

Diffusion Limit of Kinetic Equations for Multiple Species Charged Particles

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Abstract

In ionic solutions, there are multi-species charged particles (ions) with different properties like mass, charge etc. Macroscopic continuum models like the Poisson–Nernst–Planck (PNP) systems have been extensively used to describe the transport and distribution of ionic species in the solvent. Starting from the kinetic theory for the ion transport, we study a Vlasov–Poisson–Fokker–Planck (VPFP) system in a bounded domain with reflection boundary conditions for charge distributions and prove that the global renormalized solutions of the VPFP system converge to the global weak solutions of the PNP system, as the small parameter related to the scaled thermal velocity and mean free path tends to zero. Our results may justify the PNP system as a macroscopic model for the transport of multi-species ions in dilute solutions.

1. Introduction

The transport of ions in different biological environments is very important in our life and it has attracted more and more attention recently [25,26,36,37,56]. In biological problems, the ionic solutions usually consist of charged particles (ions) like sodium Na^+ , potassium K^+ , calcium Ca^{2+} and chloride Cl^- etc., which have different but comparable masses, charge valencies and sizes. These differences have dramatic effects on the dynamics of multi-species ions which produce the functions of cells in biological system, for example, the ion channels. To study the dynamics of multi-species ions, molecular dynamics simulations (MD) using microscopic models (from Newton’s laws) to describe charge particle trajectories are popular and useful but expensive because the models are usually huge and the numerical computation time is very long (see, for instance, [36, Chapter 10]).

To increase the efficiency of numerical simulations, one may use the (multi-species) Poisson–Nernst–Planck (PNP) system [19,27,28,54,62], which is a

macroscopic model to describe multi-species ion transport. Conventionally, the PNP system consists of coupled diffusion-convection equations and the Poisson equation, being represented as follows:

$$\begin{cases} \partial_t c_i = \nabla \cdot J_i, \\ J_i = d_i \left(\nabla c_i + \frac{q_i}{k_B T} c_i \nabla \phi \right), \\ -\nabla \cdot (\varepsilon \nabla \phi) = \sum_{i=1}^N q_i c_i + D(x). \end{cases} \quad (1.1)$$

Here, c_i ($i = 1, 2, \dots, N$) stand for densities of charged particles in the ionic solution and ϕ is the self-consistent electric potential. In addition, q_i are the (positive or negative) charges of particles, J_i are the ionic flux densities, d_i are their diffusion coefficients, ε is the dielectric coefficient, k_B is the Boltzmann constant, T is the temperature and $D(x)$ is the permanent charge density in the domain. The PNP system (1.1) is also one of the fundamental macroscopic models in the study of transport of carriers in semiconductors, see, for example, [31, 43, 47, 48]. Concerning the mathematical analysis, the initial value problem and the initial boundary value problem of the PNP system have been extensively studied, we refer to [2, 4, 5, 30–32, 42, 48] and the references cited therein.

The PNP system (1.1) provides a continuum description of the evolution of charged particles via macroscopic (averaged) quantities, for example, the particle density, the current density etc., which have cheaper costs for numerics. Such continuum models can be (formally) derived from kinetic models by coarse graining methods, like the moment method, the Hilbert expansion method and so on [42, 48, 50]. Although many results for the PNP systems have been obtained, it seems that none of them can reveal basic principles like gating and selectivity of ion channels. Recently, new PNP type systems have been derived and the selectivity of ion channels have been simulated successfully [27, 38–40, 46, 63]. In order to justify these continuum models, here we develop the kinetic theory for the PNP system like (1.1) as the first step of the work. Our goal in the present paper is to rigorously justify the PNP system for dilute ionic solutions consisting of multi-species charged particles by studying the diffusion limit of a suitable kinetic system. We will continue to study the kinetic theory for those new PNP type systems as in [27, 38–40, 46, 63] in the near future.

In this paper, we consider the case that the motion of multi-species charged particles is governed by the electrostatic force coming from their (self-consistent) Coulomb interaction. We also assume that the momentum of charged particles with collision is small and ignorable. Then the collision term in the kinetic equation may be approximated by the Fokker–Planck operator that describes the Brownian force [18], and the resulting kinetic system becomes the Vlasov–Poisson–Fokker–Planck (VPFP) system as follows:

$$\partial_t f_i + v \cdot \nabla_x f_i - \frac{z_i q}{m_i} \nabla_x \phi \cdot \nabla_v f_i = \frac{1}{\tau_i} \mathcal{L}_{\text{FP}}^i(f_i), \quad (1.2)$$

$$-\varepsilon_0 \Delta_x \phi = q \left(\sum_{i=1}^N z_i \int_{\mathbb{R}^d} f_i \, dv + D(x) \right), \quad (1.3)$$

where $\mathcal{L}_{\text{FP}}^i$ ($i = 1, \dots, N$) are the Fokker–Planck operators such that

$$\mathcal{L}_{\text{FP}}^i(f_i) = \nabla_v \cdot (vf_i + \theta_i \nabla_v f_i).$$

Here, $\varepsilon_0 > 0$ is the vacuum permittivity, $q > 0$ is the positive elementary charge. For $i = 1, \dots, N$, the state of each species is given by a distribution function $f_i(t, x, v) \geq 0$, that is, a probability density in the (x, v) -phase space at time t ($f_i dx dv$ is the number of the i -th species charged particles at time t located at a volume element dx about the position x and having velocities in a volume dv about the value v). Furthermore, $z_i \in \mathbb{Z}$ are the valencies for the N -species charged particles, m_i are the masses, τ_i are relaxation time due to collisions of the particles with the thermal bath, $\sqrt{\theta_i}$ are the thermal velocities given by $\sqrt{\theta_i} = \sqrt{2k_B T_b m_i^{-1}}$ and T_b is the temperature of the thermal bath.

In plasma physics, the VPPF system (1.2)–(1.3) with $N = 1$ (that is, the single species case) is reasonable because the mass ratio between the ions and electrons is huge; only the evolution of the electrons is described in terms of a distribution function in the resulting system, and those ‘heavy’ ions are supposed to be static. For such a case, the existence and uniqueness of solutions to the initial value problem or the initial boundary value problem of the VPPF system have been investigated in the literature. We refer to [7, 58, 61] for results on the classical solutions and to [8, 10, 11, 60] for weak solutions and their regularity. Concerning the long-time behavior of the VPPF system, we refer to [6, 9, 12]. Instead of the single species case with $N = 1$, here we study the system (1.2)–(1.3) with $N \geq 2$ for multi-species charged particles, which is more complicated due to the (nonlocal) interactions between particles via the Poisson equation (1.3) (that is, the Coulomb interaction).

Suitable scalings of the VPPF system should be introduced in order to study its diffusion limit. Let L be the characteristic length. We denote by N_0 the characteristic value for the concentration of particles and by Φ_0 the characteristic variation of the electric potential over L . Since we have to treat multiple species of charged particles that have different masses and charges, it is convenient to introduce a ‘reference particle’ with mass m_{ref} , electric charge $z_{\text{ref}}q$ (with $z_{\text{ref}} = 1$), relaxation time τ_{ref} and thermal velocity θ_{ref} . The microscopic variation as well as the drift velocity for the reference particle are given by $V_{\text{ref}} = \sqrt{\theta_{\text{ref}}}$, $U_{\text{ref}} = \tau_{\text{ref}} \frac{q}{m_{\text{ref}}} \frac{\Phi_0}{L}$, respectively. Choosing the following scaling (with respect to the reference particle) $t \rightarrow T_0 t'$, $x \rightarrow Lx'$, $v \rightarrow V_{\text{ref}} v'$, $T_0 = \frac{L}{U_{\text{ref}}}$ and the change of unknowns $f_i(t, x, v) = N_0 V_{\text{ref}}^{-d} f'_i(t', x', v')$, $\phi(t, x, v) = \Phi_0 \phi'(t', x', v')$, $D(x) = N_0 D'(x')$, we obtain the rescaled VPPF equations (drop the prime for simplicity):

$$\begin{aligned} \partial_t f_i + v v \cdot \nabla_x f_i - \frac{\kappa_i z_i}{\varepsilon} \nabla_x \phi \cdot \nabla_v f_i &= \frac{\zeta_i v}{\varepsilon} \nabla_v \cdot (vf_i + \kappa_i \nabla_v f_i), \quad i = 1, \dots, N, \\ -\varpi \Delta_x \phi &= \sum_{i=1}^N z_i \int_{\mathbb{R}^d} f_i dv + D(x), \end{aligned}$$

where the dimensionless parameters v (the ‘scaled’ thermal velocity), ε (the ‘scaled’ thermal mean free path), ϖ and the ratios κ_i , ζ_i are given by

$$v = \frac{V_{\text{ref}}}{U_{\text{ref}}}, \quad \varepsilon = \frac{\tau_{\text{ref}} V_{\text{ref}}}{L}, \quad \varpi = \frac{\varepsilon_0 \Phi_0}{q N_0 L^2}, \quad \kappa_i = \frac{m_{\text{ref}}}{m_i}, \quad \zeta_i = \frac{\tau_{\text{ref}}}{\tau_i}.$$

