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Hydrodynamic Limit for a Hamiltonian System with Boundary Conditions and Conservative Noise

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Abstract

We study the hyperbolic scaling limit for a chain of N coupled anharmonic oscillators. The chain is attached to a point on the left and there is a force (tension) τ acting on the right. In order to provide good ergodic properties to the system, we perturb the Hamiltonian dynamics with random local exchanges of velocities between the particles, so that momentum and energy are locally conserved. We prove that in the macroscopic limit the distributions of the elongation, momentum and energy converge to the solution of the Euler system of equations in the smooth regime.

1. Introduction

The aim of this paper is to study the hydrodynamic limit for a non-equilibrium system subject to an exterior time dependent force at the boundary. We consider the most simple mechanical model with non-linear interaction, that is, a one dimensional chain of N anharmonic oscillators. The left side is attached to a fixed point, while on the right side is acting a force τ (tension). For each value of τ there is a family of equilibrium (Gibbs) measures parametrized by the temperature (and by the tension τ). It turns out that these Gibbs measures can be written as a product.

We are interested in the macroscopic non-equilibrium behavior of this system as N tends to infinity, after rescaling space and time with N in the same way (*hyperbolic scaling*). We also consider situations in which the tension τ depends slowly on time, such that it changes in the macroscopic time scale. In this way we can also take the system originally at equilibrium at a certain tension τ_0 and push out of equilibrium by changing the exterior tension.

The goal is to prove that the three conserved quantities (elongation, momentum and energy) satisfy in the limit an autonomous closed set of hyperbolic equations given by the Euler system.

We approach this problem by using the *relative entropy method* (cf. [11]) as already done in [8] for a system of interacting particles moving in $\mathbb{R}^3$ (*gas dynamics*).

The relative entropy method permits one, in general, to obtain such a hydrodynamic limit if the system satisfies certain conditions:

(A) The dynamic should be *ergodic* in the sense that the only conserved quantities that *survive* the limit as $N \to \infty$ are those that we are looking for the macroscopic autonomous behavior (in this case elongation, momentum and energy). More precisely, the only stationary measures for the infinite system, with finite local entropy, are given by the Gibbs measures.
(B) The macroscopic equations have smooth solutions.
(C) Microscopic currents of the conserved quantities should be bounded by the local energy of the system.

We do not know any deterministic Hamiltonian system that satisfies condition (A), and this is a major, challenging open problem in statistical mechanics. Stochastic perturbation of the dynamics that conserves energy and momentum can give such an ergodic property and have been used in [8] (cf. also [2,3,7]). We use here a simpler stochastic mechanism than in [8]: at random independent exponential times we exchange the momentum of nearest neighbor particles, as if they were performing an elastic collision. Under this stochastic dynamics, every stationary measure has the property of being exchangeable in the velocity coordinates, and this is sufficient to characterize it as a convex combination of Gibbs measures (cf. [2] and [1]).

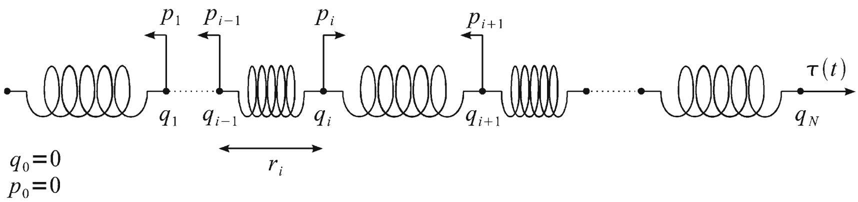
Regarding condition (B), it is well known that nonlinear hyperbolic equations in general develop shocks also starting from smooth initial conditions. The characterization and uniqueness of weak solutions in the presence of shock is a challenging problem in the theory of hyperbolic equations. We expect that a shock will increase the thermodynamic entropy associated with the profiles of the conserved quantities.

The relative entropy method compares the microscopic Gibbs entropy production (associated to the probability distribution of the system at a given time) with the macroscopic (thermodynamic) entropy production. If no shocks are present both entropy productions are small. The presence of the boundary force changes this balance a bit, since one should take into account the (macroscopic) change of entropy due to the work performed by the force. It turns out that the right choice of the boundary conditions in the macroscopic equation does compensate this large entropy production, keeping the time derivative of the relative entropy small. It would be interesting to prove similar cancellation of entropy productions when this is caused by shocks, as it would allow one to prove the hydrodynamical limit in these cases, and provide a microscopic derivation of irreversible thermodynamic **adiabatic** transformations, between thermodynamic equilibrium states that increase the thermodynamic entropy. Recent efforts in this direction use different methods (cf. [4]). Similar results on isothermal transformation are mathematically easier (cf. [9]).

