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The Ginzburg–Landau Functional with Vanishing Magnetic Field

Bernard Helffer & Ayman Kachmar

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Abstract

We study the infimum of the Ginzburg–Landau functional in a two dimensional simply connected domain and with an external magnetic field allowed to vanish along a smooth curve. We obtain energy asymptotics which are valid when the Ginzburg–Landau parameter is large and the strength of the external field is below the third critical field. Compared with the known results when the external magnetic field does not vanish, we show in this regime a concentration of the energy near the zero set of the external magnetic field. Our results complete former results obtained by K. Attar and X.B. Pan–K.H. Kwek.

1. Introduction

The Ginzburg–Landau functional is a model describing the response of a superconducting material to an applied magnetic field through the qualitative behavior of the minimizing/critical configurations. The mathematically rigorous analysis of such configurations led to a vast literature and to many mathematically challenging questions, with the aim of recovering what physicists had already observed through experiments or heuristic computations. (See [16] for an introduction to the physics of superconductivity, and the two monographs [8,31] for the mathematical progress on this subject.)

Much of the mathematical literature concerns samples in the form of a long cylinder or a thin film subject to a constant magnetic field. The direction of the magnetic field is parallel to the cylinder's axis (for cylindrical samples) or perpendicular to the plane of the thin film (for thin film samples). For such samples, we have the following behavior (this is thoroughly reviewed in the two monographs [8,31]):

- For very large values of the intensity of the magnetic field, the magnetic field penetrates the sample which is in a normal (non-superconducting) state.

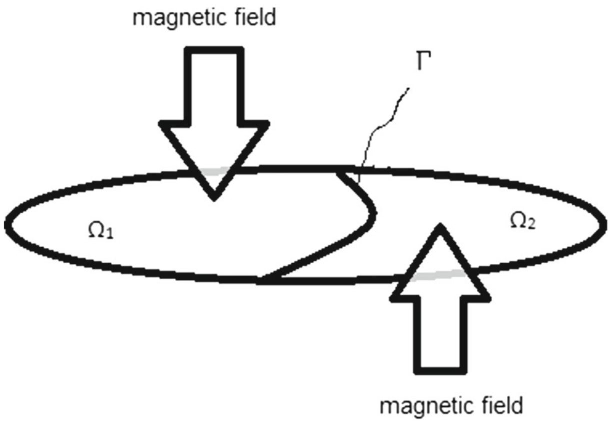

Fig. 1. Sample subject to a variable magnetic field that vanishes along the curve Γ

- Decreasing the intensity of the magnetic field gradually past a critical value H_{c_3}, superconductivity nucleates along the boundary of the sample; the bulk of the sample remains in a normal state ; this is the phenomenon of surface superconductivity (see [33]).
- Decreasing the field further, superconductivity is restored in the bulk of the sample; the magnetic field may penetrate the sample along point defects called *vortices*; such vortices indicate regions of the sample that remain in the normal state (see [31]).

In this paper, we will consider samples submitted to a variable magnetic field (both the direction and the intensity of the field will be variable). Samples submitted to variable magnetic fields are considered in the physical literature, see [22,34].

For the sake of illustrating the results in this paper, let us consider a thin film sample placed horizontally (see Fig. 1). The region occupied by the sample is decomposed into two sub-regions Ω_1 and Ω_2 separated by a smooth curve Γ. Now, we let the sample be subjected to a non-constant magnetic field such that the field is applied on Ω_1 from above, while it is applied from below on Ω_2. We suppose that the magnetic field varies smoothly, hence it has to vanish along the smooth curve Γ. In such a situation, we have the following picture:

- For very large values of the intensity of the magnetic field, the sample is in a normal state [27].
- Decreasing the intensity of the magnetic field gradually past a critical value H_{c_3}, superconductivity nucleates along the *curve* Γ; the rest of the sample remains in a normal state ; this is in contrast of the phenomenon of surface superconductivity observed for samples subject to a constant magnetic field (see [3] and the results in this paper).
- There are two regimes describing the concentration of the superconductivity along the curve Γ. In a first regime, the distribution of the superconductivity is displayed via a *new* limiting function $E(\cdot)$; this limiting function is defined via a simplified Ginzburg–Landau type functional with a magnetic field vanishing

along a line. In another regime, the distribution of the superconductivity is displayed via a *known* limiting function $g(\cdot)$; surprisingly, the limiting function $g(\cdot)$ is defined via a simplified Ginzburg–Landau type functional with a *constant* magnetic field; the function $g(\cdot)$ displays the distribution of (bulk) superconductivity for samples submitted to a constant magnetic field [13,32].

The rest of this introduction is devoted to precise statements displaying the picture that we have sketched previously.

In a two dimensional bounded and simply connected domain Ω with smooth boundary, the Ginzburg–Landau functional is defined over configurations $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ by,

$$\mathcal{E}(\psi, \mathbf{A}) = \int_\Omega e_{\kappa,H}(\psi, \mathbf{A})\,\mathrm{d}x \tag{1.1}$$

where

$$e_{\kappa,H}(\psi, \mathbf{A}) := |(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + (\kappa H)^2|\operatorname{curl}\mathbf{A} - B_0|^2.$$

The modulus of the wave function ψ measures the density of the superconducting electrons; the curl of the vector field $\mathbf{A}$ measures the induced magnetic field; the parameter H measures the intensity of the external magnetic field and the parameter κ ($\kappa > 0$) is a characteristic of the superconducting material; $\mathrm{d}x$ is the Lebesgue measure $\mathrm{d}x_1\,\mathrm{d}x_2$. The function B_0 represents the profile of the external magnetic field in Ω and is allowed to vanish non-degenerately on a smooth curve. We suppose that B_0 is defined and C^∞ in a neighborhood of $\overline{\Omega}$ and satisfies,

$$|B_0| + |\nabla B_0| \geqq c > 0 \quad \text{in } \overline{\Omega}, \tag{1.2}$$

and that the set

$$\Gamma = \{x \in \overline{\Omega} \ :\ B_0(x) = 0\} \tag{1.3}$$

consists of a **finite number of simple smooth curves**. We also assume that:

$$\Gamma \cap \partial\Omega \ \text{ is a } \textbf{finite} \text{ set}. \tag{1.4}$$

The assumptions on Γ, together with (1.2), force the function B_0 to change sign. In physical terms, the set Γ splits the domain Ω into two parts $\Omega_1 = \{B_0(x) > 0\}$ and $\Omega_2 = \{B_0(x) < 0\}$ such that the magnetic field applied on Ω_1 is along the opposite direction of the magnetic field applied on Ω_2 (compare with Fig. 1). The results in this paper do not cover the potentially interesting case where the magnetic field B_0 vanishes on *isolated* points; such an assumption displays different physics since the magnetic field can *not* change sign here.

The ground state energy of the functional is,

$$\mathrm{E_{gs}}(\kappa, H) = \inf\{\mathcal{E}(\psi, \mathbf{A}) \ :\ (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)\}. \tag{1.5}$$

We focus on the regime where H satisfies

$$H = \sigma\kappa^2, \quad \sigma \in (0, \infty). \tag{1.6}$$

Our results allow for σ to be a function of κ satisfying $\sigma \gg \kappa^{-1}$. Earlier results corresponding to vanishing magnetic fields have been obtained recently in [3,4]. The assumption on the strength of the magnetic field was $H \leqq C\kappa$, where C is a constant. In the regime of large κ, K. Attar has obtained, in [3,4], parallel results to those known for the constant magnetic field in [32]. However, it is proved in [3] that if

$$H = b\kappa, \tag{1.7}$$

and b is a constant, then when b is large enough, the energy and the superconducting density are concentrated near the set Γ with a length scale $\frac{1}{b}$. Essentially, that is a consequence of the following asymptotics of the energy ($\kappa \to \infty$),

$$\mathrm{E}_{\mathrm{gs}}(\kappa, H) = \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x + o\left(\kappa H \left|\ln \frac{\kappa}{H}\right| + 1\right), \tag{1.8}$$

which is valid under the relaxed assumption that

$$\Lambda_1 \kappa^{1/3} \leqq H \leqq \Lambda_2 \kappa, \tag{1.9}$$

Λ_1 and Λ_2 being positive constants.
In particular, the assumption (1.9) covers the situation in (1.7). The function g appears in the analysis of the two and three dimensional Ginzburg–Landau functional with constant magnetic field, [13,32]. It is associated with some effective model energy. The function g will play a central role in this paper and its definition will be recalled later in this text (see (3.49)).

One purpose of this paper is to give a precise description of the aforementioned concentration of the order parameter and the energy when $\sigma \gg 1$, thereby leading to the assumption in (1.6).

The leading order term of the ground state energy in (1.5) is expressed via the quantity $E(\cdot)$ introduced in Theorem 3.8 below. The function $(0, \infty) \ni L \mapsto E(L)$ is a continuous function satisfying the following properties:

- $E(L)$ is defined via a reduced Ginzburg–Landau energy in the strip (this energy is introduced in (3.14)).
- $E(L) = 0$ iff $L \geqq \lambda_0^{-3/2}$, where λ_0 is a universal constant defined as the bottom of the spectrum of a Montgomery operator, see (3.4).
- As $L \to 0_+$, the expected asymptotic behavior of $E(L)$ is like $L^{-4/3}$.

Throughout this text, we use the following notation. If A and B are two positive quantities, then

- $A \ll B$ means $A = \delta(\kappa)B$ and $\delta(\kappa) \to 0$ as $\kappa \to \infty$;
- $A \lesssim B$ means $A \leqq CB$ and $C > 0$ is a constant independent of κ;
- $A \gg B$ means $B \ll A$, and $A \gtrsim B$ means $B \lesssim A$;
- $A \approx B$ means $c_1 B \leqq A \leqq c_2 B$, $c_1 > 0$ and $c_2 > 0$ are constants independent of κ.

The main result in this paper is:

Theorem 1.1. *Suppose that the function B_0 satisfies Assumptions* (1.2) *and* (1.3). *Let $b : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying*

$$\lim_{\kappa\to\infty} b(\kappa) = \infty \quad \text{and} \quad \limsup_{\kappa\to\infty} \kappa^{-1} b(\kappa) < \infty. \tag{1.10}$$

Suppose that

$$H = b(\kappa)\kappa.$$

Then, as $\kappa \to \infty$, the ground state energy in (1.5) *satisfies:*

(1) *If $b(\kappa) \gg \kappa^{1/2}$, then*

$$\mathrm{E_{gs}}(\kappa, H) = \kappa\left(\int_\Gamma \left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \mathrm{d}s(x)\right) + o\left(\frac{\kappa^3}{H}\right), \tag{1.11}$$

where ds *denotes the arc-length measure in* Γ.

(2) *If $b(\kappa) \lesssim \kappa^{1/2}$, then*

$$\mathrm{E_{gs}}(\kappa, H) = \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x + o\left(\frac{\kappa^3}{H}\right). \tag{1.12}$$

Remark 1.2. (About the critical field H_{c_3})
As we shall see in Section 2, Pan and Kwek [27] prove that if H is larger than a critical value $H_{c_3}(\kappa)$, then the minimizers of the functional in (1.1) are trivial and the ground state energy is $\mathrm{E_{gs}}(\kappa, H) = 0$. Furthermore, the value of $H_{c_3}(\kappa)$ as given in [27] admits, as $\kappa \to \infty$, the following asymptotics

$$H_{c_3}(\kappa) \sim c_0\kappa^2, \tag{1.13}$$

where c_0 is an explicit constant (determined by the function B_0). As such, the assumption on the magnetic field in Theorem 1.1 is significant when $b(\kappa)\kappa \leqq H \leqq M\kappa^2$ and $M \in (0, c_0]$ is a constant. Note also that our theorem gives a bridge between the situations studied by Attar in [3,4] and Pan and Kwek in [27].

Remark 1.3. (The remainder terms in Theorem 1.1)
As long as the intensity of the external magnetic field satisfies $\kappa \ll H \leqq M\kappa^2$ and $M \in (0, c_0)$, the remainder term appearing in Theorem 1.1 is of lower order compared with the principal term. The function $g(b)$ is bounded and vanishes when $b \geqq 1$. Accordingly,

$$\int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x = \int_{\{|B_0(x)|<\frac{\kappa}{H}\}} g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x \approx \frac{\kappa}{H}.$$

We shall see in Theorem 3.12 that,

$$\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \approx \frac{\kappa^2}{H}.$$

Remark 1.4. (The two regimes in Theorem 1.1)
Theorem 1.1 display two regimes governing the behavior of the ground state energy. The two regimes appear as follows. In the regime $H \lesssim \kappa^{3/2}$, if we estimate $\mathrm{E}_{\mathrm{gs}}(\kappa, H)$ using the limiting function $E(\cdot)$, then we cannot manage to prove that the error terms are of lower order compared to the term

$$\kappa\left(\int_\Gamma \left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \mathrm{d}s(x)\right).$$

Surprisingly, when $H \lesssim \kappa^{3/2}$, the leading order behavior of the ground state energy $\mathrm{E}_{\mathrm{gs}}(\kappa, H)$ is governed by the limiting function $g(\cdot)$.

As such, there is a small gap between the two regimes considered in Theorem 1.1. Hence it would be interesting to show that the two asymptotics match in this intermediate zone. A necessary step would be to inspect whether there exists a relationship between the limiting functions $E(\cdot)$ and $g(\cdot)$.

Remark 1.5. (Curvature effects)
By analogy with the existing results for the case of a constant magnetic field in [6,7,11], one expects that the ground state energy $\mathrm{E}_{\mathrm{gs}}(\kappa, H)$ behaves as follows. Let c_0 be the value in (1.13). We expect that:

- If $H = c_0\kappa^2 + o(\kappa^2)$, then the curvature of $\Gamma = \{B_0(x) = 0\}$ will contribute to the leading order behavior of $\mathrm{E}_{\mathrm{gs}}(\kappa, H)$.
- If $\kappa^{3/2} \ll H \leqq M\kappa^2$ and $0 < M < c_0$, then the second correction term in the aymptotics in Theorem 1.1 involves the curvature of Γ.

Along with the proof of Theorem 1.1, we obtain:

Theorem 1.6. *Suppose that the function B_0 satisfies Assumptions (1.2) and (1.3). Let $b : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying (1.10). Suppose that*

$$H = b(\kappa)\kappa$$

and that $(\psi, \mathbf{A})$ is a minimizer of the functional in (1.1).
Then, as $\kappa \to \infty$, the following items hold:

1. **Estimate of the magnetic energy**.

$$\kappa^2 H^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2\,\mathrm{d}x = \frac{\kappa^3}{H} o(1)\,.$$

2. **Estimate of the local energy**.
Let $D \subset \Omega$ be an open set with a smooth boundary such that $\partial D \cap \Gamma$ is a finite set.

(a) *If* $b(\kappa) \gg \kappa^{1/2}$, *then*

$$\begin{aligned}\mathcal{E}_0(\psi, \mathbf{A}; D) &:= \int_D \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right) \mathrm{d}x \\ &= \kappa \left(\int_{D\cap\Gamma} \left(|\nabla B_0(x)| \frac{H}{\kappa^2} \right)^{1/3} E\left(|\nabla B_0(x)| \frac{H}{\kappa^2} \right) \mathrm{d}s(x) \right) \\ &\quad + \frac{\kappa^3}{H} o(1) . \end{aligned} \tag{1.14}$$

(b) *If* $1 \ll b(\kappa) \lesssim \kappa^{1/2}$, *then*

$$\mathcal{E}_0(\psi, \mathbf{A}; D) = \kappa^2 \int_D g\left(\frac{H}{\kappa} |B_0(x)| \right) \mathrm{d}x + \frac{\kappa^3}{H} o(1) .$$

3. **Concentration of the order parameter**.
Let $D \subset \Omega$ *be an open set with a smooth boundary such that* $\partial D \cap \Gamma$ *is a finite set.*
(a) *If* $b(\kappa) \gg \kappa^{1/2}$, *then*

$$\begin{aligned}\int_D |\psi(x)|^4 \,\mathrm{d}x = &-\frac{2}{\kappa}\left(\int_{D\cap\Gamma} \left(|\nabla B_0(x)| \frac{H}{\kappa^2} \right)^{1/3} E\left(|\nabla B_0(x)| \frac{H}{\kappa^2} \right) \mathrm{d}s(x) \right) \\ &+ \frac{\kappa}{H} o(1) .\end{aligned}$$

(b) *If* $1 \ll b(\kappa) \lesssim \kappa^{1/2}$, *then*

$$\int_D |\psi(x)|^4 \,\mathrm{d}x = -\int_D g\left(\frac{H}{\kappa} |B_0(x)| \right) \mathrm{d}x + \frac{\kappa}{H} o(1) .$$

Remark 1.7. In Theorem 1.6, the functions $o(1)$ are controlled independently of the choice of the minimizer $(\psi, \mathbf{A})$. In the first assertion, the expression of $o(1)$ depends only on the domain Ω and the function B_0, while in the second and third assertions, the expression depends additionally on the domain D.

Remark 1.8. In the two regimes displayed in Theorem 1.6, the main term in the asymptotic expansions vanish when $D \cap \Gamma = \emptyset$. It could be interesting to improve the remainder terms. In Theorem 6.3, we will prove that the L^2-norm of the order parameter ψ is concentrated near the set Γ, and that ψ exponentially decays as $\kappa \to \infty$, away from Γ.

2. Critical Fields

The identification of the critical *magnetic* fields is an important question regarding the functional in (1.1). This question has an early appearance in physics (see for example [16]) and was the subject of a vast mathematical literature in the past two decades. The two monographs [8,31] contain an extensive review of many important results. In this section, we give a brief *informal* description of critical fields and highlight the importance of the case of a vanishing applied magnetic field.

2.1. Reminder: The Constant Field Case

When the magnetic field B_0 is a (non-zero) constant, three critical values are assigned to the magnetic field H, namely H_{c_1}, H_{c_2} and H_{c_3}. The behavior of minimizers (and critical points) of the functional in (1.1) changes as the parameter H (that is magnetic field) crosses the values H_{c_1}, H_{c_2} and H_{c_3}. The identification of these critical values is not easy, especially the value H_{c_2}, which is still *loosely* defined.

Let us recall that a critical point $(\psi, \mathbf{A})$ of the functional in (1.1) is said to be *normal* if $\psi = 0$ everywhere. The critical field $H_{c_3}(\kappa)$ is then defined as the value at which the transition from *normal* to *non-normal* critical points takes place.

The identification of the critical value $H_{c_3}(\kappa)$ is strongly related to the spectral analysis of the magnetic Schrödinger operator with a constant magnetic field and Neumann boundary condition. Suppose that $\Omega \subset \mathbb{R}^2$ is connected, open, has a smooth boundary and the boundary consists of a finite number of connected components, A_0 a vector field satisfying $\text{curl} A_0 = B_0$, the function B_0 is constant and positive, and $\lambda(H\kappa A_0)$ the lowest eigenvalue of the magnetic Schrödinger operator

$$-\Delta_{\kappa H A_0} = -(\nabla - i\kappa H A_0)^2 \quad \text{in} \quad L^2(\Omega), \tag{2.1}$$

with Neumann boundary conditions. It has been proved that the function $t \mapsto \lambda(tA_0)$ is monotonic for large values of t, see [8] and the references therein. Grosso modo, the critical field H_{c_3} is the unique solution of the equation,[1]

$$\lambda(H_{c_3}(\kappa)\kappa A_0) = \kappa^2. \tag{2.2}$$

In this case, it was shown by LU and PAN [23] that,

$$\lambda(H\kappa A_0) \sim (H\kappa) B_0 \Theta_0, \quad \text{when} \quad H\kappa \gg 1. \tag{2.3}$$

Further improvements of (2.3) are available, see [8] for the state of the art in 2009 and references therein.

As a consequence of (2.2) and (2.3), we get for κ sufficiently large,

$$H_{c_3}(\kappa) \sim \kappa/(\Theta_0 B_0). \tag{2.4}$$

The second critical field $H_{c_2}(\kappa)$ is usually defined as follows

$$H_{c_2}(\kappa) = \kappa/B_0. \tag{2.5}$$

Notice that this definition of H_{c_2} is asymptotically matching with the following definition,

$$\lambda^D(H_{c_2}(\kappa)\kappa A_0) = \kappa^2, \tag{2.6}$$

[1] Initially (see [23]), one should start by defining four critical values according to locally or globally minimizing solutions. Following the terminology of [8], these are upper or lower, global or local fields. The four fields are proved to be equal in [8].

where λ^D is the first eigenvalue of the operator in (2.1), but with Dirichlet boundary condition[2].

Near $H_{c_2}(\kappa)$, a transition takes place between surface and bulk superconductivity. At the level of the energy, this transition is described in [14]. The bulk distribution of the superconductivity near H_{c_2} is computed in [21].

We recall that $\Theta_0 < 1$. Hence, as expected, $H_{c_2}(\kappa) < H_{c_3}(\kappa)$ for κ sufficiently large. For the identification of the critical field $H_{c_1}(\kappa)$, we refer to Sandier and Serfaty [31]. A natural question is to extend this discussion in the variable magnetic field case (that is, where B_0 is a non-constant function).

2.2. The Case of a Non Vanishing Exterior Magnetic Field

Here we discuss the situation where the magnetic field B_0 is a non-constant function such that $B_0(x) \neq 0$ everywhere in $\overline{\Omega}$. In this case, it is proved by Lu-Pan [24, Theorem 1] that,

$$\lambda(H\kappa A_0) \sim (H\kappa)\min\left(\inf_{x\in\overline{\Omega}} |B_0(x)|, \Theta_0 \inf_{x\in\partial\Omega} |B_0(x)|\right), \tag{2.7}$$

as $H\kappa \to \infty$. Basically, this leads to the consideration of two cases as follows.

Surface superconductivity First, we assume that

$$\inf_{x\in\overline{\Omega}} |B_0(x)| > \Theta_0 \inf_{x\in\partial\Omega} |B_0(x)|. \tag{2.8}$$

In this case, the phenomenon of surface superconductivity observed in the constant magnetic field case is preserved. More precisely, superconductivity starts to appear at the points where $(B_0)_{/\partial\Omega}$ is minimal. The critical value $H_{c_3}(\kappa)$ is still defined by (2.2). If the minima of $(B_0)_{/\partial\Omega}$ are non-degenerate, then the monotonicity of the eigenvalue $\lambda(t\,A_0)$ for large values of t is established in [29, Section 6]. Consequently, we get when κ is sufficiently large,

$$H_{c_3}(\kappa) \sim \frac{\kappa}{\Theta_0 \inf_{x\in\partial\Omega} |B_0(x)|}. \tag{2.9}$$

Tentatively, one could think to define $H_{c_2}(\kappa)$ either by

$$H_{c_2}(\kappa) = \frac{\kappa}{\inf_{x\in\overline{\Omega}} |B_0(x)|}, \tag{2.10}$$

or by

$$\lambda^D(H_{c_2}(\kappa)\kappa A_0) = \kappa^2, \tag{2.11}$$

where λ^D is the first eigenvalue of the operator in (2.1) with Dirichlet boundary condition.

[2] Assuming the monotonicity of $t \mapsto \lambda^D(tA_0)$ for t large.

Notice that both formulas agree with their analogues in the constant magnetic field case (see (2.5) and (2.6)). Also, the values of $H_{c_2}(\kappa)$ given in (2.10) or (2.11) asymptotically match as $\kappa \to \infty$.

In order that the definition of $H_{c_2}(\kappa)$ in (2.11) is consistent, one should prove monotonicity of $t \mapsto \lambda^D(tA_0)$ for large of values of t. This will ensure that (2.11) assigns a unique value of $H_{c_2}(\kappa)$. However, such a monotonicity is not proved yet. The definition in (2.10) was proposed in [8].

Interior onset of superconductivity Here we assume that

$$\inf_{x\in\overline{\Omega}} |B_0(x)| < \Theta_0 \inf_{x\in\partial\Omega} |B_0(x)|. \tag{2.12}$$

In this case, the onset of superconductivity near the surface of the domain disappears. If one decreases gradually the intensity of the magnetic field H from ∞, then superconductivity will start to appear near the minima of the function $|B_0|$, that is inside a compact subset of Ω.

In this situation, we need not distinguish between the critical fields $H_{c_2}(\kappa)$ and $H_{c_3}(\kappa)$, since surface superconductivity is absent here. Consequently, we expect that,

$$H_{c_2}(\kappa) = H_{c_3}(\kappa) \sim \frac{\kappa}{\inf_{x\in\overline{\Omega}} |B_0(x)|}. \tag{2.13}$$

A partial justification of this fact can be done using the linearized Ginzburg–Landau equation near a normal solution. Actually, we may also define $H_{c_3}(\kappa)$ and $H_{c_2}(\kappa)$ as the values verifying (2.2) and (2.6). It should be noticed here that the vector field A_0 satisfies curl $A_0 = B_0$ and B_0 cannot be constant. Under the assumption (2.12), the known spectral asymptotics (which are actually the same in this case) of the Dirichlet and Neumann eigenvalues will lead us to the asymptotics given in the righthand side of (2.13). Under the additional assumption that $\inf_{x\in\overline{\Omega}} |B_0(x)|$ is attained at a unique minimum in Ω and that this minimum is non degenerate, a complete asymptotics of $\lambda^N(tA_0)$ can be given (see HELFFER and MOHAMED [20], HELFFER and KORDYUKOV [18,19], RAYMOND and VU NGOC [30]) and the monotonicity/strong diamagnetism property holds for large values of t (see Chapter 3 in [8]). Hence the definition of $H_{c_3}(\kappa)$ is clear in this case.

Besides the aforementioned linearized calculations, the results of [3] can be used to justify the equality of the critical fields $H_{c_2}(\kappa)$ and $H_{c_3}(\kappa)$ as well as their definition in (2.13).

First, we observe that if C is a positive constant such that $C < \dfrac{1}{\inf_{x\in\overline{\Omega}} |B_0(x)|}$, and if $H \leqq C\kappa$, then the open set $D = \{x \in \Omega\ :\ |B_0(x)| < \frac{1}{C}\} \neq \emptyset$ is non-empty. Now, Theorem 1.4 of [3] asserts that,

$$\exists\, \kappa_0 > 0,\ \exists\, \epsilon_D > 0, \quad \int_D |\psi(x)|^4\, \mathrm{d}x \geqq \epsilon_D > 0,$$

for any $\kappa \geqq \kappa_0$ and any *minimizer* $(\psi, \mathbf{A})$ of the functional in (1.1). Consequently, a minimizer cannot be a *normal solution*.