The case we are interested in this paper is called the low field limit (or the parabolic limit), which means that the drift velocity is small compared with the thermal velocity, while the thermal velocity is small compared to the relaxation velocity, and the two ratios have the same order of magnitude (cf. [1, 29, 33, 57]):

$$v \simeq \varepsilon^{-1} \quad \text{and} \quad \varepsilon \ll 1.$$

For $\varepsilon > 0$, taking $v = \varepsilon^{-1}$ (just for the sake of simplicity), we arrive at the rescaled VFPF system under low field scaling, which will be investigated in the remaining part of this paper:

$$\partial_t f_i^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_i^\varepsilon - \frac{\kappa_i z_i}{\varepsilon} \nabla_x \phi^\varepsilon \cdot \nabla_v f_i^\varepsilon = \frac{\zeta_i}{\varepsilon^2} L_{\text{FP}}^i(f_i^\varepsilon), \tag{1.4}$$

$$-\varpi \Delta_x \phi^\varepsilon = \sum_{i=1}^N z_i \int_{\mathbb{R}^d} f_i^\varepsilon(t, x, v) \, dv + D(x), \tag{1.5}$$

where the rescaled Fokker–Planck operators are given by

$$L_{\text{FP}}^i(f_i^\varepsilon) = \nabla_v \cdot (v f_i^\varepsilon + \kappa_i \nabla_v f_i^\varepsilon). \tag{1.6}$$

We recall that the diffusion limit of the VFPF system has been studied extensively in the literature (cf. [29, 33, 34, 55, 57] and the references therein). In [29, 33, 57], the authors studied the low field limit and proved the convergence of suitable solutions to the single species VFPF system towards a solution to the drift–diffusion–Poisson model in the whole space. In [57], under a suitable regularity assumption on the initial data, the convergence result was obtained globally in time in two dimensions and locally in time for the three dimensional case. Later, the author proved in [33] a global convergence result in the two dimensional case, without any restriction on the time interval and the assumptions on the initial data were weakened with bounds only on the associated entropy and energy. Quite recently, in [29] the authors established a global convergence result, without any restriction on the time interval or on the spatial dimensions, by working with the renormalized solutions (or free energy solutions, cf. [22, 24]). As pointed out in [29], the notion of renormalized solutions is natural for the problem, because the free energy of the VFPF system seems to be the only quantity that is uniformly bounded with respect to the small parameter ε (that is, the ‘scaled’ mean free path). Even one works with more regular initial data such that the solutions can be defined in the usual weak sense without the need of renormalizing, one still has to use renormalization techniques to pass to the limit as $\varepsilon \rightarrow 0$. Besides, the use of renormalization techniques together with an averaging lemma helps to remove the restriction on spatial dimensions and treat the nonlinear term $\nabla_x \phi \cdot \nabla_v f$, where the main difficulty comes from (we refer to [29] for more details).

In this paper, we rigorously prove that for the multi-species case, the VFPF system (1.4)–(1.5) converges to a rescaled PNP system as ε tends to zero in the low field limit. We generalized the techniques introduced in the previous works [29, 49, 53], to the case involving multiple species of charged particles in a bounded region with reflection boundary conditions [10, 15, 53]. The specific boundary conditions

recover the classical no-flux boundary conditions of the PNP system. Different from the single species case in the literature, the previous arguments have to be modified in order to deal with the nonlocal interactions between different species of particles through the Poisson equation for the electric potential ϕ . In order to deal with the integrals on the boundary, we shall make use of the Darrozès–Guiraud information [21], which helps to obtain the energy dissipation. Finally, effects of different but comparable quantities like masses and valencies of the charged particles will become obvious in our mathematical analysis.

Our results support the PNP system (1.1) as a suitable model for multi-species charged particles in a dilute solution. As we mentioned before, several variants of the PNP system (1.1) have recently been derived by using the energetic variational approaches [41] to model important physical ingredients such as size (steric) effects for non-diluted solutions (cf. for example, [27, 38–40, 46, 63]) that are crucial in the study of the selectivity of ion channels in cell membranes [19, 36, 45]. The total energy for these modified PNP systems consists of the entropic energy induced by the Brownian motion of ions, the electrostatic potential energy representing the coulomb interaction between the charged ions, and in particular, the repulsive potential energy caused by the excluded volume effect (for example, the singular Lennard–Jones potential). Our result can be viewed as a starting point for the further investigation on the case of crowded ions. It would be interesting to study the diffusion-limit of suitable kinetic systems to obtain the modified PNP systems [27, 39].

The remaining part of this paper is organized as follows. In Section 2, we present the definition of renormalized solutions and state the main result on the diffusion limit of the VFFP system (1.4)–(1.5) (Theorem 2.1). In Section 3, after deriving the energy dissipation of the VFFP system in bounded domain (Proposition 3.1), which yields the necessary uniform estimates (Lemmas 3.1, 3.2, 3.3), we proceed to prove our main result by using the renormalization techniques.

2. Preliminaries and Main Result

2.1. Boundary and Initial Conditions

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a sufficiently smooth bounded domain. For instance, the outward unit normal vector $\mathbf{n}(x)$ at $x \in \partial\Omega$ satisfies $\mathbf{n} \in W^{2,\infty}(\Omega, \mathbb{R}^d)$. The Lebesgue surface measure on $\partial\Omega$ will be denoted by dS .

Then we introduce the boundary conditions for the distribution functions. As in Cercignani’s work [13–16] (see also [6, 53]), we define the sets of outgoing (Σ_+^x) and incoming (Σ_-^x) velocities at point $x \in \partial\Omega$ such that $\Sigma_\pm^x := \{v \in \mathbb{R}^d : \pm v \cdot \mathbf{n}(x) > 0\}$ and denote the boundary sets $\Sigma_\pm = \{(x, v) : x \in \Omega, v \in \Sigma_\pm^x\}$. Let γh be the trace of function h and $\gamma_\pm h = \mathbf{1}_{(0,+\infty) \times \Sigma_\pm} \gamma h$. Reflection boundary conditions for the kinetic equations take the form of integral (balance) relations between the densities of the particles on the outgoing and incoming velocity subsets of the boundary $\partial\Omega$ at a given time [14–16]. For instance, given $x \in \partial\Omega$ and $t > 0$, we have (cf. [6]):

$$\gamma_- f(t, x, v) = \int_{\Sigma_+^x} R(t, x; v, v^*) \gamma_+ f(t, x, v^*) dv^*, \quad v \in \Sigma_-^x, \quad (2.1)$$

where R represents the probability that a particle with velocity v^* at time t striking the boundary on x reemerges at the same instant and location with velocity v . If we consider $v' = -v$ for any $v \in \Sigma_-^x$ and take $R(t, x; v, v^*) = \delta_{v'}$ being the Dirac measure centered at $v^* = v'$, then we have $\gamma_- f(t, x, v) = \gamma_- f(t, x, -v)$ on Σ_- , which is the classical (local) inverse reflection boundary condition. Similarly, if we take $v' = v - 2(v \cdot \mathbf{n}(x))\mathbf{n}(x)$, then we arrive at the classical (local) specular reflection boundary condition, see [6, 10]. We refer to [6] for possible minimal assumptions on R such that (2.1) is well-defined, that is, R is nonnegative and it verifies the normalization condition as well as the reciprocity principle. Detailed discussions on the boundary conditions can be found in [14–16].

