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Quasi-Static Brittle Damage Evolution in Elastic Materials with Multiple Damaged States

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Abstract

We present energetic and strain-threshold models for the quasi-static evolution of brutal brittle damage for geometrically-linear elastic materials. By allowing for anisotropic elastic moduli and multiple damaged states we present the issues for the first time in a truly elastic setting, and show that the threshold methods developed in (Garroni, A., Larsen, C. J., Threshold-based quasi-static brittle damage evolution, Archive for Rational Mechanics and Analysis 194 (2), 585–609, 2009) extend naturally to elastic materials with non-interacting damage. We show the existence of solutions and that energetic evolutions are also threshold evolutions.

1. Introduction

Many irreversible phenomena in mechanics have been studied through variational models, plasticity (c.f., e.g., [10,12,13]) and fracture [6] being prominent examples. Variational formulations enable the use of the powerful tools of calculus of variations, for instance it is typically easy to show the existence of global minimizers albeit perhaps only for a relaxed energy.

Mechanical phenomena have also been understood through threshold criteria. In the examples given above, plastic behavior is triggered when stress reaches a yield surface and fracture occurs where the stress has a sufficiently large singularity. An attractive feature of these models is that these criteria are spatially local, which is physically natural, expresses engineering intuition and facilitates modelling. On the other hand it is often unclear what correspondence (if any) there is between variational formulations and threshold formulations of the same phenomenon.

In this paper we present, first, energetic (i.e., variational) and threshold models for the quasi-static evolution of brutal brittle damage in geometrically-linear elastic materials. This part of our work may be viewed as an extension to true elasticity (i.e., with vector-valued displacement fields and possibly anisotropic elastic moduli) of an earlier model [9] which was restricted to anti-plane shear (a scalar setting) with essentially scalar moduli (multiples of the identity). Moreover we allow for multiple damage processes, and thus multiple damaged states; to the best of our knowledge this is the first model to do so. Our model, both in the energetic and threshold versions, allows for microstructure formation, due to both elasticity and damage; we expand on the significance of this below. For the energetic formulation we show the existence of solutions under reasonable hypotheses.

These two approaches to damage are formulated independently but the question arises as to whether they are related for any given material and, if yes, how. In the second part of our work we relate these formulations for a broad class of materials which includes classical slip-plane plasticity without strain hardening. We show that energetic evolutions are also threshold evolutions, for a threshold that is related to the energetic cost of damage (i.e., the energy dissipated per unit volume due to damage). Thus energetic evolutions also have a spatially-local description.

To place our work in context in Section 1.2 we briefly summarise three energetic formulations [4,5,9] that have been proposed in the literature, in Section 1.3 we summarise a threshold formulation [9], and in Section 1.4 we summarise the link between them. It is convenient to introduce our notation before we do so.

1.1. Notation

Let $\mathbb{D} := \{0, 1\}$ and s be the dimension of space. Let $\Omega \subset \mathbb{R}^s$ be Lipschitz and $\mathbb{P}(\Omega) := 2^{\Omega}$, the set of subsets of Ω. $|\cdot|$ denotes either the Euclidean norm on $\mathbb{R}$ or the Lebesque measure on $\mathbb{R}^s$. We denote the Euclidean inner product in $\mathbb{R}^s$ by $\cdot$. For $a, b \in \mathbb{R}^s$,

$$a \otimes_s b := \frac{1}{2}(a \otimes b + b \otimes a)$$

where $a \otimes b$ is the tensor product of a and b.

Let $\mathbb{S} := \left\{M \in \mathbb{R}^{s\times s} \mid M = M^T\right\}$ be the linear space of symmetric matrices. We denote the standard inner product in $\mathbb{S}$ by $\langle\cdot,\cdot\rangle$. $\mathcal{P}$ is the set of all orthogonal projections on $\mathbb{S}$ and $\mathcal{M}$ is the set of all elastic modulli (i.e., positive-definite self-adjoint linear operators) on $\mathbb{S}$. We use the standard operator norm on $\mathcal{M}$:

$$\|\cdot\| := \sup_{\epsilon\in\mathbb{S}} \frac{\langle\cdot\epsilon, \epsilon\rangle}{\|\epsilon\|^2},$$

and the partial order that is defined through quadratic forms: $\forall\alpha_1, \alpha_2 \in \mathcal{M}$,

$$\alpha_1 \leqslant \alpha_2 \iff \forall\epsilon \in \mathbb{S}, \langle\alpha_1\epsilon, \epsilon\rangle \leqslant \langle\alpha_2\epsilon, \epsilon\rangle.$$

We set

$$\mathcal{M}(c_1, c_2) := \{\alpha \in \mathcal{M} \mid c_1 \leqslant \|\alpha\| \leqslant c_2\}$$

for $0 < c_1 < c_2$.

The map $e\colon H^1_0(\Omega, \mathbb{R}^s) \to L^2(\Omega, \mathbb{S})$ is defined through

$$e(\cdot) := \frac{1}{2}\left(D\cdot + D\cdot^T\right),$$

so $e(u)$ is the strain corresponding to the deformation u.

For $\alpha \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$ and body force $f \in H^{-1}(\Omega, \mathbb{R}^s)$ by $u(\alpha, f)$ we denote the solution in $H_0^1(\Omega, \mathbb{R}^s)$ of

$$-\operatorname{div}(\alpha e(\cdot)) = f. \tag{1.1}$$

In addition we set $e(\alpha, f) := e\,(u(\alpha, f)) \in L^2(\Omega, \mathbb{S})$. The corresponding elastic energy on $S \subset \Omega$ is

$$\mathcal{E}_S(\alpha, f) := \int_S \left(\frac{1}{2} \langle \alpha\, e(\alpha, f), e(\alpha, f) \rangle - f \cdot u(\alpha, f) \right) \mathrm{d}x.$$

When $S = \Omega$ we drop the subscript and write $\mathcal{E}$. Note that

$$\mathcal{E}(\alpha, f) = \inf_{u \in H_0^1(\Omega, \mathbb{R}^s)} \int_\Omega \left(\frac{1}{2} \langle \alpha\, e(u), e(u) \rangle - f \cdot u \right) \mathrm{d}x \tag{1.2}$$

and the minimisers $u \in H_0^1(\Omega, \mathbb{R}^s)$ of $\mathcal{E}(\alpha, f)$ satisfy (1.1).

Since our deformations $u \in H_0^1(\Omega, \mathbb{R}^s)$ vanish on the boundary the quasi-static evolutions that we consider are driven only by time-dependent body-forces $f \in H^{-1}(\Omega, \mathbb{R}^s)$. In fact there is no loss of generality here, see [4, Remark 5].

Unless explicitly indicated otherwise, by $\overset{\star}{\rightharpoonup}$ we denote weak* convergence in $L^\infty(\Omega, \mathbb{R}^m)$, where m would be clear from the context.

1.2. Energetic Formulations

In the model for damage proposed by [5], two states, undamaged and damaged, are characterised by two elastic moduli $A_0 \in \mathcal{M}$ and $A_0 - \Delta A_1 \in \mathcal{M}$, respectively.[1] The elastic moduli are well-ordered: $A_0 > A_0 - \Delta A_1$. The energy of each displacement $u \in H_0^1(\Omega, \mathbb{R}^s)$ and body-force $f \in H^{-1}(\Omega, \mathbb{R}^s)$ is given by

$$\int_\Omega W(e(u)) - f \cdot u \, \mathrm{d}x,$$

for an energy density

$$W(\cdot) = \min \left\{ \frac{1}{2} \langle A_0 \cdot, \cdot \rangle, \frac{1}{2} \langle (A_0 - \Delta A_1) \cdot, \cdot \rangle + k \right\}.$$

Here k is the energetic cost of damage. This energy density is not quasi-convex, thus we expect microstructure formation, which necessitates relaxation. The quasi-convex envelope of W, QW, is given by

$$QW(\cdot) := \min_{\theta \in [0,1]} Q_\theta W(\cdot).$$

$$Q_\theta W(\cdot) := \min_{A \in \mathcal{G}_\theta(\{A_0, \Delta A_1\})} \frac{1}{2} \langle A \cdot, \cdot \rangle + k\theta,$$

[1] For consistency with the rest of the paper our notation differs from [5].

where $\mathcal{G}_\theta(\{A_0, \Delta A_1\})$ is the G-closure of A_0 and $A_0 - \Delta A_1$ mixed with volume fractions $1-\theta$ and θ respectively (see Section 2.2 below). $Q_\theta W$ is known in variety of situations, see [1].

Given a time-parametrised external loading $f \in H^{-1}(\Omega, \mathbb{R}^s)$, a relaxed quasi-static evolution for this model that also includes irreversibility of damage was constructed in [4]. It was proved that there exists a time-parametrized family of elastic moduli $A(t) \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$ and a time-parametrized family of damage volume fraction $\theta(t) \in L^\infty(\Omega, [0, 1])$ such that,

$$A(t) \in \mathcal{G}_{\theta(t)}(\{A_0, \Delta A_1\})$$

(which, at $x \in \Omega$, mix A_0 and $A_0 - \Delta A_1$ with proportion $1 - \theta(t, x)$ and $\theta(t, x)$ respectively) and satisfying

1. monotonicity conditions (related to irreversibility of the damage): A is non-increasing and θ is non-decreasing,
2. an energy balance, and
3. a minimality condition: At every time t,

$$\mathcal{E}(A(t), f(t)) + \int_\Omega k\theta(t)\,\mathrm{d}x \leqslant \mathcal{E}(\tilde{A}, f(t)) + \int_\Omega k\left((1-\tilde{\theta})\theta(t) + \tilde{\theta}\right)\,\mathrm{d}x$$

   for every $(\tilde{A}, \tilde{\theta})$ such that $\tilde{A}$ is in G-closure of $A(t)$ and $A_0 - \Delta A_1$ mixed with volume fractions $1 - \tilde{\theta}$ and $\tilde{\theta}$ respectively.

It was observed in [9, Remark 4 and Example 1] that if a sequence of mixtures of undamaged and damaged materials corresponds to a homogenised elasticity tensor $A \in \mathcal{M}$, and another sequence of mixtures with more (in the sense of set inclusion) damaged material corresponds to $A' \in \mathcal{M}$ then it is not necessarily true that A' is obtainable as a mixture of A and the damaged material. Yet it is only with respect to such mixtures that the evolutions in [4] are minimal. By enforcing minimality with respect to G-closures of $A(t)$ with the damaged material [4] imposes minimality only with regard to further damage on larger length scales while neglecting the possibility of additional damage on the same length scale.

GARRONI ET AL. [9] overcomes this through an energetic formulation that is expressed explicitly in terms of sequences of sets that generate microstructure (rather than in terms of the effective behaviour of the microstructure); this leads to the notion of constrained G-closure (see Definition 2.5 in Section 2.2 below) and the notion of weak energy-minimizing evolutions in Definition 1.1 below. Their analysis is restricted to scalar (i.e., anti-plane) deformations with elastic moduli being isotropic (i.e., multiples of the identity): $A_0 = \beta I$ and $\Delta A_1 = \Delta\beta I$ for some $\beta > 0$ and $\Delta\beta \in (0, \beta)$.

Definition 1.1. ([9, *Definition* 3]) Let $f \in W^{1,1}([0, T], H^{-1}(\Omega, \mathbb{R}))$. An evolution

$$[0, T] \ni t \mapsto (A(t), \theta(t)) \in L^\infty(\Omega, \mathcal{M}(c_1, c_2)) \times L^\infty(\Omega, [0, 1])$$

with

$$A(t) \in \mathcal{G}_{\theta(t)}(\{\beta I, \Delta\beta I\})$$

(see Definition 2.5 in Section 2.2 below) is a weak energy-minimizing evolution if the following hold:

1. *Monotonicity*: The map $t \mapsto A(t)$ is non-increasing and the map $t \mapsto \theta(t)$ is non-decreasing.
2. *Energy balance*: The energy satisfies

$$\mathcal{E}(A(t), f(t)) + \int_\Omega k \cdot \theta(t)\, \mathrm{d}x = \mathcal{E}(A(0), f(0)) + \int_\Omega k \cdot \theta(0)\, \mathrm{d}x - \int_0^t \dot{f}(s) \cdot u(A, f)(s)\, \mathrm{d}s.$$

3. *Minimality*: There exists a sequence $\{D^n(t)\} \subset \Omega$, non-decreasing in t for each n, such that for every $t \in [0, T]$,

$$\beta I - \Delta\beta \chi_{D^n(t)} I \xrightarrow{G} A(t),$$
$$\chi_{D^n(t)} \overset{\star}{\rightharpoonup} \theta(t)$$

(see Section 2.1 below for the definition of G-convergence) and for every $(\tilde{A}, \tilde{\theta})$ such that

$$\tilde{A} \in \mathcal{G}_{\tilde{\theta}(t)}(\{D^n(t)\}, \{\beta I, \Delta\beta I\})$$

(see Section 2.2 below for the definition of constrained G-closure) we have

$$\mathcal{E}(A(t), f(t)) + \int_\Omega k \cdot \theta(t)\, \mathrm{d}x \leqslant \mathcal{E}(\tilde{A}, f(t)) + \int_\Omega k \cdot \tilde{\theta}\, \mathrm{d}x.$$

Garroni et al. [9] also shows the existence of weak energy-minimizing evolutions for every $f \in W^{1,1}([0, T], H^{-1}(\Omega, \mathbb{R}^s))$.

1.3. A Threshold Formulation

We now describe the model introduced in [9] for damage evolution based explicitly on a strain threshold without any reference to an energetic cost for damage. As before this analysis is restricted to scalar deformations with $A_0 = \beta I$ and $\Delta A_1 = \Delta\beta I$ for some $\beta > 0$ and $\Delta\beta \in (0, \beta)$.