As for condition (C), it created a problem in [8]: in the usual gas dynamics the energy current has the convecting term cubic in the velocities, while energy is quadratic. This was fixed in [8] by modifying the kinetic energy of the model: if the kinetic energy grows linearly as a function of the velocity, the energy current will also grow linearly. Since we work here in Lagrangian coordinates, our energy current does not have the cubic convecting term. This allows us to work with the usual quadratic kinetic energy.

2. The Model and the Main Theorem

We will study a system of $N + 1$ coupled oscillators in one dimension. Each particle has the same mass that we set equal to 1. The position of atom i ($i = 0, \dots, N$) is denoted by $q_i \in \mathbb{R}$, while its momentum is denoted by $p_i \in \mathbb{R}$. We assume that particle 0 is attached to a fixed point and it does not move, that is, $(q_0, p_0) \equiv (0, 0)$, while on particle N we apply a force $\tau(t)$ depending on time. Observe that only the particle 0 is constrained not to move, and that q_i can also assume negative values.



Denote by $\mathbf{q} := (q_0, \dots, q_N)$ and $\mathbf{p} := (p_0, \dots, p_N)$. The interaction between two particles i and $i-1$ will be described by the potential energy $V(q_i - q_{i-1})$ of an anharmonic spring relying on the particles. We assume V to be a positive smooth function that grows quadratically at infinity, that is, there exist strictly positive constants C_+ and C_- such that for any $r \in \mathbb{R}$:

$$V'(r)^2 \leqq C_+(1 + V(r)), \quad r^2 \leqq C_-(1 + V(r)), \quad V(0) = 0. \tag{1}$$

Energy is defined by the following Hamiltonian:

$$\mathcal{H}_N(\mathbf{q}, \mathbf{p}) := \sum_{i=1}^{N} \left(\frac{p_i^2}{2} + V(q_i - q_{i-1}) \right).$$

Since we focus on a nearest neighbors interaction, we define the distance between particles by

$$r_i = q_i - q_{i-1}, \quad i = 1, \dots, N.$$

Consequently the phase space is given by $(\mathbb{R}^2)^N$. We define the energy of particle $i \in \{1, \dots, N\}$ as

$$e_i := \frac{p_i^2}{2} + V(r_i)$$

so that $\mathcal{H}_N(\mathbf{r}, \mathbf{p}) = \sum_{i=1}^{N} e_i$, where $\mathbf{r} := (r_1, \dots, r_N)$.

Given a smooth function $\tau(s)$ that represents the force applied to particle N at the macroscopic time s, the dynamics of the system is determined by the generator

$$NG_N^{\tau(t)} := NL_N^{\tau(t)} + N\gamma S_N. \tag{2}$$

Here the Liouville operator L_N^τ is given by

$$\begin{aligned} L_N^\tau = \sum_{i=1}^{N}(p_i - p_{i-1})\frac{\partial}{\partial r_i} + \sum_{i=1}^{N-1}\left(V'(r_{i+1}) - V'(r_i)\right)\frac{\partial}{\partial p_i} \\ + \left(\tau - V'(r_N)\right)\frac{\partial}{\partial p_N}, \end{aligned} \tag{3}$$

where we used the fact that $p_0 \equiv 0$. Notice that the time scale in the tension is chosen such that it changes smoothly on the macroscopic scale.

The symmetric operator S_N is the generator of the stochastic part of the dynamics that exchanges at random times the velocities of nearest neighbors particles. For any smooth function f, we define the operator $\Upsilon_{i,i+1}$ by

$$\Upsilon_{i,i+1} = f\left(\mathbf{r}, \mathbf{p}^{i,i+1}\right) - f\left(\mathbf{r}, \mathbf{p}\right) \tag{4}$$

where $\mathbf{p}^{i,i+1} \in \mathbb{R}^N$ is defined from $\mathbf{p} \in \mathbb{R}^N$ by exchanging the coordinates p_j and p_{j+1}

$$p_j^{i,i+1} = \begin{cases} p_j & \text{if } j \neq i, i+1 \\ p_{i+1} & \text{if } j = i \\ p_i & \text{if } j = i+1. \end{cases}$$

Then S_N is defined through

$$S_N f(\mathbf{r}, \mathbf{p}) := \sum_{i=1}^{N-1}\left(f\left(\mathbf{r}, \mathbf{p}^{i,i+1}\right) - f(\mathbf{r}, \mathbf{p})\right) = -\frac{1}{2}\sum_{i=1}^{N-1}\Upsilon_{i,i+1}^2 f(\mathbf{r}, \mathbf{p}). \tag{5}$$

With this choice of the noise, the three balanced quantities, that is, locally conserved, are given by r_i, p_i, e_i.