Here, we are more interested in the so-called diffuse reflection according to a Maxwellian with temperature of the thermal bath, which is nonlocal. Denote by $M_i(v)$ the Maxwellians for charged particles

$$M_i(v) = \frac{1}{(2\pi)^{\frac{d-1}{2}} \kappa_i^{\frac{d+1}{2}}} e^{-\frac{1}{2\kappa_i} |v|^2}, \quad i = 1, \dots, N. \quad (2.2)$$

We note that M_i are the zeros of the rescaled Fokker–Planck operators L_{FP}^i given in (1.6), that is, $L_{\text{FP}}^i(M_i) = 0$, ($i = 1, \dots, N$). Then we can choose a special form of R in (2.1) and propose the following boundary conditions for the distribution functions (cf. [15, 53]), which are special cases of the so-called Maxwell boundary condition [50, 53]: for given $x \in \partial\Omega$ and $t > 0$,

$$\gamma_- f_i^\varepsilon = \frac{M_i(v)}{\int_{v \cdot \mathbf{n}(x) < 0} |v \cdot \mathbf{n}(x)| M_i(v) dv} \int_{v^* \cdot \mathbf{n}(x) > 0} (\gamma_+ f_i^\varepsilon) v^* \cdot \mathbf{n}(x) dv^*, \quad \text{on } \Sigma_-^x. \quad (2.3)$$

For the electric potential ϕ^ε , we simply impose the zero-outward electric field condition such that

$$\nabla_x \phi^\varepsilon \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega. \quad (2.4)$$

In summary, below we will consider the rescaled VPFP system (1.4)–(1.5) on $(0, T) \times \Omega \times \mathbb{R}^d$ subject to boundary conditions (2.3)–(2.4) and the initial data (depending on the parameter ε):

$$f_i^\varepsilon(t, x, v)|_{t=0} = f_{i0}^\varepsilon(x, v). \quad (2.5)$$

We remark that the boundary conditions (2.3) allow us to preserve mass conservation and obtain proper energy and entropy balance laws of the VPFP system (1.4)–(1.5). Denote by

$$n_i^\varepsilon(t, x) = \int_{\mathbb{R}^d} f_i^\varepsilon(t, x, v) dv \quad \text{and} \quad J_i^\varepsilon = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} v f_i^\varepsilon dv, \quad (2.6)$$

the densities as well as the current densities associated to the distribution functions for the i -th species, respectively. Multiplying (2.3) by $v \cdot \mathbf{n}(x)$ and integrating over Σ_x^- , we easily deduce the (macroscopic) boundary conditions for the fluxes such that

$$J_i^\varepsilon \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{2.7}$$

which imply that all the particles that reach the boundary are reflected (no particle goes out nor enters in the domain Ω) and thus the mass $\int_\Omega n_i^\varepsilon(t, x) dx$ is conserved for all time. On the other hand, in order to uniquely determine the solution ϕ^ε to the Poisson equation (1.5) with homogeneous Neumann boundary condition (2.4), we require the global neutrality condition

$$\sum_{i=1}^N z_i \int_\Omega \int_{\mathbb{R}^d} f_i^\varepsilon dv dx + \int_\Omega D(x) dx = 0 \tag{2.8}$$

and the zero-mean constraint $\int_\Omega \phi^\varepsilon dx = 0$.

2.2. Main Result

We first introduce the definition of renormalized solutions in the spirit of [29, 49]:

Definition 2.1. The set $(f_i^\varepsilon, \phi^\varepsilon) \in L^\infty(0, T; (L^1(\Omega \times \mathbb{R}^d))^N \times H^1(\Omega))$ is a renormalized solution to the VFPF system (1.4)–(1.5) with initial and boundary conditions (2.3)–(2.5), if

(1) For all functions $\beta_i \in C^2(\mathbb{R}), i = 1, \dots, N$ satisfying

$$|\beta_i(s)| \leq C(s^{\frac{1}{2}} + 1), \quad |\beta_i'(s)| \leq C(1 + s)^{-\frac{1}{2}}, \quad |\beta_i''(s)| \leq C(1 + s)^{-1}, \quad \forall s \geq 0,$$

the set $(\beta_i(f_i^\varepsilon), \phi^\varepsilon)$ is a weak solution to the system

$$\varepsilon \partial_t \beta_i(f_i^\varepsilon) + v \cdot \nabla_x \beta_i(f_i^\varepsilon) - \kappa_i z_i \nabla_x \phi^\varepsilon \cdot \nabla_v \beta_i(f_i^\varepsilon) = \frac{\zeta_i}{\varepsilon} L_{\text{FP}}^i(f_i^\varepsilon) \beta_i'(f_i^\varepsilon), \tag{2.9}$$

$$-\varpi \Delta_x \phi^\varepsilon = \sum_{i=1}^N z_i \int_{\mathbb{R}^d} f_i^\varepsilon dv + D(x), \tag{2.10}$$

with initial data

$$\beta_i(f_0^\varepsilon)|_{t=0} = \beta_i(f_{i0}^\varepsilon) \tag{2.11}$$

and boundary conditions

$$\gamma_- \beta_i(f_i^\varepsilon) = \frac{M_i(v)}{\int_{v \cdot \mathbf{n}(x) < 0} |v \cdot \mathbf{n}(x)| M_i(v) dv} \int_{v^* \cdot \mathbf{n}(x) > 0} \gamma_+ \beta_i(f_i^\varepsilon) v^* \cdot \mathbf{n}(x) dv^*, \tag{2.12}$$

$$\nabla_x \phi^\varepsilon \cdot \mathbf{n} = 0. \tag{2.13}$$

(2) For any $\lambda > 0$, the functions $\theta_{\varepsilon,\lambda}^i = (f_i^\varepsilon + \lambda \tilde{M}_i)^{\frac{1}{2}}$ satisfy

$$\varepsilon \partial_t \theta_{\varepsilon,\lambda}^i + v \cdot \nabla_x \theta_{\varepsilon,\lambda}^i - \kappa_i z_i \nabla_v \cdot (\nabla_x \phi^\varepsilon \theta_{\varepsilon,\lambda}^i) = \frac{\zeta_i}{2\varepsilon \theta_{\varepsilon,\lambda}^i} L_{\text{FP}}^i(f_i^\varepsilon) + \frac{z_i \lambda \tilde{M}_i}{2\theta_{\varepsilon,\lambda}^i} v \cdot \nabla_x \phi^\varepsilon, \tag{2.14}$$

where \tilde{M}_i are the normalized Maxwellians [comparing with (2.2)]

$$\tilde{M}_i(v) = \left(\frac{\kappa_i}{2\pi}\right)^{\frac{1}{2}} M_i(v) \quad \text{such that} \quad \int_{\mathbb{R}^d} \tilde{M}_i(v) \, dv = 1, \quad i = 1, \dots, N. \tag{2.15}$$

Remark 2.1. Due to the regularity of renormalized functions β_i , the corresponding boundary conditions (2.12) for the renormalized distribution functions make sense. We refer to [3,17,59] (see also [6,53]) for more detailed discussions about the traces of distribution functions on the boundary.

Next, we consider the rescaled version of the PNP system (1.1):

$$\partial_t n_i + \nabla_x \cdot J_i = 0, \tag{2.16}$$

$$-\varpi \Delta_x \phi = \sum_{i=1}^N z_i n_i + D(x), \tag{2.17}$$

with density currents given by

$$J_i = -\frac{1}{\zeta_i} \nabla_x n_i - \frac{z_i}{\zeta_i} n_i \nabla_x \phi \tag{2.18}$$

and subject to the following boundary conditions and initial conditions:

$$J_i \cdot \mathbf{n} = \nabla_x \phi \cdot \mathbf{n} = 0, \quad \text{on } (0, T) \times \partial\Omega, \tag{2.19}$$

$$n_i|_{t=0} = n_{i0}, \quad \text{in } \Omega. \tag{2.20}$$

Moreover, we require that

$$\int_{\Omega} \phi \, dx = 0 \quad \text{and} \quad \int_{\Omega} \left(\sum_{i=1}^N z_i n_i + D(x) \right) \, dx = 0.$$

Then we introduce the weak formulation of the PNP system (2.16)–(2.20).

Definition 2.2. We say that the set (n_i, ϕ) is a weak solution to the initial boundary value problem of the PNP system (2.16)–(2.20), if

$$\begin{aligned} n_i &\in L^\infty(0, T; L \log L(\Omega)), \quad \sqrt{n_i} \in L^2(0, T; H^1(\Omega)), \\ \partial_t n_i &\in L^1(0, T; W^{-1,1}(\Omega)), \quad \phi \in L^2(0, T; H^1(\Omega)), \end{aligned}$$

where the function space $L \log L(\Omega)$ is given by

$$L \log L(\Omega) := \left\{ n : n \geq 0, \int_{\Omega} n(1 + |\log n|) \, dx < +\infty \right\}$$

and the PNP system (2.16)–(2.17) is satisfied in the weak sense: for any $u \in C^\infty([0, T]; C^\infty(\bar{\Omega}))$, $\psi \in L^2(0, T; (H^1(\Omega))'$,

$$\begin{aligned} & \int_{\Omega} n_i(t, \cdot)u(t, \cdot) \, dx - \int_{\Omega} n_{i0}u(0, \cdot) \, dx \\ &= \int_0^t \int_{\Omega} n_i \partial_t u \, dx \, d\tau - \frac{1}{\zeta_i} \int_0^t \int_{\Omega} (\nabla_x n_i + z_i n_i \nabla_x \phi) \cdot \nabla_x u \, dx \, d\tau, \quad t \in [0, T], \\ \varpi \int_0^T \int_{\Omega} \nabla_x \phi \cdot \nabla_x \psi \, dx \, dt &= \int_0^T \int_{\Omega} \left(\sum_{i=1}^N z_i n_i + D(x) \right) \psi \, dx \, dt. \end{aligned}$$

Moreover, the weak solution (n_i, ϕ) satisfies the following energy inequality

$$\begin{aligned} e(t) + \sum_{i=1}^N \int_0^t \int_{\Omega} \frac{1}{\zeta_i} n_i \left| \nabla \left(\ln n_i + z_i \phi \right) \right|^2 \, dx \, dt &\leq e(0), \quad t \in [0, T], \\ \text{with } e(t) := \int_{\Omega} \left(\sum_{i=1}^N n_i \ln n_i + \frac{\varpi}{2} |\nabla \phi|^2 \right) \, dx. \end{aligned}$$

Now we are in a position to state the main result of this paper.

