Arch. Rational Mech. Anal. 216 (2015) 313–357
Digital Object Identifier (DOI) 10.1007/s00205-014-0810-5

Trace Formula for Linear Hamiltonian Systems with its Applications to Elliptic Lagrangian Solutions

Xijun Hu, Yuwei Ou & Penghui Wang

Communicated by P. Rabinowitz

Abstract

In the present paper, we build up trace formulas for both the linear Hamiltonian systems and Sturm–Liouville systems. The formula connects the monodromy matrix of a symmetric periodic orbit with the infinite sum of eigenvalues of the Hessian of the action functional. A natural application is to study the non-degeneracy of linear Hamiltonian systems. Precisely, by the trace formula, we can give an estimation for the upper bound such that the non-degeneracy preserves. Moreover, we could estimate the relative Morse index by the trace formula. Consequently, a series of new stability criteria for the symmetric periodic orbits is given. As a concrete application, the trace formula is used to study the linear stability of elliptic Lagrangian solutions of the classical planar three-body problem, which depends on the mass parameter $\beta \in [0, 9]$ and the eccentricity $e \in [0, 1)$. Based on the trace formula, we estimate the stable region and hyperbolic region of the elliptic Lagrangian solutions.

1. Introduction

In the study of symmetric periodic solutions or quasi-periodic solutions in n-body problem, it is natural to consider the S-periodic solution in Hamiltonian system

$$\dot{z}(t) = JH'(t, z(t)), \quad z(0) = Sz(T), \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, S is a symplectic orthogonal matrix on $\mathbb{R}^{2n}$, and $H(t, x) \in C^2(\mathbb{R}^{2n+1}; \mathbb{R})$. Please refer to [5,6,10] and references therein for the background

X. Hu was partially supported by NSFC, NCET. Y. Ou was partially supported by NSFC (No.11131004). P. Wang was partially supported by NSFC (No.11101240, No.11471189) and Excellent Young Scientist Foundation of Shandong Province.

of S-periodic orbits in n-body problems. For the solution z of (1.1), let $\gamma \equiv \gamma_z(t)$ be the corresponding fundamental solution, that is $\dot{\gamma}(t) = JB(t)\gamma(t)$, $\gamma(0) = I_{2n}$, where $B(t) = B(t)^T = H''(t, z(t))$. $\gamma(T)$ is called the monodromy matrix.

The linear stability of S-periodic solution $z(t)$ depends on the location of eigenvalues of $S\gamma(T)$(see for example [16]), but due to the non-commutativity, in general, the fundamental solution could not be obtained directly. In the present paper, we obtain a kind of trace formula for the linear Hamiltonian system. Using the trace formula, we can estimate the relative Morse index, and hence, based on the theory of the Maslov-type index [24], we give some new stability criteria for Hamiltonian system. Finally, the trace formula will be used to study the stable region and hyperbolic region of Lagrangian solutions in the planar three body problem.

For $k \in \mathbb{N}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $\mathbb{M}(k, \mathbb{F})$ be the set of $k \times k$ matrices on $\mathbb{F}^k$. We denote by $\mathrm{Sp}(2k) = \{\mathcal{P} \in \mathbb{M}(2k, \mathbb{R}), \mathcal{P}^T J \mathcal{P} = J\}$ the symplectic group, $\mathcal{S}(k)$ the set of $k \times k$ real symmetric matrices and $\mathcal{B}(k) = C([0, T]; \mathcal{S}(k))$, the space of continuous paths on $[0, T]$ of matrices in $\mathcal{S}(k)$. For $B, D \in \mathcal{B}(2n)$, consider the eigenvalue problem of the following linear Hamiltonian systems,

$$\dot{z}(t) = J(B(t) + \lambda D(t))z(t), \quad z(0) = Sz(T). \tag{1.2}$$

Denote by $A = -J\frac{\mathrm{d}}{\mathrm{d}t}$, which is densely defined in the Hilbert space $E = L^2([0, T]; \mathbb{C}^{2n})$ with the domain

$$D_S = \left\{ z(t) \in W^{1,2}([0, T]; \mathbb{C}^{2n}) \,\middle|\, z(0) = Sz(T) \right\}.$$

B is a bounded linear operator defined by $(Bz)(t) = B(t)z(t)$ on E. Then A is a self-adjoint operator with compact resolvent; moreover for $\lambda \in \rho(A)$, the resolvent set of A, $(\lambda - A)^{-1}$ is Hilbert–Schmidt.

As above, let $\gamma_\lambda(t)$ be the fundamental solution of (1.2). To state the trace formula for the Hamiltonian system, we need some notations. Write $M = S\gamma_0(T)$ and $\hat{D}(t) = \gamma_0^T(t)D(t)\gamma_0(t)$. For $k \in \mathbb{N}$, let

$$M_k = \int_0^T J\hat{D}(t_1) \int_0^{t_1} J\hat{D}(t_2) \cdots \int_0^{t_{k-1}} J\hat{D}(t_k)\,\mathrm{d}t_k \cdots \mathrm{d}t_2\,\mathrm{d}t_1, \tag{1.3}$$

and

$$\mathcal{M}(\nu) = M\left(M - \mathrm{e}^{\nu T} I_{2n}\right)^{-1}, \quad G_k(\nu) = M_k \cdot \mathcal{M}(\nu).$$

Moreover, for $\nu \in \mathbb{C}$ such that $A - B - \nu J$ is invertible, we set

$$\mathcal{F}(\nu, B, D) = D\,(A - B - \nu J)^{-1}.$$

In what follows, $\mathcal{M}(\nu)$, $G_k(\nu)$ and $\mathcal{F}(\nu, B, D)$ will be written in short form as $\mathcal{M}$, G_k and $\mathcal{F}$, respectively, if there is no confusion.

Theorem 1.1. *For any positive integer m, we have*

$$Tr\left(\mathcal{F}^m\right) = mTr(\mathcal{G}_m) \tag{1.4}$$

where $\mathcal{G}_m = \sum_{k=1}^m \frac{(-1)^k}{k}\left[\sum_{j_1+\cdots+j_k=m} (G_{j_1}\cdots G_{j_k})\right]$.

There are two reasons why we consider the parameter ν in Theorem 1.1. Firstly, for a given $B \in \mathcal{B}(2n)$, we cannot expect that $A - B$ is invertible. However, for every $\nu \in \mathbb{C}$ except countable points, $A - B - \nu J$ is invertible. Secondly, the operator $\mathcal{F}$ comes from the following boundary value problem naturally

$$\dot{z}(t) = J(B(t) + \lambda D(t))z(t), \quad z(0) = \omega S z(T), \tag{1.5}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ and $\omega = \mathrm{e}^{\nu T}$. In fact, if we set $A_\omega = -J\frac{\mathrm{d}}{\mathrm{d}t}$ with the domain $D_{\omega S}$, then $\mathrm{e}^{-\nu t} A_\omega \mathrm{e}^{\nu t} = A - \nu J$. Thus $z \in \ker(A_\omega - B - \lambda D)$ if and only if $\mathrm{e}^{-\nu T} z(t) \in \ker(A - \nu J - B - \lambda D)$, which is equivalent to $\frac{1}{\lambda}$ being an eigenvalue $\mathcal{F}$ provided that $A - \nu J - B$ is invertible.

Remark 1.2. (1). For $m = 1$, $\mathcal{F}$ is not a trace class operator but a Hilbert–Schmidt operator, and hence $Tr(\mathcal{F})$ is not the usual trace but a kind of conditional trace [16].

(2). For $m \geqq 2$, $\mathcal{F}^m$ are trace class operators. By the preceding argument, λ is a nonzero eigenvalue of system (1.5) if and only if $\frac{1}{\lambda}$ is an eigenvalue of $\mathcal{F}$. Hence, if we let $\{\lambda_i\}$ be the set of nonzero eigenvalues of the system (1.5),

$$Tr(\mathcal{F}^m) = \sum_j \frac{1}{\lambda_j^m} = m Tr(\mathcal{G}_m), \tag{1.6}$$

where the sum is taken for the eigenvalue $\frac{1}{\lambda_j}$ of $\mathcal{F}$ counting the algebraic multiplicity.

For large m, the right hand side of (1.4) is a little complicated. However, for $m = 1, 2$, we can write it down more precisely.

Corollary 1.3.

$$Tr\,(\mathcal{F}) = -Tr(M_1 \mathcal{M}), \tag{1.7}$$

and

$$Tr\left(\mathcal{F}^2\right) = Tr\left[(M_1\mathcal{M})^2 - 2M_2\mathcal{M}\right]. \tag{1.8}$$

In the case that $M = \pm I_{2n}$,

$$Tr(\mathcal{F}^2) = \frac{\pm \mathrm{e}^{\nu T}}{(1 \mp \mathrm{e}^{\nu T})^2} Tr(M_1^2). \tag{1.9}$$

In some concrete problems, such as the estimation of hyperbolic region of elliptic Lagrangian solution, the trace formula for the Lagrangian system is more convenient to use. In order to introduce the trace formula for the Lagrangian system, it is natural to consider the following eigenvalue problem of the Sturm–Liouville system with $\bar{S}$-periodic boundary condition

$$-(P\dot{y} + Qy)^{\cdot} + Q^T\dot{y} + (R + \lambda R_1)y = 0, \quad y(0) = \bar{S}y(T), \quad \dot{y}(0) = \bar{S}\dot{y}(T), \tag{1.10}$$

where $\bar{S}$ is an orthogonal matrix on $\mathbb{R}^n$, $P, R, R_1 \in \mathcal{B}(n)$, $Q \in C([0,T]; \mathbb{M}(n,\mathbb{R}))$. Instead of the Legendre convexity condition, we assume that for any $t \in [0,T]$, $P(t)$ is invertible. Moreover we assume

$$\bar{S}P(T) = P(0)\bar{S} \quad \text{and} \quad \bar{S}Q(T) = Q(0)\bar{S}. \tag{1.11}$$

Such a boundary value problem with condition (1.11) comes naturally from the study of symmetric periodic orbits in the n-body problem.

By the standard Legendre transformation, the linear system (1.10) corresponds to the linear Hamiltonian system,

$$\dot{z} = JB_\lambda(t)z, \quad z(0) = \bar{S}_d z(T), \tag{1.12}$$

with

$$\bar{S}_d = \begin{pmatrix} \bar{S} & 0_n \\ 0_n & \bar{S} \end{pmatrix}, \quad \text{and} \quad B_\lambda(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}Q(t) \\ -Q(t)^T P^{-1}(t) & Q(t)^T P^{-1}(t)Q(t) - R(t) - \lambda R_1(t) \end{pmatrix}. \tag{1.13}$$

Obviously, $\bar{S}_d$ is a symplectic orthogonal matrix on $\mathbb{R}^{2n}$, and the eigenvalue problem (1.12) is a special case of the eigenvalue problem (1.2). Without confusion, for the Lagrangian system, denote by $\gamma_\lambda(t)$ the fundamental solution of (1.12).

Using the notations in Theorem 1.1, take $D = \begin{pmatrix} 0_n & 0_n \\ 0_n & -R_1 \end{pmatrix}$. Temporarily, we assume that the unperturbed system is non-degenerate, that is, 0 is not the eigenvalue of (1.10), which is equivalent to saying that 1 is not the eigenvalue of $M = \bar{S}_d\gamma_0(T)$.

Theorem 1.4. *Let $\{\lambda_j\}$ be the eigenvalues for the boundary value problem (1.10), then*

$$\sum_j \frac{1}{\lambda_j^m} = mTr(\mathcal{G}_m), \quad \forall m \in \mathbb{N}, \tag{1.14}$$

especially for $m = 1$,

$$\sum_j \frac{1}{\lambda_j} = -Tr(M_1\mathcal{M}). \tag{1.15}$$

It should be pointed out that from Proposition 3.5, for $m \geqq 2$, the trace formula (1.14) is a special case of the formula (1.6). However, for $m = 1$, the meanings of the formula (1.7) and (1.15) are totally different. In fact, $Tr(\mathcal{F})$ is a kind of conditional trace. Details can be found in Remark 3.6. The formula (1.15) is proved for the Sturm–Liouville system, and we do not know for the general Hamiltonian system whether it holds true or not. Fortunately, (1.15) is easy to calculate.

During the study of the above trace formula, thanks to Chongchun Zeng's suggestion, we can find the original work by KREIN [19,20] from the 1950's. In fact, Krein considered the following system

$$\dot{z}(t) = \lambda JD(t)z(t), \quad z(0) = -z(T), \tag{1.16}$$

where $D \geqq 0$ and $\int_0^T D(t)\,\mathrm{d}t > 0$. The system (1.16) is a special case of our system (1.2). For the system (1.16), Krein proved that $\lim_{r\to\infty} \sum_{|\lambda_j|<r} \frac{1}{\lambda_j} = 0$, and

$$\sum \frac{1}{\lambda_j^2} = \frac{T^2}{2} Tr(A_{11}A_{22} - A_{12}^2), \tag{1.17}$$

where λ_j are the eigenvalues for the system (1.16), and $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{1}{T}\int_0^T D(t)\,\mathrm{d}t$. Moreover, under the condition $D \geqq 0$, $\int_0^T D(t)\,\mathrm{d}t > 0$, Krein gave an interesting stability criteria:

$$\frac{T^2}{2} Tr(A_{11}A_{22} - A_{12}^2) < 1.$$

Obviously, by taking $\nu = 0$ and $M = -I_{2n}$ in the formula (1.9), it is easy to see that Theorem 1.1 generalizes Krein's formula (1.17).

Remark 1.5. Krein considered the simplest Hamiltonian system with some special conditions such as $D \geqq 0$ and $\int_0^T D(t)\,\mathrm{d}t > 0$. For the system coming from the n-body problem, the conditions are not satisfied. Hence, Krein's trace formula cannot be used to study the n-body problem. However, Krein's trace formula is a powerful tool to study the stability. It is surprising that, to the best of our knowledge, there are no further studies along these lines.

Next, we will introduce some applications of the trace formula. As one application, we will give some estimations on the non-degeneracy of the linear system. It is well-known that the system preserves the non-degeneracy under small perturbations. A natural question will arise: can we give an upper bound for the perturbation such that, under the smaller perturbation, the systems preserve the non-degeneracy? By the trace formula, we can answer this question partly. Details can be found in Section 4. As another application, the trace formula could be used to estimate the relative Morse index for Hamiltonian systems and the Morse index for Lagrangian systems. It is well-known that the relative Morse index (or Morse index) is equal to the Maslov-type index for the path of symplectic matrices and the Maslov-type index is a successful tool in judging the linear stability [14,24]. In Section 4, by using the trace formula, we can give some new stability criteria.

Before giving further applications of the trace formula on the n-body problem, we want to interpret the proof of the trace formula intuitively. For a matrix F, to calculate the trace $Tr F^m$ for $m > 0$, the most effective method is to consider the determinant $\det(I + \alpha F)$, where I is the identity matrix and α is a parameter. In the case of a trace formula of differential equations, the idea does work too. From this viewpoint, the Hill-type formula is the cornerstone to get the trace formula. The study of such a formula begins with the original work of Hill [12] in 1877. In his study of the motion of lunar perigee, Hill considered the following equation:

$$\ddot{x}(t) + \theta(t)x(t) = 0, \tag{1.18}$$

where $\theta(t) = \sum_{j\in\mathbb{Z}} \theta_j e^{2j\sqrt{-1}t}$ with $\theta_0 \neq 0$ is a real π-periodic function. Let $\gamma(t)$ be the fundamental solution of the associated first order system of (1.18), that is, $\dot{\gamma}(t) = \begin{pmatrix} 0 & -\theta(t) \\ 1 & 0 \end{pmatrix} \gamma(t)$, $\gamma(0) = I_2$. Suppose $\rho = \mathrm{e}^{c\sqrt{-1}\pi}$, $\rho^{-1} = \mathrm{e}^{-c\sqrt{-1}\pi}$ are the eigenvalues of the monodromy matrix $\gamma(\pi)$. In order to compute c, Hill obtained the following formula which connects the infinite determinant, corresponding to the differential operator, and the characteristic polynomial:

$$\frac{\sin^2(\frac{\pi}{2}c)}{\sin^2(\frac{\pi}{2}\theta_0)} = \det\left[\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \theta_0\right)^{-1}\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \theta\right)\right], \tag{1.19}$$

where the right hand side of (1.19) is the Fredholm determinant. We should point out that the right hand side of the original formula of Hill [12] is a determinant of an infinite matrix. In Hill [12] did not prove the convergence of the infinite determinant, and the convergence was proved by Poincaré [29]. The Hill formula for a periodic solution of Lagrangian systems on a manifold was given by Bolotin [3]. In Bolotin and Treschev [4] studied the Hill-type formula for both continuous and discrete Lagrangian systems with a Legendre convexity condition. For the periodic solution of ODE, the Hill-type formula was given by Denk [9].

The Hill-type formula for an S-periodic orbit of the Hamiltonian system was given by the first and the third authors [16]. In this paper, for $B, D \in \mathcal{B}(2n)$, we always set

$$p(\alpha) = \det[(A - (B + \alpha D) - \nu J)(A + P_0)^{-1}];$$

the Hill-type formula says

$$p(\alpha) = C(S)\mathrm{e}^{-n\nu T}\det(S\gamma_\alpha(T) - \mathrm{e}^{\nu T} I_{2n}), \tag{1.20}$$

where $C(S) > 0$ is a constant depending only on S. The equality (1.20) is our starting point to get the trace formula of the Hamiltonian system. In fact, both sides of (1.20) are analytic functions on α. Then, by taking the Taylor expansion and comparing the coefficients on both sides of (1.20), we get the trace formula in Theorem 1.1. Based on this idea, in order to obtain the trace formula for the Lagrangian system, in the present paper we will get the following Hill-type formula:

Theorem 1.6. *Let* $\{\lambda_j\}$ *be the nonzero eigenvalues for the boundary value problem* (1.10)*, then*

$$\prod_j \left(1 - \frac{1}{\lambda_j}\right) = \det(\bar{S}_d\gamma_1(T) - I_{2n}) \cdot \det(\bar{S}_d\gamma_0(T) - I_{2n})^{-1}, \tag{1.21}$$

where γ_λ *is the fundamental solution of the system* (1.12).

Remark 1.7. The Hill-type formula for periodic orbits of the Lagrangian system with the Legendre convex condition was given by Bolotin [3] in 1988, and Theorem 1.6 can be considered as a generalization of Bolotin's work to indefinite Lagrangian systems. Recently, the Hill formula for the g-periodic trajectories was given by Davletshin [8].

At the end of this paper, we will study the stability of Lagrangian orbits in planar three body problems. In 1772, LAGRANGE [21] discovered some celebrated periodic solutions, now named after him, to the planar three-body problem, namely that three bodies form an equilateral triangle at any instant of the motion and at the same time each body travels along a specific Keplerian elliptic orbit about the center of masses of the system. All these orbits are homographic solutions. MEYER and SCHMIDT [27] heavily relied upon the central configuration nature of the elliptic Lagrangian orbits and decomposed the fundamental solution of the elliptic Lagrangian orbit into two parts symplectically, one of which is the same as that of the Keplerian solution and the other is the essential part for the stability.

