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A Derivation of the Magnetohydrodynamic System from Navier–Stokes–Maxwell Systems

Diogo Arsénio, Slim Ibrahim & Nader Masmoudi

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Abstract

We provide a full and rigorous derivation of the standard viscous magnetohydrodynamic system (MHD) as the asymptotic limit of Navier–Stokes–Maxwell systems when the speed of light is infinitely large. We work in the physical setting provided by the natural energy bounds and therefore mainly consider Leray solutions of fluid dynamical systems. Our methods are based on a direct analysis of frequencies and we are able to establish the weak stability of a crucial nonlinear term (the Lorentz force), neither assuming any strong compactness of the components nor applying standard compensated compactness methods (which actually fail in this case).

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1. Introduction

In this work, we provide a full and rigorous macroscopic derivation of the viscous incompressible magnetohydrodynamic system:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u = -\nabla p + (\nabla\times B)\times B, & \operatorname{div} u = 0, \\ \partial_t B - \frac{1}{\sigma}\Delta B = \nabla\times(u\times B), & \operatorname{div} B = 0, \end{cases} \tag{1.1}$$

subject to some initial data

$$u|_{t=0} = u^0, \qquad B|_{t=0} = B^0,$$

which is a standard model for an electrically conducting fluid or a plasma. Here, $u, B : \mathbb{R}^+_t \times \mathbb{R}^d_x \to \mathbb{R}^3$ are vector fields, with $d = 2$ or 3. More precisely, the field $u = (u_1, \dots, u_d)$ denotes the velocity of the fluid and $\mu > 0$ its viscosity, while $B = (B_1, \dots, B_d)$ is a magnetic field interacting with the fluid and $\sigma > 0$ is the electrical conductivity of the fluid. Finally, the scalar function $p : \mathbb{R}^+_t \times \mathbb{R}^d_x \to \mathbb{R}$ stands for the pressure. Note that the pressure p can be recovered from u and B via an explicit Calderón-Zygmund singular integral operator (see [4] for instance).

In the two-dimensional case $d = 2$, the functions u and B are defined on the whole space $\mathbb{R}^2$ with values in $\mathbb{R}^3$. In this setting, the operator ∇ is given by

$$\nabla = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ 0 \end{pmatrix},$$

so that

$$\operatorname{div} u = \nabla\cdot u := \partial_{x_1} u_1 + \partial_{x_2} u_2, \qquad \nabla p := \begin{pmatrix} \partial_{x_1} p \\ \partial_{x_2} p \\ 0 \end{pmatrix},$$

and

$$\operatorname{curl} B = \nabla\times B := \begin{pmatrix} \partial_{x_2} B_3 \\ -\partial_{x_1} B_3 \\ \partial_{x_1} B_2 - \partial_{x_2} B_1 \end{pmatrix}.$$

It is well-known that, for the two-dimensional incompressible Navier–Stokes equations, a two-component initial structure (that is an initial velocity with zero third component) is propagated by the flow. However, for the system (1.1), this is not likely to be true. Indeed, the forcing term $(\nabla\times B)\times B$ induces the fluid to move in all directions. Nevertheless, one observes that if we also assume $B_1^0 \equiv B_2^0 \equiv 0$ initially, then this initial structure is propagated and the movement of the fluid remains planar.

In this case, one easily verifies that $(\nabla\times B)\times B = -\frac{1}{2}\nabla(B_3^2)$, which is a pressure gradient, and that $\nabla\times(u\times B) = -(u\cdot\nabla)B$. Therefore, the two-dimensional magnetohydrodynamic system (1.1) can be recast as the decoupled system:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u = -\nabla p, & \operatorname{div} u = 0, \\ \partial_t B + u\cdot\nabla B - \frac{1}{\sigma}\Delta B = 0, & \operatorname{div} B = 0. \end{cases} \tag{1.2}$$

Back to the general case (1.1), that is $d = 2$ or 3, a standard energy estimate yields the following formal conservation of energy:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L_x^2}^2 + \|B\|_{L_x^2}^2\right) + \mu\|\nabla u\|_{L_x^2}^2 + \frac{1}{\sigma}\|\nabla B\|_{L_x^2}^2 = 0.$$

The a priori estimates implied by this energy identity show that, for any initial data

$$(u^0, B^0) \in L_x^2,$$

it is possible to establish the existence of global weak solutions

$$(u, B) \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1,$$

following the method of Leray [17] (see [16] for a recent account of the subject). Indeed, the dissipation on both u and B is clearly sufficient to establish the weak stability of the nonlinear terms $u \cdot \nabla u$, $(\nabla \times B) \times B$ and $\nabla \times (u \times B)$ from (1.1).

As for the physical theory of the above magnetohydrodynamic system (1.1), we refer to Biskamp [3] and Davidson [7] for a detailed introduction.

The standard macroscopic derivation of (1.1) (see [3,7]) consists in considering first the Navier–Stokes equations for a viscous incompressible fluid subject to some force field $F : \mathbb{R}_t^+ \times \mathbb{R}_x^d \to \mathbb{R}^3$ to be specified:

$$\partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + F, \qquad \mathrm{div} u = 0. \tag{1.3}$$

Then, since the fluid is made of electrically charged particles, their motion induces an electromagnetic field $(E, B) : \mathbb{R}_t^+ \times \mathbb{R}_x^d \to \mathbb{R}^3 \times \mathbb{R}^3$, where the magnetic field B is assumed to be given by Ampère's law:

$$\nabla \times B = j, \tag{1.4}$$

where $j : \mathbb{R}_t^+ \times \mathbb{R}_x^d \to \mathbb{R}^3$ is the electric current in the fluid, and the electric field E is determined by Faraday's law:

$$\nabla \times E = -\partial_t B, \tag{1.5}$$

which also determines the dynamics of the magnetic field B.

For a fluid at rest, the electric field and current are linked through Ohm's law $j = \sigma E$. However, in general, one has to take into account the moving reference frame of the fluid, which yields the more general Ohm's law:

$$j = \sigma\left(E + u \times B\right). \tag{1.6}$$

Note that this correction to Ohm's law is deduced by keeping Faraday's law invariant under Galilean transformations.

Finally, the electromagnetic field acts back on the fluid through the action of the Lorentz force $F = nE + j \times B$, where $n : \mathbb{R}_t^+ \times \mathbb{R}_x^d \to \mathbb{R}$ denotes the electric charge density. However, in dense fluids, electrostatic fields enforce charge neutrality over macroscopic distances, which is known as quasi-neutrality. Therefore, we may assume that $n = 0$, which yields the force

$$F = j \times B. \tag{1.7}$$

On the whole, recalling that a magnetic field is always solenoidal, that is $\operatorname{div} B = 0$, and noticing that

$$\nabla \times (\nabla \times B) = -\Delta B,$$

combining (1.3), (1.4), (1.5), (1.6) and (1.7) clearly yields (1.1).

Nevertheless, the preceding formal derivation of the magnetohydrodynamic system (1.1) remains unsatisfactory, for it appeals to the quasi-static approximation, without justifying it, to account for the use of Ampère's law (1.4). Indeed, in general, electromagnetic effects require the use of Ampère's equation with Maxwell's correction:

$$\nabla \times B = j + \partial_t E.$$

In the quasi-static approximation, the so-called displacement current $\partial_t E$ can be neglected but not the electric field E.

In this paper, we give a full and rigorous derivation of the magnetohydrodynamic system (1.1) starting from a two-fluid system which takes into account the full electromagnetic phenomenon as described by the complete set of Maxwell's equations. More precisely, we consider the asymptotic behavior, as the speed of light $c > 0$ tends to infinity and the momentum transfer coefficient $\varepsilon > 0$ tends to zero, of the two-fluid incompressible Navier–Stokes–Maxwell system:

$$\begin{cases} \partial_t u^+ + u^+ \cdot \nabla u^+ - \mu \Delta u^+ \\ \qquad + \dfrac{1}{2\sigma\varepsilon^2}\left(u^+ - u^-\right) = -\nabla p^+ + \dfrac{1}{\varepsilon}\left(cE + u^+ \times B\right), & \operatorname{div} u^+ = 0, \\ \partial_t u^- + u^- \cdot \nabla u^- - \mu \Delta u^- \\ \qquad - \dfrac{1}{2\sigma\varepsilon^2}\left(u^+ - u^-\right) = -\nabla p^- - \dfrac{1}{\varepsilon}\left(cE + u^- \times B\right), & \operatorname{div} u^- = 0, \\ \qquad \dfrac{1}{c}\partial_t E - \nabla \times B = -\dfrac{1}{2\varepsilon}\left(u^+ - u^-\right), & \operatorname{div} E = 0, \\ \qquad \dfrac{1}{c}\partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0, \end{cases} \tag{1.8}$$

subject to some initial data

$$u^\pm|_{t=0} = u^{\pm 0}, \qquad E|_{t=0} = E^0, \qquad B|_{t=0} = B^0.$$

This system governs the evolution of a plasma of oppositely charged ions (that is positively charged cations and negatively charged anions), of approximately equal mass. Thus, the vector fields $u^+, u^- : \mathbb{R}_t^+ \times \mathbb{R}_x^d \to \mathbb{R}^3$ represent the velocities of cations and anions, respectively.

Multiplying the Navier–Stokes equations for u^+ and u^- in (1.8) by u^+ and u^-, respectively, and Maxwell's equations by $\begin{pmatrix} E \\ B \end{pmatrix}$, and then integrating in space (using the divergence-free condition of the velocities), one obtains the following formal energy conservation law:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|u^+\right\|_{L_x^2}^2+\frac{1}{2}\left\|u^-\right\|_{L_x^2}^2+\|E\|_{L_x^2}^2+\|B\|_{L_x^2}^2\right)$$
$$+\frac{\mu}{2}\left(\left\|\nabla u^+\right\|_{L_x^2}^2+\left\|\nabla u^-\right\|_{L_x^2}^2\right)+\frac{1}{\sigma}\left\|\frac{u^+-u^-}{2\varepsilon}\right\|_{L_x^2}^2=0. \tag{1.9}$$

This identity shows that the energy dissipates due to the effects of both the viscosity and the electric conductivity.

As before, the a priori estimates implied by this energy conservation show that, for any initial data

$$(u^{\pm 0}, E^0, B^0) \in L_x^2,$$

it is possible to establish the existence of global weak solutions

$$u^{\pm} \in L_t^{\infty} L_x^2 \cap L_t^2 \dot{H}_x^1, \qquad (E, B) \in L_t^{\infty} L_x^2, \qquad \frac{u^+-u^-}{2\varepsilon} \in L_{t,x}^2,$$

following the method of Leray [17]. Indeed, the dissipation on $u^{\pm}$ is clearly sufficient to establish the weak stability of the nonlinear terms $u^{\pm}\cdot\nabla u^{\pm}$ and $u^{\pm}\times B$ in (1.8).

Finally, note that the local well-posedness of the two-fluid incompressible Navier–Stokes–Maxwell system (1.8) has been studied in [12], while the existence of small global solutions has been recently established in [11]. Also, a microscopic derivation of (1.1) and (1.8) is provided by [1,15].

As explained in the next section, it is possible to show, at least formally, that letting $c \to \infty$ and $\varepsilon \to 0$ simultaneously in (1.8) yields the magnetohydrodynamic system (1.1). The objective of this paper is to establish this asymptotic result rigorously. Our main result in this direction follows.

Theorem 1.1. *Let $d = 2, 3$ and, for each $c > 0$, consider $\left(u^{\pm c}, E^c, B^c\right)$ a global and finite energy weak solution of the two-fluid incompressible Navier–Stokes–Maxwell system (1.8), where $\varepsilon = \varepsilon(c) > 0$ satisfies $\lim_{c\to\infty}\varepsilon = 0$ such that*

$$\lim_{c\to\infty}\phi(\delta c)\varepsilon = \infty, \quad \textit{for any } \delta > 0, \tag{1.10}$$

with

$$\begin{aligned}\phi(c) &= c^2, && \textit{if} \quad d = 3,\\ \phi(c) &= \exp\left(c^2\right), && \textit{if} \quad d = 2,\end{aligned} \tag{1.11}$$

for some initial data

$$(u^{\pm 0c}, E^{0c}, B^{0c}) \in L_x^2, \quad \textit{such that} \quad \mathrm{div} u^{\pm 0c} = \mathrm{div} E^{0c} = \mathrm{div} B^{0c} = 0,$$

converging weakly, as $c \to \infty$, towards some

$$(u^{\pm 0}, E^0, B^0) \in L_x^2, \quad \textit{such that} \quad \mathrm{div} u^{\pm 0} = \mathrm{div} E^0 = \mathrm{div} B^0 = 0.$$

Then, as $c \to \infty$, up to extraction of a subsequence, $\left(u^{\pm c}, B^c\right)$ converges weakly towards $\left(u^{\pm}, B\right)$, where $u^+ = u^- = u$ and (u, B) is a global and finite *energy weak solution of the magnetohydrodynamic system* (1.1)*, with initial data precisely given by*

$$u|_{t=0} = \frac{u^{+0} + u^{-0}}{2}, \qquad B|_{t=0} = B^0.$$

Remark. The non-relativistic limit $c \to \infty$ is by now very standard both in the quantum and classical cases (see for instance [20]). It will be clear from its proof, that the above theorem remains valid for any choice of asymptotic parameter $\varepsilon = \varepsilon(c) > 0$ satisfying $\lim_{c\to\infty} \varepsilon = 0$ if, instead of requiring (1.10), one asks that

$$\limsup_{c\to\infty} \frac{1}{c} \left\| \left(u^{+c} + u^{-c}\right) \times B^c \right\|_{L^2_{t,x,\mathrm{loc}}} = 0, \tag{1.12}$$

and

$$\limsup_{c\to\infty} \frac{1}{c} \left\| u^{+c} \otimes u^{+c} - u^{-c} \otimes u^{-c} \right\|_{L^2_{t,x,\mathrm{loc}}} = 0. \tag{1.13}$$

Note that condition (1.12) almost holds true in two dimensions, for $u^{\pm c}$ barely fails to belong uniformly to $L^2_t L^\infty_x$ (by Sobolev embedding of $L^2_t \dot{H}^1_x$), whereas condition (1.13) is always verified in two dimensions, for $u^{\pm c}$ lies uniformly in $L^4_t L^4_x$ in this case.

The proof of this result, which we defer to Section 5, is based on a priori estimates and on the study of the spectral properties of the damped wave operator defined by Maxwell's system in (1.8).

In the two-dimensional case, we do not know whether it is possible to exploit the simpler decoupled structure of system (1.2) to improve our result.

Prior to addressing the core of the proof, we give in the next section a complete formal asymptotic analysis of (1.8) (containing some intermediate rigorous results), which will shed light on its links with (1.1). Next, in Section 3, we provide a spectral analysis of the linear system determined by Maxwell's equations in (1.8), which will be crucial for the refined estimates in the later parts of the proof. Then, in Section 4, we discuss an intermediate conditional result, simpler than Theorem 1.1, whose understanding will lead us to the proof of our main result. And finally, in Section 5, we present the full demonstration of Theorem 1.1.

2. Preliminary Asymptotic Analysis

In this section, we give a formal asymptotic analysis of (1.8), which will shed light on its links with (1.1).

Thus, as $c \to \infty$ and $\varepsilon \to 0$, according to the conservation of energy (1.9), we expect the electrical current $\frac{u^+ - u^-}{2\varepsilon}$, which we henceforth denote by j, to be asymptotically uniformly bounded. Further denoting the bulk velocity $u = \frac{u^+ + u^-}{2}$, we rewrite the system (1.8), considering the sum and the difference of the equations for cations and anions, as

$$\begin{cases} \partial_t u + u \cdot \nabla u + \varepsilon^2 j \cdot \nabla j \\ \qquad\qquad -\mu \Delta u = -\nabla p + j \times B, & \operatorname{div} u = 0, \\ \varepsilon^2 \partial_t j + \varepsilon^2 (u \cdot \nabla j + j \cdot \nabla u) \\ \qquad\qquad -\varepsilon^2 \mu \Delta j + \dfrac{1}{\sigma} j = -\nabla \bar{p} + cE + u \times B, & \operatorname{div} j = 0, \\ \qquad\qquad \dfrac{1}{c} \partial_t E - \nabla \times B = -j, & \operatorname{div} E = 0, \\ \qquad\qquad \dfrac{1}{c} \partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0, \end{cases} \tag{2.1}$$

where we have denoted the pressures $p = \frac{1}{2}\left(p^+ + p^-\right)$ and $\bar{p} = \frac{\varepsilon}{2}\left(p^+ - p^-\right)$. Also, the formal energy conservation (1.9) is now rewritten as

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L_x^2}^2 + \varepsilon^2 \|j\|_{L_x^2}^2 + \|E\|_{L_x^2}^2 + \|B\|_{L_x^2}^2\right) \\ &\qquad + \mu\left(\|\nabla u\|_{L_x^2}^2 + \varepsilon^2 \|\nabla j\|_{L_x^2}^2\right) + \frac{1}{\sigma}\|j\|_{L_x^2}^2 = 0. \end{aligned} \tag{2.2}$$

Since we are considering a degenerate Gauss's law, that is $\operatorname{div} E = 0$, which consistently reflects quasi-neutrality, Faraday's equation implies that

$$cE = -\partial_t A, \quad \text{where} \quad B = \nabla \times A \quad \text{and} \quad \operatorname{div} A = 0.$$

Therefore, cE enjoys some kind of uniform weak bound. Thus, for convenience and later use, we define the adjusted electric field

$$\widetilde{E} = cE = -\partial_t A. \tag{2.3}$$

Now, fixing the constant $c > 0$ and letting $\varepsilon \to 0$ in (2.1), we formally obtain the incompressible Navier–Stokes–Maxwell system with solenoidal Ohm's law:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + j \times B, & \operatorname{div} u = 0, \\ \qquad\qquad j = \sigma\left(-\nabla \bar{p} + cE + u \times B\right), & \operatorname{div} j = 0, \\ \dfrac{1}{c} \partial_t E - \nabla \times B = -j, & \operatorname{div} E = 0, \\ \dfrac{1}{c} \partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0, \end{cases} \tag{2.4}$$

which enjoys the formal energy conservation law:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L_x^2}^2 + \|E\|_{L_x^2}^2 + \|B\|_{L_x^2}^2\right) + \mu\|\nabla u\|_{L_x^2}^2 + \frac{1}{\sigma}\|j\|_{L_x^2}^2 = 0. \tag{2.5}$$

Another very similar system presenting the same difficulties as (2.4) is the following incompressible Navier–Stokes–Maxwell system with Ohm's law:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + j \times B, & \mathrm{div} u = 0, \\ \frac{1}{c}\partial_t E - \nabla \times B = -j, & j = \sigma\left(cE + u \times B\right), \\ \frac{1}{c}\partial_t B + \nabla \times E = 0, & \mathrm{div} B = 0, \end{cases} \tag{2.6}$$

whose associated formal energy conservation law is also given by (2.5). It is to be emphasized that (2.6) seems physically less apposite than (2.4) because quasi-neutrality of the fluid (that is $n \approx 0$) is used to neglect the electric component of the Lorentz force (that is $nE \approx 0$) whereas Gauss's law may be non-trivial (that is $0 \neq \mathrm{div} E = n$). Nevertheless, (2.6) and (2.4) are mathematically very much alike and, thus, present the same difficulties.

In fact, the well-posedness theories of both systems (2.4) and (2.6) are very challenging, because these essentially couple the incompressible Navier–Stokes equations with a hyperbolic system, and there still remains several open questions, such as the existence of global weak solutions in the energy space in both dimensions $d = 2, 3$, which are otherwise easily answered for the system (1.8) (or (2.1), equivalently).