The formulation rests on three principles:

1. *Irreversibility of damage*: The damaged region is non-decreasing in time (in the sense of set inclusion).
2. *Presence of a threshold*: There exists a (positive) damage threshold which is not exceeded by the absolute value of the strain in the undamaged region.
3. *Necessity of damage*: Damage only occurs as is necessary in order to maintain condition (2).

The first two principles are straightforward to formulate, but the third is more subtle (see Remark 1.3 below). These principles lead to:

Definition 1.2. ([9, *Definition* 8]) Let $f \in W^{1,1}([0, T], H^{-1}(\Omega, \mathbb{R}))$. For $D \subset \Omega$ we set

$$\alpha_D := \beta I - \Delta\beta \chi_D I.$$

An evolution

$$[0, T] \ni t \mapsto D(t) \subset \Omega$$

is a strong threshold evolution with threshold $\lambda > 0$ if

1. *Monotonicity*: The damage evolution $t \mapsto D(t)$ is non-decreasing.
2. *Threshold*: The threshold is not exceeded outside the damage set:

$$|\nabla u(\alpha_{D(t)}, f(t))| \leqslant \lambda$$

a.e. in $\Omega \setminus D(t)$.
3. *Necessity of damage*:
   (a) For every $E \subset D(T)$ with $|E| > 0$ and every sufficiently small $\Delta\tau$, there exists $\tau < T - \Delta\tau$ such that, with

$$\Delta E := E \cap (D(\tau + \Delta\tau) \setminus D(\tau)),$$

we have

$$|\nabla u(\alpha_{D(\tau+\Delta\tau)\setminus \Delta E}, f(\tau + \Delta\tau))| > \lambda$$

in a subset of ΔE with positive measure.
   (b) (Trivially satisfied if D is continuous from below at T.) For every

$$E \subset D(T) \setminus \bigcup_{\tau<T} D(\tau)$$

with positive measure, we have

$$|\nabla u(\alpha_{D(T)\setminus E}, f(T))| > \lambda$$

in a subset of E with positive measure.

Remark 1.3. The monotonicity condition in Definition 1.2 requires that damage be irreversible and the threshold condition requires that the threshold be exceeded unless damage occurs.

The necessity condition imposes the converse restriction, locally in space: from Item (3a), were a region not included in the damage set then the threshold would have been exceeded in a (measurable) subset of *that* region. Item (3b) asks that the damage set jump (increases discontinuously) to include a region only if the alternative would have been to exceed the threshold in *that* region.

Note that it suffices to impose the necessity condition at the final time T since all earlier times are included.

1.4. Link between Variational and Threshold Formulations

Garroni et al. [9, Theorem 9] shows that there is a correspondence between the damage cost k in [4] and the threshold λ such that a strong energy-minimimising evolution (in the sense of [4]) is a strong threshold evolution in the sense of Definition 1.2.

If the energy needs to be relaxed, due to the development of microstructure, then so does the threshold criterion. That is, there might be no solutions to the threshold formulation, only approximate solutions that develop microstructure. A weak form of the threshold criterion then needs to be formulated. This is done in [9, Definition 10]. Since we will present a more general definition in Section 4 below we do not repeat their definition here.

However we will highlight the key feature: any threshold model must involve pointwise properties of (symmetrised) gradients of the deformation. It follows that it cannot be formulated only in terms of weak limits of approximating sequences. The solution used in [9] is to use pointwise properties of deformation gradients corresponding to sequences of damage sets; we shall use the same approach.

The resulting definition of "weak threshold evolutions" then suggests a formulation for "weak energy-minimising evolutions". Again, it cannot be formulated only in terms of weak limits of approximating sequences but must consider the sequences of damage sets that generate the relaxed solution. This formulation, presented in [9], was described above (Definition 1.1). It maintains the correspondence between energy and threshold formulations, that is, it guarantees that weak energy-minimising evolutions are also weak threshold evolutions.

1.5. Outline of Paper

In Section 2 we remind the reader of the notions of G-convergence and G-closure. This is followed, in Section 3, by a description of elastic damage with illustrative examples. In Sections 4 and 5 we explore threshold and energetic formulations of elastic damage, respectively. We also prove the existence of quasi-static evolutions for the energetic formulation, even in the presence of microstructure.

In Section 6 we relate these two formulations, as in [9], by showing that, for a broad class of materials, all energetic (quasistatic) evolutions are also threshold (quasistatic) evolutions. As a consequence we obtain the existence of quasi-static evolutions for the threshold formulation, even in the presence of microstructure. In addition this shows that energetic (quasistatic) evolutions satisfy a local threshold. Moreover this link between variational and threshold formulations enables us to show in Section 7 that all local minimisers of energy are global minimisers.

2. G-Convergence and G-Closures

2.1. G-convergence

We recall, for the readers' convenience, the notion of G-convergence; for detailed introductions we suggest [2,14].

Definition 2.1. (*G-convergence*, $\stackrel{G}{\to}$) A sequence $A^n \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$ G-converges to $A \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$, $A^n \stackrel{G}{\to} A$, iff for every $f \in H^{-1}(\Omega, \mathbb{R}^s)$,

$$u(A^n, f) \rightharpoonup u(A, f) \text{ weakly in } H_0^1(\Omega, \mathbb{R}^s). \tag{2.1a}$$

Remark 2.2. (*Convergence of stresses*) As a consequence of the symmetries of the linear operators in $\mathcal{M}$, if $A^n \stackrel{G}{\to} A$ then

$$A^n e(A^n, f) \rightharpoonup Ae(A, f) \text{ weakly in } L^2(\Omega, \mathbb{S}). \tag{2.1b}$$

The following properties of G-convergence are worth noting:

Remark 2.3. For sequences $A^n, B^n \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$,

1. *Compactness*: There exists $A \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$ such that $A^n \stackrel{G}{\to} A$ upto subsequence.
2. *Convergence of energy*: If $A^n \stackrel{G}{\to} A$ then, $\forall f \in H^{-1}(\Omega, \mathbb{R}^s)$,
$$\mathcal{E}(A^n, f) \to \mathcal{E}(A, f).$$
3. *Metrizability*: G-convergence is associated to a metrizable topology on $L^\infty(\Omega, \mathcal{M}(c_1, c_2))$.
4. *Ordering*: If $A^n \stackrel{G}{\to} A$, $B^n \stackrel{G}{\to} B$ and $B^n \leqslant A^n$ then $B \leqslant A$.
5. *Locality*: If $A^n \stackrel{G}{\to} A$, $B^n \stackrel{G}{\to} B$, and χ is a characteristic function on Ω, then
$$\chi A^n + (1-\chi)B^n \stackrel{G}{\to} \chi A + (1-\chi)B.$$
6. *Periodicity*: Let $A \in L^\infty([0,1]^s, \mathcal{M}(c_1, c_2))$ be periodic and $A^n(x) := A(nx)$. Then $A^n \stackrel{G}{\to} A^0$ where A^0 satisfies
$$\langle A^0 e, e\rangle = \inf_{\varphi\,:\,\text{periodic}} \int_{[0,1]^s} \langle A(x)(e + e(\varphi)), (e + e(\varphi))\rangle \, \mathrm{d}x.$$

2.2. *G-closures*

Next we introduce two notions of G-closure. While the concept of G-closure (Definition 2.4) is standard (c.f., e.g., [2,14]) the specific notation here has been chosen to suit our purposes. Our definition of Constrained G-closures (Definition 2.5) extends the corresponding definition in [9]. For these definitions we set $A_0 \in \mathcal{M}(c_1, c_2)$, $m \in \mathbb{N}$ and

$$\mathbf{A} := \{A_0\} \cup \{\Delta A_i \in \mathcal{M}(c_1, c_2) \mid i = 1, \ldots, m\}, \tag{2.2}$$

(viewed as a $(m+1)$-tuple) while requiring

$$A_0 - \sum_{i=1}^{m} \Delta A_i \in \mathcal{M}(c_1, c_2). \tag{2.3}$$

Definition 2.4. (G-*closure*, $\mathcal{G}$.) Let $\theta \in L^\infty(\Omega, [0,1]^m)$ and let $\chi^n : \Omega \to \mathbb{D}^m$ be such that $\chi^n \stackrel{\star}{\rightharpoonup} \theta$. Then the G-closure of $\mathbf{A}$, $\mathcal{G}_\theta(\mathbf{A})$, is the set of all possible G-limits of

$$A_0 - \sum_{i=1}^{m} \chi_i^n \Delta A_i.$$

We also set

$$\mathcal{G}(\mathbf{A}) := \left\{\alpha \mid \exists \theta \in L^\infty(\Omega, [0,1]^m), \alpha \in \mathcal{G}_\theta(\mathbf{A})\right\}.$$

Definition 2.5. (*Constrained G-closure*) Let $\xi^n : \Omega \to \mathbb{D}^m$ be weak-* convergent characteristic functions on Ω. When the sequence χ in Definition 2.4 is picked such that $\chi_i^n \geqslant \xi_i^n$, $i = 1, \ldots, m$, then the set of all possible G-limits of

$$A_0 - \sum_{i=1}^{m} \chi_i^n \Delta A_i$$

is the constrained G-closure of $\mathbf{A}$ (with phase fraction θ and constraint $\{\xi^n\}$), $\mathcal{G}_\theta(\{\xi^n\}, \mathbf{A})$. By abuse of notation we also denote this by $\mathcal{G}_\theta(\{D^n\}, \mathbf{A})$ where $\xi_i^n = \chi_{D_i^n}$, $i = 1, \ldots, m$, for some sequence $\{D^n\} = \{(D_i^n, D_2^n, \ldots D_m^n)\} \subset \Omega^m$.

3. Damage

We consider a geometrically-linear elastic material which in the undamaged state has elastic modulus $\alpha_{\{0\}^m} \in \mathcal{M}$ (the reason for this notation will become clear in a moment) and, thus, energy density $W_{\{0\}^m} : \mathbb{S} \to \mathbb{R}$ given by

$$W_{\{0\}^m}(\cdot) = \frac{1}{2}\left\langle \alpha_{\{0\}^m} \cdot, \cdot \right\rangle.$$

This material is capable of undergoing $m \geqslant 1$ damage processes, any combination of which can occur simultaneously in both space and time. The (pointwise) damage state of the material is denoted by $d \in \mathbb{D}^m$ where

$$d_i = \begin{cases} 1 & \text{if} \quad i\text{-damage has occurred,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for example, $\{0\}^m$ denotes the undamaged material and $\{1\}^m$ the fully damaged material.

The ith damage process ("i-damage") costs $k_i > 0$ and weakens the material by diminishing the elastic modulus by $\Delta\alpha_i \in \mathcal{M}$ where $\Delta\alpha_i \geqslant 0$. Thus, the elastic modulus and energy density corresponding to damage $d \in \mathbb{D}^m$ are

$$\alpha_d := \alpha_{\{0\}^m} - \sum_{i=1}^{m} d_i \, \Delta\alpha_i \tag{3.1a}$$

$$W_d := \frac{1}{2} \langle \alpha_d \cdot, \cdot \rangle + k \cdot d \tag{3.1b}$$

and the possible elastic moduli are

$$\boldsymbol{\alpha} = \left\{ \alpha_d \mid d \in \mathbb{D}^m \right\} \subset \mathcal{M}.$$

The weakest elastic modulus corresponds to the material being damaged in all m ways; we require this to be positive-definite:

$$\alpha_{\{1\}^m} = \alpha_{\{0\}^m} - \sum_{i=1}^{m} \Delta\alpha_i > 0.$$

For convenience we set $\mathbf{M} := \{1, \ldots, m\}$.

We pause to introduce two examples after which we will be ready for the threshold and energy formulations of damage.

3.1. Examples

For the examples we present it is convenient to decompose $\mathbb{S}$ into hydrostatic, diagonal shear and off-diagonal shear subspaces. Thus, in two dimensions, let

$$\begin{aligned}\mathcal{H} &:= \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\},\\ \mathcal{D} &:= \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\},\\ \mathcal{O} &:= \operatorname{span}\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\};\\ \mathcal{S} &:= \mathcal{D}\oplus\mathcal{O}.\end{aligned}$$

Similarly, in three dimensions, let

$$\begin{aligned}\mathcal{H} &:= \operatorname{span}\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right\},\\ \mathcal{D} &:= \operatorname{span}\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\right\},\\ \mathcal{O} &:= \operatorname{span}\left\{\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right\};\\ \mathcal{S} &:= \mathcal{D}\oplus\mathcal{O}.\end{aligned}$$

In either case we denote the projections on these spaces by Λ_H, Λ_D, Λ_O and Λ_S respectively.

A cubic elastic modulus is an element of $\mathcal{M}$ that is of the form $\kappa\Lambda_H + \mu\Lambda_D + \eta\Lambda_O$ for some $\kappa, \mu, \eta > 0$. When $\mu = \eta$ we obtain an isotropic elastic modulus, $\kappa\Lambda_H + \mu\Lambda_S$.

Example 3.1. (*Isotropic damage*) Consider an isotropic material which in the undamaged state has elastic modulus $\kappa\Lambda_H + \mu\Lambda_S$. Suppose this material is susceptible to two damage processes:

1. $\mathcal{H}$-damage which reduces the elastic modulus by $\Delta\kappa\,\Lambda_H$ with $\Delta\kappa \in (0, \kappa)$,
2. $\mathcal{S}$-damage which reduces the elastic modulus by $\Delta\mu\,\Lambda_S$ with $\Delta\mu \in (0, \mu)$.

