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On the Cauchy Problem for Axi-Symmetric Vortex Rings

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Abstract

We consider the classical Cauchy problem for the three dimensional Navier–Stokes equation with the initial vorticity ω_0 concentrated on a circle, or more generally, a linear combination of such data for circles with common axis of symmetry. We show that natural approximations of the problem obtained by smoothing the initial data satisfy good uniform estimates which enable us to conclude that the original problem with the singular initial distribution of vorticity has a solution. We impose no restriction on the size of the initial data.

1. Introduction

Let us consider the classical Cauchy problem for the Navier–Stokes equation in $\mathbb{R}^3 \times (0, \infty)$:

$$\left.\begin{aligned} u_t + \operatorname{div}(u \otimes u) + \nabla p - \nu \Delta u = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \times (0, \infty), \tag{1.1}$$

$$u(\,\cdot\,, 0) = u_0 \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

We will consider the initial data u_0 with vorticity $\omega_0 = \operatorname{curl} u_0$ which is supported on a circle. In terms of the geometric measure theory, ω_0 is a *1-current* of strength κ supported on a smooth circle γ. This means that for any smooth compactly supported test vector field (or, more precisely, 1-form) $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ we can write

$$\int_{\mathbb{R}^3} \omega_0 \cdot \varphi \, \mathrm{d}x = \kappa \int_\gamma \varphi_i(x) \, \mathrm{d}x_i, \tag{1.3}$$

where the last integral is the classical curve integral (summation over the repeated indices is understood). We will use the notation

$$\omega_0 = \kappa \delta_\gamma \tag{1.4}$$

in this situation. The initial velocity field is recovered from ω_0 via the Biot–Savart law

$$u_0(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\wedge\omega_0(y)}{|x-y|^3}\,\mathrm{d}y = -\frac{\kappa}{4\pi}\int_\gamma \frac{(x-y)\wedge \mathrm{d}y}{|x-y|^3}. \tag{1.5}$$

We note that such u_0 has infinite kinetic energy:

$$\int_{\mathbb{R}^3} \frac{1}{2}|u_0|^2\,\mathrm{d}x = +\infty, \tag{1.6}$$

due to the contributions from the immediate neighborhood of γ. The initial datum of this form and its regularized variants are usually referred to as a *vortex ring*. Their study goes back to Kelvin. If γ is the circle $(r_0\cos\theta, r_0\sin\theta, 0)$ (with $-\pi \leqq \theta < \pi$) and $\kappa > 0$, we expect from Kelvin's calculations and the regularization due to the viscosity that at time t the ring $\kappa\delta_\gamma$ will "fatten" to thickness $\sim \sqrt{\nu t}$ and will be moving up along the z-axis at speed roughly

$$\frac{\kappa}{4\pi r_0}\log\frac{a}{\sqrt{\nu t}}, \tag{1.7}$$

where a is a suitable reference length.

Our goal here is to establish the existence of such a solution, although we will not verify rigorously the detailed behavior suggested by Kelvin's calculations. Our estimates will be less precise. On the other hand, our method will be quite robust, and can handle not only one vortex ring, but also a finite or even continuous combination (with coefficients of the same sign) of such as long as they have a common axis of symmetry. The last condition is crucial; our method relies on the rotational symmetry of the situation.

It is instructive to compare our problem with the situation of parallel recti-linear vorticies. When the initial vorticity is supported on a line l,

$$\omega_0 = \kappa\delta_l, \tag{1.8}$$

the solution of the problem is given simply by the "heat extension" of the initial data. When l is the x_3-axis, one has the text-book solution

$$\omega(x,t) = (0, 0, \kappa\Gamma_2(x_1, x_2, \nu t)), \tag{1.9}$$

where $\Gamma_2(x_1, x_2, \nu t) = \frac{1}{4\pi\nu t}e^{-\frac{x_1^2+x_2^2}{4\nu t}}$ is the two dimensional heat kernel. The non-linear term vanishes identically on these solutions. Uniqueness is a subtle problem. The uniqueness has been proved in the class of the solutions of the form

$$u = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0) \tag{1.10}$$

(two dimensional Navier–Stokes solutions), see [4,5], but uniqueness among the three dimensional solutions seems to be open.

When the line l is replaced by a collection of parallel lines l_i and

$$\omega_0 = \sum_i \kappa_i\delta_{l_i} \tag{1.11}$$

or possibly

$$\omega_0 = \int \kappa_i \delta_{l_i} \, \mathrm{d}\mu(i), \tag{1.12}$$

where μ is a probability measure, one no longer has explicit solutions. The existence problem becomes more difficult and was solved only in the 1980s in [2,7], see also [1,9]. Uniqueness is again a subtle issue and is known only in the class (1.10) of two dimensional solutions, see [6].

Another class of existence results was obtained in [8] for small data, see also [17]. In those papers the authors proved both existence and uniqueness (in suitable classes of functions) of the Cauchy problem (1.1), (1.2) for example in the case when the initial data u_0 is

$$\omega_0 = \kappa \delta_\gamma, \tag{1.13}$$

where γ is a smooth closed curve and κ is sufficiently small (with the notion of smallness depending on γ). These results are proved by perturbation theory, and also follow from later works based on perturbation theory, such as [10].

Our main result in this paper is the following:

Theorem 1.1. *Let γ be a circle, $\kappa \in \mathbb{R}$ and $\omega_0 = \kappa\delta_\gamma$. Then the Cauchy problem* (1.1), (1.2) *for the initial data u_0 given by ω_0 has a global solution which is smooth for $t > 0$. The initial condition for the vorticity is satisfied in the following weak sense: for any $\varphi \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$,*

$$\lim_{t\to 0} \int_{\mathbb{R}^3} \omega(x,t) \cdot \varphi(x) \, \mathrm{d}x = \int_{\mathbb{R}^3} \omega_0(x) \cdot \varphi(x) \, \mathrm{d}x, \tag{1.14}$$

where $\omega = \operatorname{curl} u$ is the vorticity field.

At the level of the velocity field we have

$$\lim_{t\to 0} ||u(\cdot,t) - u_0||_{L^p(\mathbb{R}^3)} \to 0 \tag{1.15}$$

for each $p \in (1,2)$.

Remark.

1. Our method can be used to show that the same results hold when $\omega_0 = \int \kappa(\gamma)\delta_\gamma \, \mathrm{d}\mu(\gamma)$, where μ is a probability measure supported on the set of the circles with a given axis of symmetry, and $\kappa(\gamma) \geqq 0$ is an integrable function with respect to μ.
2. The sense in which the initial condition u_0 is assumed is somewhat weak, see (1.1). A more precise analysis than ours is needed to determine the optimal convergence of $\omega(\cdot,t) \to \omega_0$ as $t \to 0_+$.

We now outline the main ideas involved in the proof. By using the following transformation

$$u(x,t) \mapsto \nu u(x,\nu t), \quad p(x,t) \mapsto \nu^2 p(x,\nu t), \tag{1.16}$$

we can change the first equation in (1.1) to

$$u_t + \operatorname{div}(u \otimes u) + \nabla p - \Delta u = 0. \tag{1.17}$$

Therefore, without loss of generality, we can assume $\nu = 1$. Let us work with the vorticity equation (obtained by taking the curl of the Navier–Stokes equations)

$$\omega_t + u\nabla\omega - \omega\nabla u = \Delta\omega, \tag{1.18}$$

which simplifies significantly for the axi-symmetric velocity fields with no swirl which we will be considering. The precise definition is as follows.

Definition 1.2. (*Axi-symmetric vector field*). A vector field u in $\mathbb{R}^3$ is axi-symmetric if there is a coordinate frame in which it can be written as

$$u = u_r(r,z)e_r + u_\theta(r,z)e_\theta + u_z(r,z)e_z, \tag{1.19}$$

where

$$e_r = (x_1/r, x_2/r, 0), \quad e_\theta = (-x_2/r, x_1/r, 0), \quad e_z = (0,0,1) \tag{1.20}$$

and (r,θ,z) are the usual cylindrical coordinates associated with the frame. The components u_r, u_θ and u_z are independent of θ. The component u_θ is referred to as the swirl component of the vector field u (in the given frame). If u_θ vanishes, we say that u has no swirl.

It is easy to check that the curl of an axi-symmetric vector field $u = u_r e_r + u_z e_z$ with no swirl is of the form

$$\omega = \operatorname{curl} u = \left(u_{r,z} - u_{z,r}\right) e_\theta, \tag{1.21}$$

which has only the e_θ component, where $u_{r,z}$ denotes the partial derivative $\partial u_r/\partial z$, etc. We will seek the solution of (1.18) in the form $\omega = \omega_\theta(r,z,t)e_\theta$ and the velocity field in the form $u = u_r(r,z,t)e_r + u_z(r,z,t)e_z$. The vorticity equation (1.18) simplifies to

$$\left(\frac{\omega_\theta}{r}\right)_t + u\nabla\left(\frac{\omega_\theta}{r}\right) = \Delta\left(\frac{\omega_\theta}{r}\right) + \frac{2}{r}\left(\frac{\omega_\theta}{r}\right)_{,r}. \tag{1.22}$$

The right hand side of (1.22) can be interpreted as the Laplacian in $\mathbb{R}^5 = \{(y_1, \dots, y_4, z)\}$ on functions which depend only on $r = \sqrt{y_1^2 + \cdots + y_4^2}$ and z. Therefore the quantity $\frac{\omega_\theta}{r}$ satisfies a maximum principle, see Lemma 3.4.

There are three main ingredients of the proof:

1. Nash-type estimates for the quantity $\frac{\omega_\theta}{r}$ based on equation (1.22) and the div-free nature of the field u. These estimates give a good decay of $\left\| \frac{\omega_\theta(t)}{r} \right\|_{L^\infty_x(\mathbb{R}^3)}$ in terms of $t^{-\alpha}$ for suitable $\alpha > 0$, even when the initial condition for ω_θ is a Dirac distribution, see (3.28).
2. The use of the conservation of the vorticity flux and momentum, which are, respectively, the quantities $\int \omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z$ and $\int r^2\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z$.
3. Weighted inequalities for axi-symmetric fields with no swirl, such as

$$\|u\|_{L^\infty_x(\mathbb{R}^3)} \leqq C\,\|r\omega_\theta\|^{\frac14}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega_\theta}{r}\right\|^{\frac14}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega_\theta}{r}\right\|^{\frac12}_{L^\infty_x(\mathbb{R}^3)}. \tag{1.23}$$

Step 1 is achieved by application of Nash's techniques [14] for estimates of equations with div-free drift. In our case they cannot quite be used directly, due to the singular behavior of the coefficients of $\frac{2}{r}\left(\frac{\omega_\theta}{r}\right)_{,r}$ near the z-axis which give extra terms in the Nash-type estimates. Fortunately, the terms have a good sign, see the second term on line 6 in (3.25) in the proof of Lemma 3.8. Inequality (1.23) seems to be of independent interest, and it gives information about u in terms of ω_θ, the quantity for which we have the most control.

Combining the results 1–3, we can then proceed along similar lines as [7]. The uniqueness of the solutions from the above theorem seems to be a difficult open problem. We conjecture that it is possible to prove uniqueness in some natural classes of axi-symmetric solutions without swirl, but uniqueness in the class of all reasonable three dimensional vector fields may be much harder to prove and one might perhaps even have counter-examples. We plan to consider these topics in a future work.

2. Weighted Inequalities

In this section, we present some weighted inequalities. We will have uniform upper bounds on three quantities related to the vorticity: $\|r\omega\|_{L^1_x(\mathbb{R}^3)}$, $\left\|\frac{\omega}{r}\right\|_{L^1_x(\mathbb{R}^3)}$, $\left\|\frac{\omega}{r}\right\|_{L^\infty_x(\mathbb{R}^3)}$, and our aim is to obtain further estimates on the velocity u from these bounds. The inequalities presented in this section will be sufficient for our purposes in this paper.

Proposition 2.1. *Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be such that $\|rf\|_{L^1_x(\mathbb{R}^3)}$, $\left\|\frac{f}{r}\right\|_{L^1_x(\mathbb{R}^3)}$ and $\left\|\frac{f}{r}\right\|_{L^\infty_x(\mathbb{R}^3)}$ are finite, where $r = \sqrt{x_1^2 + x_2^2}$. Then for every $1 \leqq p \leqq 2$, $f \in L^p_x(\mathbb{R}^3)$ and*

$$\|f\|_{L^p_x(\mathbb{R}^3)} \leqq \|rf\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{\frac{1}{p}-\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{1-\frac{1}{p}}_{L^\infty_x(\mathbb{R}^3)}.$$

Proof. We first prove the two cases of $p = 1$ and $p = 2$ and then use interpolation to prove the other cases. We can write

$$\begin{aligned}\int_{\mathbb{R}^3} |f|\,\mathrm{d}x &= \int_{\mathbb{R}^3} r^{\frac{1}{2}}\,|f|^{\frac{1}{2}}\,\frac{|f|^{\frac{1}{2}}}{r^{\frac{1}{2}}}\mathrm{d}x \\ &\leqq \left(\int_{\mathbb{R}^3} r\,|f|\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3} \frac{|f|}{r}\mathrm{d}x\right)^{\frac{1}{2}} = \|rf\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)}\left\|\frac{f}{r}\right\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)},\end{aligned}$$

which proves the case $p = 1$.

Next we consider

$$\left(\int_{\mathbb{R}^3} |f|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} r\,|f|\,\frac{|f|}{r}\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\leqq \left(\int_{\mathbb{R}^3} r\,|f|\,\left\|\frac{f}{r}\right\|_{L_x^\infty(\mathbb{R}^3)} \mathrm{d}x\right)^{\frac{1}{2}} = \|rf\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{2}} \left\|\frac{f}{r}\right\|_{L_x^\infty(\mathbb{R}^3)}^{\frac{1}{2}},$$

which proves the case $p = 2$.

Let $1 < p < 2$. We have

$$\|f\|_{L_x^p(\mathbb{R}^3)} \leqq \|f\|_{L_x^1(\mathbb{R}^3)}^{\frac{2}{p}-1} \|f\|_{L_x^2(\mathbb{R}^3)}^{2-\frac{2}{p}}$$
$$\leqq \left(\|rf\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{2}} \left\|\frac{f}{r}\right\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{2}}\right)^{\frac{2}{p}-1} \left(\|rf\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{2}} \left\|\frac{f}{r}\right\|_{L_x^\infty(\mathbb{R}^3)}^{\frac{1}{2}}\right)^{2-\frac{2}{p}}$$
$$= \|rf\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{2}} \left\|\frac{f}{r}\right\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{p}-\frac{1}{2}} \left\|\frac{f}{r}\right\|_{L_x^\infty(\mathbb{R}^3)}^{1-\frac{1}{p}}.$$

□

Remark 2.2. Under the assumption of Proposition 2.1, one cannot control $\|f\|_{L_x^p(\mathbb{R}^3)}$ for $p > 2$. It is not hard to exhibit counterexamples.