We define $\boldsymbol{\zeta}(r, p) = (r, p, -e(r, p))^{\mathrm{T}} \in \mathbb{R}^2 \times \mathbb{R}_-$, and the Gibbs thermodynamic potential:

$$\Theta(\boldsymbol{\lambda}) := \log \int_{\mathbb{R}^2} e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}(r,p)}\, \mathrm{d}r\, \mathrm{d}p. \tag{6}$$

By the condition imposed on V, this function is always finite.

For $\boldsymbol{\zeta} \in \mathbb{R}^2 \times \mathbb{R}_-$ we define $\Phi : \mathbb{R}^2 \times \mathbb{R}_- \to \mathbb{R}$ the Legendre transform of $\Theta(\boldsymbol{\lambda})$:

$$\Phi(\boldsymbol{\zeta}) := \sup_{\boldsymbol{\eta}\in\mathbb{R}^2\times\mathbb{R}_+} \left\{\boldsymbol{\eta}\cdot\boldsymbol{\zeta} - \Theta(\boldsymbol{\eta})\right\}. \tag{7}$$

We denote by $\boldsymbol{\lambda}(\tilde{\mathfrak{u}})$ and $\tilde{\mathfrak{u}}(\boldsymbol{\lambda}) := (\mathfrak{r}, \mathfrak{p}, -E)^{\mathrm{T}}$ the corresponding convex conjugate variables that satisfy

$$\boldsymbol{\lambda} = D\Phi(\tilde{\mathfrak{u}}) \quad \text{and} \quad \tilde{\mathfrak{u}} = D\Theta(\boldsymbol{\lambda}), \tag{8}$$

where the operator D is defined by

$$Df(\mathbf{a}) := \left(\frac{\partial f}{\partial a_1}, \frac{\partial f}{\partial a_2}, \frac{\partial f}{\partial a_3} \right) \tag{9}$$

for any C^1 function $f : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{a} := (a_1, a_2, a_3) \in \mathbb{R}^3$.

On the one particle state space $\mathbb{R}^2$ we define a family of probability measure

$$\nu_{\boldsymbol{\lambda}}(\mathrm{d}r, \mathrm{d}p) = e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}(r,p) - \Theta(\boldsymbol{\lambda})} \mathrm{d}r \mathrm{d}p. \tag{10}$$

Observe that

$$E_{\nu_{\boldsymbol{\lambda}}}[\boldsymbol{\zeta}(r, p)] = \tilde{\mathfrak{u}}$$

so we can identify $\tilde{\mathfrak{u}} = (\mathfrak{r}, \mathfrak{p}, -E)^{\mathrm{T}}$ as, respectively, the average distance, velocity and (negative) energy. We also define the *internal energy* $\mathfrak{e} = E - \mathfrak{p}^2/2$. We have the relations

$$E_{\nu_{\boldsymbol{\lambda}}}(p^2) - \mathfrak{p}^2 = \lambda_3^{-1} := \beta^{-1}, \qquad P(\mathfrak{r}, \mathfrak{e}) := E_{\nu_{\boldsymbol{\lambda}}}[V'(r)] = \frac{\lambda_1}{\lambda_3} := \tau$$

that identify β^{-1} as temperature and τ as tension. This thermodynamic terminology is justified by observing that, for constant τ in the dynamics, and any $\beta > 0$, with the choice $\boldsymbol{\lambda} = (\beta\tau, 0, \beta)$ the family of product measures given by:

$$\nu^N_{(\tau\beta,0,\beta)}(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) = \prod_{i=1}^{N} \nu_{(\tau\beta,0,\beta)}(\mathrm{d}r_i, \mathrm{d}p_i), \quad \beta \in \mathbb{R}^+$$

is stationary for the dynamics. These are the Gibbs measures at an average temperature β^{-1}, pressure τ and velocity 0. In what follows we also need a Gibbs measure with average velocity different from 0, and we will use the following notation:

$$\nu^N_{\boldsymbol{\lambda}} := \prod_{i=1}^{N} e^{\boldsymbol{\lambda}\cdot\zeta_i - \Theta(\boldsymbol{\lambda})} \mathrm{d}r_i \mathrm{d}p_i := g^N_{\boldsymbol{\lambda}}(\mathbf{r}, \mathbf{p}) \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p},$$

where $\boldsymbol{\zeta}_i := (\zeta_{i,1}, \zeta_{i,2}, \zeta_{i,3})^{\mathrm{T}} := (r_i, p_i, -e_i)^{\mathrm{T}}$.