Schematically:

$$\begin{array}{rcl} \kappa\Lambda_H + \mu\Lambda_S & & (\kappa - \Delta\kappa)\Lambda_H + \mu\Lambda_S \\ =: \alpha_{(0,0)} & \xrightarrow{\mathcal{H}-\text{damage}} & =: \alpha_{(1,0)} \\ \mathcal{S}-\text{damage}\downarrow & & \downarrow\mathcal{S}-\text{damage} \\ \kappa\Lambda_H + (\mu - \Delta\mu)\Lambda_S & & (\kappa - \Delta\kappa)\Lambda_H + (\mu - \Delta\mu)\Lambda_S \\ =: \alpha_{(0,1)} & \xrightarrow{\mathcal{H}-\text{damage}} & =: \alpha_{(1,1)}. \end{array}$$

Note that

$$\alpha_{(0,0)} > \alpha_{(0,1)}, \alpha_{(1,0)} > \alpha_{(1,1)},$$

but $\alpha_{(0,1)}$ and $\alpha_{(1,0)}$ are not well-ordered.

Example 3.2. (*Cubic shear damage*) Consider a cubic material which in the undamaged state has elastic modulus $\kappa\Lambda_H + \mu\Lambda_D + \eta\Lambda_O$. Suppose this material is susceptible to two damage processes:

1. $\mathcal{D}$-damage which reduces the elastic modulus by $\Delta\mu\,\Lambda_D$ with $\Delta\mu \in (0, \mu)$,
2. $\mathcal{O}$-damage which reduces the elastic modulus by $\Delta\eta\,\Lambda_O$ with $\Delta\eta \in (0, \eta)$.

Schematically:

$$\begin{array}{rcl}
\kappa\Lambda_H + \mu\Lambda_D + \eta\Lambda_O & \mathcal{D}-\text{damage} & \kappa\Lambda_H + (\mu - \Delta\mu)\Lambda_D + \eta\Lambda_O \\
=: \alpha_{(0,0)} & \xrightarrow{\hspace{2cm}} & =: \alpha_{(1,0)} \\
\mathcal{O} - \text{damage} \downarrow & & \downarrow \mathcal{O} - \text{damage} \\
\kappa\Lambda_H + \mu\Lambda_D + (\eta - \Delta\eta)\Lambda_O & \mathcal{D}-\text{damage} & \kappa\Lambda_H + (\mu - \Delta\mu)\Lambda_D + (\eta - \Delta\eta)\Lambda_O \\
=: \alpha_{(0,1)} & \xrightarrow{\hspace{2cm}} & =: \alpha_{(1,1)}
\end{array}$$

As before

$$\alpha_{(0,0)} > \alpha_{(0,1)}, \alpha_{(1,0)} > \alpha_{(1,1)},$$

but $\alpha_{(0,1)}$ and $\alpha_{(1,0)}$ are not well-ordered.

3.2. Two Formulations of Damage

Let $i \in \mathbf{M}$.

Formulation 3.3. (Threshold criterion for damage) *The ith damage process is (pointwise) sensitive only to the strain ϵ and only through an orthogonal projection Λ_i on a subspace of $\mathbb{S}$: At $x \in \Omega$, i-damage occurs only if otherwise,*

$$\|\Lambda_i \epsilon(x)\| > \lambda_i \tag{3.2}$$

for some specified threshold $\lambda_i > 0$. We refer to $\text{range}(\Lambda_i)$ *as the ith damage subspace.*

Note that (3.2) presents a necessary but not sufficient condition for damage. The sufficient condition is more subtle and is stated in Section 4 (Definitions 4.1(3) and 4.3(3) and Remark 4.2).

Formulation 3.4. (Energetic description of damage) *The ith damage process costs $k_i > 0$: The energy density corresponding to damage $d \in \mathbb{D}^m$ is*

$$W_d(\cdot) := \frac{1}{2}\langle \alpha_d \cdot, \cdot \rangle + d \cdot k \tag{3.3}$$

where α_d is given by (3.1).

We explore these formulations in Sections 4 and 5 respectively. In Section 6 (See proofs of Lemmata 6.4 and 6.5) we explore the relationship between these formulations for materials that possesses the following property:

Property 3.5. 1. The damage subspaces are strain compatible: For each $i \in \mathbf{M}$,

$$\epsilon \in \mathrm{Range}(\Lambda_i) \implies \exists a, b \in \mathbb{R}^s \quad \text{such that} \quad \epsilon = a \otimes_s b. \tag{3.4a}$$

2. The undamaged elastic modulus is a multiple of the identity on each damage subspace: For each $i \in \mathbf{M}$,

$$\alpha_{\{0\}^m} \Lambda_i = \beta_i \Lambda_i \tag{3.4b}$$

for some (scalar) $\beta_i > 0$. (That is, the damage subspaces are eigenspaces of the undamaged elastic modulus with β_i being the corresponding positive eigenvalue.)
3. The elastic modulus is weakened only on the relevant damage subspace, and uniformly on the subspace: for $i \in \mathbf{M}$,

$$\Delta\alpha_i = \Delta\beta_i \Lambda_i, \tag{3.4c}$$

for some $\Delta\beta_i \in (0, \beta_i)$.
4. The damage subspaces are orthogonal: for $i, j \in \mathbf{M}$ with $i \neq j$,

$$\Lambda_i \Lambda_j = 0. \tag{3.4d}$$

Remark 3.6. Equations (3.4b), (3.4c) and (3.4d) imply that for $d \in \mathbb{D}^m$,

$$\alpha_d = \sum_{i=1}^{m} (\beta_i - d_i \Delta\beta_i)\, \Lambda_i + \beta' \tag{3.5}$$

where $\beta' \in \mathcal{M}$ satisfies $\beta' \Lambda_i = 0$ for all $i \in \mathbf{M}$.

Remark 3.7. (*Relation to crystal plasticity*) Property 3.5 is suggestive of classical slip-plane perfect-plasticity, albeit imperfectly (especially post-yielding) since damage softens the material, whilst plasticity induces residual deformations:

When (3.4a) is replaced by the stronger condition

$$\exists \hat{n} \in \mathbb{R}^s, \qquad \epsilon \in \mathrm{Range}(\Lambda_i) \implies \exists a \in \mathbb{R}^s \text{ such that } \epsilon = a \otimes_s \hat{n}$$

then the damage subspaces are precisely slip planes with plane normal $\hat{n}$ and slip direction a. Equation (3.4d) asks that the slip systems be independent (in fact pairwise orthogonal); thus $m \leqslant 6$, or, if the hydrostatic subspace is excluded, $m \leqslant 5$. The threshold condition (3.2) is the yield condition and involves the resolved shear strain on the slip system.

3.3. Notation

In the rest of the paper we adopt the following notation:

Let $D_i \subset \Omega$, $i \in \mathbf{M}$, denote the region in which i-damage has occurred, and let χ_{D_i} be the corresponding characteristic functions. We define:

$$D := (D_1, \ldots, D_m) \in \mathbb{P}(\Omega)^m,$$
$$\chi_D := \left(\chi_{D_1}, \ldots, \chi_{D_m}\right).$$

By abuse of notation we set,

$$\alpha_D(x) := \alpha_{\chi_{D(x)}},$$
$$u(D, \cdot) := u(\alpha_D, \cdot),$$
$$e(D, \cdot) := e(\alpha_D, \cdot),$$
$$\mathcal{E}(D, \cdot) := \mathcal{E}(\alpha_D, \cdot).$$

Set-theoretic operations on $\mathbb{P}(\Omega)^m$ are performed component-wise, e.g., for $D, D' \in \mathbb{P}(\Omega)^m$,

$$D \cup D' = \left(D_1 \cup D_1', \ldots, D_m \cup D_m'\right),$$
$$D \cap D' = \left(D_1 \cap D_1', \ldots, D_m \cap D_m'\right),$$
$$D \setminus D' = \left(D_1 \setminus D_1', \ldots, D_m \setminus D_m'\right);$$

and likewise for set-theoretic statements on $\mathbb{P}(\Omega)^m$:

$$D \subset D' \iff D_i \subset D_i', \quad \forall i \in \mathbf{M}.$$

For $D \in \mathbb{P}(\Omega)^m$, we define $|D| \in [0, \infty)^m$ by:

$$|D|_i = |D_i|, \quad i \in \mathbf{M}.$$

4. Threshold Formulation

First we formulate a definition for the classical situation in which the damage occurs in a set:

Definition 4.1. (*Strong threshold evolution*) Let $f : [0, T] \to H^{-1}(\Omega, \mathbb{R}^s)$. An evolution

$$[0, T] \ni t \mapsto D(t) \in \mathbb{P}(\Omega)^m$$

is a strong threshold evolution with thresholds

$$\{(\Lambda_i, \lambda_i) \in \mathcal{P} \times (0, \infty) \mid i \in \mathbf{M}\} \tag{4.1}$$

if the following hold for each $i \in \mathbf{M}$:

1. *Monotonicity*: The damage evolution $t \mapsto D_i(t)$ is non-decreasing.

2. *Threshold*: The threshold is not exceeded outside the damage set: $\forall t \in [0, T]$,

$$\|\Lambda_i\, e\,(D(t), f(t))\| \leqslant \lambda_i \quad \text{a.e. in } \Omega \setminus D_i(t).$$

3. *Necessity of damage*:
   (a) For every $E \subset D_i(T)$ with $|E| > 0$ and every sufficiently small $\Delta\tau$, there exists $\tau < T - \Delta\tau$ such that, with $\Delta E \in \mathbb{P}(\Omega)^m$,

$$\Delta E_j := \begin{cases} E \cap D_i(\tau + \Delta\tau) \setminus D_i(\tau) & \text{if } j = i, \\ \emptyset & \text{if } j \neq i, \end{cases}$$

   we have

$$\|\Lambda_i\, e\,(D(\tau + \Delta\tau) \setminus \Delta E, f(\tau + \Delta\tau))\| > \lambda_i \tag{4.2a}$$

   in a subset of ΔE_i with positive measure.
   (b) (Trivially satisfied if D_i is continuous from below at T.) For every $E \in \mathbb{P}(\Omega)^m$ satisfying

$$E_j \subset \begin{cases} D_i(T) \setminus \bigcup_{\tau < T} D_i(\tau) & \text{if } j = i, \\ \emptyset, & \text{if } j \neq i, \end{cases}$$

   with $|E_i| > 0$, we have

$$\|\Lambda_i\, e(\, D(T) \setminus E, f(T)\,)\| > \lambda_i \tag{4.2b}$$

   in a subset of E_i with positive measure.

Remark 4.2. In addition to being local in space, the necessity condition is "local" also with respect to the damage mode: (4.2a) and (4.2b) are stronger than requiring only

$$\exists j \in \mathbf{M}, \quad \left\|\Lambda_j\, e\,(D(\tau + \Delta\tau) \setminus \Delta E, f(\tau + \Delta\tau))\right\| > \lambda_j$$

and

$$\exists j \in \mathbf{M}, \quad \left\|\Lambda_j\, e\big(\, D(T) \setminus E, f(T)\,)\right\| > \lambda_j$$

(in a subset of E_i with positive measure), respectively.

The extension to the situation where there is damage microstructure is as follows:

Definition 4.3. (*Weak threshold evolution*) Let $f : [0, T] \to H^{-1}(\Omega, \mathbb{R}^s)$. An evolution

$$[0, T] \ni t \mapsto (A(t), \theta(t)) \in L^\infty(\Omega, \mathcal{M}(c_1, c_2)) \times [0, 1]^m$$

is a weak threshold evolution with thresholds (4.1) if: For every $t \in [0, T]$ there exists a sequence $\{D^n(t)\} \subset \mathbb{P}(\Omega)^m$ such that

$$\alpha_{D^n(t)} \xrightarrow{G} A(t),$$

$$\chi_{D^n(t)} \overset{\star}{\rightharpoonup} \theta(t),$$

and the following hold for each $i \in \mathbf{M}$:

1. *Monotonicity*: The damage evolution $t \mapsto D_i^n(t)$ is non-decreasing.
2. *Threshold*: For each $\delta > 0$ the sets in which there is no i-damage but the threshold is exceeded by at least δ converge in measure to the empty set: $\forall t \in [0, T]$,
$$\left|U_i^n(\delta)\right| \to 0 \quad \text{as } n \to \infty,$$
where
$$U_i^n(\delta) := \left\{x \notin D_i^n(t) \mid \left\|\Lambda_i\, e(D^n(t), f(t))(x)\right\| > \lambda_i + \delta\right\}. \tag{4.3}$$
3. *Necessity of the damage*: For each $\delta > 0$,
   (a) For every $E^n \subset D_i^n(T)$ with $\liminf |E^n| > 0$ and every sufficiently small $\Delta\tau$, there exists $\tau < T - \Delta\tau$ such that, with $\Delta E^n \in \mathbb{P}(\Omega)^m$,
$$\Delta E_j^n := \begin{cases} E^n \cap D_i^n(\tau + \Delta\tau) \setminus D_i^n(\tau) & \text{if } j = i, \\ \emptyset & \text{if } j \neq i, \end{cases}$$
   we have
$$\liminf_{n\to\infty} \left|V_i^n(\delta)\right| > 0,$$
   where
$$V_i^n(\delta) := \left\{x \in E_i^n \mid \left\|\Lambda_i\, e^n(x)\right\| > \lambda_i - \delta\right\}, \tag{4.4a}$$
$$e^n := e(D^n(\tau + \Delta\tau) \setminus E^n, f(\tau + \Delta\tau)). \tag{4.4b}$$
   (b) (Trivially satisfied if $\int_\Omega \theta_i(x, \cdot)\,\mathrm{d}x$ is continuous from below at T.) For every $t^n \nearrow T$ and every $E^n \in \mathbb{P}(\Omega)^m$ satisfying
$$E_j^n \subset \begin{cases} D_i^n(T) \setminus D_i^n(t^n) & \text{if } j = i, \\ \emptyset, & \text{if } j \neq i, \end{cases}$$
   with $\liminf |E_i^n| > 0$ we have
$$\liminf_{n\to\infty} \left|W_i^n(\delta)\right| > 0,$$
   where
$$W_i^n(\delta) := \left\{x \in E_i^n \mid \left\|\Lambda_i\, e(D^n(T) \setminus E^n, f(T))(x)\right\| > \lambda_i - \delta\right\}. \tag{4.5}$$

Remark 4.4. Let $f\colon [0, T] \to H^{-1}(\Omega, \mathbb{R}^s)$. An evolution
$$[0, T] \ni t \mapsto D(t) \in \mathbb{P}(\Omega)^m$$
is a strong threshold evolution if the evolution
$$[0, T] \ni t \mapsto (\alpha_{D(t)}, \chi_{D(t)}) \in L^\infty(\Omega, \mathcal{M}(c_1, c_2)) \times [0, 1]^m$$
is a weak threshold evolution. (Pick the sequence $\{D^n(t)\}$ in Definition 4.3 to be a constant sequence.)

