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Supercritical Mean Field Equations on Convex Domains and the Onsager's Statistical Description of Two-Dimensional Turbulence

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Abstract

We are motivated by the study of the Microcanonical Variational Principle within Onsager's description of two-dimensional turbulence in the range of energies where the equivalence of statistical ensembles fails. We obtain sufficient conditions for the existence and multiplicity of solutions for the corresponding Mean Field Equation on convex and "thin" enough domains in the supercritical (with respect to the Moser–Trudinger inequality) regime. This is a brand new achievement since existence results in the supercritical region were previously known only on multiply connected domains. We then study the structure of these solutions by the analysis of their linearized problems and we also obtain a new uniqueness result for solutions of the Mean Field Equation on thin domains whose energy is uniformly bounded from above. Finally we evaluate the asymptotic expansion of those solutions with respect to the thinning parameter and, combining it with all the results obtained so far, we solve the Microcanonical Variational Principle in a small range of supercritical energies where the entropy is shown to be concave.

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1. Introduction

In a pioneering paper [63] Onsager proposed a statistical theory of two-dimensional turbulence based on the N-vortex model [60]. We refer to [39] for an historical review and to [58] and the introduction in [38] for a detailed discussion about this theory and its range of applicability in real world models. More recently, those physical arguments were turned into rigorous proofs [18,19,46,47]. Together with other well known physical [14,66,71,72,74,78] and geometrical [21,44,75] applications, these new results were the motivation for efforts in the understanding of the resulting mean field [18,19] Liouville-type [54] equations. We refer the reader to [3,7,12,13,16,17,20,22–30,33–37,45,49,51–53,55,56,61,62,65,67,69, 70,73,77], and more recently [4–6,8,9,11,57] and the references quoted therein.

In spite of these efforts it seems that there are some basic questions arising in [19] which have been left unanswered so far. These are our main motivations and this is why we will begin our discussion with a short review of some of the results obtained in [19] as completed in [20].

Definition 1.1. Let $\Omega \subset \mathbb{R}^2$ be any open, bounded and simply connected domain. We say that Ω is simple if $\partial\Omega$ is the support of a simple and rectifiable Jordan curve. Let Ω be a simple domain. We say that it is regular if [see also [20]]:

(–) its boundary $\partial\Omega$ is the support of a continuous and piecewise C^2 curve $\partial\Omega = \mathrm{supp}(\gamma)$ with bounded first derivative $\|\gamma'\|_\infty \leqq C$ and at most a finite number of corner-type points $\{p_1, \dots, p_m\}$, that is, the inner angle θ_j formed by the corresponding limiting tangents is well defined and satisfies $\theta_j \in (0, 2\pi)\backslash\{\pi\}$ for any $j = 1, \dots, m$;
(–) for each p_j there exists a conformal bijection from an open neighborhood U of p_j which maps $U \cap \partial\Omega$ onto a curve of class C^2.

In particular any regular domain is by definition simply connected.

We will use these definitions throughout the rest of this paper without further comment. Of course polygons of any kind are regular according to our definition. The notations $|\Omega|$ or $A(\Omega)$ will be used to denote the area of a simple domain Ω, while $L(\partial\Omega)$ will denote the length of the boundary of Ω.

Remark 1.2. We will discuss at length solutions of a Liouville-type semilinear equation with Dirichlet boundary conditions, see $P(\lambda, \Omega)$ in Section 1.1 below.

In this respect, and if Ω is regular, a solution u will be by definition an $H_0^1(\Omega)$ weak solution [40] of the problem at hand, $H_0^1(\Omega)$ being the closure of $C_c^1(\Omega)$ in the norm $\|u\|_2 + \| \,|\nabla u|\, \|_2$. It turns out that, by using the well known Brezis–Merle results [17] together with Lemma 2.1 in [20], any $H_0^1(\Omega)$ weak solution on a regular domain is also a classical $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution. In those cases where Ω is just assumed to be simple, a solution will be by definition a classical solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

Let $\Omega \subset \mathbb{R}^2$ be open, bounded and simple. We define

$$\mathcal{P} = \left\{ \omega \in L^1(\Omega) \,|\, \omega \geqq 0 \text{ almost everywhere in } \Omega,\ \int_\Omega \omega = 1 \right\},$$

and $G_\Omega(x, y)$ to be the unique solution of

$$\begin{cases} -\Delta G_\Omega(x, y) = \delta_{x=y} & \text{in } \Omega, \\ G_\Omega(x, y) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\delta_{x=y}$ is the Dirac distribution with singular point $y \in \Omega$, $G_\Omega(x, y) = -\frac{1}{2\pi}\log(|x - y|) + H_\Omega(x, y)$ and H_Ω denotes the regular part.

For any $\omega \in \mathcal{P}$ we also define the entropy and the energy of ω as

$$\mathcal{S}(\omega) = \int_\Omega s(\omega), \qquad \mathcal{E}(\omega) = \frac{1}{2}\int_\Omega \omega G[\omega],$$

respectively, where

$$s(t) = \begin{cases} -t\log t, & t > 0, \\ 0, & t = 0, \end{cases}$$

and

$$G[\omega](x) = \int_\Omega G_\Omega(x, y)\omega(y)\,\mathrm{d}y.$$

For any $E \in \mathbb{R}$ we consider the MVP (Microcanonical Variational Principle)

$$S(E) = \sup\{\mathcal{S}(\omega),\quad \omega \in \mathcal{P}_E\}, \quad \mathcal{P}_E = \{\omega \in \mathcal{P} \,|\, \mathcal{E}(\omega) = E\}. \tag{MVP}$$

The following results have been obtained in [19] (see Propositions 2.1, 2.2, 2.3 in [19]):

MVP-(i) For any $E > 0$, $S(E) < +\infty$ and there exists $\omega \in \mathcal{P}_E$ such that $S(E) = \mathcal{S}(\omega)$;
MVP-(ii) Let $\Upsilon = \frac{1}{|\Omega|}$ be the uniform density on Ω and $E_\Upsilon = \mathcal{E}(\Upsilon)$. Then Υ is a maximizer of $\mathcal{S}$ on $\mathcal{P}_{E_\Upsilon}$ and in particular if $|\Omega| = 1$, then $S(E_\Upsilon) = 0$;
MVP-(iii) If $|\Omega| = 1$ then $S(E)$ is strictly increasing and negative for $E < E_\Upsilon$ and strictly decreasing and negative for $E > E_\Upsilon$;

MVP-(iv) Let $\omega^{(E)}$ be a solution for the MVP at energy E. Then there exists $\beta = \beta_E \in \mathbb{R}$ such that

$$\omega^{(E)} = \frac{e^{-\beta G[\omega^{(E)}]}}{\int_\Omega e^{-\beta G[\omega^{(E)}]}},$$

or, equivalently, the function $\psi = G[\omega^{(E)}]$ satisfies the Mean Field Equation

$$\begin{cases} -\Delta\psi = \dfrac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}} & \text{in} \quad \Omega \\ \psi = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad \text{(MFE)};$$

MVP-(v) $S(E)$ is continuous.

We find it appropriate at this point to continue our discussion by introducing some concepts as in [19] but with the aid of slightly different mathematical arguments based on some results in [17,49,50] and in particular in [20] which were not at hand at that time.

Since solutions of the (MFE) with fixed $\beta > -8\pi$ are unique not only if Ω is simple and smooth [69] but also if Ω is regular (see [20]), and by using the Brezis–Merle [17] theory of Liouville-type equations (as later improved in [50] and then in [49]) and the boundary estimates in [20], we can divide the set of regular domains (see Definition 1.1) into two classes, first introduced in [19]:

Definition 1.3. Let Ω be regular. We say that Ω is of *first kind* if the unique (at fixed $\beta > -8\pi$ [20,69]) solution ψ_β of the (MFE) satisfies

$$\omega_{(\beta)} := \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}} \rightharpoonup \delta_{x=p}, \quad \text{as} \quad \beta \searrow (-8\pi)^+, \tag{1.2}$$

weakly in the sense of measures, for some $p \in \Omega$.

We say that Ω is of *second kind* otherwise.

We will skip the discussion of the case $\beta > 0$ since its mathematical-physical description is well understood [19].

Let $\mathcal{E}(\omega_{(\beta)})$ be the energy of the unique solution of the (MFE) with $\beta \in (-8\pi, 0]$. By using known arguments based on the results in [17,50] and [20,49] it can be shown that either ψ_β is uniformly bounded for $\beta \in (-8\pi, 0]$ or it must satisfy (1.2) and $\mathcal{E}(\omega_{(\beta)}) \to +\infty$ as $\beta \searrow (-8\pi)^+$. Of crucial interest here is Lemma 2.1 in [20], which ensures that solutions are uniformly bounded in a neighborhood of $\partial\Omega$ whenever Ω is regular.

Remark 1.4. As a consequence of an argument which we introduce in Lemma 2.1 below, we could extend this alternative (either ψ_β is bounded or the energy $\mathcal{E}(\omega_{(\beta)}) \to +\infty$ as $\beta \searrow (-8\pi)^+$) to the case where Ω is just simple, the only difference in this case being that one would have to allow (in principle) $p \in \partial\Omega$ in (1.2). However we do not know of any result claiming the uniqueness of solutions of the (MFE) with $\beta \in (-8\pi, 0)$ under such weak regularity assumptions on Ω.

As in [19] we need the following:

Definition 1.5. We set $E_c = \mathcal{E}(\omega_{(\beta)}) \mid_{\beta=(-8\pi)^+}$ if Ω is of second kind and $E_c = +\infty$ if Ω is of first kind.

It has been shown in [19] that $E_\Upsilon < E_c$ and that to each $E_\Upsilon < E < E_c$ there corresponds a unique $\omega^{(E)}$ which attains the supremum in the MVP and in particular a unique $\beta = \beta(E) \in (-8\pi, 0)$ such that the corresponding unique solution ψ_β of the (MFE) satisfies $\omega_{(\beta(E))} \equiv \omega^{(E)}$ and attains the supremum in the associated CVP (Canonical Variational Principle)

$$f(\beta) = f_\Omega(\beta) = \sup\{\mathcal{F}_\beta(\omega), \ \ \omega \in \mathcal{P} \mid -\mathcal{S}(\omega) < +\infty\}, \qquad \text{(CVP)}$$

where, for $\omega \in \mathcal{P}$,

$$\mathcal{F}_\beta(\omega) = -\frac{1}{\beta}\,\mathcal{S}(\omega) + \mathcal{E}(\omega),$$

is the free energy of ω. In particular it has been proved in [19] that $\mathcal{E}(\omega_{(\beta)})$ is continuous and decreasing in $(-8\pi, 0)$ and $S(E)$ is smooth and concave in (E_Υ, E_c). Concerning these remarkable results we refer to Theorem 3.1 and Proposition 3.3 in [19].

In particular, for domains of first kind the (mean field) thermodynamics of the system is rigorously defined for any attainable value of the energy and equivalently described by solutions of either the MVP or the CVP. Actually, this problem is closely related with another very subtle issue: the (MFE) always admits a solution for $\beta \in (-8\pi, 0]$ (a consequence of the Moser–Trudinger inequality [59]), while in general this is not the case for $\beta \leqq -8\pi$, the value $\beta = -8\pi$ being the critical threshold where the coercivity of the corresponding variational functional [that is (1.6) below] breaks down. A detailed discussion of this point is beyond the scopes of our investigations and we will just recall a few details needed in the presentation of our results, see also Section 1.1 below.

Some sufficient conditions for the existence of solutions of the (MFE) at $\beta = -8\pi$ were provided in [18] and hence used to show that, for example, any long and thin enough rectangle is of the second kind. The problem has been later solved in [20] by using a refined version of the subtle estimates in [26,27] and the newly derived uniqueness of solutions of the (MFE) with $\beta \in (-8\pi, 0]$ and, whenever they exist, for $\beta = -8\pi$ as well on regular domains. In particular, it has been proved in Proposition 6.1 in [20] that if Ω is regular, then the following facts are equivalent:

SK-(i) Ω is of second kind;
SK-(ii) There is a solution of the (MFE) with $\beta = -8\pi$, say $\psi_{-8\pi}$;
SK-(iii) The unique branch of solutions of the (MFE) ψ_β with $\beta \in (-8\pi, 0]$ is uniformly bounded and converges uniformly to $\psi_{-8\pi}$ as $\beta \searrow (-8\pi)^+$.

We conclude that if the branch of (unique) maximizers satisfies (1.2), then there is no solution of the (MFE) with $\beta = -8\pi$ and in particular that a solution of the (MFE) with $\beta = -8\pi$ exists (and is unique) if and only if *blow up for the* (MFE) *at* $\beta = 8\pi$ *occurs from the left*, that is, (1.2) occurs but with $\beta \to (-8\pi)^-$. The fact that (irrespective on the "side" which β may choose to approach 8π) there is a branch of solutions which satisfy a concentration property as in (1.2) was already proved in [19] see NEQ-(ii) below.

The full theory as exposed in [20] as well as the equivalence of statistical ensembles has been recently extended to cover the case where Ω is multiply connected in [9]. As far as one is concerned with the analytical problem of the existence for $\beta = -8\pi$ and uniqueness for $\beta \in [-8\pi, 0)$, the results in [20] have been generalized in [7,8] to the case where Dirac-type singular data are added in the (MFE).

The mean field thermodynamics for domains of the second kind when $E \geqq E_c$ is more involved.

Since it is not difficult to show that $\mathcal{F}_\beta$ is unbounded from above for $\beta < -8\pi$, then there is no solution for the CVP with $\beta < -8\pi$ and therefore no equivalence (at all) among the MVP and the CVP is at hand in this case. Nevertheless some insight about the range of energies $E \geqq E_c$ was also obtained in [19]. Let Ω be a domain of the second kind. Then we have (see Propositions 6.1, 6.2 and Theorem 6.1 in [19]):

NEQ-(i) It holds

$$-8\pi E + C_1 \leqq S(E) \leqq -8\pi E + C_2, \ \forall\, E \geqq E_c,$$

where $C_2 = S(E_c) + 8\pi E_c = 8\pi f(-8\pi)$;

NEQ-(ii) Let $\omega^{(E)}$ be a solution of MVP at energy E. Then (up to subsequences) $\omega^{(E)} \rightharpoonup \delta_{x=p}$, as $E \to +\infty$, where p is a maximum point of $H_\Omega(x, x)$;

NEQ-(iii) $S(E)$ is not concave for $E > E_c$.

Besides these facts, we do not know of any positive result about this problem for domains of the second kind when $E \geqq E_c$.

It is one of our motivations to begin here a systematic study of the statistical mechanics description of the case $E \geqq E_c$. In this paper we work out the following program:

(–) Prove the existence of solutions of the (MFE) for suitable $\beta < -8\pi$ by assuming the domain to be "thin" enough, see Section. 1.1 and Section. 1.4.

(–) Prove that the first eigenvalue of the linearized problem for the (MFE) on these solutions is strictly positive. This fact will imply that our solutions are local maximizers of $\mathcal{F}_\beta$ as well as a multiplicity result yielding another set of unstable solutions, see Section. 1.2.

(–) Prove that if the domain is "thin" enough, then there exists at most one solution of the (MFE) with β bounded from below whose energy is less than a certain threshold. This fact will imply that we have found a connected and smooth branch of solutions where the energy is well defined as a function of $\lambda := -\beta$, see Remark 1.15 and Section. 1.3.

(–) Prove that if the domain is "thin" enough and in a small enough range of energies, then the energy is monotonic increasing as a function of $\lambda = -\beta$. This fact, under an additional and probably technical assumption, will imply that there exists one and only one solution of the MFE at fixed energy (in that small range) which therefore is also the unique maximizer of the entropy for the MVP. In particular we will prove that the entropy is concave in this range, see Section. 1.4.

This is the underlying idea which will guide us in the analysis of various problems of independent mathematical interest as discussed in the rest of this introduction. We take the occasion here to provide all the motivations and/or necessary comments about the statements of the many results obtained (with the unique exception of Proposition 4.1 below) which is why the introduction is so lengthy.

1.1. Existence of Solutions for the Supercritical (MFE) on Thin Domains

Amongst other things which will be discussed below, one of the main reasons which makes things more difficult in the case $E \geqq E_c$ is the lack of a description of the solutions set for the (MFE) with $\beta < -8\pi$. Since this will be a major point in our discussion, we introduce the quantities

$$\lambda := -\beta, \qquad \text{and} \quad u = -\beta\psi = \lambda\psi,$$

and consider the following alternative but equivalent formulation of the (MFE)

$$\begin{cases} -\Delta u = \lambda \dfrac{e^u}{\int_\Omega e^u} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad P(\lambda, \Omega)$$

which we will denote by $P(\lambda, \Omega)$. The following remark will be used throughout the rest of this paper.

Remark 1.6. Clearly $P(\lambda, \Omega)$ is rotational and translational invariant. Moreover the integral in the denominator of the nonlinear datum in $P(\lambda, \Omega)$ makes the problem dilation invariant too, that is, u is a solution of $P(\lambda, \Omega)$ if and only if $v(y) = u(y_0 + d_0 R_0 y)$ is a solution of $P(\lambda, \Omega^{(0)})$, where $y_0 \in \mathbb{R}^2$, $d_0 > 0$, R_0 is an orthogonal 2×2 matrix and

$$\Omega^{(0)} := \{y \in \mathbb{R}^2 \mid y_0 + d_0 R_0 y \in \Omega\}.$$

In particular, u solves $P(\lambda, \Omega_\rho)$ with $\rho = \frac{a}{b}$ where

$$\Omega_\rho = \{(x, y) \in \mathbb{R}^2 \mid \rho^2 x^2 + y^2 \leqq 1,\ \rho \in (0, 1]\}, \tag{1.3}$$

is the canonical two dimensional ellipse whose axis lengths are $\frac{1}{\rho}$ and 1, if and only if $u_0(x', y')$ with $\{bx' = x,\ by' = y\}$ solves $P(\lambda, \mathbb{E}_{a,b})$, where

$$\mathbb{E}_{a,b} = \{(x', y') \in \mathbb{R}^2 \mid a^2 x'^2 + b^2 y'^2 \leqq 1,\ a \in (0, 1],\ b \in (0, 1],\ b \geqq a\},$$

is the canonical two dimensional ellipse whose axis lengths are $\frac{1}{a}$ and $\frac{1}{b}$.