In a similar way we may introduce the local Gibbs measures. For any continuous *profile* $\tilde{\mathfrak{u}}(x)$, $x \in [0, 1]$, we have correspondingly a *profile* of parameters $\boldsymbol{\lambda}(x)$, and we define the inhomogeneous product measure

$$\nu^N_{\boldsymbol{\lambda}(\cdot)} := \prod_{i=1}^{N} e^{\boldsymbol{\lambda}(i/N)\cdot\zeta_i - \Theta(\boldsymbol{\lambda}(i/N))} \mathrm{d}r_i \mathrm{d}p_i,$$

that we call *Local Gibbs measures*.

We are interested in the macroscopic behavior of the elongation, momentum and energy of the particles, at time t, as $N \to \infty$. Notice that t is already the macroscopic time, since we have already multiplied the generator by N. Taking advantage of the one-dimensionality of the system, we will use *lagrangian* coordinates, that is, our space variables will be given by the lattice coordinates $\{1/N, \dots, (N-1)/N, 1\}$. Also observe that at this time scale, the generator of the process is given by $N\mathcal{G}_N^{\tau(t)}$.

Consequently, we introduce the (time dependent) empirical measures representing the spatial distribution (on the interval [0, 1]) of these quantities:

$$\eta_\alpha^N(\mathrm{d}x, t) := \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{i}{N}\right) \zeta_{i,\alpha}(t)\, \mathrm{d}x, \quad \text{for} \quad \alpha = 1, 2, 3.$$

We expect the measures $\eta_\alpha^N(\mathrm{d}x, t)$, $\alpha = 1, 2, 3$ to converge, as $N \to \infty$, to measures $\mathfrak{r}(x, t)\mathrm{d}x$, $\mathfrak{p}(x, t)\mathrm{d}x$, $-E(x, t)\mathrm{d}x$ being absolutely continuous with respect to the Lebesgue measure and with density satisfying the following system of three conservation laws:

$$\begin{cases} \partial_t \mathfrak{r} - \partial_x \mathfrak{p} = 0 \\ \partial_t \mathfrak{p} - \partial_x P(\mathfrak{r}, \mathfrak{e}) = 0 \\ \partial_t E - \partial_x(\mathfrak{p} P(\mathfrak{r}, \mathfrak{e})) = 0 \end{cases}, \quad \begin{cases} \mathfrak{r}_0(x) = \mathfrak{r}(x, 0), \ \mathfrak{p}_0(x) = \mathfrak{p}(x, 0), \ E_0(x) = E(x, 0) \\ \mathfrak{p}(0, t) = 0, \ P(\mathfrak{r}(1, t), \mathfrak{e}(1, t)) = \tau(t) \end{cases} \tag{11}$$

for bounded, smooth initial data $\mathfrak{r}_0, \mathfrak{p}_0, E_0 : [0, 1] \to \mathbb{R}$ and the force $\tau(t)$ depending on time t. Here we denoted by $\mathfrak{r}$ the specific volume, $\mathfrak{p}$ the velocity, E the total energy and $\mathfrak{e} := E - \frac{1}{2}\mathfrak{p}^2$ the internal energy. We also assume that the internal energy is always positive: $\mathfrak{e}(x, 0) > 0$.

We need the solutions of the system (11) to be C^2-solutions. To assure this, the following additional compatibility conditions at the space-time edges $(x, t) = (0, 0)$ and $(x, t) = (1, 0)$ have to be satisfied:

$$\lim_{x\to 0} \mathfrak{p}_0(x) = \mathfrak{p}(0, 0) = 0, \quad \lim_{x\to 1} P(\mathfrak{r}_0(x), \mathfrak{e}_0(x)) = \tau(0) \tag{12}$$

$$\lim_{x\to 0} \frac{\mathrm{d}}{\mathrm{d}x} P(\mathfrak{r}_0(x), \mathfrak{e}_0(x)) = 0, \quad \lim_{x\to 1} \frac{\mathrm{d}}{\mathrm{d}t} P(\mathfrak{r}_0(x), \mathfrak{e}_0(x)) = \tau'(0) \tag{13}$$

$$\lim_{x\to 0} \frac{\mathrm{d}^2}{(\mathrm{d}t)^2} \mathfrak{p}_0(x) = 0, \quad \lim_{x\to 1} \frac{\mathrm{d}^2}{(\mathrm{d}t)^2} P(\mathfrak{r}_0(x), \mathfrak{e}_0(x)) = \tau''(0). \tag{14}$$

A proof of this can be adapted from Chapters 4.3, 7.5 and 3.5 of [6].