5. Energetic Formulation

Definition 5.1. (*Energy*) The energy associated with $\alpha \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$, $f \in H^{-1}(\Omega, \mathbb{R}^s)$ and $\theta \in L^\infty(\Omega, [0, 1]^m)$ is

$$\begin{aligned}\mathcal{W}(\alpha, \theta, f) &:= \mathcal{E}(\alpha, f) + \int_\Omega k \cdot \theta \, \mathrm{d}x \\ &= \int_\Omega \frac{1}{2} \langle \alpha \, e(\alpha, f), e(\alpha, f)\rangle - f \cdot u(\alpha, f) + k \cdot \theta \, \mathrm{d}x. \qquad (5.1\text{a})\end{aligned}$$

It is convenient to also define, for $v \in H^1_0(\Omega, \mathbb{R}^s)$, $\chi \in L^\infty(\Omega, \mathbb{D}^m)$, $f \in H^{-1}(\Omega, \mathbb{R}^s)$ and $S \subset \Omega$,

$$\mathcal{V}_S(v, \chi, f) := \int_S \frac{1}{2} \big\langle \alpha_\chi \, e(v), e(v)\big\rangle - f \cdot v + k \cdot \chi \, \mathrm{d}x, \qquad (5.1\text{b})$$

and, by abuse of notation, for $D \in \mathbb{P}(\Omega)^m$,

$$\mathcal{V}_S(v, D, f) := \int_S \frac{1}{2} \langle \alpha_D \, e(v), e(v)\rangle - f \cdot v + k \cdot \chi_D \, \mathrm{d}x. \qquad (5.1\text{c})$$

Finally, we define

$$\widetilde{\mathcal{V}}_S(v, \alpha, \theta, f) := \int_S \frac{1}{2} \langle \alpha \, e(v), e(v)\rangle - f \cdot v + k \cdot \theta \, \mathrm{d}x. \qquad (5.1\text{d})$$

When $S = \Omega$ we drop the subscript S write $\mathcal{V}$ or $\widetilde{\mathcal{V}}$.

In (5.1), where necessary, by f we mean the localisation of a representative of f to S, see [9, page 602] for details.

Note that

$$\mathcal{W}(\alpha, \theta, f) \leqslant \widetilde{\mathcal{V}}(v, \alpha, \theta, f).$$

Definition 5.2. (*Strong energy-minimizing evolution*) Let $f \in W^{1,1}([0, T], H^{-1}(\Omega, \mathbb{R}^s))$. An evolution

$$[0, T] \ni t \mapsto D(t) \in \mathbb{P}(\Omega)^m$$

is a strong energy-minimizing evolution if the following hold:

1. *Monotonicity*: The damage evolution $t \mapsto D_i(t)$ is non-decreasing for each $i \in \mathbf{M}$.
2. *Energy balance*: For every $t \in [0, T]$ the energy satisfies

$$\mathcal{E}(D(t), f(t),) = \mathcal{E}(D(0), f(0)) - \int_0^t \dot{f}(s) \cdot u(D, f)(s) \, \mathrm{d}s.$$

3. *Minimality*: For every $t \in [0, T]$ and every $\tilde{D} \supseteq D(t)$,

$$\mathcal{E}(D(t), f(t)) \leqslant \mathcal{E}(\tilde{D}, f(t)).$$

The extension to the situation where there is damage microstructure is as follows:

Definition 5.3. (*Weak energy-minimizing evolution*) Let $f \in W^{1,1}([0,T], H^{-1}(\Omega,\mathbb{R}^s))$. An evolution

$$[0,T] \ni t \mapsto (A(t),\theta(t)) \in L^\infty(\Omega,\mathcal{M}(c_1,c_2)) \times L^\infty(\Omega,[0,1]^m)$$

with

$$A(t) \in \mathcal{G}_{\theta(t)}(\boldsymbol{\alpha})$$

is a weak energy-minimizing evolution if the following hold:

1. *Monotonicity*: The map $t \mapsto A(t)$ is non-increasing and for each $i \in \mathbf{M}$, the map $t \mapsto \theta_i(t)$ is non-decreasing.
2. *Energy balance*: For every $t \in [0,T]$ the energy

$$\mathcal{W}(t) := \mathcal{W}(A(t),\theta(t), f(t)) \tag{5.2a}$$

satisfies

$$\mathcal{W}(t) = \mathcal{W}(0) - \int_0^t \dot{f}(s)\cdot u(A,f)(s)\,\mathrm{d}s. \tag{5.2b}$$

3. *Minimality*: There exists a sequence $\{D^n(t)\} \subset \mathbb{P}(\Omega)^m$, non-decreasing in t for each n, such that for every $t \in [0,T]$,

$$\begin{aligned} \alpha_{D^n(t)} &\xrightarrow{G} A(t),\\ \chi_{D^n(t)} &\overset{\star}{\rightharpoonup} \theta(t) \end{aligned} \tag{5.3a}$$

and for every $(\tilde{A},\tilde{\theta})$ such that $\tilde{A} \in \mathcal{G}_{\tilde{\theta}(t)}(\{D^n(t)\},\boldsymbol{\alpha})$ we have

$$\mathcal{W}(t) \leqslant \mathcal{W}(\tilde{A},\tilde{\theta}, f(t)). \tag{5.3b}$$

Remark 5.4. Let $f\colon [0,T] \to H^{-1}(\Omega,\mathbb{R}^s)$. An evolution

$$[0,T] \ni t \mapsto D(t) \in \mathbb{P}(\Omega)^m$$

is a strong energy-minimising evolution if the evolution

$$[0,T] \ni t \mapsto (\alpha_{D(t)},\chi_{D(t)}) \in L^\infty(\Omega,\mathcal{M}(c_1,c_2)) \times [0,1]^m$$

is a weak energy-minimising evolution. (Pick the sequence $\{D^n(t)\}$ in Definition 5.3(3) to be a constant sequence.)

Next we show that weak energy-minimising evolutions exist:

Theorem 5.5. *For every $f \in W^{1,1}([0,T], H^{-1}(\Omega,\mathbb{R}^s))$, there exists a weak energy-minimising evolution.*

Our proof follows [4,9] (see also [15]).

Proof. Given $n \in \mathbb{N}$ we set $N := \lceil \frac{T}{n} \rceil$ and consider a partition $\{t_j^n\}_{j=0}^N$ of $[0, T]$ such that

$$t_0^n = 0,$$
$$t_N^n = T,$$
$$\Delta t^n := t_j^n - t_{j-1}^n \leqslant \frac{1}{n}, \quad j = 1, \dots, N.$$

We set $f_j^n(\cdot) := f(t_j^n, \cdot)$ and let f^n be the piecewise constant (in time) approximation of f given by

$$f^n(t, \cdot) := f(t_j^n, \cdot) \quad \text{for } t \in [t_j^n, t_{j+1}^n), \quad j = 1, \dots, N.$$

Note that $f^n(t) \to f(t)$ in $H^{-1}(\Omega, \mathbb{R}^s)$.

We construct a piecewise constant approximation of the solution starting from the almost minimizers of an appropriate incremental variational problem:
Step 1: The first time step $t_0^n = 0$. At the first time step we almost minimize $\mathcal{V}(v, \chi, f(0))$ over $(v, \chi) \in H_0^1(\Omega, \mathbb{R}^s) \times L^\infty(\Omega, \mathbb{D}^m)$ by choosing a sequence of subsets $D^{0,l,n} \subset \mathbb{P}(\Omega)^m$ such that

$$\min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D^{0,l,n}, f(0)) \leqslant \inf_D \min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D, f(0)) + \frac{1}{2l}.$$

Step 2: The subsequent time steps. For every $j \in \{1, ..., N\}$ we choose a sequence $\{D^{j,l,n}\} \subset \mathbb{P}(\Omega)^m$ with $D^{j,l,n} \supseteq D^{j-1,l,n}$ such that

$$\min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D^{j,l,n}, f_j^n) \leqslant \inf_{D \supseteq D^{j-1,l,n}} \min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D, f_j^n) + \frac{1}{2^{j+1} l}.$$

Step 3: The discrete approximation. Let

$$D^{l,n}(t) := D^{j,l,n} \quad \text{for } t \in [\, t_j^n, t_{j+1}^n), \quad j = 1, \dots, N.$$

Note that $D^{l,n}(t)$ is piecewise constant (in time) and non-decreasing in t. We extract a subsequence in l (not relabelled) such that for every $t \in [0, T]$ we have, as $l \to \infty$,

$$\alpha_{D^{l,n}(t)} \overset{G}{\to} A^n(t),$$
$$\chi_{D^{l,n}(t)} \overset{\star}{\rightharpoonup} \theta^n(t),$$

for some $A^n(t) \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$, non-increasing in t, and some $\theta^n(t) \in L^\infty(\Omega, [0, 1]^m)$, non-decreasing in t, satisfying

$$A^n(t, x) \in \mathcal{G}_{\theta^n(t,x)}(\boldsymbol{\alpha})$$

a.e. in Ω.

Thus (see, e.g., [4, Theorem 3.1 and Remark 3.3]) we can now extract a subsequence in n (not relabelled) such that for every $t \in [0, T]$ we have, as $n \to \infty$,

$$A^n(t) \overset{G}{\to} A(t),$$
$$\theta^n(t) \overset{\star}{\rightharpoonup} \theta(t),$$

for some $A(t) \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$, non-increasing in t, and some $\theta(t) \in L^\infty(\Omega, [0, 1]^m)$, non-decreasing in t, satisfying

$$A(t, x) \in \mathcal{G}_{\theta(t,x)}(\boldsymbol{\alpha})$$

a.e. in Ω.

Finally by a diagonal argument we can find a sequence $l(n)$ with $l(n) \to \infty$ as $n \to \infty$ such that the sequence $D^n(t) := D^{l(n),n}(t)$ satisfies, as $n \to \infty$,

$$\alpha_{D^n(t)} \xrightarrow{G} A(t),$$
$$\chi_{D^n(t)} \overset{\star}{\rightharpoonup} \theta(t),$$

for every $t \in [0, T]$.

Note that, by construction, $A(\cdot)$ is non-increasing and $\theta(\cdot)$ is non-decreasing. Thus Definition 5.3(1) is satisfied.

Step 4: Minimality. Fix $t \in [0, T]$ and consider $\tilde{D}^n \supset D^n(t)$ such that

$$\begin{aligned} \alpha_{\tilde{D}^n} &\xrightarrow{G} \tilde{A}, \\ \chi_{\tilde{D}^n} &\overset{\star}{\rightharpoonup} \tilde{\theta}. \end{aligned} \tag{5.4a}$$

By the definition of $D^n(t)$ we have

$$\min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D^n(t), f^n(t)) \leqslant \min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, \tilde{D}^n, f^n(t)) + o(1)$$

where $o(1) \to 0$ as $n \to \infty$. Using the definition of G-convergence and the convergence of $f^n(t)$ to $f(t)$ in $H^{-1}(\Omega, \mathbb{R}^s)$, we obtain

$$\min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \widetilde{\mathcal{V}}(v, A(t), \theta(t), f(t)) \leqslant \min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \widetilde{\mathcal{V}}(v, \tilde{A}, \tilde{\theta}, f(t)). \tag{5.4b}$$

From (5.4) we deduce (5.3). Thus Definition 5.3(3) is satisfied.