For the planar three-body problem with masses $m_1, m_2, m_3 > 0$, it turns out that the stability of elliptic Lagrangian solutions depends on two parameters, namely the mass parameter $\beta \in [0, 9]$ defined below and the eccentricity $e \in [0, 1)$,

$$\beta = \frac{27(m_1m_2 + m_1m_3 + m_2m_3)}{(m_1 + m_2 + m_3)^2}.$$

In the current paper, the fundamental solution of the linearized Hamiltonian system of the essential part of the elliptic Lagrangian orbit is denoted by $\gamma_{\beta,e}(t)$ for $t \in [0, 2\pi]$, which is a path of 4×4 symplectic matrices starting from the identity. The Lagrangian orbits are called spectrally stable (or elliptic) if all the eigenvalues of $\gamma_{\beta,e}(2\pi)$ belong to the united circle $\mathbb{U}$, and they are called linearly stable if $\gamma_{\beta,e}(2\pi)$ is semi-simple. By contrast, Lagrangian orbits are called hyperbolic if no eigenvalue of $\gamma_{\beta,e}(2\pi)$ locates on $\mathbb{U}$.

The linear stability of relative equilibria ($e = 0$) was known more than a century ago, due to GASCHEAU ([11], 1843) and ROUTH ([32], 1875), independently. Recently, BARUTELLO et al. [2] completely solved the case of the α-homogenous potential with $\alpha \in (0, 2)$. For the elliptic relative equilibria ($e > 0$), the linear stability problem is difficult; many interesting results can be found in [25–27,31]. For the historical literature on the linear stability of Lagrangian orbits, readers are referred to [13]. Recently, Long, Sun and the first author introduced a Maslov-type index and operator theory in studying the stability in the n-body problem [13,15]. In [13], the authors gave an analytic proof for the stability bifurcation diagram of Lagrangian equilateral triangular homographic orbits in the $(\beta; e)$ rectangle $[0, 9] \times [0, 1)$ and proved that the bifurcation curve is real analytic, though it is difficult to estimate the bifurcation curve.

To the best of our knowledge, there are no previous results to estimate the stability region. For the hyperbolic region, till now, we only know of two results. The first, it was proved in [13], asserts that the Lagrangian orbits are hyperbolic for $\beta = 9$ (equal mass case) with any eccentricity $e \in [0, 1)$. The second based on the result in [13], proved (by the second author [28]) that Lagrangian orbits are hyperbolic for $\beta > 8$. However, for β near 1, we know nothing about the estimation of the hyperbolic region. In the present paper, based on works in [13,15] and via the trace formula, we estimate the stability region and hyperbolic region for the elliptic Lagrangian orbits.

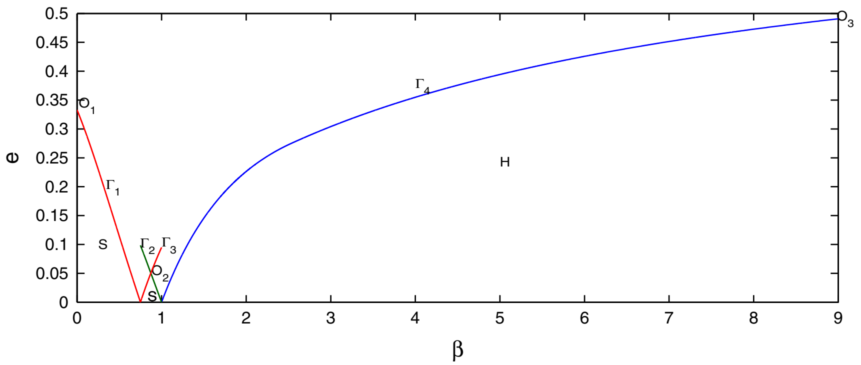

Fig. 1. The stable region **S** and hyperbolic region **H** given by Theorem 1.8

Theorem 1.8. *The elliptic Lagrangian orbits are linearly stable if*

$$e < \frac{1}{1 + f(\beta, -1)^{\frac{1}{2}}}, \quad \beta \in [0, 3/4),$$

or

$$e < min\left\{\frac{1}{\sqrt{f(\beta, -1)}}, \frac{1}{1 + \sqrt{f(\beta, \mathrm{e}^{i\sqrt{2}\pi})}}\right\}, \quad \beta \in (3/4, 1),$$

where $f(\beta, \omega)$ *is a function on* $[0, 9] \times \mathbb{U}$ *given by* (5.10)*. Let* $\hat{f}(\beta) = \sup\{f(\beta, \omega), \omega \in \mathbb{U}\}$*, then for* $\beta \in (1, 9]$*,* $\gamma_{\beta,e}$ *is hyperbolic if*

$$e < \hat{f}(\beta)^{-1/2}. \tag{1.22}$$

It will be seen that $f(\beta, \omega)$ is an elementary function determined by the trace formula. By Theorem 1.8, we can draw a picture as follows.
In Fig. 1, the points $O_1 \approx (0, 0.3333)$, $O_2 \approx (0.8730, 0.0504)$, $O_3 \approx (9, 0.4907)$. The curves

$$\Gamma_1 = \left\{(\beta, e) \;\middle|\; e = 1 \middle/ (1 + \sqrt{f(\beta, -1)}), 0 \leqq \beta \leqq 3/4 \right\},$$
$$\Gamma_3 = \left\{(\beta, e) \;\middle|\; e = 1 \middle/ \sqrt{f(\beta, -1)}, 3/4 \leqq \beta \leqq 1 \right\},$$

and

$$\Gamma_2 = \left\{(\beta, e) \;\middle|\; e = 1 \middle/ (1 + \sqrt{f(\beta, \mathrm{e}^{i\sqrt{2}\pi})}), 3/4 \leqq \beta < 1 \right\},$$
$$\Gamma_4 = \left\{(\beta, e) \;\middle|\; e = 1 \middle/ \sqrt{\hat{f}(\beta)}, 1 \leqq \beta \leqq 9 \right\}.$$

This paper is organized as follows. In Section 2, we give the proof of the trace formula for linear Hamiltonian systems. Moreover, some application of the trace formula on the identity which related to the Zeta function is given. In Section 3, we prove the Hill-type formula and trace formula for Sturm–Liouville systems. The applications of the trace formula on the study of stability for Hamiltonian systems are given in Section 4, where we estimate the relative Morse index (Morse index for Sturm–Liouville systems), and some new stability criteria will be given. The study of the stability of elliptic Lagrangian solutions will be given in Section 5.

2. Trace Formula for Linear Hamiltonian System

In this section, we will give the proof of the trace formula for the linear Hamiltonian system. As has been pointed out in the introduction, we will consider the Taylor expansion for the conditional Fredholm determinant of the Hamiltonian system and the Monodromy matrices separately in Sections 2.1 and 2.2. Based on it, we prove Theorem 1.1 in Section 2.3; some examples on infinite identity and relation with the Zeta function are discussed.

2.1. Taylor Expansion for Conditional Fredholm Determinant of the Linear Perturbation of Hamiltonian System

In this subsection, we will mainly consider the Taylor expansion of the conditional Fredholm determinant for the linearly parameterized Hamiltonian system. For $B, D \in \mathcal{B}(2n)$, notice that $(B+\alpha D)(A+P_0)^{-1}$ is not trace class but Hilbert–Schmidt. Hence $p(\alpha)$ is not the usual Fredholm determinant, but a kind of conditional Fredholm determinant. The theory of the conditional Fredholm determinant was studied in [16]. For readers convenience, we recall it briefly. For integer $N > 0$, let P_N be the projection onto the subspace

$$W_N = \bigoplus_{\nu\in\sigma(A),|\nu|\leqq N} \ker(A-\nu).$$

We need the following definition, which comes from [16]:

Definition 2.1. For a Hilbert–Schmidt operator F, it is said to have the *trace finite condition*, if the limit $\lim_{N\to\infty} Tr(P_N F P_N)$ exists, which is called the conditional trace and denoted by $Tr(F)$ without confusion.

Obviously, if F is a trace class operator, then the conditional trace coincides with the traditional trace. Moreover, if both F and $\widetilde{F}$ have the trace finite conditions, then $F+\widetilde{F}$ has the trace finite condition. Now, for a Hilbert–Schmidt operator F with trace finite condition, by [16], the limit

$$\det(id+F) = \lim_{N\to\infty} \det(id+P_N F P_N)$$

is well defined, which depends on $\{P_N\}$ and is called the *Conditional Fredholm Determinant* of $id+F$. The conditional Fredholm determinant preserves almost all the properties that the determinant of the matrix has, such as the multiplicative property of the determinant. Let $\hat{D}$ and $\hat{F}$ be two Hilbert–Schmidt operators with trace finite conditions. Then

$$\det(id+\hat{D})\det(id+\hat{F})=\det(id+\hat{D}+\hat{F}+\hat{D}\hat{F}). \tag{2.1}$$

Now, by the multiplicative property of conditional Fredholm determinant, it is obvious that

$$p(\alpha)=p(0)\det(id-\alpha\mathcal{F}).$$

Set $g(\alpha)=\det(id-\alpha\mathcal{F})$. By [16, Corollary 3.4], $p(\alpha)$ is an entire function, so is $g(\alpha)$. Let

$$\mathcal{F}_N=P_N\mathcal{F}P_N,$$

which are finite-rank operators; in particular, they are trace class operators. We set $g_N(\alpha)=\det(id-\alpha\mathcal{F}_N)$, then g_N are entire functions. Moreover, similar to [16, Proposition 3.2], $\{g_N\}$ is a normal family and there is a subsequence $\{g_{N_k}\}$ which is convergent to g uniformly on any compact subset in $\mathbb{C}$.

Since $\mathcal{F}_N$ are trace class operators, by [33, p.47, (5.12)], for α small, $\det(id-\alpha\mathcal{F}_N)=\mathrm{e}^{h_N(\alpha)}$ with

$$h_N(\alpha)=\sum_{m=1}^{\infty}-\frac{1}{m}Tr(\mathcal{F}_N^m)\alpha^m. \tag{2.2}$$

Please note that for α small enough, $id-\alpha\mathcal{F}$ is invertible, and hence $g(\alpha)$ vanishes nowhere near 0. Since $g(0)=1$, write $g(\alpha)=\mathrm{e}^{h(\alpha)}$ near 0 with

$$h(\alpha)=\sum_{m=1}^{\infty}b_m\alpha^m.$$

Obviously, h_{N_k} converges to h uniformly on any compact subset of $\mathbb{C}$, and hence

$$b_m=-\frac{1}{m}\lim_{k\to\infty}Tr(\mathcal{F}_{N_k}^m)=-\frac{1}{m}Tr(\mathcal{F}^m).$$

We get the following theorem, which is the main result in this subsection:

Theorem 2.2. *Under the above assumption, we have*

$$p(\alpha)=p(0)\exp\left\{\sum_{m=1}^{\infty}b_m\alpha^m\right\},$$

where $b_m=-\frac{1}{m}Tr(\mathcal{F}^m)$.

2.2. Taylor Expansion for Linearly Parameterized Monodromy Matrices

Set $B_\alpha=B+\alpha D$, for $\alpha\in\mathbb{C}$, let γ_α be the corresponding fundamental solutions. Fixing $\alpha_0\in\mathbb{C}$, direct computation shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma_{\alpha_0}^{-1}(t)\gamma_\alpha(t)) &= \gamma_{\alpha_0}^{-1}(t)J(B_\alpha(t)-B_{\alpha_0}(t))\gamma_\alpha(t)\\ &= J(\gamma_{\alpha_0}^T(t)(B_\alpha(t)-B_{\alpha_0}(t))\gamma_{\alpha_0}(t))\gamma_{\alpha_0}^{-1}(t)\gamma_\alpha(t))\\ &= (\alpha-\alpha_0)J(\gamma_{\alpha_0}^T(t)D(t)\gamma_{\alpha_0}(t))\gamma_{\alpha_0}^{-1}(t)\gamma_\alpha(t).\end{aligned}$$

Without loss of generality, assume $\alpha_0 = 0$. In what follows, write $\hat{\gamma}_\alpha(t) = \gamma_0^{-1}(t)\gamma_\alpha(t)$, and $\hat{D}(t) = \gamma_0^T(t)D(t)\gamma_0(t)$, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\gamma}_\alpha(t) = \alpha J\hat{D}(t)\hat{\gamma}_\alpha(t). \tag{2.3}$$

To simplify the notation, we use "$^{(k)}$" to denote the k-th derivative on α. Taking the derivative on α for both sides of (2.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\gamma}_\alpha^{(1)}(t) = J\hat{D}(t)\hat{\gamma}_\alpha(t) + \alpha J\hat{D}_\alpha(t)\hat{\gamma}_\alpha^{(1)}(t). \tag{2.4}$$

By taking $\alpha = 0$, $\hat{\gamma}_0(t) \equiv I_{2n}$, we have $\hat{\gamma}_0^{(1)}(t) = J\int_0^t \hat{D}(s)\mathrm{d}s$. Now, taking the derivative on α for both sides of (2.4), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\gamma}_\alpha^{(2)}(t) = 2J\hat{D}(t)\hat{\gamma}_\alpha^{(1)}(t) + J\alpha\hat{D}(t)\hat{\gamma}_\alpha^{(2)}(t).$$

Take $\alpha = 0$, and we get $\hat{\gamma}_0^{(2)}(t) = 2J\int_0^t \hat{D}(s)\hat{\gamma}_0^{(1)}(s)\mathrm{d}s$. By induction,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\gamma}_0^{(k)}(t) = kJ\hat{D}(t)\hat{\gamma}_0^{(k-1)}(t), \quad \hat{\gamma}_0^{(k)}(t) = kJ\int_0^t \hat{D}(s)\hat{\gamma}_0^{(k-1)}(s)\mathrm{d}s.$$

In what follows, to simplify the notation, set $M(\alpha) = \hat{\gamma}_\alpha(T)$, then $M_0 = I_{2n}$, and

$$M(\alpha) = \sum_{j=0}^{\infty}\alpha^j M_j,$$

where $M_j = \hat{\gamma}_0^{(j)}(T)/j!$ with the form (1.3). Direct computation shows that

$$M(\alpha)^T JM(\alpha) = J + \alpha C_1 + \alpha^2 C_2 + \cdots + \alpha^k C_k + \cdots$$

where $C_1 = M_1^T J + JM_1$, $C_2 = M_2^T J + JM_2 + M_1^T JM_1$, and in general

$$C_k = \sum_{j=0}^{k} M_j^T JM_{k-j}, \ \ k \in \mathbb{N}.$$

By the fact that $M(\alpha) \in \mathrm{Sp}(2n)$, $M(\alpha)^T JM(\alpha) = J$, thus $C_k = 0$ for $k \in \mathbb{N}$. We have the following proposition.

Proposition 2.3. *Under the above assumptions*

$$\sum_{j=0}^{k} M_j^T JM_{k-j} = 0, \ \ \forall k \in \mathbb{N}. \tag{2.5}$$

Please note that, by taking $k = 1$ in (2.5), we have $JM_1 + M_1^T J = 0$, which coincides with the fact that JM_1 is a symmetric matrix. Now, multiplying $-J$ on both sides of (2.5) and taking the trace, we have:

Corollary 2.4. *Under the above assumptions*

$$\sum_{j=0}^{m} Tr(-JM_j^T JM_{m-j}) = 0, \ \ \forall m \in \mathbb{N}.$$

Especially, for $m = 2$, we get

$$2Tr(M_2) = Tr(JM_1^T JM_1) = Tr(M_1^2).$$

Recall that $M = S\gamma_0(T)$ and $G_k = M_k M(M - e^{\nu T} I_{2n})^{-1}$, then $S\gamma_\alpha(T) = MM(\alpha)$, and $f(\alpha) = \det(I + \cdots + \alpha^k G_k + \cdots)$ is an entire function. For $\nu \in \mathbb{C}$ such that $e^{\nu T}$ is not an eigenvalue of M, by some easy computations, we have that

$$\det(S\gamma_\alpha(T) - e^{\nu T} I_{2n}) = \det(M - e^{\nu T} I_{2n}) f(\alpha).$$

Next, we will compute the Taylor expansion for $f(\alpha)$. Let $G(\alpha) = \sum_{k=1}^{\infty} \alpha^{k-1} G_k$, then for α small enough, by (2.2), we have

$$\begin{aligned} f(\alpha) &= \det(I + \alpha G(\alpha)) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \alpha^m Tr\big(G(\alpha)^m\big)\right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \alpha^m Tr\left[\left(\sum_{k=1}^{\infty} \alpha^{k-1} G_k\right)^m\right]\right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left[\sum_{k_1,\cdots,k_m=1}^{\infty} \alpha^{k_1+\cdots+k_m} Tr(G_{k_1} \cdots G_{k_m})\right]\right). \end{aligned} \tag{2.6}$$

Since $f(\alpha)$ vanishes nowhere near 0, we can write $f(\alpha) = e^{d(\alpha)}$, then by (2.6), some direct computation shows that

$$d^{(m)}(0)/m! = -Tr(\mathcal{G}_m). \tag{2.7}$$

For α small enough, let $d(\alpha)$ be the function satisfying

$$e^{-n\nu T} \det(S\gamma_\alpha(T) - e^{\nu T} I_{2n}) = e^{-n\nu T} \det(M - e^{\nu T} I_{2n}) \cdot \exp(d(\alpha)), \tag{2.8}$$

then the coefficients of $d^{(k)}(0)/k!$ could be determined by (2.7). Then we have the following theorem, which is the main result in this subsection.

Theorem 2.5. *Under the above assumption, let $d(\alpha)$ be the function in (2.8). Let $d(\alpha) = \sum_{m=1}^{\infty} c_m \alpha^m$ be its Taylor expansion. Then $c_m = -Tr(\mathcal{G}_m)$.*

By the definition of G_k, $Tr(G_k^m) = Tr\left[(M_k \mathcal{M})^m\right]$, and $Tr(G_{j_1} \cdots G_{j_k})$ could be given similarly.

2.3. The Proof of the Trace Formula for Hamiltonian System

In this subsection, we will give proof of Theorem 1.1.

Proof of Theorem 1.1. We begin with the Hill-type formula (1.20). On the one hand, by Theorem 2.2,

$$p(\alpha) = p(0)\exp\left\{\sum_{m=1}^{\infty} b_m\alpha^m\right\},$$

where $b_m = -\frac{1}{m}Tr\left(\mathcal{F}^m\right)$. On the other hand, by Theorem 2.5,

$$\begin{aligned} &C(S)\mathrm{e}^{-n\nu T}\det(S\gamma_\alpha(T) - \mathrm{e}^{\nu T}I_{2n}) \\ &= C(S)\mathrm{e}^{-n\nu T}\det(S\gamma(T) - \mathrm{e}^{\nu T}I_{2n})\cdot\exp\left(\sum_{m=1}^{\infty} c_m\alpha^m\right), \end{aligned}$$

where $c_m = -Tr(\mathcal{G}_m)$. Since

$$p(0) = C(S)\mathrm{e}^{-n\nu T}\det(S\gamma(T) - \mathrm{e}^{\nu T}I_{2n}),$$

we have that $b_m = c_m$. It follows that,

$$Tr\left(\mathcal{F}^m\right) = mTr(\mathcal{G}_m). \tag{2.9}$$

The proof is completed. □

By the Equation (2.9), theoretically, we can calculate the trace of $\mathcal{F}^m$, at least, numerically by computer. Notice that the right hand side of (2.9) is a kind of multiple integral, and it is a little complicated. For the first two terms, we can write it more precisely:

$$Tr[\mathcal{F}] = -Tr(G_1), \tag{2.10}$$

and

$$Tr\left(\mathcal{F}^2\right) = Tr(G_1^2) - 2Tr(G_2), \tag{2.11}$$

which are (1.7) and (1.8) in Corollary 1.3.