For any test function $J : [0, 1] \to \mathbb{R}$ with compact support in (0, 1) consider the empirical densities

$$\eta_\alpha^N(t, J) := \langle \eta_\alpha^N(\mathrm{d}x, t); J\rangle = \frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right) \zeta_{\alpha,i}(t). \tag{15}$$

Our goal is to show that, starting with an initial distribution such that there exist smooth functions $\mathfrak{r}_0$, $\mathfrak{p}_0$ and E_0 satisfying

$$\begin{aligned} &\{\eta_1^N(0, J), \eta_2^N(0, J), \eta_3^N(0, J)\} \\ &\quad \to \left\{\int J(x)\mathfrak{r}_0(x)\mathrm{d}x, \int J(x)\mathfrak{p}_0(x)\mathrm{d}x, -\int J(x)E_0(x)\mathrm{d}x\right\} \end{aligned} \tag{16}$$

in probability as $N \to \infty$, then at time $t \in [0, T]$ we have the same convergence of $\eta_\alpha^N(t, J), \alpha = 1, 2, 3$ to the corresponding profiles $\mathfrak{r}(x, t)$, $\mathfrak{p}(x, t)$ and $E(x, t)$ respectively, that satisfy (11)–(14).

Here is the precise statement of our main result, where we make a stronger assumption on the initial measure:

Theorem 1. (Main theorem) *For any time $t \leqq t_s$, t_s being the time at which the solution $\mathfrak{u}$ produces the first shock, denote by μ_t^N the probability measure on the configuration space $\mathbb{R}^{2N}$ at time t, starting from the local Gibbs measure $\nu^N_{\boldsymbol{\lambda}(\cdot,0)}$ corresponding to the initial profiles $\tilde{\mathfrak{u}}_0$. Then for any smooth function $J : [0, 1] \to \mathbb{R}$ and any $\delta > 0$*

$$\lim_{N\to\infty} \mu_t^N \left[\left| \frac{1}{N} \sum_{i=1}^{N} J\left(\frac{i}{N}\right) \boldsymbol{\zeta}_i - \int_0^1 J(x)\tilde{\mathfrak{u}}(x, t)\, \mathrm{d}x \right| > \delta \right] = 0, \tag{17}$$

where $\mathfrak{u}$ is a C^2-solution to the system of conservation laws (11)–(14).

Remark 1. As our proof is based on the relative entropy method of [11], it is only valid as long as the solution to (11) is C^2. Since even for smooth initial data it may happen that the solution develops shocks, we are forced to restrict our derivation to a time $0 < t < t_s$, where t_s is the time when the solution to the system of conservation laws enters the first shock.

Remark 2. A proof for the existence of smooth solutions to the initial-boundary-value problem (11) can be found in chapters 4.3, 7.5 and 3.5 of [6]. Notice that we can rewrite the pressure P as a function of specific volume $\mathfrak{r}$ and entropy $\mathfrak{s}$:

$$\tilde{P}(\mathfrak{r}, \mathfrak{s}) := P(\mathfrak{r}, \mathfrak{e}).$$

Then we can rewrite the initial boundary value problem (11), in the smooth regime, in terms of the unknown $\mathfrak{r}$, $\mathfrak{p}$ and $\mathfrak{s}(\mathfrak{r}, \mathfrak{e})$ as follows:

$$\begin{cases} \partial_t \mathfrak{r} - \partial_x \mathfrak{p} = 0 \\ \partial_t \mathfrak{p} - \partial_x \tilde{P}(\mathfrak{r}, \mathfrak{s}) = 0, \\ \partial_t \mathfrak{s} = 0 \end{cases} \begin{cases} \mathfrak{r}_0(x) = \mathfrak{r}(x, 0),\ \ \mathfrak{p}_0(x) = \mathfrak{p}(x, 0),\ \ \mathfrak{s}_0(x) = \mathfrak{s}(x, 0) \\ \mathfrak{p}(0, t) = 0,\ \ \tilde{P}(\mathfrak{r}(1, t), \mathfrak{s}(1, t)) = \tau(t) \end{cases}, \tag{18}$$

where we used the thermodynamic relation

$$\tilde{P}(\mathfrak{r}, \mathfrak{s}) = -\frac{\partial \mathfrak{e}(\mathfrak{r}, \mathfrak{s})}{\partial \mathfrak{r}}.$$

Hence the specific entropy $\mathfrak{s}$ does not change in time and for any $x \in [0, 1]$ is given through the initial data $\mathfrak{s}(x, 0) := \mathfrak{s}_0(x)$.