Step 5: Energy Balance. From the definition of $D^{j,l,n}$ in Step 2,

$$\begin{aligned} &\min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D^{j,l,n}, f_j^n) \\ &\leqslant \inf_{D \supset D^{j-1,l,n}} \min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D, f_j^n) + \frac{1}{2^{j+1} l} \\ &\leqslant \mathcal{V}(v^{j-1,l,n}, D^{j-1,l,n}, f_j^n) + \frac{1}{2^{j+1} l} \\ &\leqslant \min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D^{j-1,l,n}, f_{j-1}^n) + (f_{j-1}^n - f_j^n) \cdot v^{j-1,l,n} + \frac{1}{2^{j+1} l}. \end{aligned}$$

Iterating, we obtain

$$\begin{aligned} &\min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D^{j,l,n}, f_j^n) \\ &\qquad \leqslant \min_{v \in H_0^1(\Omega, \mathbb{R}^s)} \mathcal{V}(v, D^{0,l,n}, f(0)) + \sum_{k=1}^{j} \left(f_{k-1}^n - f_k^n\right) \cdot v^{k-1,l,n} + \frac{1}{l}. \end{aligned}$$

By the definition of $D^n(t)$ in Step 3, for every $t \in [t^n_j, t^n_{j+1})$ we get

$$\begin{aligned}&\mathcal{V}(v^n(t), D^n(t), f^n(t))\\&\leqslant \mathcal{V}(v^n(0), D^n(0), f(0)) - \sum_{k=1}^{j}\left(f^n_k - f^n_{k-1}\right)\cdot u(\alpha_{D^n(t^n_{k-1})}, f^n(t^n_{k-1})) + o(1).\end{aligned} \tag{5.5}$$

Notice that, as $n \to \infty$, for a.e. τ,

$$\frac{f(\tau + \Delta t^n) - f(\tau)}{\Delta t^n} \overset{H^{-1}}{\to} \dot{f}(\tau)$$

(since $\Delta t^n \leqslant \frac{1}{n}$). Thus, as $n \to \infty$, (5.5) yields

$$\mathcal{W}(t) \leqslant \mathcal{W}(0) - \int_0^t \dot{f}(\tau)\cdot u(\tau)\,\mathrm{d}\tau.$$

The proof of the inverse inequality is standard and based on the fact that from the construction of $A(t)$ it is easy to check that $(A(t), \theta(t))$ is an admissible competitor for the minimality condition at time $s < t$. In fact this is a particular case of inequality (6.22) which is proved below. Thus Definition 5.3(2) is satisfied. □

6. Relation between Threshold and Energy Formulations

Theorems 6.1 and 6.2 below show that energy-minimising evolutions are also threshold evolutions:

Theorem 6.1. *For a material satisfying Property 3.5, a weak energy-minimising evolution with damage cost $k \in \mathbb{R}^m$ is a weak threshold evolution with threshold $\lambda \in \mathbb{R}^m$ satisfying*

$$k_i = \frac{1}{2}\frac{\beta_i\,\Delta\beta_i}{\beta_i - \Delta\beta_i}\lambda_i^2, \quad i \in \mathbf{M}. \tag{6.1}$$

Before we prove Theorem 6.1 we motivate (6.1) with a one-dimensional example (Section 6.1) and state several ancillary results which we will use in the proof (Section 6.2).

A similar (but simpler) proof shows that:

Theorem 6.2. *For a material satisfying Property 3.5, a strong energy-minimizing evolution with damage cost $k \in \mathbb{R}^m$ is a strong threshold evolution with threshold $\lambda \in \mathbb{R}^m$ satisfying (6.1).*

6.1. A One-dimensional Example

The following example expands on [9, Remark 1].

Let $d = 1$, $\Omega = (0, 1)$ and $m = 1$. In this one-dimensional setting the strain corresponding to a displacement $H^1((0, 1), \mathbb{R})$ is $Du \in L^2((0, 1), \mathbb{R})$. It is simpler to consider a Dirichlet problem so we set $f \equiv 0$ and allow $u(1) \neq 0$.

Property 3.5 is trivially satisfied: Using the notation of (3.4b) and (3.4c), the undamaged elastic modulus is

$$\alpha_0 = \beta_1 > 0,$$

and the damaged elastic modulus is

$$\alpha_1 = \alpha_0 - \Delta\alpha_1 = \beta_1 - \Delta\beta_1 > 0.$$

Threshold formulation. From the threshold criterion for damage (Formulation 3.3), damage occurs at $x \in (0, 1)$ when $|Du(x)| > \lambda$. The equilibrium equation (1.1) yields

$$\alpha(x)\; Du(x) = \sigma,$$

for some constant σ. Thus we conclude:

$$Du(x) = \begin{cases} \frac{\sigma}{\beta_1} & \text{in the undamaged region,} \\ \frac{\sigma}{\beta_1 - \Delta\beta_1} & \text{in the damaged region,} \end{cases}$$

with

$$\frac{|\sigma|}{\beta_1 - \Delta\beta_1} > \lambda \geqslant \frac{|\sigma|}{\beta_1}.$$

Suppose $\frac{|\sigma|}{\beta_1} \leqslant \lambda$. Then the material is undamaged and we obtain

$$u(x) = \frac{\sigma}{\beta_1} x;$$

in particular

$$u(1) = \frac{\sigma}{\beta} \in (-\lambda, \lambda).$$

On the other hand when $|u(1)| > \lambda$, say $u(1) > \lambda$, the material is necessarily damaged in some $D \subset (0, 1)$. With no loss of generality we set $D = (0, |D|)$. Thus for a (different) constant σ,

$$Du(x) = \begin{cases} \frac{\sigma}{\beta_1 - \Delta\beta_1} & x \in (0, |D|), \\ \frac{\sigma}{\beta_1} & x \in (|D|, 1), \end{cases} \tag{6.2a}$$

with

$$\frac{\sigma}{\beta_1 - \Delta\beta_1} > \lambda \geqslant \frac{\sigma}{\beta_1}. \tag{6.2b}$$

Integrating, we obtain

$$u(x) = \begin{cases} \frac{\sigma}{\beta_1 - \Delta\beta_1} x & x \in (0, |D|), \\ \frac{\sigma}{\beta_1}(x - |D|) + \frac{\sigma}{\beta_1 - \Delta\beta_1}|D| & x \in (|D|, 1). \end{cases}$$

Thus

$$|D| = \left(\frac{u(1)}{\sigma} - \frac{1}{\beta_1}\right) \frac{\beta_1(\beta_1 - \Delta\beta_1)}{\Delta\beta_1}. \tag{6.3}$$

This is minimised when $\sigma = \lambda\beta_1$ which, from (6.2b), is the largest possible value of σ. Thus the optimal choice of $|D|$ is

$$|D| = \left(\frac{u(1)}{\lambda} - 1\right) \frac{\beta_1 - \Delta\beta_1}{\Delta\beta_1}.$$

Essentially the same holds when $u(1) < -\lambda$. Thus we conclude

$$|D| = \begin{cases} 0 & \text{if } |u(1)| \leqslant \lambda, \\ \left(\frac{|u(1)|}{\lambda} - 1\right) \frac{\beta_1 - \Delta\beta_1}{\Delta\beta_1} & \text{if } |u(1)| > \lambda \end{cases} \tag{6.4a}$$

and, from (6.2a), the optimal u satisfies

$$|Du(x)| = \begin{cases} \frac{\beta_1}{\beta_1 - \Delta\beta_1}\lambda & x \in (0, |D|), \\ \lambda & x \in (|D|, 1). \end{cases} \tag{6.4b}$$

Energetic formulation. Let us now consider the same situation from the energetic perspective (Formulation 3.4). The energy density is

$$W(\cdot) = \begin{cases} \frac{1}{2}\beta_1 \cdot^2 & \text{in the undamaged region} \\ \frac{1}{2}(\beta_1 - \Delta\beta_1) \cdot^2 + k & \text{in the damaged region} \end{cases}$$

When the material is undamaged its energy is

$$E_0 := \inf_{\substack{u \in H^1((0,1),\mathbb{R}) \\ u(0)=0}} \int_0^1 \frac{1}{2}\beta_1 \left(Du(x)\right)^2 \, dx = \frac{1}{2}\beta_1 u(1)^2. \tag{6.5}$$

On the other hand, when there is damage, with no loss of generality setting the damage set to be $(0, |D|) \subset (0, 1)$ as before, we obtain the optimal energy to be

$$\begin{aligned} E_1 &:= \inf_{|D|} \inf_{\substack{u \in H^1((0,1),\mathbb{R}) \\ u(0)=0}} \left(\int_0^{|D|} \left(\frac{1}{2}(\beta_1 - \Delta\beta_1)\left(Du(x)\right)^2 + k\right) dx + \int_{|D|}^1 \frac{1}{2}\beta_1 \left(Du(x)\right)^2 \, dx \right) \\ &= \inf_{|D|} \min_{\substack{v_1, v_2 \in R \\ |D|v_1 + (1-|D|)v_2 = u(1)}} \left(\frac{1}{2}(\beta_1 - \Delta\beta_1)v_1^2|D| + \frac{1}{2}\beta_1 v_2^2(1 - |D|)\right) + k|D| \\ &= \inf_{|D|} \frac{1}{2} \frac{(\beta_1 - \Delta\beta_1)\beta_1}{|D|\beta_1 + (1 - |D|)(\beta_1 - \Delta\beta_1)} u(1)^2 + k|D| \\ &= \beta_1 \sqrt{\frac{2(\beta_1 - \Delta\beta_1)k}{\beta_1 \Delta\beta_1}} u(1) - \frac{\beta_1 - \Delta\beta_1}{\Delta\beta_1} k \end{aligned} \tag{6.6}$$

where the optimal value of $|D|$ is

$$\left(\sqrt{\frac{\beta_1 \Delta\beta_1}{2(\beta_1 - \Delta\beta_1)k}}u(1) - 1\right)\frac{\beta_1 - \Delta\beta_1}{\Delta\beta_1}.$$

Imposing $|D| \geqslant 0$ we obtain

$$u(1) \geqslant \sqrt{\frac{2(\beta_1 - \Delta\beta_1)k}{\beta_1 \Delta\beta_1}} \tag{6.7}$$

as a necessary condition for damage. It is easy to verify from (6.5) and (6.6) that $E_1 \leqslant E_0$. Thus damage occurs whenever (6.7) is satisfied with strict inequality. We conclude:

$$|D| = \begin{cases} 0 & \text{if } |u(1)| \leqslant \sqrt{\frac{2(\beta_1 - \Delta\beta_1)k}{\beta_1 \Delta\beta_1}}, \\ \left(\sqrt{\frac{\beta_1 \Delta\beta_1}{2(\beta_1 - \Delta\beta_1)k}}u(1) - 1\right)\frac{\beta_1 - \Delta\beta_1}{\Delta\beta_1} & \text{if } |u(1)| > \sqrt{\frac{2(\beta_1 - \Delta\beta_1)k}{\beta_1 \Delta\beta_1}}. \end{cases} \tag{6.8}$$

Relating the threshold and energetic formulations. Comparing (6.4a) and (6.8) we see that the two formulations agree precisely when

$$k = \frac{\beta_1 \Delta\beta_1}{2(\beta_1 - \Delta\beta_1)}\lambda^2 \tag{6.9}$$

and (6.1) is the natural extension of this to multiple non-interacting (see Property 3.5(4)) damage processes.

For future reference we note that, from (6.6) and (6.9),

$$\begin{aligned} E_1 &= \beta_1 \lambda u(1) - \frac{1}{2}\beta_1 \lambda^2, \\ E_0 - E_1 &= \frac{1}{2}\beta_1 \left(u(1) - \lambda\right)^2. \end{aligned} \tag{6.10}$$

6.2. Ancillary Results

Remark 6.3, Lemmas 6.4 and 6.5 below prepare the way for the proof of Theorem 6.1.

Remark 6.3. (*See* [9] *Remark* 13) For $R > 0$ let Q_R be the cube in $\mathbb{R}^s$ with side R centred at 0 and oriented along the coordinate axis. Let $\{u^n\}$, $\{v^n\}$ be sequences bounded in $H^1(Q_1, \mathbb{R}^s)$ such that $u^n - v^n \to 0$ in $L^2(Q_1, \mathbb{R}^s)$. For $0 < R_1 < R_2 < 1$ let $\phi_{(R_1,R_2)}$ be a cutoff function satisfying

$$\phi_{(R_1,R_2)} = \begin{cases} 0 & \text{in } Q_{R_1}, \\ 1 & \text{in } Q_1 \setminus Q_{R_2}; \end{cases}$$

$$\|\nabla\phi_{(R_1,R_2)}\| = \frac{1}{R_2 - R_1} \quad \text{on } Q_{R_2} \setminus Q_{R_1}.$$

Then, there exists $R_2 \in (0,1)$ such that for every $R_1 \in (0, R_2)$,

$$\lim_{R_1 \to R_2} \lim_{n \to \infty} \int_{Q_{R_2} \setminus Q_{R_1}} \|\nabla(\phi_{(R_1,R_2)} u^n + (1 - \phi_{(R_1,R_2)}) v^n)\|^2 \, \mathrm{d}x = 0,$$
$$\lim_{R_1 \to R_2} \lim_{n \to \infty} \int_{Q_{R_2}} \|\nabla(\phi_{(R_1,R_2)}(u^n - v^n))\|^2 \, \mathrm{d}x = 0.$$

It suffices to choose R_1 so as to avoid concentrations in $\|\nabla u^n\|^2$ and $\|\nabla v^n\|^2$ on ∂Q_{R_2}.

Lemma 6.4. *Let $f \in H^{-1}(\Omega, \mathbb{R}^s)$, $D \in \mathbb{P}(\Omega)^m$, $\delta > 0$ and $i \in \mathbf{M}$. Assume*

$$U_i := \{x \notin D_i \mid \|\Lambda_i e(D, f)(x)\| > \lambda_i + \delta\}$$

has positive measure. For brevity we write u for $u(D, f)$.