Indeed, the natural a priori bounds deduced from (2.5) suggest that one should be able to construct a global finite energy weak solution *à la Leray* with initial data $\left(u^0, E^0, B^0\right)$ lying in L^2_x. However, in view of the hyperbolicity of Maxwell's equations, the weak stability of the nonlinear term $j \times B$ in the energy space is not known, which prevents the convergence of any standard approximating scheme. Moreover, in order to run fixed point arguments and derive closed estimates for data $\left(u^0, E^0, B^0\right)$ merely in L^2_x, controlling the term $E \times B$ presents major difficulties.

Finally, note that the method of compensated compactness of MURAT and TARTAR [21,22,25] also fails here. More precisely, this method teaches us that the linear structure of Maxwell's equations will prevent resonances in the quadratic term $E \times B$ (contained in the Lorentz force) if and only if $e \times b = 0$, for all $(e, b) \in \Lambda$, where the wave cone Λ is made up of all vectors $(e, b) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$\begin{cases} \frac{1}{c}\tau e - \xi \times b = 0, & \xi \cdot e = 0, \\ \frac{1}{c}\tau b + \xi \times e = 0, & \xi \cdot b = 0, \end{cases}$$

for some $(\tau, \xi) \in \left(\mathbb{R} \times \mathbb{R}^3\right) \setminus \{0\}$ (and such that $e_3 = b_1 = b_2 = \xi_3 = 0$ if $d = 2$). It is possible to show, however (using Proposition 3.1 below, with $\sigma = 0$, for instance), that the wave cone is then precisely given by

$$\Lambda = \left\{ (e, b) \in \mathbb{R}^3 \times \mathbb{R}^3 \ : \ b = \pm \frac{\xi}{|\xi|} \times e, \ \xi \cdot e = 0 \text{ for some } \xi \in \mathbb{R}^3 \setminus \{0\} \right\},$$

if $d = 3$, and adding the restriction $e_3 = b_1 = b_2 = \xi_3 = 0$ if $d = 2$, whereby

$$e \times b = e \times \left(\pm \frac{\xi}{|\xi|} \times e \right) = \pm |e|^2 \frac{\xi}{|\xi|} \neq 0, \quad \text{for all } (e, b) \in \Lambda,$$

which prevents the weak stability of the nonlinear term $E \times B$ according to the theory of compensated compactness.

This lack of weak stability also reflects the difficulties we encounter in the asymptotic analysis of (1.8). Loosely speaking, this can be interpreted as a consequence of the very singular perturbation of (1.8) associated with the limit $\varepsilon \to 0$, which leads to (2.4).

Nevertheless, imposing more regularity on the initial electromagnetic field, one can hope to solve (2.4) and (2.6). Thus, in the two-dimensional case and for arbitrarily large initial data $u^0 \in L^2_x$ and $E^0, B^0 \in H^s_x$, with $s > 0$, the third author [19] proved the existence and uniqueness of global solutions to (2.6). The same method can be applied to solve (2.4). The proof relies on the use of the energy inequality combined with a logarithmic inequality that enables to estimate the L^∞-norm of the velocity field by the energy norm and higher Sobolev norms.

It is also interesting to note that the method from [19] uses neither the divergence-free condition of the magnetic field, nor the decay property of the linear part coming from Maxwell's equations combined with Ohm's law, that is the damping of the electric field obtained by rewriting Maxwell's equations from (2.6) as

$$\begin{cases} \frac{1}{c}\partial_t E - \nabla \times B + \sigma c E = G, \\ \frac{1}{c}\partial_t B + \nabla \times E = 0, \\ \operatorname{div} B = 0, \end{cases} \tag{2.7}$$

where $G := -\sigma u \times B$ is treated as a nonlinear source term. In the setting of (2.4), the source is given by $G := -\sigma P(u \times B)$, where P is the Leray projector onto divergence-free vector fields.

More recently, these properties were used in the three-dimensional setting by the second author and Keraani [13] to prove the global existence of solutions for small initial data, in the spirit of Fujita and Kato [9]. The local existence of strong solutions for large data has been established in a separate work by the second author and Yoneda [14]. Finally, the well-posedness theory of (2.6) has been refined in the very recent work [10] by the second and third authors with Germain.

Back to the formal asymptotic analysis of the preceding systems, further letting $c \to \infty$ in (2.4), we obtain the quasi-static approximation of the incompressible Navier–Stokes–Maxwell system with solenoidal Ohm's law (recall that $\widetilde{E}$ is given by (2.3)):

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + j \times B, & \operatorname{div} u = 0, \\ j = \sigma \left(-\nabla \bar{p} + \widetilde{E} + u \times B \right), & \operatorname{div} j = 0, \\ \nabla \times B = j, & \operatorname{div} \widetilde{E} = 0, \\ \partial_t B + \nabla \times \widetilde{E} = 0, & \operatorname{div} B = 0. \end{cases}$$

Eliminating the unknowns j and $\widetilde{E}$, the above system is then easily rewritten as the magnetohydrodynamic system (1.1). As already seen, it is a much tamer system, since the incompressible Navier–Stokes equations are now coupled with a parabolic system on the magnetic field B.

Next, fixing $\varepsilon > 0$ and letting first $c \to \infty$ in (1.8) (or (2.1)) yields a quasi-static approximation of the two-fluid incompressible Navier–Stokes–Maxwell system (1.8) (recall that $\widetilde{E}$ is given by (2.3)):

$$\begin{cases} \partial_t u^+ + u^+ \cdot \nabla u^+ - \mu \Delta u^+ \\ \qquad + \frac{1}{2\sigma\varepsilon^2}\left(u^+ - u^-\right) = -\nabla p^+ + \frac{1}{\varepsilon}\left(\widetilde{E} + u^+ \times B\right), \quad \operatorname{div} u^+ = 0, \\ \partial_t u^- + u^- \cdot \nabla u^- - \mu \Delta u^- \\ \qquad - \frac{1}{2\sigma\varepsilon^2}\left(u^+ - u^-\right) = -\nabla p^- - \frac{1}{\varepsilon}\left(\widetilde{E} + u^- \times B\right), \quad \operatorname{div} u^- = 0, \\ \qquad \nabla \times B = \frac{1}{2\varepsilon}\left(u^+ - u^-\right), \quad \operatorname{div} \widetilde{E} = 0, \\ \qquad \partial_t B + \nabla \times \widetilde{E} = 0, \quad \operatorname{div} B = 0, \end{cases} \tag{2.8}$$

or, equivalently,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \varepsilon^2 j \cdot \nabla j \\ \qquad -\mu \Delta u = -\nabla p + j \times B, \quad \operatorname{div} u = 0, \\ \varepsilon^2 \partial_t j + \varepsilon^2 \left(u \cdot \nabla j + j \cdot \nabla u\right) \\ \qquad -\varepsilon^2 \mu \Delta j + \frac{1}{\sigma} j = -\nabla \bar{p} + \widetilde{E} + u \times B, \quad \operatorname{div} j = 0, \\ \qquad \nabla \times B = j, \quad \operatorname{div} \widetilde{E} = 0, \\ \qquad \partial_t B + \nabla \times \widetilde{E} = 0, \quad \operatorname{div} B = 0, \end{cases} \tag{2.9}$$

whose energy conservation law:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2_x}^2 + \varepsilon^2 \|j\|_{L^2_x}^2 + \|B\|_{L^2_x}^2\right) \\ &\quad + \mu\left(\|\nabla u\|_{L^2_x}^2 + \varepsilon^2 \|\nabla j\|_{L^2_x}^2\right) + \frac{1}{\sigma}\|\nabla \times B\|_{L^2_x}^2 = 0, \end{aligned} \tag{2.10}$$

provides, at least formally, a uniform bound on $\nabla \times B$.

In view of this strong dissipation on the magnetic field B, the preceding system is thus parabolic on B and not merely hyperbolic, as is the case when coupling the fluid systems with the full set of Maxwell's equations. Accordingly, the well-posedness theory of (2.9) (or (2.8)) is much simpler than it is for the systems (2.4) and (2.6), for instance. Indeed, following once again the method of LERAY [17], for any initial data

$$(u^0, j^0, B^0) \in L^2_x,$$

it is possible to establish, for fixed $\varepsilon > 0$, the existence of global weak solutions

$$(u, j, B) \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$$

of (2.9). However, it is to be emphasized that, even though the spatial compactness deduced from the preceding a priori estimate is largely sufficient to handle the convergence in x of the nonlinear terms from (2.9), the analysis of the temporal compactness of the unknowns requires some greater care, which we now address.

Indeed, standard methods from modern nonlinear analysis imply, through a routine application of a classical compactness result by Aubin and Lions [2,18] (see also [23] for a sharp compactness criterion), that the velocity u is strongly relatively compact in L^2_{loc} in all variables t and x, which is enough to prove the weak stability of the nonlinear terms $u \cdot \nabla u$, $u \cdot \nabla j$ and $j \cdot \nabla u$ in (2.9). But the same argument does not apply directly to the vector fields j and B separately because of the presence of the adjusted electric field $\widetilde{E}$ in (2.9), which, we recall, is merely defined as a distribution by (2.3). Instead, we consider the combination

$$B + \varepsilon^2 \nabla \times j = \left(1 - \varepsilon^2 \Delta\right) B,$$

which satisfies the evolution equation

$$\begin{aligned}&\left(1 - \varepsilon^2 \Delta\right) \partial_t B + \varepsilon^2 \nabla \times (u \cdot \nabla j + j \cdot \nabla u) - \varepsilon^2 \mu \Delta \nabla \times j + \frac{1}{\sigma} \nabla \times j \\ &= \nabla \times (u \times B),\end{aligned}$$

where the adjusted electric field $\widetilde{E}$ has been cancelled out. We may now apply the aforementioned standard argument based on the classical compactness result by Aubin and Lions [2,18] to deduce that B is relatively compact in L^2_{loc} in all variables t and x. It ensues that j also enjoys the same local relative compactness in all variables and, therefore, that each nonlinear term in (2.9) (or (2.8), equivalently) is weakly stable. This concludes the justification of the existence of Leray solutions to the systems (2.8) and (2.9).

Accordingly, we have the following simple result on the convergence of (1.8) towards (2.8).

Proposition 2.1. *Let $d = 2, 3$ and, for each $c > 0$, consider $\left(u^{\pm c}, E^c, B^c\right)$ a global and finite energy weak solution of the two-fluid incompressible Navier–Stokes–Maxwell system (1.8), where $\varepsilon > 0$ is fixed, for some initial data*

$$(u^{\pm 0c}, E^{0c}, B^{0c}) \in L^2_x, \quad \textit{such that} \quad \mathrm{div} u^{\pm 0c} = \mathrm{div} E^{0c} = \mathrm{div} B^{0c} = 0,$$

converging weakly, as $c \to \infty$, towards some

$$(u^{\pm 0}, E^0, B^0) \in L^2_x, \quad \textit{such that} \quad \mathrm{div} u^{\pm 0} = \mathrm{div} E^0 = \mathrm{div} B^0 = 0.$$

Then, as $c \to \infty$, up to extraction of a subsequence, $\left(u^{\pm c}, B^c\right)$ converges weakly to a global and finite energy weak solution $\left(u^{\pm}, B\right)$ of the quasi-static two-fluid incompressible Navier–Stokes–Maxwell system (2.8), with initial data precisely given by

$$\begin{aligned} u^{\pm}|_{t=0} &= u^{\pm 0} \pm \varepsilon \left(1 - \varepsilon^2 \Delta\right)^{-1} \left(\nabla \times B^0 - \frac{u^{+0} - u^{-0}}{2\varepsilon}\right), \\ B|_{t=0} &= B^0 + \varepsilon^2 \left(1 - \varepsilon^2 \Delta\right)^{-1} \nabla \times \left(\frac{u^{+0} - u^{-0}}{2\varepsilon} - \nabla \times B^0\right). \end{aligned}$$

Proof. The demonstration of this convergence poses no great difficulty except for the handling of time compactness, which is consistent with the preceding difficulties encountered in the justification of the existence of Leray solutions to the systems (2.8) or (2.9).

Thus, in order to handle temporal compactness, we establish the convergence of the equivalent system (2.1) towards (2.9).

To this end, by virtue of the natural uniform energy bounds from (2.2) and classical compactness results by Aubin and Lions [2,18], we easily deduce first that u^c is compact, up to extraction of a subsequence, in all variables in the strong topology of L^2_{loc}. This is sufficient to handle the convergence of all nonlinear terms in (2.1) except $\varepsilon^2 j^c \cdot \nabla j^c$ and $j^c \times B^c$. Indeed, the singular term $\widetilde{E}^c = cE^c$ in (2.1) prevents the temporal compactness of j^c and B^c (at least, through classical methods).

Instead, it is crucial here to treat the remaining nonlinear terms together by making use of the identity

$$j^c \times B^c - \varepsilon^2 j^c \cdot \nabla j^c = j^c \times \left(B^c + \varepsilon^2 \nabla \times j^c\right) - \frac{\varepsilon^2}{2} \nabla \left(j^c\right)^2 .$$

The last term above is then incorporated in the pressure gradient, so that we only need to establish the temporal compactness of $B^c + \varepsilon^2 \nabla \times j^c$. This, in turn, follows through standard methods from the fact that $B^c + \varepsilon^2 \nabla \times j^c$ solves the evolution equation

$$\begin{aligned} \partial_t \left(B^c + \varepsilon^2 \nabla \times j^c\right) + \varepsilon^2 \nabla \times \left(u^c \cdot \nabla j^c + j^c \cdot \nabla u^c\right) - \varepsilon^2 \mu \Delta \nabla \times j^c \\ + \frac{1}{\sigma} \nabla \times j^c = \nabla \times \left(u^c \times B^c\right), \end{aligned}$$

where the singular term $\widetilde{E}^c = cE^c$ has been cancelled out, which concludes the proof of weak stability of all nonlinear terms.

Moreover, we easily obtain from the equicontinuity of $\int_{\mathbb{R}^d} \left(B^c + \varepsilon^2 \nabla \times j^c\right) \varphi \mathrm{d}x$, for any $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$, that the initial value of the weak limit point $B + \varepsilon^2 \nabla \times j$ is precisely given by $B^0 + \varepsilon^2 \nabla \times j^0$. Since the weak limit points of j^c and B^c necessarily satisfy the quasi-static Ampère law for each time, even initially, we finally deduce that the limiting initial data is given by

$$\begin{aligned} u|_{t=0} &= u^0, \\ j|_{t=0} &= \left(1 - \varepsilon^2 \Delta\right)^{-1} \left(\nabla \times B^0 - \varepsilon^2 \Delta j^0\right) \\ &= j^0 + \left(1 - \varepsilon^2 \Delta\right)^{-1} \left(\nabla \times B^0 - j^0\right), \\ B|_{t=0} &= \left(1 - \varepsilon^2 \Delta\right)^{-1} \left(B^0 + \varepsilon^2 \nabla \times j^0\right) \\ &= B^0 + \left(1 - \varepsilon^2 \Delta\right)^{-1} \varepsilon^2 \nabla \times \left(j^0 - \nabla \times B^0\right), \end{aligned}$$

which concludes the justification of the proposition. □

Now, in order to complete the formal asymptotic analysis of (1.8), we simply notice that letting ε tend to zero in (2.8) or (2.9) yields the magnetohydrodynamic system (1.1), again. This formal convergence can easily be made rigorous. Accordingly, we have the following simple result on the convergence of (2.8) towards (1.1).

Proposition 2.2. *Let $d = 2, 3$ and, for each $\varepsilon > 0$, consider $\left(u^{\pm\varepsilon}, B^{\varepsilon}\right)$ a global and finite energy weak solution of the quasi-static two-fluid incompressible Navier–Stokes–Maxwell system* (2.8) *for some initial data*

$$(u^{\pm 0\varepsilon}, B^{0\varepsilon}) \in L^2_x, \quad \textit{such that} \quad \mathrm{div} u^{\pm 0\varepsilon} = \mathrm{div} B^{0\varepsilon} = 0,$$

converging weakly, as $\varepsilon \to 0$, towards some

$$(u^{\pm 0}, B^{0}) \in L^2_x, \quad \textit{such that} \quad \mathrm{div} u^{\pm 0} = \mathrm{div} B^{0} = 0.$$

Then, as $\varepsilon \to 0$, up to extraction of a subsequence, $\left(u^{\pm\varepsilon}, B^{\varepsilon}\right)$ converges weakly towards $\left(u^{\pm}, B\right)$, where $u^+ = u^- = u$ and (u, B) is a global and finite energy weak solution of the magnetohydrodynamic system (1.1)*, with initial data precisely given by*

$$u|_{t=0} = \frac{u^{+0} + u^{-0}}{2}, \qquad B|_{t=0} = B^0.$$

Proof. The demonstration of this convergence poses no great difficulty except for the handling of time compactness, which is consistent with the preceding difficulties encountered in the justification of the existence of Leray solutions to the systems (2.8) or (2.9).

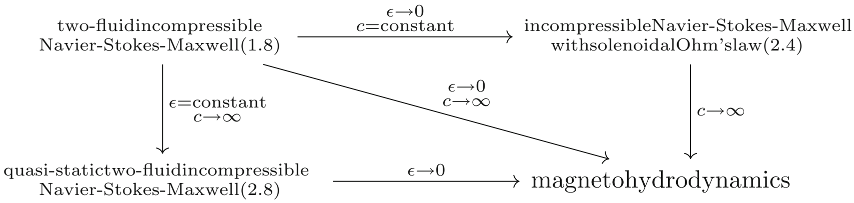
Thus, in order to handle temporal compactness, we establish the convergence of the equivalent system (2.9) towards (1.1).

To this end, by virtue of the natural uniform energy bounds from (2.10) and classical compactness results by Aubin and Lions [2, 18], we easily deduce first that u^{ε} is compact, up to extraction of a subsequence, in all variables in the strong topology of L^2_{loc}. This is sufficient to handle the convergence of all nonlinear terms in (2.9) except $j^{\varepsilon} \times B^{\varepsilon}$.

Then, similar arguments show that $B^{\varepsilon} + \varepsilon^2 \nabla \times j^{\varepsilon}$ is compact, up to extraction of a subsequence, in all variables in the strong topology of L^2_{loc}. Since $\varepsilon \nabla \times j^{\varepsilon}$ is uniformly bounded in $L^2_{t,x}$, we deduce that B^{ε} converges towards B in the strong topology of L^2_{loc}, which is sufficient to establish the weak stability of the remaining nonlinear term $j^{\varepsilon} \times B^{\varepsilon}$.

Finally, we easily obtain from the equicontinuity of $\int_{\mathbb{R}^d} \left(B^{\varepsilon} + \varepsilon^2 \nabla \times j^{\varepsilon}\right) \varphi \mathrm{d}x$, for any $\varphi \in C_c^{\infty}\left(\mathbb{R}^d\right)$, that the initial value of the weak limit point B is precisely given by B^0, which concludes the justification of the proposition. □

On the whole, we have the following formal commutative diagram representing the different asymptotic regimes:



In fact, the complete rigorous justification of the regimes

$$(1.8) \xrightarrow{\varepsilon\to 0} (2.4) \xrightarrow{c\to\infty} (1.1) \tag{2.11}$$

is far from reach, for the incompressible Navier–Stokes–Maxwell system (2.4) is very singular. Nevertheless, in Section 4, we provide a conditional result concerning the convergence of the incompressible Navier–Stokes–Maxwell system with Ohm's law (2.6), which is similar to system (2.4), towards the magnetohydrodynamic system (1.1).

On the other hand, as shown in Propositions 2.1 and 2.2 it is possible to establish rigorously the limits

$$(1.8) \xrightarrow{c\to\infty} (2.8) \xrightarrow{\varepsilon\to 0} (1.1), \tag{2.12}$$

because the transitional system (2.8) is much better behaved than (2.4).