Theorem 2.1. *Let the background charge D be independent of time and satisfy $D(x) \in L^\infty(\Omega)$. We assume that the initial data $(f_{i0}^\varepsilon, \phi_0^\varepsilon)$ satisfy the following assumptions*

$$f_{i0}^\varepsilon \geq 0, \quad \int_{\Omega} \int_{\mathbb{R}^d} f_{i0}^\varepsilon (1 + |v|^2 + |\log f_{i0}^\varepsilon|) \, dv \, dx \leq C_0, \quad (2.21)$$

$$\|\phi_0^\varepsilon\|_{H^1(\Omega)} \leq C_0, \quad \int_{\Omega} \phi_0^\varepsilon \, dx = 0, \quad (2.22)$$

for some constant $C_0 > 0$ independent of the parameter ε , and the global neutrality condition holds

$$\sum_{i=1}^N z_i \int_{\Omega} \int_{\mathbb{R}^d} f_{i0}^\varepsilon \, dv \, dx + \int_{\Omega} D(x) \, dx = 0, \quad \forall \varepsilon > 0. \quad (2.23)$$

Let $(f_i^\varepsilon, \phi^\varepsilon)$ be a free energy (renormalized) solution of the VPFP system (1.4)–(1.5) with corresponding initial and boundary conditions (2.3)–(2.5) (cf. Definition 2.1). Then, as ε tends to zero, up to a subsequence if necessary, we have the strong convergence results

$$f_i^\varepsilon(t, x, v) \rightarrow n_i(t, x) \tilde{M}_i(v) \quad \text{in } L^1(0, T; L^1(\Omega \times \mathbb{R}^d)), \quad (2.24)$$

$$\phi^\varepsilon(t, x) \rightarrow \phi(t, x) \quad \text{in } L^2(0, T; W^{1,p}(\Omega)), \quad 1 \leq p < 2. \quad (2.25)$$

Moreover, n_i^ε strongly converge in $L^1(0, T; L^1(\Omega))$ towards n_i and (n_i, ϕ) is a weak solution to the PNP system (2.16)–(2.20) (cf. Definition 2.2) with initial data $n_i|_{t=0} = n_{i0} = \int_{\mathbb{R}^d} f_{i0} \, dv$, such that f_{i0} are the weak limits of f_{i0}^ε .

Remark 2.2. We would like to mention that the PNP system (2.16)–(2.20) can also be derived from diffusion limits of other types of kinetic equations, for example, the Boltzmann–Poisson system. We refer to [49] for the one species case and we believe that their argument can also be extended to the multi-species case.

Remark 2.3. We remark that different types of scalings can be chosen for the VPFP system. For instance, if we assume that the drift and thermal velocities are comparable, but both are small comparing with the relaxation velocity, for example, $v = \mathcal{O}(1)$ and $\varepsilon \ll 1$, then we arrive at a different rescaled VPFP system

$$\begin{cases} \partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon - \frac{\kappa_i z_i}{\varepsilon} \nabla_x \phi^\varepsilon \cdot \nabla_v f_i^\varepsilon = \frac{\zeta_i}{\varepsilon} L_{\text{FP}}^i(f_i^\varepsilon), \\ -\varpi \Delta_x \phi^\varepsilon = \sum_{i=1}^n z_i \int_{\mathbb{R}^d} f_i^\varepsilon \, dv + D(x). \end{cases}$$

This is usually called drift-collision balance scaling or high field scaling in the literature. Taking the hydrodynamic limit as $\varepsilon \rightarrow 0$ (the high field limit or the hyperbolic limit), the above VPFP system will lead to a first-order hyperbolic system for the density of particles coupled with the Poisson equation, cf. for example, [1, 11, 34, 55].

3. Proof of Theorem 2.1

3.1. Uniform Estimates and Existence

The free energy of the VPFP system (1.4)–(1.5) is defined as follows

$$\mathcal{E}(t) = \sum_{i=1}^N \int_{\Omega} \int_{\mathbb{R}^d} \left(\frac{|v|^2}{2\kappa_i} f_i^\varepsilon + \mathcal{H}(f_i^\varepsilon) \right) \, dv \, dx + \frac{\varpi}{2} \int_{\Omega} |\nabla_x \phi^\varepsilon|^2 \, dx, \quad (3.1)$$

where the function \mathcal{H} takes the form $\mathcal{H}(s) = s \log s$ for $s \geq 0$. The entropy productions of the VPFP system are given by

$$\begin{aligned} \mathcal{D}^i(w) &= \int_{\Omega} \int_{\mathbb{R}^d} (v\sqrt{w} + 2\kappa_i \nabla_v \sqrt{w})^2 \, dv \, dx \\ &= 4 \int_{\Omega} \int_{\mathbb{R}^d} \left| \nabla_v \sqrt{w e^{\frac{1}{2\kappa_i} |v|^2}} \right|^2 e^{-\frac{1}{2\kappa_i} |v|^2} \, dv \, dx, \quad i = 1, \dots, N. \end{aligned} \quad (3.2)$$

Moreover, we introduce the Darrozès–Guiraud information on the boundary (cf. for example, [21]) such that

$$\mathcal{I}^i(w) = \int_{\Sigma_+^x} \mathcal{H}(w) \, d\mu_x^i - \mathcal{H} \left(\int_{\Sigma_+^x} w \, d\mu_x^i \right), \quad i = 1, \dots, N,$$

where $d\mu_x^i(v) = M_i(v)|v \cdot \mathbf{n}(x)| \, dv$ are probability measures on Σ_{\pm}^x by the particular choice of the normalized Maxwellians M_i [cf. (2.2)].

First, we derive the energy dissipation property of the VPFP system (1.4)–(1.5) with initial and boundary conditions (2.3)–(2.5).

Proposition 3.1. (Energy dissipation) *The renormalized solution of the VFPF system (1.4)–(1.5) with described initial data and boundary conditions satisfies*

$$\partial_t n_i^\varepsilon + \nabla_x \cdot J_i^\varepsilon = 0, \tag{3.3}$$

where n_i^ε and J_i^ε are given in (2.6). Moreover, the following dissipative energy inequality holds

$$\begin{aligned} \mathcal{E}(t) + \frac{1}{\varepsilon^2} \sum_{i=1}^N \frac{\zeta_i}{\kappa_i} \int_0^t \mathcal{D}^i(f_i^\varepsilon) \, ds + \frac{1}{\varepsilon} \sum_{i=1}^N \int_0^t \int_{\partial\Omega} \mathcal{I}^i \left(\frac{\gamma + f_i^\varepsilon}{M_i(v)} \right) \, dS \, ds \\ \leq \mathcal{E}(0), \quad \forall t \geq 0. \end{aligned} \tag{3.4}$$