Let $\epsilon > 0$ and $Q_i^\epsilon \subset \mathbb{R}^s$ be a cube with centre $x \in U_i$ satisfying the following properties:

1. x is a Lebesgue point for D_j, $\forall j \in \mathbf{M}$ and χ_{U_i}, u and ∇u (and thus $e(D, f)$). For brevity we write e_x for $e(D, f)(x)$.
2. Two sides of Q_i^ϵ are orthogonal to $\hat{n} \in \mathbb{R}^s$ where the unit vector $\hat{n}$ satisfies $\Lambda_i e_x = \bar{m} \otimes_s \hat{n}$ for some $\bar{m} \in \mathbb{R}^s$ (see Property 3.5(1)).
3. For $y \in Q_i^\epsilon$, let

   $$u_{\text{affine}}(y) := (y - x) \cdot \hat{n}\, \bar{m} + (\Lambda_i^\perp e_x + \nabla u(x) - \nabla^T u(x)) y + u(x)$$

   with $\bar{m}$ as in (2) above. Observe that

   $$\Lambda_i e(u_{\text{affine}})(y) = \bar{m} \otimes_s \hat{n} = \Lambda_i e_x.$$

   Then
   $$\|u - u_{\text{affine}}\|^2_{H^1(Q_i^\epsilon, \mathbb{R}^s)} \leqslant \epsilon |Q_i^\epsilon|.$$

4. The intersections of Q_i^ϵ with D satisfy:
   (a) $|D_i \cap Q_i^\epsilon| \leqslant \epsilon |Q_i^\epsilon|$.
   (b) For $j \neq i$, either $|D_j \cap Q_i^\epsilon| \leqslant \epsilon |Q_i^\epsilon|$ or $|D_j \cap Q_i^\epsilon| \geqslant (1 - \epsilon)|Q_i^\epsilon|$.

Then, for a material satisfying Property 3.5 and (6.1),

$$\inf_{\substack{v - u(D,f) \in H_0^1(Q_i^\epsilon, \mathbb{R}^s) \\ E_i \subseteq Q_i^\epsilon \\ E_j = D_j \text{ for } j \neq i}} \mathcal{V}_{Q_i^\epsilon}(v, E, f)$$
$$\leqslant \mathcal{V}_{Q_i^\epsilon}(u(D, f), D, f)|_{D_i = \emptyset} - \frac{1}{2}\beta_i \delta^2 |Q_i^\epsilon| + o(\epsilon)|Q_i^\epsilon| \quad (6.11)$$

where $o(\epsilon) \to 0$ as $\epsilon \to 0$.

Equation (6.11) states that, as $\epsilon \to 0$,

$$\frac{1}{|Q_i^\epsilon|}\mathcal{V}_{Q_i^\epsilon}(u(D,f),D,f)|_{D_i=\emptyset}-\left(\inf_{\substack{v-u(D,f)\in H_0^1(Q_i^\epsilon,\mathbb{R}^s)\\ E_i\subseteq Q_i^\epsilon\\ E_j=D_j \text{ for } j\neq i}}\frac{1}{|Q_i^\epsilon|}\mathcal{V}_{Q_i^\epsilon}(v,E,f)\right)\geqslant\frac{1}{2}\beta_i\delta^2. \tag{6.12}$$

In other words, wherever (i) the threshold for i-damage is exceeded by at least $\delta > 0$ and (ii) the material is not i-damaged, the reduction in energy density that can be achieved by introducing i-damage is at least $\frac{1}{2}\beta_i\delta^2$ where β_i is the (essentially scalar, see (3.5)) undamaged elastic modulus on the i^{th} damage subspace. (Compare with (6.10).)

Proof. *Step 1:* First we construct a test sequence of displacements and damage sets ($\{u^k_{\text{test}}\}\subset H^1(Q_i^\epsilon,\mathbb{R}^s)$ and $\{D^k_{\text{test}}\}\subset Q_i^\epsilon$ below) that reduce the energy in Q_i^ϵ; this construction is motivated by the example in Section 6.1:

We set

$$\delta' := \|\Lambda_i e_x\| - \lambda_i, \tag{6.13a}$$

$$\gamma := \frac{\Delta\beta_i}{\beta_i - \Delta\beta_i}, \tag{6.13b}$$

$$d' := \frac{1}{\gamma\lambda_i}\delta'. \tag{6.13c}$$

Let $\hat{m}\in\mathbb{R}^s$ be parallel to $\bar{m}$ but normalised such that $\|\hat{m}\otimes_s\hat{n}\| = 1$. Let $\hat{e} := \hat{m}\otimes_s\hat{n}$. Then $\Lambda_i e_x = (\lambda_i+\delta')\hat{e}$.

Let ψ be the (unique) Lipschitz function on $\mathbb{R}$, with $\psi(0) = 0$ and 1-periodic derivative $D\psi$ given by

$$D\psi(y) := \begin{cases}(1+\gamma)\lambda_i & \text{if } y\in(0,d'),\\ \lambda_i & \text{if } y\in(d',1).\end{cases} \tag{6.14}$$

(Compare (6.14) and (6.13) with (6.4b).)

Note that $\int_0^1 D\psi(y)\,\mathrm{d}y = \lambda_i+\delta'$. For $k\in\mathbb{N}$ and $y\in Q_i^\epsilon$ we set

$$u_{\text{test}}(y) := \psi(y\cdot\hat{n})\hat{m} + (\Lambda_i^\perp e_x + \nabla u - \nabla^T u)y + u(x),$$
$$u^k_{\text{test}}(y) := u_{\text{test}}(ky).$$

Then, for $y\in Q_i^\epsilon$,

$$e(u_{\text{test}})(y) = \Lambda_i^\perp e_x + \begin{cases}(1+\gamma)\lambda_i\hat{e} & \text{if } y\cdot\hat{n}\in\mathbb{Z}+(0,d'),\\ \lambda_i\hat{e} & \text{if } y\cdot\hat{n}\in\mathbb{Z}+(d',1).\end{cases}$$

Let

$$D^k_{\text{test}} := \left\{y\in Q_i^\epsilon \mid \Lambda_i e(u^k_{\text{test}})(y) = (1+\gamma)\lambda_i\hat{e}\right\}.$$

Step 2: Next we show (6.12) for the sequence $\{u^k_{\text{test}}\}$—that is, for functions that agree with u_{affine} (as opposed to u) on ∂Q^ϵ_i—and $\{D^k_{\text{test}}\}$.

As $k \to \infty$, $u^k_{\text{test}} \to u_{\text{affine}}$ in $L^2(Q^\epsilon_i, \mathbb{R}^s)$ and is bounded in $H^1(Q^\epsilon_i, \mathbb{R}^s)$. Thus, from Remark 6.3 there exists $u_{\text{test}} \in L^2(Q^\epsilon_i, \mathbb{R}^s)$ that agrees with u_{affine} on ∂Q^ϵ_i and a corresponding D_{test} for which $\mathcal{V}_{Q^\epsilon_i}(u_{\text{test}}, D_{\text{test}}, f)$ is arbitrarily close to

$$\frac{1}{2}d'\left\langle(\alpha_d - \Delta\alpha_i)(\Lambda^\perp_i e_x + (1+\gamma)\lambda_i\hat{e}), (\Lambda^\perp_i e_x + (1+\gamma)\lambda_i\hat{e})\right\rangle \\ + \frac{1}{2}(1-d')\left\langle\alpha_d(\Lambda^\perp_i e_x + \lambda_i\hat{e}), (\Lambda^\perp_i e_x + \lambda_i\hat{e})\right\rangle - \int_{Q^\epsilon_i} f \cdot u_{\text{affine}} + k_i d' \,\mathrm{d}x$$

where $d \in \mathbb{D}^m$ is defined by

$$d_j := \begin{cases} 0 & \text{if } |D_j \cap Q^\epsilon_i| \leqslant |Q^\epsilon_i|, \\ 1 & \text{if } |D_j \cap Q^\epsilon_i| \geqslant (1-\epsilon)|Q^\epsilon_i|, \end{cases} \qquad j \in \mathbf{M};$$

(thus $d_i = 0$). It follows that

$$\frac{1}{Q^\epsilon_i}\left(\mathcal{V}_{Q^\epsilon_i}(u, D, f) - \mathcal{V}_{Q^\epsilon_i}(u_{\text{test}}, D_{\text{test}}, f)\right) \tag{6.15}$$

is arbitrarily close to

$$\frac{1}{2}\left\langle\alpha_d\left(\Lambda^\perp_i e_x + (\lambda_i + \delta')\hat{e}\right), \left(\Lambda^\perp_i e_x + (\lambda_i + \delta')\hat{e}\right)\right\rangle \\ - \frac{1}{2}d'\left\langle(\alpha_d - \Delta\alpha_i)\left(\Lambda^\perp_i e_x + (1+\gamma)\lambda_i\hat{e}\right), \left(\Lambda^\perp_i e_x + (1+\gamma)\lambda_i\hat{e}\right)\right\rangle \\ - \frac{1}{2}(1-d')\left\langle\alpha_d\left(\Lambda^\perp_i e_x + \lambda_i\hat{e}\right), \left(\Lambda^\perp_i e_x + \lambda_i\hat{e}\right)\right\rangle - k_i d'. \tag{6.16}$$

We now invoke Property 3.5: A simple calculation shows that (6.16) is

$$\frac{1}{2}\delta'(\delta' - \lambda_i\gamma)\langle\alpha_d\hat{e}, \hat{e}\rangle + \frac{1}{2}d'(1+\gamma)^2\lambda_i^2\langle\Delta\alpha_i\hat{e}, \hat{e}\rangle + \frac{1}{2}d'\langle\Delta\alpha_i\Lambda^\perp_i e_x, \Lambda^\perp_i e_x\rangle \\ + d'(1+\gamma)\lambda_i\langle\Delta\alpha_i\Lambda^\perp_i e_x, \hat{e}\rangle - k_i d'.$$

Moreover, the last two terms vanish because of (3.4c); and from (3.4b), (3.4c) and the definition of $\hat{e}$,

$$\langle\alpha_d\hat{e}, \hat{e}\rangle = \beta_i, \\ \langle\Delta\alpha_i\hat{e}, \hat{e}\rangle = \Delta\beta_i.$$

Using these and (6.13) we obtain that (6.15) is arbitrarily close to

$$\frac{1}{2}\beta_i(\delta')^2 + \frac{1}{2}\beta_i\lambda_i^2\delta'\left(1 - \frac{k_i}{\lambda_i^2}\frac{2(\beta_i - \Delta\beta_i)}{\beta_i\ \Delta\beta_i}\right)$$

When (6.1) holds this equals $\frac{1}{2}\beta_i(\delta')^2$.

Step 3: Finally we observe that this conclusion holds for functions that agree with u (as opposed to u_{affine}) on ∂Q_i^ϵ. To see this we make the test sequences constructed in Step 1 admissible by adding $u(t) - u_{\text{affine}}$ to them. Notice, from properties (3) and (4) of Q_i^ϵ, that

$$\left| \mathcal{V}_{Q_i^\epsilon}(u, D, f) - \mathcal{V}_{Q_i^\epsilon}(u_{\text{affine}}, D, f)|_{D_i=\emptyset} \right| \leqslant o(\epsilon)|Q_i^\epsilon|$$

where $o(\epsilon) \to 0$ as $\epsilon \to 0$. Thus the error introduced in the modified (admissible) test sequences is $o(\epsilon)|Q_i^\epsilon|$, with $o(\epsilon) \to 0$ as $\epsilon \to 0$. Equation (6.11) follows immediately. □

Lemma 6.5, extends the corresponding scalar result in [9]:

Lemma 6.5. *Let $D \in \mathbb{P}(\Omega)^m$, $i \in \mathbf{M}$ and $S \subset \Omega \setminus D_i$.*

Define $\Delta D \in \mathbb{P}(\Omega)^m$ by

$$\Delta D_j := \begin{cases} S & \text{if} \quad j = i, \\ \emptyset & \text{if} \quad j \neq i. \end{cases}$$

Then, for a material satisfying Property 3.5,

$$\mathcal{E}(D, f) - \mathcal{E}(D \cup \Delta D, f) \leqslant \frac{\beta_i \ \Delta\beta_i}{2(\beta_i - \Delta\beta_i)} \|\Lambda_i e(D, f)\|^2_{L^2(S)}. \tag{6.17a}$$

When (6.1) holds this yields,

$$\mathcal{E}(D, f) - \mathcal{E}(D \cup \Delta D, f) \leqslant \frac{k_i}{\lambda_i^2} \|\Lambda_i e(D, f)\|^2_{L^2(S)}. \tag{6.17b}$$

In other words, wherever the material is not i-damaged, the maximum reduction in elastic energy that can be achieved by introducing i-damage is given by (6.17).

Our proof is based on [9] which in turn follows [11].