It is worth pointing out that, on the left hand side of (2.10), the trace is the conditional trace, and on the right hand side of it, it is the trace of matrix on $\mathbb{C}^{2n}$. In what follows, for the continuous path B on $[0, T]$ of the matrices, to simplify the notation, we always set

$$\mathcal{I}(B) = J\int_0^T B(t)\,\mathrm{d}t.$$

Next, we will consider some special cases.

Proposition 2.6. *Assume that $B(t) \equiv B_0$ is a constant matrix and $S = \pm I_{2n}$, then,*

$$Tr(\mathcal{F}) = -Tr\left(\mathcal{I}(D)\cdot\mathcal{M}\right).$$

Proof. Since $B(t) \equiv B_0$, obviously $\gamma_0(t) = e^{JB_0t}$, thus $\gamma_0(t)$ commutes with $\gamma_0(T)$ and also commutes with M since $S = \pm I_{2n}$. Easy computation shows that

$$Tr\left(\mathcal{I}(\hat{D}) \cdot \mathcal{M}\right) = Tr\left(\mathcal{I}(D) \cdot \mathcal{M}\right).$$

By the trace formula (1.7), the proposition is proved. □

The following proposition considers the case that $MJ = JM$, $M^T = M$.

Proposition 2.7. *If $MJ = JM$, $M^T = M$, then*

$$Tr\left(\mathcal{F}^2\right) = Tr\left[\left(\mathcal{I}(\hat{D}) \cdot \mathcal{M}\right)^2\right] - Tr\left[\mathcal{I}(\hat{D})^2\mathcal{M}\right].$$

Proof. Suppose $MJ = JM$, $M = M^T$ then

$$Tr(M_2\mathcal{M}) = Tr(-JM_2^TJ\mathcal{M}).$$

By Proposition 2.3, $M_1^TJ + JM_1 = 0$ and $-JM_2^TJ + M_2 = JM_1^TJM_1$. Thus

$$2Tr(G_2) = Tr\left[\mathcal{I}(\hat{D})^2\mathcal{M}\right].$$

By the formula (1.8), the proposition is proved. □

Some easy computation shows that, if moreover M commutes with $\mathcal{I}(\hat{D})$, then

$$Tr\left(\mathcal{F}^2\right) = e^{\nu T}Tr\left[\mathcal{I}(\hat{D})^2M(M - e^{\nu T}I_{2n})^{-2}\right]. \tag{2.12}$$

More specially, we have the following corollary.

Corollary 2.8. *If $M = \pm I_{2n}$, then*

$$Tr\left(\mathcal{F}^2\right) = \frac{\pm e^{\nu T}}{(1 \mp e^{\nu T})^2}Tr\left[\mathcal{I}(\hat{D})^2\right]. \tag{2.13}$$

Especially in the case $B = 0$, $\hat{D} = D$ and $S = \pm I_{2n}$,

$$Tr\left(\mathcal{F}^2\right) = \frac{\pm e^{\nu T}}{(1 \mp e^{\nu T})^2}Tr\left[\mathcal{I}(D)^2\right]. \tag{2.14}$$

Notice that (2.13) is just the formula (1.9) in Corollary 1.3.

Example 2.9. In the case $D(t) = I_{2n}$, then $\hat{D}(t) = \gamma_0^T(t)\gamma_0(t)$, $\mathcal{F} = (A - \nu J - B)^{-1}$ so we have

$$Tr(\mathcal{F}) = Tr\left(J\int_0^T \gamma_0^T(s)\gamma_0(s)ds \cdot \mathcal{M}\right),$$

and for $k \geqq 2$,

$$Tr\left[\mathcal{F}^k\right] = \sum_{j=-\infty}^{\infty} \frac{1}{\lambda_j^k},$$

where λ_j are eigenvalues of $A - B - \nu J$. From the trace formula, we have

$$\sum_{j=-\infty}^{\infty} \frac{1}{\lambda_j^m} = mTr(\mathcal{G}_m), \quad \forall m \geqq 2. \tag{2.15}$$

The Equation (2.15) has its own interest. In fact, we can deduce some interesting equalities from this.

Example 2.10. Let $B = \alpha I_2$, $D = I_2$, $S = I_2$ and $T = 1$, obviously $\mathcal{F} = (A - \nu J - \alpha)^{-1}$. It is easy to check that the eigenvalues for $A - \nu J - \alpha$ are $\{2k\pi \pm \sqrt{-1}\nu - \alpha | k \in \mathbb{Z}\}$. For $\nu \notin 2\pi\sqrt{-1}\mathbb{Z} - \alpha$, $A - \nu J - \alpha$ is invertible, and the left hand side of (2.15) is

$$Tr\left(\mathcal{F}^m\right) = \sum_{k\in\mathbb{Z}} \frac{1}{(2k\pi + \sqrt{-1}\nu - \alpha)^m} + \sum_{k\in\mathbb{Z}} \frac{1}{(2k\pi - \sqrt{-1}\nu - \alpha)^m}, \quad \forall m \in \mathbb{N},$$

where for $m = 1$, the infinite sum in the right side is understand by $\lim_{\beta\to\infty} \sum_{|k|\leqq\beta}$. For the right hand side, the traces $Tr(G_{j_1} \cdots G_{j_k})$ can be calculated directly. We only list the first 3 equalities. For $m = 1$, direct computation shows that $Tr(G_1) = \frac{2e^{\nu}\sin\alpha}{(\cos\alpha - e^{\nu})^2 + \sin^2\alpha}$, thus we have

$$\sum_{k\in\mathbb{Z}} \frac{1}{2k\pi + \sqrt{-1}\nu - \alpha} + \sum_{k\in\mathbb{Z}} \frac{1}{2k\pi - \sqrt{-1}\nu - \alpha} = \frac{-2e^{\nu}\sin\alpha}{(\cos\alpha - e^{\nu})^2 + \sin^2\alpha}.$$

For $m = 2$, by (2.12), direct computation shows that

$$Tr\left(\mathcal{F}^2\right) = \frac{1 - \cosh\nu\cos\alpha}{(\cos\alpha - \cosh\nu)^2}.$$

Especially in the case $\alpha = 0$,

$$\sum_{k\in\mathbb{Z}} \frac{1}{(2k\pi + \sqrt{-1}\nu)^2} = \frac{1 + \cos\sqrt{-1}\nu}{2\sin^2\sqrt{-1}\nu}.$$

Similarly, for $m = 3$, we get

$$Tr\left(\mathcal{F}^3\right) = \frac{1/2\sin\alpha(\cosh^2\nu + \cosh\nu\cos\alpha - 2)}{\cosh^3\nu - 3\cosh^2\nu\cos\alpha + 3\cosh\nu\cos^2\alpha - \cos^3\alpha}.$$

The equality in the above example can be deduced by using techniques in complex analysis. However, the above example is only a kind of easiest case. If we take a non-constant path B, then the formula will be far from trivial.

Remark 2.11. Recall that, in Atiyah et al. [1] defined a kind of zeta function for self-adjoint elliptic differential operator $\mathcal{A}$(the operator may be not positive). Let $\{\lambda\}$ be the eigenvalues for $\mathcal{A}$, then

$$\eta_{\mathcal{A}}(s) = \sum_{\lambda\neq 0} (\mathrm{sign}\lambda)|\lambda|^{-s},$$

for $Re(s)$ large, and it can be extended meromorphically to the whole s-plane. Now, for the differential operator A, if we can take some proper B, D and S in our framework, such that λ are the eigenvalues of $\mathcal{A} = D^{-1}(A - B - \nu J)$ and are real, then by the trace formula, we can obtain the values for $\eta_{\mathcal{A}}(s)$ at odd integers.

3. Hill-type Formula and Trace Formula for Sturm–Liouville Systems

In the study of $\bar{S}$-periodic orbits in Lagrangian systems, it is natural to consider the standard Sturm systems:

$$-(P\dot{y}+Qy)^{\cdot}+Q^T\dot{y}+Ry=0, \quad y(0)=\bar{S}y(T), \quad \dot{y}(0)=\bar{S}\dot{y}(T), \quad (3.1)$$

which satisfied (1.11). Denote $\hat{Q}=P^{-1}(Q^T-Q-\dot{P})$, $\hat{R}=P^{-1}(R-\dot{Q})$. Obviously, the system (3.1) is equivalent to

$$-\ddot{z}(t)+\hat{Q}(t)\dot{z}(t)+\hat{R}(t)z(t)=0,\ z(0)=\bar{S}z(T),\ \dot{z}(0)=\bar{S}\dot{z}(T). \quad (3.2)$$

Please note that if $\hat{z}(t)$ satisfies the Equation (3.2) with $\hat{z}(0)=\mathrm{e}^{-\nu T}\bar{S}\hat{z}(T)$, $\dot{\hat{z}}(0)=\mathrm{e}^{-\nu T}\bar{S}\dot{\hat{z}}(T)$, then $z(t)=\mathrm{e}^{\nu t}\hat{z}(t)$ satisfies the following second order ODE

$$\mathcal{L}(\nu,\hat{Q},\hat{R})z(t)=0,\ z(0)=\bar{S}z(T),\ \dot{z}(0)=\bar{S}\dot{z}(T), \quad (3.3)$$

where $\mathcal{L}(\nu,\hat{Q},\hat{R})=-\left(\frac{\mathrm{d}}{\mathrm{d}t}+\nu\right)^2+\hat{Q}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\nu\right)+\hat{R}(t)$. In what follows, to simplify the notation, we always use $\mathcal{L}(\nu)$ instead $\mathcal{L}(\nu,\hat{Q},\hat{R})$ and set $\mathcal{L}_0(\nu)=\mathcal{L}(\nu,0,0)$.

Let $y(t)=\dot{z}(t)+\nu z(t)$, then we can write (3.3) as the following first order ODE

$$\begin{pmatrix}\dot{y}(t)\\ \dot{z}(t)\end{pmatrix}=\begin{pmatrix}\hat{Q}(t)-\nu & \hat{R}(t)\\ I_n & -\nu\end{pmatrix}\begin{pmatrix}y(t)\\ z(t)\end{pmatrix}, \quad \begin{pmatrix}y(0)\\ z(0)\end{pmatrix}=\begin{pmatrix}\bar{S} & 0_n\\ 0_n & \bar{S}\end{pmatrix}\begin{pmatrix}y(T)\\ z(T)\end{pmatrix}. \quad (3.4)$$

For simplicity, we denote

$$\hat{B}(t)=\begin{pmatrix}I_n & 0_n\\ -\hat{Q}(t) & -\hat{R}(t)\end{pmatrix}, \quad \bar{S}_d=\begin{pmatrix}\bar{S} & 0_n\\ 0_n & \bar{S}\end{pmatrix} \quad \text{and} \quad x(t)=\begin{pmatrix}y(t)\\ z(t)\end{pmatrix}.$$

The system (3.4) can be written as the following Hamiltonian system,

$$\dot{x}(t)=J(\hat{B}(t)+\nu J)x(t), \quad x(0)=\bar{S}_{\mathrm{d}}x(T). \quad (3.5)$$

It follows that $z(t)$ is solution of (3.3) if and only if $x(t)=\begin{pmatrix}\dot{z}(t)+\nu z(t)\\ z(t)\end{pmatrix}$ is solution of (3.5). In what follows, set $\upsilon(\mathcal{A})=\dim\ker(\mathcal{A})$ for any operator $\mathcal{A}$, then we have

$$\upsilon\left(\mathcal{L}(\nu)\right)=\upsilon\left(A-\hat{B}-\nu J\right). \quad (3.6)$$

Now, we will give the Hill-type formula for indefinite Lagrangian system. For $N\in\mathbb{N}$, let

$$\hat{W}_N=\bigoplus_{\nu\in\sigma(\frac{\mathrm{d}}{\mathrm{d}t}),|\nu|\leqq N}\ker\left(\frac{\mathrm{d}}{\mathrm{d}t}-\nu I_n\right),$$

and denote by $\hat{P}_N$ the orthogonal projection onto $\hat{W}_N$. Then $\hat{Q}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\nu\right)^{-1}$ is a Hilbert–Schmidt operator with the trace finite condition with respect to $\{\hat{P}_N\}$. We define the conditional Fredholm determinant with respect to $\hat{P}_N$, $\det\left[\mathcal{L}(\nu)\mathcal{L}_0(\nu)^{-1}\right]$.

At first, we recall the Hill-type formula for linear Hamiltonian systems [16]. For $B \in C([0,T]; \mathcal{M}(2n,\mathbb{C}))$, which does not have to be real symmetric, we have that

$$p(0) = C(S)\mathrm{e}^{-\frac{T}{2}\int_0^T Tr(JB(t))\,\mathrm{d}t}\mathrm{e}^{-n\nu T}\det\left(S\gamma(T) - \mathrm{e}^{\nu T}I_{2n}\right). \tag{3.7}$$

We firstly prove the following proposition.

Proposition 3.1. *For $\nu \in \mathbb{C}$ such that $\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n$ is invertible, we have*

$$\det\left[\left(A - \hat{B} - \nu J\right)(A - \nu J)^{-1}\right] = \det\left[\mathcal{L}(\nu)\mathcal{L}_0(\nu)^{-1}\right].$$

Proof. Let $K_n = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix}$, note that $A - \nu J - K_n = \begin{pmatrix} -I_n & \frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n \\ -\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n\right) & 0_n \end{pmatrix}$. It follows that $\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n$ is invertible if and only if $A - \nu J - K_n$ is invertible; moreover

$$(A - \nu J - K_n)^{-1} = \begin{pmatrix} 0_n & -\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n\right)^{-1} \\ \left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n\right)^{-1} & -\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n\right)^{-2} \end{pmatrix}.$$

It follows that

$$\begin{aligned}&(K_n - \hat{B})(A - \nu J - K_n)^{-1} \\ &= \begin{pmatrix} 0_n & 0_n \\ \hat{R}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n\right)^{-1} & -\hat{Q}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n\right)^{-1} - \hat{R}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n\right)^{-2} \end{pmatrix}.\end{aligned}$$

Thus we have

$$\begin{aligned}\det\left[\left(A - \hat{B} - \nu J\right)(A - K_n - \nu J)^{-1}\right] &= \det\left[id - (\hat{B} - K_n)(A - \nu J - K_n)^{-1}\right] \\ &= \det\left[id - \hat{Q}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n\right)^{-1}\right. \\ &\qquad \left. - \hat{R}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n\right)^{-2}\right] \\ &= \det\left[\mathcal{L}(\nu)\mathcal{L}_0(\nu)^{-1}\right].\end{aligned} \tag{3.8}$$

Now, direct computation shows that,

$$\det\left[(A - K_n - \nu J)(A - \nu J)^{-1}\right] = 1.$$

Therefore,

$$\begin{aligned}\det\left[\left(A - \hat{B} - \nu J\right)(A - \nu J)^{-1}\right] &= \det\left[\left(A - \hat{B} - \nu J\right)(A - K_n - \nu J)^{-1}\right] \\ &\quad \cdot \det\left[(A - K_n - \nu J)(A - \nu J)^{-1}\right] \\ &= \det\left[\left(A - \hat{B} - \nu J\right)(A - K_n - \nu J)^{-1}\right].\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we have the desired result. □

For $R_1 \in \mathcal{B}(n)$, let $\hat{B}_\lambda(t) = \begin{pmatrix} I_n & 0_n \\ -\hat{Q}(t) & -\hat{R}(t) - \lambda P^{-1} R_1 \end{pmatrix}$, let $\hat{\gamma}_\lambda$ be the corresponding fundamental solutions. With the above preparation, we have the following theorem.

Theorem 3.2. *For $\nu \in \mathbb{C}$ such that $\frac{\mathrm{d}}{\mathrm{d}t} + \nu I_n$ is invertible, we have*

$$\det\left[\mathcal{L}(\nu)\mathcal{L}_0(\nu)^{-1}\right] = \mathrm{e}^{-\frac{T}{2}\int_0^T Tr(\hat{Q})\,\mathrm{d}t} \det(\bar{S}_d\hat{\gamma}_0(T) - \mathrm{e}^{\nu T} I_{2n}) \det(\bar{S}_d - \mathrm{e}^{\nu T} I_{2n})^{-1}.$$

Proof. By the multiplicative property of conditional Fredholm determinant

$$\det\left[\left(A - \hat{B} - \nu J\right)(A - \nu J)^{-1}\right] = \det\left[\left(A - \hat{B} - \nu J\right)(A + P_0)^{-1}\right] \cdot \det\left[(A + P_0)(A - \nu J)^{-1}\right]. \quad (3.10)$$

By the Hill-type formula for Hamiltonian system (3.7), we have that

$$\det\left[\left(A - \hat{B} - \nu J\right)(A + P_0)^{-1}\right] = C(\bar{S}_d)\mathrm{e}^{-\frac{T}{2}\int_0^T Tr(\hat{Q})\,\mathrm{d}t}\mathrm{e}^{-n\nu T} \det\left(\bar{S}_d\hat{\gamma}_0(T) - \mathrm{e}^{\nu T} I_{2n}\right) \quad (3.11)$$

and

$$\det[(A + P_0)(A - \nu J)^{-1}] = C(\bar{S}_d)^{-1}\mathrm{e}^{n\nu T} \det(\bar{S}_d - \mathrm{e}^{\nu T} I_{2n})^{-1}. \quad (3.12)$$

Substituting (3.12) and (3.11) into (3.10), by Proposition 3.1 we have the result. □

We come back to the Lagrangian systems. To simplify the notation, let

$$\mathcal{A}(\nu) = -\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu\right)\left(P\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu\right) + Q\right) + Q^T\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu\right) + R(t). \quad (3.13)$$

Theorem 3.3. *Under the condition (1.11), for any $\nu \in \mathbb{C}$ such that $\mathcal{A}(\nu)$ is invertible,*

$$\det\left[(\mathcal{A}(\nu) + R_1)\mathcal{A}(\nu)^{-1}\right] = \det(\bar{S}_d\gamma_1(T) - \mathrm{e}^{\nu T} I_{2n}) \cdot \det(\bar{S}_d\gamma_0(T) - \mathrm{e}^{\nu T} I_{2n})^{-1}, \quad (3.14)$$

where $\gamma_\lambda(t)$ is the fundamental solution of (1.12).

