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Linear Stability of Elliptic Lagrangian Solutions of the Planar Three-Body Problem via Index Theory

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Abstract

It is well known that the linear stability of Lagrangian elliptic equilateral triangle homographic solutions in the classical planar three-body problem depends on the mass parameter $\beta = 27(m_1m_2 + m_2m_3 + m_3m_1)/(m_1 + m_2 + m_3)^2 \in [0, 9]$ and the eccentricity $e \in [0, 1)$. We are not aware of any existing analytical method which relates the linear stability of these solutions to the two parameters directly in the full rectangle $[0, 9] \times [0, 1)$, aside from perturbation methods for $e > 0$ small enough, blow-up techniques for e sufficiently close to 1, and numerical studies. In this paper, we introduce a new rigorous analytical method to study the linear stability of these solutions in terms of the two parameters in the full (β, e) range $[0, 9] \times [0, 1)$ via the ω-index theory of symplectic paths for ω belonging to the unit circle of the complex plane, and the theory of linear operators. After establishing the ω-index decreasing property of the solutions in β for fixed $e \in [0, 1)$, we prove the existence of three curves located from left to right in the rectangle $[0, 9] \times [0, 1)$, among which two are -1 degeneracy curves and the third one is the right envelope curve of the ω-degeneracy curves, and show that the linear stability pattern of such elliptic Lagrangian solutions changes if and only if the parameter (β, e) passes through each of these three curves. Interesting symmetries of these curves are also observed. The linear stability of the singular case when the eccentricity e approaches 1 is also analyzed in detail.

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1. Introduction and Main Results

We consider the classical planar three-body problem in celestial mechanics. Denote by $q_1, q_2, q_3 \in \mathbf{R}^2$ the position vectors of three particles with masses $m_1, m_2, m_3 > 0$ respectively. By Newton's second law and the law of universal gravitation, the system of equations for this problem is

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \qquad \text{for} \quad i = 1, 2, 3, \tag{1.1}$$

where $U(q) = U(q_1, q_2, q_3) = \sum_{1 \leqq i < j \leqq 3} \frac{m_i m_j}{\|q_i - q_j\|}$ is the potential or force function by using the standard norm $\|\cdot\|$ of vector in $\mathbf{R}^2$. For periodic solutions with period 2π, the system is the Euler–Lagrange equation of the action functional

$$\mathcal{A}(q) = \int_0^{2\pi} \left[\sum_{i=1}^{3} \frac{m_i \|\dot{q}_i(t)\|^2}{2} + U(q(t)) \right] \mathrm{d}t$$

defined on the loop space $W^{1,2}(\mathbf{R}/2\pi\mathbf{Z}, \hat{\mathcal{X}})$, where

$$\hat{\mathcal{X}} := \left\{ q = (q_1, q_2, q_3) \in (\mathbf{R}^2)^3 \;\middle|\; \sum_{i=1}^{3} m_i q_i = 0, \;\; q_i \neq q_j, \;\; \forall i \neq j \right\}$$

is the configuration space of the planar three-body problem. The periodic solutions of (1.1) correspond to critical points of the action functional.

It is a well-known fact that (1.1) can be reformulated as a Hamiltonian system. Let $p_1, p_2, p_3 \in \mathbf{R}^2$ be the momentum vectors of the particles respectively. The Hamiltonian system corresponding to (1.1) is

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \;\; \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \text{for} \quad i = 1, 2, 3, \tag{1.2}$$

with Hamiltonian function

$$H(p, q) = H(p_1, p_2, p_3, q_1, q_2, q_3) = \sum_{i=1}^{3} \frac{\|p_i\|^2}{2m_i} - U(q_1, q_2, q_3). \tag{1.3}$$

In 1772, LAGRANGE ([7]) discovered some celebrated homographic periodic solutions, now named after him, to the planar three-body problem, namely that the three bodies form an equilateral triangle at any instant of the motion and at the same time each body travels along a specific Keplerian elliptic orbit about the center of masses of the system.

When $0 \leqq e < 1$, the Keplerian orbit is elliptic; following MEYER and SCHMIDT ([15]), we call such elliptic Lagrangian solutions *elliptic relative equilibria*. Especially when $e = 0$, the Keplerian elliptic motion becomes circular motion and then all the three bodies move around the center of masses along circular orbits with the same frequency, which are called *relative equilibria* traditionally.

Our main concern in this paper is the linear stability of these homographic solutions. For the planar three-body problem with masses $m_1, m_2, m_3 > 0$, it turns out that the stability of elliptic Lagrangian solutions depends on two parameters, namely the mass parameter $\beta \in [0, 9]$ defined below and the eccentricity $e \in [0, 1)$,

$$\beta = \frac{27(m_1 m_2 + m_1 m_3 + m_2 m_3)}{(m_1 + m_2 + m_3)^2}. \tag{1.4}$$

Note that besides local perturbation method or blow up technique which study only the case for small enough $e > 0$ or $e < 1$ sufficiently close to 1, we are not aware of any rigorous analytical method dealing with this problem for the major part of the full range of the (β, e) rectangle $[0, 9] \times [0, 1)$, except the recent paper [5] of the first and the third named authors. Continuing with [4] and [5], the current paper is devoted to introducing a new rigorous analytical method to study the linear stability of the elliptic Lagrangian solutions in the full range of the (β, e) rectangle $[0, 9] \times [0, 1)$ via the index theory of symplectic paths and the perturbation theory of linear operators.

The linear stability of relative equilibria was known more than a century ago and it is due to Gascheau ([2], 1843) and Routh ([17], 1875), independently. In this case, using the Floquet theory one can work out all the details explicitly by hand.

After initial considerations of Danby ([1], 1964), Roberts ([16], 2002) reduced all the symmetries of the problem and their first integrals and studied the case of sufficiently small $e \geqq 0$ by perturbation techniques. He also got a partial stability bifurcation diagram in this case, where the stability patterns are clearly presented.

In 2005, Meyer and Schmidt (cf. [15]) used heavily the central configuration nature of the elliptic Lagrangian orbits and decomposed the fundamental solution of the elliptic Lagrangian orbit into two parts symplectically, one of which is the same as that of the Keplerian solution and the other is the essential part for the stability. In the current paper, the fundamental solution of the linearized Hamiltonian system of the essential part of the elliptic Lagrangian orbit is denoted by $\gamma_{\beta,e}(t)$ for $t \in [0, 2\pi]$, which is a path of 4×4 symplectic matrices starting from the identity. They also did the stability analysis by normal form theory for small enough $e \geqq 0$.

In 2004–2006, Martínez, Samà and Simó ([12–14]) studied the stability problem when $e > 0$ is small enough by using normal form theory, and $e < 1$ and close to 1 enough by using blow-up technique in general homogeneous potential. They further gave a much more complete bifurcation diagram numerically and a beautiful figure was drawn there for the full (β, e) range, which we repeat here as Fig. 1. It is one of our primary motivations to understand this diagram globally and analytically in the present work.

Let $\mathbf{U}$ denote the unit circle in the complex plane $\mathbf{C}$. As in [13], the following notations for the different parameter regions are used in Fig. 1:

- elliptic–elliptic (EE), if $\gamma_{\beta,e}(2\pi)$ possesses two pairs of eigenvalues in $\mathbf{U}\backslash\mathbf{R}$;
- elliptic–hyperbolic (EH), if $\gamma_{\beta,e}(2\pi)$ possesses a pair of eigenvalues in $\mathbf{U}\backslash\mathbf{R}$ and a pair of eigenvalues in $\mathbf{R}\backslash\{0, \pm 1\}$;
- hyperbolic–hyperbolic (HH), if $\sigma(\gamma_{\beta,e}(2\pi)) \subset \mathbf{R}\backslash\{0, \pm 1\}$;
- complex–saddle (CS), if $\sigma(\gamma_{\beta,e}(2\pi)) \subset \mathbf{C}\backslash(\mathbf{U} \cup \mathbf{R})$.

In summary, after these authors, the following results are rigorously proved:

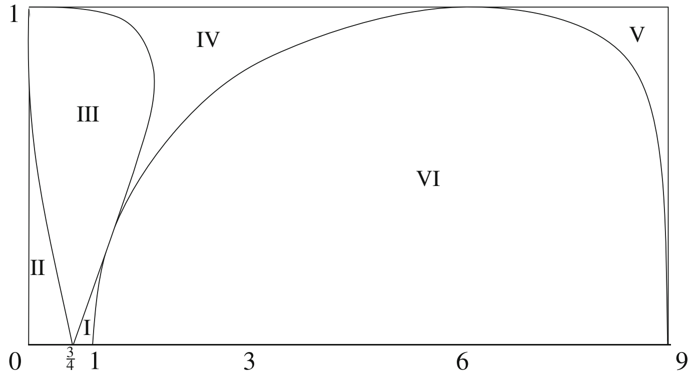

Fig. 1. Stability bifurcation diagram of Lagrangian equilateral triangular homographic orbits in the (β, e) rectangle $[0, 9] \times [0, 1)$. It is the Fig. 5 in [13]. Here the regions I, II, III, IV, V and VI are EE, EE, EH, HH, HH and CS respectively

(i) the relative equilibrium, that is, the case of $e = 0$, is linearly stable if and only if $\beta < 1$; it is only spectrally stable and not linearly stable when $\beta = 1$, and linearly unstable (in fact, CS) when $\beta > 1$. More precisely, when $e = 0$ and β goes from 0 to 1, the two pairs of elliptic characteristics $\omega_1, \overline{\omega}_1$, and $\omega_2, \overline{\omega}_2$, starting from two pairs of characteristics $+1$, and without loss of generality, we can assume that both ω_1 and ω_2 move clock-wisely around the unit circle with different speeds. One, say ω_1, moves faster and arrives at -1 of the unit circle when $\beta = 3/4$ and then continues to move forward around the unit circle. At the same time, ω_2 moves slower along the unit circle. Then ω_1 and $\overline{\omega}_2$ as well as ω_2 and $\overline{\omega}_1$ collide respectively on somewhere which is not ± 1 in the unit circle when $\beta = 1$, and then become CS when $\beta > 1$. When $\beta = 9$, they become a pair of positive double eigenvalues. So when $e = 0$ the only possible bifurcation points in the (β, e) plane are $(3/4, 0)$ and $(1, 0)$. We refer readers to Section 3.3 and Fig. 3 for more detailed discussions.

(ii) It turns out that if $\beta = 3/4$ is fixed and e increases from 0 slightly, the pair of -1 characteristics switches to a real hyperbolic pair, and two period-doubling bifurcation curves born out from the point $(\beta, e) = (3/4, 0)$. When $\beta = 1$, if e increases from 0 slightly, the two pairs of corresponding elliptic characteristics collide and a Krein collision curve bifurcates from $(\beta, e) = (1, 0)$ for such small enough $e > 0$. An interesting phenomenon occurs here, namely, when β is slightly larger than 1, some of the elliptic relative equilibrium with $e > 0$ small can be linearly stable even though the relative equilibrium with $e = 0$ is not.

(iii) When $e < 1$ and e is close to 1 enough, the relative equilibria are all HH under some non-degenerate conditions, which is not satisfied at $\beta = 6$ by numerical computations.

(iv) But the major part of the intermediate region in the (β, e) rectangle $[0, 9] \times [0, 1)$ is totally unknown theoretically, besides numerical results.

Inspired by the second named author's works on the index iteration theory of periodic orbits of Hamiltonian systems (cf. [11]), the first and the third named authors initiated the program of applying the ideas of index theory and its iteration theory in calculus of variations (cf. [11]) to the stability problem of periodic orbits in celestial mechanics, especially the elliptical Lagrangian solutions ([5]) as well as the celebrated Figure-eight periodic orbits due to Chenciner and Montgomery in the planar three-body problem ([4]). In [5], the stability of elliptical Lagrangian solutions is studied and related to the Morse indies of their iterations, that is, Theorems 2.4 and 2.5 below. But in [5], especially the 1-non-degeneracy of elliptic Lagrangian solutions is not proved, and the separation curves of different index regions and thus the stability regions in $[0, 9] \times [0, 1)$ are not identified.

In the current paper, we develop a new method using the ω-index theory for symplectic paths introduced by Conley, Zehnder and Long when $\omega = 1$ (cf. [11]) and by Long when $\omega \in \mathbf{U}\backslash\{1\}$ in [9] and linear differential operator theory to understand the linear stability diagram of elliptic Lagrangian solutions theoretically in the full range of (β, e). Especially the main purpose here is to relate such a linear stability directly to the two major parameters of the motion: the mass parameter β and the eccentricity e. For each fixed $e \in [0, 1)$, we prove that the ω-index $i_\omega(\gamma_{\beta,e})$ of the essential part of the elliptic Lagrangian solutions is non-decreasing in β for all $\omega \in \mathbf{U}$. Then we use this important property to prove all these solutions are 1-non-degenerate, find the two -1 degeneracy curves and right envelope curve of all ω-degeneracy curves for $\omega \in \mathbf{U}\backslash\{1\}$, and determine the linear stability of all sub-regions separated by these three curves. This establishes rigorously most parts of the linear stability properties observed numerically in Fig. 1, and find more interesting properties. Note that the symplectic coordinate decomposition of Meyer and Schmidt fits quite well with index theory, and our study will concentrate on the fundamental solution $\gamma_{\beta,e}(t)$ of the linearized Hamiltonian system of the essential part of the elliptic Lagrangian orbit for $(\beta, e) \in [0, 9] \times [0, 1)$.

Denote by $\mathrm{Sp}(2n)$ the symplectic group of real $2n \times 2n$ matrices. For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, let $\nu_\omega(M) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I_{2n})$, and M is called *ω-degenerate* (*ω-non-degenerate* respectively) if $\nu_\omega(M) > 0$ ($\nu_\omega(M) = 0$ respectively). When $\omega = 1$ and if there is no confusion, we shall simply omit the subindex 1 and say just *degenerate* or *non-degenerate*. Let $e(M)$ be the total algebraic multiplicity of all eigenvalues of M on $\mathbf{U}$. We call $M \in \mathrm{Sp}(2n)$ *spectrally stable* if $e(M) = 2n$, and *linearly stable* if M is spectrally stable and semi-simple. A symplectic matrix M is called *strongly linearly stable* if there is some $\varepsilon > 0$ such that all symplectic matrices N satisfying $\|M - N\| < \varepsilon$ are linearly stable. And M is *hyperbolic*, if $e(M) = 0$.

The following is the first part of our main results in this paper.

Theorem 1.1. *In the planar three-body problem with masses m_1, m_2, and $m_3 > 0$, for the elliptic Lagrangian solution $q = (q_1(t), q_2(t), q_3(t))$ with eccentricity e and mass parameter β as given in (1.4), the essential part $\gamma_{\beta,e}(2\pi) \in \mathrm{Sp}(4)$ of the monodromy matrix of the fundamental solution along this orbit is non-degenerate for all $(\beta, e) \in (0, 9] \times [0, 1)$; and when $\beta = 0$, it is degenerate. Note that the Maslov-type index satisfies $i_1(\gamma_{\beta,e}) = 0$ for all $(\beta, e) \in [0, 9] \times [0, 1)$.*

In the proof of this theorem, we consider the second order differential operators $A(\beta, e)$ (see 2.25) corresponding to the linear variation equation to the essential part $\gamma_{\beta,e}(t)$ of its fundamental solution along the orbit. The main ingredient of the proof is the non-decreasing property of ω-index proved in Lemma 4.4 and Corollary 4.5 below for all $\omega \in \mathbf{U}$, by which we further prove that the operator $A(\beta, e)$ is positive definite, and thus 1-non-degenerate.

The rest of this paper, especially Theorems 1.2 and 1.8 below, is devoted to rigorous analytical studies on the existence and properties of three distinct curves Γ_s, Γ_m and Γ_k locating from left to right in the parameter (β, e) rectangle $[0, 9) \times [0, 1)$. We prove that the linear stability of the essential part $\gamma_{\beta,e}(2\pi)$ of the monodromy matrix and thus that of the elliptic Lagrange solution changes precisely when (β, e) passes through each of these three curves, which yields a complete and rigorous understanding of the linear stability of the elliptic Lagrange solutions. Note that here $\gamma_{\beta,e}(2\pi)$ is always linearly unstable on its hyperbolic subregion in the (β, e) rectangle $[0, 9] \times [0, 1)$, and our Theorems 1.2 and 1.8 do not distinguish the regions IV, V, and VI in Fig. 1.

The main idea in the proofs of Theorems 1.2 and 1.8 is the following: by Theorem 1.1, when (β, e) changes, eigenvalues of $\gamma_{\beta,e}(2\pi)$ can leave from the unit circle $\mathbf{U}$ only at -1 or some Krein collision eigenvalues in $\mathbf{U}\backslash\{\pm 1\}$. Thus such -1 and Krein collision eigenvalues should correspond to (β, e) points which form the above mentioned three curves Γ_s, Γ_m and Γ_k. In order to find those (β, e) such that $-1 \in \sigma(\gamma_{\beta,e}(2\pi))$, we prove that the -1 index $i_{-1}(\gamma_{\beta,e})$ is non-increasing in $\beta \in [0, 9]$ for fixed $e \in [0, 1)$, and takes values 2 at $\beta = 0$ and 0 at $\beta = 9$, thus there must exist two -1 index strictly decreasing curves Γ_s and Γ_m, each of which intersects every horizontal line $e =$ constant only once for $e \in [0, 1)$, and which then yield precisely the two -1 degeneracy curves. Next we prove that the hyperbolic region of $\gamma_{\beta,e}(2\pi)$ in the (β, e) rectangle $[0, 9] \times [0, 1)$ is connected, and its boundary curve Γ_k is continuous and thus well defined, which is the third curve for determining the linear stability. Here the part of Γ_k which is different from the curve Γ_m is also the curve of Krein collision eigenvalues of $\gamma_{\beta,e}(2\pi)$. We prove also that the two curves Γ_s and Γ_m come from two real analytic curves and bifurcate out from $(3/4, 0)$, the curve Γ_k starts from $(1, 0)$, and all of them goes up and tends to the point $(0, 1)$ as e increases from 0 to 1. These three curves were observed numerically in [13] as shown in the above Fig. 1.

In this paper for any M and $N \in \mathrm{Sp}(2n)$, we write $M \approx N$ if $M = P^{-1}NP$ holds for some $P \in \mathrm{Sp}(2n)$, that is, N can be obtained from M by a symplectic coordinate change. Recall that as defined in Chapter 1 of [11], the normal form of an $M \in \mathrm{Sp}(2n)$ is the simplest matrix $N \in \mathrm{Sp}(2n)$ satisfying $N \approx M$ (cf. Theorem 1.7.3 in p. 36 of [11]). Recall also that as introduced in Definition 1.8.9 and Theorem 1.8.10 in p. 41 of [11] (cf. Definition 2.1 below), the basic normal form of an $M \in \mathrm{Sp}(2n)$ is the simplest matrix $N \in \mathrm{Sp}(2n)$ such that $\dim_{\mathbf{C}} \ker_{\mathbf{C}}(N - \omega I) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I)$ for all $\omega \in \mathbf{U}$. It yields the homotopically simplest form of M based on the normal form of M for eigenvalues in $\mathbf{U}$. Note that studies at the level of basic normal forms of $\gamma_{\beta,e}(2\pi)$ are easier and already powerful enough for determining the linear stability, but the results at the level of normal forms of $\gamma_{\beta,e}(2\pi)$ are stronger than basic normal forms and involve more demonstrations. Here we describe our main results in normal forms in Theorems 1.2 and 1.8 below. Note that here the symplectic direct sum $\diamond$ is given in (2.1) and the normal form matrices $D(\lambda)$, $R(\theta)$, $N_1(\lambda, a)$, $N_2(\omega, b)$ and $M_2(\lambda, c)$ used in Theorems 1.2 and 1.8 can be found in Section 2.1 below.

Theorem 1.2. *Using notations in the last theorem, for every* $e \in [0, 1)$*, the* -1 *index* $i_{-1}(\gamma_{\beta,e})$ *is non-increasing, and strictly decreasing only on two values of* $\beta = \beta_1(e)$ *and* $\beta = \beta_2(e) \in (0, 9)$*. Define* $\Gamma_i = \{(\beta_i(e), e) \mid e \in [0, 1)\}$ *for* $i = 1$ *and* 2*,*

$$\beta_s(e) = \min\{\beta_1(e), \beta_2(e)\} \quad \text{and} \quad \beta_m(e) = \max\{\beta_1(e), \beta_2(e)\} \qquad \text{for } e \in [0, 1),$$

and

$$\Gamma_s = \{(\beta_s(e), e) \mid e \in [0, 1)\} \quad \text{and} \quad \Gamma_m = \{(\beta_m(e), e) \mid e \in [0, 1)\}.$$

For every $e \in [0, 1)$ *we define*

$$\beta_k(e) = \sup\{\beta' \in [0, 9] \mid \sigma(\gamma_{\beta,e}(2\pi)) \cap \mathbf{U} \neq \emptyset, \quad \forall\, \beta \in [0, \beta']\}, \tag{1.5}$$

and

$$\Gamma_k = \{(\beta_k(e), e) \in [0, 9] \times [0, 1) \mid e \in [0, 1)\}. \tag{1.6}$$

Then Γ_s*,* Γ_m *and* Γ_k *form three curves which possess the following properties.*

- (i) $0 < \beta_i(e) < 9$ *for* $i = 1, 2$*, and both* $\beta = \beta_1(e)$ *and* $\beta = \beta_2(e)$ *are real analytic in* $e \in [0, 1)$*;*
- (ii) $\beta_1(0) = \beta_2(0) = 3/4$ *and* $\lim_{e\to 1} \beta_1(e) = \lim_{e\to 1} \beta_2(e) = 0$*. The two curves* Γ_1 *and* Γ_2 *are real analytic in e, and bifurcate out from* $(3/4, 0)$ *with tangents* $-\sqrt{33}/4$ *and* $\sqrt{33}/4$ *respectively, thus they are different and their intersection points must be isolated if there exist when* $e \in (0, 1)$*; Consequently,* Γ_s *and* Γ_m *are different piecewise real analytic curves;*
- (iii) *We have*

$$i_{-1}(\gamma_{\beta,e}) = \begin{cases} 2, \text{ if } \ 0 \leqq \beta < \beta_s(e), \\ 1, \text{ if } \ \beta_s(e) \leqq \beta < \beta_m(e), \\ 0, \text{ if } \ \beta_m(e) \leqq \beta \leqq 9, \end{cases} \tag{1.7}$$

  and Γ_s *and* Γ_m *are precisely the* -1 *degeneracy curves of the matrix* $\gamma_{\beta,e}(2\pi)$ *in the* (β, e) *rectangle* $[0, 9] \times [0, 1)$*;*
- (iv) *There holds* $\beta_s(e) \leqq \beta_m(e) \leqq \beta_k(e) < 9$ *for all* $e \in [0, 1)$*;*
- (v) *Every matrix* $\gamma_{\beta,e}(2\pi)$ *is hyperbolic when* $\beta \in (\beta_k(e), 9]$ *and* $e \in [0, 1)$*, and there holds*

$$\beta_k(e) = \inf\{\beta \in [0, 9] \mid \sigma(\gamma_{\beta,e}(2\pi)) \cap \mathbf{U} = \emptyset\}, \qquad \forall\, e \in [0, 1). \tag{1.8}$$

  Consequently Γ_k *is the boundary curve of the hyperbolic region of* $\gamma_{\beta,e}(2\pi)$ *in the* (β, e) *rectangle* $[0, 9] \times [0, 1)$*;*
- (vi) Γ_k *is continuous in* $e \in [0, 1)$*;*
- (vii) $\lim_{e\to 1} \beta_k(e) = 0$*;*
- (viii) *There exists a point* $\tilde{e} \in (0, 1]$ *such that* $\beta_m(e) < \beta_k(e)$ *holds for all* $e \in [0, \tilde{e})$*. Therefore the curve* Γ_k *is different from the curve* Γ_m *at least when* $e \in [0, \tilde{e})$*.*

(ix) *We have* $\gamma_{\beta,e}(2\pi) \approx R(\theta_1)\diamond R(\theta_2)$ *for some* θ_1 *and* $\theta_2 \in (\pi, 2\pi)$, *and thus it is strongly linearly stable on the segment* $0 < \beta < \beta_s(e)$;

(x) *We have* $\gamma_{\beta,e}(2\pi) \approx D(\lambda)\diamond R(\theta))$ *for some* $0 > \lambda \neq -1$ *and* $\theta \in (\pi, 2\pi)$, *and it is elliptic–hyperbolic, and thus linearly unstable on the segment* $\beta_s(e) < \beta < \beta_m(e)$.