Finally, it is now clear, at least formally, that letting both $\varepsilon \to 0$ and $c \to \infty$ simultaneously in the two-fluid system (1.8) also yields the magnetohydrodynamic system (1.1). This asymptotic regime seems therefore much less singular than (2.11), since it removes the need to deal with system (2.4) and, thus, is an intermediate regime between (2.11) and (2.12). Our main result Theorem 1.1 justifies this asymptotic regime rigorously provided the momentum transfer coefficient ε is not too small relative to the speed of light c, as expressed by the assumption (1.10). In other words, assumption (1.10) aims at staying "away" from system (2.4), which is just too singular. Relaxing the constraint (1.10) on the asymptotic parameters represents an important open problem, which we believe requires new and original ideas.

3. Spectral Properties of Maxwell's Operator

Here, we detail the linear analysis of Maxwell's system (2.7), whose spectral decomposition will be essential for the proof of Theorem 1.1.

Clearly, Maxwell's system (2.7) may be recast as

$$\partial_t \begin{pmatrix} E \\ B \end{pmatrix} = \mathcal{L}_c \begin{pmatrix} E \\ B \end{pmatrix} + c \begin{pmatrix} G \\ 0 \end{pmatrix},$$

where Maxwell's operator $\mathcal{L}_c$ is given by

$$\mathcal{L}_c := c \begin{pmatrix} -\sigma c \mathrm{Id} & \nabla\times \\ -\nabla\times & 0 \end{pmatrix}.$$

More precisely, the operator

$$\mathcal{L}_c : \mathcal{D}(\mathcal{L}_c) \subset X \to X,$$

is defined as an unbounded linear operator, where

$$X := \left\{ (E, B) \in \left(L^2 \left(\mathbb{R}^d \right) \right)^2 \, : \, \operatorname{div} B = 0 \right\},$$

whose domain is given by

$$\mathcal{D}(\mathcal{L}_c) := \left\{ (E, B) \in X \, : \, (PE, B) \in \left(H^1 \left(\mathbb{R}^d \right) \right)^2 \right\},$$

where $P : L^2 \left(\mathbb{R}^d \right) \to L^2 \left(\mathbb{R}^d \right)$ denotes the Leray projector over solenoidal vector fields. Recall that, in the case $d = 2$, one has $E_3 \equiv 0$, $B_1 \equiv B_2 \equiv 0$ and $G_3 \equiv 0$.

It is then readily seen that $\mathcal{L}_c$ is a closed operator and that $\mathcal{D}(\mathcal{L}_c)$ is dense in X. Furthermore, one verifies that, for any $\lambda > 0$, the operator

$$\lambda \mathrm{Id} - \mathcal{L}_c : \mathcal{D}(\mathcal{L}_c) \to X,$$

is bijective and that the resolvent operator $R_\lambda := (\lambda \mathrm{Id} - \mathcal{L}_c)^{-1}$ satisfies that $\|R_\lambda\| \leqq \frac{1}{\lambda}$. Therefore, by the Hille-Yosida Theorem, the unbounded operator $\mathcal{L}_c$ is the infinitesimal generator of a contraction semigroup, denoted by $e^{t\mathcal{L}_c}$, strongly continuous on X.

In particular, for any initial data $\left(E^0, B^0\right) \in \mathcal{D}(\mathcal{L}_c)$ and any source term $(G, 0) \in C\left(\mathbb{R}^+; \mathcal{D}(\mathcal{L}_c)\right)$ (or $(G, 0) \in C^1\left(\mathbb{R}^+; X\right)$), the unique (strong) solution to (2.7)

$$(E, B) \in C\left(\mathbb{R}^+; \mathcal{D}(\mathcal{L}_c)\right) \cap C^1\left(\mathbb{R}^+; X\right),$$

is given by Duhamel's formula:

$$\begin{pmatrix} E \\ B \end{pmatrix}(t) = e^{t\mathcal{L}_c} \begin{pmatrix} E^0 \\ B^0 \end{pmatrix} + c \int_0^t e^{(t-\tau)\mathcal{L}_c} \begin{pmatrix} G \\ 0 \end{pmatrix}(\tau)\, \mathrm{d}\tau. \tag{3.1}$$

Furthermore, notice that Duhamel's formula (3.1) remains well-defined whenever the initial data $\left(E^0, B^0\right)$ lies in X, while the source term G merely belongs to $L^1_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{R}^d\right)\right)$. Indeed, for any given $t > 0$, one verifies that

$$e^{(t-\tau)\mathcal{L}_c} \begin{pmatrix} G \\ 0 \end{pmatrix}(\tau) \text{ is strongly measurable,}$$

for it is weakly measurable and $L^2\left(\mathbb{R}^d\right)$ is separable, and

$$\left\| e^{(t-\tau)\mathcal{L}_c} \begin{pmatrix} G \\ 0 \end{pmatrix}(\tau) \right\|_X \leqq \|G(\tau)\|_{L^2_x} \text{ is integrable,}$$

so that

$$e^{(t-\tau)\mathcal{L}_c} \begin{pmatrix} G \\ 0 \end{pmatrix}(\tau) \text{ is Bochner integrable.}$$

In this setting, the electromagnetic field $(E, B)(t)$ given by (3.1) defines a weak solution of (2.7) in $C\left(\mathbb{R}^+; X\right)$, and one verifies (employing approximations, for instance) that it solves (2.7) in the sense of distributions and that it is unique in $C\left(\mathbb{R}^+; X\right)$ within this class of solutions.

Next, in order to refine our understanding of the action of the semigroup and the ensuing behavior of the electromagnetic field (E, B), we conduct a spectral analysis of $\mathcal{L}_c$. Since, it has constant coefficients, we use the Fourier transform, which is denoted by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\,\mathrm{d}x,$$

and its inverse by

$$\mathcal{F}^{-1}g(x) = \tilde{g}(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi)\,\mathrm{d}\xi.$$

For every $\xi \in \mathbb{R}^3 \setminus \{0\}$, such that $\xi_3 = 0$ when $d = 2$, we further define the subspace

$$\begin{aligned} \mathcal{E}(\xi) &:= \left\{(e, b) \in \mathbb{C}^3 \times \mathbb{C}^3 \,:\, \xi\cdot b = 0\right\}, && \text{if } d = 3,\\ \mathcal{E}(\xi) &:= \left\{(e, b) \in \mathbb{C}^3 \times \mathbb{C}^3 \,:\, e_3 = b_1 = b_2 = 0\right\}, && \text{if } d = 2, \end{aligned}$$

and the linear finite-dimensional operator $\hat{\mathcal{L}}_c(\xi) : \mathcal{E}(\xi) \to \mathcal{E}(\xi)$ by

$$\hat{\mathcal{L}}_c(\xi)\begin{pmatrix} e\\ b\end{pmatrix} := c\begin{pmatrix} -\sigma ce + i\xi\times b\\ -i\xi\times e\end{pmatrix}.$$

Note that, in the two-dimensional case, the vector $\xi \in \mathbb{R}^2$ is to be identified with $(\xi_1, \xi_2, 0) \in \mathbb{R}^3$, so that $\xi\times b = (\xi_2 b_3, -\xi_1 b_3, 0)$ and $\xi\times e = (0, 0, \xi_1 e_2 - \xi_2 e_1)$.

Clearly, $\mathcal{E}(\xi)$ is a $(2d-1)$-dimensional vector subspace of $\mathbb{C}^3 \times \mathbb{C}^3$ and any $(E, B) \in X$ satisfies that $\left(\hat{E}(\xi), \hat{B}(\xi)\right) \in \mathcal{E}(\xi)$, for almost every $\xi \in \mathbb{R}^d$. Finally, note that, for any $(E, B) \in X$,

$$\mathcal{F}\left(\mathcal{L}_c\begin{pmatrix} E\\ B\end{pmatrix}\right)(\xi) = c\begin{pmatrix} -\sigma c\hat{E} + i\xi\times\hat{B}\\ -i\xi\times\hat{E}\end{pmatrix} = \hat{\mathcal{L}}_c(\xi)\begin{pmatrix}\hat{E}\\ \hat{B}\end{pmatrix},$$

and

$$\mathcal{F}\left(e^{t\mathcal{L}_c}\begin{pmatrix} E\\ B\end{pmatrix}\right)(\xi) = e^{t\hat{\mathcal{L}}_c(\xi)}\begin{pmatrix}\hat{E}\\ \hat{B}\end{pmatrix}.$$

Then, we have the following properties.

Proposition 3.1. *For any $\xi \in \mathbb{R}^3 \setminus \{0\}$, such that $|\xi| \neq \frac{\sigma c}{2}$ (and $\xi_3 = 0$ when $d = 2$), the distinct eigenvalues of $\hat{\mathcal{L}}_c(\xi)$ are $\lambda_0 = -\sigma c^2$, $\lambda_+(\xi)$ and $\lambda_-(\xi)$, with*

$$\lambda_\pm(\xi) = \frac{-\sigma c^2 \pm \sqrt{\sigma^2 c^4 - 4c^2|\xi|^2}}{2}. \tag{3.2}$$

Furthermore, there exists a basis of eigenvectors (that is $\hat{\mathcal{L}}_c(\xi)$ is diagonalizable) and the eigenspaces corresponding to λ_0, $\lambda_+(\xi)$ and $\lambda_-(\xi)$ are respectively given by

$$\begin{aligned}
\mathcal{E}_0(\xi) &= \left\langle \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\rangle, \\
\mathcal{E}_+(\xi) &= \left\{ \begin{pmatrix} e \\ \frac{-ic}{\lambda_+}\xi \times e \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \, : \, e \in \mathbb{C}^3,\ \xi \cdot e = 0 \ (\textit{and } e_3 = 0 \textit{ if } d = 2) \right\} \\
&= \left\{ \begin{pmatrix} \frac{-ic}{\lambda_-}\xi \times b \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \, : \, b \in \mathbb{C}^3,\ \xi \cdot b = 0 \ (\textit{and } b_1 = b_2 = 0 \textit{ if } d = 2) \right\}, \\
\mathcal{E}_-(\xi) &= \left\{ \begin{pmatrix} e \\ \frac{-ic}{\lambda_-}\xi \times e \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \, : \, e \in \mathbb{C}^3,\ \xi \cdot e = 0 \ (\textit{and } e_3 = 0 \textit{ if } d = 2) \right\} \\
&= \left\{ \begin{pmatrix} \frac{-ic}{\lambda_+}\xi \times b \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \, : \, b \in \mathbb{C}^3,\ \xi \cdot b = 0 \ (\textit{and } b_1 = b_2 = 0 \textit{ if } d = 2) \right\}.
\end{aligned}$$

For any $\xi \in \mathbb{R}^3 \setminus \{0\}$, such that $|\xi| = \frac{\sigma c}{2}$ (and $\xi_3 = 0$ when $d = 2$), the distinct eigenvalues of $\mathcal{L}_c(\xi)$ are $\lambda_0 = -\sigma c^2$ and $\lambda_1 = -\frac{\sigma c^2}{2}$. Furthermore, $\hat{\mathcal{L}}_c(\xi)$ is not diagonalizable and the eigenspaces corresponding to λ_0 and λ_1 are respectively given by

$$\begin{aligned}
\mathcal{E}_0(\xi) &= \left\langle \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\rangle, \\
\mathcal{E}_1(\xi) &= \left\{ \begin{pmatrix} e \\ \frac{2i}{\sigma c}\xi \times e \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \, : \, e \in \mathbb{C}^3,\ \xi \cdot e = 0 \ (\textit{and } e_3 = 0 \textit{ if } d = 2) \right\} \\
&= \left\{ \begin{pmatrix} \frac{2i}{\sigma c}\xi \times b \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \, : \, b \in \mathbb{C}^3,\ \xi \cdot b = 0 \ (\textit{and } b_1 = b_2 = 0 \textit{ if } d = 2) \right\}.
\end{aligned}$$

The generalized eigenspace corresponding to λ_1 is given by

$$\mathcal{K}_1(\xi) = \left\{ \begin{pmatrix} e \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \, : \, \xi \cdot e = \xi \cdot b = 0 \ (\textit{and } e_3 = b_1 = b_2 = 0 \textit{ if } d = 2) \right\}.$$

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\hat{\mathcal{L}}_c(\xi)$. Then, there exists an eigenvector $(e, b) \in \mathcal{E}(\xi) \setminus \{0\}$ such that

$$\hat{\mathcal{L}}_c(\xi) \begin{pmatrix} e \\ b \end{pmatrix} = \lambda \begin{pmatrix} e \\ b \end{pmatrix}.$$

More precisely, one has

$$-\sigma c^2 e + ic\xi \times b = \lambda e, \tag{3.3}$$

$$-ic\xi \times e = \lambda b. \tag{3.4}$$

Notice that $e \neq 0$, otherwise $e = b = 0$.

Multiplying (3.3) by ξ (that is taking the divergence in physical variables), we get that $\left(\lambda + \sigma c^2\right) \xi \cdot e = 0$, which suggests that $\lambda = -\sigma c^2$ is a good candidate for an eigenvalue.

Thus, setting $\lambda = -\sigma c^2$, we obtain from (3.3) that necessarily $b = 0$ in this case. We then deduce from (3.4) that there exists an eigenvector exactly when e and ξ are colinear. Thus, we conclude that $\lambda_0 = -\sigma c^2$ is an eigenvalue with a corresponding eigenspace $\mathcal{E}_0(\xi)$ spanned exactly by the vector $(\xi, 0)$.

Next, since $\xi \cdot e \neq 0$ implies that $\lambda = -\sigma c^2$, there only remains to consider the case $\xi \cdot e = 0$. In particular, since $\xi \neq 0$ and $e \neq 0$, we have now that $\xi \times e \neq 0$. Thus, taking the curl of (3.3) and then using (3.4), we get

$$\left(\lambda^2 + \sigma c^2 \lambda + c^2 |\xi|^2\right) (\xi \times e) = 0,$$

whence $\left(\left(\frac{\lambda}{c}\right)^2 + \sigma\lambda + |\xi|^2\right) = 0$, whose roots $\lambda_+(\xi)$ and $\lambda_-(\xi)$ are clearly given by (3.2).

We assume now that $|\xi| \neq \frac{\sigma c}{2}$, so that $\lambda_+(\xi)$ and $\lambda_-(\xi)$ are distinct. An easy computation shows then that $\lambda_\pm(\xi)$ are eigenvalues with corresponding eigenspaces $\mathcal{E}_\pm(\xi)$ respectively defined by

$$\left\{\begin{pmatrix} e \\ -\frac{ic}{\lambda_\pm(\xi)}\xi \times e \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \ : \ e \in \mathbb{C}^3, \ \xi \cdot e = 0 \text{ (and } e_3 = 0 \text{ if } d = 2)\right\}$$
$$= \left\{\begin{pmatrix} -\frac{ic}{\lambda_\mp(\xi)}\xi \times b \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \ : \ b \in \mathbb{C}^3, \ \xi \cdot b = 0 \text{ (and } b_1 = b_2 = 0 \text{ if } d = 2)\right\}.$$

Notice that $\mathcal{E}_+(\xi)$ and $\mathcal{E}_-(\xi)$ have then maximal dimension $(d-1)$, which implies that $\hat{\mathcal{L}}_c(\xi)$ is diagonalizable.

In the case $|\xi| = \frac{\sigma c}{2}$, the eigenvalues $\lambda_+(\xi) = \lambda_-(\xi) = -\frac{\sigma c^2}{2}$ coincide and so do the eigenspaces $\mathcal{E}_+(\xi)$ and $\mathcal{E}_-(\xi)$. It is then readily seen that the eigenspace $\mathcal{E}_1(\xi)$ corresponding to the eigenvalue $\lambda_1 = -\frac{\sigma c^2}{2}$ is given by

$$\left\{\begin{pmatrix} e \\ \frac{2i}{\sigma c}\xi \times e \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \ : \ e \in \mathbb{C}^3, \ \xi \cdot e = 0 \text{ (and } e_3 = 0 \text{ if } d = 2)\right\}$$
$$= \left\{\begin{pmatrix} \frac{2i}{\sigma c}\xi \times b \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \ : \ b \in \mathbb{C}^3, \ \xi \cdot b = 0 \text{ (and } b_1 = b_2 = 0 \text{ if } d = 2)\right\},$$

which is a $(d-1)$-dimensional subspace. Therefore, since there cannot be any more eigenvalues in view of the preceding discussion, we conclude that $\hat{\mathcal{L}}_c(\xi)$ is not diagonalizable, and a complete basis of the space $\mathcal{E}(\xi)$ will be obtained by computing the generalized eigenvectors of $\hat{\mathcal{L}}_c(\xi)$.

More precisely, we are looking for generalized eigenvectors $(e, b) \in \mathcal{E}(\xi) \setminus \{0\}$ such that

$$\left(\lambda_1 \mathrm{Id} - \hat{\mathcal{L}}_c(\xi)\right)^2 \begin{pmatrix} e \\ b \end{pmatrix} = 0. \tag{3.5}$$

In fact, a straightforward calculation shows

$$\left(\lambda_1 \mathrm{Id} - \hat{\mathcal{L}}_c(\xi)\right)^2 \begin{pmatrix} e \\ b \end{pmatrix} = \begin{pmatrix} c^2 (\xi \cdot e)\, \xi \\ 0 \end{pmatrix},$$

whence the solutions to (3.5) are exactly determined by the condition $\xi \cdot e = 0$. The corresponding $(2d-2)$-dimensional generalized eigenspace $\mathcal{K}_1(\xi)$ is therefore given by

$$\left\{ \begin{pmatrix} e \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 \ : \ \xi \cdot e = \xi \cdot b = 0 \text{ (and } e_3 = b_1 = b_2 = 0 \text{ if } d = 2) \right\},$$

which concludes the proof of the proposition. □

Remark. One easily verifies that the operator $\hat{\mathcal{L}}_c(\xi)$ is not normal (that is it does not commute with its adjoint) and, therefore, the eigenspaces $\mathcal{E}_0(\xi)$, $\mathcal{E}_+(\xi)$ and $\mathcal{E}_-(\xi)$ are not all orthogonal to each other. However, it is seen that each $\mathcal{E}_\pm(\xi)$ is orthogonal to $\mathcal{E}_0(\xi)$.