Corollary 2.3. *Assume that ω is a vector field on $\mathbb{R}^3$ such that*

$$\|r\omega\|_{L_x^1(\mathbb{R}^3)} < \infty,\ \left\|\frac{\omega}{r}\right\|_{L_x^1(\mathbb{R}^3)} < \infty,\ \left\|\frac{\omega}{r}\right\|_{L_x^\infty(\mathbb{R}^3)} < \infty. \tag{2.1}$$

Let u be the vector field constructed from ω via the Biot–Savart law,

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y)\,\mathrm{d}y. \tag{2.2}$$

Then for any $\frac{3}{2} < q \leqq 6$, $u \in L_x^q(\mathbb{R}^3)$ and

$$\|u\|_{L_x^q(\mathbb{R}^3)} \lesssim \|r\omega\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{2}} \left\|\frac{\omega}{r}\right\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{q}-\frac{1}{6}} \left\|\frac{\omega}{r}\right\|_{L_x^\infty(\mathbb{R}^3)}^{\frac{2}{3}-\frac{1}{q}}. \tag{2.3}$$

Proof. By Proposition 2.1 and (2.1), for any $1 \leqq p \leqq 2$, we have

$$\|\omega\|_{L_x^p(\mathbb{R}^3)} \leqq \|r\omega\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{2}} \left\|\frac{\omega}{r}\right\|_{L_x^1(\mathbb{R}^3)}^{\frac{1}{p}-\frac{1}{2}} \left\|\frac{\omega}{r}\right\|_{L_x^\infty(\mathbb{R}^3)}^{1-\frac{1}{p}}. \tag{2.4}$$

Then by the classical Hardy-Littlewood-Sobolev inequality (see for instance [15, 19]), one can get

$$\|u\|_{L_x^q(\mathbb{R}^3)} \lesssim \|\omega\|_{L_x^p(\mathbb{R}^3)}, \quad \text{for } p \in (1,3) \text{ and } \frac{1}{q} = \frac{1}{p} - \frac{1}{3},$$

which, combining with (2.4), implies (2.3). □

Remark 2.4. By interpolation, the upper bounds (2.1) imply

$$\left\| \frac{\omega}{r} \right\|_{L^p_x(\mathbb{R}^3)} < \infty, \quad \text{for all } 1 < p < \infty.$$

What can we say about the full gradient ∇u from the above bounds and (2.1)? This question is related to the theory of singular integral operators with weights. Here we will only consider this question for vector fields which are axi-symmetric.

It is natural to ask whether we can control other $L^q_x(\mathbb{R}^3)$ norms of u except $\frac{3}{2} < q \leqq 6$ under the assumptions of Corollary 2.3. The inequality (2.5) below indicates what can be expected in this situation. We prove this inequality as a warm-up for the proof of our main inequality (1.23).

Proposition 2.5. *Assume* $f = f(x_1, x_2, z) = f\left(\sqrt{x_1^2 + x_2^2}, z\right) : \mathbb{R}^3 \to \mathbb{R}$ *is smooth and vanishes at infinity. Assume in addition that* $\|r\nabla f\|_{L^1_x(\mathbb{R}^3)}$, $\left\|\frac{\nabla f}{r}\right\|_{L^1_x(\mathbb{R}^3)}$ *and* $\left\|\frac{\nabla f}{r}\right\|_{L^\infty_x(\mathbb{R}^3)}$ *are finite. Then we have*

$$\|f\|_{L^\infty_x(\mathbb{R}^3)} \lesssim \|r\nabla f\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\nabla f}{r}\right\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\nabla f}{r}\right\|^{\frac{1}{2}}_{L^\infty_x(\mathbb{R}^3)}. \tag{2.5}$$

Proof. Assume that $|f(r, z)|$ achieves its supremum at (r_0, z_0), that is,

$$\|f\|_{L^\infty_x(\mathbb{R}^3)} = |f(r_0, z_0)|.$$

By the boundedness of $\frac{\nabla f}{r}$, ∇f must vanish at $r = 0$ (the z-axis). In particular, $\nabla_z f = 0$ along the z-axis. Thus, $f(0, z) \equiv 0$ by the assumption that f vanishes at infinity. Therefore, without loss of generality, we can assume $r_0 > 0$. By the fundamental theorem of calculus and Hölder's inequality

$$\begin{aligned}
\|f\|_{L^\infty_x(\mathbb{R}^3)} &= |f(r_0, z_0)| = \left|f(r_0, z_0)^2\right|^{\frac{1}{2}} = \left|\int_{z_0}^{\infty} \partial_z f(r_0, z)^2 \, \mathrm{d}z\right|^{\frac{1}{2}} \\
&\lesssim \left(\int_{z_0}^{\infty} |f(r_0, z)|\,|\partial_z f(r_0, z)| \, \mathrm{d}z\right)^{\frac{1}{2}} \\
&= \left(\int_{z_0}^{\infty} \left|\int_{r_0}^{\infty} \partial_r f(r, z) \, \mathrm{d}r\right| |\partial_z f(r_0, z)| \, \mathrm{d}z\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{z_0}^{\infty} \int_{r_0}^{\infty} |\partial_r f(r, z)| \, \mathrm{d}r \, |\partial_z f(r_0, z)| \, \mathrm{d}z\right)^{\frac{1}{2}} \\
&= \left(\int_{z_0}^{\infty} \int_{r_0}^{\infty} \left(r\,|\partial_r f(r, z)|^{\frac{1}{2}}\, |\partial_r f(r, z)|^{\frac{1}{2}} \frac{1}{r}\right) \mathrm{d}r \, |\partial_z f(r_0, z)| \, \mathrm{d}z\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{z_0}^{\infty} \int_{r_0}^{\infty} \left(r\,|\partial_r f(r, z)|^{\frac{1}{2}}\, |\partial_r f(r, z)|^{\frac{1}{2}}\right) \mathrm{d}r \, \frac{|\partial_z f(r_0, z)|}{r_0} \, \mathrm{d}z\right)^{\frac{1}{2}}
\end{aligned}$$

$$\lesssim \left(\int_{z_0}^{\infty}\int_{r_0}^{\infty} r\,|\partial_r f(r,z)|^{\frac{1}{2}}\,|\partial_r f(r,z)|^{\frac{1}{2}}\,\mathrm{d}r\,\mathrm{d}z\right)^{\frac{1}{2}}\left(\sup_z \frac{|\partial_z f(r_0,z)|}{r_0}\right)^{\frac{1}{2}}$$

$$\lesssim \left(\int_{-\infty}^{\infty}\int_{0}^{\infty} r^2\,|\partial_r f(r,z)|\,\mathrm{d}r\,\mathrm{d}z\right)^{\frac{1}{4}}\left(\int_{-\infty}^{\infty}\int_{0}^{\infty} |\partial_r f(r,z)|\,\mathrm{d}r\,\mathrm{d}z\right)^{\frac{1}{4}}$$

$$\times\left(\sup_{\mathbb{R}^3} \frac{|\partial_z f(r,z)|}{r}\right)^{\frac{1}{2}}$$

$$\lesssim \|r\nabla f\|_{L^1_x(\mathbb{R}^3)}^{\frac{1}{4}}\left\|\frac{\nabla f}{r}\right\|_{L^1_x(\mathbb{R}^3)}^{\frac{1}{4}}\left\|\frac{\nabla f}{r}\right\|_{L^\infty_x(\mathbb{R}^3)}^{\frac{1}{2}}.$$

□

In light of (2.5), one might ask whether the following inequality is true:

$$\|u\|_{L^\infty_x(\mathbb{R}^3)} \lesssim \|r\omega\|_{L^1_x(\mathbb{R}^3)}^{\frac{1}{4}}\left\|\frac{\omega}{r}\right\|_{L^1_x(\mathbb{R}^3)}^{\frac{1}{4}}\left\|\frac{\omega}{r}\right\|_{L^\infty_x(\mathbb{R}^3)}^{\frac{1}{2}}. \tag{2.6}$$

We do not know whether (2.6) is true for general vector fields, but we will show that it turns out to be true for the class of axi-symmetric vector fields with no swirl, which is enough for our purposes here. We will use the *axi-symmetric Biot–Savart law*. To introduce it, we start from the so-called axi-symmetric stream function.

In cylindrical coordinates, the class of axi-symmetric vector fields with no swirl is in the form $u = u_r(r,z)e_r + u_z(r,z)e_z$, see Definition 1.2, and the divergence-free condition $\operatorname{div} u = 0$ turns out to be

$$(ru_r)_{,r} + (ru_z)_{,z} = 0,$$

which means that

$$ru_r = -\psi_{,z}, \quad ru_z = \psi_{,r}$$

for a suitable function $\psi = \psi(r,z)$, called the axi-symmetric stream function, similar to the two dimensional situation. Hence

$$u_r = -\frac{1}{r}\psi_{,z}, \quad u_z = \frac{1}{r}\psi_{,r}. \tag{2.7}$$

It is easy to check that the curl of an axi-symmetric field u with no swirl is in the form

$$\operatorname{curl} u = \omega_\theta e_\theta$$

with $\omega_\theta = u_{r,z} - u_{z,r}$. Therefore, we obtain

$$L\psi :\, = -\frac{1}{r}\psi_{,rr} + \frac{1}{r^2}\psi_{,r} - \frac{1}{r}\psi_{,zz} = \omega_\theta.$$

The inverse operator L^{-1} is given by

$$\psi(\bar{r},\bar{z}) = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\bar{r}r}{4\pi}\int_0^{2\pi} \frac{\cos\varphi \; \mathrm{d}\varphi}{\left[r^2+\bar{r}^2-2\bar{r}r\cos\varphi+(z-\bar{z})^2\right]^{\frac{1}{2}}}\,\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z. \tag{2.8}$$

For the axi-symmetric stream function and the derivation of (2.8), we refer the readers to [16]. We can express (2.8) somewhat more explicitly as

$$\begin{aligned}\psi(\bar{r},\bar{z}) &= \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\sqrt{\bar{r}r}}{2\pi}\int_0^{\pi} \frac{\cos\varphi \; \mathrm{d}\varphi}{\left[2(1-\cos\varphi)+\frac{(r-\bar{r})^2+(z-\bar{z})^2}{\bar{r}r}\right]^{\frac{1}{2}}}\,\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\sqrt{\bar{r}r}}{2\pi}F\left(\frac{(r-\bar{r})^2+(z-\bar{z})^2}{\bar{r}r}\right)\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z,\end{aligned} \tag{2.9}$$

where the function $F:(0,\infty)\to\mathbb{R}$ is defined by

$$F(s) := \int_0^{\pi} \frac{\cos\varphi \; \mathrm{d}\varphi}{\left[2(1-\cos\varphi)+s\right]^{\frac{1}{2}}}. \tag{2.10}$$

Let

$$G(\bar{r},\bar{z},r,z) = \frac{\sqrt{\bar{r}r}}{2\pi}F\left(\frac{(r-\bar{r})^2+(z-\bar{z})^2}{\bar{r}r}\right). \tag{2.11}$$

Then

$$\psi(\bar{r},\bar{z}) = \int_{-\infty}^{\infty}\int_0^{\infty} G(\bar{r},\bar{z},r,z)\,\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z.$$

By (2.7) and (2.11), we get

$$\begin{aligned}u_r(\bar{r},\bar{z}) &= \int_{-\infty}^{\infty}\int_0^{\infty}\left[-\frac{1}{\bar{r}}\frac{\partial G}{\partial \bar{z}}(\bar{r},\bar{z},r,z)\right]\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{z-\bar{z}}{\pi\bar{r}^{\frac{3}{2}}\sqrt{r}}F'\left(\frac{(r-\bar{r})^2+(z-\bar{z})^2}{\bar{r}r}\right)\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z,\end{aligned} \tag{2.12}$$

$$\begin{aligned}u_z(\bar{r},\bar{z}) &= \int_{-\infty}^{\infty}\int_0^{\infty}\left[\frac{1}{\bar{r}}\frac{\partial G}{\partial \bar{r}}(\bar{r},\bar{z},r,z)\right]\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\int_0^{\infty}\mathscr{L}(\bar{r},\bar{z},r,z)\,\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z,\end{aligned} \tag{2.13}$$

where

$$\mathscr{L}(\bar{r},\bar{z},r,z) = \frac{1}{\bar{r}}\frac{\partial G}{\partial \bar{r}}(\bar{r},\bar{z},r,z).$$

The formulae (2.12) and (2.13), representing the relations between u_r, u_z and ω_θ, represent the *axi-symmetric Biot–Savart law*. We calculate the kernel $\mathscr{L}$. Let $d^2 = (r-\bar{r})^2 + (z-\bar{z})^2$. Let $\xi = \xi(\bar{r}, \bar{z}, r, z) = \frac{d}{\sqrt{\bar{r}r}}$. Then by (2.11), we have

$$G(\bar{r}, \bar{z}, r, z) = \frac{d}{2\pi\xi} F(\xi^2) = \frac{d}{2\pi} H(\xi),$$

where $H(t) = \frac{F(t^2)}{t}$. Direct calculation shows that

$$H'(t) = 2F'(t^2) - \frac{F(t^2)}{t^2}, \quad \frac{\partial \xi}{\partial \bar{r}} = \xi\Big(\frac{\bar{r}-r}{d^2} - \frac{1}{2\bar{r}}\Big), \tag{2.14}$$

$$\begin{aligned}\mathscr{L} = \frac{1}{\bar{r}}\frac{\partial G}{\partial \bar{r}} &= \frac{1}{2\pi}\frac{\bar{r}-r}{\bar{r}^{\frac{3}{2}} r^{\frac{1}{2}}}\Big[\frac{H(\xi)}{\xi} + H'(\xi)\Big] - \frac{1}{4\pi}\xi^2 H'(\xi)\frac{\sqrt{r}}{\bar{r}^{\frac{3}{2}}} \\ &= \frac{1}{\pi}\frac{\bar{r}-r}{\bar{r}^{\frac{3}{2}} r^{\frac{1}{2}}} F'(\xi^2) + \frac{1}{4\pi}\Big[F(\xi^2) - 2\xi^2 F'(\xi^2)\Big]\frac{\sqrt{r}}{\bar{r}^{\frac{3}{2}}}. \end{aligned} \tag{2.15}$$

In the sequel, we are mainly interested in $\mathscr{L}$ at $(\bar{r}, \bar{z}) = (1, 0)$. We write it down explicitly:

$$\begin{aligned}\mathscr{L}(1, 0, r, z) = &\frac{1-r}{\pi r^{\frac{1}{2}}} F'\Big(\frac{(r-1)^2 + z^2}{r}\Big) \\ &+ \frac{\sqrt{r}}{4\pi}\Big[F\Big(\frac{(r-1)^2+z^2}{r}\Big) - 2\frac{(r-1)^2+z^2}{r} F'\Big(\frac{(r-1)^2+z^2}{r}\Big)\Big]. \end{aligned} \tag{2.16}$$

At first glance, comparing with the usual Biot–Savart law (2.2), the axi-symmetric Biot–Savart law (2.12) and (2.13) look more complicated and have no advantages. But (2.12) and (2.13) indeed capture some features of axi-symmetric fields with no swirl. Although the function F in (2.10) cannot be expressed in terms of elementary functions, it has nice asymptotic properties near $s = 0$ and $s = \infty$. By (2.10), it is obvious that

$$|F(s)| \lesssim \Big(\frac{1}{s}\Big)^{\frac{1}{2}}. \tag{2.17}$$

However, F actually has a slower blow-up at $s = 0$ and a faster decay at $s = \infty$ than (2.17) as: $|F(s)| \lesssim \log\frac{1}{s}$ near $s = 0$ and $|F(s)| \lesssim \left(\frac{1}{s}\right)^{\frac{3}{2}}$ near $s = \infty$. We will use the following simple properties of F.