(xi) *We have* $\gamma_{\beta,e}(2\pi) \approx R(\theta_1)\diamond R(\theta_2)$ *for some* $\theta_1 \in (0, \pi)$ *and* $\theta_2 \in (\pi, 2\pi)$ *with* $2\pi - \theta_2 < \theta_1$, *and thus it is strongly linearly stable on the segment* $\beta_m(e) < \beta < \beta_k(e)$.

Here and below in Theorem 1.8, we write $\lambda = \lambda_{\beta,e}$ *and* $\theta = \theta_{\beta,e}$ *for short, all of which depend on the parameters* β *and* e.

By the Bott-type iteration formulas of Maslov-type indices, we can decompose $W^{2,2}([0, 2\pi], \mathbf{R}^2)$ into two subspaces E_1 and E_2 (see 7.3 and 7.4 below) according to the boundary conditions. Then using the operator $A(\beta, e)$ (see 7.9) corresponding to the variational equation, we carry out the computations of Morse indices of $A(\beta, e)|_{E_i}$ with $i = 1$ and 2 via those of $A(0, 0)|_{E_i}$ with $i = 1$ and 2.

As a corollary, we have immediately

Corollary 1.3. *For every* $e \in [0, 1)$, *the Lagrangian orbit is strongly linearly stable if* $\beta > 0$ *is small enough.*

Furthermore, we can strengthen the conclusion (v) of Theorem 1.2 to

Proposition 1.4. *For the equal mass case, that is,* $\beta = 9$, *the matrix* $\gamma_{9,e}(2\pi)$ *is always hyperbolic and possesses a pair of positive double eigenvalues* $\lambda(e)$ *and* $\lambda(e)^{-1} \neq 1$ *for every* $0 \leqq e < 1$. *Consequently, the matrix* $\gamma_{\beta,e}(2\pi)$ *is hyperbolic whenever* $\beta < 9$ *is sufficiently close to* 9.

We establish Proposition 1.4 in Section 4.1 by the Maslov-type index theory and the theory of linear differential operators. Then we further have the following:

Theorem 1.5. (i) *For every* $\omega \in \mathbf{U}\backslash\{1\}$ *and* $e \in [0, 1)$, *the* ω*-index* $i_\omega(\gamma_{\beta,e})$ *is decreasing for* $\beta \in [0, 9]$.

(ii) *There exist precisely two curves in the* (β, e) *rectangle* $[0, 9] \times [0, 1)$, *on which the Maslov-type index* $i_\omega(\gamma_{\beta,e})$ *decreases strictly. These two curves are given by* $\beta = \beta_1(e, \omega)$ *and* $\beta = \beta_2(e, \omega)$ *for* $0 \leqq e < 1$ *respectively, where both* $\beta_1(e, \omega)$ *and* $\beta_2(e, \omega)$ *are real analytic functions in* $e \in [0, 1)$ *and satisfy* $\lim_{e\to 1} \beta_1(e, \omega) = \lim_{e\to 1} \beta_2(e, \omega) = 0$.

This is proved in Theorem 6.3 below. Similar to the idea in the proof of Theorem 1.2, we know that the ω-Morse indices of $A(0, e)$ and $A(9, e)$ are 2 and 0 respectively. The existence of the two ω-index strictly decreasing curves follows from the monotonicity of the operators involved. Note that ω-index strictly decreasing is equivalent to the ω-degeneracy of the operator $A(\beta, e)$ by our Proposition 6.1 below. With the aid of Dunford–Taylor integral, the ω-degeneracy of $A(\beta, e)$ is related to the spectral problem of another compact operator $B(e, \omega)$ (see 6.5 below), and the real analyticity of the two index degeneracy curves follows from the theory of operators.

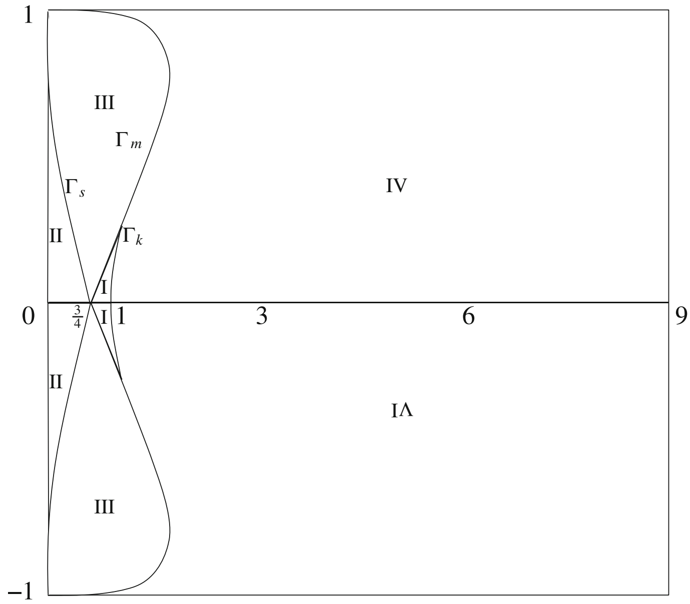

Fig. 2. Stability bifurcation diagram of Lagrangian triangular homographic orbits in the (β, e) rectangle $[0, 9] \times (-1, 1)$. It is symmetric with respect ro the β-axis. The region IV is hyperbolic (HH or CS)

Although $e < 0$ does not have physical meaning, we can extend the fundamental solution to the case $e \in (-1, 1)$ mathematically and some interesting properties of the two degeneracy curves follows. Namely we get

Theorem 1.6. (i) *The identity* $\beta_1(e, -1) = \beta_2(-e, -1)$ *holds for all* $e \in (-1, 1)$. *For fixed* $\omega \in \mathbf{U}\backslash\{-1\}$ *and* $i = 1$ *and* 2, *the function* $\beta_i(e, \omega)$ *is also even in* $e \in (-1, 1)$.

(ii) *For* $e \in (-1, 1)$, *the function* $\beta_k(e)$ *is also even in e, that is,* $\beta_k(-e) = \beta_k(e)$ *for all* $e \in (-1, 1)$. *Consequently* Γ_k *can be continuously extended to the region* $[0, 9] \times (-1, 1)$ *as a curve symmetric to the segment* $[0, 9] \times \{0\}$.

Theorem 1.6 follows from the fact that $A(\beta, e)$ is conjugate to $A(\beta, -e)$ by a unitary operator. This is proved in Theorems 6.4 and 7.2 below.

Fig. 2 represents the case of $\omega = -1$ in Theorem 1.6. Following Theorem 1.2, in Fig. 1, the curve separating the regions II and III is Γ_s and the curve separating the regions III and the union of I and IV is Γ_m. The curves Γ_1 and the mirror of the curve Γ_2 in Theorem 1.2 together give one of the analytic curves in Theorem 1.6, and the another one in the Theorem 1.6 is derived from the curves Γ_2 and the mirror of Γ_1 in Theorem 1.2 as indicated in the Fig. 2. So we see that the two seemingly unrelated index degeneracy curves in Theorem 1.2 are in fact coming from one degeneracy curve in Theorem 1.6. Note that the curve Γ_k separates the regions I, IV and VI in Fig. 1. Numerical studies on $\omega \in \mathbf{U}$ are also given in Fig. 4 in Section 10.

When $e = 1$, the operator $A(\beta, 1)$ is singular and its domain is different from that of $A(\beta, e)$ with $e < 1$, and not convenient to be used. Thus we use the corresponding sesquilinear forms to study the limiting case when $e \to 1$.

Theorem 1.7. *For any fixed* $0 < \beta \leqq 9$*, the matrix* $\gamma_{\beta,e}(2\pi)$ *is hyperbolic when* $1 - |e|$ *is small enough.*

Note that by definition, at least one pair of eigenvalues of the matrix $\gamma_{\beta,e}(2\pi)$ is located on the unit circle $\mathbf{U}$ when $\beta \in [0, \beta_k(e)]$ and $e \in [0, 1)$.

For (β, e) located on these three special curves, we have the following

Theorem 1.8. *For the normal forms of* $\gamma_{\beta,e}(2\pi)$ *when* $(\beta, e) \in \Gamma_s$, Γ_m *or* Γ_k*, we have the following results.*

(i) *If* $\beta_s(e) < \beta_m(e)$*, we have* $\gamma_{\beta_s(e),e}(2\pi) \approx N_1(-1, 1)\diamond R(\theta)$ *for some* $\theta \in (\pi, 2\pi)$*, and thus it is spectrally stable and linearly unstable;*

(ii) *If* $\beta_s(e) = \beta_m(e) < \beta_k(e)$*, we have* $\gamma_{\beta_s(e),e}(2\pi) \approx -I_2\diamond R(\theta)$ *for some* $\theta \in (\pi, 2\pi)$*, and thus it is linearly stable, but not strongly linearly stable;*

(iii) *If* $\beta_s(e) < \beta_m(e) < \beta_k(e)$*, we have* $\gamma_{\beta_m(e),e}(2\pi) \approx N_1(-1, -1)\diamond R(\theta)$ *for some* $\theta \in (\pi, 2\pi)$*, and thus it is spectrally stable and linearly unstable;*

(iv) *If* $\beta_s(e) \leqq \beta_m(e) < \beta_k(e)$*, we have* $\gamma_{\beta_k(e),e}(2\pi) \approx N_2(e^{\sqrt{-1}\theta}, b)$ *for some* $\theta \in (0, \pi)$ *and* $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ *satisfying* $(b_2 - b_3)\sin\theta > 0$*, that is,* $N_2(e^{\sqrt{-1}\theta}, b)$ *is trivial in the sense of Definition 1.8.11 in p. 41 of* [11] *(cf. Section 2.1 below). Consequently the matrix* $\gamma_{\beta_k(e),e}(2\pi)$ *is spectrally stable and linearly unstable;*

(v) *If* $\beta_s(e) < \beta_m(e) = \beta_k(e)$*, we have either* $\gamma_{\beta_k(e),e}(2\pi) \approx N_1(-1, 1)\diamond D(\lambda)$ *for some* $-1 \neq \lambda < 0$ *and is linearly unstable; or* $\gamma_{\beta_k(e),e}(2\pi) \approx M_2(-1, c)$ *with* $c_1, c_2 \in \mathbf{R}$ *and* $c_2 \neq 0$*, and it is spectrally stable and linearly unstable;*

(vi) *If* $\beta_s(e) = \beta_m(e) = \beta_k(e)$*, either* $\gamma_{\beta_k(e),e}(2\pi) \approx M_2(-1, c)$ *with* $c_1, c_2 \in \mathbf{R}$ *and* $c_2 = 0$ *which possesses basic normal form* $N_1(-1, 1)\diamond N_1(-1, 1)$*, or* $\gamma_{\beta_k(e),e}(2\pi) \approx N_1(-1, 1)\diamond N_1(-1, 1)$*. Thus* $\gamma_{\beta_k(e),e}(2\pi)$ *is spectrally stable and linearly unstable.*

Theorem 1.9. *For any fixed* $e \in [0, 1)$*, the set*

$$I_e = \{\beta \in (0, 9] \mid \text{the spectrum of } \gamma_{\beta,e}(2\pi) \text{ is CS}\}$$

is a non-empty open set.

Remark 1.10. Note that our above results yield that the two curves Γ_s and Γ_m can have only isolated intersection points, but it is not clear if they do have any when $e \in (0, 1)$. It is not clear so far whether $\tilde{e} < 1$ in Theorem 1.2 and whether Γ_m and Γ_k coincide completely when $e \in (\tilde{e}, 1)$. It is also not clear whether the (β, e) sub-region in which $\sigma(\gamma_{\beta,e}(2\pi))$ is CS is connected or not.

This paper is organized as follows. In Section 2, we give the definitions of ω-Maslov-type index to fix notations and its relation to the ω-Morse index. Some basic variational facts on the elliptic Lagrangian solutions are also recalled.

In Section 3, we study the linear stability along the three boundary segments of the (β, e) rectangle $[0, 9] \times [0, 1)$. In Section 4, we prove the hyperbolicity of the elliptic Lagrangian solutions in the case of equal masses (Proposition 1.4) and the non-degeneracy stated in Theorem 1.1. In Section 5, the stability behavior in the limit case $e \to 1$ is considered by the sesquilinear forms of linear operators, and Theorem 1.7 is proved. In Section 6, we investigate the ω degeneracy curves for general $\omega \in \mathbf{U}\backslash\{1\}$ in the unit circle and establish Theorem 1.5. In Section 7, we concentrate on the -1 degeneracy curves. In Section 8, we study the non-hyperbolic region and prove Theorem 1.6 and the first half of Theorem 1.2 including its items (i)–(iii) and (ix)–(x). Section 9 is on the hyperbolic region, and we prove the second half of Theorem 1.2 including its items (iv)–(viii) and (xi), as well as Theorems 1.8 and 1.9. Finally in the conclusion, Section 10, we will give more comparisons for the results of Martínez, Samà and Simó and our theorems as well as some possible future considerations.

2. Preliminaries

2.1. ω-Maslov-Type Indices and ω-Morse Indices

Let $(\mathbf{R}^{2n}, \Omega)$ be the standard symplectic vector space with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and the symplectic form $\Omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$. Let $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ be the standard symplectic matrix, where I_n is the identity matrix on $\mathbf{R}^n$.

As usual, the symplectic group $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n, \mathbf{R}) \mid M^T J M = J\},$$

whose topology is induced from that of $\mathbf{R}^{4n^2}$. For $\tau > 0$ we are interested in paths in $\mathrm{Sp}(2n)$:

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0, \tau], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\},$$

which is equipped with the topology induced from that of $\mathrm{Sp}(2n)$. For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, the following real function was introduced in [9]:

$$D_\omega(M) = (-1)^{n-1}\overline{\omega}^n \det(M - \omega I_{2n}).$$

Thus for any $\omega \in \mathbf{U}$ the following codimension 1 hypersurface in $\mathrm{Sp}(2n)$ is defined ([9]):

$$\mathrm{Sp}(2n)_\omega^0 = \{M \in \mathrm{Sp}(2n) \mid D_\omega(M) = 0\}.$$

For any $M \in \mathrm{Sp}(2n)_\omega^0$, we define a co-orientation of $\mathrm{Sp}(2n)_\omega^0$ at M by the positive direction $\frac{\mathrm{d}}{\mathrm{d}t} M e^{tJ}|_{t=0}$ of the path Me^{tJ} with $0 \leqq t \leqq \varepsilon$ and ε being a small enough positive number. Let

$$\begin{aligned} \mathrm{Sp}(2n)_\omega^* &= \mathrm{Sp}(2n)\backslash\mathrm{Sp}(2n)_\omega^0, \\ \mathcal{P}_{\tau,\omega}^*(2n) &= \{\gamma \in \mathcal{P}_\tau(2n) \mid \gamma(\tau) \in \mathrm{Sp}(2n)_\omega^*\}, \\ \mathcal{P}_{\tau,\omega}^0(2n) &= \mathcal{P}_\tau(2n)\backslash\mathcal{P}_{\tau,\omega}^*(2n). \end{aligned}$$

For any two continuous paths ξ and $\eta : [0,\tau] \to \mathrm{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, we define their concatenation by:

$$\eta * \xi(t) = \begin{cases} \xi(2t), & \text{if } 0 \leqq t \leqq \tau/2, \\ \eta(2t-\tau), & \text{if } \tau/2 \leqq t \leqq \tau. \end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, the symplectic sum of M_1 and M_2 is defined (cf. [9,11]) by the following $2(m_1+m_2) \times 2(m_1+m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}, \tag{2.1}$$

and $M^{\diamond k}$ denotes the k copy $\diamond$-sum of M. For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and 1, let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ for all $t \in [0,\tau]$.

As in [11], for $\lambda \in \mathbf{R}\setminus\{0\}$, $a \in \mathbf{R}$, $\theta \in (0,\pi) \cup (\pi, 2\pi)$, $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbf{R}$ for $i = 1, \ldots, 4$, and $c_j \in \mathbf{R}$ for $j = 1, 2$, we denote respectively some normal forms by

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

$$N_1(\lambda, a) = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \qquad N_2(e^{\sqrt{-1}\theta}, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix},$$

$$M_2(\lambda, c) = \begin{pmatrix} \lambda & 1 & c_1 & 0 \\ 0 & \lambda & c_2 & (-\lambda)c_2 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & -\lambda^{-2} & \lambda^{-1} \end{pmatrix}.$$

Here $N_2(e^{\sqrt{-1}\theta}, b)$ is *trivial* if $(b_2 - b_3)\sin\theta > 0$, or *non-trivial* if $(b_2 - b_3)\sin\theta < 0$, in the sense of Definition 1.8.11 in p. 41 of [11]. Note that by Theorem 1.5.1 in pp. 24–25 and (1.4.7)–(1.4.8) in p. 18 of [11], when $\lambda = -1$ there hold

$$c_2 \neq 0 \quad \text{if and only if} \quad \dim\ker(M_2(-1,c) + I) = 1,$$
$$c_2 = 0 \quad \text{if and only if} \quad \dim\ker(M_2(-1,c) + I) = 2.$$

Note that we have $N_1(\lambda, a) \approx N_1(\lambda, a/|a|)$ for $a \in \mathbf{R}\setminus\{0\}$ by symplectic coordinate change, because

$$\begin{pmatrix} 1/\sqrt{|a|} & 0 \\ 0 & \sqrt{|a|} \end{pmatrix} \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \sqrt{|a|} & 0 \\ 0 & 1/\sqrt{|a|} \end{pmatrix} = \begin{pmatrix} \lambda & a/|a| \\ 0 & \lambda \end{pmatrix}.$$

Definition 2.1. ([9,11]) For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, define

$$\nu_\omega(M) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I_{2n}). \tag{2.2}$$

For every $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, as in Definition 1.8.5 in p. 38 of [11], we define the *ω-homotopy set* $\Omega_\omega(M)$ of M in $\mathrm{Sp}(2n)$ by

$$\Omega_\omega(M) = \{N \in \mathrm{Sp}(2n) \mid \nu_\omega(N) = \nu_\omega(M)\},$$

and the *homotopy set* $\Omega(M)$ of M in $\mathrm{Sp}(2n)$ by

$$\Omega(M) = \{N \in \mathrm{Sp}(2n) \mid \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U}, \text{ and } \nu_\lambda(N) = \nu_\lambda(M) \quad \forall \lambda \in \sigma(M) \cap \mathbf{U}\}.$$

We denote by $\Omega^0(M)$ (or $\Omega^0_\omega(M)$) the path connected component of $\Omega(M)$ ($\Omega_\omega(M)$) which contains M, and call it the *homotopy component* (or *ω-homotopy component*) of M in $\mathrm{Sp}(2n)$. Following Definition 5.0.1 in p. 111 of [11], for $\omega \in \mathbf{U}$ and $\gamma_i \in \mathcal{P}_\tau(2n)$ with $i = 0, 1$, we write $\gamma_0 \sim_\omega \gamma_1$ if γ_0 is homotopic to γ_1 via a homotopy map $h \in C([0,1] \times [0,\tau], \mathrm{Sp}(2n))$ such that $h(0) = \gamma_0$, $h(1) = \gamma_1$, $h(s)(0) = I$, and $h(s)(\tau) \in \Omega^0_\omega(\gamma_0(\tau))$ for all $s \in [0,1]$. We write also $\gamma_0 \sim \gamma_1$, if $h(s)(\tau) \in \Omega^0(\gamma_0(\tau))$ for all $s \in [0,1]$ is further satisfied.

Following Definition 1.8.9 in p. 41 of [11], we call the above matrices $D(\lambda)$, $R(\theta)$, $N_1(\lambda, a)$ and $N_2(\omega, b)$ basic normal forms of symplectic matrices. As proved in [9] and [10] (cf. Theorem 1.9.3 in p. 46 of [11]), every $M \in \mathrm{Sp}(2n)$ has its basic normal form decomposition in $\Omega^0(M)$ as a $\diamond$-sum of these basic normal forms. This is very important when we derive basic normal forms for $\gamma_{\beta,e}(2\pi)$ to compute the ω-index $i_\omega(\gamma_{\beta,e})$ of the path $\gamma_{\beta,e}$ later in this paper.

We define a special continuous symplectic path $\xi_n \subset \mathrm{Sp}(2n)$ by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\diamond n} \quad \text{for} \quad 0 \leqq t \leqq \tau. \tag{2.3}$$

Definition 2.2. ([9,11]) For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)). \tag{2.4}$$

If $\gamma \in \mathcal{P}^*_{\tau,\omega}(2n)$, define

$$i_\omega(\gamma) = [\mathrm{Sp}(2n)^0_\omega : \gamma * \xi_n], \tag{2.5}$$

where the right hand side of (2.5) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}^0_{\tau,\omega}(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf\{i_\omega(\beta) \mid \beta \in U \cap \mathcal{P}^*_{\tau,\omega}(2n)\}. \tag{2.6}$$

Then

$$(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\},$$

is called the index function of γ at ω.

We refer to [11] for more details on this index theory of symplectic matrix paths and periodic solutions of Hamiltonian system.

For $T > 0$, suppose x is a critical point of the functional

$$F(x) = \int_0^T L(t, x, \dot{x})\,\mathrm{d}t, \qquad \forall\, x \in W^{1,2}(\mathbf{R}/T\mathbf{Z}, \mathbf{R}^n),$$

where $L \in C^2((\mathbf{R}/T\mathbf{Z}) \times \mathbf{R}^{2n}, \mathbf{R})$ and satisfies the Legendrian convexity condition $L_{p,p}(t, x, p) > 0$. It is well known that x satisfies the corresponding Euler–Lagrangian equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} L_p(t, x, \dot{x}) - L_x(t, x, \dot{x}) = 0, \tag{2.7}$$

$$x(0) = x(T), \qquad \dot{x}(0) = \dot{x}(T). \tag{2.8}$$

For such an extremal loop, define

$$\begin{aligned} P(t) &= L_{p,p}(t, x(t), \dot{x}(t)), \\ Q(t) &= L_{x,p}(t, x(t), \dot{x}(t)), \\ R(t) &= L_{x,x}(t, x(t), \dot{x}(t)). \end{aligned}$$

Note that

$$F''(x) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(P\frac{\mathrm{d}}{\mathrm{d}t} + Q\right) + Q^T\frac{\mathrm{d}}{\mathrm{d}t} + R. \tag{2.9}$$

For $\omega \in \mathbf{U}$, set

$$D(\omega, T) = \{y \in W^{1,2}([0, T], \mathbf{C}^n) \mid y(T) = \omega y(0)\}. \tag{2.10}$$

We define the ω-Morse index $\phi_\omega(x)$ of x to be the dimension of the largest negative definite subspace of

$$\langle F''(x)y_1, y_2\rangle, \qquad \forall\, y_1, y_2 \in D(\omega, T),$$

where $\langle\cdot,\cdot\rangle$ is the inner product in L^2. For $\omega \in \mathbf{U}$, we also set

$$\overline{D}(\omega, T) = \{y \in W^{2,2}([0, T], \mathbf{C}^n) | y(T) = \omega y(0), \dot{y}(T) = \omega\dot{y}(0)\}. \tag{2.11}$$

Then $F''(x)$ is a self-adjoint operator on $L^2([0, T], \mathbf{R}^n)$ with domain $\overline{D}(\omega, T)$. We also define

$$\nu_\omega(x) = \dim\ker(F''(x)).$$

In general, for a self-adjoint operator A on the Hilbert space $\mathscr{H}$, we set $\nu(A) = \dim\ker(A)$ and denote by $\phi(A)$ its Morse index which is the maximum dimension of the negative definite subspace of the symmetric form $\langle A\cdot,\cdot\rangle$. Note that the Morse index of A is equal to the total multiplicity of the negative eigenvalues of A.