Proof. Please note that $R_1\mathcal{A}(\nu)^{-1}$ is a trace class operator, thus $\det(id + R_1\mathcal{A}(\nu)^{-1})$ is the usual Fredholm determinant. Therefore $\det(id + R_1\mathcal{A}(\nu)^{-1}) = \det(id + P^{-1}R_1\mathcal{A}(\nu)^{-1}P)$, hence

$$\det\left[(\mathcal{A}(\nu) + R_1)\mathcal{A}(\nu)^{-1}\right] = \det\left[P^{-1}(\mathcal{A}(\nu) + R_1)\mathcal{A}(\nu)^{-1}P\right] = \det\left[\left(P^{-1}(\mathcal{A}(\nu) + R_1)\right)\left(P^{-1}\mathcal{A}(\nu)\right)^{-1}\right].$$

Easy computation shows that $P^{-1}\mathcal{A}(\nu) = \mathcal{L}(\nu)$. By the multiplicative property of Fredholm determinant (2.1),

$$\det\left[P^{-1}(\mathcal{A}(\nu)+R_1)\left(P^{-1}\mathcal{A}(\nu)\right)^{-1}\right] = \frac{\det\left[P^{-1}(\mathcal{A}(\nu)+R_1)\mathcal{L}_0(\nu)^{-1}\right]}{\det\left[\left(P^{-1}\mathcal{A}(\nu)\right)\cdot\mathcal{L}_0(\nu)^{-1}\right]} \tag{3.15}$$

Substituting (3.10) into (3.15), we have

$$\det\left[(\mathcal{A}(\nu)+R_1)\mathcal{A}(\nu)^{-1}\right] = \det(\bar{S}_d\hat{\gamma}_1(T) - e^{\nu T}I_{2n})\cdot\det(\bar{S}_d\hat{\gamma}_0(T) - e^{\nu T}I_{2n})^{-1}. \tag{3.16}$$

To prove the theorem, we will make clear the relationship between $\hat{\gamma}_\lambda(T)$ with $\gamma_\lambda(T)$. Let $\eta(t) = \begin{pmatrix} P(t) & Q(t) \\ 0_n & I_n \end{pmatrix}$, then direct computation shows that

$$\frac{d}{dt}(\eta(t)\hat{\gamma}_\lambda(t)\eta(0)^{-1}) = JB_\lambda(t)\eta(t)\hat{\gamma}_\lambda(t)\eta(0)^{-1},$$

which implies $\gamma_\lambda(t) = \eta(t)\hat{\gamma}_\lambda(t)\eta(0)^{-1}$. Moreover, from (1.11), $\bar{S}_d\eta(T) = \eta(0)\bar{S}_d$, easy computation shows that

$$\bar{S}_d\gamma_\lambda(T) = \eta(0)\bar{S}_d\hat{\gamma}_\lambda(T)\eta(0)^{-1}. \tag{3.17}$$

It follows that

$$\det(\bar{S}_d\gamma_\lambda(T) - e^{\nu T}I_{2n}) = \det(\bar{S}_d\hat{\gamma}_\lambda(T) - e^{\nu T}I_{2n}).$$

Combining with (3.16), we have the desired result. □

Obviously, by taking $\nu = 0$ in Theorem 3.3 we have Theorem 1.6.

To get the trace formula, let λR_1 take place of R_1 in the Hill-type formula (3.14), and we have

$$\det(id + \lambda R_1\mathcal{A}(\nu)^{-1}) = \det\left(\bar{S}_d\gamma_\lambda(T) - e^{\nu T}I_{2n}\right)\cdot\det\left(\bar{S}_d\gamma_0(T) - e^{\nu T}I_{2n}\right)^{-1}. \tag{3.18}$$

Almost the same as the proof of Theorem 1.1, the trace formula for Lagrangian system could be obtained by taking the Taylor expansion on the variable λ and comparing the coefficients of λ^n on both sides of (3.18), and the proof will be omitted. We have the trace formula for the Lagrangian system, for $m \in \mathbb{N}$,

$$Tr\left(\left[R_1\mathcal{A}(\nu)^{-1}\right]^m\right) = (-1)^m m Tr(\mathcal{G}_m), \tag{3.19}$$

where for the Lagrangian system, in the definition of $\mathcal{G}_m$, $D = \begin{pmatrix} 0_n & 0_n \\ 0_n & -R_1 \end{pmatrix}$.

Since $\mathcal{A}(\nu)^{-1}$ is a trace class operator, let $\{\lambda_i\}$ be the nonzero eigenvalues of $\mathcal{A}(\nu)y + \lambda R_1 y = 0$, then for positive integers m,

$$\sum_j \frac{1}{\lambda_j^m} = (-1)^m \cdot Tr\left[\left(R_1\mathcal{A}(\nu)^{-1}\right)^m\right]. \tag{3.20}$$

Combining (3.19) and (3.20) we prove Theorem 1.4.

Especially,

$$Tr[R_1\mathcal{A}(\nu)^{-1}] = Tr(G_1). \tag{3.21}$$

Comparing with the Trace formula in Hamiltonian systems, we have

Corollary 3.4. *For positive integers m,*

$$(-1)^m \cdot Tr\left[\left(R_1\mathcal{A}(\nu)^{-1}\right)^m\right] = Tr\left[\mathcal{F}(\nu, B_0, D)^m\right]. \tag{3.22}$$

where $D = \begin{pmatrix} 0_n & 0_n \\ 0_n & -R_1 \end{pmatrix}$, B_0 is defined in (1.13) and $A = -J\frac{\mathrm{d}}{\mathrm{d}t}$ with domain $D_{\bar{S}_d}$.

Obviously $\mathcal{A}(\nu) + \lambda R_1$ is degenerate if only if $A - \nu J - B_0 - \lambda D$ is degenerate, moreover, we have

Proposition 3.5. *Let $\nu \in \mathbb{C}$, such that $\mathcal{A}(\nu)$ is invertible. Then $-\frac{1}{\lambda_0}$ is an eigenvalue of $R_1\mathcal{A}(\nu)^{-1}$ of algebraic multiplicity k if and only if $\frac{1}{\lambda_0}$ is an eigenvalue of $D(A - \nu J - B_0)^{-1}$ of algebraic multiplicity k.*

Remark 3.6. 1. For $m \geqq 2$, notice that both $\left(R_1\mathcal{A}(\nu)^{-1}\right)^m$ and $\mathcal{F}^m$ are trace class, and hence by Proposition 3.5 we can get the trace formula for Lagrangian system from that of Hamiltonian system directly.

2. For $m = 1$, since the operator $\mathcal{F}$ is not trace class operator, but a Hilbert–Schmidt operator with trace finite condition. For a general Hamiltonian system, we don't know whether $Tr(\mathcal{F}) = \sum_j \frac{1}{\lambda_j}$ true or not. It follows that, the trace formula (3.21) can not be obtained by the trace formula from Hamiltonian system.

To prove Proposition 3.5, we need the following lemma, which itself is of interest.

Lemma 3.7. *Let F be a Hilbert–Schmidt operator with trace finite condition, and $\frac{1}{\lambda_0}$ is its nonzero eigenvalue. Then λ_0 is a zero point of $\det(id - \lambda F)$ of degree k if and only if $\frac{1}{\lambda_0}$ is an eigenvalue of F of algebraic multiplicity k.*

Proof. Since F is a Hilbert–Schmidt operator, so $\sigma_1 = \{\frac{1}{\lambda_0}\}$ and $\sigma_2 = \sigma(F)\backslash\sigma_1$ are two disjoint closed subsets of the spectral of F. By Riesz Decomposition Theorem for operators, let

$$P_1 = \frac{1}{2\pi i}\int_\Gamma (\lambda - F)^{-1} d\lambda,$$

where Γ is a contour in the resolvent set of F such that σ_1 in its interior and σ_2 in its exterior. Then P_1 is its Riesz projection, and let $P_2 = id - P_1$. Since $\frac{1}{\lambda_0}$ is a nonzero eigenvalue, then P_1 is a finite projection, and $P_1 F = F P_1$. Now, let $F_1 = F P_1$ and $F_2 = F P_2$, then $F_1 F_2 = 0$. By the multiplicative property of the conditional Fredholm determinant,

$$\det(id - \lambda F) = \det(id - \lambda F_1 - \lambda F_2 - \lambda^2 F_1 F_2) = \det(id - \lambda F_1)\det(id - \lambda F_2).$$

Since $\frac{1}{\lambda_0}$ is not in the spectrum of F_2, hence $\frac{1}{\lambda_0}$ is not zero point of $\det(id - \lambda F_2)$; moreover, it is not hard to see that $\det(id - \lambda F_1) = \left(1 - \frac{\lambda}{\lambda_0}\right)^k$ where k is the algebraic multiplicity of the eigenvalue $\frac{1}{\lambda_0}$ of F. The proof is complete. □

Proof of Proposition 3.5. By (3.18) and Lemma 3.7, $-\frac{1}{\lambda_0}$ is an eigenvalue of $R_1\mathcal{A}(\nu)^{-1}$ of algebraic multiplicity k if and only if it is a zero point the analytic function $\det(\bar{S}_d\gamma_\lambda(T) - e^{\nu T} I_{2n})$ of degree k. On the other hand, by (1.20) and the multiplicative property, for B_λ defined in (1.13) we have

$$\det\left(id - \lambda\mathcal{F}(\nu, B_0, D)\right) = (\bar{S}_d\gamma_\lambda(T) - e^{\nu T} I_{2n})(\bar{S}_d\gamma_0(T) - e^{\nu T} I_{2n}).$$

Again, by Lemma 3.7, $1/\lambda_0$ is an eigenvalue of $\mathcal{F}(\nu, B_0, D)$ of algebraic multiplicity k if and only if it is also a zero point $\det(\bar{S}_d\gamma_\lambda(T) - e^{\nu T} I_{2n})$ of degree k. The desired result is proved. □

Example 3.8. We will compute the simplest case, that is $\mathcal{A}(\nu) = -(\frac{d}{dt} + \nu)^2$, $R_1 = -R$. Recall that $K_n = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix}$, $D = \begin{pmatrix} 0_n & 0_n \\ 0_n & R \end{pmatrix}$. Recall that $\gamma_0(t)$ satisfied $\dot{\gamma}_0(t) = J K_n \gamma_0(t)$ with $\gamma_0(0) = I_{2n}$. Direct computation shows that $\gamma_0(t) = \begin{pmatrix} I_n & 0_n \\ tI_n & I_n \end{pmatrix}$, and obviously $\gamma_0(t)^{-1} = \begin{pmatrix} I_n & 0_n \\ -tI_n & I_n \end{pmatrix}$. Therefore,

$$\mathcal{I}(\hat{D}) = \begin{pmatrix} -\int_0^T tR dt & -\int_0^T R dt \\ \int_0^T t^2 R dt & \int_0^T tR dt \end{pmatrix}.$$

Let $\bar{S}^T$ be the transposition of $\bar{S}$, then $\bar{S}^T = \bar{S}^{-1}$. Thus

$$Tr(G_1) = e^{\nu T} Tr\left(T \int_0^T R dt \cdot \bar{S}^T (I_n - e^{\nu T}\bar{S}^T)^{-2}\right).$$

To simplify the notation, we denote by $R_{ave} = \frac{1}{T}\int_0^T R(t)\,dt$, which is a constant matrix. Then

$$Tr(R\mathcal{A}(\nu)^{-1}) = -e^{\nu T} T^2 \cdot Tr(R_{ave} \cdot \bar{S}(\bar{S} - e^{\nu T})^{-2}). \tag{3.23}$$

Please note that by take derivative with respect to ν on both sides of (3.23), we get

$$Tr\left(R\mathcal{A}(\nu)^{-2}\right) = \frac{e^{\nu T} T^4}{6} Tr\left(R_{ave}\bar{S}(\bar{S}^2 + 4e^{\nu T}\bar{S} + e^{2\nu T})(\bar{S} - e^{\nu T})^{-4}\right). \tag{3.24}$$

Remark 3.9. In [19], Krein also consider the boundary value problem

$$y'' + \lambda R(t)y = 0,\ \ y(0) + y(T) = y'(0) + y'(T) = 0, \tag{3.25}$$

where $R(t) \in \mathcal{B}(n)$. Let λ_j, $j \in \mathbb{Z}$ or $\mathbb{N}$ (assume $\lambda_j \leqq \lambda_{j+1}$), be the eigenvalues of boundary value problem (3.25), that means the system

$$y'' + \lambda_j R(t)y = 0,\ \ y(0) + y(T) = y'(0) + y'(T) = 0, \tag{3.26}$$

has a nontrivial solution. Each λ_j appears as many times as its multiplicity. To state Krein's work, set

$$X(t) = \int_0^t (R(s) - R_{ave})\mathrm{d}s + C, \tag{3.27}$$

where C is a constant matrix which is chosen such that $X_{ave} = 0$. Krein proved [19]

$$\sum \frac{1}{\lambda_j} = \frac{T}{4}\int_0^T Tr(R(t))\,\mathrm{d}t, \tag{3.28}$$

and

$$\sum \frac{1}{\lambda_j^2} = \frac{T}{2}\int_0^T Tr(X^2(t))\,\mathrm{d}t + \frac{T^2}{48}Tr\left[\left(\int_0^T R(t)\,\mathrm{d}t\right)^2\right]. \tag{3.29}$$

Please note that (3.23) is a generalization of (3.28). In the formula (1.14), the expression of $\sum \frac{1}{\lambda_j^2}$ is different from (3.29). The precise generalization with the same form as Krein's formula will be given in the forthcoming paper [17].

4. Applications

The Maslov-type index is a very useful tool in studying the multiplicity and stability of periodic solutions in Hamiltonian systems [23,24]. It is well-known that the relative Morse index for the linear Hamiltonian system equals to the Maslov-type index for the corresponding fundamental solutions. It will be seen that, by the trace formula, we could estimate the relative Morse index, and therefore the trace formula could be used to judge the linear stability via the Maslov-type index. For reader's convenience, we review the relative Morse index and stability criteria via the Maslov-type index in Section 4.1, details can be found in [14,24]. The estimation of the relative Morse index by the trace of operator is given in Section 4.2, some new criteria for the stability is given in Section 4.3, at Section 4.4, we give some estimation of the Morse index for Sturm–Liouville systems.

In the whole of this section, ν will be assumed to be an imaginary number.

4.1. Brief Review of the Relative Morse Index, Spectral Flow and Stability Criteria via Maslov-type Index

The relationship between the conditional Fredholm determinant and the relative Morse index has been given in [16], where the relative Morse index is defined by a relative dimension of negative subspace. On the other hand, the relative Morse index could be defined by spectral flow [14]. As is well known, spectral flow was introduced by ATIYAH et al. [1] in their study of index theory on manifolds with a boundary. It is a very useful tool to understand the relative Morse index. Let $\{A(\theta), \theta \in [0,1]\}$ be a continuous path of self-adjoint Fredholm operators on a Hilbert space $\mathcal{H}$. Roughly speaking, the spectral flow of path $\{A(\theta), \theta \in [0,1]\}$ counts the net change in the number of negative eigenvalues of $A(\theta)$ as θ goes from 0 to 1, where the enumeration follows from the rule that each negative eigenvalue crossing to the positive axis contributes $+1$ and each positive eigenvalue crossing to the negative axis contributes -1, and for each crossing, the multiplicity of eigenvalue is counted. More precisely, as shown in [1], let

$$\wp = \bigcup_{\theta\in[0,1]} \sigma(A(\theta)),$$

where $\sigma(A(\theta))$ is the spectrum for $A(\theta)$, then $\wp$ is a closed subset of the (θ,λ)-plane. The spectral flow $Sf(\{A(\theta)\})$ is defined to be the intersection number of $\wp$ with the line $\lambda = -\epsilon$ with respect to the usual orientation for some small positive ϵ. Obviously, $Sf(\{A(\theta)\}) = Sf(\{A(\theta) + \epsilon id\})$ if id is the identity operator on $\mathcal{H}$, and $0 \leqq \epsilon \leqq \epsilon_0$ for some sufficiently small positive number ϵ_0.

Coming back to the Hamiltonian systems, suppose $B(s,t) \in C([0,1] \times [0,T], S(2n))$. For $s \in [0,1]$, let $B_s \in \mathcal{B}(2n)$. For two such operators $A - B_0$ and $A - B_1$, we can define the relative Morse index via spectral flow. In fact, by [14], we have,

$$I(A - B_0, A - B_1) = -Sf(\{A - B(s), s \in [0,1]\}).$$

For B_0, B_1, B_2, then

$$I(A - B_0, A - B_1) + I(A - B_1, A - B_2) = I(A - B_0, A - B_2).$$

Let $D = B_1 - B_0$, and we can simply let $B(s) = B_0 + sD$. The next proposition is obvious from the definition of spectral flow.

Proposition 4.1. *Let* $\kappa = \{s_0 \in [0,1], \ker(A - B(s_0)) \neq 0\}$,

$$I(A - B_0, A - B_1) \leqq \sum_{s_0\in\kappa} \nu(A - B(s_0)).$$

It is not hard to see that, if $D > 0$, then $I(A - B, A - B - D) \geqq 0$. By careful analysis [14], the crossing form

$$I(A - B, A - B - D) = \sum_{s_0\in\kappa\cap[0,1)} \nu(A - B(s_0)). \qquad (4.1)$$

Similarly

$$I(A-B, A-B+D) = -\sum_{s_0\in\kappa\cap(0,1]} \upsilon(A-B(s_0)). \tag{4.2}$$

Thus we have:

Corollary 4.2. *Suppose $D_1 \leqq D \leqq D_2$, then*

$$I(A-B, A-B-D_1) \leqq I(A-B, A-B-D) \leqq I(A-B, A-B-D_2). \tag{4.3}$$

Notice that for pure imaginary number ν, $|e^{\nu T}| = 1$. In the remaining part of this paper, we denote $\omega = e^{\nu T}$ when $\nu \in \sqrt{-1}\mathbb{R}$. To get the stability criteria, we consider the following Hamiltonian system,

$$\dot{z}(t) = JB(t)z(t) \quad z(0) = \omega S z(T).$$

Denote A_ω, B_ω as the operators corresponding to A, B respectively under the ωS-boundary condition, then A_ω is a self-adjoint operator with the domain $D_{\omega S}$. Since ν is an imaginary number, $e^{\nu t}$ is a unitary operator on E and $e^{\nu t} D_S = D_{\omega S}$. Simple calculations show that $e^{-\nu t} A_\omega e^{\nu t} = A - \nu J$. Thus we have

$$I(A_\omega, A_\omega - B_\omega) = I(A-\nu J, A-\nu J - B). \tag{4.4}$$

To judge the stability, we use the Maslov-type index $i_\omega(\gamma)$, which is essentially same as the relative Morse index [24]. Roughly speaking, for a continuous path $\gamma(t) \in \mathrm{Sp}(2n)$, $\omega \in \mathbb{U}$, the Maslov-type index $i_\omega(\gamma)$ is defined by the intersection number of γ and $\mathrm{Sp}^0_\omega(2n) = \{M \in \mathrm{Sp}(2n) \mid \det(M - \omega I_{2n}) = 0\}$. Details could be found in [22,24], some brief review could be found in [15]. For simplicity, we assume $S = I_{2n}$. From [14, Theorem 2.5 and Lemma 4.5] and (4.4), we have the following proposition.