As mentioned above, we just miss a description of the solutions set of $P(\lambda, \Omega)$ with $\lambda > 8\pi$ and Ω regular. General existence results for $P(\lambda, \Omega)$ are at hand for $\lambda \in \mathbb{R}\backslash 8\pi\mathbb{N}$ only if Ω is a multiply connected domain, see [35,67] and the deep results in [27] (see also [5]).

This is far from being a technical problem. Indeed, a well known result based on the Pohozaev identity (see for Example [18]) shows that if Ω is strictly starshaped, then there exists $\lambda_* = \lambda_*(\Omega) \geqq 8\pi$ (see also Remark 1.9 below) such that $P(\lambda, \Omega)$ has no solutions for $\lambda \geqq \lambda_*(\Omega)$. This result is sharp since, indeed, $\lambda_*(B_R(0)) = 8\pi$, where $B_R(0) = \{x \in \mathbb{R}^2 \,:\, |x| < R\}$ for some $R > 0$.

Therefore, in particular, the Leray–Shauder degree of the resolvent operator for $P(\lambda, \Omega)$ with Ω regular vanishes identically for any $\lambda > 8\pi$, see [27].

If this was not enough we also observe that, at least in case Ω is convex, the well known results in [1,26,37,45] concerning concentrating solutions for $P(\lambda, \Omega)$ as $\lambda \to 8\pi k$, for some fixed $k \in \mathbb{N}$, are of no help, since it has been shown in [41] that in fact neither those blow-up solutions sequences exists if $k \geqq 2$.

Finally let us remark that we are concerned here just with solutions of $P(\lambda, \Omega)$. If we allow some weight to multiply the exponential nonlinearity, then other solutions exist for $\lambda > 8\pi$ on simply connected domains, see for example [2,3,12] and more recently the general results derived in [11].

As a matter of fact, the only general result we are left with is the immediate corollary of the uniqueness results in [20], which shows that:

SK-(iv) if Ω is of second kind, then the branch of unique solutions u_λ, $\lambda \in [0, 8\pi]$ of $P(\lambda, \Omega)$ can be extended (via the implicit function theorem) in a small right neighborhood of 8π.

Our first result is concerned with a sufficient condition for the existence of solutions of $P(\lambda, \Omega)$ with $\lambda > 8\pi$ on "thin" domains.

Theorem 1.7.

(a) *Let Ω be a simple domain. For any $c \in (0, 1]$ there exist $\overline{\rho}_* > \underline{\rho}_*(c) > 0$ such that if $\{\rho^2 x^2 + y^2 \leqq \beta_-^2\} \subset \Omega \subset \{\rho^2 x^2 + y^2 \leqq \beta_+^2\}$ with $c = \frac{\beta_-^2}{\beta_+^2}$ then, for any $\rho \in (0, \underline{\rho}_*(c)]$ and for any $\lambda \leqq \lambda_{\rho,c}$, there exists a solution $u^{(\lambda)}$ of $P(\lambda, \Omega)$, where $\underline{\lambda}_{\rho,c} < \lambda_{\rho,c} < \overline{\lambda}_\rho$ and $\underline{\lambda}_{\rho,c}$, $\overline{\lambda}_\rho$ are strictly decreasing (as functions of ρ) in $(0, \underline{\rho}_*(c)]$, $(0, \overline{\rho}_*]$ respectively with $\underline{\lambda}_{\underline{\rho}_*(c),c} = 8\pi = \overline{\lambda}_{\overline{\rho}_*}$ and $\underline{\lambda}_{\rho,c} \simeq \frac{4\pi c}{(8-c)\rho}$, $\overline{\lambda}_\rho \simeq \frac{11\pi}{16\rho}$ as $\rho \to 0^+$.*

(b) *There exists $\bar{N} > 4\pi$ such that if Ω is an open, bounded and convex set (therefore simple) whose isoperimetric ratio, $N \equiv N(\Omega) = \frac{L^2(\partial\Omega)}{A(\Omega)}$, satisfies $N \geqq \bar{N}$, then for any $\lambda \leqq \lambda_N$ there exists a solution $u^{(\lambda)}$ of $P(\lambda, \Omega)$, where $\underline{\Lambda}_N < \lambda_N < \overline{\Lambda}_N$ with $\underline{\Lambda}_{\bar{N}} = 8\pi$, $\underline{\Lambda}_N$ and $\overline{\Lambda}_N$ strictly increasing in N and $\underline{\Lambda}_N \simeq \frac{\pi^2 N}{496} + \mathrm{O}(1)$, $\overline{\Lambda}_N \simeq \frac{33\sqrt{3}N}{16\pi} + \mathrm{O}(1)$ as $N \to +\infty$.*

Remark 1.8. The suspicion that this result should hold was initially due to the above mentioned result in [18] (which states that if Ω is a long and thin enough rectangle then a solution of $P(8\pi, \Omega)$ exists) and to a result in [20] (which states that there exists a critical value $d_1 < 1$ such that if Ω is a rectangle whose sides lengths are $a_1 \leqq b_1$, then a solution of $P(8\pi, \Omega)$ exists if and only if $\frac{a_1}{b_1} \leqq d_1$).

Remark 1.9. Clearly $c = 1$ if and only if Ω is an ellipse, while if Ω is a rectangle it is easy to see that $c = \frac{1}{2}$ is optimal. We also have the quantitative estimate $0.0702 < \overline{\rho}_*(1)$ which could be used in principle to obtain an estimate for either d_1 (see Remark 1.8) or $\bar{N}$. We will not be insistent about this point since it seems that we are too far from optimality. In the case of the ellipse Ω_ρ, the existence lower/upper threshold values $\underline{\lambda}_\rho \simeq \frac{4\pi}{7\rho}/\overline{\lambda}_\rho \simeq \frac{11\pi}{16\rho}$ should be compared with the Pohozaev's upper bound for the existence of solutions for $P(\lambda, \Omega_\rho)$, that is

$$\lambda < \lambda_*(\Omega_\rho) := 4\int_{\partial\Omega_\rho} \frac{\mathrm{d}s}{(\underline{x}, \underline{\nu})} = \frac{4\pi}{\rho}(1+\rho^2).$$

Remark 1.10. For regular domains, the branches of solutions obtained above will be seen to be connected and smooth, see Remark 1.15 below. We will denote them by $\mathcal{G}_{\rho,c} = \{(\lambda, u^{(\lambda)}) : \lambda \in [0, \lambda_{\rho,c}]\}$ (as obtained in Theorem 1.7a) and $\mathcal{G}_N = \{(\lambda, u^{(\lambda)}) : \lambda \in [0, \lambda_N]\}$ (as obtained in Theorem 1.7b) respectively.

The proof of Theorem 1.7 is, surprisingly enough, based on the sub-supersolutions method. In particular we use the result in [31] which allows for such weak assumptions about the regularity of Ω. The underlying idea in case $\Omega = \Omega_\rho$ is:

(–) If the ellipse $\Omega = \Omega_\rho$ is "thin" enough (that is if ρ is small enough) then the branch of minimal solutions for the classical Liouville problem

$$\begin{cases} -\Delta u = \mu\, e^u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad Q(\mu, \Omega)$$

cannot be pointwise too far from the $C^2_0(\Omega_\rho)$ function

$$v_{\rho,\gamma} = 2\log\left(\frac{1+\gamma^2}{1+\gamma^2(\rho^2x^2+y^2)}\right), \quad (x, y) \in \Omega_\rho,$$

for a suitable value of γ depending on μ and ρ. Of course, the guess about $v_{\rho,\gamma}$ is inspired by the Liouville formula [54]. Therefore, for fixed μ and ρ, we seek values $\gamma_\mp$ such that $v_{\rho,\gamma_\mp}$ are sub-supersolutions respectively of $Q(\mu, \Omega_\rho)$.

(–) If the choice of $\gamma_\pm(\mu)$ is made with enough care, then, along the branch of solutions (say u_μ) for $Q(\mu, \Omega)$ found via the sub-supersolutions method, the value of λ defined as follows

$$\lambda := \mu \int_{\Omega_\rho} e^{u_\mu},$$

can be quite large whenever ρ is small enough.

Part (b) of Theorem 1.7 will be a consequence of Part (a) and Theorems 1.11 and 1.12 below.

Theorem 1.11. [43] *Let $K \subset \mathbb{R}^2$ be a convex body (that is a compact convex set with nonempty interior). Then there is an ellipsoid E (called the John ellipsoid which is the ellipsoid of maximal volume contained in K) such that, if c_0 is the center of E, then the inclusions*

$$E \subset K \subset \{c_0 + 2(x - c_0) : x \in E\}$$

hold.

Theorem 1.12. [48] *Every convex body* $K \subset \mathbb{R}^2$ *contains an ellipse of area* $\frac{\pi}{3\sqrt{3}} A(K)$.

A short proof of the previous theorem is based on a result in [15], where the existence of an affine-regular hexagon H of area at least $\frac{2}{3} A(K)$ and inscribed in K is established. Indeed, considering the concentric inscribed ellipse in H one gets the thesis.

Remark 1.13. In particular Theorem 1.12 has been used to obtain the asymptotic behaviors of $\underline{\Lambda}_N$ and $\overline{\Lambda}_N$. A more rough estimate of those asymptotics could have been obtained by using other (much worse) known estimates of the area of the enclosed ellipse. In particular, while Theorem 1.11 is well known [43], it seems that Theorem 1.12 is not and we are indebted with Prof. M. Lassak who kindly reported to us a proof of it [48] based on the cited reference [15].

Clearly, as an immediate corollary of Theorem 1.7 and the equivalence of SK-(i) and SK-(ii) we conclude that if Ω is regular and satisfies the assumptions of Theorem 1.7(a) (Theorem 1.7(b)) with $\rho \in (0, \underline{\rho}_*(c)]$ ($N(\Omega) > \bar{N}$) then it is of second kind.

1.2. Non Degeneracy and Multiplicity of Solutions of the Supercritical (MFE) on Thin Domains

Let us define the density corresponding to a solution u_λ of $P(\lambda, \Omega)$ as

$$\omega_\lambda \equiv \omega(u_\lambda) := \frac{e^{u_\lambda}}{\int_\Omega e^{u_\lambda}}. \tag{1.4}$$

A crucial tool used in the proof of the equivalence of statistical ensembles [19] is the uniqueness of solutions [20,69] (see also [9]) of $P(\lambda, \Omega)$ for $\lambda \in [0, 8\pi]$. The situation is far more involved in the case $\lambda > 8\pi$ since on domains of second kind, solutions are not anymore unique.

This fact is already clear from NEQ-(ii) and SK-(iv) above, that is, if Ω is of second kind we have a blow-up branch which satisfies

$$\omega(u_\lambda) \rightharpoonup \delta_{x=p}, \quad \text{as} \quad \lambda \searrow (8\pi)^+, \tag{1.5}$$

weakly in the sense of measures, for some critical point $p \in \Omega$ of $H_\Omega(x, x)$, and the smooth solutions of $P(\lambda, \Omega)$ in a small right neighborhood of 8π. Hence, we have at least two solutions in a right neighborhood of 8π, a well known fact that could have been also deduced by using the alternative in Theorem 7.1 in [19] together with the uniqueness result in [20].

We wish to make a further step in this direction. To this end we first study the linearized problem of $P(\lambda, \Omega)$ at $u^{(\lambda)}$, where $u^{(\lambda)}$ is the solution obtained in Theorem 1.7, showing the positivity of its first eigenvalue (see Proposition 4.1 and Remark 4.2 for details). It is worth pointing out that the above fact, which yields a multiplicity result too, is also crucial in the analysis of the solutions branches $\mathcal{G}_{\rho,c}$, $\mathcal{G}_N$, see Remarks 1.10 and 1.15. In particular we have:

Proposition 1.14. *For fixed $c \in (0, 1]$, let Ω be a regular domain satisfying $\{\rho^2 x^2 + y^2 \leqq \beta_-^2\} \subset \Omega \subset \{\rho^2 x^2 + y^2 \leqq \beta_+^2\}$, with $\frac{\beta_-^2}{\beta_+^2} = c$ and $\rho \in (0, \underline{\rho}_*(c)]$, with $\underline{\rho}_*(c)$ as found in Theorem 1.7(a). Let Ω be a convex domain with $N(\Omega) > \bar{N}$ as found in Theorem 1.7(b).*

The portions of $\mathcal{G}_{\rho,c}$, $\mathcal{G}_N$ with $\lambda \in [0, 8\pi]$ coincide with the branch of unique absolute minimizers of

$$F_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda \log\left(\int_\Omega e^u \,\mathrm{d}x\right), \quad u \in H_0^1(\Omega), \tag{1.6}$$

and for each $\lambda \in (8\pi, \lambda_{\rho,c}]$ or $\lambda \in (8\pi, \lambda_N]$ the corresponding solutions $u^{(\lambda)}$ such that $(\lambda, u^{(\lambda)}) \in \mathcal{G}_{\rho,c}$ and $(\lambda, u^{(\lambda)}) \in \mathcal{G}_N$ are strict local minimizers of F_λ.

Remark 1.15. By using the bounds provided by the sub-supersolutions method (see (3.8) in the proof of Theorem 1.7), Proposition 4.1, Theorem 1.19 below and standard bifurcation theory [32] we conclude that for any fixed $\overline{\lambda} > 8\pi$, possibly taking a smaller $\underline{\rho}_*(c)$ and a larger N, the portions of $\mathcal{G}_{\rho,c}$ and $\mathcal{G}_N$ with $\lambda \leqq \overline{\lambda}$ are smooth and connected branches with no bifurcation points.

The proof of Proposition 1.14 is a straightforward consequence of the fact that the first eigenvalue of the linearized problem for $P(\lambda, \Omega)$ is strictly positive along $\mathcal{G}_{\rho,c}$ and $\mathcal{G}_N$, see Proposition 4.1 in Section 4.

We shall see that, by virtue of Proposition 1.14, it is possible to show that for $\lambda \in (8\pi, \lambda_{\rho,c}) \setminus 8\pi\mathbb{N}$ the functional F_λ exhibits a mountain-pass type structure which in turn yields the existence of min-max type solutions to $P(\lambda, \Omega)$. More precisely we obtain the following result.

Theorem 1.16.

(a) *Let Ω, $\rho \in (0, \underline{\rho}_*(c)]$ and $\lambda_{\rho,c}$ be as in Theorem 1.7(a) and let $u^{(\lambda)}$ be a solution of $P(\lambda, \Omega)$ for $\lambda \leqq \lambda_{\rho,c}$. Then, for any $\lambda \in (8\pi, \lambda_{\rho,c}) \setminus 8\pi\mathbb{N}$ there exists a second solution $v^{(\lambda)}$ of $P(\lambda, \Omega)$ such that $F_\lambda(v^{(\lambda)}) > F_\lambda(u^{(\lambda)})$.*

(b) *Let Ω, $\bar{N} > 4\pi$, $N(\Omega)$ and λ_N be as in Theorem 1.7(b) and let $u^{(\lambda)}$ be a solution of $P(\lambda, \Omega)$ for $\lambda \leqq \lambda_N$. Then, for any $\lambda \in (8\pi, \lambda_N) \setminus 8\pi\mathbb{N}$ there exists a second solution $v^{(\lambda)}$ of $P(\lambda, \Omega)$ such that $F_\lambda(v^{(\lambda)}) > F_\lambda(u^{(\lambda)})$.*

Remark 1.17. By using well known compactness results [49] as well as those recently derived in [41], we conclude that any sequence of solutions $v^{(\lambda)}$ with $8\pi k < \lambda < 8\pi(k+1)$, $k \geqq 1$ obtained in part (b) converges as $\lambda \to 8\pi(k+1)$ to a solution $v_{8\pi(k+1)}$ of $P(8\pi(k+1), \Omega)$. We also have at least two different arguments showing that for any fixed $\overline{\lambda} > 0$, possibly taking a larger N, those $v_{8\pi k}$ which also satisfy $8\pi k \leqq \overline{\lambda}$ are distinct from those obtained in Theorem 1.7(b) for $\lambda = 8\pi k$. The first one is a standard bifurcation-type argument based on Remark 1.15 and Proposition 4.1 below. The second one is based on the uniqueness result stated in Theorem 1.19 below.

Remark 1.18. It is easy to check that if u is a solution of $P(\lambda, \Omega)$ and $\omega(u)$ is defined as in (1.4), then $\omega(u)$ is a critical point of $\mathcal{F}_{-\lambda}$ and in particular $\mathcal{F}_{-\lambda}(\omega) =$

$-\frac{1}{\lambda^2}F_\lambda(u)$. Hence, if $u^{(\lambda)}$ and $v^{(\lambda)}$ are as in Theorem 1.16, then it is readily seen that $\mathcal{F}_{-\lambda}(\omega(u^{(\lambda)})) < \mathcal{F}_{-\lambda}(\omega(v^{(\lambda)}))$. In particular $\omega(u^{(\lambda)})$ is a kind of metastable state (in the sense that it is a strict local maximizer of $\mathcal{F}_{-\lambda}$) while $\omega(v^{(\lambda)})$ is expected to be unstable (since it is a min-max type critical point of $\mathcal{F}_{-\lambda}$).

In any case, whenever Ω is regular (and since solutions of $P(8\pi, \Omega)$ are unique in this case [20]), then any sequence of solutions found in Theorem 1.16 for $P(\lambda, \Omega)$ with $\lambda \searrow 8\pi^+$ must satisfy (1.5).

1.3. Uniqueness of Solutions for the Supercritical (MFE) with Bounded Energy on Thin Domains

As a matter of fact we are still unable to define the energy as a monodrome function of λ. We explain the next step toward this goal in the case of the ellipse Ω_ρ.

Although solutions of $P(\lambda, \Omega_\rho)$ are not unique as a function of λ, what we can prove is that for fixed $\overline{\lambda} \geqq 8\pi$ and $\overline{E} \geqq 1$, then for ρ small enough there could be at most one solution $u_{\rho,\lambda}$ such that $\lambda \leqq \overline{\lambda}$ and

$$\mathcal{E}(\omega(u_{\rho,\lambda})) \leqq \overline{E}. \tag{1.7}$$

This is a major achievement since, by using also Proposition 4.1 below, it implies that (as far as ρ is small enough) the energy (see Proposition 6.1) is well defined as a function of λ, whenever $\lambda \leqq \overline{\lambda}$ and the supremum of the range of the energy itself is not greater than $\overline{E}$.