Lemma 2.6. *For every non-negative integer k, the kth-derivative of F satisfies*

$$\Big|F^{(k)}(s)\Big| \lesssim_k \frac{1}{s^{k+\frac{1}{2}}}, \tag{2.18}$$

for all $s \in (0, \infty)$.

Proof. By (2.10),

$$|F(s)| \lesssim \int_0^\pi \frac{d\varphi}{s^{\frac{1}{2}}} \lesssim \frac{1}{s^{\frac{1}{2}}}.$$

Hence (2.18) is true for the case of $k = 0$. The first derivative of F is

$$F'(s) = -\frac{1}{2}\int_0^\pi \frac{\cos\varphi \; d\varphi}{\left[2(1-\cos\varphi)+s\right]^{\frac{3}{2}}}.$$

Therefore,

$$\left|F'(s)\right| \lesssim \int_0^\pi \frac{d\varphi}{s^{\frac{3}{2}}} \lesssim \frac{1}{s^{\frac{3}{2}}}.$$

Hence the case of $k = 1$ is also true. The remaining cases can be proved similarly. □

Lemma 2.7. *There exists an absolute constant* $0 < \varepsilon_0 < 1$ *such that for all* $s \in (0, \varepsilon_0)$*, the kth-derivative of F satisfies*

$$\begin{aligned} |F(s)| &\lesssim \log\frac{1}{s} \;\; \lesssim_\tau \frac{1}{s^\tau}, \quad \text{for every } \tau > 0, \text{ if } k = 0, \\ \left|F^{(k)}(s)\right| &\lesssim_k \frac{1}{s^k}, \quad \text{if } 0 < k \in \mathbb{N}. \end{aligned} \tag{2.19}$$

Proof. $F(s)$ has the following expansion near $s = 0$, see for instance [16]

$$F(s) = \left(\log\frac{1}{s}\right)(a_0 + a_1 s + a_2 s^2 + \cdots) + (b_0 + b_1 s + b_2 s^2 + \cdots),$$

with $a_0 = \frac{1}{2}$ and $b_0 = \log 8 - 2$. Hence

$$F(s) = \frac{1}{2}\log\frac{1}{s} + \log 8 - 2 + O\left(s\log\frac{1}{s}\right), \quad s \to 0_+.$$

The estimate (2.19) follows easily from the above expansion. □

Lemma 2.8. *There exists an absolute constant* $N_0 > 1$ *such that for every non-negative integer k, the kth-derivative of F satisfies*

$$\left|F^{(k)}(s)\right| \lesssim_k \frac{1}{s^{k+\frac{3}{2}}} \tag{2.20}$$

for all $s \in (N_0, \infty)$.

Proof. This is an easy calculation. □

The estimates in Lemmas 2.7 and 2.8 are local. But those restrictions can be easily removed with the aid of Lemma 2.6. As a consequence of Lemmas 2.6, 2.7 and 2.8, we have

Corollary 2.9. *For every non-negative integer k, the kth-derivative of F satisfies*

$$|F(s)| \lesssim_\tau \min\left(\left(\frac{1}{s}\right)^{\tau}, \left(\frac{1}{s}\right)^{\frac{1}{2}}, \left(\frac{1}{s}\right)^{\frac{3}{2}}\right), \quad \text{for every } 0 < \tau < \frac{1}{2}, \text{ if } k = 0,$$

$$\left|F^{(k)}(s)\right| \lesssim_k \min\left(\left(\frac{1}{s}\right)^{k}, \left(\frac{1}{s}\right)^{k+\frac{1}{2}}, \left(\frac{1}{s}\right)^{k+\frac{3}{2}}\right), \quad \text{if } 0 < k \in \mathbb{N},$$

for all $s \in (0, \infty)$.

With the aid of Corollary 2.9, controlling the $L^\infty_x(\mathbb{R}^3)$-norm of u via the upper bounds (2.1) becomes tractable. We need the following technical lemma.

Lemma 2.10. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $\|f\|_{L^1(\mathbb{R}^2)} < \infty$ and $\|f\|_{L^\infty(\mathbb{R}^2)} < \infty$. Let $K : \mathbb{R}^2 \to \mathbb{R}$ be such that $|K(x)| \leqq \frac{C}{|x-x_0|}$ for some positive constant C, some point $x_0 \in \mathbb{R}^2$ and for all $x \in \mathbb{R}^2$. Then*

$$\left|\int_{\mathbb{R}^2} K(x) f(x)\,\mathrm{d}x\right| \leqq 2\sqrt{2\pi}\, C \, \|f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^2)} \|f\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)}.$$

Proof. For any $\rho > 0$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} K(x) f(x)\,\mathrm{d}x\right| &\leqq \int_{|x-x_0|\leqq\rho} \frac{C}{|x-x_0|}\,|f(x)|\,\mathrm{d}x + \int_{|x-x_0|>\rho} \frac{C}{|x-x_0|}\,|f(x)|\,\mathrm{d}x \\ &\leqq 2\pi C\rho\,\|f\|_{L^\infty(\mathbb{R}^2)} + \frac{C}{\rho}\,\|f\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

After minimizing the last term, we can get the desired result. □

Since an axi-symmetric vector field u with no swirl is of the form $u = u_r(r,z)e_r + u_z(r,z)e_z$, to estimate the $L^\infty_x(\mathbb{R}^3)$ norm of u, it is enough to estimate the L^∞ norms of u_r and u_z over the rz-plane $\Omega := \{r \geqq 0, z \in \mathbb{R}\}$. We will use the following simple identities.

$$\begin{aligned}\|r\omega\|_{L^1_x(\mathbb{R}^3)} = 2\pi \left\|r^2\omega_\theta\right\|_{L^1(\Omega)}, \quad \left\|\frac{\omega}{r}\right\|_{L^1_x(\mathbb{R}^3)} = 2\pi \|\omega_\theta\|_{L^1(\Omega)}, \quad \left\|\frac{\omega}{r}\right\|_{L^\infty_x(\mathbb{R}^3)} \\ = \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)}.\end{aligned}$$

We first estimate the r-component u_r.

Proposition 2.11. *Let u_r be given by the formula (2.12) with ω_θ satisfying*

$$\left\|r^2\omega_\theta\right\|_{L^1(\Omega)} < \infty, \ \|\omega_\theta\|_{L^1(\Omega)} < \infty, \ \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} < \infty.$$

Then

$$\|u_r\|_{L^\infty(\Omega)} \leqq C_1 \left\|r^2\omega_\theta\right\|^{\frac{1}{4}}_{L^1(\Omega)} \|\omega_\theta\|^{\frac{1}{4}}_{L^1(\Omega)} \left\|\frac{\omega_\theta}{r}\right\|^{\frac{1}{2}}_{L^\infty(\Omega)}, \tag{2.21}$$

where C_1 is an absolute constant.

Proof. The estimate (2.21) is invariant under the scaling and the translation in the z variable

$$u_r(r,z) \mapsto u_r(\lambda r, \lambda z + z_0), \quad \omega_\theta(r,z) \mapsto \lambda \omega_\theta(\lambda r, \lambda z + z_0)$$

for every $\lambda > 0$ and every $z_0 \in \mathbb{R}$, and therefore it is enough to prove

$$|u_r(1,0)| \lesssim \left\| r^2 \omega_\theta \right\|_{L^1(\Omega)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{4}} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\Omega)}^{\frac{1}{2}}. \tag{2.22}$$

By (2.12)

$$u_r(1,0) = \int_{-\infty}^{\infty} \int_0^{\infty} \frac{z}{\pi \sqrt{r}} F'\left(\frac{(r-1)^2 + z^2}{r} \right) \omega_\theta(r,z)\, \mathrm{d}r\, \mathrm{d}z. \tag{2.23}$$

We split the right hand side of (2.23) into two parts. One is on the region

$$I_1 = \left\{ \frac{1}{2} \leqq r \leqq 2, -1 \leqq z \leqq 1 \right\}$$

and the other on the complement $I_2 = \Omega \setminus I_1$.

On I_1, by Corollary 2.9 (using $\left|F'(s)\right| \lesssim \frac{1}{s}$), the kernel of (2.23) can be estimated as

$$\left| \frac{z}{\pi \sqrt{r}} F'\left(\frac{(r-1)^2 + z^2}{r} \right) \right| \lesssim \frac{|z|}{\sqrt{r}} \frac{r}{(r-1)^2 + z^2} \lesssim \frac{1}{\sqrt{(r-1)^2 + z^2}}$$
$$= \frac{1}{|(r,z) - (1,0)|}.$$

Therefore, by Lemma 2.10 and the fact that $r \sim 1$ on I_1, we obtain

$$\begin{aligned} &\left| \iint_{I_1} \frac{z}{\pi \sqrt{r}} F'\left(\frac{(r-1)^2 + z^2}{r} \right) \omega_\theta(r,z)\, \mathrm{d}r\, \mathrm{d}z \right| \\ &= \left| \iint \frac{z}{\pi \sqrt{r}} F'\left(\frac{(r-1)^2 + z^2}{r} \right) \omega_\theta(r,z)\, \chi_{I_1}\, \mathrm{d}r\, \mathrm{d}z \right| \\ &\lesssim \|\omega_\theta\|_{L^1(I_1)}^{\frac{1}{2}} \|\omega_\theta\|_{L^\infty(I_1)}^{\frac{1}{2}} \lesssim \left\| r^2 \omega_\theta \right\|_{L^1(I_1)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(I_1)}^{\frac{1}{4}} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(I_1)}^{\frac{1}{2}}, \end{aligned} \tag{2.24}$$

where χ_{I_1} is the characteristic function of I_1.

On I_2, by Corollary 2.9, (using $\left|F'(s)\right| \lesssim \left(\frac{1}{s}\right)^{\frac{5}{2}}$), the kernel of (2.23) can be estimated as

$$\left| \frac{z}{\pi \sqrt{r}} F'\left(\frac{(r-1)^2 + z^2}{r} \right) \right| \lesssim \frac{|z|}{\sqrt{r}} \left(\frac{r}{(r-1)^2 + z^2} \right)^{\frac{5}{2}} \lesssim \frac{1}{(r-1)^2 + z^2},$$

which is square-integrable on I_2. Therefore, noting that $|\omega_\theta| = r^{\frac{1}{2}} |\omega_\theta|^{\frac{1}{4}} |\omega_\theta|^{\frac{1}{4}} \frac{|\omega_\theta|^{\frac{1}{2}}}{r^{\frac{1}{2}}}$, by Hölder's inequality, we obtain

$$\left| \iint_{I_2} \frac{z}{\pi\sqrt{r}} F'\left(\frac{(r-1)^2+z^2}{r}\right) \omega_\theta(r,z)\, \mathrm{d}r\, \mathrm{d}z \right| \lesssim \left\| r^2\omega_\theta \right\|_{L^1(I_2)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(I_2)}^{\frac{1}{4}} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(I_2)}^{\frac{1}{2}}. \tag{2.25}$$

Clearly, (2.23), (2.24) and (2.25) imply (2.22). The proposition is proved. □

To estimate u_z, we need the following technical lemma.

Lemma 2.12. *Assume that ω_θ is a function on Ω satisfying*

$$\left\| r^2\omega_\theta \right\|_{L^1(\Omega)} < \infty, \quad \|\omega_\theta\|_{L^1(\Omega)} < \infty, \quad \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\Omega)} < \infty.$$

Then

$$\int_{z=-\infty}^{\infty} \int_{r=2}^{\infty} |\omega_\theta(r,z)| \frac{r^2}{[(r-1)^2+z^2]^{\frac{3}{2}}}\, \mathrm{d}r\, \mathrm{d}z \lesssim \left\| r^2\omega_\theta \right\|_{L^1(\Omega)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{4}} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\Omega)}^{\frac{1}{2}}. \tag{2.26}$$

We remark that the integral domain Ω of the right hand side of (2.26) can be replaced by $\{r \geqq 2\}$, where $\{r \geqq 2\}$ is shorthand for the set $\{r \geqq 2, z \in \mathbb{R}\}$. But (2.26) is enough for our purpose.