Proposition 4.3. *Suppose $S = I_{2n}$, then, for imaginary number ν such that $\omega = e^{\nu T} \in \mathbb{U} \setminus \{1\}$, we have*

$$I(A, A-B) = i_1(\gamma) + n.$$

and

$$I(A-\nu J, A-\nu J - B) = i_\omega(\gamma).$$

We will continue to review the stability criteria by the Maslov-type index. Details for the stability criteria and the Maslov-type index are given in [24]. For $\omega \in \mathbb{U}$, the unit circle, $\omega = e^{i\theta_0}$ with $\theta_0 \in [-\pi, \pi]$, let $\mathbb{U}_\omega = \{e^{i\theta}, \theta \in [-|\theta_0|, |\theta_0|]\}$, denote by $e_\omega(M)$ the total algebraic multiplicities of all eigenvalues of M in $\mathbb{U}_\omega$. We also simply denote by $e(M)$ the total algebraic multiplicities of all eigenvalues of M on $\mathbb{U}$. *Obviously, for $M = \gamma(T)$ if $e(M) = 2n$ then M is spectral stable.*

For a bounded variation function $g(w)$ defined on some closed interval $[a, b]$, we define its variation by

$$
\begin{aligned}
& var(g(w), [a, b]) \\
& = \sup \left\{ \sum_{j=0}^{k-1} |g(w_{j+1}) - g(w_j)| \,\middle|\, a = w_0 < ..... < w_k = b \;\; P \text{ is any partition} \right\}.
\end{aligned}
$$

Notice that $i_{e^{\theta\sqrt{-1}}}$ is a bounded variation function on $[0, \theta_0]$. The next proposition can be proved easily by the property of Masolv-type index (readers are referred to [24] or [14]).

Proposition 4.4. *Let γ be an arbitrary path in* $\mathrm{Sp}(2n)$ *connecting I_{2n} to M,*

$$e_\omega(M)/2 \geqq var(i_{e^{\theta\sqrt{-1}}}(\gamma), \theta \in [0, \theta_0]). \tag{4.5}$$

Corollary 4.5. *With the notations as above,*

$$e(M)/2 \geqq var(i_{e^{\theta\sqrt{-1}}}(\gamma), \theta \in [0, \pi]).$$

Obviously, for $\omega \neq \pm 1$

$$e(M)/2 \geqq |i_{-1}(\gamma) - i_\omega(\gamma)| + |i_1(\gamma) - i_\omega(\gamma)|. \tag{4.6}$$

Especially,

$$e_\omega(M)/2 \geqq |i_\omega(\gamma) - i_1(\gamma)|, \tag{4.7}$$

$$e(M)/2 \geqq |i_{-1}(\gamma) - i_1(\gamma)|. \tag{4.8}$$

Remark 4.6. All the above results, concerning the relative Morse index equaling the Maslov-type index and the stability criteria, could be proved for any S boundary condition with $S \in \mathrm{Sp}(2n) \cap O(2n)$, and details can be found in [14].

4.2. Estimate Relative Morse Index by Trace Formula

In this subsection, we will give the application of the trace formula on the estimation of the non-degeneracy. Moreover, we will estimate the Maslov-type index by using the trace formula. Suppose $A - \nu J - B$ is non-degenerate; we will estimate the relative Morse index $I(A - \nu J - B, A - \nu J - B - D)$. Firstly assume $D > 0$, thus $I(A - \nu J - B, A - \nu J - B - D) \geqq 0$.

Lemma 4.7. *Suppose $D > 0$, ν is an imaginary number, then all the eigenvalues of $\mathcal{F}$ are real.*

Proof. Let $D^{1/2}$ be the unique positive operator such that $D^{1/2}D^{1/2} = D$, then $\mathcal{F}$ is similar to $D^{1/2}(A - \nu J - B)^{-1}D^{1/2}$, which is a self-adjoint compact operator. Hence

$$\sigma(\mathcal{F}) = \sigma(D^{1/2}(A - \nu J - B)^{-1}D^{1/2}) \subset \mathbb{R}.$$

□

Let $\frac{1}{\lambda_j}$ be the eigenvalues of $\mathcal{F}$. By Lemma 4.7, $\lambda_j \in \mathbb{R}$, we can make the order such that

$$\cdots \leqq \lambda_2^- \leqq \lambda_1^- < 0 < \lambda_1^+ \leqq \lambda_2^+ \leqq \cdots .$$

Moreover, we have

Lemma 4.8. *Suppose $D > 0$, then $\lim_{j\to\infty} \lambda_j^+ = +\infty$ and $\lim_{j\to\infty} \lambda_j^- = -\infty$.*

Proof. We will use the contradiction argument. Suppose there is λ_0^+ such that, for each $j \in \mathbb{N}$, $\lambda_j^+ < \lambda_0^+$. We claim that

$$\sigma(A - \nu J - B - \lambda_0^+ D) \subset (-\infty, 0]. \tag{4.9}$$

In fact, (4.9) is equivalent to $\sigma(D^{-\frac{1}{2}}(A-\nu J-B)D^{-\frac{1}{2}}-\lambda_0^+) \subset (-\infty, 0]$. Moreover, it is easy to see that $\sigma(D^{-\frac{1}{2}}(A - \nu J - B)D^{-\frac{1}{2}}) = \{\lambda_j\}$, and hence $\sigma(D^{-\frac{1}{2}}(A - \nu J - B)D^{-\frac{1}{2}} - \lambda_0^+) \subset (-\infty, 0]$. Now, notice that A is an unbounded operator $\pm\infty$ is the limitation of its eigenvalues, and $\nu J - B - \lambda_0^+$ is a bounded operator. By the spectral theory for unbounded operator with perturbation by bounded operator, we have that

$$\sigma(A) \subset \left\{\lambda \,\middle|\, |\lambda - \lambda_0| \leqq \|\nu J - B - \lambda_0^+\|, \text{ for some } \lambda_0 \in \sigma(A - \nu J - B - \lambda_0^+ D)\right\}.$$

This is a contradiction. The other part of the lemma can be proved similarly. □

Proposition 4.9. *Suppose $D > 0$, we have that, for $\forall k \in \mathbb{N}$*

$$I(A - \nu J - B, A - \nu J - B - D) + \upsilon(A - \nu J - B - D) < Tr\left(\mathcal{F}^{2k}\right).$$

Proof. From Lemma 4.7, λ_j are real numbers, and hence $\lambda_j^{2k} > 0$. By Lemma 4.8 and (1.6), we have

$$Tr\left(\mathcal{F}^{2k}\right) > \sum_{|\lambda_j|\leqq 1} \frac{1}{\lambda_j^{2k}}, \quad \forall k \in \mathbb{N}.$$

Obviously, $\sum_{|\lambda_j|\leqq 1} \frac{1}{\lambda_j^{2k}}$ is no less than the total multiplicity of eigenvalues with $|\lambda_j| \leqq 1$. Please note that $\lambda_j \in \mathcal{F}$ if and only if $\ker(A - \nu J - B - \lambda_j D)$ is degenerate. Moreover, the multiplicity of the eigenvalue for $\mathcal{F}$ at λ_j is equal to $\upsilon(A - \nu J - B - \lambda_j D)$. By Proposition 4.1 and (4.1), the proposition is proved. □

Similar to Proposition 4.9, we have the following proposition.

Proposition 4.10. *Suppose $D > 0$, then*

$$-Tr\left(\mathcal{F}^{2k}\right) < I(A - \nu J - B, A - \nu J - B + D) \leqq 0, \quad \forall k \in \mathbb{N}.$$

We have the following corollary.

Corollary 4.11. *Suppose $D > 0$, if for some $k \in \mathbb{N}$, $Tr\left(\mathcal{F}^{2k}\right) \leqq 1$, then*

$$I(A - \nu J - B, A - \nu J - B + D) = I(A - \nu J - B, A - \nu J - B - D) + \upsilon(A - \nu J - B - D) = 0.$$

Now we can give the estimation on the upper bound that preserves the non-degeneracy.

Theorem 4.12. *Suppose $A - B - \nu J$ is non-degenerate. Suppose that there are $D_1, D_2 \in \mathcal{B}(2n)$ such that $D_1 < D < D_2$, with $D_1 < 0$, $D_2 > 0$, if there exists $k \in 2\mathbb{N}$, such that $Tr\left(\mathcal{F}(\nu, B, D_j)^k\right) \leqq 1$ for $j = 1, 2$, then $A - B - D - \nu J$ is non-degenerate.*

Proof. By the condition $Tr\left(\mathcal{F}(\nu, B, D_j)^{2k}\right) \leqq 1$, for $j = 1, 2$, applying Corollary 4.11, we have that, for any $s \in [0, 1]$,

$$I(A - \nu J - B, A - \nu J - B - sD_1) = I(A - \nu J - B, A - \nu J - B - sD_2) + \upsilon(A - \nu J - B - sD_2) = 0. \quad (4.10)$$

Next, we will prove the result by contradiction argument. Assume that $A - \nu J - B - D$ is degenerate. Now, let

$$s_0 = \inf\{s \in [0, 1], \upsilon(A - \nu J - B - sD) \neq 0\}.$$

Notice that $A - \nu J - B$ is non-degenerate, thus $s_0 > 0$. From the spectral theory of self-adjoint operators [18], the eigenvalues of $A - \nu J - B - sD$ can be considered as a smooth function on s. Denote the eigenvalue functions by $\lambda_j(s)$. Since $A - B - \nu J - s_0 D$ is degenerate, there is some $\lambda_j(s_0) = 0$. We may assume that $\lambda_j(s_0) = 0$ for $j = 1, ..., m$. By the definition of s_0, $\lambda_j(s) \neq 0$ on $[0, s_0)$ for $j = 1, ..., m$. Without loss of generality, assume $\lambda_j(s) > 0$ on $[0, s_0)$ for $j = 1, ..., m_1$ and $\lambda_j(s) < 0$ on $[0, s_0)$ for $j = m_1 + 1, ..., m$, where m_1 can take value 0 or m.

Firstly, if $m_1 > 0$, by the property of relative morse index, we have

$$I(A - \nu J - B, A - \nu J - B - s_0 D_2) = I(A - \nu J - B, A - \nu J - B - s_0 D) + I(A - \nu J - B - s_0 D, A - \nu J - B - s_0 D_2),$$

and $I(A - \nu J - B, A - \nu J - B - s_0 D) = m_1 - m$ by the definition of s_0. On the other hand, since $D_2 > 0$ form (4.1),

$$\begin{aligned} &I(A - \nu J - B, A - \nu J - B - s_0 D_2) \\ &\quad = m_1 - m + I(A - \nu J - B - s_0 D, A - \nu J - B - s_0 D - s_0(D_2 - D)) \\ &\quad \geqq m_1 - m + \upsilon(A - \nu J - B - s_0 D) = m_1 > 0, \end{aligned}$$

which contradicts (4.10).

Next, if $m_1 = 0$, noting that $D_1 < D$, by the property of spectral flow, $I(A - \nu J - B, A - \nu J - B - s_0 D) = -m$. By a similar discussion to that above, we get

$$\begin{aligned} &I(A - \nu J - B, A - \nu J - B - s_0 D_1) \\ &\quad = -m + I(A - \nu J - B - s_0 D, A - \nu J - B - s_0 D - s_0(D_1 - D)) \leqq -m < 0, \end{aligned}$$

which also contradicts to (4.10). The proof is complete. □

Next, we are going to give the estimation of the relative Morse index by the trace formula.

Theorem 4.13. *Suppose* $A-B-\nu J$ *is non-degenerate and* $D_1 \leqq D \leqq D_2$*, where* $D_1 < 0$*,* $D_2 > 0$*. Let*

$$m^- = \inf\{[Tr(\mathcal{F}(\nu, B, D_1)^k)], k \in 2\mathbb{N}\} \quad and \quad m^+ = \inf\{[Tr(\mathcal{F}(\nu, B, D_2)^k)], k \in 2\mathbb{N}\},$$

then

$$-m^- \leqq I(A-B-\nu J, A-B-D-\nu J) \leqq m^+.$$

Proof. Firstly, we will prove that $I(A-B-\nu J, A-B-D_2-\nu J) \leqq m^+$. In fact, by Proposition 4.9, we have that, for any $k \in 2\mathbb{N}$, $I(A-B-\nu J, A-B-D_2-\nu J) < Tr\left(\mathcal{F}^{2k}\right)$. It follows that

$$I(A-B-\nu J, A-B-D_2-\nu J) \leqq m^+.$$

By Proposition 4.10 and some similar reasoning, we have

$$I(A-B-\nu J, A-B-D-\nu J) \geqq -m^-.$$

Since $D_1 \leqq D \leqq D_2$, we get the result by (4.3). □

Motivated by KREIN's work [19], we consider the symmetric case, that is, $D(t) = D(T-t)$. Suppose first that D is real and invertible. A nonzero $\lambda \in \sigma(D(A-\nu J)^{-1}) = \sigma((A-\nu J)^{-1}D)$ if and only if $\bar{\lambda} \in \sigma(D(A+\nu J)^{-1})$, therefore

$$\sigma(D(A+\nu J)^{-1}) = \{\bar{\lambda} \mid \lambda \in \sigma(D(A-\nu J)^{-1})\}. \tag{4.11}$$

Now, suppose $D > 0$. Then, $\sigma(D(A-\nu J)^{-1}) \subset \mathbb{R}$, and hence $\sigma(D(A-\nu J)^{-1}) = \sigma(D(A+\nu J)^{-1})$. If moreover $D(t) = D(T-t)$, then, by some direct computation, $x(t) \in \ker(A-\nu J-\lambda D)$ if and only if $x(T-t) \in \ker(A+\nu J+\lambda D)$. We summarize the above reasoning as the following lemma.

Lemma 4.14. *Suppose* $D > 0$ *and* $D(t) = D(T-t)$*, then* $\lambda \in \sigma(D(A-\nu J)^{-1})$ *if and only if* $-\lambda \in \sigma(D(A-\nu J)^{-1})$*, and with the same multiplicity.*

As an application, we have

Proposition 4.15. *Suppose* $S = I_{2n}$*,* $B = 0$*,* $D > 0$*, and* $\omega \neq 1$*, if one of the following conditions holds*

(1) $\frac{\omega}{(1-\omega)^2} Tr\left[\mathcal{I}(D)^2\right] \leqq 1$

(2) $D(t) = D(T-t)$, $\frac{\omega}{2(1-\omega)^2} Tr\left[\mathcal{I}(D)^2\right] \leqq 1$,

then $i_\omega(\gamma) = 0$*, where* γ *is the fundamental solution with respect to* D*.*

Proof. Since $M = S = I_{2n}$, by (2.14), $Tr\left(\mathcal{F}(\nu, 0, D)^2\right) = \frac{\omega}{(1-\omega)^2} Tr\left[\mathcal{I}(D)^2\right]$. The proofs of both cases are similar, we only list the proof under the second condition. By Lemma 4.14 and the (1.6),

$$Tr\left[\mathcal{F}(\nu, 0, D)^2\right] = 2\sum_{j\in\mathbb{N}} \frac{1}{\lambda_j^2}.$$

Thus we have

$$\frac{\omega}{2(1-\omega)^2} Tr\left[\mathcal{I}(D)^2\right] = \sum_{j\in\mathbb{N}} \frac{1}{\lambda_j^2}.$$

Notice that $\frac{\omega}{2(1-\omega)^2} Tr\left[\mathcal{I}(D)^2\right] \leqq 1$. By the same discussion as in the proof of Proposition 4.9, we have $I(A - \nu J, A - \nu J - D) = 0$. By Proposition 4.3, $i_\omega(\gamma) = I(A - \nu J, A - \nu J - D) = 0$. The proof is complete. □

4.3. Stability Criteria

In this subsection, we only consider the case $S = I_{2n}$, and the general case is similar. Recall that γ is the fundamental solution with respect B and $M = \gamma(T)$, we denote $\tilde{\gamma}$ be the fundamental solution with respect to $B + D$, and write $\widetilde{M} = \tilde{\gamma}(T)$.

Proposition 4.16. *Suppose $D_1 \leqq D \leqq D_2$, where $D_1 < 0$, $D_2 > 0$. If for j=1,2,*

$$Tr\left(\mathcal{F}(0, B, D_j)^2\right) \leqq 1 \quad \textit{and} \quad Tr\left(\mathcal{F}(\nu, B, D_j)^2\right) \leqq 1$$

hold true for any $\nu \in \sqrt{-1}\mathbb{R}$, then $i_\omega(\gamma) = i_\omega(\tilde{\gamma})$, and

$$e_\omega(\widetilde{M})/2 \geqq |i_1(\gamma) - i_\omega(\gamma)|.$$

Especially, if for $j = 1, 2$, $Tr\left(\mathcal{F}(\sqrt{-1}\pi/T, B, D_j)\right) \leqq 1$, then

$$e(\widetilde{M})/2 \geqq |i_1(\gamma) - i_{-1}(\gamma)|.$$

Proof. Since $Tr((D_j(A - B)^{-1})^2) \leqq 1$ for $j = 1, 2$, by Corollary 4.11, $I(A - B, A - B - D_j) = 0$. Hence, for $j = 1, 2$,

$$I(A, A - B - D_j) = I(A, A - B) + I(A - B, A - B - D_j) = I(A, A - B),$$

thus by (4.3), $I(A, A - B - D) = I(A, A - B)$, and from Proposition 4.3, we have $i_1(\gamma) = i_1(\tilde{\gamma})$. Similarly, $Tr((D_j(A - \nu J - B)^{-1})^2) \leqq 1$ for $j = 1, 2$ implies $i_\omega(\gamma) = i_\omega(\tilde{\gamma})$. From (4.7),

$$e_\omega(\widetilde{M})/2 \geqq |i_1(\tilde{\gamma}) - i_\omega(\tilde{\gamma})| = |i_1(\gamma) - i_\omega(\gamma)|.$$

The desired result is proved. □

$Tr((D_j(A-B)^{-1})^2)$ could be estimated by using the trace formula. If moreover $MJ = JM$ and $M^T = M$, we could have a more simple estimation.