Let us think of the results obtained in Section. 1.1 and Section. 1.2 in terms of the $(\lambda, \|u_{\rho,\lambda}\|_\infty)$ bifurcation diagram. To fix the ideas, we propose the following naive description. As ρ gets smaller and smaller, we have:

(–) The portion with $\lambda \leqq \overline{\lambda}$ and $\mathcal{E}(u_{\rho,\lambda}) \leqq \overline{E}$ of the (smooth, see Remark 1.15) branches of solutions $\mathcal{G}_{\rho,c}, \mathcal{G}_N$ obtained in Theorem 1.7 gets lower and flatter, that is, $\|u_{\rho,\lambda}\|_\infty \searrow 0^+$. See also Remark 1.23 below.
(–) In the same time the portion with $\lambda \leqq \overline{\lambda}$ of the branches obtained in Theorem 1.16 (as well as any other possible solution) gets higher and higher the corresponding energies getting greater and finally greater than $\overline{E}$.
(–) Any bifurcation/bending point one should possibly meet along $\mathcal{G}_{\rho,c}, \mathcal{G}_N$ moves in the region $\lambda > \overline{\lambda}$.

It is understood that the value 1 in the condition $\overline{E} \geqq 1$ could have been substituted by any other fixed positive number. More exactly we have the following:

Theorem 1.19. *Fix $\overline{\lambda} \geqq 8\pi$ and $\overline{E} \geqq 1$. Then:*

(a) *Let Ω be a simple domain and suppose that there exists $c \in (0, 1]$ such that $\{\rho^2x^2 + y^2 \leqq \beta_-^2\} \subseteq \Omega \subseteq \{\rho^2x^2 + y^2 \leqq \beta_+^2\}$ with $c = \frac{\beta_-^2}{\beta_+^2}$.*
Then there exists $\widetilde{\rho}_1 = \widetilde{\rho}_1(c, \overline{E}, \overline{\lambda}) > 0$ such that for any $\rho \in (0, \widetilde{\rho}_1]$, there exists at most one solution u_λ of $P(\lambda, \Omega)$ with $\lambda \leqq \overline{\lambda}$ which satisfies (1.7).

(b) *Let Ω be any open, bounded and convex (therefore simple) domain. There exists $\widetilde{N} = \widetilde{N}(\overline{\lambda}, \overline{E}) \geqq 4\pi$ such that for any such Ω satisfying*

$$N(\Omega) := \frac{L^2(\partial\Omega)}{A(\Omega)} \geqq \widetilde{N},$$

there exists at most one solution u_λ of $P(\lambda, \Omega)$ with $\lambda \leqq \overline{\lambda}$ which satisfies (1.7).

The proof of Theorem 1.19 is based on two main tools.

The first one is an a priori estimate for solutions of $P(\lambda, \Omega)$ [which satisfy $\lambda \leqq \overline{\lambda}$ and (1.7)] with a uniform constant $\overline{C}$ which does not depend neither on u nor on the domain Ω, but only on $\overline{\lambda}$ and $\overline{E}$. Roughly speaking, in case $\Omega = \Omega_\rho$, this kind of uniformity with respect to the domain is needed since we consider the limit in which ρ gets very small, that is, we seek uniqueness for all domains which are "thin" in the sense specified in the statement of Theorem 1.19. We refer to Lemma 2.1 and the discussion about it in Section 2 for further details.

The second tool is a careful use of the dilation invariance (see Remark 1.6) to be used together with an estimate about the first eigenvalue of the Laplace-Dirichlet problem on a "thin" domain, see (2.12) below for more details.

1.4. Uniqueness of Solutions for the Supercritical (MFE) on Ω_ρ with Fixed Energy and Concavity of the Entropy

In this subsection we fix $\Omega = \Omega_\rho$.

As observed above, by using Theorem 1.19 and Proposition 4.1 below we can prove that (as far as ρ is small enough) the energy (see Proposition 6.1) is well defined as a function of λ (along the branch $\mathcal{G}_{\rho,1}$ found in Theorem 1.7(a), see Remark 1.15) whenever $\lambda \leqq \overline{\lambda}$ and the supremum of the range of the energy itself is not greater than $\overline{E}$. It is tempting at this point to say that the entropy maximizers of the MVP are those solutions of the (MFE) obtained in Theorem 1.7(a). However we still don't know whether or not this is true, since obviously there could be many solutions on $\mathcal{G}_{\rho,1}$ (that is with different values of λ) corresponding to a fixed energy $E \leqq \overline{E}$ (see for example fig. 5 in [19]). In such a situation it would be difficult to detect which is, (or worst, which are) the one which really maximizes the entropy. A possible solution to this problem could be obtained if we would be able to understand the monotonicity of the energy as a function of λ on $\mathcal{G}_{\rho,1}$. The first step toward this goal is to show that the solutions of $P(\lambda, \Omega_\rho)$ obtained in Theorem 1.7(a) can be expanded in powers of ρ with the leading order taking up an explicit and simple form [see also (6.3), (6.5) below], that is

$$\phi_0(x, y; \lambda, \rho) = \mu_0(\lambda, \rho)\psi_0(x, y; \rho), \quad (x, y) \in \Omega_\rho, \tag{1.8}$$

where μ_0 satisfies (1.12)–(1.13) below and

$$\psi_0(x, y; \rho) = \frac{1}{2(1 + \rho^2)} \Big(1 - (\rho^2 x^2 + y^2)\Big), \quad (x, y) \in \Omega_\rho. \tag{1.9}$$

Of course, we could have used the fact that we already knew about the existence of the branch $\mathcal{G}_{\rho,1}$ and managed to expand those solutions as a function of ρ.

Instead we decided to make the argument self-contained by pursuing another proof of independent interest of the existence of solutions of $P(\lambda, \Omega_\rho)$. It shows that there exists ρ_0 small enough (depending on $\overline{\lambda}$) such that for any $\rho < \rho_0$ and for each $\lambda \in [0, \overline{\lambda})$ a solution u_λ for $P(\lambda, \Omega_\rho)$ exists whose leading order with respect to ρ takes up the form (1.8). There is no problem in checking that these solutions coincide with those on the branch $\mathcal{G}_{\rho,1}$ obtained in Theorem 1.7(a). Indeed this is at this point an easy consequence of Theorem 1.19.

We still face the problem of how to handle the term $\int_{\Omega_\rho} e^{u_\lambda}$ in the denominator of the nonlinear term in $P(\lambda, \Omega_\rho)$. This time we will solve this issue by seeking solutions v_ρ of $Q(\mu_0\rho, \Omega_\rho)$ which satisfy the following identity in a suitable set of values of λ,

$$\lambda = \mu_0 \rho \int_{\Omega_\rho} e^{u_\lambda}. \tag{1.10}$$

This is the content of Theorem 1.20 below. More exactly, by setting

$$D_\lambda^{(k)} = \frac{\partial^k}{\partial \lambda^k}, \quad k = 0, 1, 2,$$

we have the following:

Theorem 1.20. *Let $\overline{\lambda} \geqq 8\pi$ be fixed. There exists $\rho_0 > 0$ depending on $\overline{\lambda}$ such that for any $\rho < \rho_0$ and for each $\lambda \in [0, \overline{\lambda})$ there exists a solution u_λ for $P(\lambda, \Omega_\rho)$ which satisfies*

$$u_\lambda(x, y; \lambda) = \rho\phi_0(x, y; \lambda) + \rho^2\phi_1(x, y; \lambda) + \rho^3\phi_2(x, y; \lambda), \quad (x, y) \in \Omega_\rho, \tag{1.11}$$

where $\{\phi_0, \phi_1, \phi_2\} \subset C_0^2(\Omega)$. Moreover ϕ_0 takes the form (1.8) with μ_0 a smooth function which satisfies

$$\mu_0(\lambda, \rho) = \frac{\lambda}{\pi} - \frac{\lambda^2}{4\pi^2}\rho + \mathrm{O}(\rho^2), \tag{1.12}$$

and

$$D_\lambda^{(1)}\mu_0(\lambda, \rho) = \frac{1}{\pi} - \frac{\lambda}{2\pi^2}\rho + \mathrm{O}(\rho^2), \quad D_\lambda^{(2)}\mu_0(\lambda, \rho) = -\frac{1}{2\pi^2}\rho + \mathrm{O}(\rho^2). \tag{1.13}$$

In particular the following uniform estimates hold

$$\|D_\lambda^{(k)}\phi_0\|_{C_0^2(\Omega)} + \|D_\lambda^{(k)}\phi_1\|_{C_0^2(\Omega)} + \|D_\lambda^{(k)}\phi_2\|_{C_0^2(\Omega)} \leqq \overline{M}_k, \quad k = 0, 1, 2, \tag{1.14}$$

for suitable constants $\overline{M}_k$, $k = 0, 1, 2$ depending only on $\overline{\lambda}$. Finally these solutions' set is a smooth branch which coincides with a portion of $\mathcal{G}_{\rho,1}$.

Remark 1.21. In the proof of Theorem 1.20 and therefore in all the expansions in powers of ρ what we really use is the fact that solutions v_ρ of $Q(\mu_0\rho, \Omega_\rho)$ can be expanded in powers of ρ and in particular that $\lambda_0(\mu_0, \rho) := \mu_0\rho\int_{\Omega_\rho} e^{v_\rho}$ is smooth, see Lemma 6.2 below. Here we need some estimates about the first eigenvalue of the linearization of $Q(\mu, \Omega)$ as obtained in Proposition 4.1 below (see Remark 4.2).

By using Theorem 1.20 we can prove the following result. Let $\widetilde{\rho}_1$ be fixed as in Theorem 1.19(a). Then we have:

Theorem 1.22. *Let $\overline{\lambda} \geqq 8\pi$ and let $\widehat{E}_\rho$ be defined by*

$$\widehat{E}_\rho := \frac{\rho}{8\pi} + \frac{\rho^2}{50\pi^2}\overline{\lambda}.$$

For each $\rho < \widetilde{\rho}_1$ and $E \in \left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$ there exists one and only one solution u_λ for $P(\lambda, \Omega_\rho)$ such that $\lambda \leqq \overline{\lambda}$ and whose energy is $\mathcal{E}(\omega(u_\lambda)) = E$. Let $\widehat{\lambda}_\rho$ be defined by $\mathcal{E}(\omega(u_{\widehat{\lambda}_\rho})) = \widehat{E}_\rho$. Then in particular the identities

$$\widehat{E}(\lambda) = \mathcal{E}(\omega(u_\lambda)), \quad \mathcal{E}(\omega(u_{\widehat{\lambda}(E)})) = E,$$

define:

$\widehat{E}(\lambda) : [0, \widehat{\lambda}_\rho] \to \left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$ as a smooth and strictly increasing function of λ and

$\widehat{\lambda}(E) : \left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right] \to [0, \widehat{\lambda}_\rho]$ as a smooth and strictly increasing function of E. Moreover we have

$$\widehat{E}(\lambda) = \frac{\rho}{8\pi} + \frac{\rho^2}{48\pi^2}\lambda + \mathrm{O}(\rho^3), \quad \widehat{\lambda}(E) = \frac{48\pi^2}{\rho^2}\left(E - \frac{\rho}{8\pi}\right) + \mathrm{O}(\rho). \tag{1.15}$$

$$\frac{d}{d\lambda}\widehat{E}(\lambda) = \frac{\rho^2}{48\pi^2} + \mathrm{O}(\rho^3), \quad \frac{d}{dE}\widehat{\lambda}(E) = \frac{48\pi^2}{\rho^2} + \mathrm{O}(\rho), \tag{1.16}$$

$$\frac{d^2}{d\lambda^2}\widehat{E}(\lambda) = \mathrm{O}(\rho^3), \quad \frac{d^2}{dE^2}\widehat{\lambda}(E) = \mathrm{O}(\rho). \tag{1.17}$$

Remark 1.23. The notation $\mathrm{O}(\rho^m)$, $m \in \mathbb{N}$ is used here and in the rest of this paper to denote various quantities uniformly bounded by $C_m \rho^m$ with $C_m > 0$ a suitable constant depending only on $\overline{\lambda}$.

This result is consistent with the underlying idea that, as ρ gets smaller and smaller, then the energies of the entropy maximizers (which are solutions of $P(\lambda, \Omega_\rho)$) with values of λ uniformly bounded from above have to approach the energy of the uniform density distribution $\Upsilon = \frac{1}{|\Omega_\rho|}$, that is

$$\begin{aligned} E_{\Upsilon,\rho} &:= \mathcal{E}\left(\frac{1}{|\Omega_\rho|}\right) = \frac{1}{2}\int_{\Omega_\rho} \frac{1}{|\Omega_\rho|} G_\rho\left[\frac{1}{|\Omega_\rho|}\right] \\ &= \frac{\rho}{2\pi}\int_{\Omega_\rho} \frac{1}{|\Omega_\rho| 2(1+\rho^2)}\left(1 - (\rho^2 x^2 + y^2)\right) = \frac{\rho}{8\pi(1+\rho^2)}. \end{aligned}$$

Here we used the easily derived explicit expression of the function $G_\rho\left[\frac{1}{|\Omega_\rho|}\right]$ see also (1.8), (1.9) and (6.3), (6.5) below.

Remark 1.24. In particular (1.15) yields $\widehat{\lambda}_\rho = \frac{48}{50}\overline{\lambda} + \mathrm{O}(\rho)$ and since $\overline{\lambda} \geqq 8\pi$ can be chosen at wish and (see Definition 1.5) $E_c = \mathcal{E}(\omega(u_{8\pi}))$, then of course $E_{\Upsilon,\rho} < E_c < \widehat{E}_\rho$ and we succeed in the description of the energy as a function of (minus) the inverse temperature $\lambda = -\beta$ in a very small range of energies above E_c.

Let us observe that (1.15) is in perfect agreement with the discussion in §1.3, that is, the portion with $\lambda \leqq \overline{\lambda}$ of the branch of solutions obtained in Theorem 1.7 gets lower and flatter as ρ gets smaller and smaller. We will prove Theorem 1.22 by some explicit evaluations. This is why our concern in Theorem 1.20 was with respect to the exact expression of solutions of $P(\lambda, \Omega_\rho)$ with $\lambda \leqq \overline{\lambda}$ and ρ small and not just with the estimates one can get by using the sub-supersolutions just found in Theorem 1.7.

At this point (see Section 7 for details), by using an additional and probably technical assumption, we can conclude that indeed $S(E) \equiv \mathcal{S}(\omega(u_\lambda))\,|_{\lambda=\widehat{\lambda}(E)}$ in $\left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$. In particular we conclude that $S(E)$ is also smooth in $\left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$ and by using the asymptotic expansions (1.15), (1.16) and (1.17) and the above mentioned explicit expressions (1.8) and (1.9) we are eventually able to evaluate $\frac{\mathrm{d}^2 S(E)}{\mathrm{d}E^2}$ in the case $\Omega = \Omega_\rho$ and $E \in \left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$. Indeed, we have

Proposition 1.25. *Let $\Omega = \Omega_\rho$, $E \in \left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$ and $\rho < \tilde{\rho}_1$ as defined in Theorem 1.22. Assume that there is no solutions of $P(\lambda, \Omega_\rho)$ such that $\lambda > \overline{\lambda}$ and whose energy is $\mathcal{E}(\omega(u_\lambda)) = E$. Then we have $S(E) \equiv \mathcal{S}(\omega(u_\lambda))\,|_{\lambda=\widehat{\lambda}(E)}$ in $\left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$ and in particular it holds*

$$\frac{\mathrm{d}^2 S(E)}{\mathrm{d}E^2} = -11\left(\frac{48\pi^2}{\rho^2}\right) + \mathrm{O}\left(\frac{1}{\rho}\right).$$

In other words, we conclude that the branch of "small energy" solutions of $P(\lambda, \Omega_\rho)$ with $\lambda \leqq \overline{\lambda}$ is, for ρ small enough, a branch of Entropy maximizers in a range where S is concave.

1.5. Open Problems

Obviously it will be interesting to remove that assumption in Proposition 1.25 about the non existence of solutions of $P(\lambda, \Omega_\rho)$ with $\lambda > \overline{\lambda}$ and whose energy is $\mathcal{E}(\omega(u_\lambda)) = E$. It is likely to be just a technical assumption as we are going to discuss in the following conjecture and open problem.

It is well known that $S(E)$ is not concave (see NEQ-(iii) above) for $E > E_c$ and that solutions of the MVP (see NEQ-(iii) and (1.5) above) blow up as $E \to +\infty$. Concerning this point we have the following:

Conjecture Let Ω be a convex domain of the second kind. There exists one and only one branch of solutions u_λ which satisfies (1.5) and in particular there exists $E_\Omega > E_c$ such that $S(E)$ is convex in $(E_\Omega, +\infty)$.

In particular, the uniqueness of blow-up solutions would imply that they coincide (at least in a small right neighborhood of 8π) with the set of mountain-pass type solutions found in Theorem 1.16, see Remark 1.18.

Then we pose the following problem:

Open Problems Let us assume that either the above conjecture is true or that Ω is a convex domain of the second kind for which we can find $E_\Omega > E_c$ such that $S(E)$ is convex in $(E_\Omega, +\infty)$. Is it true that the entropy have only one inflection point? If not, under which conditions (if any) does the entropy have only one inflection point?

In particular, is it true that the global branch of solutions of $P(\lambda, \Omega_\rho)$ with ρ small enough has just one bending point, no bifurcation points and it is connected with the blow-up solution's branch as $\lambda \searrow (8\pi)^+$? Can we answer this question at least on some convex domains?

Of course, these properties do not hold on general simply connected domains. For example, there should be no reason to expect the energy to be a generally injective function of λ (see for example Fig. 5 in [19]). Moreover, some well known numerical results [64] suggest that bifurcation points can exist on the bifurcation diagram of $P(\lambda, \Omega)$ on (symmetric and/or non symmetric) non convex domains. It seems however that the very rich structure of those bifurcation diagrams [64] is inherited by solutions sharing either multiple peaks or just a single peak but which may be located at different points. The typical example of such kind of blow-up behavior is observed on dumbbell shaped domains, see for example [37].