Now, suppose that the constant C satisfies

$$C > \frac{1}{\inf_{x\in\overline{\Omega}} |B_0(x)|}.$$

If $H \geqq C\kappa$, then Theorem 1.4 of [3] asserts that any critical point $(\psi, \mathbf{A})$ of (1.1) satisfies,

$$\lim_{\kappa\to\infty} \int_\Omega |\psi(x)|^4 \, dx = 0,$$

hence, loosely speaking, critical points are *nearly* normal solutions. However, repeating the proof given in [8, Section 10.4] and using the asymptotics of the first eigenvalue in (2.7), one can get that such critical points are indeed normal solutions.

The foregoing discussion shows that the value appearing in the right hand side of (2.13) is indeed critical.

2.3. The Case of a Vanishing Exterior Magnetic Field

We now discuss the case when B_0 vanishes along a curve, first considered in [27] and then in [3]. We assume that

$$|B_0| + |\nabla B_0| \neq 0 \quad \text{in } \overline{\Omega}, \tag{2.14}$$

which ensures that B_0 vanishes *non-degenerately*.

At each point of $B_0^{-1}(0) \cap \Omega$, Pan and Kwek [27] introduce a reduced model (a Montgomery operator parameterized by the intensity of the magnetic field at this point) whose ground state energy, denoted by λ_0, captures the 'local' ground state energy of the Schrödinger operator in (2.1).

Similarly, at every point x of $B_0^{-1}(0) \cap \partial\Omega$, a toy operator is defined on $\mathbb{R}^2_+$ parameterized (up to unitary equivalence) by the intensity of $B_0(x)$ and the angle $\theta(x) \in [0, \pi/2)$ between the unit normal of the boundary and $\nabla B_0(x)$. The ground state energy of this toy operator is denoted by $\lambda_0(\mathbb{R}_+, \theta(x))$.

The leading order behavior of the ground state energy of the operator in (2.1) is now described as follows [27],

$$\lambda(H\kappa A_0) \sim (H\kappa)^{2/3} \alpha_1^{2/3}, \tag{2.15}$$

as $H\kappa \to \infty$.
Here

$$\alpha_1 = \min\left(\lambda_0^{3/2} \min_{x\in\Gamma_{\rm blk}} |\nabla B_0(x)|, \min_{x\in\Gamma_{\rm bnd}} \lambda_0(\mathbb{R}_+, \theta(x))|\nabla B_0(x)|\right), \tag{2.16}$$

$$\Gamma_{\rm blk} = \{x \in \Omega \ : \ B_0(x) = 0\} \tag{2.17}$$

and

$$\Gamma_{\rm bnd} = \{x \in \partial\Omega \ : \ B_0(x) = 0\}. \tag{2.18}$$

The critical value $H_{c_3}(\kappa)$ could tentatively be defined as the solution of the equation in (2.2). However, when $B_0 = \operatorname{curl} A_0$ vanishes, monotonicity of $t \mapsto \lambda(tA_0)$ is not a direct application of Chapter 3 in [8] (see the discussion below). Nevertheless, for the various definitions of $H_{c_3}(\kappa)$ proposed in [27], one always gets that, for large values of κ,

$$H_{c_3}(\kappa) \sim \frac{\kappa^2}{\alpha_1}. \tag{2.19}$$

Surface superconductivity (near H_{c_3}) is absent if

$$\lambda_0^{3/2} \min_{x \in \Gamma_{\text{blk}}} |\nabla B_0(x)| < \min_{x \in \Gamma_{\text{bnd}}} \lambda_0(\mathbb{R}_+, \theta(x)) |\nabla B_0(x)|,$$

and in this case, we do not distinguish between H_{c_2} and H_{c_3}. However, if

$$\lambda_0^{3/2} \min_{x \in \Gamma_{\text{blk}}} |\nabla B_0(x)| > \min_{x \in \Gamma_{\text{bnd}}} \lambda_0(\mathbb{R}_+, \theta(x)) |\nabla B_0(x)|, \tag{2.20}$$

the phenomenon of surface superconductivity is observed in decreasing magnetic fields. Superconductivity will nucleate near the minima of the function

$$\Gamma_{\text{bnd}} \ni x \mapsto \lambda_0(\mathbb{R}_+, \theta(x)) |\nabla B_0(x)|.$$

In this case, a natural definition of $H_{c_2}(\kappa)$ can be

$$H_{c_2}(\kappa) := \frac{\kappa^2}{\alpha_2}, \tag{2.21}$$

for large values of κ.
Here

$$\alpha_2 = \lambda_0^{3/2} \min_{x \in \Gamma_{\text{blk}}} |\nabla B_0(x)|.$$

The methods in [9] suggest that the monotonicity of the eigenvalue $\lambda(tA_0)$ for large values of t can be obtained in the case when (2.20) is satisfied[3]. A necessary step is to find the second correction term in (2.15). The work in [25] is along this direction. (Recall that $\lambda(tA_0)$ is the eigenvalue of the operator in (2.1) with Neumann condition.)

Clearly, the condition in (1.9) is violated when the intensity of the magnetic field H is comparable with the critical value $H_{c_3}(\kappa) \approx \kappa^2$, thereby preventing the application of the results of ATTAR [3]. The case with pinning will be analyzed in [5].

3. The Limiting Problems

In this section, we define the two limiting functions $E(\cdot)$ and $g(\cdot)$ appearing in Theorem 1.1. The limiting function $g(\cdot)$, that we might call the bulk energy, is defined previously in [13,32]. It is a characteristic of superconducting samples subject to a constant magnetic field. The limiting function $E(\cdot)$ arises as the limit of a certain simplified Ginzburg–Landau functional with a magnetic field vanishing

[3] Personal communication of S. Fournais.

along a line. The construction of the limiting function $E(\cdot)$ occupies most of this section.

This section contains three important theorems:

- Theorem 3.8 contains the definition of the limiting function $E(\cdot)$.
- Theorem 3.11 displays a relationship between the limiting function $E(\cdot)$ and the energy of a simplified functional defined in a disc domain.
- Theorem 3.12 contains a remarkable property of the function $E(\cdot)$. The proof of this property uses the function $g(\cdot)$.

The conclusions in the above three theorems will be used throughout the rest of this paper.

3.1. The Montgomery Operator

Consider the self-adjoint operator in $L^2(\mathbb{R}^2)$

$$P = -\left(\partial_{x_1} - i\frac{x_2^2}{2}\right)^2 - \partial_{x_2}^2. \tag{3.1}$$

The ground state energy

$$\lambda_0 = \inf \sigma(P) \tag{3.2}$$

of the operator P is described using the Montgomery operator as follows.
If $\tau \in \mathbb{R}$, let $\lambda(\tau)$ be the first eigenvalue of the Montgomery operator [26],

$$P(\tau) = -\frac{\mathrm{d}^2}{\mathrm{d}x_2^2} + \left(\frac{x_2^2}{2} + \tau\right)^2, \quad \text{in} \quad L^2(\mathbb{R}). \tag{3.3}$$

Notice that the eigenvalue $\lambda(\tau)$ is positive, simple and has a unique positive eigenfunction φ^τ of L^2 norm 1. There exists a **unique** $\tau_0 \in \mathbb{R}$ such that

$$\lambda_0 = \lambda(\tau_0). \tag{3.4}$$

Hence $\lambda_0 > 0$. We write

$$\varphi_0 = \varphi^{\tau_0}.$$

Clearly, the function

$$\psi_0(x_1, x_2) = e^{-i\tau_0 x_1}\varphi_0(x_2), \tag{3.5}$$

is a bounded (generalized) eigenfunction of the operator P with eigenvalue λ_0. Moreover (see [17] and references therein) the minimum of λ at τ_0 is non-degenerate.

We collect some important properties of the family of operators $P(\tau)$.

Theorem 3.1. ([17])

(1) $\tau_0 < 0$.
(2) $\lim_{\tau\to\pm\infty} \lambda(\tau) = \infty$.
(3) *The function* $\lambda(\tau)$ *is increasing on the interval* $[0, \infty)$.

3.2. A One Dimensional Energy

Let $b > 0$ and $\alpha \in \mathbb{R}$. Consider the functional

$$\mathcal{E}^{1D}_{\alpha,b}(f) = \int_{-\infty}^{\infty} \left(|f'(t)|^2 + \left(\frac{t^2}{2} + \alpha\right)^2 |f(t)|^2 - b\,|f(t)|^2 + \frac{b}{2}\,|f(t)|^4 \right) \mathrm{d}t, \tag{3.6}$$

defined over configurations in the space

$$B^1(\mathbb{R}) = \{f \in H^1(\mathbb{R};\mathbb{R}) \ :\ t^2 f \in L^2(\mathbb{R})\}.$$

In light of Theorem 3.1, we may define two functions $z_1(b)$ and $z_2(b)$ satisfying,

$$z_1(b) < \tau_0 < z_2(b), \quad \lambda^{-1}\big([\tau_0, b)\big) = (z_1(b), z_2(b)). \tag{3.7}$$

Notice that, if $b < \lambda(0)$, then $z_2(b) < 0$. This follows from (3) in Theorem 3.1.

Theorem 3.2. ([8, Sec. 14.2])

(1) *The functional $\mathcal{E}^{1D}_{\alpha,b}$ has a non-trivial minimizer in the space $B^1(\mathbb{R})$ if and only if*
$\lambda(\alpha) < b$. Furthermore, a non-trivial minimizer f_α can be found which is a positive function and $\pm f_\alpha$ are the only real-valued minimizers.

(2) *Let*

$$\mathfrak{b}(\alpha, b) = \inf\{\mathcal{E}^{1D}_{\alpha,b}(f) \ :\ f \in B^1(\mathbb{R})\}. \tag{3.8}$$

There exists $\alpha_0 \in (z_1(b), z_2(b))$ such that,

$$\mathfrak{b}(\alpha_0, b) = \inf_{\alpha\in\mathbb{R}} \mathfrak{b}(\alpha, b). \tag{3.9}$$

(3) *If $b < \lambda(0)$, then $\alpha_0 < 0$.*

(4) (*Feynman–Hellmann*)

$$\int_{-\infty}^{\infty} \left(\frac{t^2}{2} + \alpha_0\right) |f_{\alpha_0}(t)|^2\, \mathrm{d}t = 0. \tag{3.10}$$

The proof of this theorem can be obtained by adapting the analysis of [8, Sec. 14.2] devoted to the functional

$$\mathcal{F}^{1D}_{\alpha,b}(f) = \int_0^{\infty} \left(|f'(t)|^2 + (t+\alpha)^2\,|f(t)|^2 - b|f(t)|^2 + \frac{b}{2}|f(t)|^4 \right) \mathrm{d}t. \tag{3.11}$$

We note for future use that a minimizer of $\mathcal{E}^{1D}_{\alpha,b}$ satisfies the Euler-Lagrange equation:

$$-\,f''(t) + \left(\frac{t^2}{2} + \alpha\right)^2 f(t) - bf(t) + bf(t)^3 = 0, \tag{3.12}$$

and that $f \in \mathcal{S}(\mathbb{R})$.

According to Theorem 3.2, we observe that the functional $\mathcal{E}^{1D}_{\alpha,b}$ has non-trivial minimizers if and only if $\alpha \in (z_1(b), z_2(b))$.

3.3. Reduced Ginzburg–Landau Functional

Let $L > 0$, $R > 0$, $\mathcal{S}_R = (-R, R) \times \mathbb{R}$ and

$$\mathbf{A}_{\text{app}}(x) = \left(-\frac{x_2^2}{2}, 0\right), \quad (x = (x_1, x_2) \in \mathcal{S}_R = (-R, R) \times \mathbb{R}). \tag{3.13}$$

Consider the functional

$$\mathcal{E}_{L,R}(u) = \int_{\mathcal{S}_R} \left(|(\nabla - i\mathbf{A}_{\text{app}})u|^2 - L^{-2/3}|u|^2 + \frac{L^{-2/3}}{2}|u|^4 \right) \mathrm{d}x, \tag{3.14}$$

and the ground state energy

$$\begin{aligned} \mathfrak{e}_{\text{gs}}(L; R) = \inf\{\mathcal{E}_{L,R}(u) \ : \ (\nabla - i\mathbf{A}_{\text{app}})u \in L^2(\mathcal{S}_R), \\ u \in L^2(\mathcal{S}_R), \ \text{and } u = 0 \text{ on } \partial\mathcal{S}_R\}. \end{aligned} \tag{3.15}$$

Following the analysis in [28] and [15, Theorem 3.6], we can prove that the functional in (3.14) has a minimizer. If $\varphi_{L,R}$ denotes such a minimizer, then we will prove in Theorem 3.3,

$$\begin{aligned} &|x_2|^{3/2}(\nabla - i\mathbf{A}_{\text{app}})\varphi_{L,R} \in L^2(\mathcal{S}_R), \quad |x_2|^{1/2}\varphi_{L,R} \in L^2(\mathcal{S}_R) \quad \text{and} \\ &|x_2|^{3/2}|\varphi_{L,R}|^2 \in L^2(\mathcal{S}_R). \end{aligned} \tag{3.16}$$

Useful properties of the minimizer $\varphi_{L,R}$ are collected in the next theorem. They give a rough description of the decay of the minimizer $\varphi_{L,R}$ at infinity. Most importantly, the estimates in (3.20) and (3.18) describe the decay at infinity and are valid when $R \to \infty$ and $L \to 0$.

The estimates obtained in Theorem 3.3 will serve in computing various quantities involving $\varphi_{L,R}$. With these estimates in hand, one can cut the domain of the variable x_2 at the price of a small controlled error (see the proof of Theorem 3.11).

Theorem 3.3. *Let $L > 0$, $R > 0$ and $\varphi_{L,R}$ be a minimizer of the functional $\mathcal{E}_{L,R}$ in (3.14). It holds that*

$$\|\varphi_{L,R}\|_\infty \leqq 1. \tag{3.17}$$

Furthermore, there exists a universal positive constants C such that the minimizer $\varphi_{L,R}$ satisfies the following inequalities:

$$\int_{\mathcal{S}_R \cap \{|x_2| \geqq 4L^{-2/3}\}} |x_2|\,|\varphi_{L,R}|^2\,\mathrm{d}x \leqq CL^{2/3}R, \tag{3.18}$$

$$\int_{\mathcal{S}_R \cap \{|x_2| \geqq 6L^{-2/3}\}} |x_2|^3|\varphi_{L,R}|^4\,\mathrm{d}x \leqq CL^{4/3}R, \tag{3.19}$$

and

$$\int_{\mathcal{S}_R \cap \{|x_2| \geqq 6L^{-2/3}\}} |x_2|^3\,|(\nabla - i\mathbf{A}_{\text{app}})\varphi_{L,R}|^2\,\mathrm{d}x \leqq C\,L^{2/3}\,R. \tag{3.20}$$

Proof. The minimizer $\varphi_{L,R}$ satisfies the Ginzburg–Landau equation,

$$-(\nabla - i\mathbf{A}_{\mathrm{app}})^2\varphi_{L,R} = L^{-2/3}(1 - |\varphi_{L,R}|^2)\varphi_{L,R}. \tag{3.21}$$

Hence (3.17) results from the strong maximum principle.

In the sequel, if x denotes a variable point in $\mathcal{S}_R$, then the coordinates of x will be denoted (x_1, x_2) so that $x_1 \in (-R, R)$ and $x_2 \in (-\infty, \infty)$.

For $m > 4L^{-2/3}$, we construct $\eta_m \in C_c^\infty(\mathbb{R})$ and ζ as functions satisfying

$$\operatorname{supp}\eta_m \subset (-2m, 2m), \quad \eta_m = 1 \quad \text{in } (-m, m), \quad |\eta_m'| \leqq \frac{C}{m} \quad \text{in } \mathbb{R},$$

and

$$\zeta = 0 \quad \text{in } [-2L^{-2/3}, 2L^{-2/3}], \quad \zeta = 1 \quad \text{in } \mathbb{R} \setminus [-4L^{-2/3}, 4L^{-2/3}],$$
$$|\zeta'| \leqq CL^{2/3} \quad \text{in } \mathbb{R}^2,$$

where C is a constant independent of m and L. Below, ζ and η_m will denote the associated functions on $\mathbb{R}^2$, that is $(x_1, x_2) \mapsto \eta_m(x_2)$ and $(x_1, x_2) \mapsto \zeta(x_1, x_2)$. We have the simple decomposition formula,

$$\begin{aligned}\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\mathrm{app}})(\zeta\eta_m\varphi_{L,R})|^2 - L^{-2/3}|\zeta\eta_m\varphi_{L,R}|^2 + \frac{L^{-2/3}}{2}\zeta^2\eta_m^2|\varphi_{L,R}|^4\,\mathrm{d}x \\ = \int_{\mathcal{S}_R} |(\zeta\eta_m)'\varphi_{L,R}|^2\,\mathrm{d}x \leqq \frac{CR}{m} + CRL^{2/3}.\end{aligned} \tag{3.22}$$

The upper bound for the integral in the right-hand side follows from the condition on the support of η_m', ζ', the bounds $|\eta_m'| \leqq C/m$, $|\zeta'| \leqq CL^{2/3}$ and $\|\varphi_{L,R}\|_\infty \leqq 1$ (see (3.17)). Since $\zeta\eta_m\varphi_{L,R} \in H_0^1(\mathcal{S}_R)$ and $\zeta = 0$ in $\{|x_2| \leqq 2L^{-2/3}\}$, then we can write,

$$\begin{aligned}\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\mathrm{app}})(\zeta\eta_m\varphi_{L,R})|^2\,\mathrm{d}x &\geqq \int_{\mathcal{S}_R} |\operatorname{curl}\mathbf{A}_{\mathrm{app}}|\,|\zeta\eta_m\varphi_{L,R}|^2\,\mathrm{d}x \\ &= \int_{\{|x_2| \geqq 2L^{-2/3}\}} |x_2|\,|\zeta\eta_m\varphi_{L,R}|^2\,\mathrm{d}x.\end{aligned} \tag{3.23}$$

In that way, we infer from (3.22),

$$\int_{\{x \in \mathcal{S}_R\,:\,|x_2| \geqq 2L^{-2/3}\}} \frac{|x_2|}{2}\,|\zeta\eta_m\varphi_{L,R}|^2\,\mathrm{d}x \leqq \frac{CR}{m} + CRL^{2/3}.$$

Sending m to ∞ (and using monotone convergence), we arrive at

$$\int_{\{x \in \mathcal{S}_R\,:\,|x_2| \geqq 2L^{-2/3}\}} \frac{|x_2|}{2}\,|\zeta\varphi_{L,R}|^2\,\mathrm{d}x \leqq CL^{2/3}R.$$

Since $\zeta = 1$ in $\{|x_2| \geqq 4L^{-2/3}\}$, we get further,

$$\int_{\{x \in \mathcal{S}_R\,:\,|x_2| \geqq 4L^{-2/3}\}} \frac{|x_2|}{2}\,|\varphi_{L,R}|^2\,\mathrm{d}x \leqq CL^{2/3}R. \tag{3.24}$$

This proves that $|x_2|^{1/2}\varphi_{L,R} \in L^2(\mathcal{S}_R)$ and the estimate in (3.18).

Next we prove that $|x_2|^{3/2}|\varphi_{L,R}|^2 \in L^2(\mathcal{S}_R)$. To that end, let $\chi \in C^\infty(\mathbb{R})$ be a function (of the variable x_2) satisfying,

$$\chi(x_2) = 0 \quad \text{if } |x_2| \leqq 4L^{-2/3}, \quad \chi(x_2) = |x_2|^{3/2} \quad \text{if } |x_2| \geqq 6L^{-2/3},$$
$$\text{and} \quad |\chi'(x_2)| \leqq CL^{-1/3} \quad \text{if } |x_2| \leqq 6L^{-2/3}.$$

We have the decomposition formula,

$$\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\rm app})(\eta_m\chi\varphi_{L,R})|^2 - L^{-2/3}|\eta_m\chi\varphi_{L,R}|^2 + \frac{L^{-2/3}}{2}\eta_m^2\chi^2|\varphi_{L,R}|^4\,\mathrm{d}x$$
$$= \int_{\mathcal{S}_R} |(\eta_m\chi)'\varphi_{L,R}|^2\,\mathrm{d}x. \tag{3.25}$$

Using the bounds satisfied by η_m', χ' and η_m, the condition on the support of η_m and the inequality in (3.24), we may write, for all $m > 6L^{-4/3}$,

$$\int_{\mathcal{S}_R} |(\eta_m\chi)'\varphi_{L,R}|^2\,\mathrm{d}x$$
$$\leqq C\int_{\{x\in\mathcal{S}_R\,:\,m\leqq|x_2|\leqq 2m\}} |x_2|^3(\eta_m'(x_2))^2|\varphi_{L,R}|^2\,\mathrm{d}x + C\int_{\mathcal{S}_R}|\chi'(x_2)|^2\,\eta_m^2\,|\varphi_{L,R}|^2\,\mathrm{d}x$$
$$\leqq C\left(\int_{\{x\in\mathcal{S}_R\,:\,|x_2|\geqq 4L^{-2/3}\}} |x_2|\,|\varphi_{L,R}|^2\,\mathrm{d}x + L^{-2/3}\int_{\{x\in\mathcal{S}_R\,:\,|x_2|\geqq 4L^{-2/3}\}}|\varphi_{L,R}|^2\,\mathrm{d}x\right)$$
$$\leqq CL^{2/3}R. \tag{3.26}$$

Next, we use the inequality

$$\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\rm app})(\eta_m\chi\varphi_{L,R})|^2\,\mathrm{d}x \geqq \int_{\mathcal{S}_R} |\,\mathrm{curl}\,\mathbf{A}_{\rm app}|\,|\eta_m\chi\varphi_{L,R}|^2\,\mathrm{d}x$$
$$= \int_{\mathcal{S}_R} |x_2|\,|\eta_m\chi\varphi_{L,R}|^2\,\mathrm{d}x,$$

and the fact that $\chi = 0$ for $|x_2| \leqq 4L^{-2/3}$ to infer from (3.25) and (3.26),

$$\frac{L^{-2/3}}{2}\int_{\mathcal{S}_R} \eta_m^2\chi^2|\varphi_{L,R}|^4\,\mathrm{d}x \leqq CL^{2/3}R.$$

Sending m to ∞, we get that $|x_2|^{3/2}|\varphi_{L,R}|^2 \in L^2(\mathcal{S}_R)$ and the estimate in (3.19). Also, we infer from (3.25) that

$$\frac{3}{4}\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\rm app})(\eta_m\chi\varphi_{L,R})|^2\,\mathrm{d}x \leqq CL^{2/3}R. \tag{3.27}$$

Using a simple commutator argument, we write,

$$\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\rm app})(\eta_m\chi\varphi_{L,R})|^2\,\mathrm{d}x \geqq \frac{1}{2}\int_{\mathcal{S}_R}(\eta_m\chi)^2|(\nabla - i\mathbf{A}_{\rm app})\varphi_{L,R}|^2\,\mathrm{d}x$$
$$-4\int_{\mathcal{S}_R} |(\eta_m\chi)'\varphi_{L,R}(x)|^2\,\mathrm{d}x.$$

By (3.26), we get, further, that

$$\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\mathrm{app}})(\eta_m\chi\varphi_{L,R})|^2\,\mathrm{d}x \geqq \frac{1}{2}\int_{\mathcal{S}_R} (\eta_m\chi)^2|(\nabla - i\mathbf{A}_{\mathrm{app}})\varphi_{L,R}|^2\,\mathrm{d}x - CL^{2/3}R.$$

We insert this into (3.27) to get

$$\frac{3}{4}\int_{\mathcal{S}_R} (\eta_m\chi)^2|(\nabla - i\mathbf{A}_{\mathrm{app}})\varphi_{L,R}|^2\,\mathrm{d}x \leqq CL^{2/3}R.$$

Sending m to ∞, we deduce that $|x_2|^{3/2}(\nabla - i\mathbf{A}_{\mathrm{app}})\varphi_{L,R} \in L^2(\mathcal{S}_R)$ and the estimate in (3.20). ◻

Remark 3.4. We can bootstrap the argument in the proof of Theorem 3.3 to get the following improvement of (3.18): for all $n \in \mathbb{N}$, there exists $C_n > 0$ such that, for all $L > 0$,

$$\int_{\mathcal{S}_R\cap\{|x_2|\geqq 4nL^{-2/3}\}} |x_2|\,|\varphi_{L,R}|^2\,\mathrm{d}x \leqq C_nL^{2n/3}R.$$

As a consequence of Theorem 3.3, we can obtain a uniform estimate of the energy components of a minimizer $\varphi_{L,R}$.

Proposition 3.5. *Let $\Lambda > 0$. There exists a positive constant C_Λ such that, for all $L \in (0, \Lambda)$ and $R > 0$,*

$$\int_{\mathcal{S}_R} |\varphi_{L,R}(x)|^2\,\mathrm{d}x \leqq C_\Lambda L^{-2/3}R \quad\text{and}\quad \int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\mathrm{app}})\varphi_{L,R}|^2\,\mathrm{d}x \leqq C_\Lambda L^{-4/3}R.$$

Proof. As a consequence of the inequalities in (3.17) and (3.18), we have,

$$\begin{aligned}\int_{\mathcal{S}_R} |\varphi_{L,R}(x)|^2\,\mathrm{d}x &= \int_{\mathcal{S}_R\cap\{|x_2|\leqq 4L^{-2/3}\}} |\varphi_{L,R}(x)|^2\,\mathrm{d}x \\ &\quad + \int_{\mathcal{S}_R\cap\{|x_2|\geqq 4L^{-2/3}\}} |\varphi_{L,R}(x)|^2\,\mathrm{d}x \\ &\leqq 8L^{-2/3}R + CL^{4/3} = L^{-2/3}(8 + CL^2).\end{aligned}$$

Since $L^{-2/3}(8 + CL^2) \sim 8L^{-2/3}$ when $L \to 0_+$, we get a constant $C_\Lambda > 0$ such that, for all $L \in (0, \Lambda)$,

$$\int_{\mathcal{S}_R} |\varphi_{L,R}(x)|^2\,\mathrm{d}x \leqq C_\Lambda L^{-2/3}.$$

To finish the proof of the theorem, we multiply both sides of (3.21) by $\overline{\varphi_{L,R}}$ and integrate by parts. In that way we obtain

$$\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\mathrm{app}})\varphi_{L,R}|^2\,\mathrm{d}x \leqq L^{-2/3}\int_{\mathcal{S}_R} |\varphi_{L,R}(x)|^2\,\mathrm{d}x \leqq C_\Lambda L^{-4/3}\,R.$$

◻

The value of the ground state energy defined in (3.15) is related to the eigenvalue λ_0 in (3.4). We will find that the energy vanishes when $L \geqq \lambda_0^{-3/2}$, and we will give a rough estimate of the energy when $L \to \lambda_0^{-3/2}$.