Corollary 4.17. *Under the condition of Proposition 4.16, if moreover $MJ = JM$, $M^T = M$, for $j = 1, 2$,*

$$Tr\left[\left(\mathcal{I}(\hat{D}_j)\mathcal{M}(\nu)\right)^2\right] - Tr\left[\mathcal{I}(\hat{D}_j)^2\mathcal{M}(\nu)\right] \leqq 1, \tag{4.12}$$

and

$$Tr\left[\left(\mathcal{I}(\hat{D}_j)\mathcal{M}(0)\right)^2\right] - Tr\left[\mathcal{I}(\hat{D}_j)^2\mathcal{M}(0)\right] \leqq 1, \tag{4.13}$$

where $\hat{D}_j(t) = \gamma_0^T(t)D_j(t)\gamma_0(t)$, then

$$e_\omega(\widetilde{M})/2 \geqq |i_1(\gamma) - i_\omega(\gamma)|.$$

Proof. From Proposition 2.7, in case $MJ = JM$, $M^T = M$, the equality (4.12) implies

$$Tr\left(\mathcal{F}(\nu, B, D_j)^2\right) \leqq 1.$$

By Proposition 4.16, $i_\omega(\gamma) = i_\omega(\tilde{\gamma})$. Similarly, by (4.13), $i_1(\gamma) = i_1(\tilde{\gamma})$. The result is from (4.7). □

Theorem 4.18. *If $M = I_{2n}$, $D > 0$ (or $D < 0$), $\frac{\omega}{(1-\omega)^2}Tr\left[\mathcal{I}(\hat{D})^2\right] \leqq 1$ then*

$$e_\omega(\widetilde{M})/2 = n. \tag{4.14}$$

Proof. Firstly, we will prove the result in the case of $D > 0$. Since $M = I_{2n}$, by [24, Chapter 9], we have $i_\omega(\gamma) = i_1(\gamma) + n$. On the other hand, since $D > 0$, by (4.1)

$$I(A, A - B - D) \geqq I(A, A - B) + \upsilon(A - B) = I(A, A - B) + 2n.$$

Thus $i_1(\tilde{\gamma}) \geqq i_1(\gamma) + 2n$. By the condition $\frac{\omega}{(1-\omega)^2}Tr\left(\mathcal{I}(\hat{D})^2\right) \leqq 1$, we have $i_\omega(\tilde{\gamma}) = i_\omega(\gamma)$. The result follows from (4.7).

In the case $D < 0$, we have $I(A, A - B - D) \leqq I(A, A - B)$, this is equivalent to $i_1(\tilde{\gamma}) \leqq i_1(\gamma)$. On the other hand, we have $i_\omega(\tilde{\gamma}) = i_\omega(\gamma)$. The result follows from (4.7). The proof is complete. □

By taking $\omega = -1$, we have

Corollary 4.19. *If $M = I_{2n}$, $D > 0$ (or $D < 0$), $-\frac{1}{4}Tr\left[\mathcal{I}(\hat{D})^2\right] \leqq 1$, then $e(M)/2 = n$, that is $\tilde{M}$ is elliptic.*

In the special case $B(t) \equiv 0$, then $\gamma(t) \equiv I_{2n}$ is a constant path, it is well known $i_1(\gamma) = -n$, and $i_\omega(\gamma) = 0$ for $\omega \in \mathbb{U} \setminus \{1\}$(see [24]).

Corollary 4.20. *Suppose* $B = 0$ *and* $D > 0$ (*or* $D < 0$) *if one of the following conditions satisfies:*

(i) $\frac{\omega}{(1-\omega)^2} Tr\left[\mathcal{I}(D)^2\right] \leqq 1$,
(ii) $D(t) = D(T-t)$ *and* $\frac{\omega}{2(1-\omega)^2} Tr\left[\mathcal{I}(D)^2\right] \leqq 1$,

then

$$e_\omega(\widetilde{M})/2 = n.$$

Proof. The result under condition (i) comes directly from Theorem 4.18, since $\hat{D} = D$ for $B = 0$. For condition (ii), by Proposition 4.15, $i_\omega(\tilde{\gamma}) = i_\omega(\gamma) = 0$. In this case $\gamma \equiv I_{2n}$ is a constant solution. By some similar argument to the proof of Theorem 4.18, we prove the result. □

We will give some hyperbolic criteria

Proposition 4.21. *Suppose M is hyperbolic,* $Tr\left(\mathcal{F}^2\right) \leqq 1$ *for* $\nu \in \left[0, \frac{\sqrt{-1}\pi}{T}\right]$, *then* $\widetilde{M}$ *is hyperbolic.*

Proof. Please note that $Tr\left(\mathcal{F}^2\right) \leqq 1$, thus $A - \nu J - B - sD$ is non-degenerate for $s \in [0, 1]$. This is equivalent to $A_\omega - B_\omega - sD_\omega$ is non-degenerate. Therefore $\widetilde{M} - \omega I_{2n}$ is nonsingular for $\omega \in \mathbb{U}$, thus $\widetilde{M}$ is hyperbolic. □

When B is constant path, our stability criteria can be easily used. The next example will give a new stability criteria.

Example 4.22. Suppose $B(t) \equiv B$ is constant path of matrices, $JB = BJ$ and $\exp(JBT) = I_{2n}$. This happens when $B = diag(\alpha_1, \alpha_2, ...\alpha_n, \alpha_1, \alpha_2, ...\alpha_n)$, and $\alpha_j T/2\pi \in \mathbb{Z}$ for $j = 1, ..., n$. Consider the linear Hamiltonian systems $\dot{z}(t) = J(B + D(t))z(t)$, with $D(t) = D(t+T) \geqq 0$ and $\int_0^T D(t)\,\mathrm{d}t > 0$. Let $\lambda(t) = \lambda_{max}(D(t))$ which is the largest eigenvalue of $B(t)$, then the linear system is spectrally stable if

$$\int_0^T \lambda(t)\,\mathrm{d}t < 2. \tag{4.15}$$

In fact, noting that $D(t) \leqq \lambda(t) I_{2n}$, let $\tilde{\gamma}(t)$ and $\tilde{\gamma}_1(t)$ be the fundamental solutions corresponding to $B + D(t)$ and $B + \lambda(t) I_{2n}$ respectively, then

$$i_\omega(\tilde{\gamma}) \leqq i_\omega(\tilde{\gamma}_1), \quad \forall \omega \in \mathbb{U}.$$

By some easy computation, the condition (4.15) implies $i_{-1}(\tilde{\gamma}_1) = i_{-1}(\gamma)$. On the other hand, by the proof of Theorem 4.18, we have $i_1(\tilde{\gamma}) \geqq i_1(\gamma) + 2n$ and $i_{-1}(\gamma) = i_1(\gamma) + n$, which yields the result by (4.8). Please note that, in the case $B = 0$, if we instead (4.15) by the condition (i) of Corollary 4.20, we also get $e_\omega(\widetilde{M})/2 = n$, which is a generalization of Krein's stability criteria.

4.4. Estimate the Morse Index for $\bar{S}$-periodic Orbits in Lagrangian System

In this subsection, we will estimate the Morse index of $\bar{S}$-periodic orbits in Lagrangian systems by using the trace formula. For $T > 0$, suppose $x(t)$ is a critical point of the functional

$$F(x)=\int_0^T L(t,x,\dot{x})\,\mathrm{d}t, \forall\, x\in E=\left\{x \;\middle|\; x\in W^{1,2}(\mathbb{R},\mathbb{R}^n), x(t)=\bar{S}x(t+T)\right\},$$

where $L\in C^2(\mathbb{R}\times\mathbb{R}^{2n},\mathbb{R})$ and satisfies circle type symmetry [14]

$$L(t,x,\xi)=L(t+T,\bar{S}^T x,\bar{S}^T\xi). \tag{4.16}$$

It is well known that $x(t)$ is a solution of the corresponding Euler–Lagrangian equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}L_p(t,x,\dot{x})-L_x(t,x,\dot{x})=0, x(0)=\bar{S}x(T), \dot{x}(0)=\bar{S}\dot{x}(T). \tag{4.17}$$

For such an extremal loop, define

$$P(t)=L_{p,p}(t,x(t),\dot{x}(t)), Q(t)=L_{x,p}(t,x(t),\dot{x}(t)), R(t)=L_{x,x}(t,x(t),\dot{x}(t)).$$

For $\omega=\mathrm{e}^{\nu T}\in\mathbb{U}$, recall $\mathcal{A}(\nu)$ defined in (3.13). We set $D_{\omega\bar{S}}(n)=\{y\in W^{1,2}([0,T];\mathbb{C}^n)\mid y(0)=\omega\bar{S}y(T)\}$, and denote by $\phi_\omega(\mathcal{A})$ the ω-Morse index of $\mathcal{A}$, which is defined to be the dimension of the negative definite space of $\langle\mathcal{A}y_1,y_2\rangle$, $y_1,y_2\in D_{\omega\bar{S}}(n)$. Obviously,

$$\phi_\omega(\mathcal{A}(0))=\phi_1(\mathcal{A}(\nu)), \tag{4.18}$$

and it can be considered as the ω-Morse index of x.

The next lemma is obvious.

Lemma 4.23. *Suppose $R_1\geqq 0$, then*

$$\phi_1(\mathcal{A}(\nu)+R_1)\leqq\phi_1(\mathcal{A}(\nu)). \tag{4.19}$$

When we transform the Sturm–Liouville system to linear Hamiltonian system, it is obvious that

$$\upsilon(A-\nu J-B)=\upsilon(\mathcal{A}(\nu)). \tag{4.20}$$

Moreover, the Morse index is essentially same as the relative Morse index (Maslov-type index) (see [24] or [14]). We have the following proposition.

Proposition 4.24.

$$I(A-\nu J-B,A-\nu J-B_1)=\phi_1(\mathcal{A}(\nu)+R_1)-\phi_1(\mathcal{A}(\nu))=i_\omega(\gamma_1)-i_\omega(\gamma_0). \tag{4.21}$$

Proof. Let γ_λ be the fundamental solution corresponding to B_λ, then from [24, P172], we have

$$\phi_1(\mathcal{A}(\nu) + \lambda R_1) = i_\omega(\gamma_\lambda). \tag{4.22}$$

Thus

$$\phi_1(\mathcal{A}(\nu) + R_1) - \phi_1(\mathcal{A}(\nu)) = i_\omega(\gamma_1) - i_\omega(\gamma_0). \tag{4.23}$$

This result is from Proposition 4.3. □

By (4.20) and (4.21), all the results in Section 4.2 can be used to estimate the Morse index and non-degenerate linear Lagrangian systems, however, there are some new estimations for the Lagrangian system.

Theorem 4.25. *Let $\nu \in \mathbb{C}$, assume $\mathcal{A}(\nu) > 0$, if $R_1 \geqq -K$, where $K \in \mathcal{B}(n)$ and $K > 0$. Then*

$$\phi_1(\mathcal{A}(\nu) + R_1) \leqq \inf\{Tr((K\mathcal{A}(\nu)^{-1})^k), k \in \mathbb{N}\}. \tag{4.24}$$

Proof. Please note that in this case, all the eigenvalues $\{1/\lambda_j\}$ of $D\mathcal{A}(\nu)^{-1}$ are positive, and $K\mathcal{A}(\nu)^{-1}$ is a trace class operator. Hence for any positive integers l,

$$Tr\left[\left(K\mathcal{A}(\nu)^{-1}\right)^l\right] > \sum_{|\lambda_j|\leqq 1} \frac{1}{\lambda_j^l}.$$

A similar argument to the proof of Proposition 4.9 implies the result. □

Corollary 4.26. *Under the conditions of Theorem 4.25, if $Tr(D\mathcal{A}(\nu)^{-1}) < 1$, then*

$$\phi_1(\mathcal{A}(\nu) + R_1) = \phi_1(\mathcal{A}(\nu)) = 0$$

and $\mathcal{A}(\nu) + R_1$ is non-degenerate.

Next, we will consider some special case that $\mathcal{A}(\nu) = -\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu\right)^2 - R(t)$. Let $R^+(t) = \frac{1}{2}(R(t) + |R(t)|)$, then $R^+(t) \geqq 0$, and $R(t) \leqq R^+(t)$, we have

Theorem 4.27. *For imaginary number ν, such that $-\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu\right)^2$ is invertible,*

$$\phi_1\left(\mathcal{A}(\nu)\right) \leqq -\omega T \cdot Tr\left[\int_0^T R^+(t)\,\mathrm{d}t \cdot S(S-\omega)^{-2}\right], \tag{4.25}$$

where $\omega = \mathrm{e}^{\nu T}$.

Proof. For any $\varepsilon > 0$, $R^+(t) + \varepsilon I_n > 0$, and $\phi_1(-(\mathcal{A}(\nu)) \leqq \phi_1(-(\frac{\mathrm{d}}{\mathrm{d}t} + \nu)^2 - (R^+(t) + \varepsilon I_n)$. The result follows from (3.23) and Theorem 4.25. □

5. Stability of Lagrangian Orbits

In this section, we will give the application of the trace formula on the stability for elliptic Lagrangian orbits. To do this, in Section 5.1 we will recall some elementary results on the Maslov-type index and the Morse index of Lagrangian orbits. In Section 5.2, we will prove Theorem 1.8. Details on the function $f(\beta, \omega)$ in Theorem 1.8 via the trace formula (1.8) will be listed in Section 5.3. At last, in Section 5.4, by the first order trace formula (1.15) we will give another estimation for the hyperbolic region which is not too sharp but with a more simple estimation.

5.1. A Brief Review on Lagrangian Orbits

Following Meyer and Schmidt [27], the linear variational equation of the elliptic equilibria is decoupled into three subsystems, the first and second subsystems are from the first integral and the third is the essential part. The essential part $\gamma = \gamma_{\beta,e}(t)$ of the fundamental solution of the Lagrangian orbit [27, P.275] satisfies $\dot{\gamma}(t) = JB_{\beta,e}(t)\gamma(t)$, $\gamma(0) = I_4$, with

$$B_{\beta,e}(t) = \begin{pmatrix} I_2 & -J_2 \\ J_2 & I_2 - \hat{K}_{\beta,e}(t) \end{pmatrix}, \tag{5.1}$$

where e is the eccentricity, t is the truly anomaly, and

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{K}_{\beta,e}(t) = \frac{1}{2(1+e\cos t)} \begin{pmatrix} 3+\sqrt{9-\beta} & 0 \\ 0 & 3-\sqrt{9-\beta} \end{pmatrix}.$$

The corresponding Sturm–Liouville system is $-\ddot{y} - 2J_2\dot{y} + \hat{K}_{\beta,e}y = 0$. For $(\beta, e) \in [0, 9) \times [0, 1)$, $\omega \in \mathbb{U}$, set

$$\overline{D}(\omega, 2\pi) = \{y \in W^{2,2}([0, 2\pi]; \mathbb{C}^n) \mid y(0) = \omega y(2\pi),\ \dot{y}(0) = \omega\dot{y}(2\pi)\}$$

and $\mathcal{A}(\beta, e, \nu) = -\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu\right)^2 - 2J_2\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu\right) + \hat{K}_{\beta,e}(t)$. Then for pure imaginary number ν, $\mathcal{A}(\beta, e, \nu)$ are self-adjoint operators on $L^2([0, 2\pi], \mathbb{C}^n)$ with domain $\overline{D}(\omega, 2\pi)$ and dependence on the parameters β and e. We denote this simply by $\mathcal{A}_\omega(\beta, e, \nu)$ and omit ω when $\omega = 1$. Let $\phi(\mathcal{A}_\omega) = \phi_1(\mathcal{A}_\omega)$ be the Morse index of $\mathcal{A}_\omega$. It is obvious that $\mathcal{A}_\omega > 0$ if and only if $\phi(\mathcal{A}_\omega) = \upsilon(\mathcal{A}_\omega) = 0$.

For any $x(t) \in \overline{D}(1, 2\pi)$, direct computations show that

$$\mathrm{e}^{-t\nu}\mathcal{A}(\beta, e, 0)\mathrm{e}^{t\nu}x(t) = \mathcal{A}(\beta, e, \nu)x(t), \tag{5.2}$$

thus for $\omega = \mathrm{e}^{2\pi\nu}$, we have

$$\phi(\mathcal{A}_\omega(\beta, e, 0)) = \phi(\mathcal{A}(\beta, e, \nu)) \quad \text{and} \quad \upsilon(\mathcal{A}_\omega(\beta, e, 0)) = \upsilon(\mathcal{A}(\beta, e, \nu)). \tag{5.3}$$

Obviously, $\phi(\mathcal{A}_\omega(\beta, e, 0)) = I\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}, \mathcal{A}_\omega(\beta, e, 0)\right)$. By the relationship between the Morse index and the Maslov-type index [24, p.172], we have that for any β and e the Morse index $\phi(\mathcal{A}_\omega(\beta, e, 0))$ and nullity $\upsilon(\mathcal{A}_\omega(\beta, e, 0))$ satisfy

$$\phi(\mathcal{A}_\omega(\beta, e, 0)) = i_\omega(\gamma_{\beta,e}), \quad \text{and} \quad \upsilon(\mathcal{A}_\omega(\beta, e, 0)) = \upsilon_\omega(\gamma_{\beta,e}), \qquad \forall\, \omega \in \mathbb{U},$$

where $\upsilon_\omega(\gamma) = \upsilon(\gamma(T) - \omega I_{2n})$. In particular, by (55) and (58) in [15, Lemma 4.1], we obtain

$$i_1(\gamma_{\beta,e}) = \phi(\mathcal{A}(\beta, e, 0)) = i_1(\gamma_{\beta,e}) = 0, \qquad \forall\, (\beta, e) \in [0, 9] \times [0, 1). \tag{5.4}$$

In the case $e = 0$, $B_{\beta,0}(t)$ is a constant matrix and $i_\omega(\gamma_{\beta,0})$, $\upsilon_\omega(\gamma_{\beta,0})$ could be computed directly. We list the results for $\omega = -1$ and $\omega = \mathrm{e}^{i\sqrt{2}\pi}$ below.