On the other side, there are easier situations, such as on convex domains, where $k-$peaks solutions with $k \geqq 2$ do not exist (as shown in [41]). Moreover it is well known (see for example [42]) that if Ω is convex then the Robin function $H_\Omega(x, x)$ is strictly concave and thus admits one and only one critical point, which of course coincides with the absolute maximum. This rules out the possibility of having more than one single peak blow-up solution.

So far, it seems that, in particular, the global structure of the solution's branch is known only for domains which are close in C^2-norm to a disk, see [68].

Of course, if (say in case $\Omega = \Omega_\rho$ with ρ small enough) the entropy really has just one inflection point, then it will coincide with the point on the continuation of $\mathcal{G}_{\rho,1}$ where the first eigenvalue of the linearized problem for $P(\lambda, \Omega_\rho)$ will finally vanish. However, in this situation we cannot use the standard results (see for example [70]) which in the classical cases show that this point must necessarily be a bending point. This is due to the peculiar form of the linearized problem for $P(\lambda, \Omega)$, see (4.1) below, which implies for example that the first eigenfunction changes sign and that in general the first eigenvalue is not simple. Indeed an explicit example of a changing sign and non simple first eigenfunction in a similar situation can be found in Appendix D in [3].

In any case we think that this topic deserves a separate discussion and that it should be already very interesting to set up the problem on some symmetric and convex domain of the second kind such as thin ellipses and/or rectangles.

This paper is organized as follows. In Section 2 we prove Theorem 1.19. In Section 3 we prove Theorem 1.7. In Section 4 we prove Proposition 1.14 by using a result concerning the first eigenvalue of the linearization of $P(\lambda, \Omega)$ around those solutions found in Theorem 1.7, see Proposition 4.1. Section 5 is devoted to the proof of Theorem 1.16. Section 6 is concerned with the proofs of Theorems 1.20 and

1.22. Finally Section 7 is devoted to the proof of Proposition 1.25. Some technical evaluations are left to the Appendix.

2. A Uniqueness Result for Solutions of $P(\lambda, \Omega)$

The aim of this section is to obtain a uniqueness result for solutions of $P(\lambda, \Omega)$ with finite energy $\mathcal{E}(\omega_\lambda) \leqq \overline{E}$ [see (1.4)] on domains chosen as in Theorem 1.19.

The proof of Theorem 1.19. We will need an a priori estimate for solutions of $P(\lambda, \Omega)$ with a uniform constant $\overline{C}$ which does not depend neither on u nor on the domain Ω. This is why we do not follow the standard route which is widely used (under some additional regularity assumption on $\partial\Omega$, see for example [20]) in case where the domain is fixed. In that case in fact one needs to prove that blow-up points (in the sense of Brezis-Merle [17]) cannot converge to the boundary. A detailed discussion of this point in our situation would be not only more tricky (since we do not fix Ω) but also really counterproductive, since instead, by using the energy bound (1.7), our argument yields the needed estimate with the weakest possible regularity assumptions about $\partial\Omega$ (that is Ω simple) see Definition 1.1.

The underlying idea is to use the dilation invariance (see Remark 1.6) of $P(\lambda, \Omega)$ to show that even if a blow-up "bubble" converges to the boundary, then its energy must be unbounded. More exactly we have:

Lemma 2.1. *Let $\overline{\lambda} \geqq 8\pi$ and $\overline{E} \geqq 1$ be fixed. There exists $\overline{C} = \overline{C}(\overline{\lambda}, \overline{E})$ such that for any simple domain Ω and for all solutions of $P(\lambda, \Omega)$ such that $\lambda \leqq \overline{\lambda}$ and $\mathcal{E}(\omega_\lambda) \leqq \overline{E}$ it holds $\|u_\lambda\|_\infty \leqq \overline{C}$. In particular $\overline{C}$ does not depend neither on u nor on Ω.*

Proof. In view of Remark 1.2 we can assume u to be a classical solution of $P(\lambda, \Omega)$.

We argue by contradiction and suppose that there exists a sequence of simple domains $\{\Omega_n\}$ and a sequence of positive numbers $\{\lambda_n\}$ such that $\sup_{\mathbb{N}} \lambda_n \leqq \overline{\lambda}$ and there exists a sequence of solutions $\{u_n\}$ for $P(\lambda_n, \Omega_n)$ such that

$$\mathcal{E}(\omega(u_n)) \leqq \overline{E},$$

and there exists a sequence of points $\{x_n\}$ such that $x_n \in \Omega_n \; \forall\, n \in \mathbb{N}$ and

$$u_n(x_n) = \max_{\Omega_n} u_n \to +\infty.$$

Of course, we have used here the fact that the maximum principle ensures that any solution for $P(\lambda_n, \Omega_n)$ is nonnegative.

Since the problem is translation invariant we can assume without loss of generality that

$$x_n \equiv 0, \;\; \forall\, n \in \mathbb{N}.$$

Let us set

$$d_n := \operatorname{dist}(0, \partial\Omega_n),$$

and define

$$w_{n,0}(y) = u_n\left(\frac{d_n}{2}y\right), \quad y \in \Omega_{n,0} := \left\{y \in \mathbb{R}^2 \ : \ \frac{d_n}{2}y \in \Omega_n\right\}.$$

Clearly we have

$$B_1(0) \Subset \Omega_{n,0} \tag{2.1}$$

and in particular (see Remark 1.6) $w_{n,0}$ is a solution of $P(\lambda_n, \Omega_{n,0})$ which therefore satisfies

$$w_{n,0}(0) = u_n(0) = \max_{\Omega_{n,0}} w_{n,0} \to +\infty. \tag{2.2}$$

Let us set

$$\mu_{n,0} := \lambda_n\left(\int_{\Omega_{n,0}} e^{w_{n,0}}\right)^{-1}.$$

We claim that:

Claim: $w_{n,0}(0) + \log \mu_{n,0} \to +\infty$.

We argue by contradiction and observe that if the claim was false, then we would find

$$\begin{cases} -\Delta w_{n,0} \leqq C_0 & \text{in} \quad \Omega_{n,0} \\ w_{n,0} = 0 & \text{on} \quad \partial\Omega_{n,0} \end{cases}$$

for some $C_0 > 0$. For any $n \in \mathbb{N}$ we can choose $R_n > 0$ such that $\Omega_{n,0} \subset B_{R_n}$ and let

$$\varphi_n(y) = \frac{C_0}{R_n^2}(R_n^2 - |y|^2), \;\; y \in B_{R_n}$$

be the unique solution of

$$\begin{cases} -\Delta\varphi_n = C_0 & \text{in} \quad B_{R_n} \\ \varphi_n = 0 & \text{on} \quad \partial B_{R_n} \end{cases}.$$

Clearly, by the maximum principle we have $w_{n,0}(0) \leqq \varphi_n(0) = C_0$, which is a contradiction to (2.2). This proves the claim.

Therefore we see that the function $w_{n,1}(y) = w_{n,0}(y) + \log \mu_{n,0}$ satisfies

$$\begin{cases} -\Delta w_{n,1} = e^{w_{n,1}} & \text{in} \quad B_1 \\ \int_{B_1} e^{w_{n,1}} \leqq \overline{\lambda} \\ w_{n,1}(0) = \max\limits_{B_1} w_{n,1} \to +\infty \end{cases}.$$

Hence we can apply the Brezis-Merle's result [17] as further improved by Li and Shafrir [50] to conclude that there exists $r_0 \in (0, 1]$ such that

$$e^{w_{n,1}} \rightharpoonup 8\pi m \delta_{p=0}, \quad \text{in} \quad B_{2r_0},$$

weakly in the sense of measures, where m is a positive integer which satisfies $1 \leqq m \leqq \frac{\overline{\lambda}}{8\pi}$. We remark that with a little extra work we could also prove that the oscillation of $w_{n,1}$ is bounded on (say) ∂B_{r_0} and hence in particular obtain the desired contradiction by using the Li's result [49]. We will not purse this approach here since we can come up with the desired conclusion just setting

$$\delta_{n,0}^2 := e^{-w_{n,1}(0)} \to 0, \tag{2.3}$$

and use the by now standard blow-up argument in [50]. It shows that there exists a subsequence (which we will not relabel) such that

$$w_n(z) = w_{n,1}(\delta_{n,0} z) - w_{n,1}(0), \quad |z| < (\delta_{n,0})^{-1},$$

satisfies

$$w_n(z) \to w(z), \quad \text{in} \quad C^2_{\mathrm{loc}}(\mathbb{R}^2), \tag{2.4}$$

where

$$w(z) = 2\log \frac{1}{(1 + \frac{1}{8}|z|^2)}, \qquad \int_{\mathbb{R}^2} e^w = 8\pi. \tag{2.5}$$

At this point we observe that, in view of the translation and dilation invariance of the energy we have

$$\int_{\Omega_{n,0}} \left|\nabla w_{n,1}\right|^2 = \int_{\Omega_{n,0}} \left|\nabla w_{n,0}\right|^2 = \int_{\Omega_n} |\nabla u_n|^2 = 2\lambda_n^2 \mathcal{E}(\omega(u_n)) \leqq 2\overline{\lambda}^{-2}\overline{E},$$

so that, by using (2.2) and (2.3), we should have,

$$\begin{aligned}
2\overline{\lambda}^{-2}\overline{E} &\geqq \int_{\Omega_n} |\nabla u_n|^2 = \lambda_n \int_{\Omega_n} \omega(u_n) u_n = \lambda_n \int_{\Omega_{n,0}} \omega(w_{n,0}) w_{n,0} > \lambda_n \int_{B_{R\delta_{n,0}}} \omega(w_{n,0}) w_{n,0} \\
&= \int_{B_{R\delta_{n,0}}} e^{w_{n,1}} (w_{n,1} - \log \mu_{n,0}) = \int_{B_R} e^{w_n} (w_n + w_{n,1}(0) - \log \mu_{n,0}) \\
&= \int_{B_R} e^{w_n} w_n + u_n(0) \int_{B_R} e^{w_n},
\end{aligned}$$

for any $R \geqq 1$ and for any $n \in \mathbb{N}$, which is clearly in contradiction with (2.2) and (2.4), (2.5). We refer to Lemma 3.1 in [7] for a proof of the fact that the Gauss-Green formula $\int_{\Omega_n} |\nabla u_n|^2 = \lambda_n \int_{\Omega_n} \omega(u_n) u_n$ holds on domains which are only assumed to be simple. □

The proof of Theorem 1.19 completed.

We first prove part (b).

We argue by contradiction and suppose that there exists a sequence of open, bounded and convex domains $\{\Omega_{n,0}\}$ such that

$$N(\Omega_{n,0}) = \frac{L^2(\partial\Omega_{n,0})}{A(\Omega_{n,0})} > n, \tag{2.6}$$

and a sequence of positive numbers $\{\lambda_n\}$ such that $\sup\limits_{\mathbb{N}} \lambda_n \leqq \overline{\lambda}$, such that for any $n \in \mathbb{N}$ there exist at least two solutions $u_{n,1}$ and $u_{n,2}$ for $P(\lambda_n, \Omega_{n,0})$ such that

$$\mathcal{E}(\omega(u_{n,i})) \leqq \overline{E}, \quad i = 1, 2. \tag{2.7}$$

In view of Theorem 1.11 we see that for each $n \in \mathbb{N}$ there exist two concentric and omotetic ellipses such that

$$\mathbb{E}_{n,-} \subseteq \Omega_{n,0} \subseteq \mathbb{E}_{n,+} \tag{2.8}$$

and

$$\frac{A(\mathbb{E}_{n,+})}{A(\mathbb{E}_{n,-})} = 4. \tag{2.9}$$

Since $P(\lambda, \Omega)$ and (2.7) are both rotational, translational and dilation invariant, then, in view of Remark 1.6, we can assume without loss of generality that for each $n \in \mathbb{N}$

$$\mathbb{E}_{n,+} = \Omega_{\rho_n}, \quad \text{for some} \quad \rho_n > 0. \tag{2.10}$$

By (2.9) and the convexity of $\Omega_{n,0}$ we have

$$N(\mathbb{E}_{n,+}) = \frac{L^2(\partial\mathbb{E}_{n,+})}{A(\mathbb{E}_{n,+})} = \frac{Ł^2(\partial\mathbb{E}_{n,+})}{4A(\mathbb{E}_{n,-})} \geqq \frac{L^2(\partial\mathbb{E}_{n,+})}{4A(\Omega_{n,0})} \geqq \frac{1}{4}N(\Omega_{n,0}) > \frac{n}{4}.$$

Therefore, since in view of (2.10) we have $L^2(\partial\mathbb{E}_{n,+}) \leqq \frac{4\pi^2}{\rho_n^2}$ and $A(\mathbb{E}_{n,+}) = \frac{\pi}{\rho_n}$, then we also conclude that

$$\frac{n}{4} < N(\mathbb{E}_{n,+}) \leqq \frac{4\pi^2}{\rho_n^2}\frac{\rho_n}{\pi},$$

that is

$$\rho_n < \frac{16\pi}{n}. \tag{2.11}$$

At this point we observe that

$$\sigma_{n,0} := \inf\left\{ \frac{\int_{\Omega_{n,0}} |\nabla\varphi|^2 \,\mathrm{d}x}{\int_{\Omega_{n,0}} \varphi^2 \,\mathrm{d}x} \;\middle|\; \varphi \in H_0^1(\Omega_{n,0}) \right\} \geqq 2(1+\rho_n^2) > 2, \tag{2.12}$$

which easily follows from the fact that $\sigma_{n,0} \geqq \sigma_n$, where

$$\sigma_n := \inf\left\{ \frac{\int_{\Omega_{\rho_n}} |\nabla\varphi|^2 \,\mathrm{d}x}{\int_{\Omega_{\rho_n}} \varphi^2 \,\mathrm{d}x} \;\middle|\; \varphi \in H_0^1(\Omega_{\rho_n}) \right\},$$

see (4.5) and (4.7) below for further details.

Hence, by using (2.12), we conclude that

$$\begin{aligned} 2\int_{\Omega_{n,0}} \left|u_{n,1} - u_{n,2}\right|^2 &\leqq \int_{\Omega_{n,0}} \left|\nabla(u_{n,1} - u_{n,2})\right|^2 \\ &= \lambda_n \int_{\Omega_{n,0}} (\omega(u_{n,1}) - \omega(u_{n,2}))(u_{n,1} - u_{n,2}). \end{aligned}$$

Let us write

$$\int_{\Omega_{n,0}} (\omega(u_{n,1}) - \omega(u_{n,2}))(u_{n,1} - u_{n,2}) = I_{1,n} + I_{2,n},$$

where

$$I_{1,n} = \int_{\Omega_{n,0}} \frac{e^{u_{n,1}} - e^{u_{n,2}}}{\int_{\Omega_{n,0}} e^{u_{n,1}}}(u_{n,1} - u_{n,2}),$$

$$I_{2,n} = \int_{\Omega_{n,0}} e^{u_{n,2}} \left(\frac{1}{\int_{\Omega_{n,0}} e^{u_{n,1}}} - \frac{1}{\int_{\Omega_{n,0}} e^{u_{n,2}}} \right)(u_{n,1} - u_{n,2}).$$

It follows from Lemma 2.1 (which of course can be applied since any open, bounded and convex domain is simple according to Definition 1.1) and the fact that solutions of $P(\lambda, \Omega)$ are non negative that, by using also (2.9), we can estimate these two integrals as follows

$$\begin{aligned}\left|I_{1,n}\right| &\leqq \int_{\Omega_{n,0}} \frac{e^{\overline{u}_n}}{A(\Omega_{n,0})}|u_{n,1} - u_{n,2}|^2 \leqq \int_{\Omega_\rho} \frac{e^{\overline{C}}}{A(\mathbb{E}_{n,-})}|u_{n,1} - u_{n,2}|^2 \\ &= \frac{4e^{\overline{C}}}{\pi}\rho_n \int_{\Omega_{n,0}} |u_{n,1} - u_{n,2}|^2,\end{aligned}$$

and similarly,

$$\begin{aligned}\left|I_{2,n}\right| &\leqq \int_{\Omega_{n,0}} e^{u_{n,2}}|u_{n,1} - u_{n,2}| \left(\int_{\Omega_{n,0}} \frac{e^{\overline{u}_n}}{\left(\int_{\Omega_{n,0}} e^{\overline{u}_n}\right)^2}(u_{n,1} - u_{n,2}) \right) \\ &\leqq \frac{e^{2\overline{C}}}{A^2(\Omega_{n,0})} \left(\int_{\Omega_{n,0}} |u_{n,1} - u_{n,2}| \right)^2 \\ &\leqq \frac{4e^{2\overline{C}}}{\pi A(\Omega_{n,0})}\rho_n \left(\int_{\Omega_{n,0}} |u_{n,1} - u_{n,2}| \right)^2 \leqq \frac{4e^{2\overline{C}}}{\pi}\rho_n \int_{\Omega_{n,0}} |u_{n,1} - u_{n,2}|^2,\end{aligned}$$

where $\overline{u}_n$ is a suitable function which satisfies $\overline{u}_n \in (\min\{u_{n,1}, u_{n,2}\}, \max\{u_{n,1}, u_{n,2}\})$.

Collecting these estimates we conclude that

$$\int_{\Omega_{n,0}} |u_{n,1} - u_{n,2}|^2 \leqq \lambda_n \rho_n \frac{8e^{2\overline{C}}}{\pi} \int_{\Omega_{n,0}} |u_{n,1} - u_{n,2}|^2,$$

which is of course a contradiction to (2.11). This contradiction shows that in fact there exists at most one solution under the given assumptions and concludes the proof of part (b) of the statement.