Lemma 3.2. *Let $\xi \in \mathbb{R}^3$ and consider the eigenvalues $\lambda_\pm(\xi)$ defined by (3.2). Then, if $|\xi| \leqq \frac{\sigma c}{2}$,*

$$-\sigma c^2 \leqq \lambda_-(\xi) \leqq -\frac{\sigma c^2}{2} \leqq -c|\xi| \leqq -\frac{2|\xi|^2}{\sigma} \leqq \lambda_+(\xi) \leqq -\frac{|\xi|^2}{\sigma},$$
$$\text{and} \quad \sqrt{1 - \left(\frac{2|\xi|}{\sigma c}\right)^2} \leqq \frac{\lambda_-(\xi) - \lambda_+(\xi)}{\lambda_-(\xi)} \leqq 2\sqrt{1 - \left(\frac{2|\xi|}{\sigma c}\right)^2},$$

and, if $|\xi| \geqq \frac{\sigma c}{2}$,

$$\Re\left(\lambda_-(\xi)\right) = \Re\left(\lambda_+(\xi)\right) = -\frac{\sigma c^2}{2}, \qquad |\lambda_+(\xi)| = |\lambda_-(\xi)| = c|\xi|,$$
$$\text{and} \quad \left|\frac{\lambda_-(\xi) - \lambda_+(\xi)}{\lambda_-(\xi)}\right| = 2\sqrt{1 - \left(\frac{\sigma c}{2|\xi|}\right)^2}.$$

Proof. We first examine the case $|\xi| \leqq \frac{\sigma c}{2}$. Writing

$$\lambda_\pm(\xi) = -\frac{\sigma c^2}{2}\left(1 \mp \sqrt{1 - \left(\frac{2|\xi|}{\sigma c}\right)^2}\right) \in \mathbb{R},$$

and using the elementary inequalities $x \leqq \sqrt{1+x} - 1 \leqq \frac{x}{2}$, valid for all $-1 \leqq x \leqq 0$, we infer that

$$-\frac{2|\xi|^2}{\sigma} \leqq \lambda_+(\xi) = \frac{\sigma c^2}{2}\left(\sqrt{1 - \left(\frac{2|\xi|}{\sigma c}\right)^2} - 1\right) \leqq -\frac{|\xi|^2}{\sigma}.$$

Moreover, we easily find

$$-\sigma c^2 \leqq \lambda_-(\xi) = -\frac{\sigma c^2}{2}\left(1+\sqrt{1-\left(\frac{2|\xi|}{\sigma c}\right)^2}\right) \leqq -\frac{\sigma c^2}{2}.$$

Finally, since

$$\lambda_-(\xi) - \lambda_+(\xi) = -\sigma c^2\sqrt{1-\left(\frac{2|\xi|}{\sigma c}\right)^2},$$

we infer, in view of the preceding estimates on $\lambda_-(\xi)$, that

$$\sqrt{1-\left(\frac{2|\xi|}{\sigma c}\right)^2} \leqq \frac{\lambda_-(\xi)-\lambda_+(\xi)}{\lambda_-(\xi)} \leqq 2\sqrt{1-\left(\frac{2|\xi|}{\sigma c}\right)^2}.$$

Next, considering the case $|\xi| \geqq \frac{\sigma c}{2}$ and writing

$$\lambda_\pm(\xi) = -\frac{\sigma c^2}{2} \pm i\sqrt{c^2|\xi|^2 - \left(\frac{\sigma c^2}{2}\right)^2},$$

we obviously deduce that $\Re\left(\lambda_-(\xi)\right) = \Re\left(\lambda_+(\xi)\right) = -\frac{\sigma c^2}{2}$ and a straightforward calculation shows that

$$|\lambda_\pm(\xi)|^2 = \Re\left(\lambda_\pm(\xi)\right)^2 + \Im\left(\lambda_\pm(\xi)\right)^2 = c^2|\xi|^2.$$

Finally, further noticing that

$$\bar{\lambda}_\pm(\xi) = \lambda_\mp(\xi) \qquad \text{and} \qquad \lambda_+(\xi)^2 + \lambda_-(\xi)^2 = \sigma^2 c^4 - 2c^2|\xi|^2,$$

we find

$$\begin{aligned}\left|\frac{\lambda_-(\xi)-\lambda_+(\xi)}{\lambda_-(\xi)}\right|^2 &= \frac{|\lambda_-(\xi)|^2 + |\lambda_+(\xi)|^2 - \lambda_-(\xi)^2 - \lambda_+(\xi)^2}{|\lambda_-(\xi)|^2} \\ &= 4\left(1-\left(\frac{\sigma c}{2|\xi|}\right)^2\right),\end{aligned}$$

which concludes the proof of the lemma. □

4. A Preliminary Conditional Convergence Result

In this section, we establish a result concerning the convergence of the incompressible Navier–Stokes–Maxwell system with Ohm's law (2.6) toward the viscous incompressible magnetohydrodynamic system (1.1). This result is conditional because, first, as discussed in Section 2, we do not know whether the system (2.6) has global weak solutions uniformly bounded in the energy space and, second, it requires rather restrictive assumptions on the control of the high frequencies of the electrical current which are unnatural. Nevertheless, the proof of this result will also provide some crucial understanding of the asymptotic behavior of the two-fluid incompressible Navier–Stokes–Maxwell system (1.8) and will serve as a primer to Theorem 1.1. Indeed, the proof of Theorem 1.1, which is presented in the next section, will closely follow the steps and ideas of the proof below. Our main result concerning system (2.6) follows. A similar result on the convergence of (2.4) towards (1.1) can also be obtained.

Proposition 4.1. *Let $d = 2, 3$ and, for each $c > 0$, consider (u^c, E^c, B^c) a global and finite energy weak solution (provided they exist, which is unknown) of the incompressible Navier–Stokes–Maxwell system with Ohm's law* (2.6) *for some initial data*

$$(u^{0c}, E^{0c}, B^{0c}) \in L^2_x, \quad \textit{such that} \quad \operatorname{div} u^{0c} = \operatorname{div} B^{0c} = 0,$$

converging weakly, as $c \to \infty$, towards some

$$(u^0, E^0, B^0) \in L^2_x, \quad \textit{such that} \quad \operatorname{div} u^0 = \operatorname{div} B^0 = 0.$$

Let us further assume that the very high frequencies of j^c [*given by Ohm's law in* (2.6)] *defined by*

$$j^c_{\gg} := \mathcal{F}^{-1} \mathbb{1}_{\{|\xi| > \phi(\delta c)\}} \mathcal{F} j^c,$$

for any $\delta > 0$, where

$$\begin{aligned} \phi(c) &= c^2, && \textit{if} \quad d = 3, \\ \phi(c) &= \exp\left(c^2\right), && \textit{if} \quad d = 2, \end{aligned} \tag{4.1}$$

are asymptotically uniformly controlled in the sense that, for any $\delta > 0$,

$$\limsup_{c \to \infty} \left\| j^c_{\gg} \right\|_{L^2_{t,x,\mathrm{loc}}} = 0. \tag{4.2}$$

Then, as $c \to \infty$, up to extraction of a subsequence, (u^c, B^c) converges weakly to a global and finite energy weak solution (u, B) of the magnetohydrodynamic system (1.1)*, with initial data precisely given by*

$$u|_{t=0} = u^0, \qquad B|_{t=0} = B^0.$$

Remark. It will be clear from its proof that the above proposition remains valid if, instead of requiring the control (4.2) on the very high frequencies of the electrical current, one asks that

$$\limsup_{c\to\infty} \frac{1}{c} \left\| u^c \times B^c \right\|_{L^2_{t,x,\mathrm{loc}}} = 0.$$

This condition almost holds true in two dimensions, for u^c barely fails to belong uniformly to $L^2_t L^\infty_x$ (by Sobolev embedding of $L^2_t \dot{H}^1_x$). In two dimensions, one can use the global solutions constructed in [19] instead of weak solutions; however, it is not clear whether they satisfy the high frequency assumption on j^c. In three dimensions, one can hope to construct stronger solutions to system (2.6) using Besov spaces techniques (see for instance [5]); however, we were not able to make the time of existence uniform in c.

Proof. We are thus considering here a family of weak solutions (u^c, E^c, B^c) (provided they exist, which is unknown) of the incompressible Navier–Stokes–Maxwell system (2.6). Accordingly, the energy inequality stemming from (2.5) merely provides uniform bounds on the weak solutions in

$$u^c \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x, \qquad E^c \in L^\infty_t L^2_x, \qquad B^c \in L^\infty_t L^2_x, \qquad j^c \in L^2_t L^2_x, \quad (4.3)$$

where j^c denotes the corresponding electric current determined through Ohm's law.

The proof will proceed through a weak compactness method. The convergence in Maxwell's system and in the incompressibility relation will be readily obtained, for these equations are linear. Moreover, in view of the uniform a priori bounds (4.3), we will easily show through standard compactness arguments that u^c is relatively compact in the strong topology of $L^2_{t,x,\mathrm{loc}}$ and the convergence of Ohm's law will ensue.

Thus, the most difficult convergence will concern the equation of conservation of momentum in (2.6), where one has to justify the stability of the term given by the Lorentz force $j^c \times B^c$. To this end, standard compensated compactness methods plainly fail (see discussion on page 774) and we will have to show that the magnetic field B^c enjoys some kind of relative compactness in the strong topology of $L^2_{t,x,\mathrm{loc}}$. The convergence of the Lorentz force will then follow employing the conditional hypothesis (4.2).

Let us move on now to the actual core of the proof. Up to extraction of subsequences, we have the following weak convergences, as $c \to \infty$:

$$\begin{aligned} \left(u^{0c}, E^{0c}, B^{0c}\right) &\rightharpoonup \left(u^0, E^0, B^0\right), && \text{in} \quad L^2_x, \\ \left(u^c, E^c, B^c\right) &\rightharpoonup^* (u, E, B), && \text{in} \quad L^\infty_t L^2_x, \\ j^c &\rightharpoonup j, && \text{in} \quad L^2_t L^2_x. \end{aligned}$$

Then, from Ampère's equation, we deduce, letting $c \to \infty$, that

$$j = \nabla \times B \in L^2_t L^2_x,$$

whence, since $\operatorname{div} B = 0$,

$$B \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1.$$

Moreover, since u^c is uniformly bounded in

$$\begin{aligned} &L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1 \subset L_t^\infty L_x^2 \cap L_t^2 L_x^6 \subset L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}, \quad \text{if} \quad d = 3, \\ &L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1 \subset \cap_{2<p\leqq\infty} L_t^p L_x^{\frac{2p}{p-2}} \subset L_t^4 L_x^4, \quad \text{if} \quad d = 2, \end{aligned}$$

and $\partial_t u^c$ is bounded in $L_t^2 H_x^{-2}$, we conclude, invoking a classical compactness result by Aubin and Lions [2,18] (see also [23] for a sharp compactness criterion), that

$$u^c \to u \quad \text{in } L^3_{t,x,\mathrm{loc}},$$

which is sufficient to justify the convergence of the nonlinear advection term $u^c \cdot \nabla u^c$ towards $u \cdot \nabla u$, in the sense of distributions, say. In fact, in the two-dimensional case only, employing the optimal constants for Sobolev embeddings evaluated in [6,24], we have the following refined estimate on u^c, for any $2 < p \leqq \infty$:

$$\begin{aligned} \left\| u^c \right\|_{L_t^p L_x^{\frac{2p}{p-2}}} &\leqq C \left(\frac{p}{p-2} \right)^{\frac{1}{2}} \left\| u^c \right\|_{L_t^p \dot{H}_x^{\frac{2}{p}}} \\ &\leqq C \left(\frac{p}{p-2} \right)^{\frac{1}{2}} \left\| u^c \right\|_{L_t^\infty L_x^2}^{1-\frac{2}{p}} \left\| u^c \right\|_{L_t^2 \dot{H}_x^1}^{\frac{2}{p}}, \end{aligned} \tag{4.4}$$

for some constant $C > 0$ independent of p.

Next, note that the bounds on u^c and B^c imply that

$$\begin{aligned} &u^c \times B^c \in L_t^\infty L_x^1 \cap L_t^2 L_x^{\frac{3}{2}}, \qquad \text{if} \quad d = 3, \\ &u^c \times B^c \in \cap_{2<p\leqq\infty} L_t^p L_x^{\frac{p}{p-1}}, \quad \text{if} \quad d = 2, \end{aligned} \tag{4.5}$$

whence, for any $2 < p < \infty$ and $d = 2, 3$,

$$G^c := -\sigma u^c \times B^c \rightharpoonup G := -\sigma u \times B \quad \text{in } L_t^p L_x^{\frac{dp}{dp-2}}.$$

Again, in the two-dimensional case only, utilizing (4.4), we have the refined estimate, for any $2 < p \leqq \infty$,

$$\begin{aligned} \left\| G^c \right\|_{L_t^p L_x^{\frac{p}{p-1}}} &\leqq \sigma \left\| u^c \right\|_{L_t^p L_x^{\frac{2p}{p-2}}} \left\| B^c \right\|_{L_t^\infty L_x^2} \\ &\leqq C \left(\frac{p}{p-2} \right)^{\frac{1}{2}} \left\| u^c \right\|_{L_t^\infty L_x^2}^{1-\frac{2}{p}} \left\| u^c \right\|_{L_t^2 \dot{H}_x^1}^{\frac{2}{p}} \left\| B^c \right\|_{L_t^\infty L_x^2}, \end{aligned} \tag{4.6}$$

for some constant $C > 0$ independent of p.

Now, utilizing Ohm's law to estimate cE^c yields

$$cE^c \in L_t^2 L_x^2 + \left(L_t^\infty L_x^1 \cap L_t^2 L_x^{\frac{3}{2}} \right), \qquad \text{if} \quad d = 3,$$

$$cE^c \in L_t^2 L_x^2 + \left(\cap_{2<p\leqq\infty} L_t^p L_x^{\frac{p}{p-1}} \right), \quad \text{if} \quad d = 2.$$

Therefore, for any $2 < p < \infty$,

$$cE^c \rightharpoonup \frac{1}{\sigma} \nabla \times B - u \times B \quad \text{in } L_t^2 L_x^2 + L_t^p L_x^{\frac{dp}{dp-2}},$$

and, from Faraday's equation,

$$B^c \in C\left(\mathbb{R}^+; w\text{-}L^2\left(\mathbb{R}^d\right)\right) \text{ is equi-continuous uniformly in } c,$$

where the prefix w- denotes that $L^2\left(\mathbb{R}^d\right)$ is endowed with its weak topology (in fact, recall that $B^c \in C\left(\mathbb{R}^+; L^2\left(\mathbb{R}^2\right)\right)$, but the equi-continuity in the strong topology of $L^2\left(\mathbb{R}^d\right)$ is not implied). Consequently, passing to the weak limit in Faraday's equation in (2.6), we find that $B \in L_t^\infty L_x^2$ solves

$$\partial_t B - \frac{1}{\sigma} \Delta B = -\frac{1}{\sigma} \nabla \times G,$$

with initial data $B^0 \in L_x^2$, where $G = -\sigma u \times B$ belongs to

$$L_t^\infty L_x^1 \cap L_t^1 L_x^3 \subset L_t^{\frac{4}{3}} L_x^2, \quad \text{if} \quad d = 3,$$

$$\cap_{1<p\leqq\infty} L_t^p L_x^{\frac{p}{p-1}} \subset L_t^2 L_x^2, \quad \text{if} \quad d = 2,$$

and that one has $B \in C\left(\mathbb{R}^+; w\text{-}L^2\left(\mathbb{R}^2\right)\right)$. Finally, writing Duhamel's formula for the above equation, we obtain

$$B(t) = e^{t\frac{1}{\sigma}\Delta} B^0 - \int_0^t e^{(t-\tau)\frac{1}{\sigma}\Delta} \frac{1}{\sigma} \nabla \times G(\tau)\, \mathrm{d}\tau,$$

or, equivalently, employing the Fourier transform,

$$\hat{B}(t) = e^{-\frac{t}{\sigma}|\xi|^2} \hat{B}^0 - \int_0^t e^{-\frac{1}{\sigma}(t-\tau)|\xi|^2} \frac{i}{\sigma} \xi \times \hat{G}(\tau)\, \mathrm{d}\tau. \tag{4.7}$$

There only remains to establish the convergence of the Lorentz force $j^c \times B^c$ towards $j \times B$, in the sense of distributions, which will follow from a precise analysis of the frequency distribution of B^c. To this end, we first decompose the initial data $\left(E^{0c}, B^{0c}\right) \in X$ and the source terms $G^c \in L^1_{\text{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{R}^d\right)\right)$ using the eigenspaces of $\hat{\mathcal{L}}_c(\xi)$.

Thus, for almost every $\xi \in \mathbb{R}^3$ (with $\xi_3 = 0$ when $d = 2$), we have, according to Proposition 3.1, that

$$\begin{pmatrix} \hat{E}^{0c} \\ \hat{B}^{0c} \end{pmatrix} = \begin{pmatrix} \frac{\xi \cdot \hat{E}^{0c}}{|\xi|^2} \xi \\ 0 \end{pmatrix} + \begin{pmatrix} e^{0c} \\ \frac{-ic}{\lambda_-} \xi \times e^{0c} \end{pmatrix} + \begin{pmatrix} \frac{-ic}{\lambda_-} \xi \times b^{0c} \\ b^{0c} \end{pmatrix}, \tag{4.8}$$

where $\xi \cdot e^{0c} = \xi \cdot b^{0c} = 0$ (and $e_3^{0c} = b_1^{0c} = b_2^{0c} = 0$ if $d = 2$), and

$$\begin{pmatrix} \hat{G}^c \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\xi \cdot \hat{G}^c}{|\xi|^2} \xi \\ 0 \end{pmatrix} + \begin{pmatrix} e^c \\ \frac{-ic}{\lambda_-} \xi \times e^c \end{pmatrix} + \begin{pmatrix} \frac{-ic}{\lambda_-} \xi \times b^c \\ b^c \end{pmatrix}, \tag{4.9}$$

where $\xi \cdot e^c = \xi \cdot b^c = 0$ (and $e_3^c = b_1^c = b_2^c = 0$ if $d = 2$). In particular, we compute straightforwardly that

$$\begin{aligned} -\frac{1}{|\xi|^2} \xi \times \left(\xi \times \hat{E}^{0c} \right) + \frac{ic}{\lambda_-} \xi \times \hat{B}^{0c} &= \left(\frac{\lambda_- - \lambda_+}{\lambda_-} \right) e^{0c}, \\ \frac{ic}{\lambda_-} \xi \times \hat{E}^{0c} + \hat{B}^{0c} &= \left(\frac{\lambda_- - \lambda_+}{\lambda_-} \right) b^{0c}, \end{aligned}$$

and

$$\begin{aligned} -\frac{1}{|\xi|^2} \xi \times \left(\xi \times \hat{G}^c \right) &= \left(\frac{\lambda_- - \lambda_+}{\lambda_-} \right) e^c, \\ \frac{ic}{\lambda_-} \xi \times \hat{G}^c &= \left(\frac{\lambda_- - \lambda_+}{\lambda_-} \right) b^c, \end{aligned}$$

whence, thanks to Lemma 3.2, almost everywhere,

$$\begin{aligned} \left| \gamma e^{0c} \right| &\leqq \left| \hat{E}^{0c} \right| + \left| \hat{B}^{0c} \right|, & \left| \gamma b^{0c} \right| &\leqq \left| \hat{E}^{0c} \right| + \left| \hat{B}^{0c} \right|, \\ \left| \gamma e^c \right| &\leqq \left| \hat{G}^c \right|, & \left| \gamma b^c \right| &\leqq \left| \hat{G}^c \right|, \end{aligned} \tag{4.10}$$

where we have written, for mere convenience of notation,

$$\gamma(\xi) = \frac{\lambda_-(\xi) - \lambda_+(\xi)}{\lambda_-(\xi)}.$$

Note that $\gamma(\xi) \neq 0$ almost everywhere, according to Lemma 3.2.

We deduce that γe^{0c} and γb^{0c} are uniformly bounded in L_ξ^2, whereas, by the Hausdorff-Young inequality, γe^c and γb^c remain uniformly bounded in $L_t^p L_\xi^{\frac{d}{2}p}$, for any $2 < p < \infty$ and $d = 2, 3$. Consequently, writing

$$\lambda_+(\xi) = -\frac{|\xi|^2}{\sigma} \left(\frac{2}{1 + \sqrt{1 - \frac{4|\xi|^2}{\sigma^2 c^2}}} \right) \quad \text{and} \quad \lambda_-(\xi) = -\sigma c^2 \left(\frac{1 + \sqrt{1 - \frac{4|\xi|^2}{\sigma^2 c^2}}}{2} \right),$$

it is seen, taking weak limits in (4.8) and (4.9), that

$$\begin{aligned} \left(\gamma e^{0c}, \gamma b^{0c} \right) &\rightharpoonup \left(-\frac{1}{|\xi|^2} \xi \times \left(\xi \times \hat{E}^0 \right), \hat{B}^0 \right), \quad \text{in } L_\xi^2, \\ \left(\gamma e^c, \gamma b^c \right) &\rightharpoonup \left(-\frac{1}{|\xi|^2} \xi \times \left(\xi \times \hat{G} \right), 0 \right), \quad \text{in } L_t^p L_\xi^{\frac{d}{2}p}, \end{aligned}$$

for any $2 < p < \infty$ and $d = 2, 3$.