Proof. We can't use Hölder's inequality directly to get (2.26) because on the region $\{r \geqq |z|\}$, the weight $\frac{r^2}{[(r-1)^2+z^2]^{\frac{3}{2}}} \sim \frac{1}{[(r-1)^2+z^2]^{\frac{1}{2}}}$, which is not square-integrable on that region. We introduce some notations. Let $d^2 = r^2+z^2$ and $f(r,z) = \frac{\omega_\theta(r,z)}{r}$. To prove (2.26), it is enough to show

$$\int_{z=-\infty}^{\infty} \int_{r=2}^{\infty} |f| \frac{r^3}{d^3}\, \mathrm{d}r\, \mathrm{d}z \lesssim \left\| r^3 f \right\|_{L^1(\Omega)}^{\frac{1}{4}} \|rf\|_{L^1(\Omega)}^{\frac{1}{4}} \|f\|_{L^\infty(\Omega)}^{\frac{1}{2}}. \tag{2.27}$$

By the Cauchy–Schwartz inequality, we have

$$\left\| r^2 f \right\|_{L^1(\Omega)} \leqq \left\| r^3 f \right\|_{L^1(\Omega)}^{\frac{1}{2}} \|rf\|_{L^1(\Omega)}^{\frac{1}{2}}.$$

Therefore, to prove (2.27), it is enough to prove

$$\int_{z=-\infty}^{\infty} \int_{r=2}^{\infty} |f| \frac{r^3}{d^3}\, \mathrm{d}r\, \mathrm{d}z \lesssim \left\| r^2 f \right\|_{L^1(\{r\geqq 2\})}^{\frac{1}{2}} \|f\|_{L^\infty(\{r\geqq 2\})}^{\frac{1}{2}}, \tag{2.28}$$

since $\{r \geqq 2\} \subset \Omega$. We may assume that f is a function supported in $\{r \geqq 2\}$ and vanishing elsewhere in Ω, otherwise, we can just replace f by $f\chi_{\{r\geqq 2\}}$. Under this assumption, it is enough to prove

$$\left\| f\frac{r^3}{d^3}\right\|_{L^1(\Omega)} \lesssim \left\| r^2 f\right\|_{L^1(\Omega)}^{\frac{1}{2}} \|f\|_{L^\infty(\Omega)}^{\frac{1}{2}}. \tag{2.29}$$

For $\lambda > 0$, let $f_\lambda(r,z) = \lambda^2 f(\lambda r, \lambda z)$. Clearly, f_λ is supported on $\{r \geqq \frac{2}{\lambda}\}$. It is easy to check that for every $\lambda > 0$, we have

$$\begin{aligned}\left\| f_\lambda \frac{r^3}{d^3}\right\|_{L^1(\Omega)} = \left\| f\frac{r^3}{d^3}\right\|_{L^1(\Omega)}, \|f_\lambda\|_{L^\infty(\Omega)} = \lambda^2 \|f\|_{L^\infty(\Omega)}, \left\| r^2 f_\lambda\right\|_{L^1(\Omega)} \\ = \lambda^{-2}\left\| r^2 f\right\|_{L^1(\Omega)}.\end{aligned}$$

We find $\lambda_0 > 0$ so that $\|f_{\lambda_0}\|_{L^\infty(\Omega)} = \|r^2 f_{\lambda_0}\|_{L^1(\Omega)}$. By calculation,

$$\lambda_0 = \left(\frac{\|r^2 f\|_{L^1(\Omega)}}{\|f\|_{L^\infty(\Omega)}}\right)^{\frac{1}{4}}.$$

To prove (2.29), it is enough to prove

$$\left\| f_{\lambda_0}\frac{r^3}{d^3}\right\|_{L^1(\Omega)} \lesssim \left\| r^2 f_{\lambda_0}\right\|_{L^1(\Omega)} + \|f_{\lambda_0}\|_{L^\infty(\Omega)}. \tag{2.30}$$

We distinguish two cases $0 < \lambda_0 \leqq 1$ and $\lambda_0 > 1$.

Case 1. $0 < \lambda_0 \leqq 1$.

By definition, f_{λ_0} is supported on $\left\{r \geqq \frac{2}{\lambda_0}\right\}$, which lies in $\{r \geqq 1\}$. On the support of f_{λ_0}, it is clear that $\frac{r^3}{d^3} \leqq 1 \leqq r^2$ and hence (2.30) is true.

Case 2. $\lambda_0 > 1$.

In this case, we have

$$\begin{aligned}\left\| f_{\lambda_0}\frac{r^3}{d^3}\right\|_{L^1(\Omega)} &\lesssim \int_{-\infty}^{\infty}\int_2^{\infty} |f_{\lambda_0}|\,\mathrm{d}r\,\mathrm{d}z + \|f_{\lambda_0}\|_{L^\infty(\Omega)} \int_{-\infty}^{\infty}\int_{\frac{2}{\lambda_0}}^{2} \frac{r^3}{d^3}\,\mathrm{d}r\,\mathrm{d}z \\ &\lesssim \left\| r^2 f_{\lambda_0}\right\|_{L^1(\Omega)} + \|f_{\lambda_0}\|_{L^\infty(\Omega)}.\end{aligned}$$

Therefore (2.30) is true. The lemma is proved. □

We now estimate the z-component u_z. The work for u_z is similar to that for u_r in Proposition 2.11 but some parts have to be treated differently.

Proposition 2.13. *Let u_z be given by the formula (2.13) with ω_θ satisfying*

$$\left\| r^2\omega_\theta\right\|_{L^1(\Omega)} < \infty, \quad \|\omega_\theta\|_{L^1(\Omega)} < \infty, \quad \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} < \infty.$$

Then

$$\|u_z\|_{L^\infty(\Omega)} \leqq C_2 \left\| r^2\omega_\theta\right\|_{L^1(\Omega)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)}^{\frac{1}{2}}, \tag{2.31}$$

where C_2 is an absolute constant.

Proof. Since the estimate (2.31) is invariant under the scaling and the translation in the z variable, it is enough to prove

$$|u_z(1,0)| \lesssim \left\| r^2\omega_\theta \right\|_{L^1(\Omega)}^{\frac14} \|\omega_\theta\|_{L^1(\Omega)}^{\frac14} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\Omega)}^{\frac12}. \tag{2.32}$$

By (2.13),

$$u_z(1,0) = \int_{-\infty}^{\infty}\int_0^{\infty} \mathscr{Z}(1,0,r,z)\,\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z, \tag{2.33}$$

where $\mathscr{Z}(1,0,r,z)$ is given by (2.16) as

$$\begin{aligned}\mathscr{Z}(1,0,r,z) &= \frac{1-r}{\pi r^{\frac12}} F'\left(\frac{(r-1)^2+z^2}{r}\right)\\ &\quad + \frac{\sqrt{r}}{4\pi}\left[F\left(\frac{(r-1)^2+z^2}{r}\right) - 2\frac{(r-1)^2+z^2}{r}F'\left(\frac{(r-1)^2+z^2}{r}\right)\right]\\ &:= \mathscr{Z}_1(r,z) + \mathscr{Z}_2(r,z). \end{aligned}\tag{2.34}$$

We split the right hand side of (2.33) into two parts. One is on the region

$$I_1 = \left\{\frac12 \leqq r \leqq 2, -1 \leqq z \leqq 1\right\}$$

and the other on the complement $I_2 = \Omega \setminus I_1$.

On I_1, by Corollary 2.9, $\mathscr{Z}_1$ can be estimated as (using $\left|F'(s)\right| \lesssim \frac1s$)

$$|\mathscr{Z}_1(r,z)| \lesssim \frac{|1-r|}{r^{\frac12}} \frac{r}{(r-1)^2+z^2} \lesssim \frac{1}{|(r,z)-(1,0)|}$$

and $\mathscr{Z}_2$ can be estimated as (using $|F(s)| \lesssim \left(\frac1s\right)^{\frac12}$ and $\left|F'(s)\right| \lesssim \left(\frac1s\right)^{\frac32}$)

$$\begin{aligned}|\mathscr{Z}_2(r,z)| &\lesssim \sqrt{r}\left[\left(\frac{r}{(r-1)^2+z^2}\right)^{\frac12} + \frac{(r-1)^2+z^2}{r}\left(\frac{r}{(r-1)^2+z^2}\right)^{\frac32}\right]\\ &\lesssim \frac{1}{|(r,z)-(1,0)|}.\end{aligned}$$

Therefore, by Lemma 2.10 and the fact that $r \sim 1$ on I_1, we obtain

$$\begin{aligned}&\left|\iint_{I_1} \mathscr{Z}(1,0,r,z)\,\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z\right|\\ &\quad\lesssim \|\omega_\theta\|_{L^1(I_1)}^{\frac12}\|\omega_\theta\|_{L^\infty(I_1)}^{\frac12} \lesssim \left\|r^2\omega_\theta\right\|_{L^1(I_1)}^{\frac14}\|\omega_\theta\|_{L^1(I_1)}^{\frac14}\left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(I_1)}^{\frac12}. \end{aligned}\tag{2.35}$$

On I_2, by Corollary 2.9, $\mathscr{Z}_1$ can be estimated as (using $\left|F'(s)\right| \lesssim \left(\frac1s\right)^{\frac52}$)

$$|\mathscr{Z}_1(r,z)| \lesssim \frac{|1-r|}{r^{\frac12}}\left(\frac{r}{(r-1)^2+z^2}\right)^{\frac52} \lesssim \frac{1}{(r-1)^2+z^2},$$

which is square-integrable on I_2. Therefore, by Hölder's inequality, we obtain

$$\left|\iint_{I_2} \mathscr{Z}_1(r,z)\,\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z\right| \lesssim \left\|r^2\omega_\theta\right\|_{L^1(I_2)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(I_2)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(I_2)}^{\frac{1}{2}}. \tag{2.36}$$

Unfortunately, the foregoing argument of $\mathscr{Z}_1$ does not work for $\mathscr{Z}_2$ because $\mathscr{Z}_2$ is not square-integrable on the region I_2. By Corollary 2.9, the best estimate for $\mathscr{Z}_2$ on I_2 is (using $|F(s)| \lesssim \left(\frac{1}{s}\right)^{\frac{3}{2}}$ and $\left|F'(s)\right| \lesssim \left(\frac{1}{s}\right)^{\frac{5}{2}}$)

$$\begin{aligned}|\mathscr{Z}_2(r,z)| &\lesssim \sqrt{r}\left[\left(\frac{r}{(r-1)^2+z^2}\right)^{\frac{3}{2}} + \frac{(r-1)^2+z^2}{r}\left(\frac{r}{(r-1)^2+z^2}\right)^{\frac{5}{2}}\right] \\ &\sim \frac{r^2}{[(r-1)^2+z^2]^{\frac{3}{2}}}.\end{aligned} \tag{2.37}$$

To overcome this difficulty, we split the region I_2 into two parts: "good" part $I_{21} := I_2 \cap \{r \leqq 2\}$ and "bad" part $I_{22} := I_2 \cap \{r > 2\} = \{r > 2\}$. By (2.37), $\mathscr{Z}_2$ is clearly square-integrable on I_{21} and therefore by Hölder's inequality, we obtain

$$\left|\iint_{I_{21}} \mathscr{Z}_2(r,z)\,\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z\right| \lesssim \left\|r^2\omega_\theta\right\|_{L^1(I_{21})}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(I_{21})}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(I_{21})}^{\frac{1}{2}}. \tag{2.38}$$

On the "bad" part I_{22}, by Lemma 2.12 and (2.37), we have

$$\begin{aligned}\left|\iint_{I_{22}} \mathscr{Z}_2(r,z)\,\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z\right| &\lesssim \int_{z=-\infty}^{\infty}\int_{r=2}^{\infty} |\omega_\theta(r,z)|\,\frac{r^2}{[(r-1)^2+z^2]^{\frac{3}{2}}}\,\mathrm{d}r\,\mathrm{d}z \\ &\lesssim \left\|r^2\omega_\theta\right\|_{L^1(\Omega)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)}^{\frac{1}{2}}.\end{aligned} \tag{2.39}$$

Clearly, (2.33), (2.34), (2.35), (2.36), (2.38) and (2.39) imply (2.32). The proposition is proved. □

The following proposition concerns the decay as $|x| \to \infty$.

Proposition 2.14. *Let $u = u_r e_r + u_z e_z$ with u_r given by (2.12) and u_z given by (2.13) and with ω_θ satisfying*

$$\left\|r^2\omega_\theta\right\|_{L^1(\Omega)} < \infty, \quad \|\omega_\theta\|_{L^1(\Omega)} < \infty, \quad \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} < \infty.$$

Then for every $\varepsilon > 0$, there exists a $R > 0$ such that for every $x \in \mathbb{R}^3$ with $|x| > R$, we have

$$|u(x)| \leqq \frac{\|r^2\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{2}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{2}}}{2(|x|-R)^2} + \frac{\varepsilon}{2}.$$

In particular, we have

$$\lim_{|x|\to\infty} |u(x)| = 0.$$

Proof. We can assume

$$\left\|r^2\omega_\theta\right\|_{L^1(\Omega)} > 0, \quad \|\omega_\theta\|_{L^1(\Omega)} > 0, \quad \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} > 0,$$

otherwise, $u \equiv 0$ and the assertions are obviously true. For any $\varepsilon > 0$, we can find a $R > 0$ so that $\omega_1 := \omega_\theta \chi_{\{r^2+z^2 \geqq R^2\}}$ satisfies

$$\|\omega_1\|_{L^1(\Omega)} < \frac{\varepsilon^4}{16(C_1^2+C_2^2)^2 \left\|r^2\omega_\theta\right\|_{L^1(\Omega)} \left\|\frac{\omega_\theta}{r}\right\|^2_{L^\infty(\Omega)}},$$

where C_1 and C_2 are the constants from Propositions 2.11 and 2.13. Let $\omega_2 = \omega_\theta - \omega_1$. Let u_1 and u_2 be the vector fields constructed from ω_1 and ω_2 via (2.12) and (2.13), respectively. Clearly, $u = u_1 + u_2$. By Propositions 2.11 and 2.13, we have

$$\|u_1\|_{L^\infty_x(\mathbb{R}^3)} \leqq \sqrt{C_1^2+C_2^2}\left\|r^2\omega_1\right\|^{\frac{1}{4}}_{L^1(\Omega)} \|\omega_1\|^{\frac{1}{4}}_{L^1(\Omega)} \left\|\frac{\omega_1}{r}\right\|^{\frac{1}{2}}_{L^\infty(\Omega)} \leqq \frac{\varepsilon}{2}. \tag{2.40}$$

We can also express u_2 in terms of ω_2 via the Biot–Savart law in Cartesian coordinates

$$u_2(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega_2\, e_\theta \,\mathrm{d}y.$$

Since ω_2 is supported in the ball $B_R(0)$, for any $|x| > R$, we have

$$|u_2(x)| \leqq \frac{1}{4\pi}\frac{\|\omega_2\|_{L^1_x(\mathbb{R}^3)}}{(|x|-R)^2} = \frac{1}{2}\frac{\|r\omega_2\|_{L^1(\Omega)}}{(|x|-R)^2} \leqq \frac{\left\|r^2\omega_\theta\right\|^{\frac{1}{2}}_{L^1(\Omega)} \|\omega_\theta\|^{\frac{1}{2}}_{L^1(\Omega)}}{2(|x|-R)^2}. \tag{2.41}$$