Proof. We just present a formal calculation which leads to (3.4). For $i = 1, \dots, N$, multiplying the i -th equation in (1.4) of the VFPF system by $\frac{1}{2}|v|^2$ and integrating the result with respect to v and x , we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{2} |v|^2 f_i^\varepsilon \, dv \, dx + \int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} |v|^2 v \cdot \nabla_x f_i^\varepsilon \, dv \, dx \\ - \int_{\Omega} \int_{\mathbb{R}^d} \frac{\kappa_i z_i}{2\varepsilon} |v|^2 \nabla_x \phi^\varepsilon \cdot \nabla_v f_i^\varepsilon \, dv \, dx \\ = \int_{\Omega} \int_{\mathbb{R}^d} \frac{\zeta_i}{2\varepsilon^2} |v|^2 L_{FP}^i(f_i^\varepsilon) \, dv \, dx, \end{aligned}$$

integrating by parts, we see that

$$\begin{aligned} \int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} |v|^2 v \cdot \nabla_x f_i^\varepsilon \, dv \, dx &= \frac{1}{2\varepsilon} \int_{\partial\Omega} \int_{\mathbb{R}^d} (v \cdot \mathbf{n}) |v|^2 \gamma f_i^\varepsilon \, dv \, dS, \\ - \int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} |v|^2 \nabla_x \phi^\varepsilon \cdot \nabla_v f_i^\varepsilon \, dv \, dx &= \frac{1}{\varepsilon} \int_{\Omega} \int_{\mathbb{R}^d} (v \cdot \nabla_x \phi^\varepsilon) f_i^\varepsilon \, dv \, dx \\ &= - \int_{\Omega} \phi^\varepsilon \nabla_x \cdot J_i^\varepsilon \, dx + \int_{\partial\Omega} \gamma \phi^\varepsilon J_i^\varepsilon \cdot \mathbf{n} \, dS \\ &= \int_{\Omega} \phi^\varepsilon \partial_t n_i^\varepsilon \, dx, \end{aligned}$$

and

$$\int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{2\varepsilon^2} |v|^2 L_{FP}^i(f_i^\varepsilon) \, dv \, dx = - \frac{1}{\varepsilon^2} \int_{\Omega} \int_{\mathbb{R}^d} (v f_i^\varepsilon + \kappa_i \nabla_v f_i^\varepsilon) \cdot v \, dv \, dx.$$

As a result, for $i = 1, \dots, N$ we obtain that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{2\kappa_i} |v|^2 f_i^\varepsilon \, dv \, dx + z_i \int_{\Omega} \phi^\varepsilon \partial_t n_i^\varepsilon \, dx \\ = - \frac{1}{2\kappa_i \varepsilon} \int_{\partial\Omega} \int_{\mathbb{R}^d} (v \cdot \mathbf{n}) |v|^2 \gamma f_i^\varepsilon \, dv \, dS - \frac{\zeta_i}{\kappa_i \varepsilon^2} \int_{\Omega} \int_{\mathbb{R}^d} (v f_i^\varepsilon + \kappa_i \nabla_v f_i^\varepsilon) \cdot v \, dv \, dx. \end{aligned} \tag{3.5}$$

Next, multiplying the i -th Equation (1.4) of the VFPF system by $\log f_i^\varepsilon$ and integrating the result with respect to v and x , we get

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}^d} \mathcal{H}(f_i^\varepsilon) \, dv \, dx + \int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{\varepsilon} (v \cdot \nabla_x f_i^\varepsilon) \log f_i^\varepsilon \, dv \, dx \\ & \quad - \int_{\Omega} \int_{\mathbb{R}^d} \frac{\kappa_i z_i}{\varepsilon} (\nabla_x \phi^\varepsilon \cdot \nabla_v f_i^\varepsilon) \log f_i^\varepsilon \, dv \, dx \\ & = \int_{\Omega} \int_{\mathbb{R}^d} \frac{\zeta_i}{\varepsilon^2} L_{\text{FP}}^i(f_i^\varepsilon) \log f_i^\varepsilon \, dv \, dx, \end{aligned}$$

after integrating by parts, we see that

$$\begin{aligned} & \int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{\varepsilon} (v \cdot \nabla_x f_i^\varepsilon) \log f_i^\varepsilon \, dv \, dx \\ & = -\frac{1}{\varepsilon} \int_{\Omega} \int_{\mathbb{R}^d} (v \cdot \nabla_x f_i^\varepsilon) \, dv \, dx + \frac{1}{\varepsilon} \int_{\partial\Omega} \int_{\mathbb{R}^d} (v \cdot \mathbf{n}) \gamma f_i^\varepsilon \log \gamma f_i^\varepsilon \, dv \, dS \\ & = -\int_{\partial\Omega} J_i^\varepsilon \cdot \mathbf{n} \, dS + \frac{1}{\varepsilon} \int_{\partial\Omega} \int_{\mathbb{R}^d} (v \cdot \mathbf{n}) \gamma f_i^\varepsilon \log \gamma f_i^\varepsilon \, dv \, dS \\ & = \frac{1}{\varepsilon} \int_{\partial\Omega} \int_{\mathbb{R}^d} (v \cdot \mathbf{n}) \gamma f_i^\varepsilon \log \gamma f_i^\varepsilon \, dv \, dS, \\ & \quad - \int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{\varepsilon} (\nabla_x \phi^\varepsilon \cdot \nabla_v f_i^\varepsilon) \log f_i^\varepsilon \, dv \, dx = 0, \end{aligned}$$

and

$$\int_{\Omega} \int_{\mathbb{R}^d} \frac{\zeta_i}{\varepsilon^2} L_{\text{FP}}^i(f_i^\varepsilon) \log f_i^\varepsilon \, dv \, dx = -\frac{\zeta_i}{\varepsilon^2} \int_{\Omega} \int_{\mathbb{R}^d} (v f_i^\varepsilon + \kappa_i \nabla_v f_i^\varepsilon) \cdot \frac{\nabla_v f_i^\varepsilon}{f_i^\varepsilon} \, dv \, dx.$$

As a consequence, we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}^d} \mathcal{H}(f_i^\varepsilon) \, dv \, dx & = -\frac{1}{\varepsilon} \int_{\partial\Omega} \int_{\mathbb{R}^d} (v \cdot \mathbf{n}) \gamma f_i^\varepsilon \log \gamma f_i^\varepsilon \, dv \, dS \\ & \quad - \frac{\zeta_i}{\varepsilon^2} \int_{\Omega} \int_{\mathbb{R}^d} (v f_i^\varepsilon + \kappa_i \nabla_v f_i^\varepsilon) \cdot \frac{\nabla_v f_i^\varepsilon}{f_i^\varepsilon} \, dv \, dx. \end{aligned} \tag{3.6}$$

Moreover, we see that for f_i^ε

$$\int_{\Omega} \int_{\mathbb{R}^d} (v f_i^\varepsilon + \kappa_i \nabla_v f_i^\varepsilon) \cdot \left(v + \frac{\kappa_i \nabla_v f_i^\varepsilon}{f_i^\varepsilon} \right) \, dv \, dx = \mathcal{D}^i(f_i^\varepsilon). \tag{3.7}$$

Due to the Poisson equation (1.5), we have

$$\int_{\Omega} \phi^\varepsilon \partial_t \left(\sum_{i=1}^N z_i n_i^\varepsilon \right) \, dx = -\varpi \int_{\Omega} \phi^\varepsilon \partial_t \Delta_x \phi^\varepsilon \, dx = \frac{\varpi}{2} \frac{d}{dt} \int_{\Omega} |\nabla_x \phi^\varepsilon|^2 \, dx. \tag{3.8}$$

Then we conclude from (3.5)–(3.8) that

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) + \sum_{i=1}^N \frac{\zeta_i}{\kappa_i \varepsilon^2} \mathcal{D}^i(f_i^\varepsilon) + \frac{1}{\varepsilon} \sum_{i=1}^N \int_{\partial\Omega} \int_{\mathbb{R}^d} (v \cdot \mathbf{n}) \\ \times \left(\frac{1}{2\kappa_i} |v|^2 + \log \gamma f_i^\varepsilon \right) \gamma f_i^\varepsilon \, dv \, dS = 0. \end{aligned} \tag{3.9}$$

Recall that $d\mu_x^i(v) = M_i(v)|v \cdot \mathbf{n}(x)|dv$ are probability measures on Σ_\pm^x [see the definition of $M_i(v)$ (2.2) and (2.15)]. Then for the boundary terms, we can apply the Darrozès–Guiraud inequality [21], namely, thanks to (2.3), the convexity of $\mathcal{H}(s) = s \log s$ and the Jensen inequality we deduce that (see also [53])

$$\begin{aligned} & \int_{\partial\Omega} \int_{\mathbb{R}^d} (v \cdot \mathbf{n}) \left(\frac{1}{2\kappa_i} |v|^2 + \log \gamma f_i^\varepsilon \right) \gamma f_i^\varepsilon \, dv \, dS \\ &= \int_{\partial\Omega} \int_{\Sigma_+^x} \mathcal{H} \left(\frac{\gamma_+ f_i^\varepsilon}{M_i(v)} \right) d\mu_x^i \, dS - \int_{\partial\Omega} \int_{\Sigma_-^x} \mathcal{H} \left(\frac{\gamma_- f_i^\varepsilon}{M_i(v)} \right) d\mu_x^i \, dS \\ &= \int_{\partial\Omega} \int_{\Sigma_+^x} \mathcal{H} \left(\frac{\gamma_+ f_i^\varepsilon}{M_i(v)} \right) d\mu_x^i \, dS - \int_{\partial\Omega} \mathcal{H} \left(\int_{\Sigma_+^x} \frac{\gamma_+ f_i^\varepsilon}{M_i(v)} d\mu_x^i \right) dS \\ &= \int_{\partial\Omega} \mathcal{I}^i \left(\frac{\gamma_+ f_i^\varepsilon}{M_i(v)} \right) dS \\ &\geq 0. \end{aligned}$$