On the other hand, $\tilde{x}(t) = (\partial L/\partial \dot{x}(t), x(t))^T$ is the solution of the corresponding Hamiltonian system of (2.7)–(2.8), and its fundamental solution $\gamma(t)$ is given by

$$\dot{\gamma}(t) = JB(t)\gamma(t), \tag{2.12}$$

$$\gamma(0) = I_{2n}, \tag{2.13}$$

with

$$B(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}(t)Q(t) \\ -Q(t)^T P^{-1}(t) & Q(t)^T P^{-1}(t)Q(t) - R(t) \end{pmatrix}. \tag{2.14}$$

Lemma 2.3. (Long [11, p. 172]) *For the ω-Morse index $\phi_\omega(x)$ and nullity $\nu_\omega(x)$ of the solution $x = x(t)$ and the ω-Maslov-type index $i_\omega(\gamma)$ and nullity $\nu_\omega(\gamma)$ of the symplectic path γ corresponding to $\tilde{x}$, for any $\omega \in \mathbf{U}$ we have*

$$\phi_\omega(x) = i_\omega(\gamma), \qquad \nu_\omega(x) = \nu_\omega(\gamma). \tag{2.15}$$

A generalization of the above lemma to arbitrary boundary conditions is given in [4]. For more information on these topics, we refer to [11].

2.2. Stability Criteria via Morse Indices

In this subsection we recall some results of [5].

Gordon's classical theorem (cf. [3]) says that the Keplerian solution is a minimizer in the loop space under some topological constraint. Regarding the essential part, a theorem of Venturelli in [18] as well as Zhang and Zhou in [19] tells us that the Lagrangian solution is the minimizer among the loops in its some homology class. Note that, up to now, these are the only known variational facts under topological constraints on the loop spaces.

By these theorems, we got criteria for the stability in terms of Morse indices. Let $\phi^k = \phi_1(q^k)$ be the Morse index of the kth iteration q^k of the Lagrangian solution q in the variational problem, and according to [18] and [19], $\phi^1 = 0$ holds.

The Lagrangian solution is called linearly stable (spectrally stable) if $\gamma(2\pi)$ is linearly stable(spectrally stable). The first and the third named authors proved the following:

Theorem 2.4. (Hu–Sun [5]) *For the monodromy matrix M corresponding to the elliptic Lagrangian solution $q = (q_1(t), q_2(t), q_3(t))$, $2 \leqq \phi^2 \leqq 4$ and,*

$$e(M)/2 \geqq \phi^2. \tag{2.16}$$

Moreover

(i) *If $\phi^2 = 4$, then the Lagrangian solution is spectrally stable;*
(ii) *If $\phi^2 = 3$, then the Lagrangian solution is linearly unstable;*
(iii) *If $\phi^2 = 2$, then the Lagrangian solution is spectrally stable if there exists some integer $k \geqq 3$, such that $\phi^k > 2(k-1)$.*
(iv) *If $\phi^k = 2(k-1)$, for all $k \in \mathbf{N}$, then the Lagrangian solution is linearly unstable.*

Moreover, if the essential part $\gamma = \gamma_{\beta,e}(t)$ (cf. Section 2.3 below) of the monodromy matrix at $t = 4\pi$ is non-degenerate, we can get the normal forms of $\gamma_{\beta,e}(2\pi)$.

Theorem 2.5. (HU–SUN [5]) *In the same setting of the above theorem, suppose $\gamma_{\beta,e}(4\pi) = \gamma_{\beta,e}(2\pi)^2$ is non-degenerate.*

(i) *If $\phi^2 = 4$, then $\gamma_{\beta,e}(2\pi) \approx R(2\pi - \theta_1)\diamond R(2\pi - \theta_2)$ holds for some θ_1 and $\theta_2 \in (0, \pi)$, and is linearly stable;*

(ii) *If $\phi^2 = 3$, then $\gamma_{\beta,e}(2\pi) \approx D(\lambda)\diamond R(2\pi - \theta)$ for some $\lambda < 0$ and $\theta \in (0, \pi)$, and is linearly unstable;*

(iii) *If $\phi^2 = 2$ and there exists some integer $k \geqq 3$ such that $\phi^k > 2(k - 1)$, then $\gamma_{\beta,e}(2\pi) \approx R(2\pi - \theta_1)\diamond R(\theta_2)$ holds with $0 < \theta_1 < \theta_2 < \pi$, and is linearly stable;*

(iv) *If $\phi^k = 2(k - 1)$ for all $k \in \mathbf{N}$, then $\gamma_{\beta,e}(2\pi)$ is hyperbolic or spectrally stable and linearly unstable.*

2.3. The Essential Part of the Fundamental Solution of the Elliptic Lagrangian Orbit

Following MEYER and SCHMIDT (cf. p. 275 of [15]), the essential part $\gamma = \gamma_{\beta,e}(t)$ of the fundamental solution of the Lagrangian orbit satisfies

$$\dot{\gamma}(t) = JB(t)\gamma(t), \tag{2.17}$$

$$\gamma(0) = I_4, \tag{2.18}$$

with

$$B(t) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \frac{2e\cos(t)-1-\sqrt{9-\beta}}{2(1+e\cos(t))} & 0 \\ 1 & 0 & 0 & \frac{2e\cos(t)-1+\sqrt{9-\beta}}{2(1+e\cos(t))} \end{pmatrix}, \tag{2.19}$$

where e is the eccentricity, and t is the truly anomaly.

Let

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad K_{\beta,e}(t) = \begin{pmatrix} \frac{3+\sqrt{9-\beta}}{2(1+e\cos(t))} & 0 \\ 0 & \frac{3-\sqrt{9-\beta}}{2(1+e\cos(t))} \end{pmatrix}, \tag{2.20}$$

and set

$$L(t, x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2 + J_2 x(t)\cdot\dot{x}(t) + \frac{1}{2}K_{\beta,e}(t)x(t)\cdot x(t), \quad \forall\, x \in W^{1,2}(\mathbf{R}/2\pi\mathbf{Z}, \mathbf{R}^2), \tag{2.21}$$

where $a\cdot b$ denotes the inner product in $\mathbf{R}^2$. Obviously the origin in the configuration space is a solution of the corresponding Euler–Lagrange system. By Legendrian transformation, the corresponding Hamiltonian function is

$$H(t, z) = \frac{1}{2}B(t)z\cdot z, \qquad \forall\, z \in \mathbf{R}^4.$$

Note first that the elliptical Lagrangian solution is a local minimizer of the action functional $\mathcal{A}$ on the homology class of curves with winding number $(1, 1, 1)$ or $(-1, -1, -1)$ in $H_1(\hat{\mathcal{X}})$ (cf. [18,19], and Lemma 4.1 of [5]). Note secondly that curves with winding number not equal to ± 1 can not approximate curves with winding number 1 or -1 in $C(\mathbf{R}/\mathbf{Z}, \mathbf{R}^2)$. Therefore the Morse index of $\mathcal{A}$ at the elliptical Lagrangian solution in the whole space $\hat{\mathcal{X}}$ and that restricted in the homology class of curves with winding number $(1, 1, 1)$ or $(-1, -1, -1)$ in $H_1(\hat{\mathcal{X}})$ coincide and take the value zero.

Here note that let γ_1 be the fundamental solution of the Kepler orbit. Then $\gamma_1 \diamond \gamma_{\beta,e}$ is the fundamental solution of the Lagrangian orbit. By Theorem 7.3.1 in p. 168 of [11] and the additivity of the index theory (cf. (ii) of Theorem 6.2.7 in p. 147 of [11]), we obtain

$$\phi^k = i_1(\gamma_1^k) + i_1(\gamma_{\beta,e}^k), \qquad \forall\, k \in \mathbf{N}, \tag{2.22}$$

where ϕ^k is defined at the beginning of Section 2.2. Note that by Proposition 3.6 in p. 110 of [5], we have

$$i_1(\gamma_1^k) = 2(k-1), \qquad \forall\, k \in \mathbf{N}. \tag{2.23}$$

Thus by our above discussions and (2.22)–(2.23) with $k = 1$, we obtain

$$i_1(\gamma_{\beta,e}) = \phi^1 = 0, \qquad \forall\, (\beta, e) \in [0, 9] \times [0, 1). \tag{2.24}$$

2.4. A Modification on the Path $\gamma_{\beta,e}(t)$

In order to transform the Lagrangian system (2.19) to a simpler linear operator corresponding to a second order Hamiltonian system with the same linear stability as $\gamma_{\beta,e}(2\pi)$, using $R(t)$ and $R_4(t) \equiv N_2(e^{t\sqrt{-1}}, 0)$ defined in Section 2.1, we let

$$\xi_{\beta,e}(t) = R_4(t)\gamma_{\beta,e}(t), \qquad \forall\, t \in [0, 2\pi], (\beta, e) \in [0, 9] \times [0, 1). \tag{2.25}$$

One can show by direct computation that

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_{\beta,e}(t) = J \begin{pmatrix} I_2 & 0 \\ 0 & R(t)(I_2 - K_{\beta,e}(t))R(t)^T \end{pmatrix} \xi_{\beta,e}(t). \tag{2.26}$$

Note that $R_4(0) = R_4(2\pi) = I_4$, so $\gamma_{\beta,e}(2\pi) = \xi_{\beta,e}(2\pi)$ holds and the linear stabilities of the systems (2.18) and (2.26) are precisely the same.

By (2.25) the symplectic paths $\gamma_{\beta,e}$ and $\xi_{\beta,e}$ are homotopic to each other via the homotopy $h(s, t) = R_4(st)\gamma_{\beta,e}(t)$ for $(s, t) \in [0, 1] \times [0, 2\pi]$. Because $R_4(s)\gamma_{\beta,e}(2\pi)$ for $s \in [0, 1]$ is a loop in $\mathrm{Sp}(4)$ which is homotopic to the constant loop $\gamma_{\beta,e}(2\pi)$, we have $\gamma_{\beta,e} \sim_1 \xi_{\beta,e}$ by the homotopy h. Then by Lemma 5.2.2 in p. 117 of [11], the homotopy between $\gamma_{\beta,e}$ and $\xi_{\beta,e}$ can be realized by a homotopy which fixes the end point $\gamma_{\beta,e}(2\pi)$ all the time. Therefore by the homotopy invariance of the Maslov-type index (cf. (i) of Theorem 6.2.7 in p. 147 of [11]) we obtain

$$i_\omega(\xi_{\beta,e}) = i_\omega(\gamma_{\beta,e}), \quad \nu_\omega(\xi_{\beta,e}) = \nu_\omega(\gamma_{\beta,e}), \quad \forall\, \omega \in \mathbf{U},\ (\beta, e) \in [0, 9] \times [0, 1). \tag{2.27}$$

On the other hand, the first order linear Hamiltonian system (2.26) corresponds to the following second order linear Hamiltonian system

$$\ddot{x}(t) = -x(t) + R(t)K_{\beta,e}(t)R(t)^T x(t). \tag{2.28}$$

For $(\beta, e) \in [0, 9) \times [0, 1)$, the second order differential operator corresponding to (2.28) is given by

$$\begin{aligned} A(\beta, e) &= -\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_2 - I_2 + R(t)K_{\beta,e}(t)R(t)^T \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_2 - I_2 + \frac{1}{2(1 + e\cos t)}\big(3I_2 + \sqrt{9-\beta}\,S(t)\big), \end{aligned} \tag{2.29}$$

where $S(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix}$, defined on the domain $\overline{D}(\omega, 2\pi)$ in (2.11). Then it is self-adjoint and depends on the parameters β and e. By Lemma 2.3, we have for any β and e, the Morse index $\phi_\omega(A(\beta, e))$ and nullity $\nu_\omega(A(\beta, e))$ of the operator $A(\beta, e)$ on the domain $\overline{D}(\omega, 2\pi)$ satisfy

$$\phi_\omega(A(\beta, e)) = i_\omega(\xi_{\beta,e}), \quad \nu_\omega(A(\beta, e)) = \nu_\omega(\xi_{\beta,e}), \qquad \forall\, \omega \in \mathbf{U}. \tag{2.30}$$

Especially by Lemma 4.1, (55) and (58) of [5] and the above (2.24), we obtain

$$i_1(\xi_{\beta,e}) = \phi_1(A(\beta, e)) = i_1(\gamma_{\beta,e}) = 0, \qquad \forall\, (\beta, e) \in [0, 9] \times [0, 1), \tag{2.31}$$

In the rest part of this paper, we shall use both of the paths $\gamma_{\beta,e}$ and $\xi_{\beta,e}$ to study the linear stability of $\gamma_{\beta,e}(2\pi) = \xi_{\beta,e}(2\pi)$. Because of (2.27), in many cases and proofs below, we shall not distinguish these two paths.

3. Stability on the Three Boundary Segments of the Rectangle $[0, 9] \times [0, 1)$

We need more precise information on stabilities and indices of the three boundary segments of the (β, e) rectangle $[0, 9] \times [0, 1)$.

3.1. The Boundary Segment $\{0\} \times [0, 1)$

When $\beta = 0$, this is the case with two zero masses, and the essential part of the fundamental solution of Lagrangian orbit is also the fundamental solution of the Keplerian orbits. In fact, when $\beta = 0$, without loss of generality, by the definition (1.4) of β we may assume $m_2 = m_3 = 0$ and $m_1 > 0$, and then every elliptic Lagrangian solution becomes the motion along two Keplerian solutions of the two points q_2 and q_3 with zero masses going along their elliptic orbits around fixed point $q_1(t) \equiv 0$ with mass $m_1 > 0$. When $\beta = 0$, the matrix $B(t)$ in (2.19) becomes

$$B(t) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -\frac{2-e\cos(t)}{1+e\cos(t)} & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

which coincides with the coefficient matrix $\tilde{B}(t)$ in (17) in p. 275 of [15]. Note that by the different sign choice of the standard symplectic matrices in p. 259 of [15], the order of the independent variables there are different from that in our system (2.17)–(2.19).

(A) *The case of* $e = 0$.

In this case, the matrix $B(t)$ becomes independent of t:

$$B \equiv B(t) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

Then one can find the fundamental solution $\gamma_{0,0}(t)$ of the corresponding system (2.1) with constant coefficient JB explicitly:

$$\gamma_{0,0}(t) = \begin{pmatrix} 2-\cos t & 3t-2\sin t & 3t-\sin t & 1-\cos t \\ -\sin t & 2\cos t - 1 & \cos t - 1 & -\sin t \\ \sin t & 2-2\cos t & 2-\cos t & \sin t \\ 2\cos t - 2 & 4\sin t - 3t & 2\sin t - 3t & 2\cos t - 1 \end{pmatrix}. \tag{3.3}$$

Letting

$$P = \begin{pmatrix} 1 & 0 & 0 & 6\pi \\ 0 & -1/(6\pi) & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -6\pi \end{pmatrix},$$

we then obtain

$$\check{\gamma}(t) \equiv P^{-1}\gamma_{0,0}(t)P = \begin{pmatrix} \cos t & -\frac{2\sin t}{6\pi} & -\sin t & 0 \\ 0 & 1 & 0 & 0 \\ \sin t & \frac{2\cos t - 2}{6\pi} & \cos t & 0 \\ \frac{2-2\cos t}{6\pi} & \frac{4\sin t - 3t}{36\pi^2} & \frac{2\sin t}{6\pi} & 1 \end{pmatrix}.$$

Next for $\varepsilon \in [0, 1]$ we consider the following homotopy path $\check{\gamma}_\varepsilon(t)$:

$$\check{\gamma}_\varepsilon(t) = \begin{pmatrix} \cos t & -\varepsilon\frac{2\sin t}{6\pi} & -\sin t & 0 \\ 0 & 1 & 0 & 0 \\ \sin t & \varepsilon\frac{2\cos t - 2}{6\pi} & \cos t & 0 \\ s\varepsilon\frac{2-2\cos t}{6\pi} & \frac{4\sin t - 3t}{36\pi^2} & \varepsilon\frac{2\sin t}{6\pi} & 1 \end{pmatrix}.$$

Then $\check{\gamma}_\varepsilon(t) \in \mathrm{Sp}(4)$ and $\check{\gamma}_\varepsilon(0) = I_4$ hold for all $t \in \mathbf{R}$ and $\varepsilon \in [0, 1]$. We have $\check{\gamma}_1(t) = \check{\gamma}(t)$ and

$$\check{\gamma}_0(t) = \begin{pmatrix} \cos t & 0 & -\sin t & 0 \\ 0 & 1 & 0 & 0 \\ \sin t & 0 & \cos t & 0 \\ 0 & \frac{4\sin t - 3t}{36\pi^2} & 0 & 1 \end{pmatrix} = R(t)\diamond\begin{pmatrix} 1 & 0 \\ \frac{4\sin t - 3t}{36\pi^2} & 1 \end{pmatrix} \sim R(t)\diamond\begin{pmatrix} 1 & \frac{t}{2\pi} \\ 0 & 1 \end{pmatrix},$$

and especially we obtain

$$\gamma_{0,0}(2\pi) = \begin{pmatrix} 1 & 6\pi & 6\pi & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -6\pi & -6\pi & 1 \end{pmatrix} = P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{6\pi} & 0 & 1 \end{pmatrix} P^{-1} \approx I_2 \diamond N_1(1, 1). \tag{3.4}$$

(B) *The case of $e \in [0, 1)$.*

Note that it is well known that the matrix $\Phi(t)$ with $t = f$ in Lemma 3.1 in p. 271 of [15] consists of two parts, one of which corresponds to the Keplerian elliptic solution of the two-body problem and the other of which corresponds to the coefficient matrix $B(t)$ of the essential part $\gamma_{\beta,e}(t)$ in our notations. Especially when $\beta = 0$ (that is, $\sigma = 0$ in [15]), both the two parts of the matrix $\Phi(t)$ coincide to each other, and yields precisely $\Phi(t) = \tilde{B}(t) \diamond \tilde{B}(t)$ there. Therefore when $\beta = 0$, the essential part $\gamma_{\beta,e}(t)$ corresponds to the Keplerian elliptic solution of the two-body problem.

Thus by Lemma 3.3 and the discussion on (46) of [5], the matrix $\gamma_{0,e}(2\pi)$ satisfies:

$$\gamma_{0,e}(2\pi) \approx I_2 \diamond N_1(1, 1), \qquad \forall\, e \in [0, 1), \tag{3.5}$$

and it includes (3.4) as a special case.

(C) *The indices $i_\omega(\gamma_{0,e})$ for $\omega \in \mathbf{U}$.*

By (2.31) we obtain

$$\phi_1(A(0, e)) = i_1(\xi_{0,e}) = i_1(\gamma_{0,e}) = 0. \tag{3.6}$$

By (3.5) and properties of splitting numbers in Chapter 9 of [11], as in (56) of [5], for $\omega \in \mathbf{U} \backslash \{1\}$ and $M = \gamma_{0,e}(2\pi)$ we obtain

$$\begin{aligned} i_\omega(\gamma_{0,e}) &= i_\omega(\xi_{0,e}) \\ &= i_1(\xi_{0,e}) + S_M^+(1) - S_M^-(\omega) \\ &= 0 + S_{I_2}^+(1) + S_{N_1(1,1)}^+(1) - 0 \\ &= 2. \end{aligned} \tag{3.7}$$

For every $e \in [0, 1)$, note that (3.5) yields also

$$\nu_\omega(\gamma_{0,e}) = \nu_\omega(\xi_{0,e}) = \begin{cases} 3, & \text{if } \omega = 1, \\ 0, & \text{if } \omega \in \mathbf{U} \backslash \{1\}. \end{cases} \tag{3.8}$$

3.2. The Boundary Segment $\{9\} \times [0, 1)$

This is studied in our Proposition 1.4 and more precisely in Section 4 below. Especially we have

$$\sigma(\gamma_{9,e}(2\pi)) \subset \mathbf{R}_+ \backslash \{1\}, \qquad \forall\, e \in [0, 1)$$

and it possesses a pair of double positive real hyperbolic eigenvalues. By (2.31), we have $i_1(\gamma_{9,e}) = 0$ for all $e \in [0, 1)$. By our studies in Section 4.1, the matrix $\gamma_{9,e}(2\pi)$ always possesses two double positive eigenvalues not equal to 1, thus by the definition of the splitting number in Section 9.1 of [11], we have $S_M^{\pm}(\omega) = 0$ for $M = \gamma_{9,e}(2\pi)$ with $e \in [0, 1)$ and all $\omega \in \mathbf{U}$. Then for all $e \in [0, 1)$ and $\omega \in \mathbf{U}$ this yields

$$i_\omega(\gamma_{9,e}) = i_\omega(\gamma_{9,0}) = 0, \qquad \nu_\omega(\gamma_{9,e}) = \nu_\omega(\gamma_{9,0}) = 0. \tag{3.9}$$

3.3. The Boundary Segment $[0, 9] \times \{0\}$

In this case $e = 0$. It is considered in (A) of Section 3.1 when $\beta = 0$. When $\beta \in (0, 9]$, this is the case of circular orbits with three positive masses. It was studied in Section 4 of [16] by Roberts and in pp. 275–276 of [15] by Meyer and Schmidt. Below, we shall first recall the properties of eigenvalues of $\gamma_{\beta,0}(2\pi)$. Then we carry out the computations of normal forms of $\gamma_{\beta,0}(2\pi)$, and ± 1 indices $i_{\pm 1}(\gamma_{\beta,0})$ of the path $\gamma_{\beta,0}$ for all $\beta \in [0, 9]$, which are new.

In this case, the essential part of the motion (2.17)–(2.19) becomes an ODE system with constant coefficients:

$$B = B(t) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -\frac{\sqrt{9-\beta}+1}{2} & 0 \\ 1 & 0 & 0 & \frac{\sqrt{9-\beta}-1}{2} \end{pmatrix}. \tag{3.10}$$

The characteristic polynomial $\det(JB - \lambda I)$ of JB is given by

$$\lambda^4 + \lambda^2 + \frac{\beta}{4} = 0. \tag{3.11}$$

Letting $\alpha = \lambda^2$, the two roots of the quadratic polynomial $\alpha^2 + \alpha + \frac{\beta}{4}$ are given by $\alpha = \frac{1}{2}(-1 \pm \sqrt{1-\beta})$. Therefore the four roots of the polynomial (3.11) are given by

$$\alpha_{1,\pm} = \pm\sqrt{\frac{1}{2}(-1 + \sqrt{1-\beta})}, \qquad \alpha_{2,\pm} = \pm\sqrt{\frac{1}{2}(-1 - \sqrt{1-\beta})}. \tag{3.12}$$

(A) Eigenvalues of $\gamma_{\beta,0}(2\pi)$ for $\beta \in [0, 9]$.

When $0 \leqq \beta \leqq 1$, by (3.12), we get the four well-known characteristic multipliers of the matrix $\gamma_{\beta,0}(2\pi)$

$$\rho_{i,\pm}(\beta) = e^{2\pi\alpha_{i,\pm}} = e^{\pm 2\pi\sqrt{-1}\theta_i(\beta)}, \qquad \text{for } \ i = 1, 2, \tag{3.13}$$

where

$$\theta_1(\beta) = \sqrt{\frac{1}{2}(1 - \sqrt{1-\beta})}, \qquad \theta_2(\beta) = \sqrt{\frac{1}{2}(1 + \sqrt{1-\beta})}. \tag{3.14}$$

When $1 < \beta \leqq 9$, from (3.12) by direct computation the four characteristic multipliers of the matrix $\gamma_{\beta,0}(2\pi)$ are given by

$$\rho_{\pm,\pm} = e^{\pm\pi\sqrt{\sqrt{\beta}-1}} e^{\pm\pi\sqrt{-1}\sqrt{\sqrt{\beta}+1}}. \tag{3.15}$$

In particular, we obtain the following results:

When $\beta = 0$, we have $\sigma(\gamma_{0,0}(2\pi)) = \{1, 1, 1, 1\}$, which coincides with (3.5).