Proposition 3.6. *For $R > 0$, $L > 0$, if $\mathfrak{e}_{\rm gs}(L; R)$ denotes the ground state energy in (3.15), it holds:*

(1) *If $L \geqq \lambda_0^{-3/2}$, then $\mathfrak{e}_{\rm gs}(L; R) = 0$.*
(2) *There exist positive constants C_1, C_2 and C_3 such that, if $L < \lambda_0^{-3/2}$ and $R > 0$, then*

$$- C_1 L^{-4/3} R \leqq \frac{\mathfrak{e}_{\rm gs}(L; R)}{(1 - \lambda_0 L^{2/3})} \leqq -C_2 L^{-2/3} R + \frac{C_3}{R}. \tag{3.28}$$

Proof. Suppose that $L \geqq \lambda_0^{-3/2}$. Let $u \in H_0^1(\mathcal{S}_R)$. The min-max principle and the condition on L tell us that $\mathcal{E}_{L,R}(u) \geqq 0$ and consequently $\mathfrak{e}_{\rm gs}(L; R) \geqq 0$. But $\mathfrak{e}_{\rm gs}(L; R) \leqq \mathcal{E}_{L,R}(0) = 0$. This proves the statement in (1).
Now, suppose that $L < \lambda_0^{-3/2}$. Let $\theta \in C_c^\infty(\mathbb{R})$ be a function satisfying,

$$\operatorname{supp}\theta \subset (-1, 1), \quad 0 \leqq \theta \leqq 1, \quad \theta = 1 \text{ in } \quad (-1/2, 1/2),$$

and let

$$\theta_R(x) := \theta(x/R).$$

Let $t > 0$ and

$$u(x_1, x_2) = t\, \theta_R(x_1)\, \psi_0(x_1, x_2),$$

where ψ_0 is the function in (3.5).
Recall that ψ_0 satisfies $-(\nabla - i\mathbf{A}_{\rm app})^2\, \psi_0 = \lambda_0 \psi_0$. An integration by parts yields,

$$\begin{aligned}
&\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\rm app})u|^2\, \mathrm{d}x \\
&= t^2 \left(\left\langle \theta_R(x_1)^2 \psi_0,\, -(\nabla - i\mathbf{A}_{\rm app})^2 \psi_0 \right\rangle + \int_{\mathcal{S}_R} |\phi_0(x_2)\theta_R'(x_1)|^2\, \mathrm{d}x \right) \\
&= t^2 \left(\left\langle \theta_R(x_1)^2 \psi_0,\, -(\nabla - i\mathbf{A}_{\rm app})^2 \psi_0 \right\rangle + \frac{1}{R} \int \theta'(x_1)^2\, \mathrm{d}x_1 \right) \\
&= t^2 \left(\lambda_0 \int_{\mathcal{S}_R} |\theta_R(x_1)\psi_0(x)|^2\, \mathrm{d}x + \frac{C}{R} \right).
\end{aligned}$$

As a consequence, we get that,

$$\begin{aligned}
\mathfrak{e}_{\rm gs}(L, R) \leqq \mathcal{E}_{L,R}(u) &\leqq t^2 \left(\lambda_0 - L^{-2/3} \right) \int_{\mathcal{S}_R} |\theta_R \psi_0|^2\, \mathrm{d}x \\
&\quad + t^2 \frac{C}{R} + t^4 \frac{L^{-2/3}}{2} \int_{\mathcal{S}_R} |\theta_R(x_1)\psi_0(x)|^4 \mathrm{d}x. \\
&\leqq t^2 \left(R \left(\lambda_0 - L^{-2/3} \right) + \frac{C}{R} \right) + R\nu L^{-2/3}\, t^4.
\end{aligned}$$

Here

$$\nu = \int_{\mathbb{R}} |\varphi_0(x_2)|^4 \,\mathrm{d}x_2$$

and φ_0 is the L^2-normalized function introduced in (3.17).

Selecting t such that

$$(\lambda_0 - L^{-2/3}) + \nu L^{-2/3} t^2 = \frac{1}{2}(\lambda_0 - L^{-2/3})$$

finishes the proof of the upper bound.

The lower bound is obtained as follows. Let $\varphi_{L,R}$ be the minimizer in Theorem 3.3. It follows from the min-max principle that,

$$\mathfrak{e}_{\mathrm{gs}}(L;R) = \mathcal{E}_{L,R}(\varphi_{L,R}) \geqq L^{-2/3}(\lambda_0 L^{2/3} - 1)\int_{\mathcal{S}_R} |\varphi_{L,R}(x)|^2 \,\mathrm{d}x.$$

Under the assumption $L < \lambda_0^{-2/3}$, Proposition 3.5 tells us that

$$\int_{\mathcal{S}_R} |\varphi_{L,R}(x)|^2 \,\mathrm{d}x \leqq C_1 L^{-2/3} R.$$

As a consequence, we get the lower bound. □

Remark 3.7. In light of Propositions 3.5 and 3.6, we observe that:

(1) If $L \geqq \lambda_0^{-2/3}$, then $\varphi_{L,R} = 0$ is the minimizer of the functional in (3.14) realizing the ground state energy in (3.15).

(2) If $L \leqq \lambda_0^{-2/3}$, every minimizer $\varphi_{L,R}$ satisfies,

$$\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A}_{\mathrm{app}})\varphi_{L,R}(x)|^2 \,\mathrm{d}x \leqq C L^{-4/3}, \quad \int_{\mathcal{S}_R} |\varphi_{L,R}(x)|^2 \,\mathrm{d}x \leqq C L^{-2/3} R, \tag{3.29}$$

where C is a universal constant.

Notice that the energy $\mathcal{E}_{L,R}(u)$ in (3.14) is invariant under translation along the x_1-axis. This allows us to follow the approach in [13,28] and obtain that the limit of $\frac{\mathfrak{e}_{\mathrm{gs}}(L;R)}{R}$ as $R \to \infty$ exists. The precise statement is:

Theorem 3.8. *Given $L > 0$, there exists $E(L) \leqq 0$ such that,*

$$\lim_{R\to\infty} \frac{\mathfrak{e}_{\mathrm{gs}}(L;R)}{2R} = E(L).$$

The function $(0,\infty) \ni L \mapsto E(L) \in (-\infty, 0]$ is continuous, monotone increasing and

$$E(L) = 0 \quad \textit{if and only if } L \geqq \lambda_0^{-3/2}.$$

Furthermore,

$$\forall\, R > 0, \forall L > 0, \quad E(L) \leqq \frac{\mathfrak{e}_{\rm gs}(L;\, R)}{2R}, \tag{3.30}$$

and there exists a constant C such that

$$\forall\, R \geqq 2, \forall L > 0, \quad \frac{\mathfrak{e}_{\rm gs}(L;\, R)}{2R} \leqq E(L) + C(1 + L^{-2/3})R^{-2/3}. \tag{3.31}$$

Proof. There is nothing to prove when $L \geqq \lambda_0^{-3/2}$, hence we assume that

$$0 < L < \lambda_0^{-3/2}.$$

Step 1. Let $n \geqq 2$ be a natural number, $a \in (0, 1)$ and consider the family of strips

$$\begin{aligned} I_j = \Big(& -n^2 - 1 - a + (2j-1)\left(1 + \frac{a}{2}\right), \\ & -n^2 - 1 + (2j+1)\left(1 + \frac{a}{2}\right)\Big) \times \mathbb{R}, \quad (j \in \mathbb{Z}). \end{aligned}$$

Notice that the width of each strip in the family (I_j) is $2(1 + a)$, and if two strips in the family overlap, then the width of the overlapping region is a. Consider a partition of unity of $\mathbb{R}^2$ such that

$$\sum_j |\chi_j|^2 = 1, \quad 0 \leqq \chi_j \leqq 1, \quad \sum_j |\nabla \chi_j|^2 \leqq \frac{C}{a^2}, \quad \operatorname{supp} \chi_j \subset I_j,$$

where C is a universal constant.
Define $\chi_{R,j}(x) = \chi_j(x/R)$. That way we obtain the new partition of unity,

$$\sum_j |\chi_{R,j}|^2 = 1, \quad 0 \leqq \chi_{R,j} \leqq 1, \quad \sum_j |\nabla \chi_{R,j}|^2 \leqq \frac{C}{a^2 R^2}, \quad \operatorname{supp} \chi_{R,j} \subset I_{R,j},$$

where $I_{R,j} = \{Rx \ : \ x \in I_j\}$.

Notice that $(I_{R,j})_{j \in \{1,2,\cdots,n^2\}}$ is a covering of $\mathcal{S}_{n^2R} = (-n^2R, n^2R) \times \mathbb{R}$ by n^2 strips, each having side-length $2(1 + a)R$.

Let $\varphi_{L,n^2R} \in H_0^1(\mathcal{S}_{n^2R})$ be the minimizer in Theorem 3.3. It holds the decomposition

$$\begin{aligned} \mathfrak{e}_{\rm gs}(L;\, n^2R) &= \mathcal{E}_{L,n^2R}(\varphi_{L,n^2R}) \\ &\geqq \sum_{j=1}^{n^2} \left(\mathcal{E}_{L,n^2R}(\chi_{R,j}\varphi_{L,n^2R}) - \| \, |\nabla \chi_{R,j}| \, \varphi_{L,n^2R} \|^2_{L^2(\mathcal{S}_{n^2R})} \right) \\ &= \left(\sum_{j=1}^{n^2} \mathcal{E}_{L,n^2R}(\chi_{R,j}\varphi_{L,n^2R}) \right) - \int_{\mathcal{S}_{n^2R}} \left(\sum_{j=1}^{n^2} |\nabla \chi_{R,j}|^2 \right) |\varphi_{L,n^2R}|^2 \, dx \\ &\geqq \left(\sum_{j=1}^{n^2} \mathcal{E}_{L,n^2R}(\chi_{R,j}\varphi_{L,n^2R}) \right) - \frac{Cn^2L^{-2/3}}{a^2R} \quad \text{[By Remark 3.7]}. \end{aligned}$$

The function $\chi_{R,j}\varphi_{L,n^2R}$ is supported in an infinite strip of width $2(1+a)R$. Since the energy $\mathcal{E}_{L,R}(u)$ is (magnetic) translation-invariant along the x_1 direction, we get

$$\forall\, j, \quad \mathcal{E}_{L,n^2R}(\chi_{R,j}\varphi_{L,n^2R}) \geqq \mathfrak{e}_{\mathrm{gs}}(L;\,(1+a)R)$$

and consequently,

$$\mathfrak{e}_{\mathrm{gs}}(L;\,n^2R) \geqq n^2\mathfrak{e}_{\mathrm{gs}}(L;\,(1+a)R) - C\,\frac{n^2L^{-2/3}}{a^2R}.$$

Dividing both sides of the above inequality by n^2R and using the estimate in Proposition 3.6, we get

$$\frac{\mathfrak{e}_{\mathrm{gs}}(L;\,n^2R)}{n^2R} \geqq \frac{\mathfrak{e}_{\mathrm{gs}}(L;\,(1+a)R)}{R} - C\left(aL^{-2/3} + \frac{L^{-2/3}}{a^2R^2}\right).$$

Using the trivial inequality $(1+a) \leqq (1+a)^2$, we finally obtain:

$$\frac{\mathfrak{e}_{\mathrm{gs}}(L;\,n^2R)}{n^2R} \geqq \frac{\mathfrak{e}_{\mathrm{gs}}(L;\,(1+a)^2R)}{(1+a)^2R} - C\left(aL^{-2/3} + \frac{L^{-2/3}}{a^2R^2}\right). \tag{3.32}$$

Step 2. Let $\ell > 0$. Let us define

$$d(\ell, L) = \frac{\mathfrak{e}_{\mathrm{gs}}(L;\,\ell^2)}{2}, \quad f(\ell, L) = \frac{d(\ell, L)}{\ell^2}.$$

Clearly, the function $\ell \mapsto d(\ell, L)$ is decreasing. Thanks to Proposition 3.6, we observe that $d(\ell, L) \leqq 0$ and $f(\ell, L)$ is bounded. Furthermore, (3.32) used with $R = \ell^2$ tells us that

$$f(n\ell, L) \geqq f((1+a)\ell, L) - C\left(aL^{-2/3} + \frac{1}{a^2\ell^2}\right).$$

By [15, Lemma 3.10], we get the existence of $E(L)$ such that

$$\lim_{\ell\to\infty} f(\ell, L) = E(L).$$

The simple change of variable $\ell = \sqrt{R}$ gives us

$$\lim_{R\to\infty} \frac{\mathfrak{e}_{\mathrm{gs}}(L;\,R)}{2R} = E(L).$$

Step 3. Using a comparison argument and the translation invariance of the energy $\mathcal{E}_{L,R}(u)$, we observe that

$$\forall\, n \in \mathbb{N}, \quad \mathfrak{e}_{\mathrm{gs}}(L;\,n^2R) \leqq n^2\mathfrak{e}_{\mathrm{gs}}(L;\,R).$$

Dividing both sides of the above inequality by $2n^2R$ and taking $n \to \infty$, we get

$$E(L) = \lim_{n\to\infty} \frac{\mathfrak{e}_{\mathrm{gs}}(L;\,n^2R)}{2n^2R} \leqq \frac{\mathfrak{e}_{\mathrm{gs}}(L;\,R)}{2R}.$$

The matching lower bound for $E(L)$ is obtained by taking $n \to \infty$ in (3.32), selecting $a = R^{-2/3}$ and replacing R by $(1+a)^2 R$.

Step 4. Proposition 3.6 tells us that $E(L) = 0$ if and only if $L \geqq \lambda_0^{-3/2}$. The continuity and monotonicity properties of $E(L)$ are easily obtained through the study of the energy $\mathcal{E}_{L,R}(u)$ as a function of L along the same methods used in [15, Thm 3.13]. □

It would be desirable to establish a simpler expression of $E(L)$ when $L \in (\lambda(0)^{-\frac{3}{2}}, \lambda_0^{-\frac{3}{2}})$:

Conjecture 3.9. *Let λ be the function introduced in (3.3). If*

$$\lambda_0 < L^{-2/3} < \lambda(0), \tag{3.33}$$

then

$$E(L) = E^{1\mathrm{D}}(L^{-2/3}).$$

Here, for $b > 0$, $E^{1\mathrm{D}}(b) = \mathfrak{b}(\alpha_0, b)$ and $\mathfrak{b}(\alpha_0, b)$ is defined in (3.9).

In the case of a constant magnetic field, a similar statement to Conjecture 3.9 has been conjectured in [28]. A partial affirmative answer was given in [2,12]. The conjecture has been proved recently in [6,7]. The methods in [6,7] do not yield an affirmative answer for Conjecture 3.9.

Remark 3.10. In [17], the following numerical estimate is given: $\lambda_0 \approx 0.57$. Furthermore, the lower bound:

$$\lambda(0) \leqq \left(\frac{3}{4}\right)^{\frac{4}{3}} < 1, \tag{3.34}$$

is proved. Finally the strict inequality $\lambda_0 < \lambda(0)$ is a consequence of the uniqueness of the point of minimum of the function $\lambda(\tau)$.

3.4. The Approximate Functional

Let $\nu \in [0, 2\pi)$ be a given angle. Define the magnetic potential:

$$\mathbf{A}_{\mathrm{app},\nu}(x) = -\frac{|x|^2}{2}\mathbf{n}, \quad \mathbf{n} = (\cos\nu, \sin\nu), \quad (x = (x_1, x_2) \in \mathbb{R}^2). \tag{3.35}$$

Clearly, ν is the angle between the x_1-axis and the line $\{\mathrm{curl}\,\mathbf{A}_{\mathrm{app},\nu} = 0\}$.

Let $\kappa > 0$, $\ell \in (0, 1)$, $\mathcal{D}_\ell = D(0, \ell)$ the disc centered at 0 and of radius ℓ, and $L > 0$. Consider the functional:

$$\mathcal{G}(\psi) = \int_{\mathcal{D}_\ell} \left(|(\nabla - iL\kappa^3 \mathbf{A}_{\mathrm{app},\nu})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right) \mathrm{d}x, \tag{3.36}$$

together with the ground state energy

$$\mathrm{E}_{\mathrm{gs,r}}(\kappa, L, \nu; \ell) = \inf\{\mathcal{G}(\psi) \ : \ \psi \in H_0^1(\mathcal{D}_\ell)\}. \tag{3.37}$$

The change of variable $x \mapsto \sqrt{m}\,\kappa\, x$ yields

$$\mathrm{E}_{\mathrm{gs,r}}(\kappa, L; \ell) = \mathfrak{e}_{\mathrm{gs,disc}}(\nu, L; R), \tag{3.38}$$

where $m = L^{2/3}$, $R = \sqrt{m}\,\kappa\,\ell$, $\mathcal{D}_R = D(0, R)$,

$$\mathcal{E}_{\nu,L,R}(u) = \int_{\mathcal{D}_R} \left(|(\nabla - i\mathbf{A}_{\mathrm{app},\nu})u|^2 - L^{-2/3}|u|^2 + \frac{L^{-2/3}}{2}|u|^4 \right) \mathrm{d}x, \tag{3.39}$$

and

$$\mathfrak{e}_{\mathrm{gs,disc}}(\nu, L; R) = \inf\{\mathcal{E}_{\nu,L,R}(u) \ : \ u \in H_0^1(\mathcal{D}_R)\}. \tag{3.40}$$

We now show that the ground state energy $\mathfrak{e}_{\mathrm{gs,disc}}(\nu, L; R)$ is independent of ν. Let u be a given function in $H_0^1(\mathcal{D}_R)$. We perform the rotation

$$(x_1, x_2) \mapsto (x_1 \cos\nu + x_2 \sin\nu,\ -x_1 \sin\nu + x_2 \cos\nu),$$

which transforms the function u to a new function $\widetilde{u}$, then the gauge transformation $\widetilde{u} \mapsto v = e^{ix_1^3/6}\widetilde{u}$ and get

$$\mathcal{E}_{\nu,L,R}(u) = \int_{\mathcal{D}_R} \left(|(\nabla - i\mathbf{A}_{\mathrm{app}})v|^2 - L^{-2/3}|v|^2 + \frac{L^{-2/3}}{2}|v|^4 \right) \mathrm{d}x =: G_{L,R}(v),$$

where $\mathbf{A}_{\mathrm{app}}$ is introduced in (3.13).
Hence we get,

$$\mathfrak{e}_{\mathrm{gs,disc}}(\nu, L; R) = \inf\{G_{L,R}(v) \ : \ v \in H_0^1(\mathcal{D}_R)\}. \tag{3.41}$$

This simple observation allows us to prove the following Theorem 3.11 below, which indicates a situation where the energies $\mathfrak{e}_{\mathrm{gs}}(L; R)$ and $\mathfrak{e}_{\mathrm{gs,disc}}(\nu, L; R)$ match.

Theorem 3.11. *For $\nu \in [0, 2\pi)$, $L > 0$ and $R > 0$, we have,*

$$\mathfrak{e}_{\mathrm{gs,disc}}(\nu, L; R) = \mathfrak{e}_{\mathrm{gs,disc}}(0, L; R) \geqq \mathfrak{e}_{\mathrm{gs}}(L; R), \tag{3.42}$$

where $\mathfrak{e}_{\mathrm{gs}}(L; R)$ and $\mathfrak{e}_{\mathrm{gs,disc}}(\nu, L; R)$ are the ground state energies introduced in (3.15) *and* (3.40)*.*
Moreover, there exists a constant C such that, for $L > 0$, $a \in (0, 1/2)$, and

$$R > 4\max(a^{-1/2}L^{-2/3},\ 1),$$

we have

$$\mathfrak{e}_{\mathrm{gs,disc}}(0, L; R) \leqq \mathfrak{e}_{\mathrm{gs}}(L; (1-a)R) + \frac{C}{a^{1/2}}\left(1 + \frac{L^{2/3}}{aR^2}\right). \tag{3.43}$$

Proof. The independence of ν was observed in (3.41). From now on we can take $\nu = 0$.

Lower bound. Let $u \in H_0^1(\mathcal{D}_R)$ be a minimizer of the functional $G_{L,R}$. The function u can be extended by 0 to a function in $H_0^1(\mathcal{S}_R)$. Thus,

$$\mathfrak{e}_{\mathrm{gs,disc}}(0, L; R) = G_{L,R}(u) = \mathcal{E}_{L,R}(u) \geqq \mathfrak{e}_{\mathrm{gs}}(L; R).$$

Upper bound. Let $a \in (0, \frac{1}{2})$, $\widetilde{R} = (1-a)R$ and

$$v = \varphi_{L,\widetilde{R}} \in H_0^1(\mathcal{S}_{\widetilde{R}})$$

a minimizer of $G_{L,\widetilde{R}}$. Remember that $\varphi_{L,\widetilde{R}} = 0$ when $L \geqq \lambda_0^{-3/2}$ (Proposition 3.6). We impose the condition

$$\sqrt{a}\, R > 4L^{-2/3}. \tag{3.44}$$

Consider a test function $\chi \in C_c^\infty(\mathbb{R})$ such that

$$\begin{cases} 0 \leqq \chi \leqq 1, \quad \mathrm{supp}\chi \subset (-\sqrt{a(2-a)}\, R,\ \sqrt{a(2-a)}\, R), \\ \chi = 1 \text{ in } (-\sqrt{a(1-a)}\, R,\ \sqrt{a(1-a)}\, R), \\ |\chi'| \leqq \frac{C}{\sqrt{a}\, R} \quad \text{and} \quad |\chi''| \leqq \frac{C}{a\, R^2}. \end{cases}$$

Let

$$u(x_1, x_2) = \chi(x_2)\, v(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Clearly, $u \in H_0^1(\mathcal{D}_R)$. Thus,

$$\begin{aligned} \mathfrak{e}_{\mathrm{gs,disc}}(0, L; R) &\leqq G_{L,R}(u) = \mathcal{E}_{L,R}(u) \\ &= \int_{\mathcal{S}_R} \Big(\chi(x_2)^2 |(\nabla - i\mathbf{A}_{\mathrm{app}})v|^2 - \chi(x_2)\chi''(x_2)|v|^2 \\ &\quad - L^{-2/3}|\chi(x_2)v|^2 + \frac{L^{-2/3}}{2}|\chi(x_2)v|^4\Big)\,\mathrm{d}x \\ &\leqq \int_{\mathcal{S}_R} \left(|(\nabla - i\mathbf{A}_{\mathrm{app}})v|^2 - L^{-2/3}|v|^2 + \frac{L^{-2/3}}{2}|v|^4\right)\mathrm{d}x \\ &\quad + L^{-\frac{2}{3}} \int_{\mathcal{S}_R} (1-\chi^2)|v|^2\,\mathrm{d}x + \frac{CL^{2/3}}{a^{3/2}R^2} \\ &= \int_{\mathcal{S}_{\widetilde{R}}} \left(|(\nabla - i\mathbf{A}_{\mathrm{app}})v|^2 - L^{-2/3}|v|^2 + \frac{L^{-2/3}}{2}|v|^4\right)\mathrm{d}x \\ &\quad + L^{-2/3} \int_{\mathcal{S}_R} (1-\chi^2)|v|^2\,\mathrm{d}x + \frac{CL^{2/3}}{a^{3/2}R^2} \\ &\leqq \mathfrak{e}_{\mathrm{gs}}(L; \widetilde{R}) + \frac{C}{a^{1/2}} + \frac{CL^{2/3}}{a^{3/2}R^2}. \end{aligned}$$

The two terms $L^{-\frac{2}{3}} \int_{\mathcal{S}_R} (1 - \chi(x_2)^2)|v(x_1, x_2)|^2\,\mathrm{d}x$ and $\int_{\mathcal{S}_R} \chi(x_2)\chi''(x_2)$ $|v(x_1, x_2)|^2\,\mathrm{d}x$ have been controlled by using the decay of $v = \varphi_{L,\widetilde{R}}$ established in

Theorem 3.3 (Formula (3.18)). Here, we have used Assumption (3.44) (and (3.18)) to write

$$\int_{\{|x_2|\geqq\sqrt{a}\,R\}} |v(x_1,x_2)|^2\,\mathrm{d}x \leqq CL^{2/3}R\times(\sqrt{a}\,R)^{-1} = \frac{CL^{2/3}}{a^{1/2}}.$$

Remembering that $\widetilde{R} = (1-a)R$, this achieves the proof of Theorem 3.11. □

3.5. Bulk Energy

In this subsection, we recall the construction of a function that describes the energy of the Ginzburg–Landau model with constant magnetic field [1,13,32]. Consider $b\in(0,\infty)$, $r>0$, and $Q_r=(-r/2,r/2)\times(-r/2,r/2)$. Define the functional

$$F_{b,Q_r}(u) = \int_{Q_r}\left(b|(\nabla - i\mathbf{A}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4\right)\mathrm{d}x,\quad \text{for } u\in H^1(Q_r). \tag{3.45}$$

Here, $\mathbf{A}_0$ is the magnetic potential

$$\mathbf{A}_0(x) = \frac{1}{2}(-x_2,x_1),\quad \big(x=(x_1,x_2)\in\mathbb{R}^2\big). \tag{3.46}$$

Define the two ground state energies

$$e_D(b,r) = \inf\{F_{b,Q_r}(u)\ :\ u\in H_0^1(Q_r)\}, \tag{3.47}$$

$$e_N(b,r) = \inf\{F_{b,Q_r}(u)\ :\ u\in H^1(Q_r)\}. \tag{3.48}$$

It is known [4,13,32] that

$$\forall\, b>0,\quad g(b) = \lim_{r\to\infty}\frac{e_D(b,r)}{|Q_r|} = \lim_{r\to\infty}\frac{e_N(b,r)}{|Q_r|}, \tag{3.49}$$

where $|Q_r|$ denotes the area of Q_r ($|Q_r|=r^2$) and g is a continuous function such that

$$g(0) = -\frac{1}{2}\ \text{and}\ g(b)=0\quad\text{when}\quad b\geqq 1. \tag{3.50}$$

Furthermore, there exists a constant C such that, for all $r\geqq 1$ and $b>0$,

$$g(b) - C\frac{\sqrt{b}}{r} \leqq \frac{e_N(b,r)}{|Q_r|} \leqq \frac{e_D(b,r)}{|Q_r|} \leqq g(b) + C\frac{\sqrt{b}}{r}. \tag{3.51}$$

We will use the function $g(\cdot)$ to prove Theorem 3.12 below. This theorem concerns the limiting function $E(\cdot)$. It contains sharp bounds on $E(\cdot)$ in the regime $L\to 0_+$ (compare with Proposition 3.6).