As a consequence,

$$\frac{d}{dt} \mathcal{E}(t) + \frac{1}{\varepsilon^2} \sum_{i=1}^N \frac{\zeta_i}{\kappa_i} \mathcal{D}^i(f_i^\varepsilon) + \frac{1}{\varepsilon} \sum_{i=1}^N \int_{\partial\Omega} \mathcal{I}^i \left(\frac{\gamma_+ f_i^\varepsilon}{M_i(v)} \right) dS \leq 0. \tag{3.10}$$

Integrating (3.10) with respect to time, we arrive at our conclusion (3.4). \square

The energy dissipation (3.4) yields the following global estimates that are uniform in the parameter ε , which enable us to take the diffusion limit as $\varepsilon \rightarrow 0$:

Lemma 3.1. *For any $T > 0$, there exists a constant C depending on $C_0, \zeta_i, \kappa_i, \varpi$, but independent of ε and $t \in [0, T]$ such that*

$$\begin{aligned} & \int_{\Omega} \int_{\mathbb{R}^d} (1 + |v|^2 + |\log(f_i^\varepsilon)|) f_i^\varepsilon \, dv \, dx \leq C, \\ & \int_{\Omega} |\nabla_x \phi^\varepsilon|^2 \, dx \leq C, \\ & \frac{1}{\varepsilon^2} \int_0^t \mathcal{D}^i(f_i^\varepsilon) \, ds \leq C, \\ & \frac{1}{\varepsilon} \int_0^t \int_{\partial\Omega} \mathcal{I}^i \left(\frac{\gamma_+ f_i^\varepsilon}{M_i(v)} \right) dS \, ds \leq C. \end{aligned}$$

The functions f_i^ε are weakly relatively compact in $L^1((0, T) \times \Omega \times \mathbb{R}^d)$ and fulfill

$$\|\nabla_v \sqrt{f_i^\varepsilon}\|_{L^2((0,T) \times \Omega \times \mathbb{R}^d)} \leq C.$$

Concerning the fluxes, we have

$$\|J_i^\varepsilon(t, \cdot)\|_{L^1(\Omega)} \leq \frac{1}{2\varepsilon^2} \mathcal{D}^i(f_i^\varepsilon) + \frac{1}{2} \|f_{i0}^\varepsilon\|_{L^1(\Omega \times \mathbb{R}^d)}.$$

Proof. The proof is similar to [29, Propositions 5.1, 5.2, 5.3], based on the energy inequality (3.4). Since we are now dealing with the bounded domain, we do not need to estimate terms like $\int_\Omega \int_{\mathbb{R}^d} |x| f_i^\varepsilon \, dv \, dx$ as in [29]. The L^1 weak compactness of f_i^ε follows from the well-known Dunford–Pettis theorem. \square

We recall that the initial boundary value problem of a full Vlasov–Poisson–Fokker–Planck–Boltzmann system (subject to more general reflection boundary conditions for the distribution function but only for one species of charged particles) has been studied in the recent paper [53]. The author proved the existence of DiPerna–Lions renormalized solutions by using the approximation procedure in [51] with some crucial trace theorems previously introduced by the same author for the Vlasov equations [52] and some new results concerning weak-weak convergence (the renormalized convergence and the biting L^1 -weak convergence). For the current case with multiple species of charged particles, the coupling between different species is somewhat weak, that is, only via the Poisson equation. As a result, based on the energy dissipation property Proposition 3.1 and Lemma 3.1, we are able to prove the following existence result on renormalized solutions to the VPFP system (1.4)–(1.5), by adapting the argument in [53] (see also [9, 49, 51]) with minor modifications. The details are thus omitted.

Theorem 3.1. (Existence of renormalized solution) *Suppose that the assumptions (2.21)–(2.23) on the initial data are satisfied. For arbitrary but fixed $\varepsilon > 0$, the initial boundary value problem of the VPFP system (1.4)–(1.5) admits at least one (renormalized) solution $(f_i^\varepsilon, \phi^\varepsilon)$ in the sense of Definition 2.1, which satisfies Proposition 3.1.*

3.2. Low Field Limit as $\varepsilon \rightarrow 0$

The proof of Theorem 2.1 mainly follows the arguments in [29] for the VPFP system that concerns only one single species of particles in the whole space. However, for the present problem involving multiple species of charged particles, we need to modify the previous argument to deal with nonlocal interactions between particles as well as the boundary conditions. In what follows, we state the essential steps and point out the possible differences in the proof.

Step 1. Strong convergence of the electric potential ϕ^ε .

Based on the uniform estimates in Lemma 3.1, it is straightforward to argue as [49, Proposition 3.3] to conclude that

Lemma 3.2. *The renormalized solution $(f_i^\varepsilon, \phi^\varepsilon)$ satisfies the following properties:*

- (1) for $i = 1, \dots, N$, $n_i^\varepsilon(t, x) = \int_{\mathbb{R}^d} f_i^\varepsilon(t, x, v) \, dv$ are weakly relatively compact in $L^1((0, T) \times \Omega)$,
- (2) $\phi^\varepsilon(t, x)$ is relatively compact in $L^2(0, T; W^{1,p}(\Omega))$ with $1 \leq p < 2$.

Therefore, the strong convergence of ϕ^ε (2.25) (up to a subsequence) is a direct consequence of Lemma 3.2.

Step 2. Strong convergence of the charge densities n_i^ε .

Lemma 3.2 also implies the weak compactness of densities n_i^ε . Indeed, we can show the convergence of density functions in the strong sense. By using the definition of renormalized solutions (cf. Definition 2.1) and a velocity averaging lemma (cf. [49, Lemma 4.2], also [23]), we are able to obtain the compactness of the densities (cf. [29, Proposition 6.1]) such that the densities n_i^ε are relatively compact in $L^1((0, T) \times \Omega)$, namely, there exist $n_i \in L^1((0, T) \times \Omega)$ and up to a subsequence if necessary,

$$n_i^\varepsilon \rightarrow n_i, \quad \text{in } L^1((0, T) \times \Omega) \text{ and almost everywhere as } \varepsilon \rightarrow 0. \quad (3.11)$$

The above result and the simple inequality $(\sqrt{a} - \sqrt{b})^2 \leq |a - b|$ imply that

$$\sqrt{n_i^\varepsilon} \rightarrow \sqrt{n_i}, \quad \text{in } L^2((0, T) \times \Omega) \text{ and almost everywhere as } \varepsilon \rightarrow 0. \quad (3.12)$$

Step 3. Strong convergence of the distribution functions f_i^ε .

We recall the classical Csiszar–Kullback inequality (cf. [20, Theorem 3.1, Section 4, pp. 314], see also [44]) that for all non-negative $u \in L^1(\mathbb{R}^d, d\mu)$ (where $d\mu$ is a probability measure) with $\int_{\mathbb{R}^d} u \, d\mu = 1$, it holds

$$\|u - 1\|_{L^1(\mathbb{R}^d, d\mu)} \leq 2 \left(\int_{\mathbb{R}^d} (u \log u - u + 1) \, d\mu \right)^{\frac{1}{2}}.$$

Choose in the above inequality

$$u = \frac{f_i^\varepsilon}{n_i^\varepsilon \tilde{M}_i(v)}, \quad d\mu = \tilde{M}_i(v) \, dv,$$

which easily implies

$$\begin{aligned} & \left(\int_{\Omega} \int_{\mathbb{R}^d} |f_i^\varepsilon - n_i^\varepsilon \tilde{M}_i(v)| \, dv \, dx \right)^2 \\ & \leq 4 \left(\int_{\Omega} n_i^\varepsilon \, dx \right) \int_{\Omega} \int_{\mathbb{R}^d} f_i^\varepsilon \log \left(\frac{f_i^\varepsilon}{n_i^\varepsilon \tilde{M}_i(v)} \right) \, dv \, dx. \end{aligned} \quad (3.13)$$