As for part (a) it is easy to adapt the argument by contradiction used above just by replacing the assumption of divergent isoperimetric ratio in (2.6) with that of the existence of $\rho_n \searrow 0^+$ and $0 < \beta_{-,n} \leqq \beta_{+,n} < +\infty$ such that

$$\mathbb{E}_{n,-} := \{\rho_n^2 x^2 + y^2 \leqq \beta_{-,n}^2\} \subseteq \Omega_{n,0} \subseteq \{\rho_n^2 x^2 + y^2 \leqq \beta_{+,n}^2\}$$
$$=: \mathbb{E}_{n,+}, \quad \frac{\beta_{-,n}^2}{\beta_{+,n}^2} = c, \quad \forall\, n \in \mathbb{N}.$$

In particular we see that this time we already have (by assumption) the needed concentric omotetic ellipses [as in (2.8)] which in this case satisfy

$$\frac{A(\mathbb{E}_{n,+})}{A(\mathbb{E}_{n,-})} = \frac{\beta_{+,n}^2}{\beta_{-,n}^2} = c.$$

At this point, since of course Lemma 2.1 can be applied to the situation at hand, the proof can be worked out as above with minor changes. □

3. Solutions of Supercritical Mean Field Equations on Thin Domains

In this section we prove Theorem 1.7. Indeed, we will construct a branch of solutions of $P(\lambda, \Omega_\rho)$ which for ρ small enough extends up to some value $\lambda_\rho \geqq \frac{4\pi}{7\rho}$, and more generally we obtain the same statement on any domain Ω lying between two concentric and similar "thin" ellipses. In particular we recover the result for convex domains having a large isoperimetric ratio. To achieve our goal, we consider the auxiliary problem $Q(\mu, \Omega)$ (see §1.1) and make use of a well known result [31] whose statement calls up for the following:

Definition 3.1. A function u is said to be a subsolution(supersolution) of $Q(\mu, \Omega)$ if $u \in C^0(\overline{\Omega})$ and

$$\begin{cases} \int_\Omega (-\Delta\varphi)u \leqq (\geqq)\mu\, e^u \varphi & \text{in} \quad \Omega \\ u \leqq (\geqq) 0 & \text{on} \quad \partial\Omega \end{cases}, \qquad \forall\, \varphi \in C_0^\infty(\Omega), \varphi \geqq 0. \tag{3.1}$$

Theorem 3.2. (Sub-Supersolutions method, [31]). *Let Ω be simple. Suppose that, for fixed $\mu > 0$, there exist a subsolution $\underline{u}_\mu$ and a supersolution $\overline{u}_\mu$ of $Q(\mu, \Omega)$. If $\underline{u}_\mu \leqq \overline{u}_\mu$ in Ω, then $Q(\mu, \Omega)$ admits a classical solution $u = u_\mu \in C^2(\Omega) \cap C^0(\overline{\Omega})$ which moreover satisfies $\underline{u}_\mu \leqq u_\mu \leqq \overline{u}_\mu$.*

Proof. We use the existence Theorem in [31], where the domain Ω is just assumed to be regular with respect to the Laplacian (see [40], p. 25). It is well known that any simple domain satisfies this assumption (see [40], p. 26). Therefore we can apply the result in [31] which yields the existence of a function $u_\mu \in C^0(\overline{\Omega})$ which satisfies $\underline{u}_\mu \leqq u_\mu \leqq \overline{u}_\mu$ and moreover satisfies (3.1) for all $\varphi \in C_0^\infty(\Omega)$ with the equality sign replacing the corresponding inequalities. Hence in particular u_μ is a distributional solution of the equation in $Q(\mu, \Omega)$. Therefore the Brezis-Merle [17] theory of distributional solutions of Liouville type equations shows that it is also locally bounded and then standard elliptic regularity theory shows that $u_\mu \in C^2(\Omega)$ is a classical solution of $Q(\mu, \Omega)$ as well. We insist about the fact that the continuity up to the boundary is a byproduct of the result in [31], which indeed yields a distributional solution $u_\mu \in C^0(\overline{\Omega})$. □

The proof of Theorem 1.19. For fixed $c \in (0,1]$ and in view of Remark 1.6 we can assume without loss of generality that

$$\Omega_{\rho,c} := \{\rho^2 x^2 + y^2 \leqq c\} \subseteq \Omega \subseteq \{\rho^2 x^2 + y^2 \leqq 1\} =: \Omega_\rho.$$

Let us define

$$v_{\rho,\gamma} = 2\log\left(\frac{1+\gamma^2}{1+\gamma^2(\rho^2x^2+y^2)}\right), \quad (x,y) \in \Omega_\rho. \tag{3.2}$$

A straightforward evaluation shows that $v_{\rho,\gamma}$ satisfies

$$\begin{cases} -\Delta v_{\rho,\gamma} = V_{\rho,\gamma} e^{v_{\rho,\gamma}} & \text{in} \quad \Omega_\rho \\ v_{\rho,\gamma} = 0 & \text{on} \quad \partial\Omega_\rho, \end{cases} \tag{3.3}$$

where

$$V_{\rho,\gamma}(x,y) = \frac{4\gamma^2}{(1+\gamma^2)^2}\Big(1+\rho^2+\gamma^2(1-\rho^2)(\rho^2x^2-y^2)\Big) \tag{3.4}$$

Since

$$V_{\rho,\gamma}(x,y) \geqq g_+(\gamma,\rho) := \frac{4\gamma^2}{(1+\gamma^2)^2}\Big(1+\rho^2+\gamma^2(\rho^2-1)\Big), \quad \forall (x,y) \in \Omega_\rho,$$

we easily verify that $v_{\rho,\gamma}$ is a classical supersolution and in particular a supersolution (according to the above definition) of $Q(\mu,\Omega)$ whenever

$$\mu \leqq g_+(\gamma,\rho). \tag{3.5}$$

For fixed $\rho \in (0,1)$, the function $h_\rho(t) = g_+(\sqrt{t},\rho)$ satisfies $h_\rho(0) = 0 = h_\rho\left(\frac{1+\rho^2}{1-\rho^2}\right)$, is strictly increasing in $\left(0, \frac{1+\rho^2}{3-\rho^2}\right)$ and strictly decreasing in $\left(\frac{1+\rho^2}{3-\rho^2}, \frac{1+\rho^2}{1-\rho^2}\right)$. Therefore, putting $\overline{\gamma}_\rho^2 = \frac{1+\rho^2}{3-\rho^2}$ and $\overline{\mu}_\rho := h_\rho\left(\overline{\gamma}_\rho^2\right) \equiv g_+(\overline{\gamma}_\rho,\rho) \equiv \frac{(\rho^2+1)^2}{2}$, we see in particular that for each $\mu \in (0,\overline{\mu}_\rho]$ there exists a unique $\gamma_\rho^+ \in \big(0,\overline{\gamma}_\rho\big]$ such that $g_+(\gamma_\rho^+,\rho) = \mu$ and v_{ρ,γ_ρ^+} is a supersolution of $Q(\mu,\Omega)$. Indeed we have

$$\left(\gamma_\rho^+\right)^2 = \left(\gamma_\rho^+(\mu)\right)^2 = \frac{2(1+\rho^2)-\mu-2\sqrt{(1+\rho^2)^2-2\mu}}{\mu+4(1-\rho^2)}.$$

On the other hand let us consider

$$v_{\rho,\gamma,c} = \begin{cases} 2\log\left(\frac{1+\gamma^2}{1+\frac{\gamma^2}{c}(\rho^2x^2+y^2)}\right), & (x,y) \in \Omega_{\rho,c} \\ 0, & (x,y) \in \Omega\backslash\Omega_{\rho,c}. \end{cases} \tag{3.6}$$

Again a straightforward computation shows that $v_{\rho,\gamma,c}$ satisfies

$$\begin{cases} -\Delta v_{\rho,\gamma,c} = V_{\rho,\gamma,c} e^{v_{\rho,\gamma,c}} & \text{in} \quad \Omega_{\rho,c} \\ v_{\rho,\gamma,c} = 0 & \text{on} \quad \partial\Omega_{\rho,c}, \end{cases}$$

where

$$V_{\rho,\gamma,c}(x,y)=\begin{cases}\frac{4\gamma^2}{c(1+\gamma^2)^2}\left(1+\rho^2+\frac{\gamma^2}{c}(1-\rho^2)(\rho^2x^2-y^2)\right) & \text{in } \Omega_{\rho,c}\\ 0 & \text{in } \Omega\setminus\Omega_{\rho,c}.\end{cases}$$

Since

$$V_{\rho,\gamma,c}(x,y)\leqq g_-(\gamma,\rho,c):=\frac{4\gamma^2}{c(1+\gamma^2)^2}\left(1+\rho^2+\gamma^2(1-\rho^2)\right),\quad \forall(x,y)\in\Omega,$$

it is not difficult to check that $v_{\rho,\gamma,c}$ is a subsolution (according to the above definition) of $Q(\mu,\Omega)$ whenever

$$\mu\geqq g_-(\gamma,\rho,c). \tag{3.7}$$

For fixed $\rho\in(0,1)$, the function $f_{\rho,c}(t)=g_-(\sqrt{t},\rho,c)$, $t\in(0,\overline{\gamma}_\rho^2]$ is strictly increasing and satisfies $f_{\rho,c}(t)>h_\rho(t)$. Therefore, for each $\mu\in(0,\overline{\mu}_\rho]$ there exists a unique $\gamma^-_{\rho,c}\in\left(0,\overline{\gamma}_\rho\right)$ such that $g_-(\gamma^-_{\rho,c},\rho,c)=\mu$, $\gamma^-_{\rho,c}<\gamma^+_\rho$ and $v_{\rho,\gamma^-_{\rho,c},c}$ is a subsolution of $Q(\mu,\Omega)$. Indeed we have

$$\left(\gamma^-_{\rho,c}\right)^2=\left(\gamma^-_{\rho,c}(\mu)\right)^2=\frac{\mu c-2(1+\rho^2)+2\sqrt{(1+\rho^2)^2-2\rho^2\mu c}}{4(1-\rho^2)-\mu c}.$$

In conclusion, since $\gamma^-_{\rho,c}(\mu)\leqq\gamma^+_\rho(\mu)$ implies $v_{\rho,\gamma^-_{\rho,c},c}\leqq v_{\rho,\gamma^+_\rho}$, for fixed $\rho\in(0,1)$ and for each $\mu\in(0,\overline{\mu}_\rho]$ we can set

$$\underline{u}_\mu=v_{\rho,\gamma^-_{\rho,c}(\mu),c},\quad \overline{u}_\mu=v_{\rho,\gamma^+_\rho(\mu)},$$

to obtain (through Theorem 3.2) a solution $u_{\rho,\mu,c}$ for $Q(\mu,\Omega)$ which satisfies

$$v_{\rho,\gamma^-_{\rho,c}(\mu),c}\leqq u_{\rho,\mu,c}\leqq v_{\rho,\gamma^+_\rho(\mu)},\quad \forall(x,y)\in\Omega. \tag{3.8}$$

Any such a solution $u_{\rho,\mu,c}$ therefore solves $P(\lambda,\Omega)$ with $\lambda=\lambda_{\rho,c}(\mu)$ satisfying

$$\lambda=\lambda_{\rho,c}(\mu)=\mu\int_\Omega e^{u_{\rho,\mu,c}}\geqq\mu\int_{\Omega_{\rho,c}}e^{v_{\rho,\gamma^-_{\rho,c}(\mu),c}}=\mu c\frac{\pi}{\rho}\,(1+(\gamma^-_{\rho,c}(\mu))^2), \tag{3.9}$$

and

$$\lambda=\lambda_{\rho,c}(\mu)=\mu\int_\Omega e^{u_{\rho,\mu,c}}\leqq\mu\int_{\Omega_\rho}e^{v_{\rho,\gamma^+_\rho(\mu)}}=\mu\frac{\pi}{\rho}\,(1+(\gamma^+_\rho(\mu))^2). \tag{3.10}$$

In the particular case $\mu=\overline{\mu}_\rho$ we have $(\gamma^-_{\rho,c}(\overline{\mu}_\rho))^2\equiv\underline{\gamma}^2_{\rho,c}=(1+\rho^2)\frac{c-4+c\rho^2+4\sqrt{1-c\rho^2}}{8(1-\rho^2)-c(1+\rho^2)^2}$, $\underline{\gamma}^2_{\rho,c}<\overline{\gamma}^2_\rho$, $(\gamma^+_\rho(\overline{\mu}_\rho))^2\equiv\overline{\gamma}^2_\rho=(1+\rho^2)\frac{3-\rho^2}{8(1-\rho^2)+(1+\rho^2)^2}$ and $u_{\rho,\overline{\mu}_\rho,c}$ is a solution for $P(\lambda_{\rho,c}(\overline{\mu}_\rho),\Omega)$, where

$$\lambda_{\rho,c}:=\lambda_{\rho,c}(\overline{\mu}_\rho)\geqq\underline{\lambda}_{\rho,c}=\frac{c(1+\rho^2)^2}{2}\frac{\pi}{\rho}(1+\underline{\gamma}^2_{\rho,c})\simeq\frac{4\pi c}{(8-c)\rho}, \tag{3.11}$$

and

$$\lambda_{\rho,c} := \lambda_{\rho,c}(\overline{\mu}_\rho) \leqq \overline{\lambda}_\rho = \frac{(1+\rho^2)^2}{2}\frac{\pi}{\rho}(1+\overline{\gamma}_\rho^2) \simeq \frac{11\pi}{16\rho} \tag{3.12}$$

as $\rho \to 0^+$. Moreover it is easy to verify that $\underline{\lambda}_{\rho,c}$ is strictly decreasing at least for $\rho \in (0, \frac{1}{2\sqrt{10}}]$ and that there exists $\underline{\rho}_*(c) < \frac{1}{2\sqrt{10}}$ such that $\underline{\lambda}_{\rho,c} \geqq 8\pi$ for any $\rho \in (0, \underline{\rho}_*(c)]$. We also see that $\overline{\lambda}_\rho \to 4\pi^-$ as $\rho \to 1^-$, is strictly decreasing for $\rho \in (0, \rho_p]$ and strictly increasing for $\rho \in [\rho_p, 1)$ for some $\rho_p \simeq 0.5$ and then it is straightforward to check that there exists $\overline{\rho}_* > \underline{\rho}_*(c)$ satisfying $0.0702 < \overline{\rho}_* < 0.0703$ such that $\overline{\lambda}_\rho \geqq 8\pi$ for any $\rho \in (0, \overline{\rho}_*]$.

Finally, since $\lambda_{\rho,c}(\mu)$ is continuous in μ and by using (3.9) and (3.10)

$$0 < \lambda_{\rho,c}(\mu) \leqq \mu\frac{\pi}{\rho}\left(1+(\gamma_\rho^+(\mu))^2\right) \overset{\text{as } \mu\to 0}{\longrightarrow} 0,$$

we obtain the existence of a solution for $P(\lambda, \Omega)$ not only for $\lambda = \lambda_{\rho,c}$, but for any $\lambda \in (0, \lambda_{\rho,c}]$ as well. □

The proof of Theorem 1.19. If $\bar{N}$ exists, then it must be strictly greater than 4π, since any ball is of first kind. In view of Remark 1.6 we can assume without loss of generality that $L(\partial\Omega) = 1$. Let E_1 be the John maximal ellipse of Ω, then by Theorem 1.11 $E_2 := \{c_0 + 2(x - c_0) : x \in E_1\}$, where c_0 is the center of E_1, contains Ω. Again by using Remark 1.6 we can also assume that $c_0 = 0$ and in particular that E_1 and E_2 have the following form

$$E_1 = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}, \quad E_2 = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4\right\},$$

where clearly we can suppose that $0 < b \leqq a$.

By virtue of Ramanujan's estimate of the perimeter of the ellipse [76], namely:

$$L(\partial E_1) \geqq \pi\left\{(a+b) + \frac{3(a-b)^2}{10(a+b) + \sqrt{a^2 + 14ab + b^2}}\right\},$$

being $E_1 \subset \Omega$, Ω convex, and since $N(\Omega) = \frac{L^2(\partial\Omega)}{A(\Omega)}$, we derive the following inequalities:

$$1 = L(\partial\Omega) \geqq L(\partial E_1) \geqq (a+b)\pi; \quad \frac{1}{N(\Omega)} = \frac{A(\Omega)}{L^2(\partial\Omega)} = A(\Omega) \geqq A(E_1) = \pi ab. \tag{3.13}$$

Moreover since $\Omega \subset E_2 \subset R_{a,b} := \{(x, y) \in \mathbb{R}^2 \mid |x| \leqq 2a,\ |y| \leqq 2b\}$ we get

$$1 = L(\partial\Omega) \leqq L(\partial E_2) \leqq L(R_{a,b}) = 8(a+b), \tag{3.14}$$

and by using Theorem 1.12

$$\frac{1}{N(\Omega)} = A(\Omega) \leqq \frac{3\sqrt{3}}{\pi}A(E_1) = 3\sqrt{3}ab. \tag{3.15}$$

To simplify the notation we set $N = N(\Omega)$. Collecting (3.13), (3.14) and (3.15) we have

$$\begin{cases} \frac{1}{3\sqrt{3}N} \leqq ab \leqq \frac{1}{\pi N} \\ -b + \frac{1}{8} \leqq a \leqq -b + \frac{1}{\pi}, \end{cases} \tag{3.16}$$

which in turn implies

$$\begin{cases} b^2 - \frac{b}{\pi} + \frac{1}{3\sqrt{3}N} \leqq 0 \\ b^2 - \frac{b}{8} + \frac{1}{\pi N} \geqq 0. \end{cases}$$

It is worth to notice that, since $a \geqq b$ and $ab \leqq \frac{1}{\pi N}$, if $N > \frac{64}{\pi}$ then $b < \frac{1}{8}$. Therefore solving the above system of inequalities, with $N > \frac{64}{\pi}$, we get

$$\frac{1 - \sqrt{1 - \frac{4\pi^2}{3\sqrt{3}N}}}{2\pi} \leqq b \leqq \frac{1 - \sqrt{1 - \frac{256}{\pi N}}}{16}.$$

Next, for $N > \frac{512}{\pi}$, considering the Taylor formula of the square root and estimating the second order reminder we derive

$$\frac{\pi}{3\sqrt{3}N} \leqq \frac{\frac{2\pi^2}{3\sqrt{3}N} + \frac{1}{8}(\frac{4\pi^2}{3\sqrt{3}N})^2}{2\pi} \leqq b \leqq \frac{\frac{128}{\pi N} + \frac{1}{2\sqrt{2}}(\frac{256}{\pi N})^2}{16} = \frac{8}{\pi N} + \frac{1024\sqrt{2}}{\pi^2 N^2}, \tag{3.17}$$

thus

$$\frac{1}{8} - \frac{8}{\pi N} - \frac{1024\sqrt{2}}{\pi^2 N^2} \leqq a \leqq \frac{1}{\pi} - \frac{\pi}{3\sqrt{3}N}. \tag{3.18}$$

Combining (3.17) and (3.18), we have

$$\psi(N) := \frac{\pi^2}{3\sqrt{3}N - \pi^2} \leqq \frac{b}{a} \leqq \frac{64 + \frac{8192\sqrt{2}}{\pi N}}{\pi N - 64 - \frac{8192\sqrt{2}}{\pi N}} =: \varphi(N).$$

By definition of E_1 and E_2 we are in position to apply point (a) of this theorem with $c = \frac{1}{4}$. Let us fix $\bar{N}$ such that $\frac{64+\frac{8192\sqrt{2}}{\pi \bar{N}}}{\pi \bar{N} - 64 - \frac{8192\sqrt{2}}{\pi \bar{N}}} = \underline{\rho}_*(\frac{1}{4})$. We point out that since $\underline{\rho}_*(\frac{1}{4}) \simeq 0,0161$, $\bar{N} > \frac{512}{\pi}$.