Theorem 5.1. ([13]) *For any* $\omega = \mathrm{e}^{2\pi\nu} \in \mathbb{U}$, $\beta \in (1, 9]$, $\mathcal{A}(\beta, 0, \nu) > 0$ *or equivalently*

$$i_\omega(\gamma_{\beta,0}) = \phi(\mathcal{A}(\beta, 0, \nu)) = \upsilon(\mathcal{A}(\beta, 0, \nu)) = 0. \tag{5.5}$$

For $\omega = \mathrm{e}^{i\sqrt{2}\pi/2}$, $\upsilon(\mathcal{A}(1, 0, i\sqrt{2}\pi/2)) = 1$, *and*

$$i_{\mathrm{e}^{i\sqrt{2}\pi/2}}(\gamma_{\beta,0}) = \phi(\mathcal{A}(\beta, 0, i\sqrt{2}\pi/2)) \geqq 1, \quad for \;\; \beta \in [0, 1). \tag{5.6}$$

For $\omega = -1$, $\upsilon(\mathcal{A}(3/4, 0, i/2)) = 2$ *and* $\upsilon(\mathcal{A}(\beta, 0, i/2)) = 0$ *if* $\beta \neq 3/4$,

$$i_{-1}(\gamma_{\beta,0}) = \phi(\mathcal{A}(\beta, 0, i/2)) = \begin{cases} 2 & if \;\; \beta \in [0, 3/4), \\ 0, & if \;\; \beta \in [3/4, 9]. \end{cases} \tag{5.7}$$

5.2. Stability Analysis via Trace Formula

Set

$$D_{\beta,e}(t) = B_{\beta,e}(t) - B_{\beta,0}(t) = \frac{e\cos(t)}{1 + e\cos(t)} K_\beta,$$

where $K_\beta = diag\left(0, 0, \frac{3+\sqrt{9-\beta}}{2}, \frac{3-\sqrt{9-\beta}}{2}\right)$, then $A - B_{\beta,e} = A - B_{\beta,0} - D_{\beta,e}$. Let $\cos^\pm(t) = (\cos(t) \pm |\cos(t)|)/2$, and denote

$$K_\beta^\pm = \cos^\pm(t) K_\beta,$$

which can be considered as two bounded self-adjoint operators on $L^2([0, 2\pi], \mathbb{C}^4)$; moreover $K_\beta^+ \geqq 0$ and $K_\beta^- \leqq 0$. It is obvious that

$$A - \nu J - B_{\beta,0} - \frac{e}{1-e} K_\beta^- \geqq A - \nu J - B_{\beta,e} \geqq A - \nu J - B_{\beta,0} - eK_\beta^+, \tag{5.8}$$

equivalently,

$$\mathcal{A}(\beta, 0, \nu) - \frac{e}{1-e} \cos^-(t) \hat{K}_{\beta,0} \geqq \mathcal{A}(\beta, e, \nu) \geqq \mathcal{A}(\beta, 0, \nu) - e\cos^+(t) \hat{K}_{\beta,0}. \tag{5.9}$$

Lemma 5.2. *For an imaginary number ν, such that $A - \nu J - B_{\beta,0}$ is invertible, we have*

$$Tr\left[\mathcal{F}(\nu, B_{\beta,0}, K_\beta^+)^2\right] = Tr\left[\mathcal{F}(\nu, B_{\beta,0}, K_\beta^-)^2\right]$$

Proof. Define an operator $G : x(t) \to x(t+\pi)$ on the domain $\overline{D}(1, 2\pi)$, then $G^2 = id$. Direct calculation shows that

$$\left(A - \nu J - B_{\beta,0}\right)^{-1} G = G\left(A - \nu J - B_{\beta,0}\right)^{-1}.$$

Moreover, $K_{\beta,0}G = GK_{\beta,0}$ because $K_{\beta,0}$ is a constant matrix. Therefore,

$$\begin{aligned}
&Tr\left[\left(G\cos^+(t)K_\beta\left(A - \nu J - B_{\beta,0}\right)^{-1} G\right)^2\right]\\
&\quad = Tr\left[\left(G\cos^+(t)GK_\beta\left(A - \nu J - B_{\beta,0}\right)^{-1}\right)^2\right]\\
&\quad = Tr\left[\left(\cos^-(t)K_\beta\left(A - \nu J - B_{\beta,0}\right)^{-1}\right)^2\right].
\end{aligned}$$

□

Under the assumption of Lemma 5.2, we denote

$$f(\beta, \omega) = Tr\left[\mathcal{F}(\nu, B_{\beta,0}, K_\beta^-)^2\right] = Tr\left(\mathcal{F}(\nu, B_{\beta,0}, K_\beta^+)^2\right), \tag{5.10}$$

which is a positive function. The following theorem holds true.

Theorem 5.3. *For $\beta \in [0, 3/4)$, $\gamma_{\beta,e}$ is spectrally stable if*

$$0 \leqq e < \frac{1}{1 + \sqrt{f(\beta, -1)}}. \tag{5.11}$$

Proof. Obviously,

$$Tr\left(\mathcal{F}(\frac{\sqrt{-1}}{2}, B_{\beta,0}, \frac{e}{1-e}K_\beta^-)^2\right) = \frac{\mathrm{e}^2}{(1-e)^2} f(\beta, -1).$$

Thus, (5.11) is equivalent to $\frac{\mathrm{e}^2}{(1-e)^2} f(\beta, -1) < 1$ which implies $Tr\Big(\mathcal{F}(\frac{\sqrt{-1}}{2}, B_{\beta,0}, \frac{e}{1-e}K_\beta^-)^2\Big) < 1$. By the continuity of the trace, for $\epsilon > 0$ small enough, $Tr\Big(\mathcal{F}(\frac{\sqrt{-1}}{2}, B_{\beta,0}, \frac{e}{1-e}K_\beta^- - \epsilon I_{2n})^2\Big) < 1$. Obviously, $\frac{e}{1-e}K_\beta^- - \epsilon I_{2n} < 0$. By Theorems 4.12 and 4.13, $A - \frac{\sqrt{-1}}{2}J - B_{\beta,0} - \frac{e}{1-e}K_\beta^-$ is non-degenerate and

$$I\left(A - \frac{\sqrt{-1}}{2}J - B_{\beta,0}, A - \frac{\sqrt{-1}}{2}J - B_{\beta,0} - \frac{e}{1-e}K_\beta^-\right) = 0.$$

From (5.8), $I\left(A-\frac{\sqrt{-1}}{2}J-B_{\beta,0}-\frac{e}{1-e}K_\beta^-, A-\frac{\sqrt{-1}}{2}J-B_{\beta,e}\right) \geqq 0$, consequently,

$$I\left(A-\frac{\sqrt{-1}}{2}J-B_{\beta,0}, A-\frac{\sqrt{-1}}{2}J-B_{\beta,e}\right) \geqq 0.$$

By (5.7), $i_{-1}(\gamma_{\beta,e}) \geqq i_{-1}(\gamma_{\beta,0}) = 2$. By (5.4) and (4.8), $e(\gamma_{\beta,e})/2 = 2$. The desired result is proved. □

Theorem 5.4. *For $\beta \in (3/4, 1)$, $\gamma_{\beta,e}$ is spectrally stable if*

$$0 \leqq e < f(\beta,-1)^{-\frac{1}{2}}, \tag{5.12}$$

and

$$0 \leqq e < \frac{1}{1+f(\beta, \mathrm{e}^{i\sqrt{2}\pi})^{\frac{1}{2}}}. \tag{5.13}$$

Proof. Firstly, we'll show that (5.12) implies

$$i_{-1}(\gamma_{\beta,e}) = 0, \tag{5.14}$$

and the proof is similar to the proof of Theorem5.3. In fact, please note

$$Tr\left(\mathcal{F}(\frac{\sqrt{-1}}{2}, B_{\beta,0}, eK_\beta^+)^2\right) = \mathrm{e}^2 f(\beta,-1).$$

Thus, (5.12) implies $Tr\left(\mathcal{F}(\frac{\sqrt{-1}}{2}, B_{\beta,0}, eK_\beta^+)^2\right) < 1$, then for $\epsilon > 0$ small enough,

$$Tr\left(\mathcal{F}(\frac{\sqrt{-1}}{2}, B_{\beta,0}, eK_\beta^+ + \epsilon I_{2n})^2\right) < 1.$$

Obviously, $eK_\beta^+ + \epsilon I_{2n} > 0$. Again, by Theorems 4.12 and 4.13, $A-\frac{\sqrt{-1}}{2}J-B_{\beta,0}-eK_\beta^+$ is non-degenerate and $I(A-\frac{\sqrt{-1}}{2}J-B_{\beta,0}, A-\frac{\sqrt{-1}}{2}J-B_{\beta,0}-eK_\beta^+) = 0$. By (5.8),

$$I\left(A-\frac{\sqrt{-1}}{2}J-B_{\beta,0}-eK_\beta^+, A-\frac{\sqrt{-1}}{2}J-B_{\beta,e}\right) \leqq 0.$$

Therefore

$$I\left(A-\frac{\sqrt{-1}}{2}J-B_{\beta,0}, A-\frac{\sqrt{-1}}{2}J-B_{\beta,e}\right) \leqq 0.$$

By (5.7), we have (5.14).

On the other hand, almost the same proof as that of Theorem 5.3 shows that (5.13), (5.6) implies

$$i_{\mathrm{e}^{i\sqrt{2}\pi}}(\gamma_{\beta,e}) \geqq i_{\mathrm{e}^{i\sqrt{2}\pi}}(\gamma_{\beta,0}) \geqq 1. \tag{5.15}$$

The result comes from (5.14), (5.15), (5.4) and (4.6). □

Remark 5.5. It has been proved in [13,15] that $\gamma_{\beta,e}(2\pi)$ is linear stable when (β, e) is in the stable region and not on the bifurcation curves. This implies that under the condition in Theorem 5.3 and Theorem 5.4, $\gamma_{\beta,e}$ is linear stable. Moreover, the normal form of $\gamma_{\beta,e}(2\pi)$ was given in [13,15]. Precisely, for (β, e) in the stable region given in Theorem 5.3, $\gamma_{\beta,e}(2\pi) \approx R(\theta_1) \diamond R(\theta_2)$ for some $\theta_1, \theta_2 \in (\pi, 2\pi)$; for (β, e) in the stable region given in Theorem 5.4, $\gamma_{\beta,e}(2\pi) \approx R(\theta_1) \diamond R(\theta_2)$ for some $\theta_1 \in ((2-\sqrt{2})\pi, \pi)$, $\theta_2 \in (\sqrt{2}\pi, 2\pi)$.

To estimate the hyperbolic region, denote

$$\hat{f}(\beta) = \sup\{f(\beta, \omega), \omega \in \mathbb{U}\}, \tag{5.16}$$

and we have:

Theorem 5.6. *For $\beta \in (1, 9]$, $\gamma_{\beta,e}$ is hyperbolic if*

$$e < \hat{f}(\beta)^{-1/2}. \tag{5.17}$$

Proof. Similar to the proof of Theorem 5.4, the condition (5.17) implies that for any $\omega \in \mathbb{U}$, $i_\omega(\gamma_{\beta,e}) \leqq i_\omega(\gamma_{\beta,0}) = 0$, and $\nu(A_\omega(\beta, e, 0)) = \nu(A_\omega(\beta, 0, 0)) = 0$, which implies that $\gamma_{\beta,e}$ is hyperbolic. □

Combining Theorems 5.3 and 5.4 with Theorem 5.6 and Remark 5.5, we have Theorem 1.8. The function $f(\beta, \omega)$ will be dealt with in the next subsection, and based on this, with the help of Mathlab, we can draw a picture of the stable region and hyperbolic region in Fig. 1.

5.3. The Precise Form of $f(\beta, \omega)$

In this subsection, we compute $f(\beta, \omega)$ by trace formula (1.8). In order to make the calculation easier, we need to use some transformation first. For $\beta \in (0, 9] \setminus \{1\}$, let $P_\beta = \begin{pmatrix} 0 & q_1 & p_2 & 0 \\ p_1 & 0 & 0 & q_2 \\ p_3 & 0 & 0 & q_4 \\ 0 & q_3 & p_4 & 0 \end{pmatrix}$ be the 4×4 transformation matrices, where

$$\begin{cases} p_1 = \frac{-(2-2\sqrt{1-\beta})^{3/4}(2+\sqrt{9-\beta}-\sqrt{1-\beta})}{2(1-\beta)^{1/4}(3+\sqrt{9-\beta})\sqrt{4-\sqrt{1-\beta}-\sqrt{9-\beta}}}, & q_1 = \frac{(2+2\sqrt{1-\beta})^{1/4}(\sqrt{9-\beta}-\sqrt{1-\beta})}{2(1-\beta)^{1/4}\sqrt{4+\sqrt{1-\beta}-\sqrt{9-\beta}}}, \\ p_2 = \frac{-(2-2\sqrt{1-\beta})^{1/4}(\sqrt{9-\beta}+\sqrt{1-\beta})}{2(1-\beta)^{1/4}\sqrt{4-\sqrt{1-\beta}-\sqrt{9-\beta}}}, & q_2 = \frac{(2+2\sqrt{1-\beta})^{3/4}(2+\sqrt{9-\beta}+\sqrt{1-\beta})}{2(1-\beta)^{1/4}(3+\sqrt{9-\beta})\sqrt{4+\sqrt{1-\beta}-\sqrt{9-\beta}}}, \\ p_3 = \frac{(2-2\sqrt{1-\beta})^{3/4}(4+\sqrt{9-\beta}+\sqrt{1-\beta})}{2(1-\beta)^{1/4}(3+\sqrt{9-\beta})\sqrt{4-\sqrt{1-\beta}-\sqrt{9-\beta}}}, & q_3 = \frac{-2(2+2\sqrt{1-\beta})^{1/4}}{(1-\beta)^{1/4}\sqrt{4+\sqrt{1-\beta}-\sqrt{9-\beta}}}, \\ p_4 = \frac{2(2-2\sqrt{1-\beta})^{1/4}}{(1-\beta)^{1/4}\sqrt{4-\sqrt{1-\beta}-\sqrt{9-\beta}}}, & q_4 = \frac{-(2+2\sqrt{1-\beta})^{3/4}(\sqrt{9-\beta}+4-\sqrt{1-\beta})}{2(1-\beta)^{1/4}(3+\sqrt{9-\beta})\sqrt{4+\sqrt{1-\beta}-\sqrt{9-\beta}}}. \end{cases} \tag{5.18}$$

In fact, P_β is obtained with the help of matlab. Direct computation shows that $P_\beta^T J P_\beta = J$. For $\beta \in (0, 1)$, P_β is real, thus it is a symplectic matrix, and for $\beta \in (1, 9]$, P_β is complex matrix. To continue, we need the notation of symplectic sum, which was introduced by Long [22,24]. Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, the symplectic sum of M_1 and M_2 is defined by

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Now, write $\theta_1(\beta) = -\sqrt{\frac{1}{2}(1-\sqrt{1-\beta})}$, and $\theta_2(\beta) = \sqrt{\frac{1}{2}(1+\sqrt{1-\beta})}$. For $j = 1, 2$, let $B_j(\beta) = J\theta_j(\beta)$ and denote by $S_\beta = B_1(\beta)\diamond B_2(\beta)$. Let $B_\beta = B_{\beta,0}$, which is defined in (5.1). Direct computation shows that

$$P_\beta^{-1} J B_\beta P_\beta = J P_\beta^T B_\beta P_\beta = S_\beta, \quad \beta \in (0,1) \cup (1,9]. \tag{5.19}$$

Obviously

$$\exp(B_k(\beta)t) = R(\theta_k t) := \begin{pmatrix} \cos(\theta_k t) & -\sin(\theta_k t) \\ \sin(\theta_k t) & \cos(\theta_k t) \end{pmatrix}, \quad k = 1, 2,$$

and hence $P_\beta^{-1}\gamma_{\beta,0}(t)P_\beta = R(\theta_1 t)\diamond R(\theta_2 t)$. In order to diagonalize $P_\beta^{-1}\gamma_{\beta,0}(t)P_\beta$, we use the unitary matrix $U = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & \sqrt{-1}I_2 \\ I_2 & -\sqrt{-1}I_2 \end{pmatrix}$, and we have

$$U P_\beta^{-1}\gamma_{\beta,0}(t)P_\beta U^{-1} = \mathrm{e}^{i\Theta t}. \tag{5.20}$$

where $\Theta = diag\,(\theta_1, \theta_2, \theta_3, \theta_4)$ with $\theta_3 = -\theta_1, \theta_4 = -\theta_2$. Change the basis by $P_\beta U^{-1}$, by (1.8) and some careful calculation, we have

$$f(\beta,\omega) = Tr\left(\mathcal{F}(\nu, B_{\beta,0}, K_\beta^-)^2\right) = 2f_1(\beta,\omega) - f_2(\beta,\omega) \tag{5.21}$$

for

$$f_1(\beta,\omega) = \int_0^{2\pi}\int_0^t Tr\Big[\cos^-(t)\mathrm{e}^{i\Theta(s-t)}JF_\beta \cdot \cos^-(s)\mathrm{e}^{i\Theta(t-s)}JF_\beta \cdot M_\beta(\omega)\Big]\,\mathrm{d}s\,\mathrm{d}t,$$

and

$$f_2(\beta,\omega) = \int_0^{2\pi}\int_0^{2\pi} Tr\Big[\cos^-(t)\mathrm{e}^{i\Theta(s-t)}JF_\beta M_\beta(\omega) \cdot \cos^-(s)\mathrm{e}^{i\Theta(t-s)}JF_\beta M_\beta(\omega)\Big]\,\mathrm{d}s\,\mathrm{d}t,$$

where $F_\beta = U^{-T}P_\beta^T K_\beta P_\beta U^{-1}$, and for $\omega = \mathrm{e}^{2\pi iu}$, $M_\beta(\omega) = diag\Big(\frac{\mathrm{e}^{2\pi i\theta_1}}{\mathrm{e}^{2\pi i\theta_1}-\mathrm{e}^{2\pi iu}}, \frac{\mathrm{e}^{2\pi i\theta_2}}{\mathrm{e}^{2\pi i\theta_2}-\mathrm{e}^{2\pi iu}}, \frac{\mathrm{e}^{2\pi i\theta_3}}{\mathrm{e}^{2\pi i\theta_3}-\mathrm{e}^{2\pi iu}}, \frac{\mathrm{e}^{2\pi i\theta_4}}{\mathrm{e}^{2\pi i\theta_4}-\mathrm{e}^{2\pi iu}}\Big)$. To calculate $f_1(\beta,\omega)$ and $f_2(\beta,\omega)$, it suffices to calculate JF_β. Write $JF_\beta = \frac{1}{2}\left(D_{i,j}\right)_{4\times 4}$, and denote $d_1 = \frac{3+\sqrt{9-\beta}}{2}$, $d_2 = \frac{3-\sqrt{9-\beta}}{2}$, by some calculations, we have

$$\begin{cases} D_{11} = -D_{33} = -(p_3^2 d_1 + p_4^2 d_2), \quad D_{22} = -D_{44} = -(q_3^2 d_2 + q_4^2 d_1), \\ D_{12} = -D_{21} = D_{34} = -D_{43} = -i(q_3 p_4 d_2 - p_3 q_4 d_1), \\ D_{23} = D_{32} = D_{14} = D_{41} = -i(q_3 p_4 d_2 + p_3 q_4 d_1), \\ D_{24} = -D_{42} = q_4^2 d_1 - q_3^2 d_2, \quad D_{13} = -D_{31} = p_4^2 d_2 - p_3^2 d_1, \end{cases} \tag{5.22}$$

where q_i, p_i are given in (5.18). Direct computation shows that

$$f_1(\beta,\omega) = \frac{1}{4}\sum_{\substack{n=1\\m=1}}^{4} D_{nm}D_{mn}\frac{e^{2\pi i\theta_n}}{e^{2\pi i\theta_n}-e^{2\pi iu}}\frac{2e^{\pi(\theta_m-\theta_n)i}+\pi i(\theta_m-\theta_n)[(\theta_m-\theta_n)^2-1]}{2[(\theta_m-\theta_n)^2-1]^2}, \tag{5.23}$$

and

$$f_2(\beta,\omega) = \frac{1}{4}\sum_{\substack{n=1\\m=1}}^{4} D_{nm}D_{mn}\frac{e^{2\pi i\theta_n}}{e^{2\pi i\theta_n}-e^{2\pi iu}}\frac{e^{2\pi i\theta_m}}{e^{2\pi i\theta_m}-e^{2\pi iu}}\frac{2+e^{\pi(\theta_m-\theta_n)i}+e^{-\pi(\theta_m-\theta_n)i}}{[(\theta_m-\theta_n)^2-1]^2}, \tag{5.24}$$

where the blocks D_{nm} are defined by (5.22). Thus $f_1(\beta,\omega)$, $f_2(\beta,\omega)$ and $f(\beta,\omega)$ are elementary functions. Based on the precise form of the above functions, we can draw the curves Γ_i, $i = 1, ..., 4$ in Fig. 1 with the help of Matlab.