Next, in view of Proposition 3.1, the semigroup $e^{t\hat{\mathcal{L}}_c}$ acts on (4.8) as

$$e^{t\hat{\mathcal{L}}_c}\begin{pmatrix}\hat{E}^{0c}\\ \hat{B}^{0c}\end{pmatrix} = e^{-\sigma c^2 t}\begin{pmatrix}\frac{\xi\cdot\hat{E}^{0c}}{|\xi|^2}\xi\\ 0\end{pmatrix} + e^{t\lambda_-}\begin{pmatrix}e^{0c}\\ \frac{-ic}{\lambda_-}\xi\times e^{0c}\end{pmatrix} + e^{t\lambda_+}\begin{pmatrix}\frac{-ic}{\lambda_-}\xi\times b^{0c}\\ b^{0c}\end{pmatrix},$$

and on (4.9) as

$$\begin{aligned}\int_0^t e^{(t-\tau)\hat{\mathcal{L}}_c}\begin{pmatrix}\hat{G}^c\\ 0\end{pmatrix}(\tau)\,\mathrm{d}\tau &= \int_0^t e^{-\sigma c^2(t-\tau)}\begin{pmatrix}\frac{\xi\cdot\hat{G}^c}{|\xi|^2}\xi\\ 0\end{pmatrix}\mathrm{d}\tau\\ &\quad+\int_0^t e^{(t-\tau)\lambda_-}\begin{pmatrix}e^c\\ \frac{-ic}{\lambda_-}\xi\times e^c\end{pmatrix}\mathrm{d}\tau\\ &\quad+\int_0^t e^{(t-\tau)\lambda_+}\begin{pmatrix}\frac{-ic}{\lambda_-}\xi\times b^c\\ b^c\end{pmatrix}\mathrm{d}\tau.\end{aligned}$$

Therefore, Duhamel's formula (3.1) yields that

$$\begin{aligned}\hat{B}^c(t) &= \frac{-ic}{\lambda_-}e^{t\lambda_-}\xi\times e^{0c} + e^{t\lambda_+}b^{0c}\\ &\quad+\int_0^t\left(\frac{-ic^2}{\lambda_-}e^{(t-\tau)\lambda_-}\xi\times e^c + ce^{(t-\tau)\lambda_+}b^c\right)\mathrm{d}\tau.\end{aligned}$$

Further substituting

$$\frac{ic}{\lambda_-}\xi\times e^{0c} = b^{0c} - \hat{B}^{0c},$$

$$b^c = \frac{ic}{\lambda_-}\xi\times e^c,$$

which is deduced from the second components of (4.8) and (4.9), we obtain

$$\begin{aligned}\hat{B}^c(t) &= e^{t\lambda_-}\hat{B}^{0c} + \left(e^{t\lambda_+} - e^{t\lambda_-}\right)b^{0c}\\ &\quad+\frac{ic^2}{\lambda_-}\int_0^t\left(e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-}\right)\xi\times e^c\,\mathrm{d}\tau.\end{aligned}\tag{4.11}$$

Heuristically, taking weak limits in (4.11) yields (4.7), as expected.

Now, recall that we want to improve the convergence of (4.11) towards (4.7) in $L^2_{t,x,\mathrm{loc}}$. To this end, we decompose the magnetic fields as follows:

$$B^c = B^c_{\ll} + B^c_{<} + B^c_{\sim} + B^c_{>} + B^c_{\gg},$$

where, for some fixed parameter $1 < K < 2$ to be determined later on, for any large radius $0 < R < \frac{\sigma c}{2K}$ and for some small parameter $\delta > 0$,

$$\begin{aligned}\hat{B}^c_{\ll} &= \mathbb{1}_{\{|\xi|\leqq R\}}\hat{B}^c,\\ \hat{B}^c_{<} &= \mathbb{1}_{\{R<|\xi|\leqq\frac{\sigma c}{2K}\}}\hat{B}^c,\\ \hat{B}^c_{\sim} &= \mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\hat{B}^c,\\ \hat{B}^c_{>} &= \mathbb{1}_{\left\{\frac{\sigma cK}{2}<|\xi|\leqq\phi(\delta c)\right\}}\hat{B}^c,\\ \hat{B}^c_{\gg} &= \mathbb{1}_{\{|\xi|>\phi(\delta c)\}}\hat{B}^c,\end{aligned}$$

with $\phi(c)$ defined in (4.1). We estimate now each of the above terms separately.

Thus, using Lemma 3.2 and the estimates (4.10), we first find from (4.11) that

$$
\begin{aligned}
\left\| B^c_> \right\|_{L^2_{t,\mathrm{loc}} L^2_x} \lesssim & \left\| e^{-t\frac{\sigma c^2}{2}} \right\|_{L^2_t} \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} \right) \\
& + \left\| c \int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}} \left| \hat{G}^c \right| d\tau \mathbb{1}_{\left\{ \frac{\sigma c K}{2} < |\xi| \leqq \phi(\delta c) \right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
\lesssim & \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} \\
& + \left\| c e^{-t\frac{\sigma c^2}{2}} \right\|_{L^1_t} \left\| \hat{G}^c \mathbb{1}_{\left\{ \frac{\sigma c K}{2} < |\xi| \leqq \phi(\delta c) \right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
\lesssim & \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| \hat{G}^c \mathbb{1}_{\left\{ \frac{\sigma c K}{2} < |\xi| \leqq \phi(\delta c) \right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} .
\end{aligned}
$$

Therefore, when $d = 3$, employing the definition (4.1) of $\phi(c)$,

$$
\begin{aligned}
\left\| B^c_> \right\|_{L^2_{t,\mathrm{loc}} L^2_x} & \lesssim \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \frac{\phi(\delta c)^{\frac{1}{2}}}{c} \left\| G^c \right\|_{L^2_{t,\mathrm{loc}} \dot{H}_x^{-\frac{1}{2}}} \\
& \lesssim \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \delta \left\| G^c \right\|_{L^2_t L_x^{\frac{3}{2}}} ,
\end{aligned}
$$

whence, as $c \to \infty$, in view of the bounds (4.5) on G^c,

$$
\limsup_{c \to \infty} \left\| B^c_> \right\|_{L^2_{t,\mathrm{loc}} L^2_x} \lesssim \delta. \tag{4.12}
$$

When $d = 2$, for any $2 < p \leqq \infty$, we have

$$
\begin{aligned}
\left\| B^c_> \right\|_{L^2_{t,\mathrm{loc}} L^2_x} & \lesssim \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \frac{\phi(\delta c)^{1-\frac{2}{p}}}{c} \left\| G^c \right\|_{L^2_{t,\mathrm{loc}} \dot{H}_x^{\frac{2}{p}-1}} \\
& \lesssim \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \frac{\phi(\delta c)^{1-\frac{2}{p}}}{c} \left\| G^c \right\|_{L^p_t L_x^{\frac{p}{p-1}}} .
\end{aligned}
$$

Thus, defining $2 < p \leqq \infty$ by $\frac{2}{p} + \frac{1}{2 \log \phi(\delta c)} = 1$ so that, employing the definition (4.1) of $\phi(c)$,

$$
\frac{\phi(\delta c)^{1-\frac{2}{p}}}{c} \left(\frac{p}{p-2} \right)^{\frac{1}{2}} = \left(\frac{2e \log \phi(\delta c)}{c^2} \right)^{\frac{1}{2}} = (2e)^{\frac{1}{2}} \delta,
$$

we find, as $c \to \infty$, in view of the bounds (4.6) on G^c, that

$$
\limsup_{c \to \infty} \left\| B^c_> \right\|_{L^2_{t,\mathrm{loc}} L^2_x} \lesssim \delta, \tag{4.13}
$$

as well.

Next, in order to handle $B^c_\sim$, we further decompose (4.11) into

$$\hat{B}^c(t) = e^{t\lambda_-}\hat{B}^{0c} + \left(\frac{1-e^{t(\lambda_- - \lambda_+)}}{t(\lambda_- - \lambda_+)}\right) te^{t\lambda_+}\lambda_- \gamma b^{0c} \\ + ic^2\int_0^t \left(\frac{1-e^{(t-\tau)(\lambda_- - \lambda_+)}}{(t-\tau)(\lambda_- - \lambda_+)}\right)(t-\tau)e^{(t-\tau)\lambda_+}\gamma\xi \times e^c\,\mathrm{d}\tau.$$

Then, utilizing Lemma 3.2 to deduce that $|\lambda_- - \lambda_+| \leqq 2\sigma c^2\sqrt{K^2-1}$ whenever $\frac{\sigma c}{2K} \leqq |\xi| \leqq \frac{\sigma c K}{2}$, and that $\left|\frac{e^z-1}{z}\right| \leqq 2e^{|z|}$, for any $z \in \mathbb{C}$, by the Mean Value Theorem (see [8] for a complex version), we infer

$$\begin{aligned}
&\left\|B^c_\sim\right\|_{L^2_{t,\mathrm{loc}}L^2_x} \\
&\lesssim \left\|e^{-t\frac{\sigma c^2}{2}}\right\|_{L^2_t}\left\|B^{0c}\right\|_{L^2_x} \\
&\quad + c^2\left\|e^{t|\lambda_- - \lambda_+|}te^{t\lambda_+}\left|\gamma b^{0c}\right|\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_tL^2_\xi} \\
&\quad + c^3\left\|\int_0^t e^{(t-\tau)|\lambda_- - \lambda_+|}(t-\tau)\left|e^{(t-\tau)\lambda_+}\right|\left|\gamma e^c\right|\mathrm{d}\tau\,\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi} \\
&\lesssim \left\|e^{-t\frac{\sigma c^2}{2}}\right\|_{L^2_t}\left\|B^{0c}\right\|_{L^2_x} \\
&\quad + c^2\left\|e^{t|\lambda_- - \lambda_+|}te^{-t\frac{\sigma c^2}{4}}\left|\gamma b^{0c}\right|\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_tL^2_\xi} \\
&\quad + c^3\left\|\int_0^t e^{(t-\tau)|\lambda_- - \lambda_+|}(t-\tau)e^{-(t-\tau)\frac{\sigma c^2}{4}}\left|\gamma e^c\right|\mathrm{d}\tau\,\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi} \\
&\lesssim \frac{1}{c}\left\|B^{0c}\right\|_{L^2_x} + c^2\left\|te^{-t\frac{\sigma c^2}{4}\left(1-8\sqrt{K^2-1}\right)}\right\|_{L^2_t}\left\|\gamma b^{0c}\right\|_{L^2_\xi} \\
&\quad + c^3\left\|\int_0^t (t-\tau)e^{-(t-\tau)\frac{\sigma c^2}{4}\left(1-8\sqrt{K^2-1}\right)}\left|\gamma e^c\right|\mathrm{d}\tau\,\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}.
\end{aligned}$$

Therefore, fixing $1 < K < 2$ such that $1 - 8\sqrt{K^2-1} > 0$ (that is $K^2 < \frac{65}{64}$, it is to be emphasized that this is the unique restriction on K in the present proof) and using (4.10), we obtain, for all $2 < p \leqq \infty$ (when $d = 3$, one can also take the endpoint case $p = 2$),

$$\begin{aligned}
\left\|B^c_\sim\right\|_{L^2_{t,\mathrm{loc}}L^2_x} &\lesssim \frac{1}{c}\left\|E^{0c}\right\|_{L^2_x} + \frac{1}{c}\left\|B^{0c}\right\|_{L^2_x} \\
&\quad + c^3\left\|\int_0^t (t-\tau)e^{-(t-\tau)\frac{\sigma c^2}{4}\left(1-8\sqrt{K^2-1}\right)}\left|\hat{G}^c\right|\mathrm{d}\tau\,\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi} \\
&\lesssim \frac{1}{c}\left\|E^{0c}\right\|_{L^2_x} + \frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}
\end{aligned}$$

$$
\begin{aligned}
&+c^3 \left\| t e^{-t\frac{\sigma c^2}{4}\left(1-8\sqrt{K^2-1}\right)} \right\|_{L^1_t} \left\| \hat{G}^c \mathbb{1}_{\left\{\frac{\sigma c}{2K} < |\xi| \leqq \frac{\sigma c K}{2}\right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
&\lesssim \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| \hat{G}^c \mathbb{1}_{\left\{\frac{\sigma c}{2K} < |\xi| \leqq \frac{\sigma c K}{2}\right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
&\lesssim \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \frac{1}{c^{1-\frac{d}{2}+\frac{2}{p}}} \left\| G^c \right\|_{L^2_{t,\mathrm{loc}} \dot{H}_x^{\frac{2}{p}-\frac{d}{2}}} \\
&\lesssim \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \frac{1}{c^{1-\frac{d}{2}+\frac{2}{p}}} \left\| G^c \right\|_{L^p_t L_x^{\frac{dp}{dp-2}}} .
\end{aligned}
$$

Finally, taking $2 < p < \frac{4}{d-2}$ in the preceding estimate, we conclude, in view of the uniform bounds (4.5) on G^c, that, as $c \to \infty$,

$$
B^c_\sim \to 0, \qquad \text{in} \quad L^2_{t,\mathrm{loc}} L^2_x. \tag{4.14}
$$

We focus now on $B^c_<$. From (4.11), we obtain, by Lemma 3.2 and (4.10)

$$
\begin{aligned}
\left\| B^c_< \right\|_{L^2_{t,\mathrm{loc}} L^2_x} &\lesssim \left\| e^{-t\frac{\sigma c^2}{2}} \right\|_{L^2_t} \left\| \hat{B}^{0c} - b^{0c} \right\|_{L^2_\xi} + \left\| e^{-t\frac{R^2}{\sigma}} \right\|_{L^2_t} \left\| b^{0c} \right\|_{L^2_\xi} \\
&+ \left\| \left\| \int_0^t e^{-(t-\tau)\frac{|\xi|^2}{\sigma}} \left| \hat{G}^c \right| \mathrm{d}\tau \right\|_{L^2_{t,\mathrm{loc}}} |\xi| \mathbb{1}_{\left\{R < |\xi| \leqq \frac{\sigma c}{2K}\right\}} \right\|_{L^2_\xi} \\
&+ \left\| \left\| \int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}} \left| \hat{G}^c \right| \mathrm{d}\tau \right\|_{L^2_{t,\mathrm{loc}}} |\xi| \mathbb{1}_{\left\{R < |\xi| \leqq \frac{\sigma c}{2K}\right\}} \right\|_{L^2_\xi} \\
&\lesssim \left(\frac{1}{c} + \frac{1}{R} \right) \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} \right) \\
&+ \left\| \frac{1}{|\xi|} \left\| \hat{G}^c \right\|_{L^2_{t,\mathrm{loc}}} \mathbb{1}_{\left\{R < |\xi| \leqq \frac{\sigma c}{2K}\right\}} \right\|_{L^2_\xi} + \left\| \frac{|\xi|}{c^2} \left\| \hat{G}^c \right\|_{L^2_{t,\mathrm{loc}}} \mathbb{1}_{\left\{R < |\xi| \leqq \frac{\sigma c}{2K}\right\}} \right\|_{L^2_\xi} ,
\end{aligned}
$$

whereby, for any $2 < p < \frac{4}{d-2}$,

$$
\begin{aligned}
\left\| B^c_< \right\|_{L^2_{t,\mathrm{loc}} L^2_x} &\lesssim \left(\frac{1}{c} + \frac{1}{R} \right) \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} \right) \\
&\quad + \left(\frac{1}{R^{1-\frac{d}{2}+\frac{2}{p}}} + \frac{1}{c^{1-\frac{d}{2}+\frac{2}{p}}} \right) \left\| G^c \right\|_{L^2_{t,\mathrm{loc}} \dot{H}_x^{\frac{2}{p}-\frac{d}{2}}} \\
&\lesssim \left(\frac{1}{c} + \frac{1}{R} \right) \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} \right) \\
&\quad + \left(\frac{1}{R^{1-\frac{d}{2}+\frac{2}{p}}} + \frac{1}{c^{1-\frac{d}{2}+\frac{2}{p}}} \right) \left\| G^c \right\|_{L^p_t L_x^{\frac{dp}{dp-2}}} .
\end{aligned}
$$

We conclude, in view of the uniform bounds (4.5) on G^c that

$$
\limsup_{c \to \infty} \left\| B^c_< \right\|_{L^2_{t,\mathrm{loc}} L^2_x} \lesssim \frac{1}{R^{1-\frac{d}{2}+\frac{2}{p}}} . \tag{4.15}
$$

Finally, we deal with $B^c_{\ll}$. For any small $h > 0$, using Lemma 3.2 and the estimates (4.10), we have

$$\begin{aligned}
&\left\| B^c_{\ll}(t+h) - B^c_{\ll}(t) \right\|_{L^2_{t,\mathrm{loc}} L^2_x} \\
&\lesssim \left\| e^{-t\frac{\sigma c^2}{2}} \right\|_{L^2_t} \left\| \hat{B}^{0c} - b^{0c} \right\|_{L^2_\xi} + \left\| \left(e^{h\lambda_+} - 1 \right) e^{-t\frac{|\xi|^2}{\sigma}} b^{0c} \mathbb{1}_{\{|\xi| \leqq R\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
&\quad + \left\| \left\| \int_t^{t+h} e^{-(t+h-\tau)\frac{|\xi|^2}{\sigma}} \left| \hat{G}^c \right| \mathrm{d}\tau \right\|_{L^2_{t,\mathrm{loc}}} |\xi| \mathbb{1}_{\{|\xi| \leqq R\}} \right\|_{L^2_\xi} \\
&\quad + \left\| \left\| \int_0^t \left| e^{h\lambda_+} - 1 \right| e^{-(t-\tau)\frac{|\xi|^2}{\sigma}} \left| \hat{G}^c \right| \mathrm{d}\tau \right\|_{L^2_{t,\mathrm{loc}}} |\xi| \mathbb{1}_{\{|\xi| \leqq R\}} \right\|_{L^2_\xi} \\
&\quad + \left\| \left\| \int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}} \left| \hat{G}^c \right| \mathrm{d}\tau \right\|_{L^2_{t,\mathrm{loc}}} |\xi| \mathbb{1}_{\{|\xi| \leqq R\}} \right\|_{L^2_\xi} \\
&\lesssim \left(\frac{1}{c} + hR \right) \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} \right) \\
&\quad + \left\| \left\| e^{-t\frac{|\xi|^2}{\sigma}} \mathbb{1}_{\{t \in [0,h]\}} \right\|_{L^1_t} \left\| \hat{G}^c \right\|_{L^2_{t,\mathrm{loc}}} |\xi| \mathbb{1}_{\{|\xi| \leqq R\}} \right\|_{L^2_\xi} \\
&\quad + h \left\| \left\| \hat{G}^c \right\|_{L^2_{t,\mathrm{loc}}} |\xi| \mathbb{1}_{\{|\xi| \leqq R\}} \right\|_{L^2_\xi} + \frac{1}{c^2} \left\| \left\| \hat{G}^c \right\|_{L^2_{t,\mathrm{loc}}} |\xi| \mathbb{1}_{\{|\xi| \leqq R\}} \right\|_{L^2_\xi}.
\end{aligned}$$

It follows that, for any $2 < p < \frac{4}{d-2}$,

$$\begin{aligned}
\left\| B^c_{\ll}(t+h) - B^c_{\ll}(t) \right\|_{L^2_{t,\mathrm{loc}} L^2_x} &\lesssim \left(\frac{1}{c} + hR \right) \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} \right) \\
&\quad + hR^{\frac{d}{2} - \frac{2}{p} + 1} \left\| G^c \right\|_{L^2_{t,\mathrm{loc}} \dot{H}_x^{\frac{2}{p} - \frac{d}{2}}} + \frac{R}{c^2} \left\| G^c \right\|_{L^2_{t,\mathrm{loc}} L^2_x} \\
&\lesssim \left(\frac{1}{c} + hR \right) \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} \right) \\
&\quad + hR^{\frac{d}{2} - \frac{2}{p} + 1} \left\| G^c \right\|_{L^p_t L_x^{\frac{dp}{dp-2}}} + \frac{R}{c^2} \left\| G^c \right\|_{L^2_{t,\mathrm{loc}} L^2_x}.
\end{aligned}$$

Therefore, by the uniform bounds (4.5) on G^c, we deduce that, for any fixed radius $0 < R < \frac{\sigma c}{2K}$,

$$\limsup_{h \to 0} \sup_c \left\| B^c_{\ll}(t+h) - B^c_{\ll}(t) \right\|_{L^2_{t,\mathrm{loc}} L^2_x} = 0.$$

Consequently, by the Riesz–Fréchet–Kolmogorov Compactness Criterion (see [26, Chapter X]), we infer that $B^c_{\ll}$ is relatively compact in the strong topology of $L^2_{t,x,\mathrm{loc}}$, whence

$$B^c_{\ll} \to B_{\ll}, \qquad \text{in } L^2_{t,x,\mathrm{loc}}, \tag{4.16}$$

where $B_{\ll} = \mathcal{F}^{-1}\mathbb{1}_{\{|\xi|\leqq R\}}\mathcal{F}B$.