Clearly, (2.40) and (2.41) imply the first assertion. The second assertion follows immediately from the first one. □

Remark 2.15. In the statement of Proposition 2.14, the R depends not only on the norms

$$\left\|r^2\omega_\theta\right\|_{L^1(\Omega)}, \quad \|\omega_\theta\|_{L^1(\Omega)}, \quad \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} \tag{2.42}$$

but also on the distribution of ω_θ. For example, let $\omega_\theta(r,z) = \chi_{\{1\leqq r\leqq 2, |z|\leqq 1\}}$. Let $\omega_\theta^{z_0}(r,z) = \omega_\theta(r, z-z_0)$. Let $u^{z_0} = u_r^{z_0} e_r + u_z^{z_0} e_z$ be the vector field constructed from $\omega_\theta^{z_0}$ via (2.12) and (2.13). Obviously, we have

$$\begin{aligned}\left\|r^2\omega_\theta^{z_0}\right\|_{L^1(\Omega)} &= \left\|r^2\omega_\theta\right\|_{L^1(\Omega)}, \ \left\|\omega_\theta^{z_0}\right\|_{L^1(\Omega)} = \|\omega_\theta\|_{L^1(\Omega)}, \ \left\|\frac{\omega_\theta^{z_0}}{r}\right\|_{L^\infty(\Omega)} \\ &= \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)}, \\ u_r^{z_0}(r,z) &= u_r(r, z-z_0), \ u_z^{z_0}(r,z) = u_z(r, z-z_0),\end{aligned}$$

but u and $\{u^{z_0}\}_{z_0 \in \mathbb{R}}$ do not have a uniform decay since the profile of u^{z_0} is just the translation of that of u by z_0 in the z-direction. Nevertheless, they have the uniform decay rate in the r-direction. Actually, we can prove the following result that for any $0 < \varepsilon < \frac{1}{2}$ and any $x \in \mathbb{R}^3$ with $r = \sqrt{x_1^2 + x_2^2} \geqq 1$,

$$|u(x)| \leqq \frac{C}{r^{\frac{1}{2}-\varepsilon}}, \tag{2.43}$$

where the constant C depends only on the size of the norms in (2.42). But it is not clear whether (2.43) is optimal.

3. Uniform Estimates for Regularized Solutions

In this section, we present the uniform estimates for natural approximate solutions obtained by regularizing the initial data, before which, we introduce the notations used. The superscript "(ε)" indicates that the quantity (scalar or vector or tensor-valued) is induced by regularized initial data. Sometimes we use a function $f = f(r, z)$ defined on $[0, \infty) \times \mathbb{R}$ as a function defined on $\mathbb{R}^3$ in the following way:

$$f(x_1, x_2, z) = f\left(\sqrt{x_1^2 + x_2^2}, z\right), \quad \text{for } (x_1, x_2, z) \in \mathbb{R}^3.$$

Let us get back to our problem. The initial vorticity is

$$\omega_0 = \kappa \delta_\gamma, \tag{3.1}$$

where $\kappa \in \mathbb{R}$ and γ is a circle. Without loss of generality, we assume that γ is $(r_0 \cos\theta, r_0 \sin\theta, z_0)$ for some $r_0 > 0$, $z_0 \in \mathbb{R}$ and $-\pi \leqq \theta < \pi$. Then (3.1) is equivalent, in the sense of distribution, to

$$\omega_0 = \kappa \delta_{r_0, z_0} e_\theta, \tag{3.2}$$

where δ_{r_0,z_0} is the Dirac mass at (r_0, z_0) in the rz-plane. We will search for a solution in the class of axi-symmetric velocity fields with no swirl, which have the form

$$u = u_r(r, z, t)e_r + u_z(r, z, t)e_z. \tag{3.3}$$

The related vorticity fields have the form

$$\omega = \omega_\theta(r, z, t)e_\theta \tag{3.4}$$

with $\omega_\theta = u_{r,z} - u_{z,r}$. Note that a solution of the form (3.4) is formally compatible to the initial condition (3.2). The equation for ω_θ is

$$\partial_t \omega_\theta + u_r \omega_{\theta,r} + u_z \omega_{\theta,z} - \frac{u_r}{r}\omega_\theta = \omega_{\theta,rr} + \frac{1}{r}\omega_{\theta,r} - \frac{1}{r^2}\omega_\theta + \omega_{\theta,zz}, \tag{3.5}$$

which can also be written as:

$$\partial_t \omega_\theta + u \cdot \nabla \omega_\theta - \frac{u_r}{r} \omega_\theta = \Delta \omega_\theta - \frac{1}{r^2} \omega_\theta, \tag{3.6}$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$ is the scalar Laplacian in $\mathbb{R}^3$, expressed in the cylindrical coordinates. $u \cdot \nabla \omega_\theta = u_1 \omega_{\theta,1} + u_2 \omega_{\theta,2} + u_z \omega_{\theta,z}$ is equal to $u_r \omega_{\theta,r} + u_z \omega_{\theta,z}$. In terms of ω_θ, the initial condition (3.2) can be formulated as:

$$\omega_\theta(r, z, 0) = \kappa \delta_{r_0, z_0}. \tag{3.7}$$

We will not, however, use either (3.5) or (3.6) in our method because these two equations have a vortex-stretching term $-\frac{u_r}{r}\omega_\theta$. It is easier to work with the quantity $\eta = \omega_\theta / r$, which satisfies

$$\eta_t + u_r \eta_{,r} + u_z \eta_{,z} = \eta_{,rr} + \frac{3}{r}\eta_{,r} + \eta_{,zz}, \tag{3.8}$$

or

$$\eta_t + u \cdot \nabla \eta = \Delta \eta + \frac{2}{r}\eta_{,r}. \tag{3.9}$$

Remark 3.1. For a smooth vector field u, the apparent singularity of $\eta = \omega_\theta / r$ is only an artifact of the coordinate choice. The quantity η is actually a smooth function, even across the z-axis, as long as u is smooth, see [13].

3.1. Regularized Initial Data

In terms of η, the initial data (3.7) reads:

$$\eta_0(r, z) := \eta(r, z, 0) = \frac{\omega_\theta(r, z, 0)}{r} = \frac{\kappa \delta_{r_0, z_0}}{r} = \frac{\kappa}{r_0} \delta_{r_0, z_0}. \tag{3.10}$$

The last equality of (3.10) holds in the sense of distribution. If we take an arbitrary test function $\psi = \psi(r, z)$, then

$$\Big(\frac{\kappa \delta_{r_0, z_0}}{r}, \psi\Big) = \Big(\kappa \delta_{r_0, z_0}, \frac{\psi}{r}\Big) = \kappa \frac{\psi(r_0, z_0)}{r_0} = \Big(\frac{\kappa}{r_0}\delta_{r_0, z_0}, \psi\Big).$$

Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be the standard mollifier such that $\phi \in C_0^\infty(B_1(0))$, $\phi \geqq 0$ and $\int_{\mathbb{R}^2} \phi(y)\, \mathrm{d}y = 1$. Let $\phi^{(\varepsilon)}(y_1, y_2) := \varepsilon^{-2} \phi(\frac{y_1}{\varepsilon}, \frac{y_2}{\varepsilon})$. Here and in the sequel, we assume $0 < \varepsilon < \frac{r_0}{2}$. We define $\eta_0^{(\varepsilon)}$ by

$$\eta_0^{(\varepsilon)}(r, z) := \big(\phi^{(\varepsilon)} * \eta_0\big)(r, z) = \frac{\kappa}{r_0} \varepsilon^{-2} \phi\Big(\frac{r - r_0}{\varepsilon}, \frac{z - z_0}{\varepsilon}\Big). \tag{3.11}$$

Clearly, for every $0 < \varepsilon < \frac{r_0}{2}$, $\eta_0^{(\varepsilon)}$ has a compact support which stays away from the z-axis at least $\frac{r_0}{2}$. It is easy to check that

$$\pi\, |\kappa| \leqq \Big\| \eta_0^{(\varepsilon)} \Big\|_{L^1_x} \leqq 3\pi\, |\kappa|,$$
$$\frac{\pi}{4} |\kappa|\, r_0^2 \leqq \frac{2\pi\, |\kappa|}{r_0} (r_0 - \varepsilon)^3 \leqq \Big\| r^2 \eta_0^{(\varepsilon)} \Big\|_{L^1_x} \leqq \frac{2\pi\, |\kappa|}{r_0} (r_0 + \varepsilon)^3 \leqq \frac{27\pi}{4} |\kappa|\, r_0^2. \tag{3.12}$$

Remark 3.2. Note that $\left\|\eta_0^{(\varepsilon)}\right\|_{L_x^1} \sim |\kappa|$ and $\left\|r^2\eta_0^{(\varepsilon)}\right\|_{L_x^1} \sim |\kappa|\, r_0^2$. The bounds for $\left\|\eta_0^{(\varepsilon)}\right\|_{L_x^1}$ depend only on the strength $|\kappa|$ of the ring $\kappa\delta_{r_0,z_0}e_\theta$ but the bounds for $\left\|r^2\eta_0^{(\varepsilon)}\right\|_{L_x^1}$ depend on both the strength and r_0. Nevertheless, they are both independent of ε and will serve the uniform bounds. The inequalities in (3.12) are dimensionally consistent.

Corresponding to $\eta_0^{(\varepsilon)}$, the initial vorticity field $\omega_0^{(\varepsilon)}$ and velocity field $u_0^{(\varepsilon)}$ are

$$\omega_0^{(\varepsilon)} := r\,\eta_0^{(\varepsilon)}\,e_\theta \quad \text{and} \quad u_0^{(\varepsilon)}(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega_0^{(\varepsilon)}(y)\,\mathrm{d}y, \tag{3.13}$$

respectively and $\omega_0^{(\varepsilon)}$ has compact support.

3.2. Approximate Solutions for Regularized Initial Data

Obviously the velocity $u_0^{(\varepsilon)}$ in (3.13) is axi-symmetric and swirl-free, and for each ε, $u_0^{(\varepsilon)} \in H_x^k(\mathbb{R}^3)$ for any $k \geqq 0$ and satisfies

$$\operatorname{div} u_0^{(\varepsilon)} = 0, \quad \operatorname{curl} u_0^{(\varepsilon)} = \omega_0^{(\varepsilon)}. \tag{3.14}$$

Remark 3.3. We don't have a uniform bound for $H_x^k(\mathbb{R}^3)$ norms of $u_0^{(\varepsilon)}$, not even for the $L_x^2(\mathbb{R}^3)$ norms of $u_0^{(\varepsilon)}$.

Then by the result of [11,12,20], there exists a unique global-in-time smooth solution $u^{(\varepsilon)}$ for three dimensional Navier–Stokes equations satisfying the initial condition

$$u^{(\varepsilon)}(0) = u_0^{(\varepsilon)}. \tag{3.15}$$

Moreover $u^{(\varepsilon)}$ is axi-symmetric with no swirl, that is, in cylindrical coordinates,

$$u^{(\varepsilon)} = u_r^{(\varepsilon)}(r,z,t)e_r + u_z^{(\varepsilon)}(r,z,t)e_z.$$

We shall show that a subsequence of $\left\{u^{(\varepsilon)}\right\}_{0<\varepsilon<\frac{r_0}{2}}$ converges to a smooth solution with the ring $\kappa\delta_{r_0,z_0}e_\theta$ as initial vorticity. Corresponding to $u^{(\varepsilon)}$, the vorticity field $\omega^{(\varepsilon)}$ and the scalar quantity $\eta^{(\varepsilon)}$ are

$$\omega^{(\varepsilon)} = \operatorname{curl} u^{(\varepsilon)} = \left(u_{r,z}^{(\varepsilon)} - u_{z,r}^{(\varepsilon)}\right)e_\theta \quad \text{and} \quad \eta^{(\varepsilon)} = \frac{u_{r,z}^{(\varepsilon)} - u_{z,r}^{(\varepsilon)}}{r}, \tag{3.16}$$

respectively. As a result of (3.13), (3.14), (3.15) and (3.16), $\omega^{(\varepsilon)}$ and $\eta^{(\varepsilon)}$ satisfy the initial data in (3.13)

$$\omega^{(\varepsilon)}(0) = \omega_0^{(\varepsilon)}, \quad \eta^{(\varepsilon)}(0) = \eta_0^{(\varepsilon)}. \tag{3.17}$$

By (3.9) and Remark 3.1, $\eta^{(\varepsilon)}$ is a smooth solution of the following equation:

$$\eta_t^{(\varepsilon)} + u^{(\varepsilon)}\cdot\nabla\eta^{(\varepsilon)} = \Delta\eta^{(\varepsilon)} + \frac{2}{r}\eta_{,r}^{(\varepsilon)}, \quad \text{in } \mathbb{R}^3\times(0,\infty). \tag{3.18}$$

3.3. Uniform Estimates for Approximate Solutions

The following lemma says that $\eta^{(\varepsilon)}$ enjoys the strong maximum priciple, which is crucial for our arguments of obtaining the uniform estimates.

Lemma 3.4. *If* $\kappa > 0$ *(or,* < 0*), then* $\eta^{(\varepsilon)}(r, z, t) > 0$ *(or,* < 0*) for any* $r \geqq 0$, $z \in \mathbb{R}$ *and* $t > 0$.