Then, for any $N \geqq \bar{N}$, $\rho_N := \frac{b}{a} \leqq \underline{\rho}_*(\frac{1}{4})$ and so we get the existence of a solution $u^{(\lambda)}$ to $P(\lambda, \Omega)$ for any $\lambda \leqq \lambda_N$ where

$$\underline{\Lambda}_N := \underline{\lambda}_{\varphi(N), \frac{1}{4}} \leqq \underline{\lambda}_{\rho_N, \frac{1}{4}} < \lambda_N < \overline{\lambda}_{\rho_N} \leqq \overline{\lambda}_{\psi(N)} =: \overline{\Lambda}_N.$$

At last from (3.11) and (3.12) we obtain the desired estimates on $\underline{\Lambda}_N$ and $\overline{\Lambda}_N$:

$$\underline{\Lambda}_N \simeq \frac{\pi^2 N}{496} + O(1) \qquad \overline{\Lambda}_N \simeq \frac{33\sqrt{3}N}{16\pi} + O(1) \quad \text{as } N \to +\infty.$$

□

4. The Eigenvalue Problem

The aim of this section is to prove Proposition 4.1 below which yields positivity of the first eigenvalue for the linearization of $P(\lambda, \Omega)$. Among other things, with the aid of Proposition 4.1 we have:

The proof of Theorem 1.19. Let $\mathcal{G}_{\rho,c}, \mathcal{G}_N$ denote the set of pairs of parameter-solutions for $P(\lambda, \Omega)$ found in Theorem 1.7. Since the linearized problem for $P(\lambda, \Omega)$ corresponds to the kernel equation for the second variation of J_λ, then the conclusions of Proposition 1.14 are an immediate consequence of Proposition 4.1 below and the uniqueness results in [20]. □

Putting

$$\omega = \omega(u) = \frac{e^u}{\int_\Omega e^u}, \quad \text{and} \quad < f >_\omega = \int_\Omega \omega(u) f,$$

then the linearized problem for $P(\lambda, \Omega)$ takes the form

$$\begin{cases} -\Delta\varphi - \lambda\omega(u)\varphi + \lambda\omega(u) < \varphi >_\omega = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Proposition 4.1. *For fixed $c \in (0, 1]$, let Ω be a regular domain such that $\{\rho^2 x^2 + y^2 \leqq \beta_-^2\} \subset \Omega \subset \{\rho^2 x^2 + y^2 \leqq \beta_+^2\}$, with $\frac{\beta_-^2}{\beta_+^2} = c$. For any $\rho \in (0, \underline{\rho}_*(c)]$ let $u = u^{(\lambda)} \equiv u_{\rho,\mu,c}$ be a solution of $P(\lambda, \Omega)$ and of $Q(\mu, \Omega)$ with $\lambda = \mu \int_\Omega e^u$ as obtained in Theorem 1.7(a) for $\lambda \in [0, \lambda_{\rho,c}]$. Then (4.1) has only the trivial solution and in particular the first eigenvalues of the linearized problems for $P(\lambda, \Omega)$ and $Q(\mu, \Omega)$ at $u = u^{(\lambda)} \equiv u_{\rho,\mu,c}$ respectively are strictly positive.*

Moreover, let Ω be a regular and convex domain with $N(\Omega) > \bar{N}$ as defined in Theorem 1.7(b) and $u^{(\lambda)}$ be a solution of $P(\lambda, \Omega)$ and of $Q(\mu, \Omega)$ for $0 \leqq \lambda = \mu \int_\Omega e^{u^{(\lambda)}} \leqq \lambda_N$ as obtained therein. Then the first eigenvalues of the linearized problems for $P(\lambda, \Omega)$ and $Q(\mu, \Omega)$ at $u = u^{(\lambda)}$ are strictly positive.

Remark 4.2. As far as one is concerned with problem $Q(\mu, \Omega)$, then it is well known (see for example [70]) that it is well defined (and unique) that the extremal (classical) solution v_* which corresponds to the extremal value μ_* such that the bifurcation diagram has a bending point at (μ_*, v_*). In particular the first eigenvalue of the linearized problem for $Q(\mu, \Omega)$ is zero at μ_*.

The reasons why we have strictly positive first eigenvalues for $\lambda \leqq \lambda_{\rho,c}$ are:

(–) As it will be shown in the proof below, the first eigenvalue of the linearized problem for $P(\lambda, \Omega)$ (say τ_1) is always greater or equal to the first eigenvalue (which we will denote by ν_0) of the linearized problem for $Q(\mu, \Omega)$ and we will use the latter to estimate both;
(–) The value of μ corresponding to $\lambda_{\rho,c}$, which is defined implicitly via $\lambda = \mu \int_{\Omega_\rho} e^{u_\mu}$, is less than μ_*.

Proof of Proposition 1.14 We will use the fact that (see [17,20] and Remark 1.2 above) if u solves $P(\lambda, \Omega)$ then there exists $C = C(\Omega, \lambda, u) > 0$ such that

$$\frac{1}{C} \leqq \omega(u) \leqq C.$$

Letting $H \equiv H_0^1(\Omega)$ and

$$\mathcal{L}(\phi, \psi) = \int_\Omega (\nabla\phi \cdot \nabla\psi) - \lambda \int_\Omega \omega(u)\phi\psi \\ +\lambda \left(\int_\Omega \omega(u)\phi\right)\left(\int_\Omega \omega(u)\psi\right), \ (\phi, \psi) \in H \times H,$$

then by definition $\varphi \in H$ is a weak solution of (4.1) if

$$\mathcal{L}(\varphi, \psi) = 0, \quad \forall\, \psi \in H.$$

We define $\tau \in \mathbb{R}$ to be an eigenvalue of the operator

$$L[\varphi] := -\Delta\varphi - \lambda\omega(u)(\varphi - <\varphi>_\omega), \quad \varphi \in H,$$

if there exists a weak solution $\phi_0 \in H \setminus \{0\}$ of the linear problem

$$-\Delta\phi_0 - \lambda\omega(u)\phi_0 + \lambda\omega(u) <\phi_0>_\omega = \tau\omega(u)\phi_0 \ \text{ in } \ \Omega, \tag{4.2}$$

that is, if

$$\mathcal{L}(\phi_0, \psi) = \tau \int_\Omega \omega(u)\phi_0\psi, \quad \forall\, \psi \in H.$$

Standard arguments show that the eigenvalues form an unbounded (from above) sequence

$$\tau_1 \leqq \tau_2 \leqq \cdots \leqq \tau_n \cdots,$$

with finite dimensional eigenspaces (although the first eigenfunction changes sign and cannot be assumed to be simple in this situation).

Let us define

$$Q(\phi) = \frac{\mathcal{L}(\phi, \phi)}{<\phi^2>_\omega} = \frac{\int_\Omega |\nabla\phi|^2 - \lambda <\phi^2>_\omega + \lambda <\phi>_\omega^2}{<\phi^2>_\omega}, \quad \phi \in H.$$

In particular it is not difficult to prove that the first eigenvalue can be characterized as follows

$$\tau_1 = \inf\{Q(\phi) \,|\, \phi \in H \setminus \{0\}\}.$$

At this point we argue by contradiction and assume that (4.1) admits a non trivial solution. Hence, in particular, $\tau_1 \leqq 0$ and we readily conclude that

$$\tau_0 := \inf\{Q_0(\phi) \,|\, \phi \in H \setminus \{0\}\} \leqq 0, \quad \text{where} \quad Q_0(\phi) = \frac{\mathcal{L}_0(\phi, \phi)}{<\phi^2>_\omega}$$

and

$$\mathcal{L}_0(\phi,\psi)=\int_\Omega(\nabla\phi\cdot\nabla\psi)-\lambda\int_\Omega\omega(u)\phi\psi,\ \ (\phi,\psi)\in H\times H.$$

Clearly τ_0 is attained by a simple and positive eigenfunction φ_0 which satisfies

$$\begin{cases}-\Delta\varphi_0-\lambda\omega(u)\varphi_0=\tau_0\omega(u)\varphi_0 & \text{in }\ \Omega\\ \varphi_0=0 & \text{on }\ \partial\Omega.\end{cases}\tag{4.3}$$

Let us recall that we have obtained solutions for $P(\lambda,\Omega)$ as solutions of $Q(\mu,\Omega)$ in the form $u=u_{\rho,\mu,c}$, for some $\mu=\mu(\rho)\leqq\overline{\mu}_\rho$ whose value of $\lambda=\lambda(\mu,\rho,c)$ was then estimated as a function of ρ. Therefore, at this point, it is more convenient to look at the linearized problem in the other way, that is, to go back to $\mu=\lambda\left(\int_\Omega e^u\right)^{-1}$. Hence, let us observe that for a generic value $\mu\leqq\overline{\mu}_\rho$ (4.3) takes the form

$$\begin{cases}-\Delta\varphi_0-\mu K_{\rho,\mu,c}\varphi_0=\nu_0K_{\rho,\mu,c}\varphi_0 & \text{in }\ \Omega\\ \varphi_0=0 & \text{on }\ \partial\Omega,\end{cases}\tag{4.4}$$

where

$$K_{\rho,\mu,c}=e^{u_{\rho,\mu,c}}\quad\text{and}\quad\nu_0=\mu\frac{\tau_0}{\lambda}\leqq 0.$$

Remark 4.3. Of course, the assertion about the positivity of the first eigenvalues corresponds to the positivity of τ_1 and ν_0 respectively. Therefore that part of the statement will be automatically proved once we get the desired contradiction.

Since also the linearized problem (4.1) is rotational, translational and dilation invariant, by arguing exactly as in the proof of Theorem 1.7 we can assume without loss of generality that

$$\Omega_{\rho,c}:=\{\rho^2x^2+y^2\leqq c\}\subset\Omega\subset\{\rho^2x^2+y^2\leqq 1\}=:\Omega_\rho.$$

We observe that, by defining

$$K^{(-)}_{\rho,\mu,c}:=e^{v_{\rho,\gamma^-_{\rho,c}(\mu),c}}=\begin{cases}\left(\dfrac{1+\gamma^-_{\rho,c}(\mu)^2}{1+\frac{\gamma^-_{\rho,c}(\mu)^2}{c}(\rho^2x^2+y^2)}\right)^2 & (x,y)\in\Omega_{\rho,c}\\ 1 & (x,y)\in\Omega\backslash\Omega_{\rho,c},\end{cases}$$

$$K^{(+)}_{\rho,\mu}:=e^{v_{\rho,\gamma^+_\rho(\mu)}}=\left(\frac{1+\gamma^+_\rho(\mu)^2}{1+\gamma^+_\rho(\mu)^2(\rho^2x^2+y^2)}\right)^2,\qquad(x,y)\in\Omega_\rho$$

we have

$$K^{(-)}_{\rho,\mu,c}\leqq K_{\rho,\mu,c}\leqq K^{(+)}_{\rho,\mu}\qquad\text{for any }(x,y)\in\Omega.$$

In particular, since

$$K^{(+)}_{\rho,\mu}\leqq(1+\gamma^+_\rho(\mu)^2)^2\quad\text{and}\quad 1\leqq K^{(-)}_{\rho,\mu,c}\leqq(1+\gamma^-_{\rho,c}(\mu)^2)^2\quad\text{in}\quad\Omega,$$

and

$$\Omega \subset T_\rho := \{(x, y) \in \mathbb{R}^2 \,|\, |x| \leqq (\rho)^{-1}, \ |y| \leqq 1\}, \tag{4.5}$$

then, by using the fact that

$$\nu_0 = \inf \left\{ \frac{\int_\Omega |\nabla \varphi|^2 \, dx - \mu \int_\Omega K_{\rho,\mu} \varphi^2 \, dx}{\int_\Omega K_{\rho,\mu} \varphi^2 \, dx} \,\middle|\, \varphi \in H \right\} \leqq 0,$$

it is not difficult to check that, for any $\mu \leqq \overline{\mu}_\rho = \frac{(1+\rho^2)^2}{2}$, the following inequality holds:

$$\inf \left\{ \frac{\int_{T_\rho} |\nabla \varphi|^2 \, dx - \mu(1 + \gamma_\rho^+(\mu)^2)^2 \int_{T_\rho} \varphi^2 \, dx}{\int_{T_\rho} \varphi^2 \, dx} \,\middle|\, \varphi \in H \right\} \leqq 0. \tag{4.6}$$

Hence, there exists $\overline{\mu}_0 \leqq 0$ such that, putting $\sigma = \sigma(\mu, \rho) = \mu(1+\gamma_\rho^+(\mu)^2)^2 + \overline{\mu}_0$, there exists a weak solution $\phi_0 \in H$ of

$$\begin{cases} -\Delta \phi_0 - \sigma \phi_0 = 0 & \text{in } T_\rho, \\ \phi_0 = 0 & \text{on } \partial T_\rho. \end{cases} \tag{4.7}$$

It is well known that the minimal eigenvalue $\sigma_{\min}$ of (4.7) satisfies $\sigma_{\min} = \frac{\pi^2}{4}\rho^2 + \frac{\pi^2}{4} > 2(1 + \rho^2)$ and we conclude that

$$2(1 + \rho^2) \leqq \sigma(\mu, \rho) = \mu(1 + \gamma_\rho^+(\mu)^2)^2 + \overline{\mu}_0. \tag{4.8}$$

Next, since $\underline{\rho}_*(c) < \frac{1}{2\sqrt{10}}$, it is not difficult to check that $\sigma = \sigma(\mu, \rho)$ satisfies

$$\sigma(\mu, \rho) \leqq 1,$$

for any $\rho \leqq \underline{\rho}_*(c)$, which is of course a contradiction to (4.8). This fact concludes the first part of the proof. As for the second one it can be derived by arguing as above with some minor changes as in the proof of Theorem 1.7(b). □

5. A Multiplicity Result

This section is devoted to the proof of Theorem 1.16.

The proof of Theorem 1.19. *(a).* Let us fix $\lambda \in (8\pi, \lambda_{\rho,c}) \setminus 8\pi\mathbb{N}$, then there exists $k \in \mathbb{N}^*$ such that $\lambda \in (8k\pi, 8(k + 1)\pi)$. Let us fix now k distinct points, $x_1, \ldots, x_k$, in the interior of $\Omega_{\rho,\beta_-} = \{\rho^2 x^2 + y^2 \leqq \beta_-^2\}$. Next we fix $\bar{d} > 0$ such that $\operatorname{dist}(x_i, x_j) > 4\bar{d}$ for any $i \neq j$ and such that $\operatorname{dist}(x_i, \partial\Omega_{\rho,\beta_-}) > 2\bar{d}$ for any $i \in \{1, \ldots, k\}$.

Following [36] we introduce some notations. For $d \in (0, \bar{d})$ we consider a smooth non-decreasing cut-off function $\chi_d : [0, +\infty) \to \mathbb{R}$ satisfying the following properties:

$$\begin{cases} \chi_d(t) = t & \text{for } t \in [0, d] \\ \chi_d(t) = 2d & \text{for } t \geqq 2d \\ \chi_d(t) \in [d, 2d] & \text{for } t \in [d, 2d]. \end{cases}$$

Then, given $\mu > 0$, we define the function $\varphi_{\mu,d} \in H_0^1(\Omega)$ by

$$\varphi_{\mu,d}(y) = \begin{cases} \log \sum_{j=1}^{k} \frac{1}{k}\left(\frac{8\mu^2}{(1+\mu^2\chi_d^2(|y-x_j|))^2}\right) - \log\left(\frac{8\mu^2}{(1+4d^2\mu^2)^2}\right) & y \in \Omega_{\rho,\beta_-} \\ 0 & y \in \Omega\setminus\Omega_{\rho,\beta_-}. \end{cases}$$

By arguing exactly as in Section 5 of [36] we have

$$F_\lambda(\varphi_{\mu,d}) \leqq (16k\pi - 2\lambda + o_d(1))\ln(\mu) + O(1) + C_d$$

where C_d is a constant independent of μ and $o_d(1) \to 0$ as $d \to 0$.

Then, there exist d_0 sufficiently small and μ_0 sufficiently large such that

$$F_\lambda(\varphi_{\mu_0,d_0}) < F_\lambda(u^{(\lambda)}) - 1.$$

Next we define

$$\mathcal{D} = \{\gamma : [0,1] \to H_0^1(\Omega) \,:\, \gamma \text{ is continuous}, \gamma(0) = u^{(\lambda)}, \gamma(1) = \varphi_{\mu_0,d_0}\}$$

and, for any $\eta \in (8k\pi, 8(k+1)\pi) \cap (8\pi, \lambda_{\rho,c})$, we set

$$c_\eta = \inf_{\gamma\in\mathcal{D}} \max_{s\in[0,1]} F_\eta(\gamma(s)).$$

Since $u^{(\lambda)}$ is a strict local minimum for F_λ, there exists $\varepsilon_\lambda > 0$ such that $c_\lambda \geqq F_\lambda(u^{(\lambda)}) + \varepsilon_\lambda$. Besides, since F_λ is continuous and the branch $\mathcal{G}_{\rho,c}$ is smooth, we have that a bound on the min-max levels applies uniformly in a small neighborhood of λ. More precisely the following straightforward fact holds true.