When $0 < \beta < 3/4$, in (3.14) the angle $\theta_1(\beta)$ increases strictly from 0 to 1/2 as β increases from 0 to 3/4. Therefore $\rho_{1,+}(\beta) = e^{2\pi\sqrt{-1}\theta_1(\beta)}$ runs from 1 to -1 counterclockwise along the upper semi-unit circle in the complex plane $\mathbf{C}$ as β increases from 0 to 3/4. Correspondingly $\rho_{1,-}(\beta) = e^{-2\pi\sqrt{-1}\theta_1(\beta)}$ runs from 1 to -1 clockwise along the lower semi-unit circle in $\mathbf{C}$ as β increases from 0 to 3/4. At the same time, because $\theta_2(\beta)$ decreases strictly from 1 to $\sqrt{3}/2$ as β increases from 0 to 3/4, therefore $\rho_{2,+}(\beta) = e^{2\pi\sqrt{-1}\theta_2(\beta)}$ runs from 1 to $e^{\sqrt{-1}\sqrt{3}\pi}$ clockwise along the lower semi-unit circle in $\mathbf{C}$ as β increases from 0 to 3/4. Correspondingly $\rho_{2,-}(\beta) = e^{-2\pi\sqrt{-1}\theta_2(\beta)}$ runs from 1 to $e^{-\sqrt{-1}\sqrt{3}\pi}$ counterclockwise along the upper semi-unit circle in $\mathbf{C}$ as β increases from 0 to 3/4. Thus we obtain $\sigma(\gamma_{\beta,0}(2\pi)) \subset \mathbf{U}\backslash\mathbf{R}$ for all $\beta \in (0, 3/4)$.

When $\beta = 3/4$, we have $\theta_1(3/4) = 1/2$ and $\theta_2(3/4) = \sqrt{3}/2$. Therefore we obtain $\rho_{1,\pm}(3/4) = e^{\pm\sqrt{-1}\pi} = -1$ and $\rho_{2,\pm}(3/4) = e^{\pm\sqrt{-1}\sqrt{3}\pi}$.

When $3/4 < \beta < 1$, the angle $\theta_1(\beta)$ increases strictly from 1/2 to $\sqrt{2}/2$ as β increase from 3/4 to 1. Thus $\rho_{1,+}(\beta) = e^{2\pi\sqrt{-1}\theta_1(\beta)}$ runs from -1 to $e^{\sqrt{-1}\sqrt{2}\pi}$ counterclockwise along the lower semi-unit circle in $\mathbf{C}$ as β increases from 3/4 to 1. Correspondingly $\rho_{1,-}(\beta) = e^{-2\pi\sqrt{-1}\theta_1(\beta)}$ runs from -1 to $e^{-\sqrt{-1}\sqrt{2}\pi}$ clockwise along the upper semi-unit circle in $\mathbf{C}$ as β increases from 3/4 to 1. Because $\theta_2(\beta)$ decreases strictly from $\sqrt{3}/2$ to $\sqrt{2}/2$ as β increases from 3/4 to 1, we obtain that $\rho_{2,+}(\beta) = e^{2\pi\sqrt{-1}\theta_2(\beta)}$ runs from $e^{\sqrt{-1}\sqrt{3}\pi}$ to $e^{\sqrt{-1}\sqrt{2}\pi}$ clockwise along the lower semi unit circle in $\mathbf{C}$ as β increases from 3/4 to 1. Correspondingly $\rho_{2,-}(\beta) = e^{-2\pi\sqrt{-1}\theta_2(\beta)}$ runs from $e^{-\sqrt{-1}\sqrt{3}\pi}$ to $e^{-\sqrt{-1}\sqrt{2}\pi}$ counterclockwise along the upper semi unit circle in $\mathbf{C}$ as β increases from 3/4 to 1. Thus we obtain $\sigma(\gamma_{\beta,0}(2\pi)) \subset \mathbf{U}\backslash\mathbf{R}$ for all $\beta \in (3/4, 1)$.

When $\beta = 1$, we obtain $\theta_1(1) = \theta_2(1) = \sqrt{2}/2$, and then we have double eigenvalues $\rho_{1,\pm}(1) = \rho_{2,\pm}(1) = e^{\pm\sqrt{-1}\sqrt{2}\pi}$.

When $1 < \beta < 9$, using notations defined in (3.15), the four characteristic multipliers of $\gamma_{\beta,0}(2\pi)$ satisfy $\sigma(\gamma_{\beta,0}(2\pi)) \subset \mathbf{C}\backslash(\mathbf{U} \cup \mathbf{R})$ for all $\beta \in (1, 9)$.

When $\beta = 9$, $\sqrt{\sqrt{9}+1}\pi = 2\pi$. By (3.15), we get the two positive double characteristic multipliers of $\gamma_{9,0}(2\pi)$ given by $\rho_{\pm,\pm} = e^{\pm\sqrt{2}\pi} e^{\pm\sqrt{-1}2\pi} = e^{\pm\sqrt{2}\pi} \in \mathbf{R}_+\backslash\{1\}$, where we denote by $\mathbf{R}_+ = \{r \in \mathbf{R} \mid r > 0\}$.

(B) Normal forms of $\gamma_{\beta,0}(2\pi)$ for $\beta \in [0, 9]$.

By our above analysis, the matrix $\gamma_{\beta,0}(2\pi)$ possesses no eigenvalues ± 1 for $\beta \in (3/4, 9]$. Note also that the matrix $\gamma_{\beta,0}(2\pi)$ is homotopic to $N_2(\omega_0, b)$ with $\omega_0 = e^{\sqrt{-1}\pi\sqrt{2}}$ and $b_2 - b_3 \neq 0$ when $\beta = 1$, and is hyperbolic when $\beta \in (1, 9]$. Therefore by Definition 2.1 we have

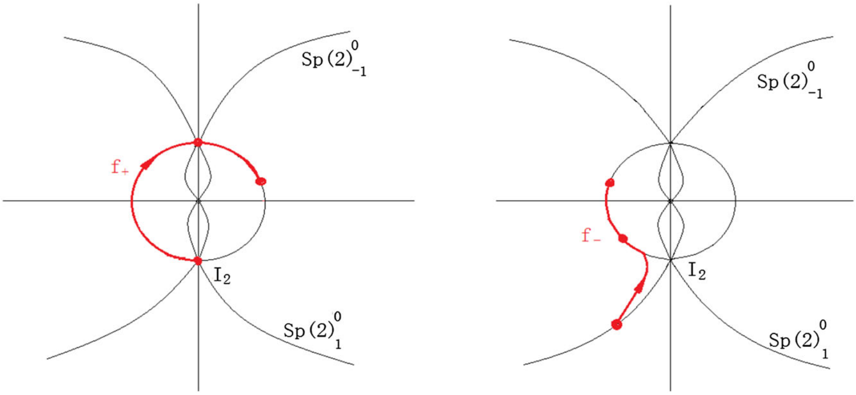

Fig. 3. The pictures of the two paths $f_+(\beta)$ and $f_-(\beta)$ in Sp(2), where the *three dark points* are their images at $\beta = 0, 3/4$ and 1

$$\gamma_{\beta,0} \sim_{\pm 1} \gamma_{\beta_0,0}, \qquad \forall\ \beta \in [1, 9] \text{ and } \beta_0 \in (3/4, 1). \tag{3.16}$$

Thus we are next especially interested in the normal forms of $\gamma_{\beta,0}(2\pi)$ with $\beta \in [0, 1)$.

For that purpose we construct a family of continuous curves $f(\beta)(t)$ in Sp(4) for $\beta \in [0, 1)$ and $t \in [0, 2\pi]$ satisfying $f(\beta)(0) = I$ such that $\gamma_{\beta,0}(2\pi) \sim_{\pm 1} f(\beta)$ for all $\beta \in [0, 1)$, which implies $\gamma_{\beta,0}(2\pi) \approx f(\beta)$ for all $\beta \in [0, 1)$. After that we shall study the normal forms of $\gamma_{\beta,0}(2\pi)$ for $\beta \in [1, 9]$.

The curve f is defined separately according to β as follows.

(i) *Normal forms of* $\gamma_{\beta,0}(2\pi)$ *when* $\beta \in [0, 3/4]$.

Here, we split f into a symplectic sum of two Sp(2)-paths:

$$f(\beta) = f_+(\beta)\diamond f_-(\beta), \qquad f_\pm(\beta) \in \text{Sp}(2). \tag{3.17}$$

Using the cylindrical coordinate representation (which is denoted by CCR for short below) of Sp(2) introduced in [8] (cf. pp. 48–50 of [11], especially Figs. 1 and 2 in pp. 49–50 there), we can describe the matrix curves in Sp(2) more precisely, which is shown below in Fig. 3.

Using notations in Section 2 we let

$$M_0 = D(2)R(2\pi - \arcsin(3/5)) = \begin{pmatrix} 8/5 & 6/5 \\ -3/10 & 2/5 \end{pmatrix}. \tag{3.18}$$

Then $M_0 \approx N_1(1, 1)$ and thus $M_0 \in \text{Sp}(2)^0_{1,-} \cap \{(r, \theta, z) \in \mathbf{R}^3 \setminus \{z\text{−axis}\} \mid z = 0\}$ in Fig. 2.1.2 in p. 50 of [11]. By (3.3) and our discussions in part (A), we define

$$f_+(0) = I_2 \qquad \text{and} \qquad f_-(0) = M_0. \tag{3.19}$$

By our analysis in part (A), when β increases from 0 to 3/4 in (0, 3/4), we define

$$f_+(\beta) = R\left(2\pi - \frac{4}{3}\pi\beta\right), \qquad \text{for} \quad \beta \in [0, 3/4]. \tag{3.20}$$

Then the matrix curve $f_+(\beta)$ runs from I_2 to $R(\pi)=-I_2$ along the left semi-circle clockwise in CCR in the left diagram of Fig. 3 below. Then we can choose some $\beta_1\in(0,3/4)$ and define

$$f_-(\beta)=D\left(2-\frac{\beta}{\beta_1}\right)$$
$$R\left((2\pi-\arcsin(3/5))\left(1-\frac{\beta}{\beta_1}\right)+\beta\frac{9\pi}{5}\right),\quad \text{for}\quad \beta\in[0,\beta_1], \tag{3.21}$$

and

$$f_-(\beta)=R\left(\sqrt{3}\pi-\frac{3/4-\beta}{3/4-\beta_1}\left(\sqrt{3}-\frac{9}{5}\right)\pi\right),\quad \text{for}\quad \beta\in[\beta_1,3/4]. \tag{3.22}$$

Thus the matrix curve $f_-(\beta)$ runs from $f_-(0)=M_0$ to $f_-(\beta_1)=R(9\pi/5)$ when β runs from 0 to β_1, and then runs from $f_-(\beta_1)=R(9\pi/5)$ to $f_-(3/4)=R(\sqrt{3}\pi)$ when β runs from β_1 to $3/4$. The image of $f_-(\beta)$ is shown along the left semi-circle clockwise in CCR in the right diagram of Fig. 3 below.

Then by adjusting the running speeds of $f_+(\beta)$ and $f_-(\beta)$ according to those of the corresponding eigenvalues in (3.13) respectively, we can have

$$f(\beta)=f_+(\beta)\diamond f_-(\beta)\approx\gamma_{\beta,0}(2\pi),\qquad \text{for}\quad \beta\in[0,3/4]. \tag{3.23}$$

Especially when $\beta=3/4$, we obtain

$$f(3/4)=-I_2\diamond R(\sqrt{3}\pi). \tag{3.24}$$

(ii) *Normal forms of $\gamma_{\beta,0}(2\pi)$ when $\beta\in[3/4,1)$.*

For $\beta\in[3/4,1)$, following part (A), we define

$$f_+(\beta)=R(4(1-\sqrt{2})\pi\beta+(3\sqrt{2}-2)\pi), \tag{3.25}$$
$$f_-(\beta)=R(4(\sqrt{2}-\sqrt{3})\pi\beta+(4\sqrt{3}-3\sqrt{2})\pi). \tag{3.26}$$

By adjusting the two curves $f_+(\beta)$ and $f_-(\beta)$ suitably when $\beta<1$ and close to 1 and then adjusting their running speeds in β suitably, we can suppose that when β increases from $3/4$ to 1 in $(3/4,1)$, the matrix curve $f_+(\beta)$ runs from $-I_2$ and tends to $R((2-\sqrt{2})\pi)$ along the right semi-circle clockwise in CCR in the left diagram of Fig. 3 below as β runs from $3/4$ and tends to 1. Simultaneously the matrix curve $f_-(\beta)$ runs from $R(\sqrt{3}\pi)$ and tends to $R(\sqrt{2}\pi)$ along the left semi-circle clockwise in CCR in the right diagram of Fig. 3 below as β runs from $3/4$ and tends to 1. Here we shall not explain how $f(\beta)=f_+(\beta)\diamond f_-(\beta)$ gets to its limit when $\beta\to 1$. Note that in this case we have also (3.23) holds when $\beta\in[3/4,1)$.

(iii) Note that for $\beta\in[0,1)$, we have $f_+(\beta)=R(\alpha(\beta))$, where the value of $\alpha(\beta)$ is uniquely given by (3.20) and (3.25). Then every matrix $f(\beta)=f_+(\beta)\diamond f_-(\beta)\in\mathrm{Sp}(4)$ in (3.17) can be reached by a path starting from I_4 in Sp(4) as follows:

$$f_+(\beta)(t)=R(\alpha(\beta)\frac{t}{2\pi}),\qquad \text{for}\quad 0\leqq t\leqq 2\pi, \tag{3.27}$$

$$f_-(\beta)(t)=\begin{cases} D(2t/\pi)R((2\pi-\arcsin(3/5))t/\pi), & \text{if}\ \ 0\leqq t\leqq\pi,\\ f_-(\beta\frac{t-\pi}{\pi}), & \text{if}\ \ \pi<t\leqq 2\pi,\end{cases} \tag{3.28}$$

for $\beta \in [0, 1)$, where we have used the expression of the matrix M_0 in (3.18).

(iv) *The normal form of* $\gamma_{\beta,0}(2\pi)$ *when* $\beta = 1$.

Firstly by Corollary 4.5 below, for fixed $e \in [0, 1)$ the index $i_\omega(\gamma_{\beta,e}(2\pi))$ is non-increasing when β increases from 0 to 9 for $\omega \in \mathbf{U}\backslash\{1\}$. By Proposition 6.1 below, the index $i_\omega(\gamma_{\beta,e}(2\pi))$ can change only when the matrix path $\gamma_{\beta,e}$ in β passes through an ω degeneracy point and there should be either one or two ω-degeneracy points and their total ω degenerate multiplicity is 2.

By part (A), $\beta = 1$ is a Krein collision point of the matrix path $\gamma_{\beta,0}(2\pi)$ for $\beta \in [0, 9]$ with $\sigma(\gamma_{1,0}(2\pi)) = \{\omega_0, \overline{\omega}_0, \omega_0, \overline{\omega}_0\}$ for $\omega_0 = e^{\sqrt{-1}\sqrt{2}\pi} \in \mathbf{U}$. Note that by our above discussions, when β increases in the open interval $(0, 3/4)$, the curve $f_+(\beta)$ passes through each ω singular surface in $\mathrm{Sp}(2)$ precisely once for all $\omega \in \mathbf{U}\backslash\mathbf{R}$, which contributes precisely a 1 to $\sum_{\beta\in[0,9]} \nu_\omega(\gamma_{\beta,0}(2\pi))$. Therefore by Proposition 6.1 below we must have

$$\nu_{\omega_0}(\gamma_{1,0}(2\pi)) \equiv \dim_{\mathbf{C}} \ker_{\mathbf{C}}(\gamma_{1,0}(2\pi) - \omega_0 I) = 1. \tag{3.29}$$

Then, we must have

$$\gamma_{1,0}(2\pi) \approx N_2(\omega_0, b) = \begin{pmatrix} R(\sqrt{2}\pi) & b \\ 0 & R(\sqrt{2}\pi) \end{pmatrix} \quad \text{with} \quad b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \tag{3.30}$$

satisfying $b_2 - b_3 \neq 0$.

(v) *Normal forms of* $\gamma_{\beta,0}(2\pi)$ *when* $\beta \in (1, 9]$.

For $\beta \in (1, 9)$, by part (A), we have

$$\gamma_{\beta,0}(2\pi) \approx \begin{pmatrix} e^{\pi\sqrt{\sqrt{\beta}-1}} R(\pi\sqrt{\sqrt{\beta}+1}) & b \\ 0 & e^{-\pi\sqrt{\sqrt{\beta}-1}} R(\pi\sqrt{\sqrt{\beta}+1}) \end{pmatrix}, \tag{3.31}$$

for some matrix $b = b(\beta)$, which is CS-hyperbolic. When $\beta = 9$ we get

$$\gamma_{9,0}(2\pi) \approx D(e^{\sqrt{2}\pi}) \diamond D(e^{\sqrt{2}\pi}). \tag{3.32}$$

Remark 3.1. Because $B(t)$ is a constant matrix depending only on β when $e = 0$, similarly to what we did for $\gamma_{0,0}(t)$ it is possible to compute out the fundamental matrix path $\gamma_{\beta,0}(t)$ explicitly when $\beta > 0$. But the computations on $\gamma_{\beta,0}(t)$ when $\beta > 0$ are rather delicate and tedious and thus are omitted here. From this computation, especially when $0 \leqq \beta < 1$ we obtain that $\gamma_{\beta,0}(t) \approx R(-2\pi\theta_1(\beta)t) \diamond R(2\pi\theta_2(\beta)t)$ for some $\theta_i(\beta)$ with $i = 1, 2$, and the β-paths $R(-2\pi\theta_1(\beta))$ and $R(2\pi\theta_2(\beta))$ are homotopic respectively to $f_+(\beta)$ and $f_-(\beta)$ which we constructed above.

(C) Indices $i_{\pm 1}(\gamma_{\beta,0})$ for $\beta \in [0, 9]$.

From the above discussions as well as Fig. 3 we obtain

$$i_1(f_+(\beta)) = 1, \quad \forall\, \beta \in [0, 1), \tag{3.33}$$

$$\nu_1(f_+(\beta)) = \begin{cases} 2, & \text{if } \ \beta = 0, \\ 0, & \text{if } \ \beta \in (0, 1). \end{cases} \tag{3.34}$$

$$i_1(f_-(\beta)) = -1, \quad \forall\, \beta \in [0, 1), \tag{3.35}$$

$$\nu_1(f_-(\beta)) = \begin{cases} 1, & \text{if } \ \beta = 0, \\ 0, & \text{if } \ \beta \in (0, 1). \end{cases} \tag{3.36}$$

Therefore by (3.17) we get

$$i_1(\gamma_{\beta,0}) = 0, \qquad \forall\, \beta \in [0, 9], \tag{3.37}$$

$$\nu_1(\gamma_{\beta,0}) = \begin{cases} 3, & \text{if } \beta = 0, \\ 0, & \text{if } \beta \in (0, 9]. \end{cases} \tag{3.38}$$

Similarly for the -1 index, we obtain

$$i_{-1}(f_+(\beta)) = \begin{cases} 2, \text{ if } \beta \in [0, 3/4), \\ 0, \text{ if } \beta \in [3/4, 1). \end{cases} \tag{3.39}$$

$$\nu_{-1}(f_+(\beta)) = \begin{cases} 0, \text{ if } \beta \in [0, 1)\backslash\{3/4\}, \\ 2, \text{ if } \beta = 3/4. \end{cases} \tag{3.40}$$

$$i_{-1}(f_-(\beta)) = 0, \qquad \nu_{-1}(f_-(\beta)) = 0, \qquad \text{if } \beta \in [0, 1). \tag{3.41}$$

Therefore by (3.16) and Proposition 6.1 below, we get

$$i_{-1}(\gamma_{\beta,0}) = \begin{cases} 2, \text{ if } \beta \in [0, 3/4), \\ 0, \text{ if } \beta \in [3/4, 9]. \end{cases} \tag{3.42}$$

$$\nu_{-1}(\gamma_{\beta,0}) = \begin{cases} 0, \text{ if } \beta \in [0, 9]\backslash\{3/4\}, \\ 2, \text{ if } \beta = 3/4. \end{cases} \tag{3.43}$$

The other ω-indices of $\gamma_{\beta,0}$ can be computed similarly for $\omega \neq e^{\pm\sqrt{-1}\sqrt{2}\pi} \equiv \omega_0$, and $i_{\omega_0}(\gamma_{\beta,0})$ can be computed using the decreasing property of the index proved in Corollary 4.5 and its values at $\beta = 0$ and $\beta = 9$ as we did in the proof of Theorem 1.2 below.

Here we point out especially that $\beta = 1$ is the only Krein collision point on the segment $[0, 9] \times \{0\}$.

4. Non-Degeneracy of Elliptic Lagrangian Solutions

Note that the complete eigenvalue 1 non-degeneracy implies that there is no linear stability change near the positive real line in the complex plane **C**.

4.1. Hyperbolicity of Elliptic Lagrangian Solutions with Equal Masses

In the equal mass case, that is $\beta = 9$, we have

$$A(9, e) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_2 - I_2 + \frac{3}{2(1 + e\cos t)} I_2. \tag{4.1}$$

Let

$$A_1(e) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 1 + \frac{3}{2(1 + e\cos t)}, \tag{4.2}$$

which is a self-adjoint operator with domain $\overline{D}(1, 2\pi) \subset W^{2,2}([0, 2\pi], \mathbf{R})$ under the periodic boundary conditions $x(t) = x(t+2\pi)$, $\dot{x}(t) = \dot{x}(t+2\pi)$ for all $t \in \mathbf{R}$. Then

$$A(9, e) = A_1(e) \oplus A_1(e). \tag{4.3}$$

By (2.31), the Morse index of $A(9, e)$ is zero, so $A_1(e)$ is a non-negative operator. Moreover, we have

Proposition 4.1. $A_1(e) > 0$ *for all* $0 \leqq e < 1$ *on its domain* $\overline{D}(1, 2\pi)$.

Proof. It suffices to show $\ker(A_1(e)) = \{0\}$ for all $0 \leqq e < 1$. We argue by contradiction. Suppose $0 \neq x \in \ker(A_1(e))$ is expressed as a Fourier series $x = x(t) = \sum_{k\in\mathbf{Z}} a_k \exp(\sqrt{-1}kt)$. Then we have

$$\begin{aligned}
0 &= 2(1 + e\cos t)A_1(e)x(t) \\
&= (2 + e\exp(\sqrt{-1}t) + e\exp(-\sqrt{-1}t)) \sum_{k\in\mathbf{Z}} a_k(k^2 - 1)\exp(\sqrt{-1}kt) \\
&\quad + \sum_{k\in\mathbf{Z}} 3a_k \exp(\sqrt{-1}kt) \\
&= \sum_{k\in\mathbf{Z}} (2a_k(k^2 - 1) \\
&\quad + ea_{k-1}((k-1)^2 - 1) + ea_{k+1}((k+1)^2 - 1) + 3a_k)\exp(\sqrt{-1}kt).
\end{aligned} \tag{4.4}$$

This implies

$$2a_k(k^2 - 1) + ea_{k-1}((k-1)^2 - 1) + ea_{k+1}((k+1)^2 - 1) + 3a_k = 0$$

holds for every $k \in \mathbf{Z}$. Let $k = 0$, we have $a_0 = 0$. So x belongs to the subspace V which is spanned by $\{\exp\sqrt{-1}kt, k \neq 0\}$. Note that $(-\frac{d^2}{dt^2} - 1)|_V \geqq 0$ and $\frac{3}{2(1+e\cos t)} > 0$, so $A_1(e)$ is positive on V, which then implies that x must be zero. This contradiction completes the proof. □

Remark 4.2. For $e = 1$, the operator $A_1(e)$ is singular. By the same argument as above one can show that $A_1(1)x = 0$ also implies that $x = 0$ in $L^2([0, 2\pi], \mathbf{R})$.