Proof. We just prove the case of $\kappa > 0$. The case of $\kappa < 0$ can be proved similarly. We cannot apply the maximum principle directly to (3.18) since the coefficient of $\frac{2}{r}\eta^{(\varepsilon)}_{,r}$ is singular. Recalling that the Laplacian of a radially symmetric function $v(r)$ defined on $\mathbb{R}^n$ is $\Delta v = v''(r) + \frac{n-1}{r}v'(r)$, the right hand side of (3.18) can be appropriately interpreted as the Laplacian in $\mathbb{R}^5$ and we can recast (3.18) in $\mathbb{R}^5 \times (0, \infty)$. To this end, we introduce some notations. Define

$$\hat{\eta}^{(\varepsilon)}(x_1, x_2, x_3, x_4, z, t) := \eta^{(\varepsilon)}\Big(\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, z, t\Big),$$

$$\hat{u}^{(\varepsilon)}(x_1, x_2, x_3, x_4, z, t) := u_r^{(\varepsilon)}\Big(\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, z, t\Big)\hat{e}_r + u_z^{(\varepsilon)}\Big(\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, z, t\Big)\hat{e}_z,$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, \quad \hat{e}_r = \Big(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}, \frac{x_4}{r}, 0\Big), \quad \hat{e}_z = \Big(0, 0, 0, 0, 1\Big).$$

Then by (3.11), (3.17) and (3.18), we have

$$\begin{cases} \hat{\eta}^{(\varepsilon)}_t + \hat{u}^{(\varepsilon)} \cdot \nabla_5 \hat{\eta}^{(\varepsilon)} = \Delta_5 \hat{\eta}^{(\varepsilon)}, & \text{in} \quad \mathbb{R}^5 \times (0, \infty), \\ \hat{\eta}^{(\varepsilon)}(0) \geqq 0, \text{ and } \not\equiv 0 & \text{in} \quad \mathbb{R}^5, \end{cases}$$

where,

$$\nabla_5 = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial z}\Big), \quad \Delta_5 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} + \frac{\partial^2}{\partial z^2}.$$

By strong maximum principle, we get

$$\hat{\eta}^{(\varepsilon)} > 0, \quad \text{in } \mathbb{R}^5 \times (0, \infty),$$

which implies

$$\eta^{(\varepsilon)} > 0.$$

Thus the lemma is proved. □

One of the important uniform estimates is the conservation of momentum.

Lemma 3.5. (Conservation of momentum). *For all* $t \geqq 0$, *we have*

$$\left\| r\omega^{(\varepsilon)}(t) \right\|_{L^1_x} = \left\| r\omega^{(\varepsilon)}(0) \right\|_{L^1_x} \leqq \frac{27\pi}{4} |\kappa| r_0^2. \tag{3.19}$$

Proof. By $\omega^{(\varepsilon)} = r\eta^{(\varepsilon)} e_\theta$, (3.19) is identical to

$$\left\| r^2 \eta^{(\varepsilon)}(t) \right\|_{L^1_x} = \left\| r^2 \eta^{(\varepsilon)}(0) \right\|_{L^1_x} \leqq \frac{27\pi}{4} |\kappa| r_0^2. \tag{3.20}$$

The "inequality" part of (3.20) follows from (3.12) and (3.17). It remains to prove the "equality" part, which is actually the conservation of momentum.

Since the initial vorticity field $\omega_0^{(\varepsilon)}$ in (3.13) is smooth and compactly supported, the vorticity field $\omega^{(\varepsilon)}$ remains Schwartz (smooth and having fast decay in all spatial derivatives) for all of the time. Therefore the momentum can be defined by using the vorticity as

$$\frac{1}{2} \int_{\mathbb{R}^3} \left(x \times \omega^{(\varepsilon)}(x, t) \right) \, \mathrm{d}x \, ,$$

and moreover, the momentum is conserved globally in time, that is

$$\frac{1}{2} \int_{\mathbb{R}^3} \left(x \times \omega^{(\varepsilon)}(x, t) \right) \, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^3} \left(x \times \omega^{(\varepsilon)}(x, 0) \right) \, \mathrm{d}x, \quad \text{for all } t > 0, \tag{3.21}$$

which can be checked by the vorticity equations (1.18), integration by parts and, the Schwartz property of the vorticity field $\omega^{(\varepsilon)}$.

By $\omega^{(\varepsilon)} = r\eta^{(\varepsilon)} e_\theta$,

$$\begin{aligned} x \times \omega^{(\varepsilon)} &= x \times r\eta^{(\varepsilon)} e_\theta \\ &= (x_1, x_2, x_3) \times \left(-x_2\eta^{(\varepsilon)}, x_1\eta^{(\varepsilon)}, 0 \right) \\ &= \left(-x_1 x_3 \eta^{(\varepsilon)}, -x_2 x_3 \eta^{(\varepsilon)}, r^2 \eta^{(\varepsilon)} \right). \end{aligned}$$

Noting that the first two components are odd in x_1 and x_2, respectively, we thus have

$$\int_{\mathbb{R}^3} \left(x \times \omega^{(\varepsilon)}(x, t) \right) \mathrm{d}x = \left(0, 0, \int_{\mathbb{R}^3} r^2 \eta^{(\varepsilon)}(x, t) \, \mathrm{d}x \right), \tag{3.22}$$

which, combining with (3.21), implies

$$\int_{\mathbb{R}^3} r^2 \eta^{(\varepsilon)}(x, t) \, \mathrm{d}x = \int_{\mathbb{R}^3} r^2 \eta^{(\varepsilon)}(x, 0) \, \mathrm{d}x, \quad \text{for all } t > 0.$$

Finally by Lemma 3.4, $\eta^{(\varepsilon)}(x, t)$ is nonnegative if $\kappa > 0$ (or, nonpositive if $\kappa < 0$) for all points $(x, t) \in \mathbb{R}^3 \times [0, \infty)$ and therefore we can get

$$\int_{\mathbb{R}^3} \left| r^2 \eta^{(\varepsilon)}(x, t) \right| \mathrm{d}x = \int_{\mathbb{R}^3} \left| r^2 \eta^{(\varepsilon)}(x, 0) \right| \mathrm{d}x.$$

We get (3.20) and the lemma is proved. □

Remark 3.6. The lemma says $\left\|r\omega^{(\varepsilon)}(t)\right\|_{L^1_x} \lesssim |\kappa|\, r_0^2$. (3.22) implies the total momentum of the fluid flow is in the z-direction. This is due to the special structure of axi-symmetric velocities with no swirl.

The following lemma claims that the L^1_x norms of $\frac{\omega^{(\varepsilon)}}{r}$ are uniformly bounded from above, which thus gives the second uniform estimate.

Lemma 3.7. *For all* $t \geqq 0$*, we have,*

$$\left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L^1_x} \leqq 3\pi\,|\kappa|\,.$$

Proof. By $\omega^{(\varepsilon)} = r\eta^{(\varepsilon)} e_\theta$, it suffices to prove

$$\left\|\eta^{(\varepsilon)}(t)\right\|_{L^1_x} \leqq 3\pi\,|\kappa|\,, \quad \text{for all} \quad t \geqq 0.$$

We just prove the case of $\kappa > 0$. The case of $\kappa < 0$ can be proved similarly. By Lemma 3.4, $\eta^{(\varepsilon)} \geqq 0$, direct calculation shows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\|\eta^{(\varepsilon)}(t)\right\|_{L^1_x(\mathbb{R}^3)} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \eta^{(\varepsilon)}\Big(x_1, x_2, z, t\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}z \\
&= \int_{\mathbb{R}^3}\Big(\Delta\eta^{(\varepsilon)} - u^{(\varepsilon)}\cdot\nabla\eta^{(\varepsilon)} + \frac{2}{r}\eta^{(\varepsilon)}_{,r}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}z \\
&= \int_{\mathbb{R}^3} \frac{2}{r}\eta^{(\varepsilon)}_{,r}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}z \\
&= 4\pi\int_{-\infty}^{\infty}\int_0^{\infty}\eta^{(\varepsilon)}_{,r}(r,z,t)\,\mathrm{d}r\,\mathrm{d}z = -4\pi\int_{-\infty}^{\infty}\eta^{(\varepsilon)}(0,z,t)\,\mathrm{d}z \\
&\leqq 0.
\end{aligned}$$

Thus $\left\|\eta^{(\varepsilon)}(t)\right\|_{L^1_x}$ is decreasing in time. Combining this with (3.12), we get

$$\left\|\eta^{(\varepsilon)}(t)\right\|_{L^1_x} \leqq \left\|\eta^{(\varepsilon)}(0)\right\|_{L^1_x} = \left\|\eta^{(\varepsilon)}_0\right\|_{L^1_x} \leqq 3\pi\,|\kappa|\,.$$

The lemma is proved. □

By Nash's method, we will now get uniform estimates of the L^p_x norms of $\frac{\omega^{(\varepsilon)}}{r}$, for all $1 \leqq p \leqq \infty$. Nash's method has been generalized in [3]. The key point in the proof below is that the drift term $\frac{2}{r}\eta^{(\varepsilon)}_{,r}$ has a good sign.

Lemma 3.8. *For every* $1 \leqq p \leqq \infty$*, we have,*

$$\left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L^p_x} \leqq C_p t^{-\frac{3}{2}(1-\frac{1}{p})}, \quad t \in (0,\infty), \tag{3.23}$$

where the constants C_p *are independent of* ε.

Proof. Note that (3.23) is valid for $p = 1$ with $C_1 = 3\pi |\kappa|$ by Lemma 3.7. Again by $\omega^{(\varepsilon)} = r\eta^{(\varepsilon)} e_\theta$, it suffices to prove

$$\left\| \eta^{(\varepsilon)}(t) \right\|_{L^p_x} \leqq C_p t^{-\frac{3}{2}(1-\frac{1}{p})}, \quad t \in (0, \infty). \tag{3.24}$$

Under the spirit of the energy method, for $p = 2^n$ with nonnegative integers n, we define

$$E^{(\varepsilon)}_p(t) := \left\| \eta^{(\varepsilon)}(t) \right\|^p_{L^p_x} = \int_{\mathbb{R}^3} \left| \eta^{(\varepsilon)}(x, t) \right|^p \mathrm{d}x.$$

For $p = 2^n$ with $n \geqq 1$, direct calculation yields that

$$\begin{aligned}
-\frac{\mathrm{d}E^{(\varepsilon)}_p}{\mathrm{d}t} &= -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \left|\eta^{(\varepsilon)}\right|^p \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \left(\eta^{(\varepsilon)}\right)^p \mathrm{d}x = -\int_{\mathbb{R}^3} p\left(\eta^{(\varepsilon)}\right)^{p-1} \eta^{(\varepsilon)}_t \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} p\left(\eta^{(\varepsilon)}\right)^{p-1} \left(\Delta\eta^{(\varepsilon)} + \frac{2}{r}\eta^{(\varepsilon)}_{,r} - u^{(\varepsilon)}\nabla\eta^{(\varepsilon)}\right) \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \left\{ p\left[\eta^{(\varepsilon)}\right]^{p-1} \Delta\eta^{(\varepsilon)} + \frac{2}{r}\left[(\eta^{(\varepsilon)})^p\right]_{,r} - u^{(\varepsilon)}\nabla\left[(\eta^{(\varepsilon)})^p\right] \right\} \mathrm{d}x \\
&= \int_{\mathbb{R}^3} p(p-1)\left[\eta^{(\varepsilon)}\right]^{p-2} \left|\nabla\eta^{(\varepsilon)}\right|^2 \mathrm{d}x - 4\pi \int_{-\infty}^{\infty}\int_0^{\infty} \left[(\eta^{(\varepsilon)})^p\right]_{,r} \mathrm{d}r\,\mathrm{d}z \\
&= \int_{\mathbb{R}^3} p(p-1) \left| \left[\eta^{(\varepsilon)}\right]^{\frac{p-2}{2}} \nabla\eta^{(\varepsilon)} \right|^2 \mathrm{d}x - 4\pi \int_{-\infty}^{\infty} \left[(\eta^{(\varepsilon)})^p\right]_{r=0}^{r=\infty} \mathrm{d}z \\
&= \int_{\mathbb{R}^3} p(p-1) \left| \frac{2}{p}\nabla\left[(\eta^{(\varepsilon)})^{\frac{p}{2}}\right] \right|^2 \mathrm{d}x + 4\pi \int_{-\infty}^{\infty} \left[(\eta^{(\varepsilon)})^p\right]_{r=0} \mathrm{d}z \\
&\geqq \frac{4(p-1)}{p} \int_{\mathbb{R}^3} \left| \nabla\left[(\eta^{(\varepsilon)})^{\frac{p}{2}}\right] \right|^2 \mathrm{d}x.
\end{aligned} \tag{3.25}$$

Recall Nash's inequality [14, P936]

$$\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x \geqq M\left(\int_{\mathbb{R}^3} |u|\right)^{-\frac{4}{3}} \left(\int_{\mathbb{R}^3} |u|^2\right)^{\frac{5}{3}}. \tag{3.26}$$

For $p = 2^n$ with $n \geqq 1$, by Nash's inequality, we get the following iteration scheme from (3.25),

$$\begin{aligned}
-\frac{\mathrm{d}E^{(\varepsilon)}_p}{\mathrm{d}t} &\geqq \frac{4(p-1)}{p} \int_{\mathbb{R}^3} \left| \nabla\left[(\eta^{(\varepsilon)})^{\frac{p}{2}}\right] \right|^2 \mathrm{d}x \\
&\geqq \frac{4(p-1)}{p} M\left(\int_{\mathbb{R}^3} \left|(\eta^{(\varepsilon)})^{\frac{p}{2}}\right|\right)^{-\frac{4}{3}} \left(\int_{\mathbb{R}^3} \left|(\eta^{(\varepsilon)})^{\frac{p}{2}}\right|^2\right)^{\frac{5}{3}} \\
&= \frac{4(p-1)}{p} M\left(E^{(\varepsilon)}_{p/2}\right)^{-\frac{4}{3}} \left(E^{(\varepsilon)}_p\right)^{\frac{5}{3}}.
\end{aligned} \tag{3.27}$$

We first prove (3.24) for $p = 2^n$ with nonnegative integers n by induction. Assume (3.24) is valid for $q = 2^k$ with $k \geqq 0$. Let $p = 2^{k+1}$. By (3.27), we have,

$$-\frac{\mathrm{d}E_p^{(\varepsilon)}}{\mathrm{d}t} \geqq \frac{4(p-1)}{p} M\left(E_q^{(\varepsilon)}\right)^{-\frac{4}{3}}\left(E_p^{(\varepsilon)}\right)^{\frac{5}{3}}$$
$$\geqq \frac{4(p-1)}{p} M\left(C_q^q t^{-\frac{3}{2}(q-1)}\right)^{-\frac{4}{3}}\left(E_p^{(\varepsilon)}\right)^{\frac{5}{3}}.$$

Thus,

$$\frac{3}{2}\left[(E_p^{(\varepsilon)})^{-\frac{2}{3}}\right]_t = -\frac{\frac{dE_p^{(\varepsilon)}}{\mathrm{d}t}}{\left(E_p^{(\varepsilon)}\right)^{\frac{5}{3}}} \geqq \frac{4(p-1)}{p} M C_q^{-\frac{4q}{3}} t^{2(q-1)}$$
$$= \frac{4(p-1)}{p} M C_q^{-\frac{2p}{3}} t^{p-2}.$$