In the non-conservative form, equation (18) reads as:

$$\partial_t \begin{pmatrix} \mathfrak{r} \\ \mathfrak{p} \\ \mathfrak{s} \end{pmatrix} - \mathbf{A}(\mathfrak{r}, \mathfrak{p}, \mathfrak{s}) \partial_x \begin{pmatrix} \mathfrak{r} \\ \mathfrak{p} \\ \mathfrak{s} \end{pmatrix} = 0$$

where the 3×3-matrix $\mathbf{A}$ is defined by

$$\mathbf{A} := \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial \tilde{P}}{\partial \mathfrak{r}} & 0 & \frac{\partial \tilde{P}}{\partial \mathfrak{s}} \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{S} \cdot \begin{pmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \mathbf{S}^{-1}$$

with $c := c(\mathfrak{r}, \mathfrak{s}) = \sqrt{\frac{\partial \tilde{P}}{\partial \mathfrak{r}}}$ and

$$\mathbf{S} := \mathbf{S}(\mathfrak{r}, \mathfrak{p}, \mathfrak{s}) = \begin{pmatrix} 1 & 1 & -\frac{1}{c}\frac{\partial \tilde{P}}{\partial \mathfrak{s}} \\ c & -c & 0 \\ 0 & 0 & c \end{pmatrix}.$$

With these notations we can rewrite (18) in the characteristic form

$$\mathbf{S}^{-1} \cdot \partial_t \begin{pmatrix} \mathfrak{r} \\ \mathfrak{p} \\ \mathfrak{s} \end{pmatrix} - \begin{pmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \mathbf{S}^{-1} \cdot \partial_x \begin{pmatrix} \mathfrak{r} \\ \mathfrak{p} \\ \mathfrak{s} \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} c(\partial_t \mathfrak{r} - c\partial_x \mathfrak{r}) + (\partial_t \mathfrak{p} - c\partial_x \mathfrak{p}) + \frac{1}{c}\frac{\partial \tilde{P}}{\partial \mathfrak{s}}(\partial_t \mathfrak{s} - c\partial_x \mathfrak{s}) = 0 \\ c(\partial_t \mathfrak{r} + c\partial_x \mathfrak{r}) - (\partial_t \mathfrak{p} + c\partial_x \mathfrak{p}) + \frac{1}{c}\frac{\partial \tilde{P}}{\partial s}(\partial_t \mathfrak{s} + c\partial_x \mathfrak{s}) = 0 \\ \partial_t \mathfrak{s} = 0. \end{cases}$$

In this way we can apply the existence proof for C^2 solutions to (11)–(14) for short times from [6].

3. The Hydrodynamic Limit

3.1. The Relative Entropy

On the phase space $(\mathbb{R}^2)^N$ we now have two time-dependent families of probability measures. One of them is the local Gibbs measure $\nu^N_{\boldsymbol{\lambda}(\cdot,t)}$ constructed from the solution of the system of conservation laws (11)–(14). We denote its density by

$$g_t^N = \prod_{i=1}^{N} e^{\boldsymbol{\lambda}(i/N,t)\cdot\boldsymbol{\zeta}_i - \Theta(\boldsymbol{\lambda}(i/N,t))}. \tag{19}$$

On the other hand we have the actual distribution μ_t^N, whose density $f_t^N(\mathbf{r}, \mathbf{p})$ is a solution, in the sense of distributions, of the Kolmogorov forward equation:

$$\begin{cases} \frac{\partial f_t^N}{\partial t}(\mathbf{r}, \mathbf{p}) = N\mathcal{G}_N^{\tau(t),\star} f_t^N(\mathbf{r}, \mathbf{p}) \\ \\ f_0^N(\mathbf{r}, \mathbf{p}) \;= g_0^N(\mathbf{r}, \mathbf{p}). \end{cases} \tag{20}$$

By $\mathcal{G}_N^{\tau,*} = L_N^{\tau,\star} + \gamma S_N$ we denote the adjoint operator of $\mathcal{G}_N^{\tau}$ with respect to the Lebesgue measure, where $L_N^{\tau,\star}$ can be computed as $L_N^{\tau,\star} = -L_N^{\tau}$.