Proof of Proposition 1.4. Let $\bar{\xi}_e(t)$ be the fundamental solution of the first order linear Hamiltonian system corresponding to $A_1(e)$. Then it satisfies

$$\frac{d}{dt}\bar{\xi}_e(t) = J_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{3}{2(1+e\cos t)} \end{pmatrix} \bar{\xi}_e(t), \tag{4.5}$$

$$\bar{\xi}_e(0) = I_2. \tag{4.6}$$

Thus we have

$$\xi_{9,e}(t) = \bar{\xi}_e(t) \diamond \bar{\xi}_e(t), \tag{4.7}$$

and so

$$i_\omega(\xi_{9,e}) = 2i_\omega(\bar{\xi}_e), \qquad \nu_\omega(\xi_{9,e}(2\pi)) = 2\nu_\omega(\bar{\xi}_e(2\pi)). \tag{4.8}$$

By Proposition 4.1, we have

$$\dim\ker(\bar{\xi}_e(2\pi)-I_2)=0, \qquad \forall\, e\in[0,1). \tag{4.9}$$

Since $\bar{\xi}_e(2\pi)$ is a 2×2 symplectic matrix, its eigenvalues are in pair $\{\lambda,\lambda^{-1}\}$ with λ real and $\lambda\neq 1$ or $\lambda\in\mathbf{U}\backslash\{1\}$. Because $i_1(\bar{\xi}_e)=0$ by Proposition 4.1, the matrix $\bar{\xi}_e(2\pi)$ must have normal form $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ with $\lambda>0$ or $\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$ by [11] (cf. pp. 179–183 there). By (4.9), 1 can not be an eigenvalue of $\bar{\xi}_e(2\pi)$ for any $e\in[0,1)$, so the only possible case is that $\bar{\xi}_e(2\pi)$ has a pair of positive real eigenvalues not equal to 1, that is λ and $\lambda^{-1}>0$ and $\lambda\neq 1$, and the eigenvalues of $\bar{\xi}_e(2\pi)$ depend continuously on e. So the eigenvalues of $\bar{\xi}_e(2\pi)$ are real positive numbers not equal to 1 for any $e\in[0,1)$. Especially from (4.7) with $t=2\pi$, (4.8) and the above discussions, we obtain

$$i_\omega(\xi_{9,e})=2i_\omega(\bar{\xi}_e)=0, \qquad \nu_\omega(\xi_{9,e}(2\pi))=2\nu_\omega(\bar{\xi}_e(2\pi))=0. \tag{4.10}$$

Thus by (2.27) and (2.30), the Proposition 1.4 is proved. □

Corollary 4.3. *$A(9,e)$ is a positive operator for any ω boundary conditions.*

Proof. When $\beta=9$, since $i_1(\bar{\xi}_e)=0$ and $\bar{\xi}_e(2\pi)$ is hyperbolic as we proved above, we have $i_\omega(\bar{\xi}_e)=0$ for any $\omega\in\mathbf{U}$ and $e\in[0,1)$. Thus the ω Morse index $i_\omega(\xi_{9,e})=\phi_\omega(A(9,e))=0$, that is, $A(9,e)\geqq 0$ in any ω boundary conditions. Then $A(9,e)>0$ follows from the fact that $A(9,e)$ is non-degenerate at any ω boundary conditions by (4.10). □

4.2. The Non-Degeneracy of Elliptic Lagrangian Solutions for $\omega=1$

For $(\beta,e)\in[0,9)\times[0,1)$, let $\bar{A}(\beta,e)=\frac{A(\beta,e)}{\sqrt{9-\beta}}$. Using (2.29) we can rewrite $A(\beta,e)$ as follows

$$\begin{aligned} A(\beta,e) &= A(9,e)+\frac{\sqrt{9-\beta}}{2(1+e\cos t)}S(t)=\sqrt{9-\beta}\left(\frac{A(9,e)}{\sqrt{9-\beta}}+\frac{S(t)}{2(1+e\cos t)}\right)\\ &=\sqrt{9-\beta}\,\bar{A}(\beta,e). \end{aligned} \tag{4.11}$$

Then we have

$$\phi_\omega(A(\beta,e))=\phi_\omega(\bar{A}(\beta,e)), \tag{4.12}$$

$$\nu_\omega(A(\beta,e))=\nu_\omega(\bar{A}(\beta,e)). \tag{4.13}$$

Following from Corollary 4.3, that is, the fact that $A(9,e)$ is positive definite for any ω boundary condition, we get the following important lemma:

Lemma 4.4. (i) *For each fixed $e\in[0,1)$, the operator $\bar{A}(\beta,e)$ is increasing with respect to $\beta\in[0,9)$ for any fixed $\omega\in\mathbf{U}$. Especially*

$$\frac{\partial}{\partial\beta}\bar{A}(\beta,e)|_{\beta=\beta_0}=\frac{1}{2(9-\beta_0)^{3/2}}A(9,e), \tag{4.14}$$

for $\beta \in [0, 9)$ *is a positive definite operator.*

(ii) *For every eigenvalue* $\lambda_{\beta_0} = 0$ *of* $\bar{A}(\beta_0, e_0)$ *with* $\omega \in \mathbf{U}$ *for some* $(\beta_0, e_0) \in [0, 9) \times [0, 1)$*, there holds*

$$\frac{d}{d\beta}\lambda_\beta|_{\beta=\beta_0} > 0. \tag{4.15}$$

Proof. It suffices to prove (ii). Let $x_0 = x_0(t)$ with unit norm such that

$$\bar{A}(\beta_0, e_0)x_0 = 0. \tag{4.16}$$

Fix e_0. Then $\bar{A}(\beta, e_0)$ is an analytic path of strictly increasing self-adjoint operators with respect to β. Following Kato ([6], p. 120 and 386), we can choose a smooth path of unit norm eigenvectors x_β with $x_{\beta_0} = x_0$ belonging to a smooth path of real eigenvalues λ_β of the self-adjoint operator $\bar{A}(\beta, e_0)$ on $\overline{D}(\omega, 2\pi)$ such that for small enough $|\beta - \beta_0|$, we have

$$\bar{A}(\beta, e_0)x_\beta = \lambda_\beta x_\beta, \tag{4.17}$$

where $\lambda_{\beta_0} = 0$. Taking inner product with x_β on both sides of (4.17) and then differentiating it with respect to β at β_0, we get

$$\begin{aligned}
\frac{\partial}{\partial\beta}\lambda_\beta|_{\beta=\beta_0} &= \left\langle \frac{\partial}{\partial\beta}\bar{A}(\beta, e_0)x_\beta, x_\beta \right\rangle|_{\beta=\beta_0} + 2\left\langle \bar{A}(\beta, e_0)x_\beta, \frac{\partial}{\partial\beta}x_\beta \right\rangle|_{\beta=\beta_0} \\
&= \left\langle \frac{\partial}{\partial\beta}\bar{A}(\beta_0, e_0)x_0, x_0 \right\rangle \\
&= \frac{1}{2(9-\beta_0)^{3/2}}\left\langle A(9, e_0)x_0, x_0 \right\rangle \\
&> 0,
\end{aligned}$$

where the second equality follows from (4.16), the last equality follows from the definition of $\bar{A}(\beta, e)$ and (4.11), the last inequality follows from the positive definiteness of $A(9, e)$ given by Corollary 4.3, and the fact $x_0 \neq 0$. Thus (4.15) is proved. □

Consequently we arrive at

Corollary 4.5. *For every fixed* $e \in [0, 1)$ *and* $\omega \in \mathbf{U}$*, the index function* $\phi_\omega(A(\beta, e))$*, and consequently* $i_\omega(\gamma_{\beta,e})$*, is non-increasing in* $\beta \in [0, 9]$*. When* $\omega \in \mathbf{U}\backslash\{1\}$*, it decreases from* 2 *to* 0*.*

Proof. For $0 \leqq \beta_1 < \beta_2 < 9$ and fixed $e \in [0, 1)$, when β increases from β_1 to β_2, it is possible that negative eigenvalues of $\bar{A}(\beta_1, e)$ pass through 0 to become positive ones of $\bar{A}(\beta_2, e)$, but it is impossible that positive eigenvalues of $\bar{A}(\beta_2, e)$ pass through 0 to become negative by (ii) of Lemma 4.4. Therefore the first claim holds. The second claim follows from (3.7), (3.9) and Corollary 4.3. □

From now on in this section, we will focus on the case of $\omega = 1$. Since $\phi_1(A(\beta, e)) = 0$, we have $\bar{A}(\beta, e) \geqq 0$ for $(\beta, e) \in [0, 9) \times [0, 1)$. Furthermore, we have

Proposition 4.6. *$A(\beta,e) > 0$ for all $(\beta,e) \in (0,9)\times[0,1)$ under the periodic boundary conditions, that is, on $\overline{D}(1,2\pi)$.*

Proof. It suffices to prove $\bar{A}(\beta,e) > 0$. This is essentially due to the fact that $\ker(\bar{A}(\beta,e)) = \{0\}$ on $(\beta,e) \in (0,9)\times[0,1)$. In fact, otherwise, there exists an $x_0 = x_0(t)$ with unit norm such that (4.16) holds. Then (4.17) implies that there exists a negative eigenvalue of $\bar{A}(\beta,e_0)$ when $\beta < \beta_0$ is sufficiently close to β_0, which contradicts to $\bar{A}(\beta,e) \geqq 0$. □

Proof of Theorem 1.1. Since $\dim\ker(A(\beta,e)) = \dim\ker(\gamma_{\beta,e}(2\pi) - I_4)$, we have proved that the elliptic Lagrangian solutions are all non-degenerate on $(\beta,e) \in (0,9]\times([0,1)$. It is degenerate when $\beta = 0$, by (3.8). The proof is complete. □

5. The Limiting Case $e \to 1$

We shall use the sesquilinear form to study the case when $e \to 1$, and please refer to Chapter 6 of [6] for the details on the sesquilinear form.

In this section we shall deal with the limiting cases $e = 1$ and -1, then the term $(1 + e\cos t)$ in the denominator of the expression of the operator $A(\beta,e)$ and the corresponding functionals will become zero when $t = (2k+1)\pi$ if $e = 1$ or $2k\pi$ if $e = -1$ with $k \in \mathbf{Z}$. Note that the boundary condition $x(2\pi) = \omega x(0)$ is equivalent to the boundary condition $x(t+2\pi) = \omega x(t)$ for all $t \in \mathbf{R}$ in the domain $D(\omega,2\pi)$ with $\omega \in \mathbf{U}$ of the corresponding functionals. Therefore in order to move the singular times to the end of the boundary points of the integral intervals, in this section we use the interval $\Theta(e) = [-\pi,\pi]$ to replace $[0,2\pi]$ when we study the case $e \to 1$, and keep $\Theta(e) = [0,2\pi]$ when we study the case $e \to -1$ for all the corresponding integrals.

For any $(\beta,e) \in [0,9]\times(-1,1)$ and $\omega \in \mathbf{U}$, we define the symmetric sesquilinear form $\Gamma(\beta,e)$ corresponding to the operator $A(\beta,e)$ in (2.29) by

$$\begin{aligned}\Gamma(\beta,e)(x,y) &= \int_{\Theta(e)} \left[\dot{x}\cdot\dot{y} - x\cdot y + \frac{1}{2(1+e\cos t)}((3I_2 + \sqrt{9-\beta}S(t))x(t))\cdot y(t)\right] dt,\\ &\qquad \forall\, x,y \in D(\omega,2\pi).\end{aligned} \tag{5.1}$$

where the domain $D(\omega,2\pi)$ is defined in (2.10). Then we denote by $\Gamma(\beta,e)(x) = \Gamma(\beta,e)(x,x)$ the corresponding quadratic form. Note that here we have extended the range of e to $(-1,1)$, even to the whole complex plan in the next section. This seems artificial from the point of view of celestial mechanics, however we can draw some interesting conclusions on the degeneracy curves in the (β,e) rectangle $[0,9]\times(-1,1)$ later. Note that our results proved for $e \in [0,1)$ in the previous and later sections hold also for $e \in (-1,1)$, which we shall not indicate explicitly later. Since $\Gamma(\beta,e) + I$ is equivalent to $\Gamma(\beta,e)$ with respect to the $W^{1,2}$-norm, $\Gamma(\beta,e)$ is closed on domain $D(\omega,2\pi)$ ([6], p. 313).

For $e = \pm 1$, we further define $\Gamma(\beta, e)$ as in (5.1), but its domain needs to be modified. More precisely, say, for $e = 1$, the domain of $\Gamma(\beta, 1)$ is defined by

$$\hat{D}(\beta) = \left\{ x \in W^{1,2}([-\pi, \pi], \mathbf{R}^2) \;\Big|\; \int_{-\pi}^{\pi} \Big(\frac{1}{2(1+\cos t)} ((3I_2 + \sqrt{9-\beta} S(t)) x(t)) \cdot x(t) \Big) \, \mathrm{d}t < \infty \right\}. \tag{5.2}$$

Since $3I_2 + \sqrt{(9-\beta)} S(t) > 0$ for $0 < \beta \leqq 9$, we have $\hat{D}(\beta) = \hat{D}(9)$ for $0 < \beta \leqq 9$ and $\hat{D}(0)$ is different from them. Note that every $x \in \hat{D}(9)$ must satisfy $x(\pi) = x(-\pi) = 0$ which is the vanishing ω boundary condition. And $\Gamma(\beta, 1)$ is closed since it is the sum of two closed symmetric forms.

For $e \in (-1, 1)$, $A(\beta, e)$ is the Friedrichs extension operator ([6], Theorem 2.1 in p. 322) of $\Gamma(\beta, e)$ under the ω boundary condition. Let $A(\beta)$ be the Friedrichs extension operator of $\Gamma(\beta, 1)$. Then it has the form

$$A(\beta) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_2 - I_2 + \frac{3}{2(1+\cos t)} I_2 + \frac{\sqrt{9-\beta} S(t)}{2(1+\cos t)}, \tag{5.3}$$

with $\mathrm{dom}(A(\beta)) \subset \hat{D}(\beta)$.

Lemma 5.1. *For each $\beta \in (0, 9]$, $A(\beta)$ is a self-adjoint operator on $L^2([-\pi, \pi], \mathbf{R}^2)$ with compact resolvent.*

Proof. Since $\Gamma(\beta, 1)$ is symmetric, $A(\beta)$ is self-adjoint. It suffices to prove that $A(\beta) + 2I_2$ has a compact inverse. Let $\Gamma'(x) = \Gamma(\beta, 1)(x) + 2\|x\|^2_{W^{1,2}}$. Then $A(\beta) + 2I_2$ is the Friedrichs extension operator of Γ'. By the second representation theorem ([6], p. 331),

$$\Gamma'(x) = \langle (A(\beta) + 2I_2)^{\frac{1}{2}} x, (A(\beta) + 2I_2)^{\frac{1}{2}} x \rangle. \tag{5.4}$$

Since $\|x\|^2_{W^{1,2}} \leqq \Gamma'(x)$, a Γ'-bounded set must have a convergent subsequence in L^2. This implies that $(A(\beta) + 2I_2)^{-\frac{1}{2}}$ is compact, so $(A(\beta) + 2I_2)^{-1}$ is also compact in L^2. □

Lemma 5.2. *For $\beta \in (0, 9]$ and $x \in \hat{D}(\beta)$, we have $\Gamma(\beta, e)(x) \to \Gamma(\beta, 1)(x)$ as $e \to 1$.*

Proof. Let $x \in \hat{D}(\beta) \subset D(\omega, 2\pi)$ and $e \in [0, 1)$. By definitions of $\Gamma(\beta, e)$ and $\Gamma(\beta, 1)$, when $\cos t > 0$, we obtain

$$\begin{aligned} &\left| \left(\frac{1}{2(1+e\cos t)} - \frac{1}{2(1+\cos t)} \right) ((3I_2 + \sqrt{9-\beta} S(t)) x(t) \cdot x(t)) \right| \\ &\quad = \frac{1}{2} \left| \frac{(1-e)\cos t}{(1+e\cos t)(1+\cos t)} ((3I_2 + \sqrt{9-\beta} S(t)) x(t) \cdot x(t)) \right| \\ &\quad \leqq (1-e) \left| \frac{1}{2(1+\cos t)} ((3I_2 + \sqrt{9-\beta} S(t)) x(t) \cdot x(t)) \right|. \end{aligned}$$

When $\cos t \leqq 0$, we have

$$\left|\frac{1}{2(1+e\cos t)}((3I_2+\sqrt{9-\beta}S(t))x(t)\cdot x(t))\right| \leqq \left|\frac{1}{2(1+\cos t)}((3I_2+\sqrt{9-\beta}S(t))x(t)\cdot x(t))\right|.$$

Therefore we get

$$\left|\frac{1}{2(1+e\cos t)}((3I_2+\sqrt{9-\beta}S(t))x(t)\cdot x(t))\right| \leqq (2-e)\left|\frac{1}{2(1+\cos t)}((3I_2+\sqrt{9-\beta}S(t))x(t)\cdot x(t))\right|, \quad (5.5)$$

for all $t \in [0, 2\pi]$. Now the lemma follows from the Lebesgue's dominated convergence theorem. □

Since under the periodic boundary condition, $\Gamma(\beta, e) > 0$ for $e \in [0, 1)$, the above lemma tells us that $\Gamma(\beta, 1) \geqq 0$. By Remark 4.2, we have $\ker(A(9, 1)) = \{0\}$, so $\Gamma(9, 1) > 0$. By completely the same reasoning as in Proposition 4.6, we have

$$\Gamma(\beta, 1) > 0, \qquad \forall \beta \in (0, 9]. \quad (5.6)$$

For the limiting case $e \to 1$ under general boundary conditions, we consider the sesquilinear form $\hat{\Gamma}(\beta, e)$, for $\beta \in [0, 9]$ and $e \in (-1, 1)$, by

$$\hat{\Gamma}(\beta, e)(x, y) = \int_{-\pi}^{\pi}\left[\dot{x}\cdot\dot{y} - x\cdot y + \left(\frac{3I_2+\sqrt{9-\beta}S(t)}{2(1+e\cos t)}x(t)\cdot y(t)\right)\right]\mathrm{d}t, \quad \forall x, y \in W^{1,2}(\mathbf{R}/2\pi\mathbf{Z}, \mathbf{R}^2). \quad (5.7)$$

We have the following

Lemma 5.3. *For $\beta \in (0, 9]$, we have $\hat{\Gamma}(\beta, e) > 0$ when $1 - e$ is small enough.*

Proof. Let $\delta(\beta, e)$ be the largest lower bound of the quadratic form $\hat{\Gamma}(\beta, e)$ for $e \in [0, 1]$. Then by (5.6) we have

$$\delta(\beta, 1) > 0 \qquad \forall\, \beta \in (0, 9]. \quad (5.8)$$

We need to show

$$\liminf_{e\to 1} \delta(\beta, e) > 0. \quad (5.9)$$

For $(\beta, e) \in (0, 9] \times [0, 1)$, we define

$$f_{\beta,e}(t) = \begin{cases} \frac{3I_2+\sqrt{9-\beta}S(t)}{2(1+e\cos t)} & \text{if} \quad \cos t \leqq 0 \\ \frac{3I_2+\sqrt{9-\beta}S(t)}{2(1+\cos t)} & \text{if} \quad \cos t > 0 \end{cases}, \quad (5.10)$$

and

$$\tilde{\Gamma}(\beta, e)(x) = \int_{-\pi}^{\pi} \left[\dot{x} \cdot \dot{x} - x \cdot x + f_{\beta,e}(t)x(t) \cdot x(t)\right] \mathrm{d}t. \tag{5.11}$$

Let $\tilde{\delta}(\beta, e)$ be the largest lower bound of $\tilde{\Gamma}(\beta, e)$ on $W^{1,2}(\mathbf{R}/(2\pi\mathbf{Z}), \mathbf{R}^2)$ for $e \in [0, 1]$.

Then by (5.5) we obtain

$$\liminf_{e\to 1} \delta(\beta, e) = \liminf_{e\to 1} \tilde{\delta}(\beta, e). \tag{5.12}$$

So it suffices to prove

$$\liminf_{e\to 1} \tilde{\delta}(\beta, e) > 0. \tag{5.13}$$

For $e_2 > e_1$, we have

$$\begin{aligned}&\tilde{\Gamma}(\beta, e_2)(x) - \tilde{\Gamma}(\beta, e_1)(x)\\ &\quad = \int_{\pi/2}^{\pi} + \int_{-\pi}^{-\pi/2} \left(\frac{(e_2 - e_1)(-\cos t)}{(1 + e_2 \cos t)(1 + e_1 \cos t)}((3I_2\right.\\ &\qquad \left. + \sqrt{9 - \beta}S(t))x(t) \cdot x(t))\right) \mathrm{d}t.\end{aligned}$$

Note that $3I_2 + \sqrt{9 - \beta}S(t)$ is positive definite whenever $\beta \in (0, 9]$. Thus we obtain

$$\tilde{\Gamma}(\beta, e_2) \geqq \tilde{\Gamma}(\beta, e_1), \qquad \text{if} \quad e_2 > e_1. \tag{5.14}$$

So $\tilde{\delta}(\beta, e)$ is increasing with respect to e. Let

$$\begin{aligned}E(\beta, e) = &\left\{x \in W^{1,2}(\mathbf{R}/(2\pi\mathbf{Z}), \mathbf{R}^2) \;\middle|\; \|x\|_{W^{1,2}} \leqq 1, \quad \tilde{\Gamma}(\beta, e)(x)\right.\\ &\left.\leqq \frac{1}{2}\delta(\beta, 1)\|x\|^2_{L^2}\right\}.\end{aligned} \tag{5.15}$$

Then $E(\beta, e)$ is closed in the Hilbert space $W^{1,2}(\mathbf{R}/2\pi\mathbf{Z}, \mathbf{R}^2)$ and $E(\beta, e_2) \subset E(\beta, e_1)$ if $e_2 > e_1$. Let $E(\beta, 1) = \bigcap_{0<e<1} E(\beta, e)$. We claim

$$E(\beta, 1) = \{0\}. \tag{5.16}$$

In fact, otherwise, there exists some $x \in E(\beta, 1)\backslash\{0\}$. We consider two cases depending on whether $x \in \hat{D}(\beta)$ or not. If $x \in \hat{D}(\beta)$, then $\tilde{\Gamma}(\beta, e)(x)$ converges to $\Gamma(\beta, 1)(x)$ when $e \to 1$ by an argument similar to that of Lemma 5.2. Therefore by the definition of $\delta(\beta, 1)$ we obtain

$$\tilde{\Gamma}(\beta, e)(x) > \frac{1}{2}\delta(\beta, 1)\|x\|^2,$$

when $1 - e$ is small enough. This contradicts the definition (5.15). On the other hand, $x \in E(\beta, 1)\backslash\hat{D}(\beta)$ implies that

$$\int_{-\pi}^{\pi} \left(\frac{3I_2 + \sqrt{9 - \beta}S(t)}{2(1 + \cos t)}x(t) \cdot x(t)\right) \mathrm{d}t$$

is infinite, which contradicts to the definitions of $\Gamma(\beta, e)$ and $\Gamma(\beta, 1)$ as well as Levi's theorem. Thus (5.16) holds.

We then further claim that there exists a constant $e_0 \in (0, 1)$ such that

$$E(\beta, e) = \{0\}, \quad \text{whenever} \quad e > e_0. \tag{5.17}$$

In fact, otherwise, there exists an increasing sequence $e_k \in (0, 1)$ with $k \in \mathbf{N}$ such that $e_k \to 1$ and there exist $x_k \in E(\beta, e_k)$ with $\|x_k\|_{W^{1,2}} = 1$. Since x_k is bounded in $W^{1,2}(\mathbf{R}/(2\pi\mathbf{Z}), \mathbf{R}^2)$, it has a weakly convergent subsequence x_{n_k}, which converges weakly to some x_0. Then we have $x_0 \in E(\beta, 1)$. In fact, by the weakly lower semi-continuity of norms, we obtain

$$\|x_0\|_{W^{1,2}} \leqq \liminf_{k\to\infty} \|x_{n_k}\|_{W^{1,2}}.$$

On the other hand, by the Sobolev compact embedding theorem, x_{n_k} converges to x_0 in the L^2 space. By definition, $x \in E(\beta, e)$ is equivalent to

$$\|x\|_{W^{1,2}} + \int_{-\pi}^{\pi} (f_{\beta,e}(t)x(t) \cdot x(t))\,\mathrm{d}t \leqq \left(\frac{1}{2}\delta(\beta, 1) + 2\right) \|x\|_{L^2}^2. \tag{5.18}$$

Thus $x_0 \in E(\beta, e_{n_k})$ for every $k \in \mathbf{N}$ implies $x_0 \in E(\beta, 1)$. On the other hand, (5.18) implies that the lower bound of $\|x_{n_k}\|_{L^2}^2$ is nonzero. So we have $x_0 \neq 0$, which contradicts to (5.16), and proves (5.17).