Integration gives

$$(E_p^{(\varepsilon)})^{-\frac{2}{3}}(t) \geqq (E_p^{(\varepsilon)})^{-\frac{2}{3}}(t) - (E_p^{(\varepsilon)})^{-\frac{2}{3}}(0) \geqq \frac{8(p-1)}{3p} M C_q^{-\frac{2p}{3}} \int_0^t s^{p-2} ds$$
$$= \frac{8M}{3p} C_q^{-\frac{2p}{3}} t^{p-1},$$

which implies

$$\left\|\eta^{(\varepsilon)}(t)\right\|_{L_x^p} = E_p^{(\varepsilon)}(t)^{\frac{1}{p}} \leqq \left(\frac{3p}{8M}\right)^{\frac{3}{2p}} C_q t^{-\frac{3}{2}(1-\frac{1}{p})}.$$

Hence (3.24) is valid for $p = 2^{k+1}$ with $C_p = \left(\frac{3p}{8M}\right)^{\frac{3}{2p}} C_q$. In fact, C_p is uniformly bounded from above:

$$C_p = \left(\frac{3}{8M}\right)^{\frac{3}{2^{k+2}}} 2^{\frac{3(k+1)}{2^{k+2}}} C_{2^k} \leqq \left(\frac{3}{8M}\right)^{\sum \frac{3}{2^{i+2}}} 2^{\sum \frac{3(i+1)}{2^{i+2}}} C_1 =: C_\infty.$$

Therefore we obtain

$$\left\|\eta^{(\varepsilon)}(t)\right\|_{L_x^\infty} \leqq C_\infty t^{-\frac{3}{2}}.$$

For other p, we can prove (3.24) by interpolation. The lemma is proved. □

Remark 3.9. From the proof of Lemma 3.8, we see the constants C_p in (3.23) linearly depend on $C_1 = 3\pi\,|\kappa|$. In particular,

$$C_\infty = \left(\frac{3}{8M}\right)^{\sum \frac{3}{2^{i+2}}} 2^{\sum \frac{3(i+1)}{2^{i+2}}} C_1 \lesssim |\kappa|,$$
$$\left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L_x^\infty} \leqq C_\infty t^{-\frac{3}{2}} \lesssim |\kappa|\, t^{-\frac{3}{2}}, \tag{3.28}$$

which gives us the third uniform estimate, where M is the absolute constant in Nash's inequality (3.26).

Remark 3.10. If the fluid is inviscid, then $\eta^{(\varepsilon)}$ satisfies

$$\eta_t + u^{(\varepsilon)} \cdot \nabla \eta = 0, \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty). \tag{3.29}$$

Since $\eta^{(\varepsilon)}$ is conserved along particle trajectories, $\eta^{(\varepsilon)}$ keeps its sign in later time. We still have the uniform estimates of the L^1_x norms:

$$\left\| \eta^{(\varepsilon)}(t) \right\|_{L^1_x} = \left\| \eta^{(\varepsilon)}(0) \right\|_{L^1_x} = \left\| \eta_0^{(\varepsilon)} \right\|_{L^1_x} \leqq 3\pi \, |\kappa| \, .$$

However, the argument in Lemma 3.8 yields: for any $1 < p \leqq \infty$,

$$\left\| \eta^{(\varepsilon)}(t) \right\|_{L^p_x} = \left\| \eta^{(\varepsilon)}(0) \right\|_{L^p_x} = \left\| \eta_0^{(\varepsilon)} \right\|_{L^p_x},$$

which will blow up as ε goes to 0. Therefore we lose uniform controls of the L^p_x norms in the inviscid case.

We now use the weighted inequalities of the previous section and the three uniform estimates from Lemmas 3.5, 3.7 and Remark 3.9 to get further estimates on vorticity, the gradient of velocity, velocity and pressure.

Lemma 3.11. *For $0 < t < \infty$, we have the following estimates:*

(i) *for any* $1 \leqq p \leqq 2$

$$\left\| \omega^{(\varepsilon)}(t) \right\|_{L^p_x} \lesssim |\kappa| \, r_0 t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}, \tag{3.30}$$

(ii) *for any* $1 < p \leqq 2$

$$\left\| \nabla u^{(\varepsilon)}(t) \right\|_{L^p_x} \lesssim |\kappa| \, r_0 t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}, \tag{3.31}$$

(iii) *for any* $\frac{3}{2} < q \leqq 6$

$$\left\| u^{(\varepsilon)}(t) \right\|_{L^q_x} \lesssim |\kappa| \, r_0 t^{-\left(1-\frac{3}{2q}\right)}, \tag{3.32}$$

(iv) *for any* $1 < q \leqq 3$

$$\left\| p^{(\varepsilon)}(t) \right\|_{L^q_x} \lesssim |\kappa|^2 \, r_0^2 t^{-\left(2-\frac{3}{2q}\right)}, \tag{3.33}$$

(v)

$$\left\| u^{(\varepsilon)}(t) \right\|_{L^\infty_x} \lesssim |\kappa| \, r_0^{\frac{1}{2}} t^{-\frac{3}{4}}. \tag{3.34}$$

Proof.

(i) By Proposition 2.1, for any $1 \leqq p \leqq 2$, we have

$$\left\|\omega^{(\varepsilon)}(t)\right\|_{L_x^p} \leqq \left\|r\omega^{(\varepsilon)}(t)\right\|_{L_x^1}^{\frac{1}{2}} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L_x^1}^{\frac{1}{p}-\frac{1}{2}} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L_x^\infty}^{1-\frac{1}{p}}. \tag{3.35}$$

Then (3.30) is an easy consequence of (3.35), Lemmas 3.5, 3.7 and (3.28) in Remark 3.9.

(ii) By $\operatorname{div} u^{(\varepsilon)}=0$, $\operatorname{curl} u^{(\varepsilon)}=\omega^{(\varepsilon)} = (\omega_1^{(\varepsilon)}, \omega_2^{(\varepsilon)}, \omega_3^{(\varepsilon)})$ and the Fourier transform, one can get

$$\nabla u^{(\varepsilon)} = \begin{bmatrix} R_1R_2\omega_3^{(\varepsilon)} - R_1R_3\omega_2^{(\varepsilon)} & R_2R_2\omega_3^{(\varepsilon)} - R_2R_3\omega_2^{(\varepsilon)} & R_2R_3\omega_3^{(\varepsilon)} - R_3R_3\omega_2^{(\varepsilon)} \\ R_1R_3\omega_1^{(\varepsilon)} - R_1R_1\omega_3^{(\varepsilon)} & R_2R_3\omega_1^{(\varepsilon)} - R_1R_2\omega_3^{(\varepsilon)} & R_3R_3\omega_1^{(\varepsilon)} - R_1R_3\omega_3^{(\varepsilon)} \\ R_1R_1\omega_2^{(\varepsilon)} - R_1R_2\omega_1^{(\varepsilon)} & R_1R_2\omega_2^{(\varepsilon)} - R_2R_2\omega_1^{(\varepsilon)} & R_1R_3\omega_2^{(\varepsilon)} - R_2R_3\omega_1^{(\varepsilon)} \end{bmatrix},$$

where R_j, $j = 1, 2, 3$ are the classical Riesz transformations, which are well-defined and continuous on $L_x^p(\mathbb{R}^3)$ for all $1 < p < \infty$, see for instance [15,19]. Therefore

$$\left\|\nabla u^{(\varepsilon)}(t)\right\|_{L_x^p} \lesssim \left\|\omega^{(\varepsilon)}(t)\right\|_{L_x^p},$$

which, combining with (3.30), implies (3.31).

(iii) By Corollary 2.3, for any $\frac{3}{2} < q \leqq 6$,

$$\left\|u^{(\varepsilon)}(t)\right\|_{L_x^q} \lesssim \left\|r\omega^{(\varepsilon)}(t)\right\|_{L_x^1}^{\frac{1}{2}} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L_x^1}^{\frac{1}{q}-\frac{1}{6}} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L_x^\infty}^{\frac{2}{3}-\frac{1}{q}}. \tag{3.36}$$

Then (3.32) is an easy consequence of (3.36), Lemmas 3.5, 3.7 and (3.28).

(iv) Recall that the pressure $p^{(\varepsilon)}$ and the velocity $u^{(\varepsilon)} = \left(u_1^{(\varepsilon)}, u_2^{(\varepsilon)}, u_3^{(\varepsilon)}\right)$ satisfy the following equation (which can be easily obtained from Navier–Stokes equations and divergence-free condition $\operatorname{div} u^{(\varepsilon)}=0$):

$$\Delta p^{(\varepsilon)} = -\partial_j\partial_k(u_j^{(\varepsilon)}u_k^{(\varepsilon)}). \tag{3.37}$$

Then by (3.32), we can use the Riesz transformation R_j to solve (3.37) to get

$$p^{(\varepsilon)} = R_jR_k(u_j^{(\varepsilon)}u_k^{(\varepsilon)}).$$

Hence

$$\left\|p^{(\varepsilon)}(t)\right\|_{L_x^q} \lesssim \left\|u^{(\varepsilon)}(t)\right\|_{L_x^{2q}}^2,$$

which, combining with (3.32), implies (3.33).

(v) By Propositions 2.11 and 2.13,

$$\left\| u^{(\varepsilon)}(t) \right\|_{L_x^\infty} \lesssim \left\| r\omega^{(\varepsilon)}(t) \right\|_{L_x^1}^{\frac{1}{4}} \left\| \frac{\omega^{(\varepsilon)}(t)}{r} \right\|_{L_x^1}^{\frac{1}{4}} \left\| \frac{\omega^{(\varepsilon)}(t)}{r} \right\|_{L_x^\infty}^{\frac{1}{2}} . \tag{3.38}$$

Then (3.34) is an easy consequence of (3.38), Lemma 3.5, Lemma 3.7 and (3.28). □

By Lemma 3.11 and the subcritical theory of Navier–Stokes equations, we can control the spatial and time derivatives of the velocity and pressure of any order pointwise.

Lemma 3.12. *For any $k, h \geqq 0$ and for any $0 < s < T$, we have the following pointwise estimate*

$$\left\| \nabla_x^k \nabla_t^h u^{(\varepsilon)} \right\|_{C^0_{x,t}(\mathbb{R}^3 \times [s,T])} \leqq C, \quad \left\| \nabla_x^k \nabla_t^h p^{(\varepsilon)} \right\|_{C^0_{x,t}(\mathbb{R}^3 \times [s,T])} \leqq C,$$

where C is independent of ε and depends only on $k, h, s, T, |\kappa|, r_0$.

Proof. This lemma is a consequence of the subcritical well-posedness theory of Navier–Stokes equations. Fix $0 < s < T$. By (3.32), we have the following subcritical estimate

$$\left\| u^{(\varepsilon)}(t) \right\|_{L_x^6} \lesssim |\kappa| \, r_0 t^{-\frac{3}{4}}, \tag{3.39}$$

since $L_x^6(\mathbb{R}^3)$ is a subcritical space for Navier–Stokes equations with respect to the scaling

$$u(x,t) \longmapsto \lambda u(\lambda x, \lambda^2 t), \quad p(x,t) \longmapsto \lambda^2 p(\lambda x, \lambda^2 t).$$

By the standard subcritical theory, see for instance [7], there exists a local-in-time unique solution $v^{(\varepsilon)}$ for Navier–Stokes equations with $u^{(\varepsilon)}\left(\frac{s}{2}\right)$ as initial velocity in the space $C\left(\left[\frac{s}{2}, T_\varepsilon\right), L_x^6(\mathbb{R}^3)\right)$ for some $\frac{s}{2} < T_\varepsilon \leqq \infty$. $v^{(\varepsilon)}$ coincides with $u^{(\varepsilon)}$ on the time interval $[\frac{s}{2}, T_\varepsilon)$ by weak-strong uniqueness. The decay property (3.39) implies $T_\varepsilon = \infty$. Hence $u^{(\varepsilon)} = v^{(\varepsilon)}$ for all $t \in [\frac{s}{2}, \infty)$. Again by the subcritical theory, $u^{(\varepsilon)}$ satisfies

$$\left\| \nabla_x^k \nabla_t^h u^{(\varepsilon)} \right\|_{L_t^\infty L_x^6(\mathbb{R}^3 \times [s,T])} \leqq C, \tag{3.40}$$

where C depends only on $k, h, s, T, \left\| u^{(\varepsilon)}\left(\frac{s}{2}\right) \right\|_{L_x^6}$. Then by Sobolev embedding, we prove the first estimate. The second estimate is a consequence of (3.40) and (3.37). □

The estimate (3.32) in Lemma 3.11 implies that the set $\left\{u^{(\varepsilon)}\right\}_{0<\varepsilon<\frac{r_0}{2}}$ has weak compactness in Lebesgue spaces. To show the strong convergence of $\left\{u^{(\varepsilon)}\right\}_{0<\varepsilon<\frac{r_0}{2}}$, we need to establish certain uniform weak continuity of $u^{(\varepsilon)}$ as functions of time t. To this end, we use the Aubin–Lions theorem, see [2,18]. Let $H_x^{-2}(\mathbb{R}^3)$ be the dual space of $H_x^2(\mathbb{R}^3)$.

Lemma 3.13. *Let* $0 < T < \infty$. *Then we have*

$$\left\| \frac{\partial u^{(\varepsilon)}}{\partial t} \right\|_{L_t^{\frac{5}{4}}\left(0,T;H_x^{-2}(\mathbb{R}^3)\right)} \leqq C, \tag{3.41}$$

where the constant C is independent of ε and depends on T.