The function $g(\cdot)$ provides us with a test function to prove:

Theorem 3.12. *There exist two positive constants C_1 and C_2 such that, if $L \in (0, \lambda_0^{-3/2}]$, then*

$$-C_1(1-\lambda_0 L^{2/3})\, L^{-4/3} \leqq E(L) \leqq -C_2(1-\lambda_0 L^{2/3})\, L^{-4/3}.$$

Proof. The lower bound follows immediately by sending R to ∞ in the lower bound in Proposition 3.6 (see also Theorem 3.8). The upper bound in the second item of Proposition 3.6 gives us the upper bound

$$E(L) \leqq -C\,(1-\lambda_0 L^{2/3})\, L^{-2/3}, \tag{3.52}$$

for all $L \in (0, \lambda_0^{-3/2})$.
We have just to improve it as $L \to 0$.
The improved upper bound with order $L^{-4/3}$ follows from the construction of a test function as follows.

Let us cover $\mathbb{R}^2$ by a lattice of squares $\overline{Q_{\ell,j}}$, where $Q_{\ell,j} = (-\ell + a_j,\ \ell + a_j)$ and

$$\ell = mL^{1/3}. \tag{3.53}$$

The choice of the positive constant m will be specified later. Notice that the magnetic potential $\mathbf{A}_{\rm app}$ (cf. (3.13)) satisfies

$$\mathbf{B}_{\rm app} = \operatorname{curl} \mathbf{A}_{\rm app} = x_2.$$

Let $\mathbf{A}_0$ be the magnetic potential in (3.46), $a_j = (a_{j,1}, a_{j,2})$ be the center of the square $Q_{\ell,j}$ and $\mathbf{F}_j(x_1,x_2) = \Big(-\frac{1}{3}(x_2-a_{j,2})^2,\, \frac{1}{3}(x_2-a_{j,2})(x_1-a_{j,1})\Big)$. It is easy to check that

$$\operatorname{curl} \mathbf{A}_{\rm app} = \operatorname{curl}(a_{j,2}\mathbf{A}_0 + \mathbf{F}_j) \quad \text{in} \quad Q_{\ell,j}.$$

Since the square $Q_{\ell,j}$ is a simply connected domain in $\mathbb{R}^2$, then there exists a real-valued smooth function ϕ_j defined in $Q_{\ell,j}$ such that

$$\mathbf{A}_{\rm app} = a_{j,2}\mathbf{A}_0 + \mathbf{F}_j - \nabla\phi_j \quad \text{in} \quad Q_{\ell,j}.$$

Thanks to the definition of $\mathbf{F}_j$, we have

$$|\mathbf{F}_j(x)| \leqq \ell^2 \quad \text{in} \quad Q_{\ell,j}.$$

Thus, for any j, we can select a gauge ϕ_j, such that, in the square $Q_{\ell,j}$, we have

$$|\mathbf{A}_{\rm app}(x) - (a_{j,2}\mathbf{A}_0(x-a_j) - \nabla\phi_j)| \leqq \ell^2.$$

Now, we define the test function as follows,

$$v(x) = \begin{cases} e^{i\phi_j(x)}\, u_r\big(\sqrt{a_{j,2}}\,(x-a_j)\big) & \text{if } a_{j,2} > 0 \text{ and} \\ & \quad x \in Q_{\ell,j} \subset \{|x_1| < R \text{ and } \frac{\epsilon}{2}L^{-2/3} < x_2 < \epsilon L^{-2/3}\}, \\ e^{i\phi_j(x)}\, \overline{u_r\big(\sqrt{|a_{j,2}|}\,(x-a_j)\big)} & \text{if } a_{j,2} < 0 \text{ and} \\ & \quad x \in Q_{\ell,j} \subset \{|x_1| < R \text{ and } \frac{\epsilon}{2}L^{-2/3} < -x_2 < \epsilon L^{-2/3}\}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.54}$$

where the function $u_r \in H_0^1(Q_r)$ is a minimizer of the ground state energy F_{b,Q_r} introduced in (3.45) and $\epsilon \in (0,1)$ is a positive constant (to be determined in (3.56)).
We impose the following condition on m and ϵ

$$m\sqrt{\frac{\epsilon}{2}} \geqq 1. \tag{3.55}$$

We will use the notation

$$\mathcal{E}_{L,R}(v;\, Q_{\ell,j}) = \int_{Q_{\ell,j}} \left(|(\nabla - i\mathbf{A}_{\mathrm{app}})v|^2 - L^{-2/3}|v|^2 + \frac{1}{2}\,L^{-2/3}\,|v|^4\right) \mathrm{d}x.$$

Notice that, if $a_{j,2} > 0$ and $Q_{\ell,j} \subset \{|x_1| < R \text{ and } \frac{\epsilon}{2}L^{-2/3} < |x_2| < \epsilon L^{-2/3}\}$, then, for all $\eta > 0$,

$$\begin{aligned}
&\mathcal{E}_{L,R}(v;\, Q_{\ell,j})\\
&\quad\leqq L^{-2/3}\Bigg(\int_{Q_{\ell,j}}\Big(L^{2/3}(1+\eta)|(\nabla - i a_{j,2}\mathbf{A}_0(x-a_j))v|^2 - L^{-2/3}|v|^2\\
&\qquad\qquad + \frac{1}{2}\,L^{-2/3}\,|v|^4\Big)\,\mathrm{d}x\Bigg)\\
&\quad\; + C\eta^{-1}\ell^6\\
&\quad= \frac{L^{-2/3}}{a_{j,2}}\Bigg(\int_{Q_{\sqrt{a_{j,2}}\ell}}\Big(L^{2/3}a_{j,2}(1+\eta)|(\nabla - i\mathbf{A}_0(x))u_r(x)|^2 - L^{-2/3}|v|^2\\
&\qquad\qquad + \frac{1}{2}\,L^{-2/3}\,|v|^4\Big)\,\mathrm{d}x\Bigg) + C\eta^{-1}\ell^6\\
&\quad\leqq \frac{L^{-2/3}}{a_{j,2}}\left(g(L^{2/3}a_{j,2}(1+\eta))|a_{j,2}|\ell^2 + C\sqrt{L^{2/3}a_{j,2}(1+\eta)}\,\sqrt{a_{j,2}}\,\ell\right) + C\eta^{-1}\ell^6.
\end{aligned}$$

To write the last inequality, (3.51) is used with $b = L^{2/3}a_{j,2}(1+\eta)$ and $r = \sqrt{a_{j,2}}\,\ell$. (Thanks to the condition (3.55), we have $r \geqq 1$).
Similarly, if $a_{j,2} < 0$ and $Q_{\ell,j} \subset \{|x_1| < R \text{ and } \frac{\epsilon}{2}L^{-1/3} < |x_2| < \epsilon L^{-2/3}\}$, then,

$$\begin{aligned}
\mathcal{E}_{L,R}(v;\, Q_{\ell,j}) \leqq \frac{L^{-2/3}}{|a_{j,2}|}&\left(g(L^{2/3}|a_{j,2}|(1+\eta))|a_{j,2}|\ell^2 + C\sqrt{L^{2/3}(1+\eta)}\,|a_{j,2}|\,\ell\right)\\
&+ C\eta^{-1}\ell^6.
\end{aligned}$$

Notice the simple decomposition of the energy of v:

$$\mathcal{E}_{L,R}(v) = \sum_{j\in\mathcal{J}} \mathcal{E}_{L,R}(v;\, Q_{\ell,j}),$$

where $\mathcal{J} = \left\{ j \ : \ Q_{\ell,j} \subset \{|x_1| < R \text{ and } \frac{\epsilon}{2} L^{-2/3} < |x_2| < \epsilon L^{-2/3}\} \right\}$.
Let

$$n = \text{Card}\, \mathcal{J}.$$

The numbers L and ℓ are small enough such that

$$\frac{\epsilon}{4} L^{-2/3} R \leqq n\, \ell^2 \leqq \frac{\epsilon}{2} L^{-2/3} R < L^{-2/3} R.$$

Now, we have the following upper bound on the energy of v,

$$\mathcal{E}_{L,R}(v) \leqq L^{-2/3} \left(\sum_j g(L^{2/3} |a_{j,2}|(1+\eta)) \ell^2 + C\, n \sqrt{L^{2/3}(1+\eta)}\, \ell \right) \\ + C\, n\, \eta^{-1} \ell^6.$$

We select $\eta = \frac{1}{2}$. Having in mind (3.50), we can select ϵ sufficiently small such that

$$g(t) \leqq -\frac{1}{4}, \quad \forall\, t \in [0, 2\epsilon]. \tag{3.56}$$

Observing

$$L^{2/3}\, |a_{j,2}|\, (1+\eta) \leqq 2\epsilon,$$

we get, for $R \geqq n\, \ell^2 L^{\frac{2}{3}}$,

$$\frac{\mathfrak{e}_{\rm gs}(L;R)}{R} \leqq \frac{\mathcal{E}_{L,R}(v)}{R} \leqq L^{-2/3} \left(-\frac{\epsilon}{16}\, L^{-2/3} + 2C L^{1/3}\, L^{-2/3}\, \ell^{-1} \right) + 2C L^{-2/3} \ell^4.$$

Sending $R \to \infty$, we deduce that

$$E(L) \leqq -\frac{\epsilon}{32} L^{-4/3} + C\, L^{-1}\, \ell^{-1} \; + C L^{-2/3} \ell^4.$$

Having in mind (3.53), we get

$$E(L) \leqq \left(-\frac{\epsilon}{32} + \frac{C}{m} \right) L^{-4/3} + C m^4 L^{2/3}. \tag{3.57}$$

Recalling (3.55) and (3.56), we select m such that

$$-\frac{\epsilon}{32} + \frac{C}{m} < 0 \quad \text{and} \quad m > \sqrt{\frac{2}{\varepsilon}}.$$

In that way, (3.57) gives us the existence of a constant $C' > 0$ such that, for sufficiently small values of L,

$$E(L) \leqq -C'\, L^{-4/3}.$$

Since L is sufficiently small, we may write

$$-C' = (1 - L\lambda_0 L^{2/3}) \frac{-C'}{1 - L\lambda_0 L^{2/3}} \leqq -C'(1 - L\lambda_0 L^{2/3}),$$

and get the upper bound in Theorem 3.12 when $L \to 0_+$. $\square$

4. A Priori Estimates and Gauge Transformation

Let $\kappa > 0$, $H > 0$ and $(\psi, \mathbf{A})$ be a critical point of the functional in (1.1), that is $(\psi, \mathbf{A})$ satisfies

$$-(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2(1 - |\psi|^2)\psi, \tag{4.1}$$

$$-\nabla^\perp \operatorname{curl}(\mathbf{A} - \mathbf{F}) = \frac{1}{\kappa H}\left(\overline{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi\right) \quad \text{in } \Omega, \tag{4.2}$$

and the two boundary conditions

$$\nu \cdot (\nabla - i\kappa H\mathbf{A})\psi = 0 \ \text{ and } \ \operatorname{curl}(\mathbf{A} - \mathbf{F}) = 0 \qquad \text{on } \partial\Omega,$$

where ν is the unit exterior normal vector of $\partial\Omega$.

We note for further use the following identity. Multiplying both the equation in (4.1) by $\overline{\psi}$ then integrating over Ω, we get

$$\begin{aligned}\mathcal{E}_0(\psi, \mathbf{A}; \Omega) &:= \int_\Omega \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\right) \mathrm{d}x \\ &= -\frac{\kappa^2}{2}\int_\Omega |\psi|^4 \,\mathrm{d}x \leqq 0. \end{aligned} \tag{4.3}$$

We need the following estimates on ψ and $\mathbf{A}$ that we take from [8]. Earlier versions of these estimates are given in [10,23] when the magnetic field is constant.

Proposition 4.1. *Let $\alpha \in (0, 1)$. There exists a constant $C = C(\alpha, \Omega) > 0$ such that, if $\kappa > 0$, $H > 0$ and $(\psi, \mathbf{A})$ a critical point of the functional in (1.1), then*

$$\|\psi\|_\infty \leqq 1, \tag{4.4}$$

$$\|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_2 \leqq \frac{C}{H}\,\|\psi\|_2, \tag{4.5}$$

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_2 \leqq \kappa\,\|\psi\|_2, \tag{4.6}$$

$$\|\mathbf{A} - \mathbf{F}\|_{C^{1,\alpha}(\overline{\Omega})} \leqq C\,\frac{1 + \kappa H + \kappa^2}{\kappa H}\,\|\psi\|_\infty\,\|\psi\|_2. \tag{4.7}$$

Using the regularity of the curl-div system, we obtain the following improved estimates of $\mathbf{A} - \mathbf{F}$.

Proposition 4.2. *Let $\alpha \in (0, 1)$. There exists a constant $C = C(\alpha, \Omega) > 0$ such that, if $\kappa > 0$, $H > 0$ and $(\psi, \mathbf{A})$ a critical point of the functional in (1.1), then,*

$$\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} \leqq C\left(\|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_2 + \frac{1}{\kappa H}\|(\nabla - i\kappa H\mathbf{A})\psi\|_2\,\|\psi\|_\infty\right).$$

Proof. Let $a = \mathbf{A} - \mathbf{F}$. Notice that a satisfies $\operatorname{div} a = 0$ in Ω and $\nu \cdot a = 0$ on $\partial\Omega$. Thus, there exists $C(\Omega) > 0$ such that for all a satisfying the previous condition

$$\|a\|_{H^2(\Omega)} \leqq C(\Omega)\|\operatorname{curl} a\|_{H^1(\Omega)}.$$

Since $(\psi, \mathbf{A})$ is a critical point of the functional in (1.1), then

$$\nabla^{\perp} \operatorname{curl} a = \frac{1}{\kappa H} \operatorname{Im}(\overline{\psi}\,(\nabla - i\kappa H)\psi).$$

Consequently, we get

$$\|a\|_{H^2(\Omega)} \leqq C\left(\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_2 + \frac{1}{\kappa H}\|(\nabla - i\kappa H\mathbf{A})\psi\|_2\,\|\psi\|_\infty\right).$$

This finishes the proof of the proposition in light of the continuous embedding of $H^2(\Omega)$ in $C^{0,\alpha}(\overline{\Omega})$. □

In the subsequent sections, we will need to approximate the magnetic potential $\mathbf{F}$ generating a non-constant magnetic field by a simpler magnetic potential generating a constant magnetic field. The approximation will be done in domains with small area and in general will lead to large errors. By applying a suitable gauge transformation, one can absorb the large errors and be left with small errors. The next proposition provides us with useful gauge transformations.

Proposition 4.3. *Given Ω and B_0 as in the introduction, there exists a constant $C > 0$ such that the following is true.*

(1) *Let $\ell > 0$, $a_j \in \Omega$, $D(a_j, \ell) \subset \Omega$ and $x_j \in D(a_j, \ell)$. There exists a function $\varphi_j \in C^1(\overline{D(a_j, \ell)})$ such that, for all $x \in D(a_j, \ell)$,*

$$|\mathbf{F}(x) - (B_0(x_j)\mathbf{A}_0(x - a_j) + \nabla\varphi_j)| \leqq C\,\ell^2. \tag{4.8}$$

(2) *Let $\ell > 0$, $a_j \in \Gamma$ and $x_j \in \overline{D(a_j, \ell)} \cap \Gamma$. There exist $\nu_j \in [0, 2\pi)$ and a function $\phi_j \in C^1(\overline{D(a_j, \ell) \cap \Omega})$ such that, for all $x \in D(a_j, \ell) \cap \Omega$,*

$$|\mathbf{F}(x) - (|\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_j}(x - a_j) + \nabla\phi_j)| \leqq C\,\ell^3. \tag{4.9}$$

Proof. The function φ_j in (1) is constructed in [3].
We give the construction of the function ϕ_j announced in (2). The vector field $\mathbf{F}$ and the function B_0 are defined in a neighborhood of $\overline{\Omega}$ (w.l.o.g. we can even assume that they are defined in $\mathbb{R}^2$). In particular, $\mathbf{F}(x)$ and $B_0(x)$ are defined in $D(a_j, \ell)$ even when $D(a_j, \ell) \not\subset \Omega$.

Select $\nu_j \in [0, 2\pi)$ such that

$$\nabla B_0(a_j) = |\nabla B_0(a_j)|(\cos\nu_j, \sin\nu_j).$$

We apply Taylor's formula to the function B_0 near a_j. Since $a_j \in \Gamma$, we get

$$B_0(x) = |\nabla B_0(a_j)|(\cos\nu_j, \sin\nu_j)\cdot(x - a_j) + f_j(x), \tag{4.10}$$

where

$$|f_j(x)| \leqq C|x - a_j|^2 \leqq C\,\ell^2, \quad (x \in D(a_j, \ell)).$$

Taylor's formula applied to the function $|\nabla B_0|$ near a_j yields

$$|\nabla B_0(x_j)| = |\nabla B_0(a_j)| + e_j,$$

where

$$|e_j| \leqq C|x_j - a_j| \leqq C\ell.$$

In that way, (4.10) becomes

$$B_0(x) = |\nabla B_0(x_j)|(\cos\nu_j, \sin\nu_j)\cdot(x-a_j) + g_j(x), \tag{4.11}$$

where $g_j(x) = f_j(x) + e_j(\cos\nu_j, \sin\nu_j)\cdot(x-a_j)$ and satisfies

$$|g_j(x)| \leqq C\,\ell^2.$$

Define the vector field:

$$\mathbf{G}_j(y) = \left(\int_0^1 s g_j(sy + a_j)\,\mathrm{d}s\right)(-y_2, y_1), \quad \text{for } y = (y_1, y_2).$$

Clearly, $|\mathbf{G}_j(y)| \leqq C\ell^3$, when $y \in D(0,\ell)$ and $y + a_j \in \Omega$.

We perform the translation $y = x - a_j$ and define

$$\widetilde{\mathbf{F}}(y) = \mathbf{F}(y + a_j), \quad \text{for } y \in D(0,\ell).$$

In that way, the formula in (4.11) reads as follows:

$$\operatorname{curl}\left(\widetilde{\mathbf{F}} - |\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_{\mathrm{j}}}\right) = \operatorname{curl}\mathbf{G}_j \quad \text{in} \quad D(0,\ell),$$

where $\mathbf{A}_{\mathrm{app},\nu_{\mathrm{j}}}$ is introduced in (3.35).
Consequently, we deduce the existence of a function $\widetilde{\phi}_j \in C^1(\overline{D(0,\ell)})$ such that,

$$\widetilde{\mathbf{F}} - |\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_{\mathrm{j}}} = \mathbf{G}_j + \nabla\widetilde{\phi}_j, \quad \text{in } D(0,\ell).$$

The function ϕ_j is defined by $\phi_j(x) = \widetilde{\phi}_j(x - a_j)$, for $x \in D(a_j,\ell)$. □

5. Energy Upper Bound

In this section, we determine an asymptotic upper bound of the energy in (1.5). The upper bound is valid under the assumptions

$$\kappa \gg 1 \quad \text{and} \quad \kappa \ll H \lesssim \kappa^2,$$

and matches with the asymptotic expansions announced in Theorem 1.1.

The conclusion in Theorem 1.1 displays two regimes for the behavior of the energy in (1.5),

Regime I : $\kappa^{3/2} \ll H \lesssim \kappa^2$,

Regime II : $\kappa \ll H \lesssim \kappa^{3/2}$.

As such, this section will present two independent constructions devoted to the aforementioned two regimes (Regime I and Regime II). Each construction will be the subject of an independent subsection.

5.1. Upper Bound: Regime I

This subsection is devoted to the proof of:

Proposition 5.1. *Let* $\Lambda > 0$ *and* $\epsilon : \mathbb{R} \to \mathbb{R}_+$ *such that* $\lim_{\kappa\to\infty} \kappa^{1/2}\epsilon(\kappa) = +\infty$ *and* $\lim_{\kappa\to\infty} \epsilon(\kappa) = 0$. *If* $\epsilon(\kappa)\kappa^2 \leqq H \leqq \Lambda\kappa^2$, *then the ground state energy in* (1.5) *satisfies,*

$$\mathrm{E}_{\mathrm{gs}}(\kappa, H) \leqq \kappa\left(\int_\Gamma \left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \mathrm{d}s(x)\right) + o\left(\frac{\kappa^3}{H}\right), \quad (\kappa \to \infty), \tag{5.1}$$

where ds *is the arc-length measure on* Γ.

The proof of Proposition 5.1 consists of computing the energy of a relevant test configuration. The construction of this test configuration hints at the actual behavior of the minimizing configurations.

The conclusion in Proposition 5.1 is a straightforward application of Lemma 5.2 below. One part of Lemma 5.2 is devoted to the construction of a test configuration. The construction requires that we cover the curves where the magnetic field vanishes by a collection of discs satisfying:

- the centers of the discs are in the set $\Gamma = \{B_0(x) = 0\}$;
- the interiors of the discs are disjoint;
- all the discs have equal radii ℓ;
- if D is a disc in this collection, then the arc-length of the curve $D \cap \Gamma$ is approximately the diameter of D.

The proof of Lemma 5.2 contains the detailed construction of these discs with precise statements of their properties.

The statement of Lemma 5.2 below requires to introduce the quantity $\psi(\mu_1, \mu_2, a)$ which is defined for $\mu_1 > 0$, $\mu_2 > 0$, and $a \in (0, 1)$ by

$$\psi(\mu_1, \mu_2, a) := \max\left(4\,a^{-1/2}\mu_1^{-1},\, 4\mu_2^{-1/3}\right).$$

Lemma 5.2. *Let* $\Lambda > 0$, $\eta \in (0, 1/2)$ *and* $b : \mathbb{R} \to \mathbb{R}_+$ *such that* $\lim_{\kappa\to\infty} b(\kappa) = \infty$ *and* $\lim_{\kappa\to\infty} \kappa^{-1/2}b(\kappa) = 0$. *There exist positive constants* C, κ_0 *and* ℓ_0 *such that the following is true.*

Suppose that $a \in (0, 1/2)$, $\ell \in (0, \ell_0)$, $\delta \in (0, 1)$, $\kappa \geqq \kappa_0$, $H > 0$ *satisfy,*

$$\kappa\ell \geqq \psi\left(\frac{H}{\kappa^2}\inf_{x\in\Gamma}|\nabla B_0(x)|,\ \frac{H}{\kappa^2}\inf_{x\in\Gamma}|\nabla B_0(x)|,\ a\right), \tag{5.2}$$

and

$$b(\kappa)\kappa^{3/2} \leqq H \leqq \Lambda\kappa^2.$$

Then, the ground state energy in (1.5) *satisfies,*

$$\mathrm{E}_{\mathrm{gs}}(\kappa, H) \leqq \kappa \int_\Gamma \left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \mathrm{d}s(x) + C(\alpha+\beta), \tag{5.3}$$

where

- $\mathrm{d}s$ *is the arc-length measure on* Γ,
- $\alpha = H^{-5/9}\kappa^{13/9}\ell^{-2/3} + a^{-1/2}(\ell^{-1} + a^{-1}\kappa^{-2}\ell^{-3}) + (\delta\kappa^2 + \delta^{-1}\kappa^2 H^2\ell^6)\ell,$
- $\beta = (\eta + \delta + a)\dfrac{\kappa^3}{H}.$

Proof. Step 1. *Existence of* ℓ_0.

This step is devoted to the definition of the constant ℓ_0 appearing in the statement of Proposition 5.1. Recall the assumption that Γ is the union of a finite number of simple smooth curves and $\Gamma \cap \partial\Omega$ is a finite set. Given $\eta > 0$, there exists a constant $\ell_1 \in (0, 1)$ such that, for all $a \in \Gamma$ and $\ell \in (0, \ell_1)$ with $\overline{D(a, \ell)} \subset \Omega$, then $D(a, \ell) \cap \Gamma$ is connected and

$$2\ell - \frac{\eta}{2}\ell \leqq \int_{D(a,\ell)\cap\Gamma} \mathrm{d}s(x) \leqq 2\ell + \frac{\eta}{2}\ell. \tag{5.4}$$

Notice that $\int_{D(a,\ell)\cap\Gamma} \mathrm{d}s(x)$ is the arc-length (along Γ) of $D(a, \ell) \cap \Gamma$. Thus, the choice of ℓ_1 is such that the arc-length of $D(a, \ell) \cap \Gamma$ is approximately 2ℓ, whenever $\ell \in (0, \ell_1)$.