5.4. Hyperbolicity Analysis via the First Order Trace Formula

Recall that in (5.16), $\hat{f}(\beta)$ is defined by taking maximum, and maybe it is not an elementary function. Another way to estimate the hyperbolic region is to use the trace formula for Lagrangian system (1.15). It will be seen that the estimation of the hyperbolic region given by the trace formula (1.8) for Hamiltonian system is sharper than that given by the trace formula (1.15) for Lagrangian system. However, the latter is more computable.

From (5.5), for $\beta \in (1, 9]$, ν is imaginary number, $\mathcal{A}(\beta, 0, \nu) > 0$. For $\omega = e^{2\pi\nu} \in \mathbb{U}$, we define

$$g(\beta,\nu) = -Tr\left(JK_\beta \cdot \gamma_{\beta,0}(2\pi)(\gamma_{\beta,0}(2\pi) - e^{2\pi\nu}I_4)^{-1}\right). \tag{5.25}$$

From (1.15) or (3.21), direct computation shows that

$$Tr\left(\frac{e\cos^+(t)}{1+e\cos(t)}\hat{K}_{\beta,0}\mathcal{A}(\beta,0,\nu)^{-1}\right) = \psi(e)g(\beta,\nu), \tag{5.26}$$

where $\psi(e) = \left(\pi - \frac{4}{\sqrt{1-e^2}}\tan^{-1}\sqrt{\frac{1-e}{1+e}}\right) \geqq 0$ for $e \in [0, 1)$. We set

$$\hat{g}(\beta) = \sup\{g(\beta,\nu), \nu \in \sqrt{-1}\mathbb{R}\}. \tag{5.27}$$

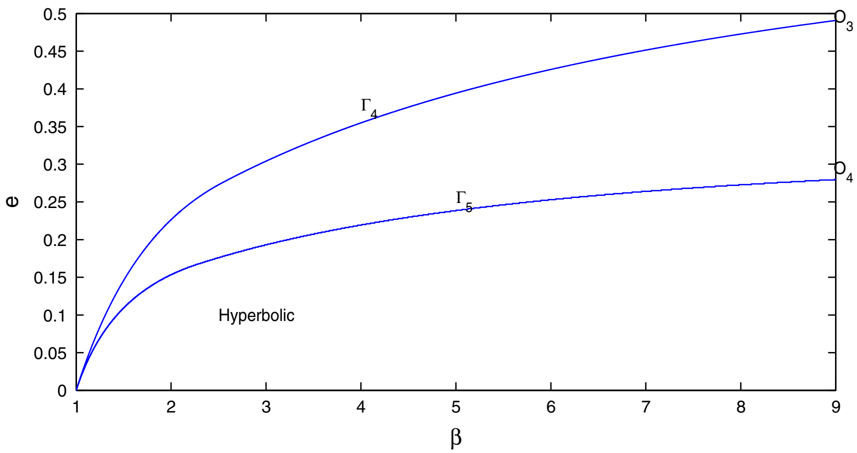

Fig. 2. The hyperbolic region given by Theorems 5.7 and 5.6

In order to calculate $g(\beta, \nu)$, we change the basis by $P_\beta U^{-1}$, then $g(\beta, \nu) = Tr(iJF_\beta M_\beta(\omega))$. From (5.22), direct computation shows that

$$g(\beta, \nu) = 2Re\left(\frac{\sqrt{2}(-3-\beta+3\sqrt{1-\beta})}{4\sqrt{1-\beta}\sqrt{-1+\sqrt{1-\beta}}}\frac{e^{-\sqrt{2}\pi\sqrt{-1+\sqrt{1-\beta}}} - e^{\sqrt{2}\pi\sqrt{-1+\sqrt{1-\beta}}}}{2\cos(2\pi u) - e^{-\sqrt{2}\pi\sqrt{-1+\sqrt{1-\beta}}} - e^{\sqrt{2}\pi\sqrt{-1+\sqrt{1-\beta}}}}\right). \tag{5.28}$$

Similar to the proof of Theorem 5.6, we have the following theorem:

Theorem 5.7. *For $\beta \in (1, 9]$, $\gamma_{\beta,e}$ is hyperbolic if $\psi(e) < 1/\hat{g}(\beta)$.*

We can use Theorem 5.7 or Theorem 5.6 to estimate the hyperbolic region. Next, we draw the following figure to compare the hyperbolic regions given by the two theorems respectively.
In Fig. 2, the points $O_3 \approx (9, 0.4907)$, $O_4 \approx (9, 0.2800)$. The curves

$$\Gamma_4 = \left\{(\beta, e) \;\middle|\; e = \hat{f}(\beta)^{-1/2}, 1 \leqq \beta \leqq 9\right\},$$
$$\Gamma_5 = \left\{(\beta, e) \;\middle|\; \psi(e) = \hat{g}(\beta)^{-1}, 1 \leqq \beta \leqq 9\right\},$$

where Γ_4 is given by theorem 5.6, which is obtained by (1.9), and Γ_5 is given by theorem 5.7, which is obtained by using (1.15).

Since $\hat{g}(\beta)$ is not easy to be computed, we will control $\hat{g}(\beta)$ by some elementary function. Let $\kappa(\beta) = \frac{\sqrt{e^{-2\sqrt{2}\pi\hat{c}} + e^{2\sqrt{2}\pi\hat{c}} - 2\cos(2\sqrt{2}\pi\hat{d})}}{|(e^{-\sqrt{2}\pi\hat{c}} - e^{\sqrt{2}\pi\hat{c}})\sin(\sqrt{2}\pi\hat{d})|}$, where $\hat{c} = Re(\sqrt{-1+\sqrt{1-\beta}})$, $\hat{d} = Im(\sqrt{-1+\sqrt{1-\beta}})$. Obviously, $\left|\sqrt{-1+\sqrt{1-\beta}}\right| = \beta^{\frac{1}{4}}$ for $\beta \in (1, 9]$, applying (5.28), we have

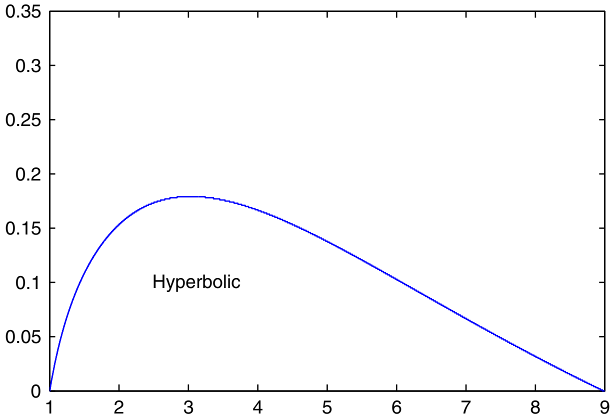

Fig. 3. The hyperbolic region given by Corollary 5.8

$$g(\beta,\nu) \leqq \left|\left(\frac{\sqrt{2}(-3-\beta+3\sqrt{1-\beta})}{2\sqrt{1-\beta}\sqrt{-1+\sqrt{1-\beta}}}\frac{e^{-\sqrt{2}\pi\sqrt{-1+\sqrt{1-\beta}}}-e^{\sqrt{2}\pi\sqrt{-1+\sqrt{1-\beta}}}}{2\cos(2\pi u)-e^{-\sqrt{2}\pi\sqrt{-1+\sqrt{1-\beta}}}-e^{\sqrt{2}\pi\sqrt{-1+\sqrt{1-\beta}}}}\right)\right|$$
$$\leqq \frac{\beta^{\frac{1}{4}}\sqrt{\beta+15}}{\sqrt{2(\beta-1)}}\kappa(\beta).$$

Hence

$$\hat{g}(\beta) \leqq \frac{\beta^{\frac{1}{4}}\sqrt{\beta+15}}{\sqrt{2(\beta-1)}}\kappa(\beta). \tag{5.29}$$

Corollary 5.8. *For $\beta \in (1,9]$, $\gamma_{\beta,e}$ is hyperbolic if*

$$\psi(e) < \frac{\sqrt{2(\beta-1)}}{\beta^{\frac{1}{4}}\sqrt{\beta+15}}\frac{1}{\kappa(\beta)}. \tag{5.30}$$

Denote $h(\beta)$ to be the right item of (5.30), and let β_0 be the point such that $h(\beta_0) = \max\{h(\beta) : \beta \in (1,9]\}$. With the help of Mathlab, we know that $\beta_0 \approx 3.0334$, correspondingly, $e \approx 0.1797$. Hence $h(\beta_0) \geqq h(3.0334) = 0.3154$. It was proved in [13] that if $\gamma_{\beta_0,e}$ is hyperbolic, then $\gamma_{\beta,e}$ is hyperbolic for any $\beta \geqq \beta_0$, then we have

Corollary 5.9. *For $\beta \in [\beta_0, 9]$, $\gamma_{\beta,e}$ is hyperbolic if $\psi(e) < 0.3154$. That is, $\gamma_{\beta,e}$ is hyperbolic if $(\beta, e) \in [3.0334, 9] \times [0, 0.1797]$.*

By using Corollary 5.8, 5.9, we can draw a picture of the hyperbolic region as follows.

Remark 5.10. From the proof of Theorems 5.3 and 5.4, $\gamma_{\beta,e}$ is -1-nondegenerate if (β,e) belongs to the set $\{(\beta,e)|0 \leqq e < 1/(1+\sqrt{f(\beta,-1)}), 0 \leqq \beta < 3/4\}$ or $\{(\beta,e)|0 \leqq e < 1/\sqrt{f(\beta,-1)}, 3/4 < \beta \leqq 9\}$. However, using (1.15), we get that $\gamma_{\beta,e}$ is -1-nondegenerate if (β,e) belongs to the set $\{(\beta,e)|\ \psi(e) < 1/g(\beta,\frac{\sqrt{-1}}{2}), 3/4 < \beta \leqq 9\}$.

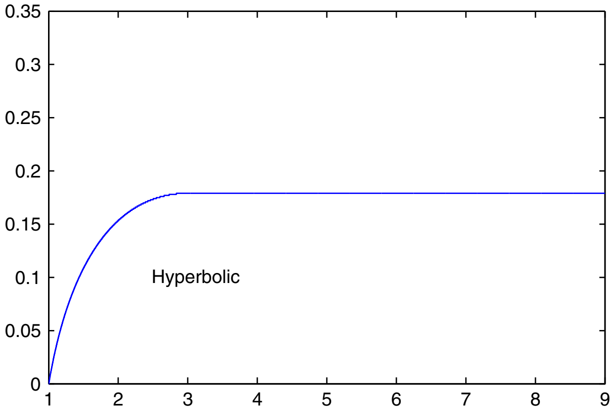

Fig. 4. The hyperbolic region given by Corollary 5.9

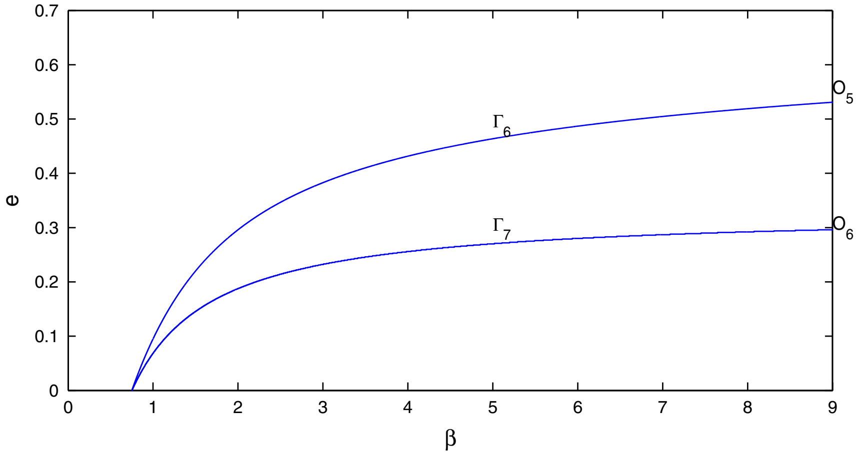

Fig. 5. The -1-nondegenerate region given by Remark 5.10

In Fig. 5, the points $O_5 \approx (9, 0.5309)$, $O_6 \approx (9, 0.2961)$. The curves

$$\Gamma_6 = \left\{ (\beta, e) \,\middle|\, e = f(\beta, -1)^{-1/2}, 1 \leqq \beta \leqq 9 \right\},$$

and

$$\Gamma_7 = \left\{ (\beta, e) \,\middle|\, \psi(e) = g\left(\beta, \frac{\sqrt{-1}}{2}\right)^{-1}, 1 \leqq \beta \leqq 9 \right\}.$$

The same reasoning as above implies that we can estimate the non-degenerate region by the trace formulas in Theorem 1.1 for k. As k is larger, the estimation of the non-degenerate region is sharper, however, the trace formula is more complex and less computable.

Acknowledgments. The authors sincerely thank Y. Long and C. Zeng for their encouragement and interest. The first author sincerely thanks Y.Long and S.Sun's helpful discussion on the Lagrangian orbits. The authors thank the referee for helpful suggestions which have made the paper more readable.

References

1. Atiyah, M. F., Patodi V. K., Singer, I. M.: Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Cambridge Philos. Soc.* **79**(1), 71–99 (1976)
2. Barutello, V., Jadanza, R. D., Portaluri, A.: Morse index and linear stability of the Lagrangian circular orbit in a three-body-type problem via index theory (2014). arXiv:1406.3519v1
3. Bolotin, S. V.: On the Hill determinant of a periodic orbit. (Russian) Vestnik Moskov. *Univ. Ser. I Mat. Mekh.* (3), 30–34, 114 (1988)
4. Bolotin, S. V., Treschev, D. V.: Hill's formula. *Russian Math. Surv.*, **65**(2), 191–257 (2010)
5. Chen, K.: Existence and minimizing properties of retrograd orbits in three-body problem with various choice of mass. *Ann. Math.* **167**(2), 325–348 (2008)
6. Chenciner, A., Montgomery, R.: A remarkable periodic solution of the three body problem in the case of equal masses. *Ann. Math.* **152**(3), 881–901 (2000)
7. Danby, J. M. A.: The stability of the triangular Lagrangian point in the general problem of three bodies. *Astron. J.* **69**, 294–296 (1964)
8. Davletshin, M.: Hill formula for g-periodic trajectories of Lagrangian systems. *Trudy MMO* **74**(1) (2013). *Trans. Moscow Math. Soc. Tom* **74**, 65–96 (2013)
9. Denk, R.: On Hilbert–Schmidt operators and determinants corresponding to periodic ODE systems. Differential and integral operators (Regensburg, 1995), pp. 57–71. Oper. Theory Adv. Appl. **102**. Birkhäuser, Basel (1998)
10. Ferrario, D. L., Terracini, S.: On the existence of collisionless equivariant minimizers for the classical n-body problem. Invent. Math. **155**(2), 305–362 (2004)
11. Gascheau, M.: Examen d'une classe d'équations différentielles et application à un cas particulier du problème des trois corps. Comptes Rend. **16**, 393–394 (1843)
12. Hill, G. W.: On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon. Acta Math. **8**(1), 1–36 (1886)
13. Hu, X., Long, Y., Sun, S.: Linear stability of elliptic Lagrangian solutions of the classical planer three-body problem via index theory. Arch. Ration. Mech. Anal. **213**(3), 993–1045 (2014)
14. Hu, X., Sun, S.: Index and stability of symmetric periodic orbits in Hamiltonian systems with its application to figure-eight orbit. *Commun. Math. Phys.* **290**(2), 737–777 (2009)
15. Hu, X., Sun, S.: Morse index and stability of elliptic Lagrangian solutions in the planar three-body problem. *Adv. Math.* **223**(1), 98–119 (2010)
16. Hu, X., Wang, P.: Conditional Fredholm determinant of S-periodic orbits in Hamiltonian systems. J. Funct. Anal. **261**(11), 3247–3278 (2011)
17. Hu, X., Wang, P.: *Hill-type formula and Krein-type trace formula for S-periodic solutions in ODEs* (Preprint 2014)
18. Kato, T.: *Perturbation Theory for Linear Operators*, 2nd edn. Springer, Berlin (1984)
19. Krein, M. G.: On tests for the stable boundedness of solutions of periodic canonical systems. *PrikL Mat. Mekh.* **19**(6), 641–680 (1955)
20. Krein, M. G.: Foundation of the theory of λ-zones of stability of a canonical systems of linear differential equations with periodic coefficients. In: *Memoriam: A. A. Andronov, Izdat.Akad.Nauk SSSR*, Moscow, pp. 413–498 (1955)
21. Lagrange, J. L.: *Essai sur le problème des trois corps. Chapitre II. Œuvres Tome 6*, pp. 272–292. Gauthier-Villars, Paris (1772)

22. Long, Y.: Bott formula of the Maslov-type index theory. *Pacific J. Math.* **187**(1), 113–149 (1999)
23. Long, Y.: Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics. *Adv. Math.* **154**(1), 76–131 (2000)
24. Long, Y.: Index theory for symplectic paths with applications. *Progress in Math.*, vol. 207. Birkhäuser, Basel (2002)
25. Martínez, R., Samà, A., Simó, C.: Stability diagram for 4D linear periodic systems with applications to homographic solutions. *J. Differ. Equ.* **226**(2), 619–651 (2006)
26. Martínez, R., Samà, A., Simó, C.: Analysis of the stability of a family of singular-limit linear periodic systems in $\mathbb{R}^4$. *Appl. J. Differ. Equ.* **226**(2), 652-686 (2006)
27. Meyer, K. R., Schmidt, D. S.: Elliptic relative equilibria in the N-body problem. J. Differ. Equ. **214**(2), 256–298 (2005)
28. Ou, Y.: Hyperbolicity of elliptic Lagrangian orbits in the planar three body problem. *Sci. China Math.* **57**(7), 1539–1544 (2014)
29. Poincaré, A.: Sur les déterminants d'ordre infini. *Bull. Soc. math. France* **14**, 77–90 (1886)
30. Reed, M., Simon, B.: *Methods of modern mathematical phisics, IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York (1978)
31. Roberts, G.E.: Linear stability of the elliptic Lagrangian triangle solutions in the three-body problem. *J. Differ. Equ.* **182**(1), 191–218 (2002)
32. Routh, E.J.: On Laplace's three particles with a supplement on the stability or their motion. *Proc. London Math. Soc.* **6**, 86–97 (1875)
33. Simon, B.: Trace ideals and their applications. 2nd edn. *Mathematical Surveys and Monographs*, vol. 120. American Mathematical Society, Providence (2005)

Department of Mathematics,
Shandong University Jinan,
Shandong 250100,
The People's Republic of China
e-mail: xjhu@sdu.edu.cn
e-mail: yuweiou@163.com
e-mail: phwang@sdu.edu.cn

(*Received July 29, 2014 / Accepted October 14, 2014*)
Published online October 29, 2014 –