Proof. For brevity we set

$$\begin{aligned}
\phi &:= u(D \cup \Delta D, f) - u(D, f), \\
e_\phi &:= e(\phi) = e(D \cup \Delta D, f) - e(D, f), \\
e_D &:= e(D, f), \\
\Delta\alpha &:= \alpha_D - \alpha_{D \cup \Delta D}, \\
\Delta\mathcal{E} &:= \mathcal{E}(D, f) - \mathcal{E}(D \cup \Delta D, f).
\end{aligned}$$

From (1.1), self-adjointness and the divergence theorem,

$$\int_\Omega \langle \alpha_D e_D, e_\phi \rangle \, \mathrm{d}x = \int_\Omega f \cdot \phi \, \mathrm{d}x.$$

Using this we easily obtain

$$\begin{aligned}\Delta\mathcal{E} &= \frac{1}{2}\int_S \langle \Delta\alpha_i\, e_D, e_D\rangle \,\mathrm{d}x + \int_S \langle \Delta\alpha_i\, e_D, e_\phi\rangle \,\mathrm{d}x \\ &\quad -\frac{1}{2}\int_S \langle (\alpha_D - \Delta\alpha_i) e_\phi, e_\phi\rangle \,\mathrm{d}x - \frac{1}{2}\int_{\Omega\setminus S} \langle \alpha_D e_\phi, e_\phi\rangle \,\mathrm{d}x \\ &\leqslant \frac{1}{2}\int_S \langle \Delta\alpha_i\, e_D, e_D\rangle \,\mathrm{d}x + \int_S \langle \Delta\alpha_i\, e_D, e_\phi\rangle \,\mathrm{d}x \\ &\quad -\frac{1}{2}\int_S \langle (\alpha_D - \Delta\alpha_i) e_\phi, e_\phi\rangle \,\mathrm{d}x. \end{aligned} \tag{6.18a}$$

We now invoke Property 3.5: using (3.4c) the first two terms become

$$\frac{1}{2}\Delta\beta_i \int_S \langle \Lambda_i e_D, e_D\rangle \,\mathrm{d}x + \Delta\beta_i \int_S \langle \Lambda_i e_D, e_\phi\rangle \,\mathrm{d}x$$

and the third term is

$$\begin{aligned} &-\frac{1}{2}\int_S \langle \alpha_D \Lambda_i e_\phi, \Lambda_i e_\phi\rangle \,\mathrm{d}x - \int_S \langle \alpha_D \Lambda_i e_\phi, \Lambda_i^\perp e_\phi\rangle \,\mathrm{d}x \\ &\qquad\qquad -\frac{1}{2}\int_S \langle \alpha_D \Lambda_i^\perp e_\phi, \Lambda_i^\perp e_\phi\rangle \,\mathrm{d}x + \frac{1}{2}\Delta\beta_i \int_S \langle \Lambda_i e_\phi, e_\phi\rangle \,\mathrm{d}x. \end{aligned}$$

Since, by hypothesis, $D_i \cap S = \emptyset$, from (3.4b) the first term here is

$$-\frac{1}{2}\beta_i \int_S \langle \Lambda_i e_\phi, e_\phi\rangle \,\mathrm{d}x.$$

Using (3.4d) also, we see that the second term is zero. Since the third term is non-positive, we obtain,

$$\begin{aligned}\Delta\mathcal{E} &\leqslant \frac{1}{2}\int_S \Delta\beta_i \langle \Lambda_i e_D, e_D\rangle \,\mathrm{d}x + \int_S \Delta\beta_i \langle \Lambda_i e_D, e_\phi\rangle \,\mathrm{d}x \\ &\quad -\frac{1}{2}(\beta_i - \Delta\beta_i)\int_S \langle \Lambda_i e_\phi, e_\phi\rangle \,\mathrm{d}x. \end{aligned} \tag{6.18b}$$

Since Λ_i is an orthogonal projection,

$$\begin{aligned}\Delta\mathcal{E} &\leqslant \frac{1}{2}\Delta\beta_i \|\Lambda_i e_D\|^2_{L^2(S)} + \Delta\beta_i \|\Lambda_i e_D\|_{L^2(S)} \|\Lambda_i e_\phi\|_{L^2(S)} \\ &\quad -\frac{1}{2}(\beta_i - \Delta\beta_i)\|\Lambda_i e_\phi\|^2_{L^2(S)}. \end{aligned}$$

The right hand side is quadratic in $\|\Lambda_i e_\phi\|_{L^2(S)}$ and has a maximum at

$$\|\Lambda_i e_\phi\|_{L^2(S)} = \frac{\Delta\beta_i}{\beta_i - \Delta\beta_i}\|\Lambda_i e_D\|_{L^2(S)}.$$

This yields (6.17a). □

Remark 6.6. Note that (6.18a) is sharp when $\Delta\mathcal{E}$ is replaced by

$$\Delta\mathcal{E}_S := \mathcal{E}_S(D, f) - \mathcal{E}_S(D \cup \Delta D, f).$$

If, in addition, $\Lambda_i^\perp e_\phi = 0$ then (6.18b) is sharp as well.

We are now ready to prove Theorem 6.1.

6.3. Proof of Theorem 6.1

By Definition 5.3(3) there exists a sequence $\{D^n(t)\} \subset \mathbb{P}(\Omega)^m$ such that $t \mapsto D_i^n(t)$ is non-decreasing. Thus Property 1 of Definition 4.3 is trivially satisfied. It remains to show that $\{D^n(t)\}$ also satisfies Properties 2 and 3 of Definition 4.3.

If Property 2 is false, then there exists $i \in \mathbf{M}$ and $\delta > 0$ such that the sets $U_i^n(\delta)$ in (4.3) do not eventually have small measure. We can then localise to nice points in $U_i^n(\delta)$ and add regions of damage as in Lemma 6.4. This creates a competitor $\tilde{D}^n$ to D^n, whose energy is lower on the order $\limsup_{n\to\infty} |U_i^n(\delta)|$, contradicting the minimality of D^n (Definition 5.3(3)).

Proof. (*that Definition* 4.3(2) *is satisfied*) Suppose, on the contrary, that there exists $i \in \mathbf{M}$ and $\delta > 0$ such that $U_i^n(\delta)$ in (4.3) satisfy

$$\limsup_{n\to\infty} |U_i^n(\delta)| = \gamma > 0. \tag{6.19}$$

Note that for every $\epsilon > 0$ the set of all cubes Q_ϵ^n that satisfy the four conditions listed in Lemma 6.4 is a fine covering of $U_i^n(\delta)$ (except possibly for a set of measure zero). Therefore for every $\epsilon > 0$ and $n \in \mathbb{N}$ we can choose a countable collection of disjoint cubes $\{Q_\epsilon^n(j) \mid j \in \mathbb{N}\}$ such that

$$\left| U_i^n(\delta) \setminus \bigcup_{j\in\mathbb{N}} Q_\epsilon^n(j) \right| = 0.$$

Using Lemma 6.4 we construct a competitor $(\tilde{u}^n, \tilde{D}^n)$ for $(u^n(t), D^n(t))$ with $\tilde{D}^n \in \mathbb{P}(\Omega)^m$ satisfying

$$\tilde{D}_j^n \begin{cases} \supset D_i^n(t) & \text{if } j = i, \\ = D_j^n(t) & \text{if } j \neq i, \end{cases}$$

that agrees with $(u^n(t), D^n(t))$ outside $\cup_{j\in\mathbb{N}} Q_\epsilon^n(j)$ and is such that

$$\mathcal{V}(\tilde{u}^n, \tilde{D}^n, f(t)) \leqslant \mathcal{V}(u^n(t), D^n(t), f(t)) - \frac{1}{2}\beta_i \delta^2 \sum_{j\in\mathbb{N}} |Q_\epsilon^n(j)| + o(\epsilon) \sum_{j\in\mathbb{N}} |Q_\epsilon^n(j)|$$

where $o(\epsilon) \to 0$ as $\epsilon \to 0$. There exist $\tilde{\theta} \in \mathbb{R}^m$ and $\tilde{A} \in G_{\tilde{\theta}}(\{D^n\}, \boldsymbol{\alpha})$ such that

$$\chi_{\tilde{D}_n} \overset{\star}{\rightharpoonup} \tilde{\theta},$$
$$\alpha_{\tilde{D}_n} \overset{G}{\to} \tilde{A}$$

where the G-convergence is unto subsequence (not relabelled). Taking the limit as $n \to \infty$,

$$\inf_{w\in H_0^1(\Omega,\mathbb{R}^s)} \mathcal{V}(w, \tilde{A}, \tilde{\theta}, f(t)) \leqslant \mathcal{V}(u(t), A(t), \theta(t), f(t)) - \frac{1}{2}\beta_i \delta^2 \gamma + o(\epsilon)\gamma$$

which contradicts the minimality of $(u(t), A(t), \theta(t))$ for sufficiently small ϵ. □

If Property 3 of Definition 4.3 is not satisfied, then there exists a sequence of sets $E_i^n \subset D_i^n(t)$ with $\liminf_{n\to\infty} |E_i^n| = \gamma > 0$, and there exist $\delta > 0$ and $\Delta\tau \searrow 0$ such that for all τ small enough, the sequence of sets $V_i^n(\delta)$ in (4.4a) satisfy $|V_i^n(\delta)| \to 0$ as $n \to \infty$. This suggests that the sets $E_i^n \subset D_i^n$ were not necessary at time τ in order for the strain in the undamaged region to remain below the threshold, and so it should not have been worth the cost to add the slices E_i^n to $D_i^n(\tau)$. Using the fact that in E_i^n we are below the threshold by δ, we show that if we consider a lower damage cost $\tilde{k}(\delta) \in \mathbb{R}^m$ given by

$$\tilde{k}_j(\delta) = \begin{cases} \frac{(\tilde{\lambda}_i(\delta))^2 \beta_i \, \Delta\beta_i}{2(\beta_i - \Delta\beta_i)} & \text{if } j = i, \\ k_j & \text{if } j \neq i, \end{cases} \tag{6.20}$$

corresponding to the lower threshold $\tilde{\lambda}(\delta) \in \mathbb{R}^m$ satisfying

$$\tilde{\lambda}_j(\delta) := \begin{cases} \lambda_i - \delta & \text{if } j = i, \\ \lambda_j & \text{if } j \neq i, \end{cases}$$

then we have an energy balance with this new coefficient on E_i^n, contradicting the energy balance with the original coefficient.

Proof. (*that Definition* 4.3(3a) *is satisfied*) Assume, on the contrary, that there exist:

1. A sequence of sets $E^n \in \mathbb{P}(\Omega)^m$ satisfying

$$E_j^n \subset \begin{cases} D_i^n(T) & \text{if } j = i, \\ \emptyset, & \text{if } j \neq i, \end{cases}$$

   with $\liminf |E_i^n| = \gamma > 0$.
2. $\delta > 0$,
3. a sequence $\Delta\tau \searrow 0$

such that for all $\tau < T$ and for each term in (3) satisfying $\Delta\tau < T - \tau$, $V_i^n(\delta)$ in (4.4a) satisfies

$$\liminf_{n\to\infty} \left| V_i^n(\delta) \right| = 0. \tag{6.21}$$

It suffices to show that u satisfies

$$\widetilde{\mathcal{V}}(t) + \gamma(\tilde{k}_i(\delta) - k_i) \geqslant \widetilde{\mathcal{V}}(0) - \int_0^t \dot{f}(\sigma) \cdot u(\sigma) \, \mathrm{d}\sigma, \tag{6.22}$$

where (for brevity, see (5.1d))

$$\widetilde{\mathcal{V}}(t) := \widetilde{\mathcal{V}}(u(t), A(t), \theta(t), f(t)).$$

Since $\tilde{k}_i(\delta) < k_i$, this together with energy balance (5.2) implies that $\gamma = 0$.

For each fixed $\Delta\tau$ we consider the partition $\{t^j \mid j = 0, \ldots, J\}$ of $[0, T]$ given by

$$t^j = j\Delta\tau,$$
$$J = \lfloor \frac{T}{\Delta\tau} \rfloor$$

(thus t^J is the last element in the partition and satisfies $0 \leqslant T - t^J \leqslant \Delta\tau$). Define

$$\Delta E^n(t^j) := E^n \cap \Big(D^n(t^j) \setminus D^n(t^{j-1}) \Big)$$

(so the only non-empty component of $\Delta E^n(t^j)$ is $\Delta E^n_i(t^j)$). From Lemma 6.5, the equi-integrability of $\|e^n\|^2$ in (4.4b) (see, e.g., [8]) and (6.21), we conclude,

$$\mathcal{E}(D^n(t^j), f(t^j)) + \tilde{k}_i(\delta)|\Delta E^n_i(t^j)| + o(1) \geqslant \mathcal{E}(D^n(t^j) \setminus \Delta E^n(t^j), f(t^j)),$$

with $o(1) \to 0$ as $n \to \infty$. Thus, up to subsequence, as $n \to \infty$,

$$\alpha_{D^n(t^j)\setminus \Delta E^n(t^j)} \xrightarrow{G} A_{\Delta\tau}(t^j)$$
$$\chi_{\Delta E^n(t^j)} \overset{\star}{\rightharpoonup} \theta_{\Delta\tau}(t^j)$$

and

$$\mathcal{E}(A(t^j), f(t^j)) + \tilde{k}(\delta) \cdot \left(\int_\Omega \theta_{\Delta\tau}(t^j)\, \mathrm{d}x \right) \geqslant \mathcal{E}(A_{\Delta\tau}(t^j), f(t^j)).$$

Using the minimality property of Definition 5.3(3) and noting that since

$$D^n(t^j) \setminus \Delta E^n(t^j) \supset D^n(t^{j-1}),$$

the pair $(A_{\Delta\tau}(t^j), \theta(t^j) - \theta_{\Delta\tau}(t^j))$ is a competitor for $(A(t^{j-1}), \theta(t^{j-1}))$ we obtain

$$\mathcal{E}(A(t^j), f(t^j)) + k \cdot \left(\int_\Omega \theta(t^j)\, \mathrm{d}x - \int_\Omega \theta_{\Delta\tau}(t^j)\, \mathrm{d}x \right) + \tilde{k}(\delta) \cdot \left(\int_\Omega \theta_{\Delta\tau}(t^j)\, \mathrm{d}x \right)$$
$$\geqslant \mathcal{E}(A(t^{j-1}), f(t^{j-1})) - (f(t^j) - f(t^{j-1})) \cdot u(A_{\Delta\tau}(t^j), f(t^j)) + k \cdot \int_\Omega \theta(t^{j-1})\, \mathrm{d}x, \tag{6.23}$$

where we have used (1.2). Let

$$\gamma_{\Delta\tau} := \lim_{n\to\infty} |E^n_i \cap D^n_i(t^I)|$$

(upto subsequence). Since

$$\sum_{j=0}^{J} \int_\Omega \chi_{\Delta E^n(t^j)}\, \mathrm{d}x \overset{\star}{\rightharpoonup} \sum_{j=0}^{J} \int_\Omega \theta_{\Delta\tau}(t^j)\, \mathrm{d}x$$

and

$$\sum_{j=0}^{J} \int_\Omega \chi_{\Delta E^n(t^i)}\, \mathrm{d}x \overset{\star}{\rightharpoonup} \gamma_{\Delta\tau},$$

summing (6.23) over j gives

$$\mathcal{E}(A(t^J), f(t^J)) + k \cdot \int_\Omega \theta(t^I)\, dx + (\tilde{k}_i(\delta) - k_i)\gamma_{\Delta\tau}$$
$$\geqslant \mathcal{E}(A(0), f(0)) + k \cdot \int_\Omega \theta(0)\, dx - \sum_{j=0}^{J} \Big(f(t^j) - f(t^{j-1})\Big) \cdot u(A_{\Delta\tau}(t^j), f(t^j)).$$