The arc-length measure of Γ is denoted by $|\Gamma|$. By assumption, Γ consists of a finite number of simple smooth curves $(\Gamma_i)_{i=1}^k$. Let

$$\ell_0 = \min\left(\frac{\eta}{16} \min_{1\leqq i\leqq k} |\Gamma_i|,\ \frac{\ell_1}{16}\left(1+\frac{\eta}{4}\right)^{-1}\right).$$

If $\ell \in (0, \ell_0)$, then, on the one hand,

$$\frac{2\ell}{|\Gamma_i|} < \frac{\eta}{4}, \quad (1+\eta)\,\ell < \frac{\ell_1}{4}, \tag{5.5}$$

and, on the other hand, $\ell < \ell_1$ and (5.4) is satisfied.

Step 2. *A covering of* Γ.

In the sequel, we suppose that $\ell \in (0, \ell_0)$. Consider $i \in \{1, \dots, k\}$ and the curve Γ_i. Let $\mathfrak{n}_i \in \mathbb{N}$ be the unique natural number satisfying

$$\frac{|\Gamma_i|}{2\ell}\left(1+\frac{\eta}{4}\right)^{-1} - 1 < \mathfrak{n}_i \leqq \frac{|\Gamma_i|}{2\ell}\left(1+\frac{\eta}{4}\right)^{-1}. \tag{5.6}$$

Select $\mathfrak{n}_i$ distinct points $(b_{j,i})_j$ on Γ_i such that

$$\forall\, j, \quad \mathrm{dist}_{\Gamma_i}(b_{j,i}, b_{j+1,i}) = \frac{|\Gamma_i|}{\mathfrak{n}_i},$$

where dist_{Γ_i} is the arc-length measure on Γ_i.

Obviously, the Euclidean distance $e_j := |b_{j+1,i} - b_{j,i}|$ satisfies $e_j \leqq \mathrm{dist}_{\Gamma_i}(b_{j,i}, b_{j+1,i}) = \frac{|\Gamma_i|}{\mathfrak{n}_i}$. Thanks to (5.6) and (5.5), we have,

$$e_j \leqq 2\ell\,(1+\eta) < \ell_1.$$

Thus, if $\overline{D(b_{j,i}, e_j)} \subset \Omega$, we can use (5.4) with $\ell = e_j$ and get,

$$2e_j\left(1 - \frac{\eta}{4}\right) \leqq 2\frac{|\Gamma_i|}{\mathfrak{n}_i} \leqq 2e_j\left(1 + \frac{\eta}{4}\right).$$

Thanks to (5.6), this leads to

$$e_j \geqq \frac{|\Gamma_i|}{\mathfrak{n}_i}\left(1 - \frac{\eta}{4}\right) \geqq 2\ell.$$

Now, define the index set

$$\mathcal{J}_i = \{j \ : \ \overline{D(b_{j,i}, e_j)} \subset \Omega\},$$

and $N_i = \mathrm{Card}\,\mathcal{J}_i$. Notice that, if $j \in \mathcal{J}_i$, then $e_j \geqq 2\ell$ and $\overline{D(b_{j,i}, \ell)} \subset \overline{D(b_{j,i}, e_j/2)} \subset \Omega$. The sets $(D(b_{j,i}, \ell))_{j\in\mathcal{J}_i}$ are pairwise disjoint.

Since $\Gamma_i \cap \partial\Omega$ is a finite set, then there exists a constant c such that,

$$\text{if } a \in \Gamma_i \text{ and } \mathrm{dist}(a, \partial\Omega) \geqq c\ell, \text{ then } \overline{D(a,\ell)} \subset \Omega.$$

Consequently, the number N_i satisfies

$$\mathfrak{n}_i - C \leqq N_i \leqq \mathfrak{n}_i,$$

where $C > 0$ is a constant. Thus, thanks to (5.6) and (5.5),

$$|\Gamma_i|\left(1 + \frac{\eta}{4}\right)^{-1} - C\ell \leqq N_i \times 2\ell \leqq |\Gamma_i|\left(1 + \frac{\eta}{4}\right)^{-1}.$$

Now, collecting the points $(b_{j,i})_{j\in\mathcal{J}_i, i\in\{1,\cdots,k\}}$, we get the collection of points on Γ,

$$(a_j)_{j\in\mathcal{J}} = (b_{j,i})_{j\in\mathcal{J}_i, i\in\{1,\cdots,k\}},$$

such that,

$$\forall\, j \in \mathcal{J}, \quad a_j \in \Gamma \quad \text{and} \quad \overline{D(a_j,\ell)} \subset \Omega,$$
$$N = \mathrm{Card}\,\mathcal{J} = \sum_{i=1}^{k} N_i \quad \text{and} \quad |\Gamma| = \sum_{i=1}^{k} |\Gamma_i|,$$
$$|\Gamma|\left(1 + \frac{\eta}{4}\right)^{-1} - C\ell \leqq N \times 2\ell \leqq |\Gamma|\left(1 + \frac{\eta}{4}\right)^{-1}. \tag{5.7}$$

Notice that

$$\bigcup_j \left(\Gamma \cap \overline{D(a_j,\ell)}\right) \subset \Gamma,$$

and the arc-length measure

$$\left|\bigcup_j \left(\Gamma\cap\overline{D(a_j,\ell)}\right)\right| = \int_{\bigcup_j\left(\Gamma\cap\overline{D(a_j,\ell)}\right)} \mathrm{d}s(x) = \sum_j \int_{\Gamma\cap\overline{D(a_j,\ell)}} \mathrm{d}s(x),$$

satisfies

$$|\Gamma| - C\eta \leqq \left|\bigcup_j \left(\Gamma\cap\overline{D(a_j,\ell)}\right)\right| \leqq |\Gamma| + C\eta.$$

Thus, the arc-length measure of the set $\Gamma\setminus\bigcup_j\left(\Gamma\cap\overline{D(a_j,\ell)}\right)$ satisfies

$$\left|\Gamma\setminus\bigcup_j \left(\Gamma\cap\overline{D(a_j,\ell)}\right)\right| \leqq C\eta. \tag{5.8}$$

Step 3. *Construction of a test configuration.* For each j, select an arbitrary point $x_j\in\overline{D(a_j,\ell)}\cap\Gamma$ and write

$$\nabla B_0(x_j) = |\nabla B_0(x_j)|(\cos\nu_j, \sin\nu_j),$$

with $\nu_j\in[0,2\pi)$.
Define

$$L = L_j = |\nabla B_0(x_j)|\,\frac{H}{\kappa^2},\quad R = R_j = L^{1/3}\kappa\ell. \tag{5.9}$$

Thanks to the assumption in (5.2), the following condition holds:

$$R \geqq 4\max(a^{-1/2}L^{-2/3},\ 1). \tag{5.10}$$

We can apply the result of Theorem 3.11.

We define a function $w\in H^1(\Omega)$ as follows. Consider the set of indices $\mathcal{J} = \{j\ :\ D(a_j,\ell)\subset\Omega\}$. Let $x\in\Omega$ and $j\in\mathcal{J}$. If $x\in D(a_j,\ell)$, define

$$w(x) = e^{i\kappa H\phi_j}u_{L,\ell,\nu_j}(x-a_j), \tag{5.11}$$

where $u_{R,L,\nu_j}\in H_0^1(D(0,\ell))$ is a minimizer of the functional in (3.36) with $\nu=\nu_j$, and ϕ_j is the function constructed in Proposition 4.3. If $x\notin\bigcup_{j\notin\mathcal{J}}\overline{D(a_j,\ell)}$, we set $w(x)=0$.
Clearly, $w\in H^1(\Omega)$.

Step 4. *Upper bound of $\mathcal{E}(w,\mathbf{F})$.*

Notice that $\operatorname{curl}\mathbf{F}=B_0$ and that the magnetic energy term in (1.1) vanishes for $\mathbf{A}=\mathbf{F}$. Thus, we have

$$\mathcal{E}(w,\mathbf{F}) = \mathcal{E}_0(w,\mathbf{F};\Omega) = \sum_j \mathcal{E}_0(w,\mathbf{F};D(a_j,\ell)), \tag{5.12}$$

where the functional $\mathcal{E}_0$ is defined in (1.14).

Recalling the definition of w, we observe that

$$\mathcal{E}_0(w,\mathbf{F};D(a_j,\ell))=\mathcal{E}_0(u_{L,\ell,\nu_j}(x-a_j),\mathbf{F}-\nabla\phi_j;D(a_j,\ell)).$$

Thanks to the choice of ϕ_j, we infer from Proposition 4.3 that

$$\Big|\,|\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_j}(x-a_j)-(\mathbf{F}-\nabla\phi_j)\Big|\leqq C\,\ell^3. \tag{5.13}$$

As a consequence, applying the Cauchy–Schwarz inequality, we get that, for any $\delta>0$,

$$\begin{aligned}\mathcal{E}_0(w,\mathbf{F};D(a_j,\ell))&\leqq(1+\delta)\mathcal{E}_0(u_{L,\ell,\nu_j}(x-a_j),\\&|\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_j}(x-a_j);D(a_j,\ell))+r_1,\end{aligned}$$

where

$$r_1=C(\delta\kappa^2+\delta^{-1}\kappa^2H^2\ell^6)\int_{D(a_j,\ell)}|u_{L,\ell,\nu_j}(x-a_j)|^2\,\mathrm{d}x.$$

Recall that u_{L,ℓ,ν_j} being a minimizer, it satisfies

$$|u_{L,\ell,\nu_j}|\leqq 1.$$

Thus,

$$r_1\leqq C(\delta\kappa^2+\delta^{-1}\kappa^2H^2\ell^6)\ell^2. \tag{5.14}$$

Now, performing the translation $x\mapsto x-a_j$, we observe that

$$\mathcal{E}_0(w,\mathbf{F};D(a_j,\ell))\leqq(1+\delta)\mathcal{E}_0(u_{L,\ell,\nu_j},|\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_j};D(0,\ell))+r_1.$$

With $L=L_j$ and $R=R_j$ in (5.9), we get, in light of Theorem 3.11,

$$\mathcal{E}_0(w,\mathbf{A};D(a_j,\ell))\leqq(1+\delta)\mathfrak{e}_{\mathrm{gs}}\left(L_j;(1-a)R_j\right)+\frac{C}{a^{1/2}}\left(1+\frac{L_j^{2/3}}{aR_j^2}\right)+r_1.$$

Thanks to Theorem 3.8, we deduce that

$$\begin{aligned}\mathcal{E}_0(w,\mathbf{F};D(a_j,\ell))\leqq\,&2(1+\delta)\,(1-a)R_j\,E(L_j)+C\,(1+L_j^{-2/3})R_j^{1/3}\\&+\frac{C}{a^{1/2}}\left(1+\frac{L_j^{2/3}}{aR_j^2}\right)+r_1.\end{aligned} \tag{5.15}$$

Recall the definition of L_j and R_j in (5.9), and that the number of discs $D(a_j,\ell)$ is inversely proportional to ℓ, that is of order ℓ^{-1}.

Substituting (5.15) into (5.12) yields

$$\begin{aligned}\mathcal{E}(w,\mathbf{F})\ \leqq\,&2\kappa\ell(1+\delta)(1-a)\left(\sum_j\left(|\nabla B_0(x_j)|\,\frac{H}{\kappa^2}\right)^{1/3}E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\right)\\&+C\,H^{-5/9}\kappa^{13/9}\ell^{-2/3}+Ca^{-1/2}(\ell^{-1}+a^{-1}\kappa^{-2}\ell^{-3})+C\,r_1\ell^{-1}.\end{aligned} \tag{5.16}$$

Thanks to (5.7) and the upper bound on $E(\cdot)$ obtained in Theorem 3.12, the term

$$\sum_j 2\ell\left(|\nabla B_0(x_j)|\,\frac{H}{\kappa^2}\right)^{1/3}\,E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right) \tag{5.17}$$

is of order κ^2/H. Thus, (5.16) becomes

$$\begin{aligned}\mathcal{E}(w,\mathbf{F}) \;&\leqq \kappa\left(\sum_j 2\ell\left(|\nabla B_0(x_j)|\,\frac{H}{\kappa^2}\right)^{1/3}\,E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\right) + C(\delta+a)\frac{\kappa^3}{H}\\ &\quad + C\,H^{-5/9}\kappa^{13/9}\ell^{-2/3} + Ca^{-1/2}(\ell^{-1}+a^{-1}\kappa^{-2}\ell^{-3}) + C\,r_1\ell^{-1}.\end{aligned} \tag{5.18}$$

In (5.17), replacing 2ℓ by the arc-length measure of $D(a_j,\ell)\cap\Gamma$ produces an error $\eta\ell/2$ and the sum becomes a Riemann sum over $V_\ell = \bigcup_{j\in\mathcal{J}}\left(\Gamma\cap\overline{D(a_j,\ell)}\right)$.

The points x_j can be selected such that the Riemann sum is a *lower* Riemann sum. Thus,

$$\begin{aligned}&\sum_j 2\ell\left(|\nabla B_0(x_j)|\,\frac{H}{\kappa^2}\right)^{1/3}\,E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\\ &\quad\leqq \int_{V_\ell}\left(|\nabla B_0(x)|\,\frac{H}{\kappa^2}\right)^{1/3}\,E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)\,\mathrm{d}s(x) + C\eta\frac{\kappa^2}{H}.\end{aligned}$$

Inserting this into (5.18), we get

$$\begin{aligned}\mathcal{E}(w,\mathbf{F}) \leqq \kappa&\left(\int_{V_\ell}\left(|\nabla B_0(x)|\,\frac{H}{\kappa^2}\right)^{1/3}\,E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)\,\mathrm{d}s(x)\right) + C(\eta+\delta+a)\frac{\kappa^3}{H}\\ &+ C\,H^{-5/9}\kappa^{13/9}\ell^{-2/3} + Ca^{-1/2}(\ell^{-1}+a^{-1}\kappa^{-2}\ell^{-3}) + C\,r_1\ell^{-1}.\end{aligned}$$

As pointed out earlier, the arc-length measure of the set $\Gamma\setminus V_\ell$ does not exceed $C\eta$. Recall the upper bound on $E(\cdot)$ obtained in Theorem 3.12. In that way, we get

$$\left|\int_{\Gamma\setminus V_\ell}\left(|\nabla B_0(x)|\,\frac{H}{\kappa^2}\right)^{1/3}\,E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)\,\mathrm{d}s(x)\right| \leqq C\eta\frac{\kappa^2}{H}.$$

Consequently, we deduce the following upper bound:

$$\begin{aligned}\mathcal{E}(w,\mathbf{F}) \leqq \kappa&\left(\int_{\Gamma}\left(|\nabla B_0(x)|\,\frac{H}{\kappa^2}\right)^{1/3}\,E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)\,\mathrm{d}s(x)\right) + C(\eta+\delta+a)\frac{\kappa^3}{H}\\ &+ C\,H^{-5/9}\kappa^{13/9}\ell^{-2/3} + Ca^{-1/2}(\ell^{-1}+a^{-1}\kappa^{-2}\ell^{-3}) + C\,r_1\ell^{-1}.\end{aligned}$$

The definition of the ground state energy in (1.5) tells us that $\mathrm{E_{gs}}(\kappa,H) \leqq \mathcal{E}(w,\mathbf{F})$. Recalling the definition of r_1 in (5.14) finishes the proof of (5.3). □

Proof of Proposition 5.1. We use the upper bound in Lemma 5.2 with the following choice of the parameters:

$$\ell = \kappa^{5/8} H^{-3/4}, \quad \delta = \kappa^{9/8} H^{-3/4}. \tag{5.19}$$

Clearly, the parameters δ and ℓ satisfy as $\kappa \to \infty$

$$\ell \ll \delta \ll 1.$$

Let us show that the two conditions in (5.2) are satisfied. Observing that the parameter $a \in (0, 1/2)$ is fixed, that is independent of κ, the conditions in (5.2) will follow from

$$\kappa\ell = \kappa^{13/8} H^{-3/4} \gg \max\left\{a^{-1/2}\kappa^2 H^{-1}, H^{-\frac{1}{3}}\kappa^{\frac{2}{3}}\right\},$$

which is a consequence of the assumption in Regime I (that is $\kappa^{3/2} \ll H \lesssim \kappa^2$).

Now, since (5.2) is satisfied, we can apply Lemma 5.2.

The remainder α in Lemma 5.2 satisfies (this simply follows by studying each individual term in α)

$$\alpha \ll \kappa^3 H^{-1}.$$

Sending κ to ∞, the upper bound in Lemma 5.2 becomes

$$\limsup_{\kappa\to\infty} \frac{H}{\kappa^3}\left\{\mathrm{E_{gs}}(\kappa, H) - 2\kappa\left(\int_\Gamma \left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)\,\mathrm{d}s(x)\right)\right\}$$
$$\leqq C(\eta + a).$$

Since this is true for all $\eta \in (0, 1/2)$ and $a \in (0, 1/2)$, we get, by sending η and a to 0,

$$\limsup_{\kappa\to\infty} \frac{H}{\kappa^3}\left\{\mathrm{E_{gs}}(\kappa, H) - 2\kappa\left(\int_\Gamma \left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)\,\mathrm{d}s(x)\right)\right\} \leqq 0,$$

and the conclusion in Proposition 5.1 follows. □

5.2. Upper Bound: Regime II

In the next proposition, we give an upper bound of the ground state energy in (1.5) valid in the regime $\kappa^{-1} \ll H \lesssim \kappa^{3/2}$.

Proposition 5.3. *Let* $\Lambda > 0$ *and* $\epsilon : \mathbb{R}+ \to \mathbb{R}_+$ *be a function satisfying* $\lim_{\kappa\to\infty} \epsilon(\kappa) = 0$ *and* $\lim_{\kappa\to\infty} \kappa\epsilon(\kappa) = \infty$.

If $\epsilon(\kappa)\kappa \leqq H \leqq \Lambda\kappa^{3/2}$*, then the ground state energy in* (1.5) *satisfies*

$$\mathrm{E_{gs}}(\kappa, H) \leqq \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right)\,\mathrm{d}x + o\left(\frac{\kappa^3}{H}\right), \tag{5.20}$$

where $g(\cdot)$ *is the function introduced in* (3.49).

Proof. Here, we construct a test function as in (5.21) below. Let $\zeta = \kappa^{-1/16}H^{-1/2}$ and $(\mathcal{Q}_{k,\zeta})$ be the lattice of squares generated by the square

$$\mathcal{Q}_\zeta = \left(-\frac{\zeta}{2}, \frac{\zeta}{2}\right) \times \left(-\frac{\zeta}{2}, \frac{\zeta}{2}\right).$$

Notice that ζ satisfies $\zeta \ll \frac{\kappa}{H} \ll 1$. Define

$$\mathcal{I} = \left\{k \;:\; \mathcal{Q}_{k,\zeta} \subset \left\{\operatorname{dist}(x,\Gamma) \leqq M\frac{\kappa}{H}\right\} \quad \text{and} \quad \operatorname{dist}(Q_{k,\zeta},\Gamma) \geqq M\zeta\right\},$$

where $M > 0$ is a constant selected sufficiently large so that, if $\operatorname{dist}(x,\Gamma) > M\frac{\kappa}{H}$, then $|B_0(x)| \geqq \frac{\kappa}{H}$. Notice that, since B_0 vanishes non-degenerately on Γ, if $k \in \mathcal{I}$, then

$$|B_0(x)| \geqq M'\zeta > 0 \text{ in } \overline{\mathcal{Q}_{k,\zeta}}.$$

For all $k \in \mathcal{I}$, let a_k be the center of the square $\mathcal{Q}_{k,\zeta}$ and select an arbitrary point $x_k \in \overline{\mathcal{Q}_{k,\zeta}}$.

If $r > 0$ and $b > 0$, let $u_r \in H_0^1(\mathcal{Q}_r)$ be the minimizer of the ground state energy $F_{b,\mathcal{Q}_r}$ introduced in (3.47). For all $k \in \mathcal{I}$, let $r_k = \zeta\sqrt{\kappa H|B_0(x_k)|}$, $b_k = \frac{H}{\kappa}|B_0(x_k)|$, $u_k = u_{r_k}$ and φ_k be the gauge function satisfying (see Proposition 4.3)

$$|\mathbf{F}(x) - (B_0(x_k)\mathbf{A}_0(x-a_k) + \nabla\varphi_k)| \leqq C\zeta^2, \quad \text{in } \overline{\mathcal{Q}_{k,\zeta}}.$$

Define the test function v as follows:

$$v(x) = \begin{cases} e^{-i\phi_k(x)}\, u_k\big(\frac{r_k}{\zeta}\,(x-a_k)\big) & \text{if } x \in \mathcal{Q}_{k,\zeta} \subset \{B_0(x) > 0\}, \\ e^{-i\phi_j(x)}\, \overline{u_k}\big(\frac{r_k}{\zeta}\,(x-a_k)\big) & \text{if } x \in \mathcal{Q}_{k,\zeta} \subset \{B_0(x) < 0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.21}$$

We outline the computation of $\mathcal{E}_0(v,\mathbf{F})$. The details of the computations are given in [3]. In every square $\mathcal{Q}_{k,\zeta}$ we have

$$\mathcal{E}_0(v,\mathbf{F};\mathcal{Q}_{k,\zeta}) \leqq (1+\kappa^{-1/16})\,\frac{F_{b_k,\mathcal{Q}_{r_k}}(u_k)}{b_k} + \kappa^{1/16}\kappa^2H^2\zeta^6.$$

Thanks to the assumption on H and the definition of $\zeta = \kappa^{-1/16}H^{-1/2}$, we have $r_k \gg 1$. Thus, we may use (3.51), and write

$$\begin{aligned}\mathcal{E}_0(v,\mathbf{F};\mathcal{Q}_{k,\zeta}) &\leqq (1+\kappa^{-1/16})\,\frac{r_k^2}{b_k}\left(g(b_k) + C\frac{\sqrt{b_k}}{r_k}\right) + \kappa^{1/16}\kappa^2H^2\zeta^6 \\ &= (1+\kappa^{-1/16})\zeta^2\kappa^2\left(g\left(\frac{H}{\kappa}|B_0(x_k)|\right) + C\frac{1}{\zeta\kappa}\right) + \kappa^{1/16}\kappa^2H^2\zeta^6.\end{aligned}$$

We sum over k and select the points x_k as follows:

$$|B_0(x_k)| = \min\{|B_0(x)| \;:\; x \in \overline{\mathcal{Q}_{k,\zeta}}\}.$$

In that way, we obtain

$$\begin{aligned}\mathcal{E}_0(v,\mathbf{F}) &= \sum_{k\in\mathcal{I}} \mathcal{E}_0(v,\mathbf{F};\mathcal{Q}_{k,\zeta})\\ &\leqq (1+\kappa^{-1/16})\kappa^2 \int_{\bigcup_{k\in\mathcal{I}}\overline{\mathcal{Q}_{k,\zeta}}} \left(g\left(\frac{H}{\kappa}|B_0(x)|\right) + C\frac{1}{\zeta\kappa} + \kappa^{1/16}H^2\zeta^4 \right) \mathrm{d}x\\ &\leqq (1+\kappa^{-1/16})\kappa^2 \int_{\bigcup_{k\in\mathcal{I}}\overline{\mathcal{Q}_{k,\zeta}}} g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x + C\kappa^{1/16}\frac{\kappa^2}{H\zeta} + M\kappa^{-1/16}\frac{\kappa^3}{H}.\end{aligned}$$

Notice that, since $g(b) = 0$ for $b \geqq 1$ and B_0 vanishes non-degenerately on Γ, then

$$\int_{\bigcup_{k\in\mathcal{I}}\overline{\mathcal{Q}_{k,\zeta}}} g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x \leqq \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x + \frac{\kappa}{H}\, o(1).$$

Thus,

$$\begin{aligned}\mathcal{E}_0(v,\mathbf{F}) \leqq (1+\kappa^{-1/16})\kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x\\ + C\kappa^{1/16}\frac{\kappa^2}{H\zeta} + M\kappa^{-1/16}\frac{\kappa^3}{H} + \frac{\kappa^3}{H}\, o(1).\end{aligned}$$

Since $H \lesssim \kappa^{3/2}$, then

$$\kappa^{1/16}\frac{\kappa^2}{H\zeta} = \kappa^{1/8}\frac{H^{1/2}}{\kappa}\frac{\kappa^3}{H} \ll 1$$

and

$$\mathcal{E}_0(v,\mathbf{F}) \leqq (1+\kappa^{-1/16})\kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x + \frac{\kappa^3}{H}\, o(1).$$

Since $\mathrm{E}_{\mathrm{gs}}(\kappa, H) \leqq \mathcal{E}(v,\mathbf{F}) = \mathcal{E}_0(v,\mathbf{F})$, then we get the upper bound in (5.20). □

6. Exponential Decay of the Order Parameter

The aim of this section is to prove that the order parameter ψ is exponentially small (in the L^2-norm) away from the curves where the magnetic field vanishes. This bound is needed in Section 7 to obtain a lower bound of the ground state energy in (1.5).

6.1. A Rough Bound

In this subsection we give a rough bound valid for any order parameter ψ.