We are now ready to justify the weak convergence of the Lorentz force $j^c \times B^c$ towards $j \times B$. In view of the hypothesis (4.2) on the very high frequencies of the electric current, one can show that, for any $\delta > 0$ and each $\varphi \in C_c^\infty\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$,

$$\limsup_{c\to\infty} \left\| \left(j^c\varphi\right)_{\gg} \right\|_{L^2_{t,x}} = 0, \tag{4.17}$$

where $(j^c\varphi)_{\gg} = \mathcal{F}^{-1}\mathbb{1}_{\{|\xi|>\phi(\delta c)\}}\mathcal{F}\left(j^c\varphi\right)$. Then, we decompose, for every $\varphi \in C_c^\infty\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$,

$$\begin{aligned}\int_{\mathbb{R}^+\times\mathbb{R}^d} \left(j^c \times B^c - j \times B\right)\varphi\, \mathrm{d}t\mathrm{d}x = &\int_{\mathbb{R}^+\times\mathbb{R}^d} j^c \times \left(B^c_< + B^c_\sim + B^c_>\right)\varphi \mathrm{d}t\mathrm{d}x \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}^d} \left(j^c \times B^c_{\ll} - j \times B_{\ll}\right)\varphi \mathrm{d}t\mathrm{d}x \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}^d} \left(j^c\varphi\right)_{\gg} \times B^c \mathrm{d}t\mathrm{d}x \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}^d} j \times \left(B_{\ll} - B\right)\varphi \mathrm{d}t\mathrm{d}x.\end{aligned}$$

It follows, utilizing the estimates (4.12), (4.13), (4.14), (4.15), (4.16) and (4.17), that

$$\begin{aligned}\limsup_{c\to\infty} \left| \int_{\mathbb{R}^+\times\mathbb{R}^d} \left(j^c \times B^c - j \times B\right)\varphi \mathrm{d}t\mathrm{d}x \right| \lesssim &\frac{1}{R^{1-\frac{d}{2}+\frac{2}{p}}} + \delta \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}^d} j \times \left(B_{\ll} - B\right)\varphi \mathrm{d}t\mathrm{d}x.\end{aligned}$$

Finally, since $B_{\ll}$ converges to B, as $R \to \infty$, in the strong topology of $L^2_{t,\mathrm{loc}}L^2_x$, we deduce, by the arbitrariness of $R > 0$ and $\delta > 0$, that

$$\lim_{c\to\infty} \int_{\mathbb{R}^+\times\mathbb{R}^d} j^c \times B^c\varphi \mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}^+\times\mathbb{R}^d} j \times B\varphi \mathrm{d}t\mathrm{d}x,$$

which concludes the proof of the proposition. □

5. Proof of Theorem 1.1

This section is devoted to the proof of the main result Theorem 1.1 of the present paper, which is obtained by building upon the proof of Proposition 4.1.

We are thus considering now a family of weak solutions $\left(u^{\pm c}, E^c, B^c\right)$ of the two-fluid incompressible Navier–Stokes–Maxwell system (1.8), subject to some uniformly bounded initial data

$$u^{\pm 0c} \in L^2\left(\mathbb{R}^d\right), \qquad E^{0c} \in L^2\left(\mathbb{R}^d\right), \qquad B^{0c} \in L^2\left(\mathbb{R}^d\right).$$

As previously explained, defining the bulk velocity $u^c = \frac{u^{+c}+u^{-c}}{2}$ and the electrical current $j^c = \frac{u^{+c}-u^{-c}}{2\varepsilon}$, we obtain a family of weak solutions (u^c, j^c, E^c, B^c) of the equivalent system (2.1), subject to some uniformly bounded initial data

$$u^{0c} \in L^2\left(\mathbb{R}^d\right), \qquad \varepsilon j^{0c} \in L^2\left(\mathbb{R}^d\right), \qquad E^{0c} \in L^2\left(\mathbb{R}^d\right), \qquad B^{0c} \in L^2\left(\mathbb{R}^d\right).$$

Accordingly, the energy inequality stemming from (2.2) only provides uniform bounds on the weak solutions in

$$\begin{aligned} u^c \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x, \qquad \varepsilon j^c \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x, \\ E^c \in L^\infty_t L^2_x, \qquad B^c \in L^\infty_t L^2_x, \qquad j^c \in L^2_t L^2_x. \end{aligned} \tag{5.1}$$

The proof will proceed through a weak compactness method. The convergence in Maxwell's system and in the incompressibility relations will be readily obtained, for these equations are linear. Moreover, in view of the uniform a priori bounds (5.1), we will easily show through standard compactness arguments that u^c is relatively compact in the strong topology of $L^2_{t,x,\mathrm{loc}}$ and the convergence of the evolution equation on j^c in (2.1) towards Ohm's law will ensue.

Thus, the most difficult convergence will concern the equation of conservation of momentum in (2.1), where one has to justify the stability of the term given by the Lorentz force $j^c \times B^c$. To this end, standard compensated compactness methods plainly fail (see discussion on page 774) and we will have to show that the magnetic field B^c and the electric current j^c enjoy some kind of relative compactness in the strong topology of $L^2_{t,x,\mathrm{loc}}$.

Let us move on to the actual core of the proof. Up to extraction of subsequences, we have the following weak convergences, as $c \to \infty$:

$$\begin{aligned} \left(u^{0c}, E^{0c}, B^{0c}\right) &\rightharpoonup \left(u^0, E^0, B^0\right), && \text{in } L^2_x, \\ \left(u^c, E^c, B^c\right) &\rightharpoonup^* (u, E, B), && \text{in } L^\infty_t L^2_x, \\ j^c &\rightharpoonup j, && \text{in } L^2_t L^2_x. \end{aligned}$$

Then, from Ampère's equation, we deduce, letting $c \to \infty$, that

$$j = \nabla \times B \in L^2_t L^2_x,$$

whence, since $\operatorname{div} B = 0$,

$$B \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x.$$

Moreover, since u^c is uniformly bounded in

$$\begin{aligned} L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x \subset L^\infty_t L^2_x \cap L^2_t L^6_x \subset L^{\frac{10}{3}}_t L^{\frac{10}{3}}_x, \quad \text{if } d = 3, \\ L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x \subset \cap_{2<p\leqq\infty} L^p_t L^{\frac{2p}{p-2}}_x \subset L^4_t L^4_x, \quad \text{if } d = 2, \end{aligned}$$

and $\partial_t u^c$ is bounded in $L^2_t H^{-2}_x$, we conclude, invoking a classical compactness result by Aubin and Lions [2,18] (see also [23] for a sharp compactness criterion), that

$$u^c \to u \quad \text{in } L^3_{t,x,\mathrm{loc}},$$

which is sufficient to justify the convergence of the nonlinear advection term $u^c \cdot \nabla u^c$ towards $u \cdot \nabla u$, in the sense of distributions, say. In fact, in the two-dimensional case only, employing the optimal constants for Sobolev embeddings evaluated in [6,24], we have the following refined estimate on u^c, for any $2 < p \leqq \infty$:

$$\begin{aligned} \left\| u^c \right\|_{L^p_t L^{\frac{2p}{p-2}}_x} &\leqq C \left(\frac{p}{p-2} \right)^{\frac{1}{2}} \left\| u^c \right\|_{L^p_t \dot{H}^{\frac{2}{p}}_x} \\ &\leqq C \left(\frac{p}{p-2} \right)^{\frac{1}{2}} \left\| u^c \right\|^{1-\frac{2}{p}}_{L^\infty_t L^2_x} \left\| u^c \right\|^{\frac{2}{p}}_{L^2_t \dot{H}^1_x}, \end{aligned} \tag{5.2}$$

for some constant $C > 0$ independent of p.

Next, note that the bounds on u^c, B^c and εj^c imply that

$$\begin{aligned} &u^c \times B^c, \varepsilon u^c \otimes j^c \in L^\infty_t L^1_x \cap L^2_t L^{\frac{3}{2}}_x, && \text{if } d = 3, \\ &u^c \times B^c, \varepsilon u^c \otimes j^c \in \cap_{2<p\leqq\infty} L^p_t L^{\frac{p}{p-1}}_x, && \text{if } d = 2, \end{aligned} \tag{5.3}$$

whence, for any $2 < p < \infty$ and $d = 2, 3$,

$$\begin{aligned} G^c_1 &:= -\sigma u^c \times B^c \rightharpoonup G_1 := -\sigma u \times B && \text{in } L^p_t L^{\frac{dp}{dp-2}}_x, \\ G^c_2 &:= \sigma\varepsilon \left(u^c \otimes j^c + j^c \otimes u^c \right) \rightharpoonup 0 && \text{in } L^p_t L^{\frac{dp}{dp-2}}_x. \end{aligned}$$

Again, in the two-dimensional case only, utilizing (5.2), we have the refined estimates, for any $2 < p \leqq \infty$,

$$\begin{aligned} \left\| G^c_1 \right\|_{L^p_t L^{\frac{p}{p-1}}_x} &\leqq \sigma \left\| u^c \right\|_{L^p_t L^{\frac{2p}{p-2}}_x} \left\| B^c \right\|_{L^\infty_t L^2_x} \\ &\leqq C \left(\frac{p}{p-2} \right)^{\frac{1}{2}} \left\| u^c \right\|^{1-\frac{2}{p}}_{L^\infty_t L^2_x} \left\| u^c \right\|^{\frac{2}{p}}_{L^2_t \dot{H}^1_x} \left\| B^c \right\|_{L^\infty_t L^2_x}, \\ \left\| G^c_2 \right\|_{L^p_t L^{\frac{p}{p-1}}_x} &\leqq 2\sigma \left\| u^c \right\|_{L^p_t L^{\frac{2p}{p-2}}_x} \left\| \varepsilon j^c \right\|_{L^\infty_t L^2_x} \\ &\leqq C \left(\frac{p}{p-2} \right)^{\frac{1}{2}} \left\| u^c \right\|^{1-\frac{2}{p}}_{L^\infty_t L^2_x} \left\| u^c \right\|^{\frac{2}{p}}_{L^2_t \dot{H}^1_x} \left\| \varepsilon j^c \right\|_{L^\infty_t L^2_x}, \end{aligned} \tag{5.4}$$

for some constant $C > 0$ independent of p.

Now, combining the evolution equation for j^c in (2.1) with Faraday's equation, we find

$$\partial_t \left(B^c + \varepsilon^2 \nabla \times j^c \right) + \frac{1}{\sigma} \nabla \times \left(j^c + G^c_1 \right) = \varepsilon^2 \mu \Delta \nabla \times j^c - \frac{\varepsilon}{\sigma} \nabla \times \operatorname{div} G^2_c,$$

whence we deduce that

$$B^c + \varepsilon^2 \nabla \times j^c \in C\left(\mathbb{R}^+; w\text{-}L^2\left(\mathbb{R}^d\right)\right) \text{ is equi-continuous uniformly in } c,$$

where the prefix w- denotes that $L^2\left(\mathbb{R}^d\right)$ is endowed with its weak topology. Consequently, passing to the weak limit in the above evolution equation, we find that $B \in L_t^\infty L_x^2$ solves

$$\partial_t B - \frac{1}{\sigma}\Delta B = -\frac{1}{\sigma}\nabla \times G_1,$$

with initial data $B^0 \in L_x^2$, where $G_1 = -\sigma u \times B$ belongs to

$$L_t^\infty L_x^1 \cap L_t^1 L_x^3 \subset L_t^{\frac{4}{3}} L_x^2, \quad \text{if} \quad d = 3,$$
$$\cap_{1<p\leqq\infty} L_t^p L_x^{\frac{p}{p-1}} \subset L_t^2 L_x^2, \quad \text{if} \quad d = 2,$$

and that one has $B \in C\left(\mathbb{R}^+; w\text{-}L^2\left(\mathbb{R}^2\right)\right)$. Finally, writing Duhamel's formula for the above equation, we obtain

$$B(t) = e^{t\frac{1}{\sigma}\Delta} B^0 - \int_0^t e^{(t-\tau)\frac{1}{\sigma}\Delta} \frac{1}{\sigma} \nabla \times G_1(\tau)\mathrm{d}\tau,$$

or, equivalently, employing the Fourier transform,

$$\hat{B}(t) = e^{-\frac{t}{\sigma}|\xi|^2} \hat{B}^0 - \int_0^t e^{-\frac{1}{\sigma}(t-\tau)|\xi|^2} \frac{i}{\sigma} \xi \times \hat{G}_1(\tau)\,\mathrm{d}\tau. \tag{5.5}$$

There only remains to establish the convergence of the Lorentz force $j^c \times B^c$ towards $j \times B$, in the sense of distributions, which will follow from a precise analysis of the frequency distribution of B^c and j^c. To this end, we first decompose the initial data $\left(E^{0c}, B^{0c}\right) \in X$ and the source terms $G^c := \sigma c E^c - j^c \in L^1_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{R}^d\right)\right)$ (this bound is not uniform as $c \to \infty$) using the eigenspaces of $\hat{\mathcal{L}}_c(\xi)$.

Thus, for almost every $\xi \in \mathbb{R}^3$ (with $\xi_3 = 0$ when $d = 2$), we have, according to Proposition 3.1, that

$$\begin{pmatrix} \hat{E}^{0c} \\ \hat{B}^{0c} \end{pmatrix} = \begin{pmatrix} e^{0c} \\ \frac{-ic}{\lambda_-}\xi \times e^{0c} \end{pmatrix} + \begin{pmatrix} \frac{-ic}{\lambda_-}\xi \times b^{0c} \\ b^{0c} \end{pmatrix}, \tag{5.6}$$

where $\xi \cdot e^{0c} = \xi \cdot b^{0c} = 0$ (and $e_3^{0c} = b_1^{0c} = b_2^{0c} = 0$ if $d = 2$), and

$$\begin{pmatrix} \hat{G}^{c} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{c} \\ \frac{-ic}{\lambda_-}\xi \times e^{c} \end{pmatrix} + \begin{pmatrix} \frac{-ic}{\lambda_-}\xi \times b^{c} \\ b^{c} \end{pmatrix}, \tag{5.7}$$

where $\xi \cdot e^c = \xi \cdot b^c = 0$ (and $e_3^c = b_1^c = b_2^c = 0$ if $d = 2$). In particular, we compute straightforwardly that

$$E^{0c} + \frac{ic}{\lambda_-}\xi \times \hat{B}^{0c} = \left(\frac{\lambda_- - \lambda_+}{\lambda_-}\right) e^{0c},$$
$$\frac{ic}{\lambda_-}\xi \times \hat{E}^{0c} + \hat{B}^{0c} = \left(\frac{\lambda_- - \lambda_+}{\lambda_-}\right) b^{0c},$$

and

$$\hat{G}^c = \left(\frac{\lambda_- - \lambda_+}{\lambda_-}\right) e^c,$$
$$\frac{ic}{\lambda_-}\xi \times \hat{G}^c = \left(\frac{\lambda_- - \lambda_+}{\lambda_-}\right) b^c,$$

whence, thanks to Lemma 3.2, almost everywhere,

$$\begin{aligned} \left|\gamma e^{0c}\right| &\leqq \left|\hat{E}^{0c}\right| + \left|\hat{B}^{0c}\right|, & \left|\gamma b^{0c}\right| &\leqq \left|\hat{E}^{0c}\right| + \left|\hat{B}^{0c}\right|, \\ \left|\gamma e^{c}\right| &\leqq \left|\hat{G}^{c}\right|, & \left|\gamma b^{c}\right| &\leqq \left|\hat{G}^{c}\right|, \end{aligned} \tag{5.8}$$

where we have written, for mere convenience of notation,

$$\gamma(\xi) = \frac{\lambda_-(\xi) - \lambda_+(\xi)}{\lambda_-(\xi)}.$$

Note that $\gamma(\xi) \neq 0$ almost everywhere, according to Lemma 3.2.

