_{S_n} |\nabla h|^2 dx dy \right)^{1/2} + \int_{S_n} |\nabla h| dx dy \\ & \leq l_n^2 + 3l_n \left(\int_{S_n} |\nabla h|^2 dx dy \right)^{1/2}, \end{aligned}$$

and thus

$$\begin{aligned} V_n &= \int_{S_n} h dx dy = \int_{x_n}^{x_n+l_n} \left(\int_{y_n}^{y_n+l_n} h(x, y) dy \right) dx \\ &\leq l_n^3 + 3l_n^2 \left(\int_{S_n} |\nabla h|^2 dx dy \right)^{1/2}. \end{aligned}$$

Since $V_n \geq 2l_n^3$, we have that $V_n^2/(36l_n^4) \leq \int_{S_n} |\nabla h|^2 dx dy$. Then (5.2) and Young's inequality imply

$$\int_{S_n \times (0, \infty) \cap \Omega_h} |\nabla u|^2 dx + \int_{S_n} |\nabla h|^2 dx dy \gtrsim l_n^3 + V_n^2 / (36l_n^4) \gtrsim V_n^{6/7}. \quad (5.4)$$

Summarizing, we get that

$$\int_{S_n \times (0, \infty) \cap \Omega_h} |\nabla u|^2 dx + \int_{S_n} |\nabla h|^2 dx dy \gtrsim \min \{ V_n^{6/7}, V_n \}.$$

Since $\bigcup_n S_n(x_n, y_n) \supset \{h \geq \varepsilon\}$ and $\int_{\{h \geq \varepsilon\}} h \geq V/2$, we have that

$$\sum_n \int_{S_n(x_n, y_n)} h dx dy = \sum_n V_n \geq V/2.$$

Hence, summing (5.4) over all the squares $S_n(x_n, y_n)$ and using the concavity of $f(t) := \min \{t^{6/7}, t\}$ yields

$$\begin{aligned} F_s^{3D}(V) &\geq \sum_n \left(\int_{S_n \times (0, \infty) \cap \Omega_h} |\nabla u|^2 dx + \int_{S_n} |\nabla h|^2 dx dy \right) \\ &\gtrsim \sum_n \min \{ V_n^{6/7}, V_n \} \gtrsim \min \{ V^{6/7}, V \}. \end{aligned}$$

□

Proof of Lemma 5.2. Since h is Lipschitz, we see that $\delta := \text{dist}(\{h \geq \varepsilon\}, \{h \leq \varepsilon/2\}) > 0$. Let $l' := (2V)^{1/3}$, and let $S_{l'}(x_n^{(0)}, y_n^{(0)})$ be a (finite) disjoint covering of the set $\{h \geq \varepsilon\}$. The choice of l' implies that $\Phi(S_{l'}(x_n^{(0)}, y_n^{(0)})) \leq 1/2$ for every n . Let us now fix one square $S_{l'}(x_n^{(0)}, y_n^{(0)})$. We aim to construct a finite collection \mathcal{N} of disjoint squares with the following properties:

- all squares in \mathcal{N} satisfy the two conditions given in Lemma 5.2,
- $S_{l'}(x_n^{(0)}, y_n^{(0)}) \cap \{h \geq \varepsilon\} \subset \bigcup_{S \in \mathcal{N}} S$.

We observe that if we construct such \mathcal{N} for each initial square $S_{l'}(x_n^{(0)}, y_n^{(0)})$, taking the union of all those \mathcal{N} will give a collection of squares which satisfies all the required conditions of Lemma 5.2.

We now describe the iterative construction. We define $\mathcal{M}_0 := \{S_{l'}(x_n^{(0)}, y_n^{(0)})\}$ and $\mathcal{N}_k := \emptyset$. Assume that \mathcal{M}_k is constructed, and let $S_l(x, y) \in \mathcal{M}_k$. We divide $S_l(x, y)$ into four disjoint squares S_1, S_2, S_3, S_4 with sidelength $l/2$, and consider two cases. If $\Phi(S_i) \leq 1/2$ for all $i = 1, 2, 3, 4$, then we add to \mathcal{M}_{k+1} all those S_i which satisfy $S_i \cap \{h \geq \varepsilon\} \neq \emptyset$. Otherwise (that is, if $\Phi(S_i) > 1/2$ for some i), we add $S_l(x, y)$ into \mathcal{N}_k .

First, we observe that at any step k_0 of the procedure we have

$$S_{l'}(x_n^{(0)}, y_n^{(0)}) \cap \{h \geq \varepsilon\} \subset \bigcup_{S \in \mathcal{M}_{k_0} \cup \bigcup_{k < k_0} \mathcal{N}_k} S. \quad (5.5)$$

We also see that any square $S \in \mathcal{N}_k$ satisfies the two conditions given in Lemma 5.2, and so we are done with the construction of $\mathcal{N} := \bigcup \mathcal{N}_k$ provided we show that $\mathcal{M}_k = \emptyset$ for large enough k . Let k be such that $l'2^{-k} \leq \min\{\delta/\sqrt{2}, \varepsilon/2\}$, and let us assume that $S \in \mathcal{M}_k$. From the way \mathcal{M}_k was constructed we see that $\Phi(S) \leq 1/2$, the length of the side of S is $l'2^{-k}$, and $S \cap \{h \geq \varepsilon\} \neq \emptyset$.

On the other hand, since $l'2^{-k} \leq \delta/\sqrt{2}$, we see that $S \subset \{h \geq \varepsilon/2\}$. Then, $l'2^{-k} \leq \varepsilon/2$ implies $h \geq l'2^{-k}$ in S , and so $\Phi(S) = 1$, which contradicts $\Phi(S) \leq 1/2$.

We showed that \mathcal{M}_k is empty for large enough k , and so the \mathcal{N} we just constructed satisfies all the requirements of Lemma 5.2. This finishes the proof of Lemma 5.2, and thus of Theorem 5.1. \square

Using the same line of proof, we can also show the scaling law for the large slope approximation, that is, for the functional

$$F_\ell^{3D}(V) := \inf \left\{ \int_{\Omega_h} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} |\nabla h| \, dx \, dy : h \in BV(\mathbb{R}^2), \right. \\ \left. h \in W^{1,\infty}(\mathbb{R}^2), u \in H_{\text{loc}}^1(\Omega_h), \int_{\mathbb{R}^2} h \, dx \, dy = V, \right. \\ \left. u(x, y, 0) = (x, y) \text{ in } \text{supp } h \right\},$$

Theorem 5.3. *There exists a positive constant c such that for every $V > 0$*

$$c \min\{V, V^{3/4}\} \leq F_\ell^{3D}(V) \leq \frac{1}{c} \min\{V, V^{3/4}\}. \quad (5.6)$$

Notice that in contrast to [18, Proposition 5.1], we do not need here any hypothesis on V .

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