Theorem 6.1. *Let $\Lambda > 0$ and $\epsilon : \mathbb{R} \to \mathbb{R}_+$ such that $\lim_{\kappa\to\infty} \kappa\,\epsilon(\kappa) = \infty$ and $\lim_{\kappa\to\infty} \epsilon(\kappa) = 0$. There exist constants C and κ_0 such that, if $(\psi, \mathbf{A})$ is a critical point of the functional in (1.1), $\kappa \geqq \kappa_0$ and*

$$\epsilon(\kappa)\kappa^2 \leqq H \leqq \Lambda\kappa^2, \tag{6.1}$$

then

$$\|\psi\|_2 \leqq C\left(\frac{\kappa}{H}\right)^{1/6}, \tag{6.2}$$

$$\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_2 \leqq \frac{C}{H}\left(\frac{\kappa}{H}\right)^{1/6}, \tag{6.3}$$

and

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_2 \leqq C\kappa\left(\frac{\kappa}{H}\right)^{1/6}. \tag{6.4}$$

An important ingredient in the proof of this theorem is:

Proposition 6.2. *Let $\Lambda > 0$ and $\epsilon : \mathbb{R} \to \mathbb{R}_+$ such that $\lim_{\kappa\to\infty} \kappa\,\epsilon(\kappa) = \infty$ and $\lim_{\kappa\to\infty} \epsilon(\kappa) = 0$. There exist positive constants C, ℓ_0 and κ_0 such that the following is true:*
For $\ell \in (0, \ell_0)$, $a \in (0, 1]$ and $h \in C_c^\infty(\Omega)$ such that

$$\operatorname{supp} h \subset \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) > \ell \;\&\; \operatorname{dist}(x, \Gamma) > \sqrt{a}\,\ell\} \text{ and } \|h\|_\infty \leqq 1,$$

if $(\psi, \mathbf{A})$ is a critical point of the functional in (1.1), $\kappa \geqq \kappa_0$ and $\epsilon(\kappa)\kappa^2 \leqq H \leqq \Lambda\kappa^2$, then

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})h\psi|^2\,\mathrm{d}x \geqq \frac{1}{C}\kappa\big(H\sqrt{a}\,\ell - C^2\big)\int_\Omega |h\psi|^2\,\mathrm{d}x \\ -C\kappa\int_\Omega (1-h^2)|\psi|^2\,\mathrm{d}x. \tag{6.5}$$

Proof. The support of the function $h\psi$ does not meet the boundary of Ω and Γ. We can use the celebrated inequality

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})h\psi|^2\,\mathrm{d}x \geqq \kappa H\int_\Omega |\operatorname{curl}\mathbf{A}|\,|h\,\psi|^2\,\mathrm{d}x.$$

The simple decomposition $\operatorname{curl}\mathbf{A} = \operatorname{curl}\mathbf{F} + \operatorname{curl}(\mathbf{A}-\mathbf{F})$ and the triangle inequality yield

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})h\psi|^2\,\mathrm{d}x \geqq \kappa H\int_\Omega |\operatorname{curl}\mathbf{F}|\,|h\psi|^2\,\mathrm{d}x \\ -\kappa H\int_\Omega |\operatorname{curl}\mathbf{A} - \operatorname{curl}\mathbf{F}|\,|h\psi|^2\,\mathrm{d}x. \tag{6.6}$$

By assumption, ∇B_0 does not vanish on Γ, hence

$$|\operatorname{curl}\mathbf{F}| = |B_0(x)| \geqq \frac{1}{M}\sqrt{a}\,\ell \quad \text{in } \{\operatorname{dist}(x,\Gamma) \geqq \sqrt{a}\,\ell\} \tag{6.7}$$

for some constant $M > 0$.
Thus,

$$\int_\Omega |\operatorname{curl}\mathbf{F}|\,|h\psi|^2\,\mathrm{d}x \geqq \frac{1}{M}\sqrt{a}\,\ell \int_\Omega |h\psi|^2\,\mathrm{d}x. \tag{6.8}$$

Next we use the Cauchy-Schwarz inequality and the inequality in (4.5) as follows

$$\begin{aligned}\kappa H \int_\Omega |\operatorname{curl}\mathbf{A} - \operatorname{curl}\mathbf{F}|\,|h\psi|^2\,\mathrm{d}x &\leqq \kappa H \|\operatorname{curl}\mathbf{A} - \operatorname{curl}\mathbf{F}\|_2 \left(\int_\Omega |h\psi|^4\,\mathrm{d}x\right)^{1/2}\\ &\leqq C\kappa\|\psi\|_2 \left(\int_\Omega |h\psi|^4\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Since $\|\psi\|_\infty \leqq 1$ and $\|h\|_\infty \leqq 1$, we get, further, that

$$\|\psi\|_2 \left(\int_\Omega |h\psi|^4\,\mathrm{d}x\right)^{1/2} \leqq \int_\Omega |\psi|^2\,\mathrm{d}x = \int_\Omega |h\psi|^2\,\mathrm{d}x + \int_\Omega (1-h^2)|\psi|^2\,\mathrm{d}x.$$

Therefore, we have

$$\kappa H \int_\Omega |\operatorname{curl}\mathbf{A} - \operatorname{curl}\mathbf{F}|\,|h\psi|^2\,\mathrm{d}x \leqq C\kappa \int_\Omega |h\psi|^2\,\mathrm{d}x + C\kappa \int_\Omega (1-h^2)|\psi|^2\,\mathrm{d}x. \tag{6.9}$$

Inserting (6.9) and (6.8) into (6.6) finishes the proof of the proposition. □

Proof of Theorem 6.1. Let $\ell > 0$ and $\Omega_\ell = \{x \in \Omega \,:\, \operatorname{dist}(x,\partial\Omega) > \ell \ \&\ \operatorname{dist}(x,\Gamma) > \ell\}$. Select a function $h \in C_c^\infty(\Omega)$ satisfying

$$0 \leqq h \leqq 1 \quad \text{in } \Omega, \quad h = 1 \quad \text{in } \Omega_{2\ell}, \quad h = 0 \quad \text{in } \Omega\setminus\Omega_\ell,$$

and

$$|\nabla h| \leqq \frac{C}{\ell} \quad \text{in} \quad \Omega,$$

where C is a constant.

Thanks to the bound $\|\psi\|_\infty \leqq 1$ and the assumptions on h, we have

$$\int_\Omega |\psi|^2 \leqq \int |h\psi|^2 + C\ell, \tag{6.10}$$

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})\psi|^2\,\mathrm{d}x \geqq \int_\Omega |h(\nabla - i\kappa H\mathbf{A})\psi|^2\,\mathrm{d}x \tag{6.11}$$

$$\geqq \frac{1}{2}\int_\Omega |(\nabla - i\kappa H\mathbf{A})h\psi|^2\,\mathrm{d}x - C\int_\Omega |\nabla h|^2\,|\psi|^2\,\mathrm{d}x. \tag{6.12}$$

Thanks to the estimate on the gradient of h, we may write

$$\frac{1}{2}\int_\Omega |(\nabla - i\kappa H\mathbf{A})h\psi|^2\,\mathrm{d}x - \kappa^2\int_\Omega |h\psi|^2\,\mathrm{d}x - C\big(\ell\kappa^2+\ell^{-1}\big) \leqq \mathcal{E}_0(\psi,\mathbf{A};\Omega) \leqq 0,$$

where $\mathcal{E}_0(\psi,\mathbf{A};\Omega)$ is introduced in (4.3).

Now, we use Proposition 6.2 with $a = 1$ and get

$$\Big(\frac{\kappa}{2C}(H\ell - C^2) - \kappa^2\Big)\int_\Omega |h\psi|^2\,\mathrm{d}x \leqq C(\ell+\ell^{-1})\kappa^2.$$

Selecting $\ell = (\kappa/H)^{1/3}$, we get for κ large and H satisfying (6.1)

$$\int_\Omega |h\psi|^2\,\mathrm{d}x \leqq C\left(\frac{\kappa}{H}\right)^{1/3}.$$

Now, thanks to (6.10), the first inequality (6.2) in Theorem 6.1 is proved. Now, the inequality (6.3) (resp. (6.4)) is simply a consequence of (4.5)) (resp. (4.6)). □

6.2. Exponential Bound

In the next theorem, we establish that every minimizing order parameter decays exponentially fast away from the set Γ where the magnetic field vanishes, provided that $\kappa \ll H \lesssim \kappa^2$.

Theorem 6.3. *Let* $\Lambda > 0$ *and* $\epsilon : \mathbb{R} \to \mathbb{R}_+$ *such that* $\lim_{\kappa\to\infty} \kappa\,\epsilon(\kappa) = \infty$ *and* $\lim_{\kappa\to\infty}\epsilon(\kappa) = 0$. *There exist positive constants* C, m_0 *and* κ_0 *such that, if* $(\psi,\mathbf{A})$ *is a critical point of the functional in* (1.1), $\kappa \geqq \kappa_0$, $\epsilon(\kappa)\kappa^2 \leqq H \leqq \Lambda\kappa^2$, *then*

$$\int_\Omega \exp\Big(2m_0\frac{H}{\kappa}\,t(x)\Big)\Big(\frac{1}{\kappa^2}|(\nabla - i\kappa H\mathbf{A})\psi|^2 + |\psi(x)|^2\Big)\,\mathrm{d}x \leqq C\int_{\{t(x)\leqq C\frac{\kappa}{H}\}} |\psi(x)|^2\,\mathrm{d}x,$$

where $t(x) = \operatorname{dist}(x,\Gamma)$.

Proof. Let

$$\zeta = (\kappa H)^{-1/3}. \tag{6.13}$$

The assumption on κ and H ensures that

$$\kappa^{-1} \lesssim \zeta \ll 1. \tag{6.14}$$

We will prove Theorem 6.3 by establishing the following two estimates (away from the boundary or in a neighborhood of the boundary):

$$\int_{\{\operatorname{dist}(x,\partial\Omega)\geqq\zeta\}} e^{2m_0\frac{H}{\kappa}t(x)}\Big(\frac{1}{\kappa^2}|(\nabla - i\kappa H\mathbf{A})\psi|^2 + |\psi(x)|^2\Big)\,\mathrm{d}x \leqq C_1\int_{\{t(x)\leqq C\frac{\kappa}{H}\}} |\psi(x)|^2\,\mathrm{d}x, \tag{6.15}$$

and

$$\int_{\{\mathrm{dist}(x,\partial\Omega)\leqq\zeta\}} e^{2m_0\frac{H}{\kappa}t(x)}\left(\frac{1}{\kappa^2}|(\nabla-i\kappa H\mathbf{A})\psi|^2+|\psi(x)|^2\right)\mathrm{d}x$$
$$\leqq C_2\int_{\{t(x)\leqq C\frac{\kappa}{H}\}}|\psi(x)|^2\,\mathrm{d}x, \tag{6.16}$$

expressing the localization of the energy of ψ near Γ.

The proof of (6.15) and (6.16) is divided into several steps.

Step 1.

Consider the parameters

$$\xi\in(1,\infty),\quad \sigma=\frac{H}{\kappa^2},\quad \ell=\frac{\xi}{\sigma\kappa}. \tag{6.17}$$

Let

$$f(x)=\chi(x)\,\exp(\ell^{-1}t(x)),$$

and

$$g(x)=\eta(x)\,\exp(\ell^{-1}t(x)).$$

The functions $\chi\in C_c^\infty(\Omega)$ and $\eta\in C^\infty(\overline{\Omega})$ satisfy

$$\begin{cases} 0\leqq\chi\leqq 1 & \text{in }\Omega,\\ \chi=1 & \text{in }\{\mathrm{dist}(x,\partial\Omega)\geqq\zeta\}\bigcup\{t(x)\geqq\ell\},\\ \chi=0 & \text{in }\{\mathrm{dist}(x,\partial\Omega)\leqq\frac{1}{2}\zeta\}\bigcup\{t(x)\leqq\frac{1}{2}\ell\},\\ |\nabla\chi|\leqq C\kappa & \text{in }\Omega. \end{cases} \tag{6.18}$$

and

$$\begin{cases} 0\leqq\eta\leqq 1 & \text{in }\Omega,\\ \eta=1 & \text{in }\{\mathrm{dist}(x,\partial\Omega)\leqq\zeta\}\bigcup\{t(x)\geqq\ell\},\\ \eta=0 & \text{in }\{\mathrm{dist}(x,\partial\Omega)\geqq2\zeta\}\bigcup\{t(x)\leqq\frac{1}{2}\ell\},\\ |\nabla\eta|\leqq C\kappa & \text{in }\Omega. \end{cases} \tag{6.19}$$

Here we have used for the control of the gradient (6.14) and that

$$\kappa^{-1}\ll\ell\lesssim 1.$$

Using the Ginzburg–Landau equation in (4.1), we write

$$\int_\Omega\left(|(\nabla-i\kappa H\mathbf{A})f\psi|^2-|\nabla f|^2|\psi|^2\right)\mathrm{d}x=\kappa^2\int_\Omega\left(|\psi|^2-|\psi|^4\right)f^2\,\mathrm{d}x$$
$$\leqq\kappa^2\int_\Omega|f\psi|^2\,\mathrm{d}x, \tag{6.20}$$

and

$$\int_\Omega \Big(|(\nabla - i\kappa H\mathbf{A}) g\psi|^2 - |\nabla g|^2 |\psi|^2 \Big)\, \mathrm{d}x = \kappa^2 \int_\Omega \Big(|\psi|^2 - |\psi|^4 \Big) g^2 \,\mathrm{d}x$$
$$\leqq \kappa^2 \int_\Omega |g\psi|^2 \,\mathrm{d}x. \tag{6.21}$$

Step 2.

In this step, we determine a lower bound of $\int_\Omega |(\nabla - i\kappa H\mathbf{A}) f\psi|^2 \,\mathrm{d}x$. Notice that $f\psi \in C_c^\infty(\Omega)$. Consequently, we may write (see (6.6))

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A}) f\psi|^2 \,\mathrm{d}x \geqq \kappa H \int_\Omega |\operatorname{curl} \mathbf{F}|\, |f\psi|^2 \,\mathrm{d}x$$
$$-\kappa H \int_\Omega |\operatorname{curl} \mathbf{A} - \operatorname{curl} \mathbf{F}|\, |f\psi|^2 \,\mathrm{d}x.$$

We use the following estimates

$$\int_\Omega |\operatorname{curl} \mathbf{F}|\, |f\psi|^2 \,\mathrm{d}x \geqq \frac{1}{M} \ell \int_\Omega |f\psi|^2 \,\mathrm{d}x \qquad \text{[by (6.7)]}$$
$$\int_\Omega |\operatorname{curl} \mathbf{A} - \operatorname{curl} \mathbf{F}|\, |f\psi|^2 \,\mathrm{d}x \leqq \frac{C}{H} \left(\frac{\kappa}{H}\right)^{1/6} \|f\psi\|_4^2 \qquad \text{[by (6.3)]},$$

and obtain

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A}) f\psi|^2 \,\mathrm{d}x \geqq \frac{1}{M} \kappa H \ell \int_\Omega |f\psi|^2 \,\mathrm{d}x - C\kappa \left(\frac{\kappa}{H}\right)^{1/6} \|f\psi\|_4^2.$$

Notice that $f\psi \in C_c^\infty(\Omega) \subset H^1(\mathbb{R}^2)$. By the continuous Sobolev embedding of $H^1(\mathbb{R}^2)$ in $L^4(\mathbb{R}^2)$ and a scaling, we get for all $\eta \in (0,1)$:

$$\|f\psi\|_4^2 = \|\, |f\psi|\, \|_4^2$$
$$\leqq C_{\mathrm{Sob}} \Big(\eta \|\nabla |f\psi|\|_2^2 + \eta^{-1} \|f\psi\|_2^2 \Big)$$
$$\leqq C_{\mathrm{Sob}} \Big(\eta \|(\nabla - i\kappa H\mathbf{A}) f\psi\|_2^2 + \eta^{-1} \|f\psi\|_2^2 \Big) \quad \text{[By the diamagnetic inequality]}.$$

We select $\eta = \frac{1}{CC_{\mathrm{Sob}}} \kappa^{-1} \left(\frac{\kappa}{H}\right)^{-\frac{1}{6}}$ and obtain

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A}) f\psi|^2 \,\mathrm{d}x \geqq \left(\frac{\kappa H \ell}{2M} - \widehat{C}\kappa^2 \left(\frac{\kappa}{H}\right)^{1/3} \right) \int_\Omega |f\psi|^2 \,\mathrm{d}x. \tag{6.22}$$

Thanks to the choice of the parameters in (6.17), the lower bound in (6.22) becomes

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A}) f\psi|^2 \,\mathrm{d}x \geqq \left(\frac{\xi\kappa^2}{2M} - \widehat{C}\kappa^2 \left(\frac{\kappa}{H}\right)^{1/3} \right) \int_\Omega |f\psi|^2 \,\mathrm{d}x. \tag{6.23}$$

Step 3.

We insert (6.23) into (6.20) and use that

$$\int_\Omega |\nabla f|^2|\psi|^2\,\mathrm{d}x \leqq 2\ell^{-2}\int_\Omega |f\psi|^2\,\mathrm{d}x + 2\int_\Omega |\nabla\chi|^2\exp(2\ell^{-1}t(x))|\psi|^2\,\mathrm{d}x$$
$$\leqq 2\ell^{-2}\int_\Omega |f\psi|^2\,\mathrm{d}x + C\kappa^2\int_\Omega |g\psi|^2\,\mathrm{d}x + C\kappa^2\int_{\{\ell^{-1}t(x)\leqq 1\}} |\psi|^2\,\mathrm{d}x, \tag{6.24}$$

to obtain

$$\int_\Omega \left(\frac{1}{2}|(\nabla - i\kappa H\mathbf{A})f\psi|^2 + \frac{1}{2}\left(\frac{\xi\kappa^2}{2M} - 2\frac{\sigma^2}{\xi^2}\kappa^2 - \widehat{C}\kappa^2\left(\frac{\kappa}{H}\right)^{1/3}\right)|f\psi|^2\,\mathrm{d}x\right)$$
$$\leqq \widehat{C}\kappa^2\int_\Omega |g\psi|^2\,\mathrm{d}x + \widehat{C}\kappa^2\int_{\{\ell^{-1}t(x)\leqq 1\}} |\psi|^2\,\mathrm{d}x. \tag{6.25}$$

Step 4.

We will determine a lower bound of $\int_\Omega |(\nabla - i\kappa H\mathbf{A})g\psi|^2\,\mathrm{d}x$. We cover the set

$$\Omega_{\zeta,\ell} = \{x \in \overline{\Omega} \ :\ \mathrm{dist}(x, \partial\Omega) \leqq 2\zeta,\ \mathrm{dist}(x, \Gamma) \geqq \ell\}$$

by a family of squares (in tubular coordinates)

$$\mathcal{K}(a_j, \zeta) = \{x \ \in \overline{\Omega} \ :\ \mathrm{dist}(x, \partial\Omega) \leqq 2\zeta,\ \mathrm{dist}_{\partial\Omega}(p(x), a_j) \leqq 2\zeta\},$$

where:

- $\mathrm{dist}_{\partial\Omega}$ is the arc-length distance along $\partial\Omega$.
- if $x \in \Omega_{\zeta,\ell}$ and ζ is sufficiently small, $p(x)$ is the unique point on $\partial\Omega$ satisfying $\mathrm{dist}(x, p(x)) = \mathrm{dist}(x, \partial\Omega)$.
- for all j, $a_j \in \partial\Omega \cap \overline{\Omega_{\zeta,\ell}}$.

Let (χ_j) be a partition of unity such that

$$\sum_j \chi_j^2 = 1, \quad \sum_j |\nabla\chi_j|^2 \leqq C\zeta^{-2}, \quad \mathrm{supp}\,\chi_j \subset \mathcal{K}(a_j, 2\zeta).$$

This holds the decomposition formula

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})g\psi|^2\,\mathrm{d}x = \sum_j\int_\Omega |(\nabla - i\kappa H\mathbf{A})\chi_j g\psi|^2\,\mathrm{d}x - \sum_j\int_\Omega |\nabla\chi_j|^2\,|g\psi|^2\,\mathrm{d}x$$
$$\geqq \sum_j\int_\Omega |(\nabla - i\kappa H\mathbf{A})\chi_j g\psi|^2\,\mathrm{d}x - C\zeta^{-2}\int_\Omega |g\psi|^2\,\mathrm{d}x. \tag{6.26}$$

Next, we define the gauge function

$$\alpha_j = (\mathbf{A}(a_j) - \mathbf{F}(a_j)) \cdot (x - a_j).$$

Let $\alpha \in (0,1)$ be a constant. Using the Cauchy-Schwarz inequality, Proposition 4.2 and Theorem 6.1, we may write

$$\begin{aligned}&\int_\Omega |(\nabla - i\kappa H\mathbf{A})\chi_j g\psi|^2\,\mathrm{d}x = \int_\Omega |(\nabla - i\kappa H(\mathbf{A} - \nabla\alpha_j))e^{-i\kappa H\alpha_j}\chi_j g\psi|^2\,\mathrm{d}x\\ &\geqq \frac{1}{2}\int_\Omega |(\nabla - i\kappa H\mathbf{F})e^{-i\kappa H\alpha_j}\chi_j g\psi|^2\,\mathrm{d}x - C\kappa^2\left(\frac{\kappa}{H}\right)^{1/3}\zeta^{2\alpha}\int_\Omega |\chi_j g\psi|^2\,\mathrm{d}x.\end{aligned} \tag{6.27}$$

Recall the definition of the magnetic potential $\mathbf{A}_0$ in (3.46). There exists a gauge function φ_j satisfying (see Proposition 4.3)

$$\left|\mathbf{F}(x) - (B_0(x_j)\mathbf{A}_0(x - a_j) + \nabla\varphi_j)\right| \leqq C\zeta^2 \quad \text{in } \mathcal{K}_j(a_j, \zeta).$$

Again, using the Cauchy-Schwarz inequality, we may write

$$\begin{aligned}&\int_\Omega |(\nabla - i\kappa H\mathbf{F})e^{-i\kappa H\alpha_j}\chi_j g\psi|^2\,\mathrm{d}x\\ &\quad\geqq \frac{1}{2}\int_\Omega |(\nabla - i\kappa HB_0(a_j)\mathbf{A}_0(x - a_j))e^{-i\kappa H\varphi_j}e^{-i\kappa H\alpha_j}\chi_j g\psi|^2\,\mathrm{d}x\\ &\quad -\kappa^2H^2\zeta^4\int_\Omega |\chi_j g\psi|^2\,\mathrm{d}x.\end{aligned} \tag{6.28}$$

Now, we are allowed to use the analysis of the Neumann realization of the Schrödinger operator with a constant magnetic field equal to $\kappa HB_0(a_j)$ in our case. In the half-plane case, the ground state energy of this operator is $\Theta_0\kappa H|B_0(a_j)|$, where the constant Θ_0 is universal and satisfies $\Theta_0 \in (\frac{1}{2}, 1)$. The result remains asymptotically true in general domains with smooth and compact boundary [20]. More precisely, there exists a function

$$\mathrm{err} : \mathbb{R}_+ \to \mathbb{R}_+,$$

such that $\lim_{|b|\to\infty} \mathrm{err}(b) = 0$ and

$$\forall\, b, \quad \lambda^N(b) \geqq \Theta_0|b| - |b|\,\mathrm{err}(b),$$

where $\lambda^N(b)$ is the lowest eigenvalue of the operator $-(\nabla - ib\mathbf{A}_0)^2$ in $L^2(\Omega)$ with Neumann boundary condition.

Notice that by the assumptions on ℓ and the points (a_j), we may use (6.7) with $x = a_j$, and get

$$\forall\, j, \quad \kappa H|B_0(a_j)| \geqq \frac{1}{M}\ell\kappa H \gg 1.$$

Moreover, the magnetic potentials $\mathbf{A}_0(x)$ and $\mathbf{A}_0(x - a_j)$ are gauge equivalent since

$$\mathbf{A}_0(x - a_j) = \mathbf{A}_0(x) - \mathbf{A}_0(a_j) = \mathbf{A}_0(x) - \nabla u_j(x),$$

with $u_j(x) = A_0(a_j) \cdot x$.

In that way, when κ is sufficiently large, we may write

$$\begin{aligned}&\int_\Omega |(\nabla - i\kappa H B_0(a_j)\mathbf{A}_0(x-a_j))e^{-i\kappa H\varphi_j} e^{-i\kappa H\alpha_j}\chi_j g\psi|^2\,\mathrm{d}x\\ &\geqq \frac{\Theta_0}{2}\kappa H|B_0(a_j)|\int_\Omega |\chi_j g\psi|^2\,\mathrm{d}x \geqq \frac{1}{4M}\,\ell\kappa H\int_\Omega |\chi_j g\psi|^2\,\mathrm{d}x.\end{aligned} \tag{6.29}$$

Collecting the estimates in (6.26), (6.27), (6.28) and (6.29), we get

$$\begin{aligned}&\int_\Omega |(\nabla - i\kappa H\mathbf{A})g\psi|^2\,\mathrm{d}x\\ &\geqq \kappa\left(\frac{H\ell}{4M} - C\kappa H^2\zeta^4 - \frac{C}{\kappa\zeta^2} - C\kappa\left(\frac{\kappa}{H}\right)^{1/3}\zeta^{2\alpha}\right)\int_\Omega |g\psi|^2\,\mathrm{d}x.\end{aligned} \tag{6.30}$$

Recall the definition of the parameters in (6.17) and (6.13):

$$\zeta = (H\kappa)^{-1/3} = \sigma^{-1/3}\kappa^{-1}.$$

We insert (6.30) into (6.21) and use that

$$\begin{aligned}\int_\Omega |\nabla g|^2|\psi|^2\,\mathrm{d}x &\leqq 2\ell^{-2}\int_\Omega |g\psi|^2\,\mathrm{d}x + 2\int_\Omega |\nabla\eta|^2\exp(2\ell^{-1}t(x))|\psi|^2\,\mathrm{d}x\\ &\leqq 2\ell^{-2}\int_\Omega |g\psi|^2\,\mathrm{d}x + C\kappa^2\int_\Omega |f\psi|^2\,\mathrm{d}x\\ &\quad + C\kappa^2\int_{\{\ell^{-1}t(x)\leqq 1\}} |\psi|^2\,\mathrm{d}x\end{aligned}$$

to obtain

$$\begin{aligned}&\int_\Omega\left(\frac{1}{2}|(\nabla - i\kappa H\mathbf{A})g\psi|^2 + \frac{1}{2}\Big(\frac{\xi\kappa^2}{4M} - 2\ell^{-2} - C\sigma^{2/3}\kappa\right.\\ &\qquad\left. - C\kappa\left(\frac{\kappa}{H}\right)^{1/3}\zeta^{2\alpha}\Big)|g\psi|^2\,\mathrm{d}x\right)\\ &\leqq C\kappa^2\int_\Omega |f\psi|^2\,\mathrm{d}x + C\kappa^2\int_{\{\ell^{-1}t(x)\leqq 1\}} |\psi|^2\,\mathrm{d}x.\end{aligned} \tag{6.31}$$

Step 5.
Summing the two inequalities in (6.25) and (6.31), we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega (|(\nabla - i\kappa H\mathbf{A})g\psi|^2 + |(\nabla - i\kappa H\mathbf{A})f\psi|^2)\,\mathrm{d}x\\ &\quad + \frac{1}{2}\int_\Omega\left(\frac{\xi\kappa^2}{4M} - C\frac{\sigma^2}{\xi^2}\kappa^2 - C\kappa^2 - C\sigma^{2/3}\kappa - C\kappa\left(\frac{\kappa}{H}\right)^{1/3}\zeta^{2\alpha}\right)\\ &\qquad (|f\psi|^2 + |g\psi|^2)\\ &\leqq C\kappa^2\int_{\{\ell^{-1}t(x)\leqq 1\}} |\psi|^2\,\mathrm{d}x.\end{aligned} \tag{6.32}$$

Recall that σ satisfies $\kappa^{-1} \ll \sigma \leqq \Lambda$. We select ξ sufficiently large such that

$$\frac{\xi}{4M} - C\frac{\Lambda^2}{\xi^2} - C > 2.$$

Since $\left(\frac{\kappa}{H}\right)^{1/3} \ll 1$ and $\zeta \ll 1$, we get,

$$\int_\Omega \left(\frac{1}{2}|(\nabla - i\kappa H\mathbf{A})f\psi|^2 + \frac{\kappa^2}{2}|f\psi|^2 \,\mathrm{d}x\right) \leqq C\kappa^2 \int_{\{\ell^{-1}t(x) \leqq 1\}} |\psi|^2 \,\mathrm{d}x,$$

and

$$\int_\Omega \left(\frac{1}{2}|(\nabla - i\kappa H\mathbf{A})g\psi|^2 + \frac{\kappa^2}{2}|g\psi|^2 \,\mathrm{d}x\right) \leqq C\kappa^2 \int_{\{\ell^{-1}t(x) \leqq 1\}} |\psi|^2 \,\mathrm{d}x.$$

Thanks to the definitions of f and g, the two aforementioned inequalities yield the inequalities in (6.15) and (6.16) with $m_0 = 1/\xi$. □

As a consequence of Theorem 6.3, we get an improvement of the bound given in Theorem 6.1.