We deduce that γe^{0c} and γb^{0c} are uniformly bounded in L^2_ξ. Consequently, writing

$$\lambda_+(\xi) = -\frac{|\xi|^2}{\sigma}\left(\frac{2}{1+\sqrt{1-\frac{4|\xi|^2}{\sigma^2c^2}}}\right) \quad \text{and} \quad \lambda_-(\xi) = -\sigma c^2\left(\frac{1+\sqrt{1-\frac{4|\xi|^2}{\sigma^2c^2}}}{2}\right),$$

it is seen, taking weak limits in (5.6), that

$$\left(\gamma e^{0c}, \gamma b^{0c}\right) \rightharpoonup \left(\hat{E}^0, \hat{B}^0\right) \qquad \text{in } L^2_\xi.$$

Next, in view of Proposition 3.1, the semigroup $e^{t\hat{\mathcal{L}}_c}$ acts on (5.6) as

$$e^{t\hat{\mathcal{L}}_c}\begin{pmatrix}\hat{E}^{0c}\\ \hat{B}^{0c}\end{pmatrix} = e^{t\lambda_-}\begin{pmatrix} e^{0c} \\ \frac{-ic}{\lambda_-}\xi \times e^{0c}\end{pmatrix} + e^{t\lambda_+}\begin{pmatrix}\frac{-ic}{\lambda_-}\xi \times b^{0c} \\ b^{0c}\end{pmatrix},$$

and on (5.7) as

$$\begin{aligned} \int_0^t e^{(t-\tau)\hat{\mathcal{L}}_c}\begin{pmatrix}\hat{G}^c\\ 0\end{pmatrix}(\tau)\mathrm{d}\tau &= \int_0^t e^{(t-\tau)\lambda_-}\begin{pmatrix} e^c \\ \frac{-ic}{\lambda_-}\xi\times e^c\end{pmatrix}\mathrm{d}\tau \\ &\quad + \int_0^t e^{(t-\tau)\lambda_+}\begin{pmatrix}\frac{-ic}{\lambda_-}\xi\times b^c \\ b^c\end{pmatrix}\mathrm{d}\tau \\ &= \int_0^t \begin{pmatrix}\left(\lambda_- e^{(t-\tau)\lambda_-} - \lambda_+ e^{(t-\tau)\lambda_+}\right)\frac{1}{\lambda_- - \lambda_+}\hat{G}^c \\ \left(e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-}\right)\frac{ic}{\lambda_- - \lambda_+}\xi\times\hat{G}^c\end{pmatrix}\mathrm{d}\tau. \end{aligned}$$

Therefore, Duhamel's formula (3.1) yields that

$$\begin{aligned} \hat{B}^c(t) &= \frac{-ic}{\lambda_-}e^{t\lambda_-}\xi\times e^{0c} + e^{t\lambda_+}b^{0c} \\ &\quad + \int_0^t \left(e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-}\right)\frac{ic^2}{\lambda_- - \lambda_+}\xi\times\hat{G}^c\,\mathrm{d}\tau. \end{aligned}$$

Further substituting

$$\frac{ic}{\lambda_-}\xi \times e^{0c} = b^{0c} - \hat{B}^{0c},$$

which is deduced from the second component of (5.6), we obtain

$$\begin{aligned}\hat{B}^c(t) = e^{t\lambda_-}\hat{B}^{0c} + \left(e^{t\lambda_+} - e^{t\lambda_-}\right) b^{0c} \\ + \int_0^t \left(e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-}\right) \frac{ic^2}{\lambda_- - \lambda_+}\xi \times \hat{G}^c \, \mathrm{d}\tau. \end{aligned} \tag{5.9}$$

Now, notice from (2.1) that

$$G^c = \sigma\varepsilon^2\partial_t j^c - \sigma\varepsilon^2\mu\Delta j^c + PG_1^c + P\left(\varepsilon \mathrm{div} G_2^c\right).$$

In particular, taking the Fourier transform,

$$\hat{G}^c = \sigma\varepsilon^2\partial_t \hat{j}^c + \sigma\varepsilon^2\mu\, |\xi|^2\, \hat{j}^c - \frac{1}{|\xi|^2}\xi \times \left(\xi \times \hat{G}_1^c\right) - \frac{i\varepsilon}{|\xi|^2}\xi \times \left(\xi \times \left(\hat{G}_2^c\xi\right)\right),$$

whence

$$\begin{aligned}&\int_0^t \left(e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-}\right) \xi \times \hat{G}^c \, \mathrm{d}\tau \\ &= -\sigma\varepsilon^2\left(e^{t\lambda_+} - e^{t\lambda_-}\right)\xi \times \hat{j}^{0c} + \sigma\varepsilon^2 \int_0^t \left(\lambda_+ e^{(t-\tau)\lambda_+} - \lambda_- e^{(t-\tau)\lambda_-}\right)\xi \times \hat{j}^c \, \mathrm{d}\tau \\ &\quad + \int_0^t \left(e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-}\right)\left(\xi \times \left(\sigma\varepsilon^2\mu\, |\xi|^2\, \hat{j}^c + \hat{G}_1^c + i\varepsilon\hat{G}_2^c\xi\right)\right) \mathrm{d}\tau.\end{aligned}$$

Combining the preceding identity with the representation formula (5.9) finally yields

$$\begin{aligned}\hat{B}^c(t) = e^{t\lambda_-}\hat{B}^{0c} + \left(e^{t\lambda_+} - e^{t\lambda_-}\right)\left(b^{0c} - \frac{i\sigma c^2\varepsilon^2}{\lambda_- - \lambda_+}\xi \times \hat{j}^{0c}\right) \\ + \frac{i\sigma c^2\varepsilon^2}{\lambda_- - \lambda_+}\int_0^t \left(\lambda_+ e^{(t-\tau)\lambda_+} - \lambda_- e^{(t-\tau)\lambda_-}\right)\xi \times \hat{j}^c \, \mathrm{d}\tau \\ + \frac{ic^2}{\lambda_- - \lambda_+}\int_0^t \left(e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-}\right)\xi \\ \times \left(\sigma\mu\varepsilon^2\, |\xi|^2\, \hat{j}^c + \hat{G}_1^c + i\varepsilon\hat{G}_2^c\xi\right) \mathrm{d}\tau.\end{aligned} \tag{5.10}$$

Heuristically, taking weak limits in (5.10) yields (5.5), as expected.

Now, recall that we want to improve the convergence of (5.10) towards (5.5) in $L^2_{t,x,\mathrm{loc}}$. To this end, we decompose the magnetic fields as follows:

$$B^c = B^c_{\ll} + B^c_{<} + B^c_{\sim} + B^c_{>} + B^c_{\gg},$$

where, for some fixed parameter $1 < K < 2$ to be determined later on, for any large radii $0 < R < \frac{\sigma c}{2K}$ and $L > 0$ and for some small parameter $\delta > 0$,

$$\begin{aligned}
\hat{B}^c_{\ll} &= \mathbb{1}_{\left\{|\xi|\leqq R,\ |\xi|\leqq \frac{L}{\varepsilon}\right\}} \hat{B}^c, \\
\hat{B}^c_{<} &= \mathbb{1}_{\left\{R<|\xi|\leqq \frac{\sigma c}{2K},\ |\xi|\leqq \frac{L}{\varepsilon}\right\}} \hat{B}^c, \\
\hat{B}^c_{\sim} &= \mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq \frac{\sigma c K}{2},\ |\xi|\leqq \frac{L}{\varepsilon}\right\}} \hat{B}^c, \\
\hat{B}^c_{>} &= \mathbb{1}_{\left\{\frac{\sigma c K}{2}<|\xi|\leqq \phi(\delta c),\ |\xi|\leqq \frac{L}{\varepsilon}\right\}} \hat{B}^c, \\
\hat{B}^c_{\gg} &= \mathbb{1}_{\left\{|\xi|>\phi(\delta c)\text{ or }|\xi|> \frac{L}{\varepsilon}\right\}} \hat{B}^c,
\end{aligned}$$

with $\phi(c)$ defined in (1.11). We estimate now each of the above terms separately.

Thus, using Lemma 3.2 and the estimates (5.8), we find from (5.10) that

$$\begin{aligned}
\left\| B^c_{>} \right\|_{L^2_{t,\mathrm{loc}} L^2_x} \lesssim{}& \left\| e^{-t\frac{\sigma c^2}{2}} \right\|_{L^2_t} \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} + c\varepsilon^2 \left\| j^{0c} \right\|_{L^2_x} \right) \\
&+ \left\| c^2\varepsilon^2 |\xi| \int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}} \left| \hat{j}^c \right| \mathrm{d}\tau \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
&+ \left\| c\varepsilon^2 |\xi|^2 \int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}} \left| \hat{j}^c \right| \mathrm{d}\tau \mathbb{1}_{\left\{|\xi|\leqq\frac{L}{\varepsilon}\right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
&+ \left\| c \int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}} \left| \hat{G}^c_1 \right| \mathrm{d}\tau \mathbb{1}_{\left\{\frac{\sigma c K}{2}<|\xi|\leqq\phi(\delta c)\right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
&+ \left\| c\varepsilon|\xi| \int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}} \left| \hat{G}^c_2 \right| \mathrm{d}\tau \mathbb{1}_{\left\{\frac{\sigma c K}{2}<|\xi|\leqq\phi(\delta c),\ |\xi|\leqq\frac{L}{\varepsilon}\right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left\| B^c_{>} \right\|_{L^2_{t,\mathrm{loc}} L^2_x} \lesssim{}& \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \varepsilon \left\| \varepsilon j^{0c} \right\|_{L^2_x} \\
&+ (c\varepsilon + L) \left\| c e^{-t\frac{\sigma c^2}{2}} \right\|_{L^1_t} \left\| \varepsilon j^c \right\|_{L^2_t \dot{H}^1_x} \\
&+ \left\| c e^{-t\frac{\sigma c^2}{2}} \right\|_{L^1_t} \left\| \hat{G}^c_1 \mathbb{1}_{\left\{\frac{\sigma c K}{2}<|\xi|\leqq\phi(\delta c)\right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
&+ L \left\| c e^{-t\frac{\sigma c^2}{2}} \right\|_{L^1_t} \left\| \hat{G}^c_2 \mathbb{1}_{\left\{\frac{\sigma c K}{2}<|\xi|\leqq\phi(\delta c)\right\}} \right\|_{L^2_{t,\mathrm{loc}} L^2_\xi} \\
\lesssim{}& \frac{1}{c} \left\| E^{0c} \right\|_{L^2_x} + \frac{1}{c} \left\| B^{0c} \right\|_{L^2_x} + \varepsilon \left\| \varepsilon j^{0c} \right\|_{L^2_x}
\end{aligned}$$

$$+\left(\varepsilon+\frac{L}{c}\right)\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}+\frac{1}{c}\left\|\hat G_1^c\mathbb{1}_{\left\{\frac{\sigma cK}{2}<|\xi|\leqq\phi(\delta c)\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}$$
$$+\frac{L}{c}\left\|\hat G_2^c\mathbb{1}_{\left\{\frac{\sigma cK}{2}<|\xi|\leqq\phi(\delta c)\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}.$$

Therefore, when $d=3$, employing the definition (1.11) of $\phi(c)$,

$$\begin{aligned}\left\|B^c_>\right\|_{L^2_{t,\mathrm{loc}}L^2_x}\lesssim{}&\frac{1}{c}\left\|E^{0c}\right\|_{L^2_x}+\frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}+\varepsilon\left\|\varepsilon j^{0c}\right\|_{L^2_x}+\left(\varepsilon+\frac{L}{c}\right)\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}\\&+\frac{\phi(\delta c)^{\frac12}}{c}\left\|G_1^c\right\|_{L^2_{t,\mathrm{loc}}\dot H_x^{-\frac12}}+L\frac{\phi(\delta c)^{\frac12}}{c}\left\|G_2^c\right\|_{L^2_{t,\mathrm{loc}}\dot H_x^{-\frac12}}\\\lesssim{}&\frac{1}{c}\left\|E^{0c}\right\|_{L^2_x}+\frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}+\varepsilon\left\|\varepsilon j^{0c}\right\|_{L^2_x}+\left(\varepsilon+\frac{L}{c}\right)\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}\\&+\delta\left\|G_1^c\right\|_{L^2_tL_x^{\frac32}}+L\delta\left\|G_2^c\right\|_{L^2_tL_x^{\frac32}},\end{aligned}$$

whence, as $c\to\infty$, in view of the bounds (5.3) on G_1^c and G_2^c,

$$\limsup_{c\to\infty}\left\|B^c_>\right\|_{L^2_{t,\mathrm{loc}}L^2_x}\lesssim(1+L)\,\delta. \tag{5.11}$$

When $d=2$, for any $2<p\leqq\infty$, we have

$$\begin{aligned}\left\|B^c_>\right\|_{L^2_{t,\mathrm{loc}}L^2_x}\lesssim{}&\frac{1}{c}\left\|E^{0c}\right\|_{L^2_x}+\frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}+\varepsilon\left\|\varepsilon j^{0c}\right\|_{L^2_x}+\left(\varepsilon+\frac{L}{c}\right)\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}\\&+\frac{\phi(\delta c)^{1-\frac2p}}{c}\left\|G_1^c\right\|_{L^2_{t,\mathrm{loc}}\dot H_x^{\frac2p-1}}+L\frac{\phi(\delta c)^{1-\frac2p}}{c}\left\|G_2^c\right\|_{L^2_{t,\mathrm{loc}}\dot H_x^{\frac2p-1}}\\\lesssim{}&\frac{1}{c}\left\|E^{0c}\right\|_{L^2_x}+\frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}+\varepsilon\left\|\varepsilon j^{0c}\right\|_{L^2_x}+\left(\varepsilon+\frac{L}{c}\right)\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}\\&+\frac{\phi(\delta c)^{1-\frac2p}}{c}\left\|G_1^c\right\|_{L^p_tL_x^{\frac{p}{p-1}}}+L\frac{\phi(\delta c)^{1-\frac2p}}{c}\left\|G_2^c\right\|_{L^p_tL_x^{\frac{p}{p-1}}}.\end{aligned}$$

Thus, defining $2<p\leqq\infty$ by $\frac2p+\frac{1}{2\log\phi(\delta c)}=1$ so that, employing the definition (1.11) of $\phi(c)$,

$$\frac{\phi(\delta c)^{1-\frac2p}}{c}\left(\frac{p}{p-2}\right)^{\frac12}=\left(\frac{2e\log\phi(\delta c)}{c^2}\right)^{\frac12}=(2e)^{\frac12}\delta,$$

we find, as $c\to\infty$, in view of the bounds (5.4) on G_1^c and G_2^c, that

$$\limsup_{c\to\infty}\left\|B^c_>\right\|_{L^2_{t,\mathrm{loc}}L^2_x}\lesssim(1+L)\,\delta, \tag{5.12}$$

as well.

Next, in order to handle $B^c_\sim$, we further decompose (5.10) into

$$\begin{aligned}
&\hat{B}^c(t) \\
&= e^{t\lambda_-}\hat{B}^{0c} + \left(\frac{1-e^{t(\lambda_- - \lambda_+)}}{t(\lambda_- - \lambda_+)}\right) te^{t\lambda_+}\lambda_- \left(\gamma b^{0c} - \frac{i\sigma c^2\varepsilon^2}{\lambda_-}\xi \times \hat{j}^{0c}\right) \\
&\quad + i\sigma c^2\varepsilon^2 \int_0^t \left(\left(\frac{1-e^{(t-\tau)(\lambda_- - \lambda_+)}}{(t-\tau)(\lambda_- - \lambda_+)}\right)(t-\tau)e^{(t-\tau)\lambda_+}\lambda_- - e^{(t-\tau)\lambda_+}\right)\xi \\
&\quad \times \hat{j}^c d\tau \\
&\quad + ic^2 \int_0^t \left(\frac{1-e^{(t-\tau)(\lambda_- - \lambda_+)}}{(t-\tau)(\lambda_- - \lambda_+)}\right)(t-\tau)e^{(t-\tau)\lambda_+}\xi \\
&\quad \times \left(\sigma\mu\varepsilon^2 |\xi|^2 \hat{j}^c + \hat{G}^c_1 + i\varepsilon\hat{G}^c_2\xi\right) d\tau.
\end{aligned}$$

Then, since $\left|\frac{e^z-1}{z}\right| \leqq 2e^{|z|}$, for any $z \in \mathbb{C}$, by the Mean Value Theorem (see [8] for a complex version), we infer

$$\begin{aligned}
&\|B^c_\sim\|_{L^2_{t,\mathrm{loc}}L^2_x} \\
&\lesssim \left\|e^{-t\frac{\sigma c^2}{2}}\right\|_{L^2_t} \|B^{0c}\|_{L^2_x} \\
&\quad + c^2 \left\|e^{t|\lambda_- - \lambda_+|}te^{t\lambda_+}\left(\left|\gamma b^{0c}\right| + \varepsilon L\left|\hat{j}^{0c}\right|\right)\mathbb{1}_{\left\{\frac{\sigma c}{2K} < |\xi| \leqq \frac{\sigma c K}{2}\right\}}\right\|_{L^2_t L^2_\xi} \\
&\quad + c^2\varepsilon \left\|\int_0^t \left|e^{(t-\tau)\lambda_+}\right| \varepsilon|\xi| \left|\hat{j}^c\right| d\tau \mathbb{1}_{\left\{\frac{\sigma c}{2K} < |\xi| \leqq \frac{\sigma c K}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi} \\
&\quad + \left(c^4\varepsilon + c^3 L\right) \\
&\times \left\|\int_0^t e^{(t-\tau)|\lambda_- - \lambda_+|}(t-\tau)\left|e^{(t-\tau)\lambda_+}\right| \varepsilon|\xi| \left|\hat{j}^c\right| d\tau \mathbb{1}_{\left\{\frac{\sigma c}{2K} < |\xi| \leqq \frac{\sigma c K}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi} \\
&\quad + c^3 \\
&\quad \times \left\|\int_0^t e^{(t-\tau)|\lambda_- - \lambda_+|}(t-\tau)\left|e^{(t-\tau)\lambda_+}\right| \left(\left|\hat{G}^c_1\right| + L\left|\hat{G}^c_2\right|\right) d\tau \mathbb{1}_{\left\{\frac{\sigma c}{2K} < |\xi| \leqq \frac{\sigma c K}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi},
\end{aligned}$$

whence, in view of Lemma 3.2,

$$\begin{aligned}
&\|B^c_\sim\|_{L^2_{t,\mathrm{loc}}L^2_x} \\
&\lesssim \left\|e^{-t\frac{\sigma c^2}{2}}\right\|_{L^2_t} \|B^{0c}\|_{L^2_x} \\
&\quad + c^2 \left\|e^{t|\lambda_- - \lambda_+|}te^{-t\frac{\sigma c^2}{4}}\left(\left|\gamma b^{0c}\right| + \varepsilon L\left|\hat{j}^{0c}\right|\right)\mathbb{1}_{\left\{\frac{\sigma c}{2K} < |\xi| \leqq \frac{\sigma c K}{2}\right\}}\right\|_{L^2_t L^2_\xi} \\
&\quad + c^2\varepsilon \left\|\int_0^t e^{-(t-\tau)\frac{\sigma c^2}{4}} \varepsilon|\xi| \left|\hat{j}^c\right| d\tau \mathbb{1}_{\left\{\frac{\sigma c}{2K} < |\xi| \leqq \frac{\sigma c K}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}
\end{aligned}$$

$$
\begin{aligned}
&+\left(c^4\varepsilon+c^3L\right)\\
&\times\left\|\int_0^t e^{(t-\tau)|\lambda_--\lambda_+|}(t-\tau)e^{-(t-\tau)\frac{\sigma c^2}{4}}\varepsilon\,|\xi|\left|\hat{j}^c\right|\mathrm{d}\tau\,\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}\\
&+c^3\\
&\times\left\|\int_0^t e^{(t-\tau)|\lambda_--\lambda_+|}(t-\tau)e^{-(t-\tau)\frac{\sigma c^2}{4}}\left(\left|\hat{G}_1^c\right|+L\left|\hat{G}_2^c\right|\right)\mathrm{d}\tau\,\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}.
\end{aligned}
$$

Consequently, utilizing Lemma 3.2 to deduce that $|\lambda_--\lambda_+|\leqq 2\sigma c^2\sqrt{K^2-1}$ whenever $\frac{\sigma c}{2K}\leqq|\xi|\leqq\frac{\sigma cK}{2}$,

$$
\begin{aligned}
&\left\|B^c_\sim\right\|_{L^2_{t,\mathrm{loc}}L^2_x}\\
&\lesssim\frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}\\
&\quad+c^2\left\|te^{-t\frac{\sigma c^2}{4}\left(1-8\sqrt{K^2-1}\right)}\right\|_{L^2_t}\left\||\gamma b^{0c}|+\varepsilon L\left|\hat{j}^{0c}\right|\right\|_{L^2_\xi}\\
&\quad+c^2\varepsilon\left\|\int_0^t e^{-(t-\tau)\frac{\sigma c^2}{4}}\varepsilon|\xi|\left|\hat{j}^c\right|\mathrm{d}\tau\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}\\
&\quad+\left(c^4\varepsilon+c^3L\right)\left\|\int_0^t(t-\tau)e^{-(t-\tau)\frac{\sigma c^2}{4}\left(1-8\sqrt{K^2-1}\right)}\varepsilon\,|\xi|\left|\hat{j}^c\right|\mathrm{d}\tau\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}\\
&\quad+c^3\left\|\int_0^t(t-\tau)e^{-(t-\tau)\frac{\sigma c^2}{4}\left(1-8\sqrt{K^2-1}\right)}\left(\left|\hat{G}_1^c\right|+L\left|\hat{G}_2^c\right|\right)\mathrm{d}\tau\,\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}.
\end{aligned}
$$