The relative entropy of f_t^N with respect to g_t^N is defined by

$$H_N(t) = \int f_t^N \; \log \frac{f_t^N}{g_t^N} \,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}. \tag{21}$$

Our main result will follow from:

Theorem 2. (Relative entropy) *Under the same assumptions as in Theorem 1, for any time $t \in [0, T]$, $T < t_s$,*

$$\lim_{N\to\infty} \frac{1}{N} H_N\,(t) = 0.$$

Remark. Recall that the relative entropy $H(\alpha|\beta)$ of a probability measure α with respect to a probability measure β can be rewritten as

$$H(\alpha|\beta) = \sup_{\varphi} \left\{ \int \varphi \,\mathrm{d}\alpha - \log \int e^{\varphi} \,\mathrm{d}\beta \right\} \tag{22}$$

where the supremum is taken over all bounded measurable functions φ. It is easy to see that the relative entropy has the following properties: $H(\alpha|\beta)$ is positive, convex, and lower semi continuous function of α. It follows that for any measurable function F and any $\sigma > 0$:

$$\int F \,\mathrm{d}\alpha \leqq \frac{1}{\sigma} \log \int e^{\sigma F} \,\mathrm{d}\beta + \frac{1}{\sigma} H(\alpha|\beta). \tag{23}$$

Proof of Theorem 1. A useful special case of the entropy inequality can be stated if we set $F := \mathbf{1}_{[A]}$ to be the indicator function on a set A. With the choice $\sigma = \log\left(1 + \frac{1}{\beta[A]}\right)$, we obtain the inequality

$$\begin{aligned} E_\alpha[\mathbf{1}_{[A]}] \;&=\; \alpha[A] \leqq \frac{1}{\sigma} \log \beta[\exp(\sigma \mathbf{1}_{[A]})] + \frac{1}{\sigma} H(\alpha|\beta) \\ &= \frac{1}{\sigma} \log \left(\beta[A](e^{\sigma} - 1) + 1\right) + \frac{1}{\sigma} H(\alpha|\beta) \\ &\Rightarrow \alpha[A] \leqq \frac{\log 2 + H(\alpha|\beta)}{\log\left(1 + \frac{1}{\beta[A]}\right)}. \end{aligned} \tag{24}$$

Thus, if we define the set A_δ to be

$$A_\delta := \left\{ \left| \frac{1}{N} \sum_{i=1}^{N} J\left(\frac{i}{N}\right) \boldsymbol{\zeta}_i - \int_0^1 J(x)\mathfrak{u}(x,t)\,\mathrm{d}x \right| > \delta \right\},$$

for any test function $J : [0, 1] \to \mathbb{R}$ with compact support in $(0, 1)$, then with inequality (24), to prove that $\lim_{N\to\infty} \mu_t^N[A_\delta] = 0$, it is enough to show that for each $\delta > 0$

$$\log\left(1 + \frac{1}{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}}\right) \geqq C(\delta) N,$$

for some constant C not depending on N, since from Theorem 2 we have that $H_N(t) = o(N)$. However this is satisfied if $\nu^N_{\boldsymbol{\lambda}(\cdot,t)}[A_\delta]$ is exponentially small, that is,

$$\nu^N_{\boldsymbol{\lambda}(\cdot,t)}[A_\delta] \leqq \frac{1}{e^{C(\delta)N}}. \tag{25}$$

This is a result of the large deviation theory which can be adapted from [1,5,10].

3.2. Time Evolution of the Relative Entropy

In this section we will prove Theorem 2. Notice that by the choice of the initial distribution

$$H_N(0) = 0.$$

The strategy is to show that for some constant C

$$H_N(t) \leqq C \int_0^t H_N(s)\, \mathrm{d}s + \int_0^t R_N(s)\, \mathrm{d}s \tag{26}$$

with

$$\lim_{N\to\infty} \frac{1}{N} \int_0^t R_N(s)\, \mathrm{d}s = 0. \tag{27}$$

Then it follows by Gronwall's inequality that $\lim_{N\to\infty} \frac{H_N(t)}{N} = 0$ which concludes the proof of Theorem 2. We first prove the following inequality:

Lemma 1.

$$H_N(t) \leqq - \int_0^t \mathrm{d}s \int f_s^N \left(N\mathcal{G}_N^{\tau(s)} + \partial_s \right) \log g_s^N \, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{p} \tag{28}$$

Proof. By convexity of the function $\phi(f) = f \log f$, since Lebesgue measure is stationary for the dynamics generated by $\mathcal{G}_N^{\tau(t)}$, we have that

$$\int f_{t+h}^N \log f_{t+h}^N \, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{p} \leqq \int f_t^N \log f_t^N \, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{p}. \tag{29}$$

Then, since g_s^N is smooth and $H_N(0) = 0$, (28) follows.