Next, we proceed to estimate the second factor in the righthand side of (3.13). Recalling the logarithmic Sobolev inequality (cf. for example, [35, Corollary 4.2])

$$\begin{aligned} & \int_{\mathbb{R}^d} |h(v')|^2 \log |h(v')| \, d\mu(v') \\ & \leq \int_{\mathbb{R}^d} |\nabla_{v'} h(v')|^2 \, d\mu(v') + \|h(v')\|_{L^2(\mathbb{R}^d, d\mu(v'))}^2 \log \|h(v')\|_{L^2(\mathbb{R}^d, d\mu(v'))}, \end{aligned}$$

where $d\mu(v')$ is the Gauss measure $d\mu(v') = (2\pi)^{-\frac{d}{2}} e^{-\frac{|v'|^2}{2}} dv$. Making the simple change of variable $v' \rightarrow \frac{v}{\sqrt{\kappa}}$ and denoting $h_\kappa(v) = h(v')$, we have

$$\begin{aligned} d\mu(v') &= \left(\frac{\kappa}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{1}{2\kappa}|v|^2} dv := d\mu_\kappa(v), \quad \|h(v')\|_{L^2(\mathbb{R}^d, d\mu(v'))} \\ &= \|h_\kappa(v)\|_{L^2(\mathbb{R}^d, d\mu_\kappa(v))}, \end{aligned}$$

which yields that

$$\begin{aligned} &\int_{\mathbb{R}^d} |h_\kappa(v)|^2 \log |h_\kappa(v)| d\mu_\kappa(v) \\ &\leq \kappa \int_{\mathbb{R}^d} |\nabla_v h_\kappa(v)|^2 d\mu_\kappa(v) + \|h_\kappa(v)\|_{L^2(\mathbb{R}^d, d\mu_\kappa(v))}^2 \log \|h_\kappa(v)\|_{L^2(\mathbb{R}^d, d\mu_\kappa(v))}. \end{aligned}$$

In the above inequality, we set

$$\kappa = \kappa_i, \quad h_\kappa(v) = \sqrt{\frac{f_i^\varepsilon}{\tilde{M}_i(v)}}, \quad d\mu_\kappa(v) = \tilde{M}_i(v) dv.$$

Then we infer from the definition (2.6) that $\|h_\kappa\|_{L^2(\mathbb{R}^d, d\mu_\kappa(v))}^2 = n_i^\varepsilon$, which yields

$$\begin{aligned} &\int_{\Omega} \int_{\mathbb{R}^d} f_i^\varepsilon \log \left(\frac{f_i^\varepsilon}{n_i^\varepsilon \tilde{M}_i(v)} \right) dv dx \\ &= \int_{\Omega} \left(2 \int_{\mathbb{R}^d} |h_\kappa(v)|^2 \log |h_\kappa(v)| d\mu_\kappa(v) - 2 \|h_\kappa(v)\|_{L^2(\mathbb{R}^d, d\mu_\kappa(v))}^2 \right. \\ &\quad \left. \times \log \|h_\kappa(v)\|_{L^2(\mathbb{R}^d, d\mu_\kappa(v))} \right) dx \\ &\leq 2\kappa_i \int_{\Omega} \int_{\mathbb{R}^d} \left| \nabla_v \sqrt{\frac{f_i^\varepsilon}{\tilde{M}_i(v)}} \right|^2 \tilde{M}_i(v) dv dx \\ &= \frac{\kappa_i}{2} \mathcal{D}^i(f_i^\varepsilon). \end{aligned} \tag{3.14}$$

As a consequence, we infer from the entropy dissipation in (3.4), the uniform estimates in Lemma 3.1 and the estimates (3.13) and (3.14) that when $\varepsilon \rightarrow 0$,

$$f_i^\varepsilon - n_i^\varepsilon \tilde{M}_i \rightarrow 0, \quad \text{in } L^1((0, T) \times \Omega \times \mathbb{R}^d) \text{ and almost everywhere}$$

Combing the above results with the convergence result of n_i^ε (3.11), we conclude that as $\varepsilon \rightarrow 0$

$$f_i^\varepsilon \rightarrow n_i \tilde{M}_i, \quad \text{in } L^1((0, T) \times \Omega \times \mathbb{R}^d) \text{ and almost everywhere} \tag{3.15}$$

Here and below, the convergence results are always understood to be up to a subsequence.

Step 4. Weak convergence of the fluxes J_i^ε .

We introduce the auxiliary functions

$$r_i^\varepsilon = \frac{\sqrt{f_i^\varepsilon} - \sqrt{n_i^\varepsilon \tilde{M}_i(v)}}{\varepsilon \sqrt{\tilde{M}_i(v)}}, \quad i = 1, \dots, N. \tag{3.16}$$

In analogy to [49, Proposition 3.4] and [29, Proposition 5.5], we have

Lemma 3.3. *For arbitrary $T > 0$, the following uniform estimates hold*

$$\int_0^T \int_\Omega \int_{\mathbb{R}^d} \left(|r_i^\varepsilon|^2 \tilde{M}_i + \varepsilon |r_i^\varepsilon|^2 |v|^2 \tilde{M}_i + \sqrt{\varepsilon} |r_i^\varepsilon|^2 |v| \tilde{M}_i \right) dv dx dt \leq C,$$

where C is a constant that may depend on $C_0, \zeta_i, \kappa_i, \varpi$, but independent of ε and $t \in [0, T]$.

Using the expressions of r_i^ε [cf. (3.16)], we have

$$f_i^\varepsilon = n_i^\varepsilon \tilde{M}_i + 2\varepsilon \tilde{M}_i \sqrt{n_i^\varepsilon} r_i^\varepsilon + \varepsilon^2 |r_i^\varepsilon|^2 \tilde{M}_i. \tag{3.17}$$

Due to the simple facts $\int_{\mathbb{R}^d} v \tilde{M}_i(v) dv = 0$, it follows from (2.6), (3.12) and Lemma 3.3 that as $\varepsilon \rightarrow 0$

$$\begin{aligned} J_i^\varepsilon &= 2\sqrt{n_i^\varepsilon} \int_{\mathbb{R}^d} r_i^\varepsilon v \tilde{M}_i dv + \sqrt{\varepsilon} \int_{\mathbb{R}^d} \sqrt{\varepsilon} |r_i^\varepsilon|^2 v \tilde{M}_i dv \\ &\rightarrow 2\sqrt{n_i} \int_{\mathbb{R}^d} r_i v \tilde{M}_i dv, \quad \text{weakly in } L^1((0, T) \times \Omega), \end{aligned}$$

where r_i are the weak limits of r_i^ε , for $i = 1, \dots, N$.

It remains to identify the limit function of J_i^ε , which can be done by using a similar argument as in [29, Proposition 7.2]. The strong convergence of f_i^ε [see (3.15)] implies that for any fixed $\lambda > 0$,

$$\theta_{\varepsilon, \lambda}^i \rightarrow \sqrt{(n_i + \lambda) \tilde{M}_i}, \quad \text{as } \varepsilon \rightarrow 0.$$

On the other hand, it follows from (3.17) that for any $\lambda > 0$, when $\varepsilon \rightarrow 0$, we have

$$\frac{\zeta_i}{2\varepsilon} L_{\text{FP}}^i(f_i^\varepsilon) = \zeta_i L_{\text{FP}}^i \left(\tilde{M}_i \sqrt{n_i^\varepsilon} r_i^\varepsilon + \frac{\varepsilon}{2} |r_i^\varepsilon|^2 \tilde{M}_i \right) \rightarrow \zeta_i \sqrt{n_i} L_{\text{FP}}^i(r_i \tilde{M}_i).$$

As a consequence, in the renormalized formula (2.14), first for any fixed $\lambda > 0$ passing to the limit as $\varepsilon \rightarrow 0$ and then letting $\lambda \rightarrow 0$, we obtain that

$$\left(\nabla_x \sqrt{n_i} + \frac{z_i}{2} \nabla_x \phi \sqrt{n_i} \right) \cdot v \tilde{M}_i = \zeta_i L_{\text{FP}}^i(r_i \tilde{M}_i), \tag{3.18}$$

where ϕ is the limit of ϕ^ε [recall (2.25)].