Proof. Let $\phi \in H_x^2(\mathbb{R}^3)$. By Navier–Stokes equations and Lemma 3.11, we have

$$\begin{aligned}
\left| \left(\frac{\partial u^{(\varepsilon)}}{\partial t}, \phi \right) \right| &= \left| \left(-\mathrm{div}(u^{(\varepsilon)} \otimes u^{(\varepsilon)}) - \nabla p^{(\varepsilon)} + \Delta u^{(\varepsilon)}, \phi \right) \right| \\
&\leqq \left| (u^{(\varepsilon)} \otimes u^{(\varepsilon)}, \nabla\phi) \right| + \left| (p^{(\varepsilon)}, \mathrm{div}\phi) \right| + \left| (u^{(\varepsilon)}, \Delta\phi) \right| \\
&\lesssim \left\| u^{(\varepsilon)}(t) \right\|_{L_x^{p_1}} \left\| u^{(\varepsilon)}(t) \right\|_{L_x^{p_2}} \|\nabla\phi\|_{L_x^{p_3}} + \left\| p^{(\varepsilon)}(t) \right\|_{L_x^{q_1}} \|\nabla\phi\|_{L_x^{q_2}} \\
&\quad + \left\| u^{(\varepsilon)}(t) \right\|_{L_x^2} \|\Delta\phi\|_{L_x^2} \\
&\lesssim |\kappa|^2 r_0^2 t^{-2+\frac{3}{2p_1}+\frac{3}{2p_2}} \|\nabla\phi\|_{L_x^{p_3}} + |\kappa|^2 r_0^2 t^{-2+\frac{3}{2q_1}} \|\nabla\phi\|_{L_x^{q_2}} + |\kappa| r_0 t^{-\frac{1}{4}} \|\phi\|_{H_x^2},
\end{aligned}$$

where

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad \frac{3}{2} < p_1, p_2 \leqq 6, \quad \frac{1}{q_1} + \frac{1}{q_2} = 1, \quad 1 < q_1 \leqq 3.$$

One can take, for example,

$$p_1 = p_2 = \frac{12}{5}, \quad p_3 = 6, \quad q_1 = \frac{6}{5}, \quad q_2 = 6.$$

Then by Sobolev embedding, we have for $0 < t \leqq T$,

$$\begin{aligned}
\left| \left(\frac{\partial u^{(\varepsilon)}}{\partial t}, \phi \right) \right| &\lesssim |\kappa|^2 r_0^2 t^{-\frac{3}{4}} \|\nabla\phi\|_{L_x^6} + |\kappa|^2 r_0^2 t^{-\frac{3}{4}} \|\nabla\phi\|_{L_x^6} + |\kappa| r_0 t^{-\frac{1}{4}} \|\phi\|_{H_x^2} \\
&\lesssim \left(|\kappa|^2 r_0^2 t^{-\frac{3}{4}} + |\kappa| r_0 t^{-\frac{1}{4}} \right) \|\phi\|_{H_x^2}.
\end{aligned}$$

Hence

$$\left\| \frac{\partial u^{(\varepsilon)}}{\partial t}(t) \right\|_{H_x^{-2}} \lesssim |\kappa|^2 r_0^2 t^{-\frac{3}{4}} + |\kappa| r_0 t^{-\frac{1}{4}}.$$

Finally, integrating with respect to time from $(0, T)$ yields the desired result. □

Lemma 3.14. *For any* $0 < T < \infty$, $\left\{ u^{(\varepsilon)} \right\}_{0<\varepsilon<\frac{r_0}{2}}$ *is precompact in* $L_t^{\frac{8}{5}}\big(0, T;$ $L_{x,\mathrm{loc}}^2(\mathbb{R}^3)\big)$.

Proof. By Lemma 3.11, one has

$$\left\|u^{(\varepsilon)}(t)\right\|_{L_x^{\frac{8}{5}}} \lesssim |\kappa|\, r_0 t^{-\frac{1}{16}}, \quad \left\|\nabla u^{(\varepsilon)}(t)\right\|_{L_x^{\frac{8}{5}}} \lesssim |\kappa|\, r_0 t^{-\frac{9}{16}},$$

which implies that

$$\left\{u^{(\varepsilon)}\right\}_{0<\varepsilon<\frac{r_0}{2}} \text{ is a bounded set of } L_t^{\frac{8}{5}}\left(0, T; W_x^{1,\frac{8}{5}}(\mathbb{R}^3)\right). \tag{3.42}$$

Then (3.42), Lemma 3.13 and Theorem 2.1 of [18, Chap. III] imply the desired result. □

4. Proof of Theorem 1.1

We first show that we can pass to the limit in the regularized solutions to get a smooth solution for the Navier–Stokes equations. (3.32) implies

$$\left\|u^{(\varepsilon)}(t)\right\|_{L_x^2} \lesssim |\kappa|\, r_0 t^{-\frac{1}{4}}, \tag{4.1}$$

which in turn implies that

$$\left\{u^{(\varepsilon)}\right\} \text{ is a bounded set in } L_t^{\frac{8}{3}} L_x^2\left(\mathbb{R}^3 \times (0, T)\right), \quad \text{for any } 0 < T < \infty. \tag{4.2}$$

Arzela–Ascoli's theorem, Lemmas 3.12, 3.14 and (4.2) allow us to extract a subsequece of $\left\{u^{(\varepsilon)}, p^{(\varepsilon)}\right\}$, still denoted as $\left\{u^{(\varepsilon)}, p^{(\varepsilon)}\right\}$ such that for a smooth vector field u and a smooth scalar function p, for any nonnegative integers k, h and for any $0 < T < \infty$, we have

$$\begin{aligned} u^{(\varepsilon)} &\to u \quad \text{in } L_t^{\frac{8}{5}}\left(0, T; L^2_{x,\text{loc}}(\mathbb{R}^3)\right), \\ u^{(\varepsilon)} &\rightharpoonup u \quad \text{in } L_t^{\frac{8}{3}} L_x^2\left(\mathbb{R}^3 \times (0, T)\right), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \nabla_x^k \nabla_t^h u^{(\varepsilon)} &\rightrightarrows \nabla_x^k \nabla_t^h u \quad \text{locally in } \mathbb{R}^3 \times (0, \infty), \\ \nabla_x^k \nabla_t^h p^{(\varepsilon)} &\rightrightarrows \nabla_x^k \nabla_t^h p \quad \text{locally in } \mathbb{R}^3 \times (0, \infty), \end{aligned} \tag{4.4}$$

which imply that the limit $\left(u, p\right)$ is a global-in-time smooth solution of the Navier–Stokes equations in $\mathbb{R}^3 \times (0, \infty)$ and that u is axi-symmetric with no swirl. Secondly we prove the initial condition (1.14). Take a $\varphi \in C_0^\infty\left(\mathbb{R}^3; \mathbb{R}^3\right)$ with its support contained in $B_R(0)$. By Navier–Stokes equations, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \left\{\left(u^{(\varepsilon)} \otimes u^{(\varepsilon)}\right) \cdot \nabla \text{curl}\varphi + u^{(\varepsilon)} \cdot \Delta \text{curl}\varphi\right\} \mathrm{d}x\, \mathrm{d}t \\ &\quad = \int_{\mathbb{R}^3} \omega^{(\varepsilon)}(x, T) \cdot \varphi(x)\, \mathrm{d}x - \int_{\mathbb{R}^3} \omega_0^{(\varepsilon)}(x) \cdot \varphi(x)\, \mathrm{d}x. \end{aligned} \tag{4.5}$$

We claim that we are able to pass to the limit in (4.5) to get

$$\int_0^T \int_{\mathbb{R}^3} \Big\{ \big(u \otimes u\big) \cdot \nabla \mathrm{curl}\varphi + u \cdot \Delta \mathrm{curl}\varphi \Big\} \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}^3} \omega(x,T) \cdot \varphi(x) \, \mathrm{d}x - \int_{\mathbb{R}^3} \kappa \delta_{r_0,z_0} e_\theta \cdot \varphi \, \mathrm{d}x. \tag{4.6}$$

To this end, it suffices to check the nonlinear term in (4.5) and (4.6). By (4.2) and (4.3), we have

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{R}^3} \big(u^{(\varepsilon)} \otimes u^{(\varepsilon)}\big) \cdot \nabla \mathrm{curl}\varphi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{\mathbb{R}^3} \big(u \otimes u\big) \cdot \nabla \mathrm{curl}\varphi \, \mathrm{d}x \, \mathrm{d}t \right| \\ &\quad \lesssim \left\| u^{(\varepsilon)} - u \right\|_{L_t^{\frac{8}{5}} L_x^2 \left(B_R(0) \times (0,T)\right)} \left(\left\| u^{(\varepsilon)} \right\|_{L_t^{\frac{8}{3}} L_x^2 \left(\mathbb{R}^3 \times (0,T)\right)} + \|u\|_{L_t^{\frac{8}{3}} L_x^2 \left(\mathbb{R}^3 \times (0,T)\right)} \right) \\ &\quad \times \|\nabla \mathrm{curl}\varphi\|_{L_x^\infty}, \end{aligned}$$

which goes to 0 as $\varepsilon \to 0$. Thus (4.6) is obtained. Fatou's lemma and (4.1) imply

$$\|u(t)\|_{L_x^2} \lesssim |\kappa| r_0 t^{-\frac{1}{4}}. \tag{4.7}$$

Hence in view of (4.6) and (4.7), we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \omega(x,T) \cdot \varphi(x) \, \mathrm{d}x - \int_{\mathbb{R}^3} \kappa \delta_{r_0,z_0} e_\theta \cdot \varphi \, \mathrm{d}x \right| \\ &\quad = \left| \int_0^T \int_{\mathbb{R}^3} \Big\{ \big(u \otimes u\big) \cdot \nabla \mathrm{curl}\varphi + u \cdot \Delta \mathrm{curl}\varphi \Big\} \, \mathrm{d}x \, \mathrm{d}t \right| \\ &\quad \lesssim \int_0^T |\kappa|^2 r_0^2 t^{-\frac{1}{2}} \, \mathrm{d}t + \int_0^T |\kappa| r_0 t^{-\frac{1}{4}} \, \mathrm{d}t \lesssim |\kappa|^2 r_0^2 T^{\frac{1}{2}} + |\kappa| r_0 T^{\frac{3}{4}}, \end{aligned} \tag{4.8}$$

which implies (1.14). This concludes the proof of the statement concerning the vorticity. The convergence of the velocity field (1.15) is a consequence of the vorticity, our uniform estimates and Lemma 4.2 below.

Remark 4.1. Theorem 1.1 is also true if we replace the initial condition by finite many vortex rings

$$\omega(\cdot,0) = \sum_{i=1}^n \kappa_i \delta_{r_i,z_i} e_\theta, \tag{4.9}$$

where all $\kappa_i > 0$ (or, all $\kappa_i < 0$), or more generally, by

$$\omega(\cdot,0) = \mu e_\theta, \tag{4.10}$$

where μ is a positive or negative finite measure with a compact support in the rz-plane. Without any modification, the preceding proof for a single vortex ring also works for the cases of (4.9) and (4.10).

Lemma 4.2. *Let γ_0 be a circle in the x_1x_2-plane with its center at the origin and let ω_0 be given by (1.4). Let u_0 be the velocity field generated by ω_0 from the Biot–Savart law. Assume that ω^k is sequence of axi-symmetric vector fields of the form $\omega^k \sim \omega^k(r,z)e_\theta$ satisfying (2.1) with bounds uniform in k such that $\omega^k \to \omega_0$ weakly, in the sense of (1.14). Then the velocity fields u^k generated from the Biot–Savart law by ω^k converge strongly to u_0 in $L^p(\mathbb{R}^3)$ for any $p \in (1,2)$.*

Proof. Let us denote by $U^{r,z}$ the velocity field generated by the Biot–Savart law via the vorticity field $\delta_{\gamma^{r,z}}$, where $\gamma^{r,z}$ is the circle $(r\cos\theta, r\sin\theta, z)$, $\theta \in [0, 2\pi)$. We will also set $U^r = U^{r,0}$ and $U = U^1$. Similarly, we will write γ^r for $\gamma^{r,0}$ and γ for γ^1. We have

$$|U(x)| = O(|x|^{-3}), \qquad |x| \to \infty, \tag{4.11}$$

and

$$|U| \sim \frac{1}{\mathrm{dist}(x,\gamma)}, \qquad x \to \gamma. \tag{4.12}$$

In particular, we see that

$$U \in L^p(\mathbb{R}^3) \qquad p \in (1,2). \tag{4.13}$$

We note that

$$U^r = \frac{1}{r}U\left(\frac{x}{r}\right). \tag{4.14}$$

Letting

$$A_p = ||U||_{L^p(\mathbb{R}^3)}, \tag{4.15}$$

we see that

$$||U^{r,z}||_{L^p(\mathbb{R}^3)} = A_p r^{\frac{3}{p}-1}, \qquad p \in (1,2). \tag{4.16}$$

In what follows we will assume (without loss of generality) that $\gamma_0 = \gamma$ and $u_0 = U$. Let

$$\mathcal{O} = \left\{x, \mathrm{dist}(x,\gamma) < \frac{1}{2}\right\}. \tag{4.17}$$

By a slight abuse of notation, we can also consider $\mathcal{O}$ as a subset of the (r,z)-coordinate plane $\Pi = \{(r,z), r > 0\}$. We claim that

$$\lim_{k\to\infty} \left\| \int_{\Pi\setminus\mathcal{O}} U^{r,z}\omega^k(r,z)\,\mathrm{d}r\,\mathrm{d}z \right\|_{L^p(\mathbb{R}^3)} = 0. \tag{4.18}$$

For this it is enough to show that

$$\lim_{k\to\infty} \int_{\Pi\setminus\mathcal{O}} ||U^{r,z}||_{L^p(\mathbb{R}^3)}\omega^k(r,z)\,\mathrm{d}r\,\mathrm{d}z = A_p \lim_{k\to\infty} \int_{\Pi\setminus\mathcal{O}} r^{\frac{3}{p}-1}\omega^k(r,z)\,\mathrm{d}r\,\mathrm{d}z = 0. \tag{4.19}$$

This follows easily from the uniform bounds on ω^k, together with the weak convergence of ω^k to δ_γ as $k \to \infty$. In view of (4.18) we can assume for the remainder of the proof that ω^k are supported in $\mathcal{O}$. Let $\varphi_\epsilon(x) = \epsilon^{-3}\varphi(\frac{x}{\epsilon})$ be a standard mollifier in $\mathbb{R}^3$ and let

$$U_\epsilon^{r,z} = U^{r,z} * \varphi_\epsilon. \tag{4.20}$$

In view of the weak convergence of ω^k to ω_0 and the assumption which we now can make that the support of ω^k is in $\mathcal{O}$, it is clear that the fields

$$u_\epsilon^k = \int U_\epsilon^{r,z} \omega^k(r, z)\, \mathrm{d}r\, \mathrm{d}z \tag{4.21}$$

converge as $k \to \infty$ to $u_{0\epsilon} = U_\epsilon$ in $L^p(\mathbb{R}^3)$ for $p \in (1, 2)$. At the same time, for $p \in (1, 2)$ we have

$$||U_\varepsilon^{r,z} - U^{r,z}||_{L^p(\mathbb{R}^3)} \to 0, \qquad \epsilon \to 0 \tag{4.22}$$

uniformly in $(r, z) \in \mathcal{O}$, and the result follows easily. □

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