Therefore, fixing $1<K<2$ such that $1-8\sqrt{K^2-1}>0$ (that is $K^2<\frac{65}{64}$, it is to be emphasized that this is the unique restriction on K in the present proof) and using (5.8), we obtain

$$
\begin{aligned}
&\left\|B^c_\sim\right\|_{L^2_{t,\mathrm{loc}}L^2_x}\\
&\lesssim\frac{1}{c}\left\|E^{0c}\right\|_{L^2_x}+\frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}+\frac{L}{c}\left\|\varepsilon j^{0c}\right\|_{L^2_x}\\
&\quad+c^2\varepsilon\left\|e^{-t\frac{\sigma c^2}{4}}\right\|_{L^1_t}\left\|\varepsilon j^c\right\|_{L^2_t\dot{H}^1_x}+\left(c^4\varepsilon+c^3L\right)\left\|te^{-t\frac{\sigma c^2}{4}\left(1-8\sqrt{K^2-1}\right)}\right\|_{L^1_t}\left\|\varepsilon j^c\right\|_{L^2_t\dot{H}^1_x}\\
&\quad+c^3\left\|te^{-t\frac{\sigma c^2}{4}\left(1-8\sqrt{K^2-1}\right)}\right\|_{L^1_t}\left\|\left(\left|\hat{G}_1^c\right|+L\left|\hat{G}_2^c\right|\right)\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}\\
&\lesssim\frac{1}{c}\left\|E^{0c}\right\|_{L^2_x}+\frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}+\frac{L}{c}\left\|\varepsilon j^{0c}\right\|_{L^2_x}+\left(\varepsilon+\frac{L}{c}\right)\left\|\varepsilon j^c\right\|_{L^2_t\dot{H}^1_x}\\
&\quad+\frac{1}{c}\left\|\left(\left|\hat{G}_1^c\right|+L\left|\hat{G}_2^c\right|\right)\mathbb{1}_{\left\{\frac{\sigma c}{2K}<|\xi|\leqq\frac{\sigma cK}{2}\right\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}.
\end{aligned}
$$

It follows that, for all $2<p\leqq\infty$ (when $d=3$, one can also take the endpoint case $p=2$),

$$
\begin{aligned}
\left\|B^c_\sim\right\|_{L^2_{t,\mathrm{loc}}L^2_x} &\lesssim \frac{1}{c}\left\|E^{0c}\right\|_{L^2_x}+\frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}+\frac{L}{c}\left\|\varepsilon j^{0c}\right\|_{L^2_x}+\left(\varepsilon+\frac{L}{c}\right)\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}\\
&\quad+\frac{1}{c^{1-\frac{d}{2}+\frac{2}{p}}}\left\|\left|G^c_1\right|+L\left|G^c_2\right|\right\|_{L^2_{t,\mathrm{loc}}\dot H_x^{\frac{2}{p}-\frac{d}{2}}}\\
&\lesssim \frac{1}{c}\left\|E^{0c}\right\|_{L^2_x}+\frac{1}{c}\left\|B^{0c}\right\|_{L^2_x}+\frac{L}{c}\left\|\varepsilon j^{0c}\right\|_{L^2_x}+\left(\varepsilon+\frac{L}{c}\right)\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}\\
&\quad+\frac{1}{c^{1-\frac{d}{2}+\frac{2}{p}}}\left\|\left|G^c_1\right|+L\left|G^c_2\right|\right\|_{L^p_t L_x^{\frac{dp}{dp-2}}}.
\end{aligned}
$$

Finally, taking $2<p<\frac{4}{d-2}$ in the preceding estimate, we conclude, in view of the uniform bounds (5.3) on G^c_1 and G^c_2, that, as $c\to\infty$,

$$
B^c_\sim \to 0, \qquad \text{in} \quad L^2_{t,\mathrm{loc}}L^2_x. \tag{5.13}
$$

We focus now on $B^c_<$. From (5.10), we obtain, by Lemma 3.2 and (5.8),

$$
\begin{aligned}
\left\|B^c_<\right\|_{L^2_{t,\mathrm{loc}}L^2_x} &\lesssim \left\|e^{-t\frac{\sigma c^2}{2}}\right\|_{L^2_t}\left\|\hat B^{0c}-b^{0c}\right\|_{L^2_\xi}+\left\|e^{-t\frac{R^2}{\sigma}}\right\|_{L^2_t}\left\|b^{0c}\right\|_{L^2_\xi}\\
&\quad+L\left\|e^{-t\frac{\sigma c^2}{2}}\right\|_{L^2_t}\left\|\varepsilon\hat j^{0c}\right\|_{L^2_\xi}+L\left\|e^{-t\frac{R^2}{\sigma}}\right\|_{L^2_t}\left\|\varepsilon\hat j^{0c}\right\|_{L^2_\xi}\\
&\quad+\varepsilon^2\left\|\left\|\int_0^t e^{-(t-\tau)\frac{|\xi|^2}{\sigma}}\left|\hat j^c\right|d\tau\right\|_{L^2_{t,\mathrm{loc}}}|\xi|^3\mathbb{1}_{\{R<|\xi|\leqq\frac{\sigma c}{2K}\}}\right\|_{L^2_\xi}\\
&\quad+c^2\varepsilon^2\left\|\left\|\int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}}\left|\hat j^c\right|d\tau\right\|_{L^2_{t,\mathrm{loc}}}|\xi|\mathbb{1}_{\{R<|\xi|\leqq\frac{\sigma c}{2K}\}}\right\|_{L^2_\xi}\\
&\quad+\left\|\left\|\int_0^t e^{-(t-\tau)\frac{|\xi|^2}{\sigma}}\left(\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right)d\tau\right\|_{L^2_{t,\mathrm{loc}}}|\xi|\mathbb{1}_{\{R<|\xi|\leqq\frac{\sigma c}{2K}\}}\right\|_{L^2_\xi}\\
&\quad+\left\|\left\|\int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}}\left(\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right)d\tau\right\|_{L^2_{t,\mathrm{loc}}}|\xi|\mathbb{1}_{\{R<|\xi|\leqq\frac{\sigma c}{2K}\}}\right\|_{L^2_\xi}\\
&\lesssim \left(\frac{1}{c}+\frac{1}{R}\right)\left(\left\|E^{0c}\right\|_{L^2_x}+\left\|B^{0c}\right\|_{L^2_x}+L\left\|\varepsilon j^{0c}\right\|_{L^2_x}\right)\\
&\quad+\left\|\varepsilon^2|\xi|\left\|\hat j^c\right\|_{L^2_{t,\mathrm{loc}}}\mathbb{1}_{\{R<|\xi|\leqq\frac{\sigma c}{2K}\}}\right\|_{L^2_\xi}\\
&\quad+\left\|\frac{1}{|\xi|}\left\|\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right\|_{L^2_{t,\mathrm{loc}}}\mathbb{1}_{\{R<|\xi|\leqq\frac{\sigma c}{2K}\}}\right\|_{L^2_\xi}\\
&\quad+\left\|\frac{|\xi|}{c^2}\left\|\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right\|_{L^2_{t,\mathrm{loc}}}\mathbb{1}_{\{R<|\xi|\leqq\frac{\sigma c}{2K}\}}\right\|_{L^2_\xi},
\end{aligned}
$$

whence, for any $2 < p < \frac{4}{d-2}$,

$$\begin{aligned}\left\|B^c_<\right\|_{L^2_{t,\mathrm{loc}}L^2_x} &\lesssim \left(\frac{1}{c}+\frac{1}{R}\right)\left(\left\|E^{0c}\right\|_{L^2_x}+\left\|B^{0c}\right\|_{L^2_x}+L\left\|\varepsilon j^{0c}\right\|_{L^2_x}\right)\\ &\quad+\varepsilon\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}+\left(\frac{1}{R^{1-\frac{d}{2}+\frac{2}{p}}}+\frac{1}{c^{1-\frac{d}{2}+\frac{2}{p}}}\right)\left\|\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right\|_{L^2_{t,\mathrm{loc}}\dot H_x^{\frac{2}{p}-\frac{d}{2}}}\\ &\lesssim \left(\frac{1}{c}+\frac{1}{R}\right)\left(\left\|E^{0c}\right\|_{L^2_x}+\left\|B^{0c}\right\|_{L^2_x}+L\left\|\varepsilon j^{0c}\right\|_{L^2_x}\right)\\ &\quad+\varepsilon\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}+\left(\frac{1}{R^{1-\frac{d}{2}+\frac{2}{p}}}+\frac{1}{c^{1-\frac{d}{2}+\frac{2}{p}}}\right)\left\|\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right\|_{L^p_t L_x^{\frac{dp}{dp-2}}}.\end{aligned}$$

We conclude, in view of the uniform bounds (5.3) on G^c_1 and G^c_2 that

$$\limsup_{c\to\infty}\left\|B^c_<\right\|_{L^2_{t,\mathrm{loc}}L^2_x}\lesssim\frac{1+L}{R^{1-\frac{d}{2}+\frac{2}{p}}}. \tag{5.14}$$

Finally, we deal with $B^c_\ll$. For any small $h>0$, using Lemma 3.2 and the estimates (5.8), we have

$$\begin{aligned}&\left\|B^c_\ll(t+h)-B^c_\ll(t)\right\|_{L^2_{t,\mathrm{loc}}L^2_x}\\ &\lesssim \left\|e^{-t\frac{\sigma c^2}{2}}\right\|_{L^2_t}\left\|\hat B^{0c}-b^{0c}\right\|_{L^2_\xi}+\left\|\left(e^{h\lambda_+}-1\right)e^{-t\frac{|\xi|^2}{\sigma}}b^{0c}\mathbb{1}_{\{|\xi|\leqq R\}}\right\|_{L^2_{t,\mathrm{loc}}L^2_\xi}\\ &\quad+\varepsilon R\left\|\varepsilon\hat j^{0c}\right\|_{L^2_\xi}+\varepsilon\left\|\varepsilon|\xi|\hat j^c\right\|_{L^2_tL^2_\xi}\\ &\quad+\left\|\left\|\int_t^{t+h}e^{-(t+h-\tau)\frac{|\xi|^2}{\sigma}}\left(\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right)\mathrm{d}\tau\right\|_{L^2_{t,\mathrm{loc}}}|\xi|\mathbb{1}_{\{|\xi|\leqq R\}}\right\|_{L^2_\xi}\\ &\quad+\left\|\left\|\int_0^t\left|e^{h\lambda_+}-1\right|e^{-(t-\tau)\frac{|\xi|^2}{\sigma}}\left(\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right)\mathrm{d}\tau\right\|_{L^2_{t,\mathrm{loc}}}|\xi|\mathbb{1}_{\{|\xi|\leqq R\}}\right\|_{L^2_\xi}\\ &\quad+\left\|\left\|\int_0^t e^{-(t-\tau)\frac{\sigma c^2}{2}}\left(\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right)\mathrm{d}\tau\right\|_{L^2_{t,\mathrm{loc}}}|\xi|\mathbb{1}_{\{|\xi|\leqq R\}}\right\|_{L^2_\xi}\\ &\lesssim\left(\frac{1}{c}+hR\right)\left(\left\|E^{0c}\right\|_{L^2_x}+\left\|B^{0c}\right\|_{L^2_x}\right)\\ &\quad+\varepsilon R\left\|\varepsilon j^{0c}\right\|_{L^2_x}+\varepsilon\left\|\varepsilon j^c\right\|_{L^2_t\dot H^1_x}\\ &\quad+\left\|\left\|e^{-t\frac{|\xi|^2}{\sigma}}\mathbb{1}_{\{t\in[0,h]\}}\right\|_{L^1_t}\left\|\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right\|_{L^2_{t,\mathrm{loc}}}|\xi|\mathbb{1}_{\{|\xi|\leqq R\}}\right\|_{L^2_\xi}\\ &\quad+h\left\|\left\|\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right\|_{L^2_{t,\mathrm{loc}}}|\xi|\mathbb{1}_{\{|\xi|\leqq R\}}\right\|_{L^2_\xi}\\ &\quad+\frac{1}{c^2}\left\|\left\|\left|\hat G^c_1\right|+L\left|\hat G^c_2\right|\right\|_{L^2_{t,\mathrm{loc}}}|\xi|\mathbb{1}_{\{|\xi|\leqq R\}}\right\|_{L^2_\xi}.\end{aligned}$$

It follows that, for any $2 < p < \frac{4}{d-2}$,

$$
\begin{aligned}
&\left\| B^c_{\ll}(t+h) - B^c_{\ll}(t) \right\|_{L^2_{t,\mathrm{loc}} L^2_x} \\
&\lesssim \left(\frac{1}{c} + hR \right) \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} \right) \\
&\quad + \varepsilon R \left\| \varepsilon j^{0c} \right\|_{L^2_x} + \varepsilon \left\| \varepsilon j^c \right\|_{L^2_t \dot{H}^1_x} \\
&\quad + \left(h + \frac{1}{c^2} \right) R^{\frac{d}{2} - \frac{2}{p} + 1} \left(\left\| G^c_1 \right\|_{L^2_{t,\mathrm{loc}} \dot{H}_x^{\frac{2}{p} - \frac{d}{2}}} + L \left\| G^c_2 \right\|_{L^2_{t,\mathrm{loc}} \dot{H}_x^{\frac{2}{p} - \frac{d}{2}}} \right) \\
&\lesssim \left(\frac{1}{c} + hR \right) \left(\left\| E^{0c} \right\|_{L^2_x} + \left\| B^{0c} \right\|_{L^2_x} \right) \\
&\quad + \varepsilon R \left\| \varepsilon j^{0c} \right\|_{L^2_x} + \varepsilon \left\| \varepsilon j^c \right\|_{L^2_t \dot{H}^1_x} \\
&\quad + \left(h + \frac{1}{c^2} \right) R^{\frac{d}{2} - \frac{2}{p} + 1} \left(\left\| G^c_1 \right\|_{L^p_t L_x^{\frac{dp}{dp-2}}} + L \left\| G^c_2 \right\|_{L^p_t L_x^{\frac{dp}{dp-2}}} \right).
\end{aligned}
$$

Therefore, by the uniform bounds (5.3) on G^c_1 and G^c_2, we deduce that, for any fixed radii $0 < R < \frac{\sigma c}{2K}$ and $L > 0$,

$$
\limsup_{h \to 0} \sup_c \left\| B^c_{\ll}(t+h) - B^c_{\ll}(t) \right\|_{L^2_{t,\mathrm{loc}} L^2_x} = 0.
$$

Consequently, by the Riesz–Fréchet–Kolmogorov Compactness Criterion (see [26, Chapter X]), we infer that $B^c_{\ll}$ is relatively compact in the strong topology of $L^2_{t,x,\mathrm{loc}}$, whence

$$
B^c_{\ll} \to B_{\ll}, \qquad \text{in } L^2_{t,x,\mathrm{loc}}, \tag{5.15}
$$

where $B_{\ll} = \mathcal{F}^{-1} \mathbb{1}_{\{|\xi| \leqq R\}} \mathcal{F} B$.

We are now ready to justify the weak convergence of the Lorentz force $j^c \times B^c$ towards $j \times B$. To this end, notice first that, for any $L > 0$,

$$
\left\| \mathcal{F}^{-1} \mathbb{1}_{\left\{|\xi| > \frac{L}{\varepsilon}\right\}} \mathcal{F} j^c \right\|_{L^2_{t,x}} \lesssim \frac{\varepsilon}{L} \left\| |\xi| \hat{j}^c \right\|_{L^2_{t,\xi}} \lesssim \frac{1}{L} \left\| \varepsilon j^c \right\|_{L^2_t \dot{H}^1_x}.
$$

Hence, defining $j^c_{\gg}$ by

$$
\hat{j}^c_{\gg} = \mathbb{1}_{\left\{|\xi| > \phi(\delta c) \text{ or } |\xi| > \frac{L}{\varepsilon}\right\}} \hat{j}^c,
$$

we find that, in view of (1.10), for any $\delta > 0$ and $L > 0$,

$$
\begin{aligned}
\limsup_{c \to \infty} \left\| j^c_{\gg} \right\|_{L^2_{t,x}} &\leqq \limsup_{c \to \infty} \left\| \mathcal{F}^{-1} \mathbb{1}_{\{|\xi| > \phi(\delta c)\}} \mathcal{F} j^c \right\|_{L^2_{t,x}} \\
&\quad + \limsup_{c \to \infty} \left\| \mathcal{F}^{-1} \mathbb{1}_{\left\{|\xi| > \frac{L}{\varepsilon}\right\}} \mathcal{F} j^c \right\|_{L^2_{t,x}} \\
&\lesssim \frac{1}{L}.
\end{aligned}
$$

It follows that, for each $\varphi \in C_c^\infty\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$,

$$\limsup_{c\to\infty} \left\| \left(j^c\varphi\right)_{\gg} \right\|_{L^2_{t,x}} \lesssim \frac{1}{L}, \tag{5.16}$$

where $(j^c\varphi)_{\gg} = \mathcal{F}^{-1}\mathbb{1}_{\left\{|\xi|>\phi(\delta c) \text{ or } |\xi|>\frac{L}{\varepsilon}\right\}}\mathcal{F}\left(j^c\varphi\right)$. Then, we decompose, for any $\varphi \in C_c^\infty\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$,

$$\begin{aligned}\int_{\mathbb{R}^+\times\mathbb{R}^d} \left(j^c \times B^c - j \times B\right)\varphi \mathrm{d}t\mathrm{d}x &= \int_{\mathbb{R}^+\times\mathbb{R}^d} j^c \times \left(B^c_< + B^c_\sim + B^c_>\right)\varphi \mathrm{d}t\mathrm{d}x \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}^d} \left(j^c \times B^c_\ll - j \times B_\ll\right)\varphi \mathrm{d}t\mathrm{d}x \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}^d} \left(j^c\varphi\right)_\gg \times B^c \mathrm{d}t\mathrm{d}x \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}^d} j \times \left(B_\ll - B\right)\varphi \mathrm{d}t\mathrm{d}x.\end{aligned}$$

It follows, utilizing the estimates (5.11), (5.12), (5.13), (5.14), (5.15) and (5.16), that

$$\begin{aligned}\limsup_{c\to\infty} \left| \int_{\mathbb{R}^+\times\mathbb{R}^d} \left(j^c \times B^c - j \times B\right)\varphi \mathrm{d}t\mathrm{d}x \right| &\lesssim \frac{1+L}{R^{1-\frac{d}{2}+\frac{2}{p}}} + (1+L)\delta + \frac{1}{L} \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}^d} j \times \left(B_\ll - B\right)\varphi \mathrm{d}t\mathrm{d}x.\end{aligned}$$

Finally, since $B_\ll$ converges to B, as $R \to \infty$, in the strong topology of $L^2_{t,\mathrm{loc}}L^2_x$, we deduce, by the arbitrariness of $R > 0$, $\delta > 0$ and $L > 0$, that

$$\lim_{c\to\infty} \int_{\mathbb{R}^+\times\mathbb{R}^d} j^c \times B^c\varphi \mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}^+\times\mathbb{R}^d} j \times B\varphi \mathrm{d}t\mathrm{d}x,$$

which concludes the proof of the theorem. □

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Univ Paris Diderot, Sorbonne Paris Cité, Sorbonne Universités,
UPMC Paris 06, CNRS, UMR 7586,
Institut de mathématiques de Jussieu-Paris Rive Gauche,
75013 Paris,
France.
e-mail: diogo.arsenio@imj-prg.fr

and

Department of Mathematics and Statistics,
University of Victoria,
Victoria,
Canada.
e-mail: ibrahim@math.uvic.ca
URL: http://www.math.uvic.ca/˜ibrahim/

and

Courant Institute of Mathematical Sciences,
New York University,
New York,
USA.
e-mail: masmoudi@cims.nyu.edu

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