Since

1. $A(t^j) \leqslant A_{\Delta\tau}(t^j) \leqslant A(t^{j-1})$,
2. $A(t)$, being monotonic, is continuous at a.e. t, and
3. $f \in W^{1,1}([0,T], H^{-1}(\Omega, \mathbb{R}^s))$,

by the continuous dependence of u on A, we obtain for a.e. τ,

$$u(A_{\Delta\tau}(\tau), f(\tau)) \overset{L^2}{\to} u(\tau).$$

Moreover, for a.e. τ,

$$\frac{f(\tau + \Delta\tau) - f(\tau)}{\Delta\tau} \overset{H^{-1}}{\to} \dot{f}(\tau)$$

and hence

$$\sum_i \big(f(t_i^n) - f(t_{i-1}^n)\big) \cdot u(A_{\Delta\tau}(t_i^n), f(t_i^n)) \to \int_0^T \dot{f}(\tau) \cdot u(\tau)\, d\tau.$$

As a consequence of the energy balance $\mathcal{E}$ is continuous and thus $\mathcal{E}(t^J) \to \mathcal{E}(t)$ as $\Delta\tau \to 0$. In addition $\gamma_{\Delta\tau} \to \gamma$ as $\Delta\tau \to 0$. This yields (6.22). □

Proof. (*that Definition 4.3(3b) is satisfied*) Assume, on the contrary, that there exist:

1. Sequences $t^n \nearrow t$ and $E^n \in \mathbb{P}(\Omega)^m$ satisfying
$$E_j^n \subset \begin{cases} D_i^n(T) \setminus D_i^n(t^n) & \text{if } j = i, \\ \emptyset & \text{if } j \neq i, \end{cases}$$
with $\liminf_{n\to\infty} |E_i^n| = \gamma > 0$ and,
2. $\delta > 0$ such that $W_i^n(\delta)$ in (4.5) satisfies
$$\liminf_{n\to\infty} |W_i^n(\delta)| = 0.$$

From Lemma 6.5 we see that, as in the previous proof,

$$\mathcal{E}(D^n(t), f(t)) + k \cdot |D^n(t) \setminus E^n| + \tilde{k}_i(\delta)|E_i^n| + \int_\Omega (f(t) - f(t^n)) \cdot u(D^n(t), f(t))\, dx$$
$$\geqslant \mathcal{E}(D^n(t^n), f(t^n)) + k \cdot |D^n(t^n)| - o(1)$$

with $o(1) \to 0$ as $n \to \infty$. Taking the limit as $n \to \infty$, we get

$$\mathcal{W}(t) - (k_i - \tilde{k}_i(\delta))\gamma \geqslant \mathcal{W}(t).$$

Since $\tilde{k}_i(\delta) < k_i$, this is a contradiction. □

7. Local Minimality and Stability

In this section we will use the threshold property of energetic solutions to show that local minimisers (defined below) of damage are also global minimisers. In other words there are no purely local minimisers of damage.

It is natural to consider an *increment* in damage to be small if, for every damage process, the $L^1(\Omega, [0, 1])$ norm of the (characteristic function of the) damage increment is small. (Of course, any $L^p(\Omega, [0, 1])$ norm for $p < \infty$ would be acceptable; whereas for $p = \infty$ there would be no small increments. Following [9] we use $p = 1$.) This leads to the definition:

Definition 7.1. (*Damage neighbourhood*) Let $\epsilon > 0$.

1. The ϵ-*damage neighbourhood* of $\theta \in L^1(\Omega, [0, 1]^m)$ is

$$B(\theta, \epsilon) := \left\{ \theta' \in L^1(\Omega, [0, 1]^m) \,\middle|\, \forall i \in \mathbf{M},\ \theta_i' \geqslant \theta_i \text{ a.e. in } \Omega \text{ and } \int_\Omega \theta_i' - \theta_i \,\mathrm{d}x < \epsilon \right\}.$$

2. The ϵ-*damage neighbourhood* of $D \in \mathbb{P}(\Omega)^m$ is

$$B(D, \epsilon) := \left\{ D' \in \mathbb{P}(\Omega)^m \mid \chi_{D'} \in B(\chi_D, \epsilon) \right\}.$$

(These would induce asymmetric metrics on $L^1(\Omega, [0, 1]^m)$ and $\mathbb{P}(\Omega)^m$ respectively; see, e.g., [3,7,12,13]).

We are now ready to define local minimisers and stable states of damage; these are one-sided (so to speak) notions since we compare only with *increments* in damage:

Definition 7.2. (*Minimmality and stability*) Let $A \in L^\infty(\Omega, \mathcal{M}(c_1, c_2))$ and $\theta \in L^\infty(\Omega, [0, 1]^m)$ with $A(x) \in \mathcal{G}_{\theta(x)}(\boldsymbol{\alpha})$ a.e. in Ω. Let the sequence $\{D^n\} \subset \mathbb{P}^m$ satisfy

$$\alpha_{D^n} \xrightarrow{G} A, \tag{7.1a}$$

$$\chi_{D^n} \overset{\star}{\rightharpoonup} \theta. \tag{7.1b}$$

Let $f \in H^{-1}(\Omega, \mathbb{R}^s)$.

1. *Local minimality*: $\{D^n\}$ is a local minimizer of $\mathcal{W}(\cdot, \cdot, f)$ if there exists $\epsilon > 0$ such that for every $A' \in \mathcal{G}_{\theta'}(\{D_n\}, \boldsymbol{\alpha})$ with $\theta' \in B(\theta, \epsilon)$, we have

$$\mathcal{W}(A, \theta, f) \leqslant \mathcal{W}(A', \theta', f). \tag{7.2}$$

2. *Global minimality*: $\{D^n\}$ is a global minimizer of $\mathcal{W}(\cdot, \cdot, f)$ if (7.2) holds for every $A' \in \mathcal{G}_{\theta'}(\{D_n\}, \boldsymbol{\alpha})$ with $\theta' \in B(\theta, \infty)$.
3. *Stability*: $\{D^n\}$ is a stable state of $\mathcal{W}(\cdot, \cdot, f)$ if

$$\limsup_{\epsilon \to 0} \sup_{\substack{\theta' \in B(\theta, \epsilon) \\ A' \in \mathcal{G}_{\theta'}(\{D^n\}, \boldsymbol{\alpha})}} \frac{\mathcal{W}(A, \theta, f) - \mathcal{W}(A', \theta', f)}{\epsilon} \leqslant 0.$$

In fact local minimality implies global minimality, and so does stability.

Theorem 7.3. (Either local minimality or stability implies global minimality) *Let $f \in H^{-1}(\Omega, \mathbb{R}^s)$ and let the sequence $\{D^n\} \subset \mathbb{P}^m$ satisfy (7.1). If $\{D^n\}$ is either a local minimiser or a stable state of $\mathcal{W}(\cdot, \cdot, f)$ then it is a global minimiser of $\mathcal{W}(\cdot, \cdot, f)$.*

The proof follows [9].

Proof. Suppose, on the contrary, that $\{D^n\}$ is not a global minimiser. We show it is neither a local minimiser nor a stable state.

Let A, θ be as in (7.1). By assumption there exists $\theta' \in L^\infty(\Omega, [0,1]^m)$, with $\theta' \geqslant \theta$, and $A' \in \mathcal{G}_{\theta'}(\{D^n\}, \boldsymbol{\alpha})$ such that

$$\mathcal{W}(A, \theta, f) > \mathcal{W}(A', \theta', f). \tag{7.3}$$

Step 1: First we show that $\theta'_i > \theta_i$ for some $i \in \mathbf{M}$:

From the definition of $\mathcal{G}(\{D^n\}, \boldsymbol{\alpha})$ there exists a sequence of sets $E^n \supseteq D^n$ such that

$$\chi_{E^n} \overset{\star}{\rightharpoonup} \theta',$$
$$\alpha_{E^n} \overset{G}{\to} A'.$$

Let u^n and v^n be minimisers of $\widetilde{\mathcal{V}}(\cdot, \alpha_{D^n}, \chi_{D^n}, f)$ and $\widetilde{\mathcal{V}}(\cdot, \alpha_{E^n}, \chi_{E^n}, f)$, respectively (see (5.1d)).

Then from (7.3) and $\theta' \geqslant \theta$ we have

$$\lim_{n\to\infty} \left(\begin{array}{l} \displaystyle\int_\Omega \frac{1}{2} \left\langle \alpha_{D^n}\, e(u^n), e(u^n) \right\rangle - f \cdot u^n \\ \qquad \displaystyle - \frac{1}{2} \left\langle \alpha_{E^n}\, e(v^n), e(v^n) \right\rangle + f \cdot v^n \,\mathrm{d}x \end{array} \right) > 0,$$

while from the minimality of u^n we have

$$\lim_{n\to\infty} \left(\begin{array}{l} \displaystyle\int_\Omega \frac{1}{2} \left\langle \alpha_{D^n}\, e(u^n), e(u^n) \right\rangle - f \cdot u^n \\ \qquad \displaystyle - \frac{1}{2} \left\langle \alpha_{D^n}\, e(v^n), e(v^n) \right\rangle + f \cdot v^n \,\mathrm{d}x \end{array} \right) \leqslant 0.$$

But if, for all $i \in \mathbf{M}$, $\lim_{n\to\infty} \left| E_i^n \setminus D_i^n \right| = 0$, since $\{|e(v^n)|^2\}$ is equi-integrable (see, e.g., [8]),

$$\begin{aligned} 0 &< \lim_{n\to\infty} \left(\int_\Omega \left\langle \alpha_{D^n}\, e(v^n), e(v^n) \right\rangle - \left\langle \alpha_{E^n}\, e(v^n), e(v^n) \right\rangle \,\mathrm{d}x \right) \\ &= \lim_{n\to\infty} \sum_{i\in\mathbf{M}} \Delta\beta_i \int_{E_i^n \setminus D_i^n} \|\Lambda_i e(v^n)\|^2 \,\mathrm{d}x \\ &= 0, \end{aligned}$$

which is a contradiction. Thus $\theta'_i > \theta_i$ for some $i \in \mathbf{M}$.

Step 2: Let $i \in \mathbf{M}$ satisfy $\theta_i' > \theta_i$. Next we show that $e(u^n)$ exceeds the threshold for i-damage somewhere in $E_i^n \setminus D_i^n$; more precisely that there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \left|\widetilde{U}_i^n(\delta)\right| > 0$$

where

$$\widetilde{U}_i^n(\delta) := \left\{x \in E_i^n \setminus D_i^n \;\middle|\; \|\Lambda_i e(u^n)(x)\| > \lambda_i + \delta\right\}.$$

Assume, on the contrary, that for every $\delta > 0$,

$$\limsup_{n\to\infty} \left|\widetilde{U}_i^n(\delta)\right| = 0. \tag{7.4}$$

Then, using (6.17a),

$$\mathcal{E}(D^n, f) - \mathcal{E}(E^n, f) \leqslant \tilde{k}(\delta) \cdot |E^n \setminus D^n| + \frac{\beta_i \ \Delta\beta_i}{2(\beta_i - \Delta\beta_i)} \|\Lambda_i e^n\|^2_{L^2(\widetilde{U}_i^n(\delta))},$$

where $\tilde{k}$ is defined in (6.20). Taking the limit as $n \to \infty$, using (7.4), the equi-integrability of $e(u^n)$ and the arbitrariness of δ, we obtain a contradiction of (7.3).
Step 3: Finally we show that $\{D^n\}$ is neither a local global minimiser not a stable state.

Continuing from Step 2, dropping to a subsequence, we have

$$\chi_{\widetilde{U}_i^n(\delta)} \overset{\star}{\rightharpoonup} \theta_i > 0.$$

Pick $x_o \in \Omega$ that is a Lebesgue point for θ with $\theta(x_o) > 0$. Now, given any $\epsilon > 0$, we can choose a ball $U \subset \Omega$, with radius $|U| \leqslant \epsilon$, such that

$$\int_U \theta \, \mathrm{d}x > \frac{|U|}{2}\theta(x_o),$$

that is,

$$\lim_{n\to\infty} \left|\widetilde{U}_i^n(\delta) \cap U\right| > \frac{|U|}{2}\theta(x_o) > 0.$$

Then, from Lemma 6.4, by adding laminates of damage within $\widetilde{U}_i^n(\delta) \cap U$, the energy of $\{D^n\}$ can be lowered in the limit by at least $\frac{1}{2}\beta_i\delta^2\frac{|U|}{2}\theta(x_o)$. The sequence $\{\tilde{D}^n\}$ generated by the union of $\{D^n\}$ and these laminates (i) has lower energy than $\{D^n\}$ and (ii) its characteristic functions have a weak-* limit that lies in $B(\theta, \epsilon)$. Thus $\{D^n\}$ is not a local minimiser.

In fact, this also shows that

$$\limsup_{\epsilon\to 0} \sup_{\substack{\theta' \in B(\theta,\epsilon) \\ A' \in \mathcal{G}_{\theta'}(\{D^n\},\boldsymbol{\alpha})}} \frac{\mathcal{W}(A, \theta, f) - \mathcal{W}(A', \theta', f)}{\epsilon} \geqslant \frac{1}{2}\beta_i\delta^2,$$

and so $(\{D_n\}, A, \theta)$ is not stable. □

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