Proposition 6.4. *Under the assumptions of Theorem 6.3, it holds,*

$$\|\psi\|_2 \leqq C\sqrt{\frac{\kappa}{H}}.$$

Combining the results in Propositions 4.1, 4.2 and 6.4, we obtain the improved estimates:

Proposition 6.5. *Under the assumptions of Theorem 6.3 and Proposition 4.2, it holds,*

$$\|\operatorname{curl}\mathbf{A} - \operatorname{curl}\mathbf{F}\|_2 \leqq \frac{C}{H}\sqrt{\frac{\kappa}{H}},$$

$$\|\mathbf{A} - \mathbf{F}\|_{C^{1,\alpha}(\overline{\Omega})} \leqq C_\alpha\sqrt{\frac{\kappa}{H}},$$

$$\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} \leqq \frac{\widehat{C}_\alpha}{H}\sqrt{\frac{\kappa}{H}}.$$

7. Energy Lower Bound

In this section, we will derive lower bounds of the following energy:

$$\mathcal{E}_0(\psi, \mathbf{A}; U) = \int_U \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\right) \mathrm{d}x, \tag{7.1}$$

where $U \subset \mathbb{R}^2$ is an open set such that $\overline{U} \subset \Omega$.

Proposition 7.1. *Let* $\Lambda > 0$ *and* $\epsilon : \mathbb{R} \to \mathbb{R}_+$ *such that* $\lim_{\kappa\to\infty} \kappa\,\epsilon(\kappa) = \infty$ *and* $\lim_{\kappa\to\infty} \epsilon(\kappa) = 0$ *. For* $\alpha \in (0,1)$*, there exist positive constants* C *and* κ_0 *such that, for* $\ell \in (0,1)$*,* $\delta \in (0,1)$*,* $a_j \in \Gamma$*,* $D(a_j,\ell) \subset \Omega$*,* $x_j \in \overline{D(a_j,\ell)} \cap \Gamma$*,* $h \in C_c^\infty(D(a_j,\ell))$ *a function satisfying* $\|h\|_\infty \leqq 1$*,* $(\psi, \mathbf{A})$ *a critical point of the functional in* (1.1)*,* $\kappa \geqq \kappa_0$*, and*

$$\epsilon(\kappa)\kappa^2 \leqq H \leqq \Lambda\kappa^2,$$

the following holds:

$$\mathcal{E}_0(h\,\psi, \mathbf{A}; D(a_j,\ell)) \geqq (1-\delta)\,2\ell\,\kappa \left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right) - r,$$

where

$$r = C\left(\delta\kappa^2 + \delta^{-1}\left(\frac{\kappa^3}{H}\ell^{2\alpha} + \kappa^2 H^2 \ell^6\right)\right)\int_{D(a_j,\ell)} |h\psi|^2\,\mathrm{d}x.$$

Proof. Let $\alpha_j = (\mathbf{A}(a_j) - \mathbf{F}(a_j)) \cdot (x - a_j)$. Thanks to Proposition 6.5, we have

$$\left|\mathbf{A} - (\mathbf{F} + \nabla\alpha_j)\right| \leqq C\,\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} |x - a_j|^\alpha \leqq \frac{C}{H}\sqrt{\frac{\kappa}{H}}\,\ell^\alpha \quad \text{in} \quad D(a_j,\ell). \tag{7.2}$$

Notice that

$$\begin{aligned}
&\mathcal{E}_0(h\,\psi, \mathbf{A}; D(a_j,\ell)) \\
&\quad = \mathcal{E}_0(h\,\psi\,e^{-i\kappa H\alpha_j}, \mathbf{A} - \nabla\alpha_j; D(a_j,\ell)) \\
&\quad \geqq (1-\delta)\mathcal{E}_0(h\,\psi\,e^{-i\kappa H\alpha_j}, \mathbf{F}; D(a_j,\ell)) \\
&\qquad -C\left(\delta\kappa^2 \int_{D(a_j,\ell)} |h\psi|^2\,\mathrm{d}x + \delta^{-1}\kappa^2 H^2 \int_{D(a_j,\ell)} |\mathbf{A} - (\mathbf{F}+\nabla\alpha_j)|^2 |h\psi|^2\,\mathrm{d}x\right).
\end{aligned} \tag{7.3}$$

Using (7.2), we get,

$$\begin{aligned}
\mathcal{E}_0(h\,\psi, \mathbf{A}; D(a_j,\ell)) &\geqq (1-\delta)\mathcal{E}_0(h\,\psi\,e^{-i\kappa H\alpha_j}, \mathbf{F}; D(a_j,\ell)) \\
&\quad -C\left(\delta\kappa^2 + \delta^{-1}\frac{\kappa^3}{H}\ell^{2\alpha}\right)\int_{D(a_j,\ell)} |h\psi|^2\,\mathrm{d}x.
\end{aligned} \tag{7.4}$$

Let

$$f_j = h\,\psi\,e^{-i\kappa H\alpha_j} e^{i\kappa H\phi_j},$$

where ϕ_j is defined in Proposition 4.3.
Notice that $f_j \in H_0^1(D(a_j,\ell))$, $\|f_j\|_\infty \leqq 1$ and, using (4.9),

$$\begin{aligned}
&\mathcal{E}_0(h\,\psi\,e^{-i\kappa H\alpha_j}, \mathbf{F}; D(a_j,\ell)) \\
&\quad = \mathcal{E}_0(f_j, \mathbf{F} - \nabla\phi_j; D(a_j,\ell)) \\
&\quad \geqq (1-\delta)\mathcal{E}_0(f_j, |\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_j}(x - a_j); D(a_j,\ell)) \\
&\qquad - C(\delta\kappa^2 + \delta^{-1}\kappa^2 H^2 \ell^6)\int_{D(a_j,\ell)} |f_j|^2\,\mathrm{d}x.
\end{aligned} \tag{7.5}$$

We will use Theorem 3.11 to get a lower bound of the energy

$$\mathcal{E}_0(f_j, |\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_j}(x-a_j);\, D(a_j,\ell)).$$

Define

$$L = L_j = |\nabla B_0(x_j)|\, \frac{H}{\kappa^2}. \tag{7.6}$$

Performing the translation $x \mapsto x + a_j$, we get that

$$\mathcal{E}_0(f_j, |\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_j}(x-a_j);\, D(a_j,\ell)) = \mathcal{G}(f_j) \geqq \mathrm{E}_{\mathrm{gs,r}}(\kappa, L; \ell). \tag{7.7}$$

Here $\mathcal{G}$ is the functional in (3.36) and $\mathrm{E}_{\mathrm{gs,r}}(\kappa, L; \ell)$ is the ground state energy in (3.37).
Let

$$R = L^{1/3}\kappa\ell.$$

Now, Theorems 3.11 and 3.8 applied successively tell us that

$$\begin{aligned}\mathcal{E}_0\big(f_j, |\nabla B_0(x_j)|\mathbf{A}_{\mathrm{app},\nu_j}(x-a_j);\, D(a_j,\ell)\big) &\geqq \mathfrak{e}_{\mathrm{gs,disc}}(\nu, L; R)\\ &\geqq 2R\, E(L) = 2L^{1/3}\kappa\ell\, E(L).\end{aligned}$$

Recall the definition of L in (7.6). We insert the aforementioned estimate into (7.7). In that way, we infer from (7.5) and (7.4) the lower bound of Proposition 7.1. □

Proposition 7.2. *For $r > 0$, $h \in C^\infty(\mathbb{R}^2)$ satisfying $\|h\|_\infty \leqq 1$, and $(\psi, \mathbf{A})$ a critical point of the functional in (1.1), the following lower bound holds:*

$$\mathcal{E}_0(h\,\psi, \mathbf{A};\, D(a_j,r)\cap\Omega) \geqq -\pi\kappa^2 r^2. \tag{7.8}$$

Proof. Notice that all terms in $\mathcal{E}_0(h\,\psi, \mathbf{A};\, D(a_j,r)\cap\Omega)$ are positive except the integral of $|h\psi|^2$. Thus,

$$\mathcal{E}_0(h\,\psi, \mathbf{A};\, D(a_j,r)\cap\Omega) \geqq -\kappa^2 \int_{\Omega\cap D(a_j,r)} |h\psi|^2\,\mathrm{d}x.$$

This finishes the proof of the proposition upon using $\|h\psi\|_\infty \leqq 1$ and $\|\psi\|_\infty \leqq 1$.
□

Theorem 7.3. *Let $\Lambda > 0$ and $\epsilon : \mathbb{R} \to \mathbb{R}_+$ such that $\lim_{\kappa\to\infty} \kappa\,\epsilon(\kappa) = \infty$ and $\lim_{\kappa\to\infty} \epsilon(\kappa) = 0$.*

There exist $\kappa_0 > 0$ and a function err $: \mathbb{R} \to \mathbb{R}$ *such that the following is true:*

(1) $\lim_{\kappa\to\infty} \mathrm{err}(\kappa) = 0.$
(2) *Let $D \subset \Omega$ be a regular open set, $h \in C^\infty(\overline{D})$, $\|h\|_\infty \leqq 1$, $(\psi, \mathbf{A})$ a critical point of the functional in (1.1), $\kappa \geqq \kappa_0$ and $\epsilon(\kappa)\kappa^2 \leqq H \leqq \Lambda\kappa^2$.*

(a) *If* $H \gg \kappa^{3/2}$, *then,*

$$\mathcal{E}_0(h\psi, \mathbf{A}; D) \geqq \kappa \left(\int_{\Gamma \cap D} \left(|\nabla B_0(x)| \frac{H}{\kappa^2} \right)^{1/3} E\left(|\nabla B_0(x)| \frac{H}{\kappa^2} \right) \mathrm{d}s(x) \right) + \frac{\kappa^3}{H} \mathrm{err}(\kappa). \tag{7.9}$$

(b) *If* $H \lesssim \kappa^{3/2}$, *then,*

$$\mathcal{E}_0(h\psi, \mathbf{A}; D) \geqq \kappa^2 \int_D g\left(\frac{H}{\kappa} |B_0(x)| \right) \mathrm{d}x + \frac{\kappa^3}{H} \mathrm{err}(\kappa). \tag{7.10}$$

Proof. Consider three parameters

$$a \in (0, 1), \quad \ell \in (0, 1), \quad \delta \in (0, 1),$$

and define the following sets:

$$D_1 = \{x \in \Omega \ : \ \mathrm{dist}(x, \Gamma) < 2\sqrt{a}\,\ell\},$$
$$D_2 = \{x \in \Omega \ : \ \mathrm{dist}(x, \Gamma) > \sqrt{a}\,\ell\}.$$

Let (χ_j) be a partition of unity satisfying

$$\sum_{j=1}^{2} \chi_j^2 = 1, \quad \sum_{j=1}^{2} |\nabla \chi_j|^2 \leqq C(a\ell^2)^{-1}, \quad \mathrm{supp}\, \chi_j \subset D_j \quad (j \in \{1, 2\}).$$

This holds the following decomposition of the energy:

$$\mathcal{E}_0(h\psi, \mathbf{A}; D) \geqq \mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1) + \mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_j) - \sum_{j=1}^{2} \int_\Omega |\nabla \chi_j|^2 \, |h\psi|^2 \, \mathrm{d}x.$$

The error terms are controlled using the pointwise bounds on $|h|$, $|\psi|$, $|\nabla \chi_j|$, and the conditions on the support of χ_j. We obtain the following lower bound:

$$\mathcal{E}_0(h\psi, \mathbf{A}; D) \geqq \mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1) + \mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2) - C(\sqrt{a}\,\ell)^{-1}. \tag{7.11}$$

The formula in (7.11) is the key to compute a lower bound of the ground state energy $\mathrm{E}_{\mathrm{gs}}(\kappa, H)$ as in Theorem 1.1. Loosely speaking, we will do the following:

- Estimate the energy $\mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1)$ using the limiting function $E(\cdot)$ (this is the energy close to the set Γ);
- Estimate the energy $\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2)$ using the limiting function $g(\cdot)$ (this is the energy which is 'relatively' away from Γ).

Here we will need to split into the two regimes displayed in Theorem 1.1. In the regime $\kappa^{3/2} \ll H \lesssim \kappa^2$, estimating $\mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1)$ via the limiting function $E(\cdot)$ will produce small errors. In the regime $H \lesssim \kappa^{3/2}$, the errors will be large, so that the main contributions in (7.11) will be captured through the term $\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2)$ via the function $g(\cdot)$. For the moment, we cannot explain this surprising behavior through intuitive/physical terms.

The Regime $H \gg \kappa^{3/2}$

In this regime, we shall see that $\mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1)$ is the leading term and $\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2)$ is an error term.

Lower bound of the term $\mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1)$. Consider a constant $a \in (0, 1)$ and distinct points (a_j) in Γ such that

$$\forall\, j, \quad 2\ell - a\ell \leqq \operatorname{dist}(a_j, a_{j+1}) \leqq 2\ell - \frac{a}{2}\ell.$$

Choose the constant a sufficiently small so that

$$D_1 = \{x \in \Omega \;:\; \operatorname{dist}(x, \Gamma) < 2\sqrt{a}\,\ell\} \subset \bigcup_j D(a_j, \ell).$$

Consider a partition of unity satisfying

$$\sum_j f_j^2 = 1 \text{ in } D_1, \quad \operatorname{supp} f_j \subset D(a_j, \ell), \quad \sum_j |\nabla f_j|^2 \leqq \frac{C}{a^2\ell^2}.$$

Notice that the support of each ∇f_j is in $D(a_j, \ell) \cap D(a_{j+1}, \ell)$. Since the points (a_j) are selected in such a manner that $\operatorname{dist}(a_j, a_{j+1}) - 2\ell \approx a\ell$, then the area of the domain $D(a_j, \ell) \cap D(a_{j+1}, \ell)$ is proportional to $\sqrt{a}\,\ell \times a\ell = a\sqrt{a}\,\ell^2$.

The partition of unity (f_j) allows us to decompose the energy as follows:

$$\begin{aligned}\mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1) &\geqq \sum_j \mathcal{E}_0(f_j\, \chi_1 h\psi, \mathbf{A}; D_1) - \sum_j \big\|\, |\nabla f_j|\, \chi_1 h\psi \big\|_2^2 \\ &\geqq \sum_j \mathcal{E}_0(h_j\, \psi, \mathbf{A}; D_1) - \frac{C}{\sqrt{a}\,\ell},\end{aligned} \tag{7.12}$$

where $h_j = f_j\, \chi_1 h\psi$ is supported in $D \cap D(a_j, \ell)$.
If $D(a_j, \ell) \cap \partial\Omega \neq \emptyset$, then we can apply Proposition 7.2. Since $\Gamma \cap \partial\Omega$ is a finite set, then we get

$$\sum_{D(a_j,\ell)\cap\partial\Omega\neq\emptyset} \mathcal{E}_0(h_j\, \psi, \mathbf{A}; D_1) \geqq -C\kappa^2\ell^2.$$

Let $\delta \in (0, 1)$ be a constant. We select the parameter ℓ as follows:

$$\ell = \delta H^{-1/3}. \tag{7.13}$$

In that way, we obtain

$$\ell \ll 1, \quad \kappa^2\ell^2 \ll \frac{\kappa^3}{H}, \quad \frac{1}{\sqrt{a}\,\ell} \ll \frac{\kappa^3}{H},$$

and

$$\sum_{D(a_j,\ell)\cap\partial\Omega\neq\emptyset} \mathcal{E}_0(h_j\, \psi, \mathbf{A}; D_1) \geqq \frac{\kappa^3}{H}\, o(1) \quad (\kappa \to \infty). \tag{7.14}$$

If $D(a_j,\ell)\subset D^c$, then $h_j=0$ and

$$\mathcal{E}_0(h_j\,\psi,\mathbf{A};\,D_1)=0.$$

Now, if $j\in\mathcal{I}=\{j\ :\ D(a_j,\ell)\subset\Omega \text{ and } D(a_j,\ell)\cap D\neq\emptyset\}$, then we can apply Proposition 7.1 and get

$$\begin{aligned}\sum_{j\in\mathcal{I}}\mathcal{E}_0(h_j\,\psi,\mathbf{A};\,D_1)&\geqq(1-\delta)\,2\ell\,\kappa\sum_{j\in\mathcal{I}}\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3}E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\\&\quad-C\left(\delta\kappa^2+\delta^{-1}\left(\frac{\kappa^3}{H}\ell^{2\alpha}+\kappa^2H^2\ell^6\right)\right)\int_\Omega|h\psi|^2\,\mathrm{d}x,\end{aligned}$$

where, for all j, x_j is an arbitrary point in $\overline{D(a_j,\ell)}$.

Thanks to Proposition 6.4 and the choice of ℓ in (7.13), we get, further, that

$$\begin{aligned}\sum_{j\in\mathcal{I}}\mathcal{E}_0(h_j\,\psi,\mathbf{A};\,D_1)\ &\geqq(1-\delta)\,2\ell\,\kappa\sum_{j\in\mathcal{I}}\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3}E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\\&\quad-C\left(\delta+\delta^{2\alpha-1}\frac{\kappa}{H}\,H^{-2\alpha/3}\right)\frac{\kappa^3}{H}.\end{aligned}$$

Theorem 3.12 allows us to write

$$\left|\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3}E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\right|\leqq C\frac{\kappa^2}{H}.\tag{7.15}$$

Consequently,

$$\begin{aligned}\sum_{j\in\mathcal{I}}\mathcal{E}_0(h_j\,\psi,\mathbf{A};\,D_1)\ &\geqq 2\ell\,\kappa\sum_{j\in\mathcal{I}}\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3}E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\\&\quad-C\left(\delta+\delta^{2\alpha-1}\frac{\kappa}{H}\,H^{-2\alpha/3}\right)\frac{\kappa^3}{H}.\end{aligned}$$

Inserting this and (7.14) into (7.12), and using the fact that $(\sqrt{a}\,\ell)^{-1}\ll\frac{\kappa^3}{H}$, we get

$$\begin{aligned}\mathcal{E}_0(\chi_1h\psi,\mathbf{A};\,D_1)&\geqq\kappa\sum_{j\in\mathcal{I}}2\ell\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3}E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\\&\quad-C\left(\delta+\delta^{2\alpha-1}\frac{\kappa}{H}\,H^{-2\alpha/3}\right)\frac{\kappa^3}{H}.\end{aligned}\tag{7.16}$$

Thanks to (7.15), the sum

$$\sum_{j\in\mathcal{I}}2\ell\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3}E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\tag{7.17}$$

is of order κ^2/H. Let $\eta \in (0,1)$. Select ℓ_0 sufficiently small such that, for all $\ell \in (0,\ell_0)$, the arc-length measure of $D(a_j,\ell)\cap\Gamma$ along Γ satisfies

$$2\ell - \ell\frac{\eta}{2} \leqq |D(a_j,\ell)\cap\Gamma| \leqq 2\ell + \ell\frac{\eta}{2}.$$

Thus, replacing 2ℓ by $|D(a_j,\ell)\cap\Gamma|$ in the sum in (7.17) produces an error of order $\eta\ell$. Now, select $x_j \in \overline{D(a_j,\ell)}$ such that

$$|\nabla B_0(x_j)|^{1/3}\, E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right) = \max_{D(a_j,\ell)} |\nabla B_0(x)|^{1/3}\, E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right).$$

In that way, the sum in (7.17) satisfies

$$\begin{aligned}&\sum_{j\in\mathcal{I}} 2\ell\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\\ &\quad\geqq \sum_{j\in\mathcal{I}}\left(\int_{D(a_j,\ell)\cap\Gamma}\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\mathrm{d}x\right) - C\eta\frac{\kappa^2}{H}.\end{aligned} \tag{7.18}$$

Recall (7.15). Since the balls $\big(D(a_j,\ell)\big)$ overlap in a region of length $\mathcal{O}(a\ell)$, and the number of these balls is inversely proportional to ℓ, then

$$\begin{aligned}&\sum_{j\in\mathcal{I}}\left(\int_{D(a_j,\ell)\cap\Gamma}\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\mathrm{d}x\right)\\ &\quad\geqq \int_{D\cap\Gamma}\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x_j)|\frac{H}{\kappa^2}\right)\mathrm{d}x - Ca\frac{\kappa^2}{H}.\end{aligned}$$

Inserting this into (7.18), then inserting the resulting inequality into (7.16), we get

$$\begin{aligned}\mathcal{E}_0(\chi_1 h\psi,\mathbf{A};D_1) &\geqq \kappa\int_{\Gamma\cap D}\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)\mathrm{d}s(x)\\ &\quad - C\left(a+\delta+\delta^{2\alpha-1}\frac{\kappa}{H}H^{-2\alpha/3}+\eta\right)\frac{\kappa^3}{H}.\end{aligned} \tag{7.19}$$

Recall that $\alpha > 0$. Taking $\kappa\to\infty$, we get

$$\begin{aligned}\liminf_{\kappa\to\infty}\frac{H}{\kappa^3}\Bigg\{&\mathcal{E}_0(\chi_1 h\psi,\mathbf{A};D_1) - \kappa\int_{\Gamma\cap D}\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3}\\ &E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)\mathrm{d}s(x)\Bigg\}\\ \geqq -C(a+\delta+\eta).&\end{aligned}$$

Taking $\eta \to 0_+$, we obtain

$$\liminf_{\kappa\to\infty} \frac{H}{\kappa^3}\left\{\mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1) - \kappa \int_{\Gamma\cap D}\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \mathrm{d}s(x)\right\} \geqq -C(a+\delta). \tag{7.20}$$

Lower bound of the term $\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2)$ Since $H \gg \kappa^{3/2}$, the parameter ℓ defined in (7.13) satisfies

$$\ell = \delta H^{-1/3} = \delta \frac{\kappa}{H}\frac{H^{2/3}}{\kappa} \gg \frac{\kappa}{H}.$$

Thanks to the exponential decay in Theorem 6.3, there holds

$$\kappa^2 \int_{D_2} |\psi|^2\, \mathrm{d}x \ll \frac{\kappa^3}{H},$$

and

$$\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2) \geqq -\frac{\kappa^3}{H} o(1) \quad (\kappa \to \infty),$$

which implies

$$\liminf_{\kappa\to\infty} \frac{H}{\kappa^3}\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2) \geqq 0. \tag{7.21}$$

Inserting (7.21) and (7.20) into (7.11), we get

$$\liminf_{\kappa\to\infty} \frac{H}{\kappa^3}\left(\mathcal{E}_0(h\psi, \mathbf{A}; D) - \kappa \int_{\Gamma\cap D}\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \mathrm{d}s(x)\right) \geqq -C(a+\delta).$$

Now, we take the limit $(a, \delta) \to (0, 0)$ to obtain

$$\liminf_{\kappa\to\infty} \frac{H}{\kappa^3}\left(\mathcal{E}_0(h\psi, \mathbf{A}; D) - \kappa \int_{\Gamma\cap D}\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \mathrm{d}s(x)\right) \geqq 0.$$

Thus, we arrive at

$$\mathcal{E}_0(h\psi, \mathbf{A}; D) \geqq \kappa \int_{\Gamma\cap D}\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \mathrm{d}s(x) - \frac{\kappa^3}{H} o(1). \tag{7.22}$$

This completes the proof of Theorem 7.3 in the case $H \gg \kappa^{3/2}$.

The Regime $H \lesssim \kappa^{3/2}$.

In this regime, we shall see that $\mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1)$ is an error term and $\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2)$ is the leading term.

Since $H \lesssim \kappa^{3/2}$, the parameter ℓ introduced in (7.13) satisfies

$$\ell \lesssim \delta \frac{\kappa}{H}.$$

Consequently, we have

$$\mathcal{E}_0(\chi_1 h\psi, \mathbf{A}; D_1) \geqq -\kappa^2 \int_\Omega |\chi_1 h\psi|^2 \,\mathrm{d}x \geqq C\kappa^2 \ell \geqq -\delta \frac{\kappa^3}{H}. \tag{7.23}$$

Unlike the regime $H \gg \kappa^{3/2}$, we can no more ignore the energy in $\{\sqrt{a}\,\ell \leqq \operatorname{dist}(x, \Gamma) \leqq \frac{\kappa}{H}\}$.