Now by (5.8), (5.14) and (5.16), for fixed $\beta \in (0, 1]$ and every $e > e_0$ we obtain $\tilde{\delta}(\beta, e) \geqq \delta(\beta, 1)/2 > 0$, which completes the proof. □

Since $D(\omega, 2\pi) \subset W^{1,2}(\mathbf{R}/2\pi\mathbf{Z}, \mathbf{C}^n)$ for any $\omega \in \mathbf{U}$, then $\hat{\Gamma}(\beta, e) > 0$ implies $\Gamma(\beta, e) > 0$ for any $\omega \in \mathbf{U}$. Thus we have

Corollary 5.4. *For any fixed $\beta \in (0, 9]$, there exists an $e_* \in (0, 1)$ such that for any $\omega \in \mathbf{U}$ there holds $\Gamma(\beta, e) > 0$ for all $e \in [e_*, 1]$.*

Proof of Theorem 1.7. The proof of the limiting case $e \to -1$ is similar and thus is omitted. The above lemmas imply that for any fixed $\beta \in (0, 9]$, we have always $A(\beta, e) > 0$ for any $\omega \in \mathbf{U}$ whenever $1 - |e|$ is small enough. Thus Theorem 1.7 is proved. □

6. The ω Degeneracy Curves of Elliptic Lagrangian Solutions

For any ω boundary condition, that is on domain $\overline{D}(\omega, 2\pi)$ of (2.11), $A(\beta, e)$ is a closed unbounded operator. If we extend e to the complex plane and denote the open unit disc by $D = \{e \in \mathbf{C} \mid |e| < 1\}$, then $A(\beta, e)$ is holomorphic with respect to $e \in D$ ([6], p. 366). It satisfies $A(\beta, \bar{e}) = A(\beta, e)^*$. In fact

$$\begin{aligned} A(\beta, e) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}I_2 - I_2 + \frac{1}{2}\Big(&3I_2 + \sqrt{9-\beta}S(t))(1 - e\cos(t) \\ &+ e^2\cos^2(t) - e^3\cos^3(t) + \cdots\Big), \end{aligned} \tag{6.1}$$

where we have

$$3I_2+\sqrt{9-\beta}S(t)\geqq 0,\qquad \text{for}\quad 0\leqq\beta\leqq 9,$$
$$3I_2+\sqrt{9-\beta}S(t)>0,\qquad \text{for}\quad 0<\beta\leqq 9.$$

Let Ω be a small narrow neighborhood of the interval $(-1, 1)$ in the complex plane. For $e\in\Omega$, if its imaginary part $\Im e$ is small enough, $A(9, e)$ is strictly accretive ([6], p. 281), that is, there exists $\delta>0$ such that the real part

$$\Re(A(9,e)x,x)\geqq\delta\|x\|^2$$

for any x in the domain $\overline{D}(\omega, 2\pi)$ of $A(9, e)$. We get $A(9,e)^{-\frac{1}{2}}$ by the Dunford–Taylor integral ([6], (3.43) in p. 282),

$$A(9,e)^{-\frac{1}{2}}=\frac{1}{\pi}\int_0^{\infty}\mu^{-\frac{1}{2}}(A(9,e)+\mu)^{-1}\,\mathrm{d}\mu, \tag{6.2}$$

which is bounded and holomorphic in e ([6], p. 398).

By the idea of the proof of Proposition 4.6, we have the following important results for all $\omega\in\mathbf{U}\backslash\{1\}$.

Proposition 6.1. (i) *For every* $(\beta, e)\in(0, 9)\times[0, 1)$ *and* $\omega\in\mathbf{U}\backslash\{1\}$*, there exists* $\varepsilon_0=\varepsilon_0(\beta,e)>0$ *sufficiently small such that for all* $\varepsilon\in(0,\varepsilon_0]$ *there holds*

$$i_\omega(\gamma_{\beta-\varepsilon,e})-i_\omega(\gamma_{\beta,e})=\nu_\omega(\gamma_{\beta,e}).$$

(ii) *For every* $e\in[0, 1)$ *and* $\omega\in\mathbf{U}\backslash\{1\}$*, the total multiplicity of* ω *degeneracy of* $\gamma_{\beta,e}(2\pi)$ *for* $\beta\in[0, 9]$ *is always precisely* 2*, that is,*

$$\sum_{\beta\in[0,9]}\nu_\omega(\gamma_{\beta,e}(2\pi))=2,\qquad\forall\,\omega\in\mathbf{U}\backslash\{1\}.$$

Proof. Firstly, for $(\beta, e)\in[0, 9)\times[0, 1)$, the operator $\bar{A}(\beta, e)$ is a bounded perturbation of the operator $-\mathrm{d}^2/\mathrm{d}t^2$. Thus as $-\mathrm{d}^2/\mathrm{d}t^2$ the operator $\bar{A}(\beta, e)$ possesses only point spectrum, finite Morse index, each of its eigenvalues has finite multiplicity, and the only accumulation point of its spectrum is $+\infty$. Consequently its eigenvalues are all isolated.

(i) Fix $\omega\in\mathbf{U}\backslash\{1\}$ and $e\in[0, 1)$. Let $\eta(\beta)$ be a unit norm eigenvector belonging to an eigenvalue $\lambda(\beta)$ of the operator $\bar{A}(\beta, e)$ on $\overline{D}(\omega, 2\pi)$ for β near some $\beta_0\in(0, 9)$. Then as in Proposition 4.6, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\beta}\lambda(\beta)\Big|_{\beta=\beta_0}&=\left\langle\frac{\partial}{\partial\beta}\bar{A}(\beta,e)\eta(\beta),\eta(\beta)\right\rangle\Big|_{\beta=\beta_0}\\&=\frac{1}{2(9-\beta_0)^{3/2}}\langle A(9,e)\eta(\beta_0),\eta(\beta_0)\rangle\\&>0.\end{aligned}\tag{6.3}$$

Therefore eigenvalues of $\bar{A}(\beta, e)$ on $\overline{D}(\omega, 2\pi)$ for $\beta\in[0, 9)$ are strictly increasing in β.

Note that by definition the Morse index $k_- = \phi_\omega(\bar{A}(\beta_0, e))$ is the total multiplicity of the negative eigenvalues of $\bar{A}(\beta_0, e)$, which is finite.

Suppose $k_0 = \dim\ker(\bar{A}(\beta_0, e)) > 0$ holds on the domain $\overline{D}(\omega, 2\pi)$. By (6.3) there is a smallest positive eigenvalue $\lambda_+(\beta_0)$ of the operator $\bar{A}(\beta_0, e)$. Because $\bar{A}(\beta, e)$ depends analytically on β, we can choose $\varepsilon > 0$ to be small enough so that all the negative eigenvalues of the operator $\bar{A}(\beta, e)$ with $\beta \in [\beta_0 - 2\varepsilon, \beta_0 + 2\varepsilon] \subset (0, 9)$ come only from perturbations of negative and zero eigenvalues of $\bar{A}(\beta_0, e)$, and are not perturbations from any eigenvalues of $\bar{A}(\beta_0, e)$ larger than or equal to $\lambda_+(\beta_0)$. Therefore by (6.3) we obtain

$$\phi_\omega(\bar{A}(\beta_0 - \varepsilon, e)) - \phi_\omega(\bar{A}(\beta_0, e)) = \dim\ker(\bar{A}(\beta_0, e)), \tag{6.4}$$

together with Lemma 2.3 and (2.30), which yields (i).

(ii) Note first that by Lemma 2.3, (2.30), (3.8) and (3.9), the operator $\bar{A}(\beta, e)$ on $\overline{D}(\omega, 2\pi)$ for $\beta = 0$ or 9 and $\omega \in \mathbf{U}\backslash\{1\}$ is non-degenerate.

Following our discussions in (i), at every $\beta_0 \in (0, 9)$ such that $\bar{A}(\beta_0, e)$ is degenerate, the ω-index must decrease strictly. But by (3.7), (3.9) and Corollary 4.5, there exist at most two values β_1 and β_2 at each of which the ω-index decreases by 1 if $\beta_1 \neq \beta_2$, or the ω-index decreases by 2 if $\beta_1 = \beta_2$. Therefore there exist at most two βs in $[0, 9]$ at which the operator $\bar{A}(\beta, e)$ degenerates by (i), which we denote by $\beta_1(e)$ and $\beta_2(e) \in (0, 9]$. Thus by Corollary 4.5 again, we can choose $\varepsilon > 0$ small enough according to $\beta_1(e)$ and $\beta_2(e)$ in the above way so that we have $\phi_\omega(\bar{A}(0, e)) = \phi_\omega(\bar{A}(\beta_1(e) - \varepsilon, e))$, $\phi_\omega(\bar{A}(\beta_1(e), e)) = \phi_\omega(\bar{A}(\beta_2(e) - \varepsilon, e))$, $\phi_\omega(\bar{A}(9, e)) = \phi_\omega(\bar{A}(\beta_2(e), e))$, and (6.4) holds for β_0 replaced by $\beta_1(e)$ and $\beta_2(e)$. Then this yields

$$\begin{aligned}
2 &= \phi_\omega(\bar{A}(0, e)) - \phi_\omega(\bar{A}(9, e)) \\
&= \phi_\omega(\bar{A}(\beta_1(e) - \varepsilon, e)) - \phi_\omega(\bar{A}(\beta_1(e), e)) \\
&\quad + \phi_\omega(\bar{A}(\beta_2(e) - \varepsilon, e)) - \phi_\omega(\bar{A}(\beta_2(e), e)) \\
&= \dim\ker(\bar{A}(\beta_1(e), e)) + \dim\ker(\bar{A}(\beta_2(e), e)) \\
&= \nu_\omega(\gamma_{\beta_1(e),e}(2\pi)) + \nu_\omega(\gamma_{\beta_2(e),e}(2\pi)) \\
&= \sum_{\beta\in[0,9]} \nu_\omega(\gamma_{\beta,e}(2\pi)),
\end{aligned}$$

which proves the proposition. □

Now set

$$B(e, \omega) = A(9, e)^{-\frac{1}{2}} \frac{1}{2(1 + e\cos(t))} S(t) A(9, e)^{-\frac{1}{2}}. \tag{6.5}$$

Be aware that $B(e, \omega)$ depends on ω, since so is $A(9, e)$ on its domain $\overline{D}(\omega, 2\pi)$. Now we have

Lemma 6.2. *For any ω boundary condition and $e \in \Omega$, $A(\beta, e)$ is ω degenerate if and only if $\frac{-1}{\sqrt{9-\beta}}$ is an eigenvalue of $B(e, \omega)$.*

Proof. Suppose for $e \in \Omega$, $A(\beta, e)x = 0$ holds for some $x \in \overline{D}(\omega, 2\pi)$. Let $y = A(9, e)^{\frac{1}{2}}x$. Then by (4.11) we obtain

$$\begin{aligned} A(9, e)^{\frac{1}{2}} &\left(\frac{1}{\sqrt{9-\beta}} + B(e, \omega) \right) y(t) \\ &= \left(\frac{1}{\sqrt{9-\beta}} A(9, e) + \frac{1}{2(1 + e\cos t)} S(t) \right) x(t) \\ &= \frac{1}{\sqrt{9-\beta}} A(\beta, e)x \\ &= 0. \end{aligned} \tag{6.6}$$

Conversely, if $\left(\frac{1}{\sqrt{9-\beta}} + B(e, \omega) \right) y = 0$, then $x = A(9, e)^{-\frac{1}{2}} y$ is an eigenfunction of $A(\beta, e)$ belonging to the eigenvalue 0 by our computations in (6.6). □

Theorem 6.3. *For any $\omega \in \mathbf{U}$, there exist two analytic ω degeneracy curves $(\beta_i(e, \omega), e)$ in $e \in (-1, 1)$ with $i = 1$ and 2. In particular, each $\beta_i(e, \omega)$ is a real analytic function in $e \in (-1, 1)$, and $0 < \beta_i(e, \omega) < 9$ and $\gamma_{\beta_i(e,\omega),e}(2\pi)$ is ω degenerate for $\omega \in \mathbf{U}\backslash\{1\}$ and $i = 1$ or 2.*

Proof. For $\omega = 1$, we have $\beta_i(e, 1) \equiv 0$ for $i = 1$ and 2, by Theorem 1.1 and (2.31), which is obviously analytic.

For $\omega \in \mathbf{U}\backslash\{1\}$, from (3.7) we have $\phi_\omega(A(0, e)) = \phi_\omega(\bar{A}(0, e)) = 2$. On the other hand, $\phi_\omega(A(9, e)) = 0$ by (3.9). Recall that $\bar{A}(\beta, e) = \frac{A(9,e)}{\sqrt{9-\beta}}$, and it is strictly increasing with respect to β by Lemma 4.4. This shows that, for fixed $e \in (-1, 1)$, there are exactly two values $\beta = \beta_1(e, \omega)$ and $\beta_2(e, \omega)$ at which (6.6) is satisfied, and then $\bar{A}(\beta, e)$ at these two β values is ω degenerate. Note that these two β values are possibly equal to each other at some e (compare with the figure in [12]), which is not needed in this proof.

Since $\beta = 9$ is ω-non-degenerate for any $\omega \in \mathbf{U}$, we must have $\beta_i(e, \omega) \neq 9$ for $i = 1$ and 2. By Lemma 6.2, $\frac{-1}{\sqrt{9-\beta_i(e,\omega)}}$ is an eigenvalue of $B(e, \omega)$. Note that $B(e, \omega)$ is a compact operator and self-adjoint when e is real. Moreover it depends analytically on e. By [6] (Theorem 3.9 in p. 392), we know that $\frac{-1}{\sqrt{9-\beta_i(e,\omega)}}$ with $i = 1$ or 2 is real analytic in e. This in turn implies that both $\beta_1(e, \omega)$ and $\beta_2(e, \omega)$ are real analytic functions of e. □

Now we can give

Proof of Theorem 1.5. By Theorem 6.3, $\beta_i(e, \omega)$ is real analytic on $e \in [0, 1)$ for $i = 1$ or 2. That $\beta_i(e, \omega) \to 0$ as $e \to 1$ for $i = 1, 2$ follows by the arguments in the proof of (ii) of Theorem 1.2 below in Section 8. And Corollary 4.5 tells us that $i_\omega(\gamma_{\beta,e})$ is decreasing with respect to $\beta \in [0, 9]$. □

Recall that the ω boundary condition is $x(t) = \omega x(t+2\pi), \dot{x}(t) = \omega\dot{x}(t+2\pi)$. Let $\psi(x)(t) = x(t+\pi)$, and obviously, ψ preserves the ω boundary condition. Also it is a unitary operator and $\psi^* = \psi^{-1}$ is given by $\psi^*(x)(t) = x(t-\pi)$. One can show that

$$\psi^* A(\beta, e)\psi = A(\beta, -e). \tag{6.7}$$

In fact,

$$\begin{aligned}
&\psi^* A(\beta, e)\psi x(t) \\
&= \psi^* \left(-\frac{d^2}{dt^2} I_2 - I_2 + \frac{1}{2(1+e\cos t)}(3I_2 + \sqrt{9-\beta}S(t))\right) x(t+\pi) \\
&= \psi^* \left(-\frac{d^2}{dt^2} I_2 - I_2 + \frac{1}{2(1-e\cos(t+\pi))}(3I_2 + \sqrt{9-\beta}S(t+\pi))\right) x(t+\pi) \\
&= A(\beta, -e)x(t). \qquad (6.8)
\end{aligned}$$

By this property, we know that the ω degeneracy curve must be symmetric with respect to $e = 0$.

When $e = 0$, the eigenvalues of $\gamma_{\beta,0}(2\pi)$ have been studied in Section 3.3. $A(\beta, 0)$, especially, has no multiple eigenvalues for $\omega \in \mathbf{U}\backslash\{\pm 1\}$ and $0 < \beta < 9$. So we have

Theorem 6.4. *For any fixed* $\omega \in \mathbf{U}\backslash\{\pm 1\}$ *and* $i = 1$ *or* 2*, the function* $\beta_i(e, \omega)$ *is real analytic and even on the interval* $(-1, 1)$.

It then follows that $\frac{\partial}{\partial e}\beta_i(e, \omega)|_{e=0} = 0$ when $\omega \in \mathbf{U}\backslash\{\pm 1\}$. But it is not the case when $\omega = -1$, to which we now turn.

7. The -1 Degeneracy Curves of Elliptic Lagrangian Solutions

7.1. The Two $\omega = -1$ Degeneracy Curves

For the $\omega = -1$ boundary condition, denote by g the following operator

$$g(z)(t) = Nz(2\pi - t), \qquad (7.1)$$

where $N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Obviously, $g^2 = 1$ and g is unitary on $L^2([0, 2\pi], \mathbf{R}^2)$. Recall that $E = \overline{D}(-1, 2\pi)$ is given by (2.11). One can check directly that

$$A(\beta, e)g = gA(\beta, e). \qquad (7.2)$$

Let $E_+ = \ker(g + I)$ and $E_- = \ker(g - I)$. Following the studies in Section 2.2 and especially the proof of Theorem 1.1 in [4], the subspaces E_+ and E_- are $A(\beta, e)$-orthogonal, and $E = E_+ \oplus E_-$. Note that element $z = (x, y)^T$ in E_- satisfies

$$x(2\pi - t) = x(t), \quad -y(2\pi - t) = y(t), \qquad \forall\, t \in [0, 2\pi].$$

Thus for all $z = (x, y)^T \in E_-$ we have

$$x(\pi + t) = x(\pi - t), \quad -y(\pi + t) = y(\pi - t), \qquad \forall\, t \in [0, 2\pi].$$

By the definition of $z \in \overline{D}(-1, 2\pi)$, we have $z(2\pi) = -z(0)$. Thus we have $x(0) = x(2\pi) = -x(0)$ which implies $x(0) = 0$ and $y(\pi) = 0$. Similarly for all $z = (x, y)^T \in E_+$ we have

$$x(\pi + t) = -x(\pi - t), \quad y(\pi + t) = y(\pi - t), \qquad \forall\, t \in [0, 2\pi],$$

and $x(\pi) = 0$ and $y(0) = 0$.

Therefore by the above discussions, the subspaces E_- and E_+ are isomorphic to the following subspaces E_1 and E_2 respectively:

$$E_1 = \{z = (x, y)^T \in W^{2,2}([0, \pi], \mathbf{R}^2) \mid x(0) = 0,\ y(\pi) = 0\}, \tag{7.3}$$

$$E_2 = \{z = (x, y)^T \in W^{2,2}([0, \pi], \mathbf{R}^2) \mid x(\pi) = 0,\ y(0) = 0\}. \tag{7.4}$$

For $(\beta, e) \in [0, 9] \times [0, 1)$, restricting $A(\beta, e)$ to E_1 and E_2 respectively, we then obtain

$$\phi_{-1}(A(\beta, e)) = \phi(A(\beta, e)|_{E_1}) + \phi(A(\beta, e)|_{E_2}), \tag{7.5}$$

$$\nu_{-1}(A(\beta, e)) = \nu(A(\beta, e)|_{E_1}) + \nu(A(\beta, e)|_{E_2}), \tag{7.6}$$

where the left hand sides are the Morse index and nullity of the operator $A(\beta, e)$ on the space $\overline{D}(-1, 2\pi)$, that is, the -1 index and nullity of $A(\beta, e)$; on the right hand sides of (7.5)–(7.6), we denote by $\phi(A(\beta, e)|_{E_i})$ and $\nu(A(\beta, e)|_{E_i})$ the usual Morse index and nullity of the operator $A(\beta, e)|_{E_i}$ on the space E_i.

By (4.10), we have $\phi_{-1}(A(9, e)) = 0$ and $\nu_{-1}(A(9, e)) = 0$. Because all the terms in both sides of (7.5) and (7.6) are the Morse indices and nullities which are nonnegative integers, we have

$$\phi(A(9, e)|_{E_1}) = \phi(A(9, e)|_{E_2}) = 0, \qquad \nu(A(9, e)|_{E_1}) = \nu(A(9, e)|_{E_2}) = 0. \tag{7.7}$$

This shows that $A(9, e)|_{E_i}$ with $i = 1$ or 2 is positive definite.

Since the operator $S(t)$ commutes also with the operator g, similarly we have

$$S(t) = S(t)|_{E_1} \oplus S(t)|_{E_2}. \tag{7.8}$$

So, for $i = 1, 2$, we obtain

$$\begin{aligned} A(\beta, e)|_{E_i} &= A(9, e)|_{E_i} + \left(\frac{\sqrt{9-\beta}}{2(1 + e\cos t)} S(t)\right)|_{E_i} \\ &= \sqrt{9-\beta}\left(\frac{A(9, e)|_{E_i}}{\sqrt{9-\beta}} + \frac{S(t)|_{E_i}}{2(1 + e\cos t)}\right). \end{aligned} \tag{7.9}$$

Now we want to compute the Morse index of $A(0, e)|_{E_i}$ for $i = 1, 2$. By (3.7) and (7.5), we have

$$\phi(A(0, e)|_{E_1}) + \phi(A(0, e)|_{E_2}) = 2, \quad \forall\, e \in (-1, 1). \tag{7.10}$$

So the possible value of $\phi(A(0, e)|_{E_i})$ can only be 0, 1 and 2. By (3.8) and (7.6), we obtain

$$\nu(A(0, e)|_{E_1}) = \nu(A(0, e)|_{E_2}) = 0, \quad \forall\, e \in (-1, 1). \tag{7.11}$$

By the property of Morse index, we have

$$\phi(A(0,e)|_{E_i}) = \phi(A(0,0)|_{E_i}), \quad \forall\, e \in (-1,1),\ i = 1,2. \tag{7.12}$$

From (3.43) and (7.6), we obtain

$$\nu(A(3/4,0)|_{E_1}) + \nu(A(3/4,0)|_{E_2}) = \nu_{-1}(A(3/4,0)) = 2. \tag{7.13}$$

From the fact that $\xi E_1 = E_2$ and $\xi E_2 = E_1$ with ξ defined as in (6.7), we have

$$\nu(A(3/4,0)|_{E_1}) = \nu(A(3/4,0)|_{E_2}).$$

Then we obtain

$$\nu(A(3/4,0)|_{E_i}) = 1, \quad \text{for } \ i = 1,2. \tag{7.14}$$

By (7.9), for any fixed $e \in (-1,1)$ and $i = 1,2$, $\frac{A(\beta,e)|_{E_i}}{\sqrt{9-\beta}}$ is increasing with respect to β as proved before. It has the same Morse index and nullity as those of $A(\beta,0)|_{E_i}$. So we get

$$\phi(A(\beta,0)|_{E_i}) = \begin{cases} 1, & \text{if } \ 0 \leqq \beta < 3/4, \\ 0, & \text{if } \ \beta \geqq 3/4, \end{cases} \tag{7.15}$$

This shows that for $-1 < e < 1$ and $i = 1,2$, by (7.12) we obtain

$$\phi(A(0,e)|_{E_i}) = 1. \tag{7.16}$$

By the same idea as in the proof of Theorem 6.3, we get

Proposition 7.1. *The $\omega = -1$ degeneracy curve $(\beta_i(e,-1),e)$ is precisely the degeneracy curve of $A(\beta,e)|_{E_i}$ for $i = 1$ or 2.*

From the results of [12,15,16], we know that the curves $(\beta_1(e,-1),e)$ and $(\beta_2(e,-1),e)$ intersect transversely at the point $(3/4,0)$. By symmetries of the ω-index gap curves, we have

Theorem 7.2. $\beta_1(e,-1) = \beta_2(-e,-1)$ *holds for all* $e \in (-1,1)$.

Remark 7.3. We can also compute $\phi(A(0,0)|_{E_1})$ via the relation between Maslov-type index and Morse index. Let $V_1 = \{(0,x,y,0) \mid x,y \in \mathbf{R}\}$ and $V_2 = \{(x,0,0,y) \mid x,y \in \mathbf{R}\}$. Then both of them are Lagrangian subspaces of the phase space $\mathbf{R}^4$ with standard symplectic structure. From Theorem 1.2 of [4], we have

$$\phi(A(0,0)|_{E_1}) = \mu(V_2, \gamma_{0,0}(t)V_1), \tag{7.17}$$

where the right hand side is the Maslov index for paths of Lagrangian subspaces. Note that for $(\beta,e) = (0,0)$, by (2.19), $B(t) \equiv B$ is a constant matrix. Then $\gamma_{0,0}(t) = \exp(JBt)$ and its Maslov-type index can be computed explicitly as we did in Section 3.3.