Lemma 5.1. *There exists $\lambda_0 > 0$ sufficiently small such that*

$$[\lambda - \lambda_0, \lambda + \lambda_0] \subset (8k\pi, 8(k+1)\pi) \cap (8\pi, \lambda_{\rho,c})$$

and for any $\eta \in [\lambda - \lambda_0, \lambda + \lambda_0]$ we have $F_\eta(\varphi_{\mu_0,d_0}) \leqq F_\eta(u^{(\lambda)}) - \frac{1}{2}$ and

$$c_\eta \geqq F_\lambda(u^{(\lambda)}) + \frac{3}{4}\varepsilon_\lambda \geqq F_\eta(u^{(\lambda)}) + \frac{1}{2}\varepsilon_\lambda.$$

If $\eta, \eta' \in (\lambda - \lambda_0, \lambda + \lambda_0)$, $\eta \leqq \eta'$, then $\frac{F_\eta}{\eta} - \frac{F'_\eta}{\eta'} = \frac{1}{2}\left(\frac{1}{\eta} - \frac{1}{\eta'}\right)\int_\Omega |\nabla u|^2 \geqq 0$, whence

$$\frac{c_\eta}{\eta} \geqq \frac{c_{\eta'}}{\eta'}. \tag{5.1}$$

Therefore we have that the function $\eta \mapsto \frac{c_\eta}{\eta}$ is non-increasing and in turn differentiable almost everywhere in $(\lambda - \lambda_0, \lambda + \lambda_0)$. Set

$$\Lambda = \left\{\eta \in (\lambda - \lambda_0, \lambda + \lambda_0) \mid \frac{c_\eta}{\eta} \text{ is differentiable at } \eta\right\}.$$

Lemma 5.2. *c_η is achieved by a critical point $v^{(\lambda)}$ of F_η provided that $\eta \in \Lambda$.*

Proof of Theorem 1.16 The proof is a step by step adaptation of the arguments of Lemma 3.2 of [35] where, with respect to their notations, we have just to choose $\delta < \frac{1}{4}\varepsilon_{\lambda}$. $\square$

Finally we state a (well known) compactness result for sequence of solutions of $P(\lambda_n, \Omega)$.

Lemma 5.3. *Let $\lambda_n \to \lambda$ and let $v^{(\lambda_n)} \in H_0^1(\Omega)$ be a solution of $P(\lambda_n, \Omega)$. If $\lambda \notin 8\pi\mathbb{N}$, then $v^{(\lambda_n)}$ admits a subsequence which converges smoothly to a solution $v^{(\lambda)}$ of $P(\lambda, \Omega)$.*

Proof of Theorem 1.16 In view of Lemma 2.1 in [20] $v^{(\lambda_n)}$ is uniformly bounded in a fixed neighborhood of the boundary. Hence the conclusion is a straightforward and well known consequence of the Brezis-Merle [17] concentration-compactness result as completed by Li and Shafrir [50]. $\square$

Now we are able to conclude the proof of Theorem 1.16(a). Indeed the thesis is an easy consequence of Lemmas 5.2 and 5.3, noticing that the solution $v^{(\lambda)}$, obtained by this procedure, does not coincide with $u^{(\lambda)}$, because by Lemma 5.1 $F_{\lambda}(v^{(\lambda)}) > F_{\lambda}(u^{(\lambda)})$.

(b). This part can be proved exactly as the previous one. $\square$

6. A Refined Estimate for Solutions on $\mathcal{G}_{\rho,1}$

Let $\mathcal{G}_{\rho,c}$, $\mathcal{G}_N$ denote the branches of parameter-solutions pairs of $P(\lambda, \Omega)$ found in Theorem 1.7. As a consequence of Theorem 1.19 and Proposition 4.1 we obtain the following:

Proposition 6.1. *Let $\overline{\lambda} \geqq 8\pi$, $\widetilde{\rho}_1$ and $\widetilde{N}$ be as in Theorem 1.19. Let either $\mathcal{G}^{(\overline{\lambda})} = \{(\lambda, u^{(\lambda)}) \in \mathcal{G}_{\rho,c} \,:\, \lambda \in [0, \overline{\lambda})\}$ or $\mathcal{G}^{(\overline{\lambda})} = \{(\lambda, u^{(\lambda)}) \in \mathcal{G}_N \,:\, \lambda \in [0, \overline{\lambda})\}$ denote that part of $\mathcal{G}_{\rho,c}, \mathcal{G}_N$ with $\lambda \in [0, \overline{\lambda})$, $\rho \in (0, \widetilde{\rho}_1]$ and $N \geqq \widetilde{N}$ respectively. Then the energy function*

$$\widehat{E}(\lambda) := \mathcal{E}(\omega(u^{(\lambda)})), \quad u^{(\lambda)} \in \mathcal{G}^{(\overline{\lambda})}, \tag{6.1}$$

is a monodrome and smooth function of $\lambda \in [0, \overline{\lambda})$.

Proof of Theorem 1.16 By using the explicit bounds (3.8) and the fact that

$$\mathcal{E}(\omega(u^{(\lambda)})) = \frac{1}{2\lambda} \int_{\Omega} \omega(u^{(\lambda)}) u^{(\lambda)},$$

then it is straightforward to show that the energy of any solution lying on $\mathcal{G}^{(\overline{\lambda})}$ is uniformly bounded from above by a suitable value $\overline{E}$, which we can assume without loss of generality to be larger than 1. Therefore Theorem 1.19 applies and we see that $\mathcal{E}(\omega(u^{(\lambda)}))$ is monodrome as a function of $\lambda \in [0, \overline{\lambda})$ and consequently $\widehat{E}(\lambda)$ is well defined. At this point Proposition 4.1 implies that it is smooth as well, see also Remark 1.15. $\square$

Our next aim is to improve Proposition 6.1 in the case $\Omega = \Omega_\rho$ to come up with a unique solution of $P(\lambda, \Omega_\rho)$ at fixed energy. Indeed, this is the content of Theorem 1.22 whose proof is the main aim of this section. To achieve this goal we have to pay a price in terms of a smallness assumption on the energy and indeed we will obtain this result by using Theorem 1.19 and the expansion of solutions as functions of ρ. Actually, we first need a more precise formula about the explicit form of solutions of $P(\lambda, \Omega_\rho)$ lying on $\mathcal{G}_{\rho,1}$, as claimed in (1.11) of Theorem 1.20. By using these expansions we will be able to calculate explicitly, at least for small ρ, their energy as a function of λ and then prove that $\widehat{E}$ is monotone. It turns out that this is enough to prove uniqueness of solutions with fixed energy. Actually we also provide another proof (still by using the sub-supersolutions method) of the existence of solutions for $P(\lambda, \Omega_\rho)$.

The Proof of Theorem 1.20. The notation $\mathrm{O}(\rho^m)$, $m \in \mathbb{N}$ will be used in the rest of this proof to denote various quantities uniformly bounded by $C_m \rho^m$ with $C_m > 0$ a suitable constant depending only on $\overline{\lambda}$.

Let us first seek solutions v_ρ of $Q(\mu_0\rho, \Omega_\rho)$ in the form

$$v_\rho = \rho\phi_0 + \rho^2\phi_{0,1}, \quad \phi_0, \phi_{0,1} \in C^2(\Omega_\rho) \cap C^0(\overline{\Omega_\rho}), \tag{6.2}$$

with the additional constraints

$$0 \leqq \|\phi_0\|_\infty \leqq M_0, \quad 0 \leqq \|\phi_{0,1}\|_\infty \leqq M_1.$$

Since v_ρ must satisfy $-\Delta v = \mu_0 \rho e^v$ then ϕ_0 and $\phi_{0,1}$ should be solutions of

$$\begin{cases} -\Delta\phi_0 = \mu_0 & \text{in } \ \Omega_\rho \\ \phi_0 = 0 & \text{on } \ \partial\Omega_\rho \end{cases} \tag{6.3}$$

and

$$\begin{cases} -\Delta\phi_{0,1} = \mu_0\rho^{-1}\left(e^{\rho\phi_0}e^{\rho^2\phi_{0,1}} - 1\right) & \text{in } \ \Omega_\rho \\ \phi_{0,1} = 0 & \text{on } \ \partial\Omega_\rho \end{cases} \tag{6.4}$$

respectively. Therefore the explicit expression of ϕ_0 is easily derived to be

$$\phi_0(x, y; \rho) = \frac{\mu_0}{2(1+\rho^2)}\left(1 - (\rho^2x^2 + y^2)\right), \quad (x, y) \in \Omega_\rho. \tag{6.5}$$

Please observe that the function $\phi_0(x, y; \lambda, \rho)$ as defined in (1.8) will be recognized to be $\phi_0(x, y; \rho)$ where $\mu_0 = \mu_0(\lambda, \rho)$.

Clearly

$$\|\phi_0\|_\infty = \frac{\mu_0}{2(1+\rho^2)},$$

and therefore, in particular we have

$$\forall\, t_0 > 1\, \exists\, \rho_1 = \rho_1(t_0) > 0 \,:\, e^{\rho\phi_0} \leqq e^{\frac{\mu_0\rho}{2(1+\rho^2)}} < 1 + t_0\frac{\mu_0\rho}{2}, \quad \forall\, \rho < \rho_1, \tag{6.6}$$

the last inequality being a trivial consequence of the convexity of $e^{\frac{\mu_0 s}{2(1+s^2)}}$ in a right neighborhood of $s = 0$.

Our next aim is to use the sub-supersolutions method to obtain solutions for (6.4). Let us define

$$f(t;\phi_0) := e^{\rho\phi_0} e^{\rho^2 t}, \quad t \geqq 0,$$

so that, in particular, we have

$$\forall\, t_1 > 1\, \exists\, \rho_2 > 0 \, : \, e^{\rho^2 t} < 1 + t_1 \rho^2 t, \quad \forall\, \rho < \rho_2, \tag{6.7}$$

with ρ_2 depending on t_1. By using (6.6) and (6.7) we conclude that

$$f(t;\phi_0) \leqq \left(1 + t_0 \frac{\mu_0 \rho}{2}\right)\left(1 + t_1 \rho^2 t\right), \quad \forall\, \rho < \min\{\rho_1, \rho_2\}.$$

Hence, setting

$$A_+ = 1 + t_0 \frac{\mu_0 \rho}{2},$$

we see that a supersolution ϕ_+ for (6.4) will be obtained whenever we will be able to solve the differential problem

$$\begin{cases} -\Delta\phi_+ \geqq t_0 \frac{\mu_0^2}{2} + t_1 \mu_0 A_+ \rho \phi_+ & \text{in} \quad \Omega_\rho \\ \phi_+ \geqq 0 & \text{on} \quad \partial\Omega_\rho \\ 0 \leqq \phi_+ \leqq M_1 & \text{in} \quad \Omega_\rho. \end{cases} \tag{6.8}$$

Let us define

$$\phi_+(x, y) = \frac{C_+}{2(1+\rho^2)}\left(1 - (\rho^2 x^2 + y^2)\right), \quad (x, y) \in \Omega_\rho,$$

with $C_+ > 0$, so that the differential inequality in (6.8) yields

$$-\Delta\phi_+ = C_+ = \frac{C_+}{2} + \frac{C_+}{2} = \frac{C_+}{2} + (1+\rho^2)\|\phi_+\|_\infty \geqq t_0 \frac{\mu_0^2}{2} + t_1 \mu_0 A_+ \rho \phi_+.$$

Therefore (6.8) will be satisfied whenever we can choose C_+ such that the following inequalities are verified

$$\begin{cases} C_+ & \geqq t_0 \mu_0^2 \\ (1+\rho^2) & \geqq t_1 \mu_0 A_+ \rho \\ C_+ & \leqq 2(1+\rho^2) M_1. \end{cases} \tag{6.9}$$

We first impose

$$C_+ = 2M_1,$$

so that the third inequality in (6.9) is automatically satisfied and then we substitute it in the first inequality obtaining

$$\mu_0^2 \leqq \min\left\{\frac{2M_1}{t_0}, (4M_0)^2\right\} = \frac{2M_1}{t_0}, \text{ for any } M_0 \text{ large enough.} \tag{6.10}$$

We conclude in particular that the second inequality is trivially satisfied for any ρ small enough. At this point Theorem 3.2 shows that there exists a solution v_ρ of $Q(\mu_0\rho, \Omega_\rho)$ taking the form (6.2), where ϕ_0 is defined as in (6.5) and $0 \leqq \phi_{0,1} \leqq M_1$ with the constraint (6.10).

Our next aim is to show that $\forall \overline{\lambda} \geqq 8\pi$ we can find ρ_0 small enough such that $\forall \rho < \rho_0$ and for any $\lambda < \overline{\lambda}$ we can choose μ_0 in such a way that v_ρ is a solution of $P(\lambda, \Omega_\rho)$. Indeed, we have

$$\lambda = \lambda_0(\mu_0, \rho) := \mu_0\rho \int_{\Omega_\rho} e^{v_\rho} = \pi\mu_0 + f_0(\mu_0, \rho), \quad \text{where} \quad |f_0(\mu_0, \rho)| \leqq C_{M_1}\rho, \tag{6.11}$$

where λ is a fixed value in the range of λ_0 and we have used $\|\phi_{0,1}\|_\infty \leqq M_1$ and

$$\int_{\Omega_\rho} e^{\rho\phi_0} = \left(1+\rho^2\right) \frac{2\pi}{\mu_0\rho^3} \left(e^{\frac{\mu_0\rho^2}{2(1+\rho^2)}} - 1 \right).$$

Lemma 6.2. $\lambda_0(\mu_0, \rho)$ *is smooth and in particular*

$$\frac{\partial \lambda_0}{\partial \mu_0}(\mu_0, \rho) = \pi + \mathrm{O}(\rho). \tag{6.12}$$

Proof of Theorem 1.16 It is straightforward to check that the energy of these solutions v_ρ is uniformly bounded from above by a suitable positive number $\overline{E}$ (possibly depending on M_1 and $\overline{\lambda}$) which we can assume without loss of generality to be larger than 1. Therefore Theorem 1.19 shows that they must coincide with some subset of the branch $\mathcal{G}^{(\overline{\lambda})}$ (see Proposition 6.1). We can use Proposition 4.1 (see Remark 4.2) at this point and conclude that $\lambda_0(\mu_0, \rho)$ is smooth as a function of μ_0. In particular standard arguments show that

$$\frac{\partial v_\rho}{\partial \mu_0} = \rho \frac{\partial \phi_0}{\partial \mu_0} + \rho^2 \frac{\partial \phi_{0,1}}{\partial \mu_0},$$

with $\frac{\partial \phi_0}{\partial \mu_0}$ and $\frac{\partial \phi_{0,1}}{\partial \mu_0}$ being both bounded in $L^\infty(\Omega_\rho)$ by some constants depending only on M_0, M_1. By using this fact, then (6.12) follows either by a straightforward evaluation or just by observing that then (6.11) holds in C^1 sense with respect to μ_0. At this point the (joint) regularity of $\lambda_0(\mu_0, \rho)$ as a function of μ_0 and ρ is derived by standard elliptic estimates. □

Hence, in particular we can always choose μ_0 and ρ_0 such that $\forall \rho < \rho_0$ we have [see (6.10)]

$$[0, \overline{\lambda}) \subset \lambda_0\left(\left[0, 2\sqrt{\frac{M_1}{t_0}}\right), \rho\right),$$

and since $\lambda_0(\mu_0, \rho)$ is also continuous, we finally obtain the desired solution for any $\lambda < \overline{\lambda}$.

At this point, let us fix a positive value $\lambda < \overline{\lambda}$ for which we seek an approximate solution u_λ of $P(\lambda, \Omega_\rho)$. As a consequence of (6.11), (6.12) we have

$$\mu_0 = \mu_0(\lambda, \rho) = \frac{\lambda}{\pi} + \mathrm{O}(\rho), \tag{6.13}$$

and then

$$\begin{aligned} u_\lambda := \rho\phi_0 + \rho^2\phi_{0,1} &= \frac{\rho\mu_0}{2\pi(1+\rho^2)}\left(1 - (\rho^2x^2 + y^2)\right)(1 + \mathrm{O}(\rho)) \\ &= \frac{\rho\lambda}{2\pi}\left(1 - (\rho^2x^2 + y^2)\right)(1 + \mathrm{O}(\rho)), \end{aligned} \tag{6.14}$$

is a solution for $P(\lambda, \Omega_\rho)$, as desired.

Remark 6.3. However, by using (1.10), (6.13) and (6.14), a straightforward explicit evaluation shows that

$$\begin{aligned} \mathcal{E}(\omega_\lambda) &= \frac{1}{2}\int_{\Omega_\rho} \omega_\lambda G_\rho[\omega_\lambda] = \frac{1}{2\lambda}\int_{\Omega_\rho} \omega_\lambda u_\lambda = \frac{\mu_0\rho}{2\lambda^2}\int_{\Omega_\rho} e^{u_\lambda}u_\lambda \\ &= \frac{\lambda\rho + \mathrm{O}(\rho^3)}{2\pi\lambda^2}\int_{\Omega_\rho} e^{u_\lambda}u_\lambda = \frac{\rho}{8\pi}\left(1 + \mathrm{O}(\rho)\right), \end{aligned}$$

see Remark 1.23. Therefore, as far as we are interested in the monotonicity of $\widehat{E}(\lambda)$, we see that the first order expansion is not enough to our purpose.