We introduce the two parameters

$$m > 1 \quad \text{and} \quad \zeta \in (0, \ell), \tag{7.24}$$

and the domain,

$$U = \left\{ x \in D_2 \;:\; \operatorname{dist}(x, \Gamma) \geqq m\frac{\kappa}{H} \right\}. \tag{7.25}$$

Thanks to the exponential decay in Theorem 6.3, we get

$$\kappa^2 \int_U |\psi|^2 \,\mathrm{d}x \leqq C e^{-2m\, m_0} \frac{\kappa^3}{H}$$

and

$$\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; U) \geqq -C e^{-2m\, m_0} \frac{\kappa^3}{H} \tag{7.26}$$

Our next task is to determine a lower bound of the energy $\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2 \setminus U)$. Consider for $\zeta \in (0,1)$ the lattice of squares $(\mathcal{Q}_{k,\zeta})_k$ generated by the square

$$\mathcal{Q}_\zeta = \left(-\frac{\zeta}{2}, \frac{\zeta}{2}\right) \times \left(-\frac{\zeta}{2}, \frac{\zeta}{2}\right).$$

Let

$$\mathcal{J}_{\mathrm{blk}} = \{k \;:\; \mathcal{Q}_{k,\zeta} \subset D_2 \setminus U \quad \text{and } \mathcal{Q}_{k,\zeta} \cap \partial\Omega = \emptyset\},$$
$$\mathcal{J}_{\mathrm{bnd},1} = \{k \;:\; k \notin \mathcal{J}_{\mathrm{blk}},\ \mathcal{Q}_{k,\zeta} \cap (D_2 \setminus U) \neq \emptyset \quad \text{and} \quad \mathcal{Q}_{k,\zeta} \cap \partial\Omega = \emptyset\},$$
$$\mathcal{J}_{\mathrm{bnd},2} = \{k \;:\; \mathcal{Q}_{k,\zeta} \subset D_2 \setminus U \quad \text{and } \mathcal{Q}_{k,\zeta} \cap \partial\Omega \neq \emptyset\}.$$

We have the obvious decomposition,

$$\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2 \setminus U) \geqq \sum_{k \in \mathcal{J}_{\mathrm{blk}}} \mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; \mathcal{Q}_{k,\zeta}$$
$$+ \sum_{j=1}^{2} \sum_{k \in \mathcal{J}_{\mathrm{bnd},j}} \mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; \mathcal{Q}_{k,\zeta} \cap (D_2 \setminus U)). \tag{7.27}$$

Since $\Gamma \cap \partial\Omega$ is a finite set, then $N = \mathrm{Card}\mathcal{J}_{\mathrm{bnd},2}$ is bounded independently of κ. Now, the terms corresponding to $k \in \mathcal{J}_{\mathrm{bnd},j}$ are easily estimated as follows:

$$\sum_{j=1}^{2}\sum_{k\in\mathcal{J}_{\mathrm{bnd},j}} \mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; \mathcal{Q}_{k,\zeta} \cap (D_2 \setminus U))$$
$$\geqq -\kappa^2 \int_{\{\mathrm{dist}(x,\partial\Gamma)\leqq C\zeta} |\psi|^2\,\mathrm{d}x - N\kappa^2\zeta^2 \geqq -C\kappa^2\zeta. \tag{7.28}$$

For all k, let x_k be the center of the square $\mathcal{Q}_{k,\zeta}$ and a_k an arbitrary point in $\overline{\mathcal{Q}_{k,\zeta}}$. Repeating the proof of Proposition 7.1, we get, for all $k \in \mathcal{J}$ and $\eta \in (0, 1)$,

$$\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; \mathcal{Q}_{k,\zeta}) \geqq (1-\eta)\mathcal{E}_0(\chi_2 h\psi\, e^{-i\kappa H u_k}, B_0(a_k)\mathbf{A}_0(x-x_k); \mathcal{Q}_{k,\zeta})$$
$$-C\left(\eta\kappa^2 + \eta^{-1}\left(\frac{\kappa^3}{H}\zeta^{2\alpha} + \kappa^2 H^2\zeta^4\right)\right)\|\psi\|^2_{L^2(\mathcal{Q}_{k,\zeta})}, \tag{7.29}$$

where u_k is a gauge function.

We select the parameter ζ as follows

$$\zeta = \eta H^{-1/2}. \tag{7.30}$$

Clearly, ζ satisfies

$$\zeta \ll \ell \lesssim \delta\frac{\kappa}{H} \ll 1, \quad \kappa^2\zeta \ll \frac{\kappa}{H}, \quad H^2\zeta^4 = \eta^4,$$
$$\zeta\sqrt{\kappa H|B_0(a_k)|} \gtrsim \zeta\sqrt{\kappa H\sqrt{a}\,\ell} \gtrsim \eta\delta^{1/2}a^{1/4}.$$

Applying a scaling and a translation, we may use (3.51) and get

$$\frac{1}{\left(\zeta\sqrt{\kappa H|B_0(a_k)|}\right)^2}\mathcal{E}_0(\chi_2 h\psi\, e^{-i\kappa H u_k}, B_0(a_k)\mathbf{A}_0(x-x_k); \mathcal{Q}_{k,\zeta})$$
$$\geqq \frac{\kappa}{H|B_0(a_k)|}\left(g\left(\frac{H}{\kappa}|B_0(a_k)|\right) - C\frac{\sqrt{\frac{H}{\kappa}|B_0(a_k)|}}{\zeta\sqrt{\kappa H|B_0(a_k)|}}\right).$$

We insert this into (7.29), sum over $k \in \mathcal{J}_{\mathrm{blk}}$ and use Proposition 6.4 to get

$$\sum_{k\in\mathcal{J}_{\mathrm{blk}}} \mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; \mathcal{Q}_{k,\zeta}) \geqq \zeta^2\kappa^2 \sum_{k\in\mathcal{J}_{\mathrm{blk}}}\left(g\left(\frac{H}{\kappa}|B_0(a_k)|\right) - \frac{C}{\zeta\kappa}\right)$$
$$-(C\eta + o(1))\frac{\kappa^3}{H}.$$

The sum in the inequality above becomes a lower Riemann sum if for each k the point (a_k) is selected in $\overline{Q_{k,\ell}}$ as follows:

$$|B_0(a_k)| = \max\{|B_0(x)| \;:\; x \in \overline{\mathcal{Q}_{k,\ell}}\}.$$

Notice that $\mathcal{N}_{\mathrm{blk}} = \mathrm{Card}\,\mathcal{J}_{\mathrm{blk}}$ satisfies $\mathcal{N}_{\mathrm{blk}} \times \zeta^2 \approx |D_2 \setminus U|$ as $\zeta \to 0$ and

$$|D_2 \setminus U| = |\{\sqrt{a}\,\ell \leqq \mathrm{dist}(x,\Gamma) \leqq m\frac{\kappa}{H}\}| \leqq Cm\frac{\kappa}{H}.$$

Consequently, we have

$$\sum_{k\in\mathcal{J}_{\mathrm{blk}}} \mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; \mathcal{Q}_{k,\zeta}) \geqq \kappa^2 \int_{\mathcal{D}_\zeta} g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x - m\frac{C\kappa}{\zeta}\frac{\kappa}{H} - (C\eta + o(1))\,\frac{\kappa^3}{H},$$

where

$$\mathcal{D}_\zeta = \bigcup_{k\in\mathcal{J}_{\mathrm{blk}}} \overline{\mathcal{Q}_{k,\zeta}} \subset D_2 \setminus U.$$

Since the function g is non-positive, then we get that

$$\begin{aligned}
&\sum_{k\in\mathcal{J}_{\mathrm{blk}}} \mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; \mathcal{Q}_{k,\zeta}) \\
&\geqq \kappa^2 \int_{\mathcal{D}_2\setminus U} g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x - Cm\,\frac{\kappa}{\zeta}\,\frac{\kappa}{H} - (C\eta + o(1))\,\frac{\kappa^3}{H} \\
&\geqq \kappa^2 \int_{\mathcal{D}_2\setminus U} g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x - C\,m\eta^{-1}\frac{H^{1/2}}{\kappa}\frac{\kappa^3}{H} - (C\eta + o(1))\,\frac{\kappa^3}{H}. \quad (7.31)
\end{aligned}$$

We insert (7.31) and (7.28) into (7.27). Since $g\left(\frac{H}{\kappa}|B_0(x)|\right) = 0$ in $\{|B_0(x)| \geqq \frac{H}{\kappa}\}$ and $H \lesssim \kappa^{3/2}$, it results in the inequality

$$\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2 \setminus U) \geqq \kappa^2 \int_D g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x - (C\eta + o(1))\,\frac{\kappa^3}{H}, \quad (\kappa \to \infty). \qquad (7.32)$$

Combining (7.32) and (7.26), we get

$$\mathcal{E}_0(\chi_2 h\psi, \mathbf{A}; D_2) \geqq \kappa^2 \int_D g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x - Ce^{-2m\,m_0}\,\frac{\kappa^3}{H} - (C\eta + o(1))\,\frac{\kappa^3}{H}.$$

Now, we insert this inequality and (7.23) into (7.11) to get

$$\mathcal{E}_0(h\psi, \mathbf{A}; D) \geqq \kappa^2 \int_D g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x - C(a + \delta + \eta + e^{-2m\,m_0} + o(1))\frac{\kappa^3}{H}.$$

By taking the successive limits,

$$\liminf_{\kappa\to\infty}, \quad \lim_{a\to 0_+}, \quad \lim_{\delta\to 0_+}, \quad \lim_{\eta\to 0_+}, \quad \lim_{m\to\infty},$$

we get

$$\mathcal{E}_0(h\psi, \mathbf{A}; D) \geqq \kappa^2 \int_D g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x - \frac{\kappa^3}{H}\,o(1),$$

which finishes the proof of Theorem 7.3 in the regime $H \lesssim \kappa^{3/2}$. □

We get, by applying Theorem 7.3 with $D = \Omega$ and $h = 1$:

Corollary 7.4. *Let* $\Lambda > 0$ *and* $\epsilon : \mathbb{R} \to \mathbb{R}_+$ *such that* $\lim_{\kappa\to\infty} \kappa\,\epsilon(\kappa) = \infty$ *and* $\lim_{\kappa\to\infty} \epsilon(\kappa) = 0$.

There exist $\kappa_0 > 0$ *and a function* err $: \mathbb{R} \to \mathbb{R}$ *such that the following is true:*

(1) $\lim_{\kappa\to\infty} \mathrm{err}(\kappa) = 0$.

(2) *Let* $\kappa \geqq \kappa_0$, $\epsilon(\kappa)\kappa^2 \leqq H \leqq \Lambda\kappa^2$ *and* $(\psi, \mathbf{A})$ *be a critical point of the functional in* (1.1).

(a) *If* $H \gg \kappa^{3/2}$, *then,*

$$\mathcal{E}_0(\psi, \mathbf{A}) \geqq \kappa \left(\int_\Gamma \left(|\nabla B_0(x)| \frac{H}{\kappa^2} \right)^{1/3} E\left(|\nabla B_0(x)| \frac{H}{\kappa^2} \right) \mathrm{d}s(x) \right) + \frac{\kappa^3}{H}\mathrm{err}(\kappa). \tag{7.33}$$

(b) *If* $H \lesssim \kappa^{3/2}$, *then,*

$$\mathcal{E}_0(\psi, \mathbf{A}) \geqq \kappa^2 \int_\Omega g\left(\frac{H}{\kappa} |B_0(x)| \right) \mathrm{d}x + \frac{\kappa^3}{H}\mathrm{err}(\kappa). \tag{7.34}$$

We conclude this section with the

Proof of Theorem 1.1. We have just to combine the conclusions of Theorem 5.1 and Corollary 7.4. □

8. Local Energy Estimates

8.1. Preliminaries

Let $D \subset \Omega$ be an open set with a smooth boundary such that $\partial D \cap \Gamma$ is a finite set. Let $\rho_0 \in (0, 1)$, $\rho \in (0, \rho_0)$ and

$$D_\rho = \{x \in \Omega \ : \ \mathrm{dist}(x, D) < \rho\}.$$

We select ρ_0 sufficiently small so that the boundary of ∂D_ρ is smooth.

Let $h_1 \in C_c^\infty(D_\rho)$ and $h_2 \in C^\infty(\mathbb{R}^2)$ be functions satisfying

$$0 \leqq h_1 \leqq 1, \quad |\nabla h_1| + |\nabla h_2| \leqq \frac{C}{\rho} \quad \text{in } \mathbb{R}^2, \quad h_1 = 1 \quad \text{in } D_\rho, \text{ and } h_1^2 + h_2^2 = 1.$$

Notice that

$$\mathrm{supp}\, h_2 \subset \overline{D}^c.$$

Let $(\psi, \mathbf{A})$ be a minimizer of (1.1). We will estimate the following energy

$$\mathcal{E}_0(\psi, \mathbf{A}; D) = \int_D \left(|\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right) \mathrm{d}x.$$

Notice that we have the following decomposition of the energy

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega) \geqq \mathcal{E}_0(h_1\psi, \mathbf{A}; D_\rho) + \mathcal{E}_0(h_2\psi, \mathbf{A}; \overline{D}^c) - \frac{C}{\rho^2}\int_\Omega |\psi|^2\,\mathrm{d}x.$$

Now we use the estimate in Proposition 6.4 and write

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega) \geqq \mathcal{E}_0(h_1\psi, \mathbf{A}; D_\rho) + \mathcal{E}_0(h_2\psi, \mathbf{A}; \overline{D}^c) - \frac{C}{\rho^2}\frac{\kappa}{H}. \tag{8.1}$$

Recall that we deal with two separate regimes:

$$\begin{cases} \textbf{Regime I}: & \kappa \ll H \lesssim \kappa^{3/2}; \\ \textbf{Regime II}: & \kappa^{3/2} \ll H \lesssim \kappa^2. \end{cases}$$

We define the quantity $C_0(\kappa, H; D)$ as follows:

$$C_0(\kappa, H; D) = \begin{cases} \kappa\left(\int_{D\cap\Gamma}\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)\mathrm{d}s(x)\right) & \text{in Regime I}, \\ \kappa^2\int_D g\left(\frac{H}{\kappa}|B_0(x)|\right)\mathrm{d}x & \text{in Regime II}. \end{cases} \tag{8.2}$$

Notice that, in Regimes I and II, the result of Theorem 1.1 reads as follows:

$$\mathrm{E}_{\mathrm{gs}}(\kappa, H) = C_0(\kappa, H; \Omega) + \frac{\kappa^3}{H}o(1), \quad (\kappa \to \infty).$$

8.2. Upper Bound

The results in this section are valid under the assumption that $(\psi, \mathbf{A})$ is a minimizer of the functional in (1.1).

We have $\mathcal{E}_0(\psi, \mathbf{A}; \Omega) \leqq \mathrm{E}_{\mathrm{gs}}(\kappa, H)$. Using $|\psi| \leqq 1$ and the upper bound in Theorem 5.1, we get

$$\mathcal{E}_0(h_1\psi, \mathbf{A}; D_\rho) + \mathcal{E}_0(h_2\psi, \mathbf{A}; \overline{D}^c) \leqq C_0(\kappa, H; \Omega) + \frac{\kappa^3}{H}\mathrm{err}(\kappa) + \frac{C}{\rho^2}\frac{\kappa}{H}.$$

Using Theorem 7.3, we may write

$$\mathcal{E}_0(h_2\psi, \mathbf{A}; D^c) \geqq C_0(\kappa, H; \overline{D}^c) + \frac{\kappa^3}{H}\mathrm{err}(\kappa).$$

As a consequence, we have

$$\mathcal{E}_0(h_1\psi, \mathbf{A}; D_\rho) \leqq C_0(\kappa, H; D) + \frac{\kappa^3}{H}\mathrm{err}(\kappa) + \frac{C}{\rho^2}\frac{\kappa}{H}.$$

Since $h_1 = 1$ in D, we get the simple decomposition of the energy

$$\mathcal{E}_0(\psi, \mathbf{A}; D) = \mathcal{E}_0(h_1\psi, \mathbf{A}; D_\rho) - \mathcal{E}_0(h_1\psi, \mathbf{A}; D_\rho \setminus D).$$

Since $\|h_1\|_\infty \leqq 1$ and the boundary of $D_\rho \setminus D$ is smooth, we get, in light of Theorem 7.3,

$$\mathcal{E}_0(h_1\psi, \mathbf{A}; D_\rho \setminus D) \geqq C_0(\kappa, H; D_\rho \setminus D) + \frac{\kappa^3}{H}\mathrm{err}_\rho(\kappa).$$

In light of the upper bound in Theorem 3.12, we have

$$\left|\left(\int_{(D_\rho\setminus D)\cap\Gamma} \left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla B_0(x)|\frac{H}{\kappa^2}\right) \mathrm{d}s(x)\right)\right| \leqq C\frac{\kappa^2}{H}\rho.$$

In the same vein, since $g(b)$ is bounded and vanishes when $b \geqq 1$, then

$$\left|\int_{D_\rho\setminus D} g\left(\frac{H}{\kappa}|B_0(x)|\right) \mathrm{d}x\right| \leqq C\rho\frac{\kappa}{H}.$$

As a consequence, we get

$$\left|C_0(\kappa, H; D_\rho \setminus D)\right| \leqq C\frac{\kappa^3}{H}\rho,$$

and

$$\mathcal{E}_0(\psi, \mathbf{A}; D) \leqq C_0(\kappa, H; D) + \frac{\kappa^3}{H}(C\rho + \mathrm{err}_\rho(\kappa)) + \frac{C}{\rho^2}\frac{\kappa}{H}.$$

Sending κ to infinity, we deduce that

$$\limsup_{\kappa\to\infty} \frac{H}{\kappa^3}\{\mathcal{E}_0(\psi, \mathbf{A}; D_\rho) - C_0(\kappa, H; D)\} \leqq C\rho.$$

Next, we send ρ to 0_+ and get

$$\limsup_{\kappa\to\infty} \frac{H}{\kappa^3}\{\mathcal{E}_0(\psi, \mathbf{A}; D) - C_0(\kappa, H; D)\} \leqq 0. \tag{8.3}$$

Notice that the upper bound in (8.3) is valid for any open set $D \subset \Omega$ with smooth boundary. In particular, it is true when D is replaced by $\overline{D}^c = \Omega \setminus \overline{D}$, that is

$$\limsup_{\kappa\to\infty} \frac{H}{\kappa^3}\{\mathcal{E}_0(\psi, \mathbf{A}; \overline{D}^c) - C_0(\kappa, H; \overline{D}^c)\} \leqq 0. \tag{8.4}$$

8.3. *Lower Bound*

We continue to assume that $(\psi, \mathbf{A})$ is a minimizer of the functional in (1.1). We will give a lower bound of the energy $\mathcal{E}_0(\psi, \mathbf{A}; D)$. We plug the lower bound in Corollary 7.4 into the following simple decomposition of the energy

$$\mathcal{E}_0(\psi, \mathbf{A}; D) + \mathcal{E}_0(\psi, \mathbf{A}; \overline{D}^c) = \mathcal{E}_0(\psi, \mathbf{A}; \Omega).$$

In that way, we get

$$\mathcal{E}_0(\psi, \mathbf{A}; D) \geqq C_0(\kappa, H; \Omega) - \mathcal{E}_0(\psi, \mathbf{A}; \overline{D}^c) + \frac{\kappa^3}{H}\mathrm{err}(\kappa).$$

Notice the following simple decomposition of the term on the right hand side:

$$\begin{aligned}\mathcal{E}_0(\psi, \mathbf{A}; D) &\geqq C_0(\kappa, H; D) + \frac{\kappa^3}{H}\mathrm{err}(\kappa)\\ &-\{\mathcal{E}_0(\psi, \mathbf{A}; \overline{D}^c) - C_0(\kappa, H; \overline{D}^c)\}.\end{aligned}$$

Now we send κ to ∞ and using (8.4), we get

$$\liminf_{\kappa\to\infty} \frac{H}{\kappa^3}\{\mathcal{E}_0(\psi, \mathbf{A}; D) - C_0(\kappa, H; D)\} \geqq 0. \tag{8.5}$$

8.4. *Conclusion for the Local Energy*

Combining (8.3) and (8.5), we get, in the two regimes we are considering, that the local energy in D of a minimizer $(\psi, \mathbf{A})$ satisfies

$$\mathcal{E}_0(\psi, \mathbf{A}; D) = C_0(\kappa, H; D) + \frac{\kappa^3}{H}o(\kappa), \tag{8.6}$$

where $C_0(\kappa, H; D)$ is introduced in (8.2).

9. Proof of Theorem 1.6

The proof of (1) in Theorem 1.6 is a simple combination of the upper bound in Theorem 5.1 and the lower bound in Theorem 7.3 (used with $D = \Omega$).

The assertion (2) in Theorem 1.6 is the conclusion of Section 8.4.

The rest of the section is devoted to the proof of statement (3) in Theorem 1.6. This will be done in three steps. Recall the definition of the quantity $C_0(\kappa, H; D)$ in (8.2) and that we work under the assumption on H described in Regimes I and II. It is sufficient to prove that the following formula is true in Regimes I and II:

$$\int_D |\psi(x)|^4\,\mathrm{d}x = -\frac{2}{\kappa^2}C_0(\kappa, H; D) + \frac{\kappa}{H}o(1), \quad (\kappa\to\infty),$$

where $(\psi, \mathbf{A})$ is a minimizer of the energy in (1.1).

Step 1: The Case $D = \Omega$

A minimizer $(\psi, \mathbf{A})$ satisfies the Ginzburg–Landau equation in (4.1). Recall the useful identity in (4.3):

$$-\frac{\kappa^2}{2}\int_\Omega |\psi(x)|^4\,\mathrm{d}x = \mathcal{E}_0(\psi, \mathbf{A}; \Omega).$$

Thanks to the formulas in Sect. 8.4 used with $D = \Omega$, we observe that (8.6) yields

$$\int_\Omega |\psi(x)|^4\,\mathrm{d}x = -\frac{2}{\kappa^2}C_0(\kappa, H; \Omega) + \frac{\kappa}{H}\,o(1), \tag{9.1}$$

where the formula is valid in Regimes I and II.

Step 2: Upper Bound

Let

$$\ell = \kappa^{-1/4}\sqrt{\frac{\kappa}{H}} \quad \text{and} \quad D_\ell = \{x \in D \,:\, \operatorname{dist}(x, \partial D) \geqq \ell\}.$$

Consider a cut-off function $\chi_\ell \in C_c^\infty(D)$ such that

$$\|\chi_\ell\|_\infty \leqq 1, \quad \|\nabla \chi_\ell\| \leqq \frac{C}{\ell}, \quad \chi_\ell = 1 \text{ in } D_\ell.$$

Multiplying both sides of the equation in (4.1) by $\chi_\ell^2\overline{\psi}$ then integrating by parts and using the estimate in Proposition 6.4 Yields,

$$\begin{aligned}\int_D \Big(|(\nabla - i\kappa H\mathbf{A})\chi_\ell\psi|^2 - \kappa^2\chi_\ell^2|\psi|^2 + \kappa^2\chi_\ell^2|\psi|^4\Big)\,\mathrm{d}x \\ = \int_D |\nabla\chi_\ell|^2|\psi|^2\,\mathrm{d}x = \mathcal{O}\Big(\frac{C}{\ell^2}\,\frac{\kappa}{H}\Big) = \frac{\kappa^3}{H}o(1).\end{aligned}$$

Since $1 \geqq \chi_\ell^2 \geqq \chi_\ell^4$, this formula implies

$$-\frac{\kappa^2}{2}\int_D \chi_\ell^2|\psi|^4\,\mathrm{d}x \geqq \mathcal{E}_0(\chi_\ell\psi, \mathbf{A}; D) - \frac{\kappa^3}{H}o(1). \tag{9.2}$$

Using the bounds $\|\psi\|_\infty \leqq 1$ and $\|\psi\|_2 \leqq C\sqrt{\frac{\kappa}{H}}$, the fact that χ_ℓ is supported in D and $\chi_\ell = 1$ in D_ℓ, we get

$$\begin{aligned}\int_D |\psi(x)|^4\,\mathrm{d}x &= \int_D \chi_\ell^2(x)|\psi(x)|^4\,\mathrm{d}x + \int_D (1 - \chi_\ell^2(x))|\psi(x)|^4\,\mathrm{d}x \\ &= \int_D \chi_\ell^2(x)|\psi(x)|^4\,\mathrm{d}x + \mathcal{O}\Big(\sqrt{\ell}\sqrt{\frac{\kappa}{H}}\Big) \\ &= \int_D \chi_\ell^2(x)|\psi(x)|^4\,\mathrm{d}x + \frac{\kappa}{H}\,o(1).\end{aligned} \tag{9.3}$$

Now, we infer from (9.2) and Theorem 7.3 that

$$\int_D |\psi(x)|^4\,\mathrm{d}x \leqq -\frac{2}{\kappa^2}C_0(\kappa, H; D) + \frac{\kappa}{H}\,o(1). \tag{9.4}$$

Step 3: Lower Bound

Notice that (9.4) is valid for any open domain $D \subset \Omega$ with a smooth boundary, in particular, it is valid when D is replaced by the complementary of $\overline{D}$ in Ω: $\overline{D}^c$. We have the simple decomposition

$$\int_D |\psi(x)|^4 \, dx = \int_\Omega |\psi(x)|^4 \, dx - \int_{\overline{D}^c} |\psi(x)|^4 \, dx$$
$$\geqq \int_\Omega |\psi(x)|^4 \, dx - \frac{2}{\kappa^2} C_0(\kappa, H; \overline{D}^c) + \frac{\kappa}{H} o(1).$$

Using the asymptotics in (9.1) obtained in Step 1, we deduce that

$$\int_D |\psi(x)|^4 \, dx \geqq -\frac{2}{\kappa^2} C_0(\kappa, H; D) + \frac{\kappa}{H} o(1).$$

Combining this lower bound and the upper bound in (9.4), we obtain the asymptotics announced in the third assertion of Theorem 1.6.

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Laboratoire de Mathématiques,
Université de Paris-Sud 11,
Bât 425, 91405 Orsay,
France.
e-mail: bernard.helffer@math.u-psud.fr

and

Laboratoire Jean Leray,
Université de Nantes,
Nantes, France

and

Department of Mathematics,
Lebanese University,
Hadat, Lebanon.
e-mail: ayman.kashmar@gmail.com

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