7.2. -1 *Degeneracy Curve Bifurcations from* $(3/4, 0)$ *as* e *Leaves from* 0

By (3.22), -1 is a double eigenvalue of the matrix $\gamma_{3/4,0}(2\pi)$. As studied by ROBERTS (cf. p. 212 in [16]) and MEYER–SCHMIDT (cf. Section 3 of [15]), there are two period doubling curves bifurcating out from $(3/4, 0)$ when $e > 0$ is sufficiently small. In [15], the tangent directions of these two curves are computed. Note that these two curves are precisely the -1 index gap curves found by our Theorem 1.2. For reader's conveniences, here we give a simple proof on these two tangent directions based on our above studies.

Proposition 7.4. *The tangent directions of the two curves* Γ_s *and* Γ_m *bifurcating from* $(3/4, 0)$ *when* $e > 0$ *is small are given by*

$$\beta_s'(e)|_{e=0} = -\frac{\sqrt{33}}{4}, \qquad \beta_m'(e)|_{e=0} = \frac{\sqrt{33}}{4}.$$

Proof. To compute the slope of -1 degeneracy curve bifurcating out from $(\beta, e) = (3/4, 0)$, let $(\beta(e), e)$ be one of the curve (say, the E_1 degeneracy curve) with $e \in (-\varepsilon, \varepsilon)$ for some small $\varepsilon > 0$, and $x_e \in E_1$ be the corresponding eigenvector, that is

$$A(\beta(e), e)x_e = 0. \tag{7.18}$$

Here the space E_1 is defined in (7.3) above. Thus there holds

$$\langle A(\beta(e), e)x_e, x_e\rangle = 0. \tag{7.19}$$

Then by direct computations, $\ker(A(3/4, 0)) \cap E_1$ is generated by $x_0 = R(t)z(t)$ with

$$z(t) = \left(\frac{7-\sqrt{33}}{4}\sin(t/2), \cos(t/2)\right)^T. \tag{7.20}$$

Differentiating both sides of (7.19) with respect to e yields

$$\beta'(e)\left\langle\frac{\partial}{\partial\beta}A(\beta(e), e)x_e, x_e\right\rangle+\left(\left\langle\frac{\partial}{\partial e}A(\beta(e), e\right)x_e, x_e\right\rangle+2\langle A(\beta(e), e)x_e, x_e'\rangle = 0,$$

where $\beta'(e)$ and x_e' denote the derivatives with respect to e. Then evaluating both sides at $e = 0$ yields

$$\beta'(0)\left\langle\frac{\partial}{\partial\beta}A(3/4, 0)x_0, x_0\right\rangle + \left\langle\frac{\partial}{\partial e}A(3/4, 0)x_0, x_0\right\rangle = 0. \tag{7.21}$$

Then by the definition (2.29) of $A(\beta, e)$ we have

$$\frac{\partial}{\partial\beta}A(\beta, e)\Big|_{(\beta,e)=(3/4,0)} = R(t)\frac{\partial}{\partial\beta}K_{\beta,e}(t)\Big|_{(\beta,e)=(3/4,0)} R(t)^T, \tag{7.22}$$

$$\frac{\partial}{\partial e}A(\beta, e)\Big|_{(\beta,e)=(3/4,0)} = R(t)\frac{\partial}{\partial e}K_{\beta,e}(t)\Big|_{(\beta,e)=(3/4,0)} R(t)^T, \tag{7.23}$$

where $R(t)$ is given in Section 2.1. By direct computations from the definition of $K_{\beta,e}(t)$ in (2.20), we obtain

$$\frac{\partial}{\partial\beta}K_{\beta,e}(t)\Big|_{(\beta,e)=(3/4,0)} = \frac{1}{2\sqrt{33}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.24}$$

$$\frac{\partial}{\partial e}K_{\beta,e}(t)\Big|_{(\beta,e)=(3/4,0)} = \frac{-\cos t}{4}\begin{pmatrix} 6+\sqrt{33} & 0 \\ 0 & 6-\sqrt{33} \end{pmatrix}. \tag{7.25}$$

Therefore from (7.20) and (7.22)–(7.25) we have

$$\begin{aligned}\left\langle \frac{\partial}{\partial\beta}A(3/4,0)x_0, x_0\right\rangle &= \left\langle \frac{\partial}{\partial\beta}K_{3/4,0}z, z\right\rangle \\ &= \int_0^\pi \left[\frac{1}{2\sqrt{33}}\cos^2(t/2) - \frac{1}{2\sqrt{33}}\left(\frac{7-\sqrt{33}}{4}\right)^2 \sin^2(t/2)\right]\mathrm{d}t \\ &= \frac{\pi}{4\sqrt{33}}\left(1-\left(\frac{7-\sqrt{33}}{4}\right)^2\right), \end{aligned}\tag{7.26}$$

and

$$\begin{aligned}&\left\langle \frac{\partial}{\partial e}A(3/4,0)x_0, x_0\right\rangle - \left\langle \frac{\partial}{\partial e}K_{3/4,0}z, z\right\rangle \\ &= \int_0^\pi \left[\frac{1}{4}(6+\sqrt{33})\left(\frac{7-\sqrt{33}}{4}\right)^2 \cos(t)\sin^2(t/2) \right.\\ &\qquad \left. +\frac{1}{4}(6-\sqrt{33})\cos(t)\cos^2(t/2)\right]\mathrm{d}t \\ &= \frac{\pi}{16}\left(6-\sqrt{33}-(6+\sqrt{33})\left(\frac{7-\sqrt{33}}{4}\right)^2\right). \end{aligned}\tag{7.27}$$

Therefore by (7.21) and (7.26)–(7.27) we obtain

$$\beta'(0) = \frac{\sqrt{33}}{4}. \tag{7.28}$$

By the above Theorem 7.2, the two -1 degenerate curves are symmetric with respect to the $e = 0$ axis. Therefore the claim of the Proposition 7.4 follows from (7.28). □

8. Study on the Non-Hyperbolic Regions

Now we give proofs of the first halves of our main Theorems 1.6 and 1.2.

Proof of (i) of Theorem 1.6. It follows from Theorems 6.4 and 7.2. □

Proof of the first half of Theorem 1.2. Here we give proofs for items (i)–(iii) and (ix)–(x) of this theorem.

(i) By Theorem 1.5, for $i = 1$ and 2 we have got the existence of two curves defined by $(\beta_i(e), e)$ and $\lim_{e\to 1}\beta_i(e) = 0$ such that $\gamma_{\beta,e}(2\pi)$ is degenerate with respect to $\omega = -1$ only on them. Note that here these two curves may coincide somewhere. In particular we define

$$\begin{aligned} 0 < \beta_s(e) \equiv \min\{\beta_1(e), \beta_2(e)\} &\leqq \beta_m(e) \\ \equiv \max\{\beta_1(e), \beta_2(e)\} < 9, &\quad \text{for} \quad e \in [0, 1). \end{aligned} \tag{8.1}$$

Thus (i) is proved.

(ii) By the studies in Section 3.3, the only -1 degenerate point in the (β, e) segment $[0, 9] \times \{0\}$ is $(\beta, e) = (3/4, 0)$, which is a 2-fold -1 degenerate point, and there hold

$$i_{-1}(\gamma_{\beta,0}) = 2, \qquad \nu_{-1}(\gamma_{\beta,0}) = 0, \quad \text{for} \quad \beta \in [0, 3/4), \tag{8.2}$$

$$i_{-1}(\gamma_{3/4,0}) = 0, \qquad \nu_{-1}(\gamma_{3/4,0}) = 2, \tag{8.3}$$

$$i_{-1}(\gamma_{\beta,0}) = 0, \qquad \nu_{-1}(\gamma_{\beta,0}) = 0, \quad \text{for} \quad \beta \in (3/4, 9]. \tag{8.4}$$

Therefore

$$\beta_i(0) = 3/4, \quad \text{for} \quad i = 1 \ \text{and} \ 2. \tag{8.5}$$

By Meyer and Schmidt in [15] or our Proposition 7.4, the two -1 degeneracy curves bifurcating out from $(\beta, e) = (3/4, 0)$ when $e > 0$ is sufficiently small must coincide with our curves Γ_s and Γ_m respectively. Because these two curves bifurcate out from $(3/4, 0)$ in different angles with tangents $-\sqrt{33}/4$ and $\sqrt{33}/4$ respectively when $e > 0$ is small, they are different from each other near $(3/4, 0)$. By our Theorem 6.4, these two curves Γ_s and Γ_m are real analytic with respect to e. Therefore they are different curves and their intersection points including the point $(3/4, 0)$ can only be isolated.

By Theorems 6.3 and 1.5, these two curves Γ_s and Γ_m must tend to the segment $[0, 9] \times \{1\}$ from $(3/4, 0)$ as e increases from 0 and tends to 1. By the proof of our Theorem 1.8 in Section 8 below, for each $e \in [0, 1)$ the function $\beta_k(e)$ defined by (1.5) satisfies $0 < \beta_s(e) \leqq \beta_m(e) \leqq \beta_k(e) < 9$, and $\lim_{e\to 1}\beta_k(e) = 0$. Therefore the two curves Γ_s and Γ_m must tend to $(0, 1)$ as $e \to 1$.

(iii) By our studies on the segments $\{0\} \times [0, 1)$ and $\{9\} \times [0, 1)$ in Section 3 and the definitions of $\beta_s(e)$ and $\beta_m(e)$, the index $i_{-1}(\gamma_{\beta,e})$ must take the claimed values 2, 1, and 0 in (1.7) respectively when $\beta \in [0, 9]\backslash\{\beta_s(e), \beta_m(e)\}$ for each $e \in [0, 1)$. Note that when $\beta = \beta_s(e)$ or $\beta_m(e)$, the -1 index claim (1.7) follows from (i) of Proposition 6.1. The last claim in (iii) follows from Proposition 6.1.

(ix)–(x) By the Bott-type formula (Theorem 9.2.1 in p. 199 of [11]), we obtain

$$i_1(\gamma^k_{\beta,e}) = \sum_{\omega^k=1} i_\omega(\gamma_{\beta,e}), \qquad \forall\, k \in \mathbf{N},$$

and

$$\nu_1(\gamma^2_{\beta,e}) = \nu_1(\gamma_{\beta,e}) + \nu_{-1}(\gamma_{\beta,e}) = \nu_{-1}(\gamma_{\beta,e}),$$

where the last equality follows from Theorem 1.1.

Therefore by (1.7), (3.7) and (3.8), for $e \in [0,1)$ we obtain

$$\phi^2 = \begin{cases} 4, & \text{if } 0 \leqq \beta < \beta_s(e), \\ 3, & \text{if } \beta_s(e) \leqq \beta < \beta_m(e), \\ 2, & \text{if } \beta_m(e) \leqq \beta \leqq 9. \end{cases} \tag{8.6}$$

Thus for $(\beta, e) \in (0,9] \times [0,1)$, the matrix $\gamma_{\beta,e}(4\pi) = \gamma_{\beta,e}^2(2\pi)$ is non-degenerate with respect to the eigenvalue 1, whenever $(\beta, e) \notin \Gamma_s \cup \Gamma_m$. Therefore by (8.6) we can apply Theorem 2.5 (that is, Theorem 1.2 of [5]) to get (ix) and (x). The rest parts of Theorem 1.2 will be proved in the next section. □

9. Study on the Hyperbolic Region

In this section we study the hyperbolic region of $\gamma_{\beta,e}(2\pi)$ in the rectangle $[0,9] \times [0,1)$. By the first halves of Theorems 1.2 and 1.6 proved in the Section 8, the function $\beta_k(e)$ defined by (1.5) satisfies

$$\beta_m(e) \leqq \beta_k(e), \qquad \forall\, e \in [0,1). \tag{9.1}$$

We have the following further results.

Lemma 9.1. (i) *If* $0 \leqq \beta_1 < \beta_2 \leqq 9$ *and* $\gamma_{\beta_1,e}(2\pi)$ *is hyperbolic, so does* $\gamma_{\beta_2,e}(2\pi)$. *Consequently, the hyperbolic region of* $\gamma_{\beta,e}(2\pi)$ *in* $[0,9] \times [0,1)$ *is connected.*

(ii) *For any fixed* $e \in [0,1)$, *every matrix* $\gamma_{\beta,e}(2\pi)$ *is hyperbolic if* $\beta_k(e) < \beta \leqq 9$ *for* $\beta_k(e)$ *defined by* (1.5). *Thus* (1.8) *holds and* Γ_k *is the boundary set of this hyperbolic region.*

(iii) *We have*

$$\sum_{\beta \in [0, \beta_k(e)]} \nu_\omega(\gamma_{\beta,e}(2\pi)) = 2, \qquad \forall\, \omega \in \mathbf{U} \backslash \{1\}. \tag{9.2}$$

Proof. (i) By Lemma 4.4, for any fixed $\omega \in \mathbf{U}$ and $e \in [0,1)$, the operator $\bar{A}(\beta, e) = \frac{1}{\sqrt{9-\beta}} A(\beta, e)$ is self-adjoint on $\bar{D}(\omega, 2\pi)$ and increasing with respect to β in the sense that

$$\bar{A}(\beta_1, e) < \bar{A}(\beta_2, e), \qquad \text{if} \quad \beta_1 < \beta_2. \tag{9.3}$$

Suppose $\gamma_{\beta_1,e}(2\pi)$ is hyperbolic. This implies that $A(\beta_1, e)$ is non-degenerate on $\bar{D}(\omega, 2\pi)$ for every $\omega \in \mathbf{U}$. By (ix)–(x) of Theorem 1.2, it also implies $\beta_m(e) < \beta_1$. Thus by (2.30), (3.7), Corollary 4.5, and Theorem 1.1, the ω-index $\phi_\omega(\bar{A}(\beta_1, e)) = 0$ for all $\omega \in \mathbf{U}$. Then $\bar{A}(\beta_1, e)$ is positive definite on $\bar{D}(\omega, 2\pi)$ for every $\omega \in \mathbf{U}$. Therefore by (9.3) the operator $\bar{A}(\beta_2, e)$ is positive definite too, and then is non-degenerate on $\bar{D}(\omega, 2\pi)$ for all $\omega \in \mathbf{U}$. Therefore $\gamma_{\beta_2,e}(2\pi)$ must be hyperbolic and so does $\gamma_{\beta,e}(2\pi)$ for all $\beta \in [\beta_1, 9)$.

Recall that along the segment $\{9\} \times [0,1)$ the matrix $\gamma_{9,e}(2\pi)$ is hyperbolic by our Proposition 1.4. Therefore the hyperbolic region of $\gamma_{\beta,e}(2\pi)$ is connected in $[0,9] \times [0,1)$.

(ii) By the definition of $\beta_k(e)$, there exists a sequence $\{\beta_i\}_{i\in\mathbf{N}}$ satisfying $\beta_i > \beta_k(e)$, $\beta_i \to \beta_k(e)$, and $\gamma_{\beta_i,e}(2\pi)$ is hyperbolic. Therefore $\gamma_{\beta,e}(2\pi)$ is hyperbolic for every $\beta \in (\beta_k(e), 9]$ by (i). Then (1.8) holds and Γ_k is the envelope curve of this hyperbolic region.

Now (iii) follows from (ii) and Proposition 6.1, and the proof is complete. □

Corollary 9.2. *For every* $e \in [0, 1)$*, we have*

$$\sum_{\beta\in(0,\beta_m(e)]} \nu_{-1}(\gamma_{\beta,e}(2\pi)) = 2 \quad \text{and} \quad \sum_{\beta\in(\beta_m(e),9]} \nu_{-1}(\gamma_{\beta,e}(2\pi)) = 0. \tag{9.4}$$

Proof. Fix an $e \in [0, 1)$. If $\beta_s(e) < \beta_m(e)$, then we obtain

$$\sum_{\beta\in(0,\beta_m(e)]} \nu_{-1}(\gamma_{\beta,e}(2\pi)) \geqq \nu_{-1}(\gamma_{\beta_s(e),e}(2\pi)) + \nu_{-1}(\gamma_{\beta_m(e),e}(2\pi)) \geqq 2.$$

Thus (9.4) follows from (ii) of Proposition 6.1.

If $\beta_s(e) = \beta_m(e)$, then by (i) of Proposition 6.1 we obtain $\nu_{-1}(\gamma_{\beta_m(e),e}(2\pi)) = 2$. Therefore we have

$$\sum_{\beta\in(0,\beta_m(e)]} \nu_{-1}(\gamma_{\beta,e}(2\pi)) \geqq \nu_{-1}(\gamma_{\beta_m(e),e}(2\pi)) = 2.$$

Thus (9.4) follows also from (ii) of Proposition 6.1. □

Now we can give the

Proof of the second half of Theorem 1.2. Here we give the proofs for the items (iv)–(viii) and (xi) of this theorem.

Note that Claims (iv) and (v) of the theorem follow from (9.1) and Lemma 9.1.

(vi) In fact, if the function $\beta_k(e)$ is not continuous in $e \in [0, 1)$, then there exist some $\hat{e} \in [0, 1)$, a sequence $\{e_i \mid i \in \mathbf{N}\} \subset [0, 1)\backslash\{\hat{e}\}$ and $\beta_0 \in [0, 9]$ such that

$$\beta_k(e_i) \to \beta_0 \neq \beta_k(\hat{e}) \quad \text{and} \quad e_i \to \hat{e} \qquad \text{as } i \to +\infty. \tag{9.5}$$

We continue in two cases according to the sign of the difference $\beta_0 - \beta_k(\hat{e})$.

By the definition of $\beta_k(e_i)$ we have $\sigma(\gamma_{\beta_k(e_i),e_i}(2\pi)) \cap \mathbf{U} \neq \emptyset$ for every e_i. By the continuity of eigenvalues of $\gamma_{\beta_k(e_i),e_i}(2\pi)$ in i and (9.5), we obtain

$$\sigma(\gamma_{\beta_0,\hat{e}}(2\pi)) \cap \mathbf{U} \neq \emptyset.$$

Then by Lemma 9.1, this would yield a contradiction if $\beta_0 > \beta_k(\hat{e})$.

Now suppose $\beta_0 < \beta_k(\hat{e})$. By Lemma 9.1 for all $i \geqq 1$ we have

$$\sigma(\gamma_{\beta,e_i}(2\pi)) \cap \mathbf{U} = \emptyset, \qquad \forall\, \beta \in (\beta_k(e_i), 9]. \tag{9.6}$$

Then by the continuity of $\beta_m(e)$ in e, (9.6) and the definition of β_0, we obtain

$$\beta_m(\hat{e}) \leqq \beta_0 < \beta_k(\hat{e}).$$

Let $\omega_0 \in \sigma(\gamma_{\beta_k(\hat{e}),\hat{e}}(2\pi)) \cap \mathbf{U}$, which exists by the definition of $\beta_k(\hat{e})$.

Let $L = \{(\beta, \hat{e}) \mid \beta \in (\beta_k(\hat{e}), 9]\}$, $V = \{(9, e) \mid e \in [0, 1)\}$, and $L_i = \{(\beta, e_i) \mid \beta \in (\beta_k(e_i), 9]\}$ for all $i \geqq 1$. Note that by (3.9), (4.10), Corollary 4.5, Proposition 6.1, Lemma 9.1, and the definitions of $\beta_k(e_i)$ and $\beta_k(\hat{e})$, we obtain

$$i_{\omega_0}(\gamma_{\beta,e}) = \nu_{\omega_0}(\gamma_{\beta,e}) = 0, \qquad \forall\, (\beta, e) \in L \cup V \cup \bigcup_{i \geqq 1} L_i. \tag{9.7}$$

In particular we have

$$i_{\omega_0}(\gamma_{\beta_k(\hat{e}),\hat{e}}) = 0 \quad \text{and} \quad \nu_{\omega_0}(\gamma_{\beta_k(\hat{e}),\hat{e}}) \geqq 1.$$

Therefore by Proposition 6.1 and the definition of ω_0, there exists $\hat{\beta} \in (\beta_0, \beta_k(\hat{e}))$ sufficiently close to $\beta_k(\hat{e})$ such that

$$i_{\omega_0}(\gamma_{\hat{\beta},\hat{e}}) = i_{\omega_0}(\gamma_{\beta_k(\hat{e}),\hat{e}}) + \nu_{\omega_0}(\gamma_{\beta_k(\hat{e}),\hat{e}}(2\pi)) \geqq 1. \tag{9.8}$$

This estimate (9.8) in fact holds for all $\beta \in [\hat{\beta}, \beta_k(\hat{e}))$ too. Note that $(\hat{\beta}, \hat{e})$ is an accumulation point of $\cup_{i \geqq 1} L_i$. Consequently for each $i \geqq 1$ there exists $(\beta_i, e_i) \in L_i$ such that $\gamma_{\beta_i,e_i} \in \mathcal{P}_{2\pi}(4)$ is ω_0 non-degenerate, $\beta_i \to \hat{\beta}$ in $\mathbf{R}$, and $\gamma_{\beta_i,e_i} \to \gamma_{\hat{\beta},\hat{e}}$ in $\mathcal{P}_{2\pi}(4)$ as $i \to \infty$. Therefore by (9.7), (9.8), the Definition 5.4.2 of the ω_0-index of ω_0-degenerate path $\gamma_{\hat{\beta},\hat{e}}$ in p. 129 and Theorem 6.1.8 in p. 142 of [11], we obtain the following contradiction

$$1 \leqq i_{\omega_0}(\gamma_{\hat{\beta},\hat{e}}) \leqq i_{\omega_0}(\gamma_{\beta_i,e_i}) = 0,$$

for $i \geqq 1$ large enough. Thus the continuity of $\beta_k(e)$ in $e \in [0, 1)$ is proved.
(vii) To prove the claim $\lim_{e \to 1} \beta_k(e) = 0$, we argue by contradiction, and suppose that there exist $e_i \to 1$ as $i \to +\infty$, $\beta_0 > 0$, such that $\lim_{i\to\infty} \beta_k(e_i) = \beta_0$. Then at least one of the following two cases must occur:

(A) *There exists a subsequence* $\{\hat{e}_i\}$ *of* $\{e_i\}$ *such that* $\beta_k(\hat{e}_{i+1}) \leqq \beta_k(\hat{e}_i)$ *for all* $i \in \mathbf{N}$*;*

(B) *There exists a subsequence* $\{\hat{e}_i\}$ *of* $\{e_i\}$ *such that* $\beta_k(\hat{e}_i) \leqq \beta_k(\hat{e}_{i+1})$ *for all* $i \in \mathbf{N}$.

If Case (A) happens, for this β_0 by Theorem 1.7 there exists $e_0 > 0$ sufficiently close to 1 such that $\gamma_{\beta_0,e}(2\pi)$ is hyperbolic for all $e \in [e_0, 1)$. Then $\gamma_{\beta,e}(2\pi)$ is hyperbolic for all (β, e) in the region $[\beta_0, 9] \times [e_0, 1)$ by Lemma 9.1. But by the monotonicity of Case (A) we obtain

$$\beta_0 \leqq \beta_k(\hat{e}_{i+m}) \leqq \beta_k(\hat{e}_i) \qquad \forall\, m \in \mathbf{N}.$$

Therefore $(\beta_k(\hat{e}_{i+m}), \hat{e}_{i+m})$ will get into this region for sufficiently large $m \geqq 1$, which contracts to the definition of $\beta_k(\hat{e}_{i+m})$ in (1.5).

If Case (B) happens, fix a subindex i, by Theorem 1.7 and the same argument as in Case (A) there exists an $e_0 > 0$ sufficiently close to 1 such that $\gamma_{\beta,e}(2\pi)$ is hyperbolic for all (β, e) in the region $[\beta_k(\hat{e}_i), 9] \times [e_0, 1)$. Then by the monotonicity of Case (B) we obtain

$$\beta_k(\hat{e}_i) \leqq \beta_k(\hat{e}_{i+m}) \leqq \beta_0 \qquad \forall\, m \in \mathbf{N}.$$

Therefore $(\beta_k(\hat{e}_{i+m}), \hat{e}_{i+m})$ will get into this region for sufficiently large $m \geqq 1$, which contracts to the definition of $\beta_k(\hat{e}_{i+m})$ in (1.5). Thus (vii) holds.