On the other hand, it follows from [29, Proposition 3.1] that $\chi_j = -v_j \tilde{M}_i$ ($i = 1, \dots, N, j = 1, \dots, d$) is the unique solution to the equation $L_{FP}^i \chi_j = v_j \tilde{M}_i$ in $R(L_{FP}^i) \cap D(L_{FP}^i)$, where

$$\begin{aligned} L_{\tilde{M}_i}^2(\mathbb{R}^d) &= L^2(\mathbb{R}^d; \tilde{M}_i^{-1} dv), \\ R(L_{FP}^i) &= \left\{ f \in L_{\tilde{M}_i}^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} f(v) dv = 0 \right\}, \\ D(L_{FP}^i) &= \left\{ f \in L_{\tilde{M}_i}^2(\mathbb{R}^d) : \nabla_v \cdot \left(e^{-\frac{1}{2\kappa_i} |v|^2} \nabla_v (e^{\frac{1}{2\kappa_i} |v|^2} f) \right) \in L_{\tilde{M}_i}^2(\mathbb{R}^d) \right\}. \end{aligned}$$

Since $-L_{FP}^i$ is a self-adjoint operator on $L_{\tilde{M}_i}^2(\mathbb{R}^d)$, using (3.18), we have

$$\begin{aligned} J_i &= 2\sqrt{n_i} \int_{\mathbb{R}^d} r_i v \tilde{M}_i dv \\ &= 2\sqrt{n_i} \int_{\mathbb{R}^d} (r_i \tilde{M}_i) L_{FP}^i(-v \tilde{M}_i) \tilde{M}_i^{-1} dv \\ &= 2\sqrt{n_i} \int_{\mathbb{R}^d} L_{FP}^i(r_i \tilde{M}_i)(-v \tilde{M}_i) \tilde{M}_i^{-1} dv \\ &= \frac{2}{\zeta_i} \sqrt{n_i} \int_{\mathbb{R}^d} \left[(\nabla_x \sqrt{n_i} + \frac{z_i}{2} \nabla_x \phi \sqrt{n_i}) \cdot v \tilde{M}_i \right] (-v \tilde{M}_i) \tilde{M}_i^{-1} dv \\ &= -\frac{2}{\zeta_i} \sqrt{n_i} \left(\int_{\mathbb{R}^d} v \otimes v \tilde{M}_i dv \right) \left(\nabla_x \sqrt{n_i} + \frac{z_i}{2} \nabla_x \phi \sqrt{n_i} \right) \\ &= -\frac{2}{\zeta_i} \sqrt{n_i} \left(\nabla_x \sqrt{n_i} + \frac{z_i}{2} \nabla_x \phi \sqrt{n_i} \right). \end{aligned}$$

where we use the fact that $\int_{\mathbb{R}^d} v \otimes v \tilde{M}_i dv = \mathbb{I}$. Therefore, we can see that as $\varepsilon \rightarrow 0$

$$J_i^\varepsilon \rightarrow J_i := -\frac{2}{\zeta_i} \sqrt{n_i} \left(\nabla_x \sqrt{n_i} + \frac{z_i}{2} \nabla_x \phi \sqrt{n_i} \right) \tag{3.19}$$

in the distribution sense.

Step 5. Passage to the limit in the PDE system.

In order to recover the PNP system (2.16)–(2.20), we state a regularity result for the density functions n_i in the spirit of [49, Lemma 7.1]:

Lemma 3.4. *Let Ω be a smooth bounded and open set in \mathbb{R}^d . Assume n_i are positive functions belonging to $L^\infty(0, T; L^1(\Omega))$ and $\phi \in L^2(0, T; H^1(\Omega))$ that satisfy*

$$\begin{aligned} \nabla_x \sqrt{n_i} + \frac{z_i}{2} \nabla_x \phi \sqrt{n_i} &= G_i \in L^2(0, T; L^2(\Omega)), \quad i = 1, \dots, N, \tag{3.20} \\ -\varpi \Delta_x \phi &= \sum_{i=1}^N z_i n_i + D(x). \end{aligned}$$

Then we have

$$\sqrt{n_i} \in L^2(0, T; H^1(\Omega)), \quad \sum_{i=1}^N z_i n_i \in L^2(0, T; L^2(\Omega)), \quad \nabla_x \phi \sqrt{n_i} \in L^2(0, T; L^2(\Omega)).$$

Proof. As in [49, Corollary 3.2], we take $\beta_\delta(s) = \delta^{-1}\beta(\delta s)$ where $\beta \in C^\infty(\mathbb{R})$ satisfying $\beta(s) = s$ for $-1 \leq s \leq 1$, $0 \leq \beta'(s) \leq 1$ for $s \in \mathbb{R}$ and $\beta(s) = 2$ for $|s| \geq 3$. Then we renormalize the equations (3.20) for $\sqrt{n_i}$ such that

$$\nabla_x \beta_\delta(\sqrt{n_i}) + \frac{z_i}{2} \nabla_x \phi \beta'_\delta(\sqrt{n_i}) \sqrt{n_i} = G_i \beta'_\delta(\sqrt{n_i}) \in L^2(0, T; L^2(\Omega)). \quad (3.21)$$

For any $\delta > 0$, due to our choice of β and the given regularity for $\nabla_x \phi$, we have

$$\|\nabla_x \phi \beta'_\delta(\sqrt{n_i}) \sqrt{n_i}\|_{L^2(0, T; L^2(\Omega))} \leq \frac{3}{\delta} \|\nabla_x \phi\|_{L^2(0, T; L^2(\Omega))},$$

which together with (3.21) implies that $\nabla_x \beta_\delta(\sqrt{n_i}) \in L^2(0, T; L^2(\Omega))$. Then we can take L^2 norm on both sides of the Equation (3.21), summing up with respect to $i = 1, \dots, N$, we have

$$\begin{aligned} & \sum_{i=1}^N \|\nabla_x \beta_\delta(\sqrt{n_i})\|_{L^2(0, T; L^2(\Omega))}^2 + \sum_{i=1}^N \frac{z_i^2}{4} \|\nabla_x \phi \beta'_\delta(\sqrt{n_i}) \sqrt{n_i}\|_{L^2(0, T; L^2(\Omega))}^2 \\ & + \sum_{i=1}^N \int_0^T \int_\Omega [z_i \beta'_\delta(\sqrt{n_i}) \sqrt{n_i} \nabla_x \beta_\delta(\sqrt{n_i})] \cdot \nabla_x \phi \, dx \, dt \\ & \leq \sum_{i=1}^N \|G_i\|_{L^2(0, T; L^2(\Omega))}^2, \end{aligned}$$

where the right-hand side is independent of δ . For the crossing term on the left hand side, using integration by parts, we have

$$\begin{aligned} & \sum_{i=1}^N \int_0^T \int_\Omega [z_i \beta'_\delta(\sqrt{n_i}) \sqrt{n_i} \nabla_x \beta_\delta(\sqrt{n_i})] \cdot \nabla_x \phi \, dx \, dt \\ & = \sum_{i=1}^N \int_0^T \int_\Omega \nabla_x [z_i \tilde{\beta}_\delta(\sqrt{n_i})] \cdot \nabla_x \phi \, dx \, dt \\ & = \frac{1}{\varpi} \int_0^T \int_\Omega \sum_{i=1}^N z_i \tilde{\beta}_\delta(\sqrt{n_i}) \cdot \left(\sum_{i=1}^N z_i n_i + D(x) \right) \, dx \, dt \end{aligned}$$

where $\tilde{\beta}$ satisfies

$$\tilde{\beta}(s) = \int_0^s \tau \beta'(\tau)^2 \, d\tau, \quad \tilde{\beta}_\delta(s) = \delta^{-2} \tilde{\beta}(\delta s), \quad \tilde{\beta}_\delta(s) \rightarrow \frac{s^2}{2}, \quad \text{as } \delta \rightarrow 0.$$

Let $\delta \rightarrow 0$, we infer from the above estimates that

$$\begin{aligned} & \sum_{i=1}^N \|\nabla_x \sqrt{n_i}\|_{L^2(0,T;L^2(\Omega))}^2 + \sum_{i=1}^N \frac{z_i^2}{4} \|\nabla_x \phi \sqrt{n_i}\|_{L^2(0,T;L^2(\Omega))}^2 \\ & \quad + \frac{1}{2\varpi} \int_0^T \int_{\Omega} \left(\sum_{i=1}^N z_i n_i \right)^2 dx dt \\ & \leq \sum_{i=1}^N \|G_i\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{1}{2\varpi} \left| \int_0^T \int_{\Omega} D(x) \left(\sum_{i=1}^N z_i n_i \right) dx dt \right| \\ & \leq \sum_{i=1}^N \|G_i\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{1}{4\varpi} \int_0^T \int_{\Omega} |D(x)|^2 + \left(\sum_{i=1}^N z_i n_i \right)^2 dx dt, \end{aligned}$$

which easily yields the required regularity estimate. The lemma is proved. \square

Finally, using the above regularity lemma and the convergence result (3.19), we are able to write the currents J_i as in (2.18). Then we can pass to the limit as $\varepsilon \rightarrow 0$ in the weak form of Equation (3.3) as well as in the Poisson equation (2.10) to conclude that the limit functions (n_i, ϕ) satisfy the rescaled PNP system (2.16)–(2.20).

The proof of Theorem 2.1 is complete.

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