Hence we make a further step to come up with an expansion of $\mathcal{E}$ at order ρ^2. Let $\phi_{0,1}$ be the solution of (6.4) determined above, we write it as

$$\phi_{0,1} = \phi_1 + \rho\phi_2,$$

so that, if ϕ_1 is the unique solution of

$$\begin{cases} -\Delta\phi_1 = \mu_0\phi_0 = \mu_0^2\psi_0 & \text{in } \Omega_\rho \\ \phi_1 = 0 & \text{on } \partial\Omega_\rho \end{cases} \tag{6.15}$$

[see (1.8)–(1.9)] then by definition ϕ_2 is a solution for

$$\begin{cases} -\Delta\phi_2 = \mu_0\rho^{-2}\left(e^{\rho\phi_0}e^{\rho^2\phi_{0,1}} - 1 - \rho\phi_0\right) & \text{in } \Omega_\rho \\ \phi_2 = 0 & \text{on } \partial\Omega_\rho \end{cases} \tag{6.16}$$

and it is not difficult to check that it also satisfies $\|\phi_2\| \leqq M_2$, for a suitable M_2 depending only M_0 and M_1.

At this point, using (6.11), (6.12) and by arguing as in Lemma 6.2, then standard elliptic estimates to be used together with the maximum principle show that $\{\phi_0, \phi_1, \phi_2\} \subset C^2_0(\Omega)$ and $\|D^{(k)}_\lambda\phi_0\|_{C^2_0(\Omega)} + \|D^{(k)}_\lambda\phi_1\|_{C^2_0(\Omega)} + \|D^{(k)}_\lambda\phi_2\|_{C^2_0(\Omega)} \leqq \overline{M}_k$ for suitable constants $\overline{M}_k > 0$ depending only on M_0, M_1, M_2, that is, depending only on $\overline{\lambda}$.

Let $\lambda_0 = \lambda_0(\mu_0, \rho)$ as defined in (6.11) above, at this point a straightforward evaluation shows that the following second order expansion holds,

$$\lambda_0(\mu_0, \rho) := \mu_0\rho \int_{\Omega_\rho} e^{v_\rho} = \mu_0\rho \int_{\Omega_\rho} (1 + \rho\phi_0 + \mathrm{O}(\rho^2))$$
$$= \pi\mu_0 + \frac{\pi\mu_0^2\rho}{4(1+\rho^2)} + \mathrm{O}(\rho^2) = \pi\mu_0 + \frac{\pi\mu_0^2\rho}{4} + \mathrm{O}(\rho^2).$$

By arguing as in Lemma 6.2, and for a fixed value λ in the range of λ_0, we can use the implicit function theorem to obtain the inverse expansion up to order ρ^2, that is

$$\lambda = \pi\mu_0 + \frac{\pi\mu_0^2\rho}{4} + \mathrm{O}(\rho^2), \quad \mu_0 = \frac{\lambda}{\pi} - \frac{\lambda^2}{4\pi^2}\rho + \mathrm{O}(\rho^2),$$

in C^2 sense and (1.12)–(1.13) follow immediately. This observation concludes the proof. □

The Proof of Theorem 1.22. The notation $\mathrm{O}(\rho^m)$, $m \in \mathbb{N}$ will be used in the rest of this proof to denote various quantities uniformly bounded by $C_m\rho^m$ with $C_m > 0$ a suitable constant possibly depending on $\overline{\lambda}$ and on the constants $\overline{M}_k$, $k = 0, 1, 2$ as obtained in Theorem 1.20.

By using (1.10) above and Theorem 1.20 we obtain the Taylor expansion

$$\mathcal{E}(\omega_\lambda) = \frac{1}{2}\int_{\Omega_\rho} \omega_\lambda G_\rho[\omega_\lambda] = \frac{1}{2\lambda}\int_{\Omega_\rho} \omega_\lambda u_\lambda = \frac{\mu_0\rho}{2\lambda^2}\int_{\Omega_\rho} e^{u_\lambda}u_\lambda$$
$$= \frac{\mu_0\rho}{2\lambda^2}\int_{\Omega_\rho} e^{u_\lambda}u_\lambda = \frac{\mu_0\rho}{2\lambda^2}\int_{\Omega_\rho} (1 + \rho\phi_0 + \mathrm{O}(\rho^2))(\rho\phi_0 + \rho^2\phi_1 + \mathrm{O}(\rho^3))$$
$$= \frac{\mu_0\rho}{2\lambda^2}\int_{\Omega_\rho} (\rho\phi_0 + \rho^2\phi_0^2 + \rho^2\phi_1 + \mathrm{O}(\rho^3))$$
$$= \frac{\mu_0\rho}{2\lambda^2}\left[\frac{\pi\mu_0}{4(1+\rho^2)} + \frac{\pi\mu_0^2\rho}{12(1+\rho^2)^2} + \frac{\pi\mu_0^2\rho}{12(1+\rho^2)^2} + \mathrm{O}(\rho^2)\right],$$

where we have used the fact that

$$\int_{\Omega_\rho} \rho^2\phi_1 = \frac{\pi\mu_0^2\rho}{12(1+\rho^2)^2}, \tag{6.17}$$

which can be obtained by using the explicit expression of ϕ_0 in (1.8) together with the fact that ϕ_1 solves (6.15), see the Appendix 8.1 below for further details.

Hence, by using Proposition 6.1 and (1.12)–(1.13) and (1.14), we have

$$\widehat{E}(\lambda) := \mathcal{E}(\omega_\lambda) = \frac{\pi\mu_0^2\rho}{8\lambda^2} + \frac{\pi\mu_0^3\rho^2}{12\lambda^2} + \mathrm{O}(\rho^3) = \frac{\pi\rho}{8\lambda^2}\left(\frac{\lambda^2}{\pi^2} - \frac{\lambda^3}{2\pi^3}\rho + \mathrm{O}(\rho^2)\right)$$
$$+ \frac{\pi\rho^2}{12\lambda^2}\frac{\lambda^3}{\pi^3} + \mathrm{O}(\rho^3) = \frac{\rho}{8\pi} + \frac{\rho^2}{48\pi^2}\lambda + \mathrm{O}(\rho^3).$$

In particular we conclude that

$$\widehat{E}(\lambda) = \frac{\rho}{8\pi} + \frac{\rho^2}{48\pi^2}\lambda + \mathrm{O}(\rho^3), \tag{6.18}$$

and, in view of (1.12)–(1.13) and (1.14),

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{E}(\lambda) = \frac{\rho^2}{48\pi^2} + \mathrm{O}(\rho^3), \tag{6.19}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\widehat{E}(\lambda) = \mathrm{O}(\rho^3).$$

At this point (6.18) shows that we may restrict the domain of $\widehat{E}$ to the preimages of $E \in \left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$. Then (6.19) implies that $\widehat{E}(\lambda)$ is monotonic increasing there. Hence the preimage of $\left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$ is exactly $[0, \widehat{\lambda}_\rho]$ and so, according to Theorem 1.19, as far as $\lambda \leqq \overline{\lambda}$ the uniqueness of u_λ as a function of λ implies that the equation $\mathcal{E}(\omega(u_{\widehat{\lambda}(E)})) = E$ defines $\widehat{\lambda}(E)$ as a monotonic increasing function of E in $\left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$. Therefore, we can use (6.18) and (6.19) together with the implicit function theorem to take the inverse up to order ρ^2, that is

$$\widehat{\lambda}(E) = \frac{48\pi^2}{\rho^2}\left(E - \frac{\rho}{8\pi}\right) + \mathrm{O}(\rho),$$

and then conclude that

$$\frac{\mathrm{d}}{\mathrm{d}E}\widehat{\lambda}(E) = \frac{48\pi^2}{\rho^2} + \mathrm{O}(\rho),$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}E^2}\widehat{\lambda}(E) = \mathrm{O}(\rho).$$

□

7. The Entropy is Concave in $E \in \left[\frac{\rho}{8\pi}, \widehat{E}_\rho\right]$

The Proof of Proposition 1.25. Let us recall that according to definition 1.4 the density corresponding to a solution u_λ of $P(\lambda, \Omega_\rho)$ is defined to be

$$\omega_\lambda \equiv \omega(u_\lambda) := \frac{e^{u_\lambda}}{\int_{\Omega_\rho} e^{u_\lambda}}.$$

As usual $\mathcal{G}_{\rho,1}$ denotes the branch of solutions obtained in Theorem 1.7(a).

When evaluated on $(\lambda, u_\lambda) \in \mathcal{G}_{\rho,1}$, of course $\mathcal{S}(\omega(u_\lambda))$ yields a function of λ defined in principle on $\lambda \in [0, \lambda_{\rho,1}]$. Then we can use MVP-(iv), that is, the fact that any entropy maximizer (at fixed E) of the MVP satisfies $P(\lambda, \Omega)$ (for a certain unknown value λ). We can then observe, however, that Theorem 1.22 states that there exists one and only one solution of $P(\lambda, \Omega)$ with $\lambda = \widehat{\lambda}(E)$ such that the energy is exactly E, $\widehat{E}(\lambda)=E$, as far as $\lambda\leqq\overline{\lambda}$ and whenever $E\in\left[\frac{\rho}{8\pi},\widehat{E}_\rho\right]$. At this point we will need the assumption in Proposition 1.25 about the non existence of solutions of $P(\lambda,\Omega)$ with $\lambda>\overline{\lambda}$ and energy $E\in\left[\frac{\rho}{8\pi},\widehat{E}_\rho\right]$. As a consequence we conclude that indeed $S(E)\equiv\mathcal{S}(\omega(u_\lambda))\,|_{\lambda=\widehat{\lambda}(E)}$ in $\left[\frac{\rho}{8\pi},\widehat{E}_\rho\right]$. Hence, when evaluated on those densities $\omega_{\widehat{\lambda}(E)}$ as obtained in Theorem 1.22, we have

$$S(E)\equiv\mathcal{S}(\omega_{\widehat{\lambda}(E)})=-2E\widehat{\lambda}(E)+\log\left(\int_{\Omega_\rho}e^{u_{\widehat{\lambda}(E)}}\right),\quad E\in\left[\frac{\rho}{8\pi},\widehat{E}_\rho\right].$$

In particular, in view of Theorem 1.22 we can set

$$\dot{u}=\frac{\mathrm{d}u_{\widehat{\lambda}(E)}}{\mathrm{d}E},\quad\text{and}\quad\ddot{u}=\frac{\mathrm{d}^2u_{\widehat{\lambda}(E)}}{\mathrm{d}E^2},$$

to obtain

$$\frac{\mathrm{d}S(E)}{\mathrm{d}E}=-2\widehat{\lambda}(E)-2E\frac{\mathrm{d}\widehat{\lambda}(E)}{\mathrm{d}E}+\int_{\Omega_\rho}\omega_{\widehat{\lambda}(E)}\dot{u},$$

and then

$$\frac{\mathrm{d}^2S(E)}{\mathrm{d}E^2}=-4\frac{\mathrm{d}\widehat{\lambda}(E)}{\mathrm{d}E}-2E\frac{\mathrm{d}^2\widehat{\lambda}(E)}{\mathrm{d}E^2}+\int_{\Omega_\rho}\omega_{\widehat{\lambda}(E)}(\dot{u})^2-\left(\int_{\Omega_\rho}\omega_{\widehat{\lambda}(E)}\dot{u}\right)^2+\int_{\Omega_\rho}\omega_{\widehat{\lambda}(E)}\ddot{u}.\tag{7.1}$$

We wish to evaluate $\frac{\mathrm{d}^2S(E)}{\mathrm{d}E^2}$ in case $\Omega=\Omega_\rho$ and $E\in\left[\frac{\rho}{8\pi},\widehat{E}_\rho\right]$.

We are going to evaluate (7.1) by using (1.12)–(1.13), Theorem 1.22 and the estimates (1.14) in Theorem 1.20. Let us set

$$\dot{\widehat{\lambda}}=\frac{\mathrm{d}}{\mathrm{d}E}\widehat{\lambda}(E),\quad\ddot{\widehat{\lambda}}=\frac{\mathrm{d}^2}{dE^2}\widehat{\lambda}(E),$$

and

$$\phi_j^{'}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_j,\quad\phi_j^{''}=\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\phi_j,\qquad j=0,1,2,$$

so that, in view of (1.14) and (1.15), (1.16), (1.17) we have

$$\dot{u}=\frac{\mathrm{d}u}{\mathrm{d}\lambda}\dot{\widehat{\lambda}}=\dot{\widehat{\lambda}}\left(\rho\phi_0^{'}+\rho^2\phi_1^{'}+\mathrm{O}(\rho^3)\right),$$

and

$$\ddot{u}=\frac{\mathrm{d}^2u}{\mathrm{d}\lambda^2}\dot{\widehat{\lambda}}^2+\frac{du}{d\lambda}\ddot{\widehat{\lambda}}=\dot{\widehat{\lambda}}^2\left(\rho\phi_0^{''}+\rho^2\phi_1^{''}+\mathrm{O}(\rho^3)\right)+\ddot{\widehat{\lambda}}\left(\rho\phi_0^{'}+\rho^2\phi_1^{'}+\mathrm{O}(\rho^3)\right),\tag{7.2}$$

where the derivatives with respect to λ will be estimated by using (1.12)–(1.13).

Hence we can introduce

$$\ddot{S}_0(E):=\int_{\Omega_\rho}\omega(u_{\widehat{\lambda}(E)})(\ddot{u}+\dot{u}^2)-\left(\int_{\Omega_\rho}\omega(u_{\widehat{\lambda}(E)})\dot{u}\right)^2,$$

to obtain, after a lengthy evaluation where we use (1.12)–(1.13) and (7.2),

$$\ddot{S}_0(E) = \frac{(48\pi)^2}{\rho^2}\left[-\int_{\Omega_\rho}\omega(u_{\widehat{\lambda}(E)})\psi_0 - \left(\int_{\Omega_\rho}\omega(u_{\widehat{\lambda}(E)})\psi_0\right)^2 + \pi^2\int_{\Omega_\rho}\omega(u_{\widehat{\lambda}(E)})\phi_1^{''}\right] + \mathrm{O}\left(\frac{1}{\rho}\right).$$

At this point we can use

$$\int_{\Omega_\rho}\omega(u_{\widehat{\lambda}(E)})\psi_0 = \frac{1}{4} + \mathrm{O}(\rho), \tag{7.3}$$

and

$$\int_{\Omega_\rho}\omega(u_{\widehat{\lambda}(E)})\phi_1^{''} = \frac{1}{6\pi^2} + \mathrm{O}(\rho), \tag{7.4}$$

whose proof is left to Appendix 8.2, and (1.16), (1.17) to obtain

$$\frac{\mathrm{d}^2 S(E)}{\mathrm{d}E^2} = -4\dot{\widehat{\lambda}} - 2E\ddot{\widehat{\lambda}} + \ddot{S}_0(E) = -4\frac{48\pi^2}{\rho^2} + \frac{(48\pi)^2}{\rho^2}\left(-\frac{1}{4} - \frac{1}{16} + \frac{1}{6}\right) + \mathrm{O}\left(\frac{1}{\rho}\right),$$

and the conclusion readily follows. □

8. Appendix

8.1. The Proof of (6.17)

To obtain (6.17) we multiply $-\Delta\phi_1$ by y^2 and integrate by parts twice to obtain

$$-\int_{\Omega_\rho} y^2\Delta\phi_1 = -\int_{\partial\Omega_\rho} y^2\partial_\nu\phi_1 - 2\int_{\Omega_\rho}\phi_1.$$

Similarly we have

$$-\int_{\Omega_\rho}\rho^2x^2\Delta\phi_1 = -\int_{\partial\Omega_\rho}\rho^2x^2\partial_\nu\phi_1 - 2\rho^2\int_{\Omega_\rho}\phi_1,$$

so that we can sum up to obtain

$$2(1+\rho^2)\int_{\Omega_\rho}\phi_1 = \int_{\Omega_\rho}(\rho^2x^2+y^2)\Delta\phi_1 - \int_{\partial\Omega_\rho}\partial_\nu\phi_1.$$

Therefore, by using the equation in (6.15) and the divergence theorem we have

$$2(1+\rho^2)\int_{\Omega_\rho}\phi_1 = \int_{\Omega_\rho}(-(\rho^2x^2+y^2)+1)\mu_0\phi_0,$$

that is

$$\int_{\Omega_\rho}\phi_1 = (\mu_0)^2\int_{\Omega_\rho}\psi_0^2, \tag{8.1}$$

and the conclusion follows by a straightforward evaluation based on the explicit expression of ψ_0 [see (1.9)]. □

8.2. The Proofs of (7.3) and (7.4)

Concerning (7.3) we just observe that

$$\int_{\Omega_\rho} \omega(u_{\widehat{\lambda}(E)})\psi_0 = \int_{\Omega_\rho} \frac{1+\mathrm{O}(\rho)}{\int_{\Omega_\rho}(1+\mathrm{O}(\rho))}\psi_0 = \frac{\rho}{\pi}(1+\mathrm{O}(\rho))\int_{\Omega_\rho}\psi_0 = \frac{1}{4}+\mathrm{O}(\rho),$$

where the last equality is obtained by a straightforward evaluation based on the explicit expression of ψ_0 [see (1.9)].

Concerning (7.4) we observe as above that

$$\int_{\Omega_\rho} \omega(u_{\widehat{\lambda}(E)})\phi_1^{''} = \frac{\rho}{\pi}(1+\mathrm{O}(\rho))\int_{\Omega_\rho}\phi_1^{''}, \tag{8.2}$$

and that in view of (6.15) and (1.8), then $\phi_1^{''}$ satisfies

$$\begin{cases} -\Delta\phi_1^{''} = (\mu_0\phi_0)^{''} \equiv (\mu_0^2)^{''}\psi_0 & \text{in } \Omega_\rho \\ \phi_1^{''} = 0 & \text{on } \partial\Omega_\rho \end{cases} \tag{8.3}$$

where $\mu_0 = \mu_0(\lambda, \rho)$ [see (1.12)–(1.13)]. In other words $\phi_1^{''}$ is a solution for the same problem as ϕ_1 [that is (6.15)] but for the fact that μ_0^2 is replaced by $(\mu_0^2)^{''}$ in (8.3). Hence the argument in Section 8.1 applies and we obtain [see (8.1)]

$$\int_{\Omega_\rho}\phi_1^{''} = (\mu_0^2)^{''}\int_{\Omega_\rho}\psi_0^2 = (\mu_0^2)^{''}\frac{\pi}{12\rho}+\mathrm{O}(\rho^2) = \frac{1}{6\pi\rho}+\mathrm{O}(\rho^2),$$

and the conclusion follows by substituting this result in (8.2). □

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