(viii) By our study in Section 3.3, we have $(1, 0) \in \Gamma_k \backslash \Gamma_m$. Thus there exists an $\tilde{e} \in (0, 1]$ such that $\beta_k(e) > \beta_m(e)$ for all $e \in [0, \tilde{e})$. Therefore Γ_k is different from Γ_m when $e \in [0, \tilde{e})$.

(xi) Let $e_0 \in [0, 1)$ and $\beta_m(e_0) < \beta_0 \leqq \beta_k(e_0)$. Then $M \equiv \gamma_{\beta_0,e_0}(2\pi)$ is not hyperbolic by Lemma 9.1 and thus at least one pair of its eigenvalues is on $\mathbf{U}$. Note also that no eigenvalues of M can be ± 1 by Theorem 1.1 and Corollary 9.2. Write

$$\sigma(M) = \{\lambda_1(\beta_0), \lambda_1(\beta_0)^{-1}, \lambda_2(\beta_0), \lambda_2(\beta_0)^{-1}\}. \tag{9.9}$$

Thus we can assume $\lambda_1 \equiv \lambda_1(\beta_0) \in \mathbf{U}\backslash\mathbf{R}$ and the other pair of eigenvalues satisfy $\lambda_2 \equiv \lambda_2(\beta_0) \in (\mathbf{U} \cup \mathbf{R})\backslash\{\pm 1, 0\}$.

Claim. $\lambda_2(\beta_0) \in \mathbf{U}\backslash\mathbf{R}$.

In fact, if not, we assume $\lambda_2(\beta_0) \in \mathbf{R}\backslash\{\pm 1, 0\}$.

In this case, M has normal form $R(\theta)\diamond D(\lambda_2) \in \Omega^0(M)$ for some $\theta \in (0, \pi) \cup (\pi, 2\pi)$. Thus by (3.6), Theorem 1.1 and (iii) of our Theorem 1.2, we obtain the following contradiction:

$$\begin{aligned} 0 &= i_{-1}(\gamma_{\beta_0,e_0}) \\ &= i_1(\gamma_{\beta_0,e_0}) + S_M^+(1) - S_M^-(e^{\pm\sqrt{-1}\theta}) + S_M^+(e^{\pm\sqrt{-1}\theta}) - S_M^-(-1) \\ &= 0 + 0 - S_{R(\theta)}^-(e^{\pm\sqrt{-1}\theta}) + S_{R(\theta)}^+(e^{\pm\sqrt{-1}\theta}) - 0 \\ &= \pm 1, \end{aligned}$$

where the last equality follows from Lemma 9.1.6 in p. 192 and ⟨5⟩ of List 9.1.12 in p. 198 of [11]. Therefore the claim is proved.

Now from $\lambda_1(\beta_0)$ and $\lambda_2(\beta_0) \in \mathbf{U}\backslash\mathbf{R}$, the matrix M has basic normal form $R(\theta_1)\diamond R(\theta_2) \in \Omega^0(M)$ for some θ_1 and $\theta_2 \in (0, \pi) \cup (\pi, 2\pi)$. Then by the study in Section 9.1 of [11], we obtain

$$\begin{aligned} 0 &= i_{-1}(\gamma_{\beta_0,e_0}) \\ &= i_1(\gamma_{\beta_0,e_0}) + S_M^+(1) - S_{R(\theta_1)}^-(e^{\pm\sqrt{-1}\theta_1}) + S_{R(\theta_1)}^+(e^{\pm\sqrt{-1}\theta_1}) \\ &\quad - S_{R(\theta_2)}^-(e^{\pm\sqrt{-1}\theta_2}) + S_{R(\theta_2)}^+(e^{\pm\sqrt{-1}\theta_2}) - S_M^-(-1) \\ &= -S_{R(\theta_1)}^-(e^{\pm\sqrt{-1}\theta_1}) + S_{R(\theta_1)}^+(e^{\pm\sqrt{-1}\theta_1}) - S_{R(\theta_2)}^-(e^{\pm\sqrt{-1}\theta_2}) + S_{R(\theta_2)}^+(e^{\pm\sqrt{-1}\theta_2}). \end{aligned} \tag{9.10}$$

By Lemma 9.1.6 in p. 192 and ⟨5⟩ of List 9.1.12 in p. 198 of [11] again, the right hand side of (9.10) would be ± 2, if both θ_1 and θ_2 are located in only one interval of $(0, \pi)$ and $(\pi, 2\pi)$. Thus we must have $\theta_1 \in (0, \pi)$ and $\theta_2 \in (\pi, 2\pi)$. Let $\omega = \exp(\sqrt{-1}\theta_1)$.

If $2\pi - \theta_2 = \theta_1$, we then obtain

$$\sum_{0 \leqq \beta \leqq \beta_0} \nu_\omega(\gamma_{\beta,e_0}) \geqq \sum_{0 \leqq \beta \leqq \beta_m(e_0)} \nu_\omega(\gamma_{\beta,e_0}) + \nu_\omega(\gamma_{\beta_0,e_0}) \geqq 1 + 2.$$

This is in contradiction to Lemma 9.1 and proves $2\pi - \theta_2 \neq \theta_1$.

By the study in Section 9.1 of [11] again, if $2\pi-\theta_2>\theta_1$, for $\omega=\exp(\sqrt{-1}\theta_1)$ we obtain

$$0\leqq i_\omega(\gamma_{\beta_0,e_0})=i_1(\gamma_{\beta_0,e_0})+S_M^+(1)-S_{R(\theta_1)}^-(e^{\sqrt{-1}\theta_1})=-S_{R(\theta_1)}^-(e^{\sqrt{-1}\theta_1})=-1.$$

This contradiction proves that the only possible case is $2\pi-\theta_2<\theta_1$.

The proof of Theorem 1.2 is complete. □

Now we give

The proof of (ii) of Theorem 1.6. By (v) of Theorem 1.2, the curve Γ_k for $e\in[0,1)$ is the boundary curve of the hyperbolic region of $\gamma_{\beta,e}(2\pi)$ in the (β,e) rectangle $[0,9]\times[0,1)$. By the definition (1.5), the curve Γ_k is also the envelope curve of the ω degeneracy curves for all $\omega\in\mathbf{U}\backslash\{1\}$ from the right hand side of the rectangle $[0,9]\times[0,1)$. Then (i) of Theorem 1.6 implies that Γ_k can be continuously extended into $[0,9]\times(-1,0]$ so that it is symmetric with respect to $[0,9]\times\{0\}$. □

The next lemma is useful in the proof of Theorem 1.8.

Lemma 9.3. *If $\gamma_{\beta_0,e}(2\pi)\approx M_2(-1,c)$ holds for some $c\in\mathbf{R}^2$, or it possesses the basic normal form $N_1(-1,a)\diamond N_1(-1,b)$ for some $(\beta_0,e)\in(0,9)\times[0,1)$ and a, $b\in\mathbf{R}$, then $\gamma_{\beta,e}(2\pi)$ is hyperbolic for all $\beta\in(\beta_0,9]$.*

Proof. Note that the basic normal form of the matrix $M_2(-1,c)$ is either $N_1(-1,\hat{a})\diamond N_1(-1,\hat{b})$ or $N_1(-1,\hat{a})\diamond D(\lambda)$ for some $\hat{a}$, $\hat{b}\in\mathbf{R}$ and $0>\lambda\neq-1$. Thus for any $\omega\in\mathbf{U}\backslash\{1\}$, by Corollary 4.5, (2.31), and the study in Section 9.1 of [11], we obtain

$$0\leqq i_\omega(\gamma_{\beta_0,e})=i_1(\gamma_{\beta_0,e})+S_M^+(1)-S_M^-(\omega)=-S_M^-(\omega)\leqq 0,$$

where $M=\gamma_{\beta_0,e}(2\pi)$. This proves $i_\omega(\gamma_{\beta_0,e})=0$ for all $\omega\in\mathbf{U}$. Note that $\phi_\omega(\bar{A}(\beta_0,e))=i_\omega(\gamma_{\beta_0,e})$ and $\nu_\omega(\bar{A}(\beta_0,e))=\nu_\omega(\gamma_{\beta_0,e}(2\pi))$ follow from (2.27), (2.30), (4.12) and (4.13).

Now from $\phi_\omega(\bar{A}(\beta_0,e))=0$ and (ii) of Lemma 4.4, we obtain $\bar{A}(\beta,e)>0$ for all $\beta\in(\beta_0,9]$ on $\overline{D}(\omega,2\pi)$ with $\omega\in\mathbf{U}$. Therefore $\nu_\omega(\gamma_{\beta,e}(2\pi))=\nu_\omega(\bar{A}(\beta,e))=0$ holds for all $\beta\in(\beta_0,9]$ and $\omega\in\mathbf{U}$, and thus the lemma follows. □

Now we can give

Proof of Theorem 1.8. **(i)** Let $e\in[0,1)$ satisfy $\beta_s(e)<\beta_m(e)$. Then Corollary 9.2 implies $\nu_{-1}(\gamma_{\beta_s(e),e})=1$. As the limiting case of cases (ix) and (x) of Theorem 1.2, the matrix $M=\gamma_{\beta_s(e),e}(2\pi)$ must have all eigenvalues in $\mathbf{U}$, and possesses its normal form either $M\approx M_2(-1,c)$ for some $c_2\neq0$, or $M\approx N_1(-1,1)\diamond R(\theta)$ for some $\theta\in(\pi,2\pi)$, where to get the second case we have used the Fig. 2.1.2 in p. 50 of [11] and the fact $N_1(-1,1)\in\mathrm{Sp}(2)^0_{-1,-}$ in that figure.

Note that $M\approx M_2(-1,c)$ can not hold for any $c_2\neq0$ by Lemma 9.3 and the fact $\beta_s(e)<\beta_m(e)$. The following is a direct proof of this fact. In this case, its basic normal form is $N_1(-1,a)\diamond D(\lambda)$ for some $a\in\{-1,1\}$ and $0>\lambda\neq-1$.

Therefore by Theorem 1.1, (iii) of Theorem 1.2, and ⟨3⟩ and ⟨4⟩ of List 9.1.12 in p. 198 of [11], we obtain the following contradiction

$$1 = i_{-1}(\gamma_{\beta_s(e),e}) = i_1(\gamma_{\beta_s(e),e}) + S_M^+(1) - S_{N_1(-1,a)}^-(-1) = -S_{N_1(-1,a)}^-(-1) \leqq 0.$$

Thus $M \approx N_1(-1,1)\diamond R(\theta)$ must hold for some $\theta \in (\pi, 2\pi)$, so M is spectrally stable and linearly unstable.

(ii) Let $e \in [0,1)$ satisfy $\beta_s(e) = \beta_m(e) < \beta_k(e)$. As the limiting case of the cases (ix) and (xi) of Theorem 1.2 and Corollary 9.2, the matrix $M = \gamma_{\beta_s(e),e}(2\pi)$ must have basic normal form either $N_1(-1,a)\diamond N_1(-1,b)$ for some a and $b \in \{-1,1\}$, or $-I_2\diamond R(\theta)$ for some $\theta \in (\pi, 2\pi)$, where we have used the Fig. 2.1.2 in p. 50 of [11]. But the first case is impossible by Lemma 9.3. Therefore $M \approx -I_2\diamond R(\theta)$ holds for some $\theta \in (\pi, 2\pi)$, and it is linear stable and not strongly linear stable.

(iii) Let $e \in [0,1)$ satisfy $\beta_s(e) < \beta_m(e) < \beta_k(e)$. As the limiting case of Cases (x) and (xi) of Theorem 1.2, the matrix $M = \gamma_{\beta_m(e),e}(2\pi)$ must satisfy either $M \approx N_1(-1,-1)\diamond R(\theta)$ for some $\theta \in (\pi, 2\pi)$, or $M \approx M_2(-1,c)$ with $c_2 \neq 0$, where we have used the Fig. 2.1.2 in p. 50 of [11] and the fact $N_1(-1,-1) \in \mathrm{Sp}(2)^0_{-1,+}$ in that figure. Here the second case is also impossible by Lemma 9.3, and the conclusion of (iii) follows.

(iv) Let $e \in [0,1)$ satisfy $\beta_s(e) \leqq \beta_m(e) < \beta_k(e)$. As the limiting case of the cases (v) and (xi) of Theorem 1.2, the matrix $M \equiv \gamma_{\beta_k(e),e}(2\pi)$ must have Krein collision eigenvalues $\sigma(M) = \{\lambda_1, \overline{\lambda}_1, \lambda_2, \overline{\lambda}_2\}$ with $\lambda_1 = \overline{\lambda}_2 = e^{\sqrt{-1}\theta}$ for some $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Here we have used Theorem 1.1 and Corollary 9.2 to exclude the possibility of eigenvalues ± 1. Therefore for this angle θ, the matrix M must have its normal form $N_2(\omega, b)$ for $\omega = e^{\sqrt{-1}\theta}$ and some 2×2 matrix $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, which is of the form (25)–(27) by Theorem 1.6.11 in p. 34 of [11]. Because $(I_2\diamond(-I_2))^{-1}N_2(e^{\sqrt{-1}\theta}, b)(I_2\diamond(-I_2)) = N_2(e^{\sqrt{-1}(2\pi-\theta)}, \hat{b})$ holds for $\hat{b} = \begin{pmatrix} b_1 & -b_2 \\ -b_3 & b_4 \end{pmatrix}$, we can always suppose $\theta \in (0,\pi)$ without changing the fact $M \approx N_2(\omega, b)$.

Note that by (3.7), (3.9), Corollary 4.5 and Proposition 6.1, we have $i_\omega(\gamma_{\beta_k(e),e}) = 0$.

Now if $b_2 - b_3 = 0$, by Lemma 1.9.2 in p. 43 of [11], we get $\nu_\omega(N_2(\omega,b)) = 2$, and then $N_2(\omega, b)$ has basic normal form $R(\theta)\diamond R(2\pi-\theta)$ by the study in case 4 in p. 40 of [11]. Thus we arrive at the following contradiction

$$0 = i_\omega(\gamma_{\beta_k(e),e}) = i_1(\gamma_{\beta_k(e),e}) + S_M^+(1) - S_{R(\theta)}^-(\omega) - S_{R(\theta)}^-(\overline{\omega}) \leqq -1,$$

by Lemma 9.1.6 in p. 192 and ⟨5⟩ of List 9.1.12 in p. 198 of [11].

Therefore $b_2 - b_3 \neq 0$ must hold. Then we obtain

$$0 = i_\omega(\gamma_{\beta_k(e),e}) = i_1(\gamma_{\beta_k(e),e}) + S_M^+(1) - S_{N_2(\omega,b)}^-(\omega) = -S_{N_2(\omega,b)}^-(\omega).$$

By ⟨6⟩ and ⟨7⟩ in List 9.1.12 in p. 199 of [11], we obtain that $N_2(\omega, b)$ must be trivial as in our discussion in Section 2.1. Then by Theorem 1 of [20], the matrix M is spectrally stable and is linearly unstable as claimed.

(v) Let $e \in [0, 1)$ satisfy $\beta_s(e) < \beta_m(e) = \beta_k(e)$. Note first that -1 must be an eigenvalue of $M = \gamma_{\beta_k(e),e}(2\pi)$ with geometric multiplicity 1 by Corollary 9.2. As the limiting case of cases (v) and (x) of Theorem 1.2, the matrix M must satisfy either $M \approx M_2(-1, b)$ with $b_1, b_2 \in \mathbf{R}$ and $b_2 \neq 0$, and thus is spectrally stable and linearly unstable; or $M \approx N_1(-1, a)\diamond D(\lambda)$ for some $a \in \{-1, 1\}$ and $-1 \neq \lambda < 0$.

Then in the later case we obtain

$$0 = i_{-1}(\gamma_{\beta_k(e),e}) = i_1(\gamma_{\beta_k(e),e}) + S_M^+(1) - S_{N_1(-1,a)}^-(-1) = -S_{N_1(-1,a)}^-(-1).$$

Then by $\langle 3\rangle$ and $\langle 4\rangle$ in List 9.1.12 in p. 199 of [11], we must have $a = 1$. This case can be seen in Fig. 2.1.2 in p. 50 of [11] with the fact $N_1(-1, 1) \in \mathrm{Sp}(2)^0_{-1,-}$. Thus M is elliptic–hyperbolic (EH) and linearly unstable.

Note that by the above argument, the matrix $M_2(-1, b)$ has also the basic normal form $N_1(-1, 1)\diamond D(\lambda)$ for some $-1 \neq \lambda < 0$.

(vi) Let $e \in [0, 1)$ satisfy $\beta_s(e) = \beta_m(e) = \beta_k(e)$. As the limiting case of cases (v) and (ix) of Theorem 1.2, -1 must be the only eigenvalue of $M = \gamma_{\beta_k(e),e}(2\pi)$ with $\nu_{-1}(M) = 2$ by Corollary 9.2. Thus the matrix M must satisfy $M \approx M_2(-1, c)$ with $c_2 = 0$ and $\nu_{-1}(M_2(-1, c)) = 2$ by Section 2.1; or $M \approx N_1(-1, \hat{a})\diamond N_1(-1, \hat{b})$ for some $\hat{a}$ and $\hat{b} \in \{-1, 1\}$. In both cases, M has basic normal form $N_1(-1, a)\diamond N_1(-1, b)$ for some a and $b \in \{-1, 1\}$. Thus we obtain

$$\begin{aligned}
0 &= i_{-1}(\gamma_{\beta_k(e),e}) \\
&= i_1(\gamma_{\beta_k(e),e}) + S_M^+(1) - S_{N_1(-1,a)}^-(-1) - S_{N_1(-1,b)}^-(-1) \\
&= -S_{N_1(-1,a)}^-(-1) - S_{N_1(-1,b)}^-(-1).
\end{aligned}$$

Then by $\langle 3\rangle$ and $\langle 4\rangle$ in List 9.1.12 in p. 199 of [11], we must have $a = b = 1$ similar to our above study for (v). Therefore it is spectrally stable and linearly unstable as claimed.

The proof is complete. □

We give finally the

Proof of Theorem 1.9. Note first that $\gamma_{\beta,e}$ is analytic in both β and e. Since the property of the spectrum for a 4×4 symplectic matrix being complex saddle is an open condition in Sp(4), the set I_e in the theorem must be open in β for any fixed $e \in [0, 1)$. Thus for any fixed $e \in [0, 1)$, it suffices to show $I_e \neq \emptyset$. We argue by contradiction and suppose $I_e = \emptyset$. Then for any $(\beta, e) \in (0, 9] \times [0, 1)$, all the eigenvalues of $\gamma_{\beta,e}$ form two pairs and are located on the union $(\mathbf{R}\backslash\{0\}) \cup \mathbf{U}$. By our Theorem 1.1, the matrix $\gamma_{\beta,e}$ is non-degenerate when $\beta > 0$, and then it has no eigenvalue 1 at all. But by our Proposition 1.4, the matrix $\gamma_{9,e}(2\pi)$ has a pair of double positive eigenvalues not equal to 1 for all $e \in [0, 1)$. Therefore fix an $e \in [0, 1)$. By the continuity of the spectrum in β, the matrix $\gamma_{\beta,e}(2\pi)$ would have only positive eigenvalues not equal to 1 for all $\beta \in (0, 9]$. This then contradicts our Theorem 1.2, which yields the existence of the eigenvalue -1 of $\gamma_{\beta,e}(2\pi)$ for some $\beta \in (0, 9)$, and completes the proof. □

10. More Observations

Here we describe briefly some more results of [14] of Martínez, Samà and Simó on the Lagrangian triangular homographic solutions in the Newton potential case.

For $e < 1$ and close to 1, the system is HH for any β except in a neighborhood of some critical value which, numerically, appears to be equal to 6. Some interesting tangencies are also observed near the corresponding boundaries.

(i) The tangency at $(\beta, e) = (0, 1)$ between the e-vertical axis and the curve which separates the EE and EH domains is of the form $e = 1 - C\beta^{\frac{2}{5}}$;
(ii) The tangency at $(\beta, e) = (0, 1)$ between the $e = 1$ horizontal line and the curve which separates the EH and HH domains is of the form $e = 1 - C\beta^4$;
(iii) For fixed $\beta \in (0, 9)$, the matrix $\gamma_{\beta,e}(2\pi)$ is HH if $1 - e > 0$ is small enough under a special "non-degenerate" condition, which is defined in their Lemma 5 in p. 663 of [14], that is, $d_g \neq 0$ and $e_g \neq 0$ there. It seems to us that it is not easy to verify this non-degenerate condition, and that the point $(\beta, e) = (6, 1)$ is a possible degenerate point only checked numerically in [14].
(iv) The tangency at $(\beta, e) = (9, 0)$ between the $\beta = 9$ vertical line and the curve which separates the HH and CS domains is of the form $e = C(9 - \beta)^{\frac{1}{4}}$.

In all the above expressions C denotes suitable constants. Furthermore there is a point of contact of four different types of domains located approximately at $(1.2091, 0.3145)$.

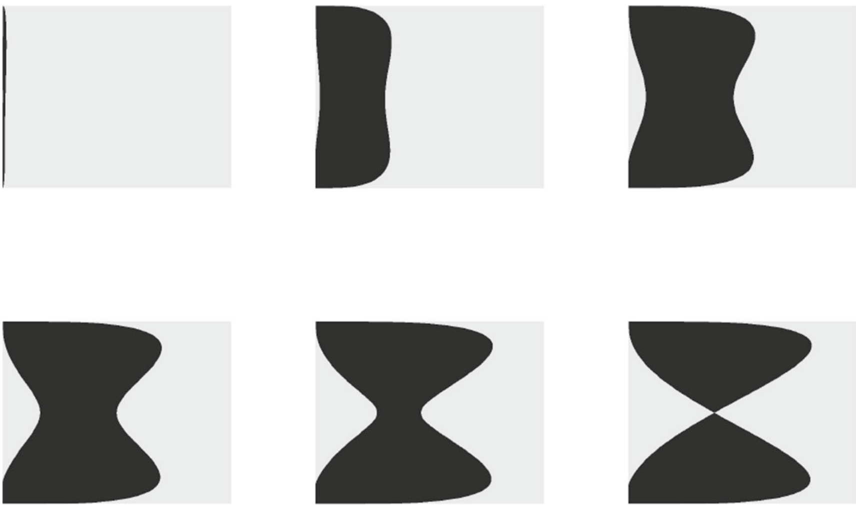

Fig. 4. ω-Degeneracy curves of Lagrangian triangular homographic orbits in the rectangle (β, e), $\beta \in (0, 2]$ and $e \in (-1, 1)$. From the *left top* to the *right bottom*, ω goes from 1 to -1 along the upper half unit circle of $\mathbf{U}$, more precisely, $\omega = \exp(\sqrt{-1}\theta)$ with $\theta = \pi/100, \pi/5, 2\pi/5, 3\pi/5, 4\pi/5, \pi$ respectively. The last one corresponds to $\omega = -1$. In the *black region*, the ω-index is 1

In the current paper, we have proved the non-degeneracy of the elliptic Lagrangian triangular solutions. We have also proved the global existence of separation curves Γ_s in (i), Γ_m and Γ_k in (ii). Our Theorem 1.7 is related to (iii).

For $\omega \in \mathbf{U}\backslash\{1\}$, we showed that there are two nontrivial degeneracy curves (possibly tangential to each other at isolated points) $\beta_1(e,\omega)$ and $\beta_2(e,\omega)$ ($\omega = -1$ in Theorem 1.2 and general ω in Theorem 1.5) which are real analytic in $e \in [0,1)$. These ω degeneracy curves actually yield a foliation of the non-hyperbolic region of $\gamma_{\beta,e}(2\pi)$ in the (β,e) rectangle $[0,9]\times(-1,1)$, when ω runs through $\mathbf{U}$. We have conducted numerical computations to see this interesting phenomenon according to our above analysis for $(\beta,e) \in [0,9]\times(-1,1)$. Especially in the Fig. 4, we pick up certain figures from such computations for readers. It is interesting to know how these degeneracy curves behave under the variation of ω, which we leave for future studies.

In summary, many problems observed numerically already deserve to be pursued further. For example, more precise properties of degeneracy curves including their asymptotic behaviors, possible intersections and variations with respect to ω, including the above mentioned interesting properties. In this paper, we have not considered separations between HH and CS either. We shall study these problems in some forthcoming papers, and we believe that the ideas and the methods we have developed here can also be applied to linear stability problems for other solutions of the n-body problems and systems with periodic coefficients.

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