Before we proceed in the proof, we have to introduce some further notations. For any C^1 function $F := (f_1, f_2, f_3)^{\mathrm{T}} : \mathbb{R}^3 \to \mathbb{R}^3$ we define

$$DF(\mathbf{a}) := ((Df_1)(\mathbf{a}), (Df_2)(\mathbf{a}), (Df_3)(\mathbf{a}))^{\mathrm{T}},$$

with $Df_i(\mathbf{a}), i = 1, 2, 3$ defined by (9). Recall that $\tilde{\mathfrak{u}} = (\mathfrak{r}, \mathfrak{p}, -E)^{\mathrm{T}}$, $\mathfrak{e} := E - \frac{1}{2}\mathfrak{p}^2$ and let us denote by

$$\tilde{\mathbf{J}}(\tilde{\mathfrak{u}}) := (\mathfrak{p}, P(\mathfrak{r}, \mathfrak{e}), -\mathfrak{p}P(\mathfrak{r}, \mathfrak{e}))^{\mathrm{T}} = \left(\mathfrak{p}, \frac{\lambda_1(\mathfrak{r}, \mathfrak{e})}{\lambda_3(\mathfrak{r}, \mathfrak{e})}, -\mathfrak{p}\frac{\lambda_1(\mathfrak{r}, \mathfrak{e})}{\lambda_3(\mathfrak{r}, \mathfrak{e})} \right)^{\mathrm{T}} \tag{30}$$

the flux of (11). Then the equation (11) can be rewritten as

$$\partial_t \tilde{\mathfrak{u}} = D\tilde{\mathbf{J}}(\tilde{\mathfrak{u}})\partial_x \tilde{\mathfrak{u}}$$

with the Jacobian

$$D\tilde{\mathbf{J}}(\tilde{\mathfrak{u}}) \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial P}{\partial \mathfrak{r}} & -\mathfrak{p}\frac{\partial P}{\partial \mathfrak{e}} & \frac{\partial P}{\partial \mathfrak{e}} \\ -\mathfrak{p}\frac{\partial P}{\partial \mathfrak{r}} & -P + \mathfrak{p}^2\frac{\partial P}{\partial \mathfrak{e}} & -\mathfrak{p}\frac{\partial P}{\partial \mathfrak{e}} \end{pmatrix}. \tag{31}$$

With the dual relation (8), $\boldsymbol{\lambda}$ is solution of the symmetric system

$$\partial_t[D\Theta(\boldsymbol{\lambda})] = \partial_x[D\Sigma(\boldsymbol{\lambda})], \tag{32}$$

where

$$\Sigma(\boldsymbol{\lambda}) = \boldsymbol{\lambda} \cdot \tilde{\mathbf{J}}(D\Theta(\boldsymbol{\lambda})).$$

Equation (32) can be rewritten as

$$(D^2\Theta)\partial_t\boldsymbol{\lambda} = (D^2\Sigma)\partial_x\boldsymbol{\lambda}.$$

Since

$$D^2\Theta(\boldsymbol{\lambda}(t,x))^{-1} = (D^2\Phi)(\tilde{\mathfrak{u}}(t,x)),$$

it follows that

$$\partial_t\boldsymbol{\lambda}(D^2\Phi) = (D^2\Sigma)\partial_x\boldsymbol{\lambda}.$$

Since

$$(D^2\Sigma) = (D^2\Phi)(D\tilde{\mathbf{J}}(\tilde{\mathfrak{u}}))$$

the following system of partial differential equations is satisfied:

$$\partial_t\boldsymbol{\lambda}(t,x) = (D\tilde{\mathbf{J}})^{\mathrm{T}}(\tilde{\mathfrak{u}})\partial_x\boldsymbol{\lambda}(t,x). \tag{33}$$

Let us define the microscopic fluxes:

$$\begin{aligned} \mathbf{J}_{i-1,i} &:= (-p_{i-1}, -V'(r_i), p_{i-1}V'(r_i))^{\mathrm{T}} \quad i = 1, \dots, N-1, \\ \mathbf{J}_{N,N+1} &:= (-p_N, -\tau(t), p_N\tau(t))^{\mathrm{T}}. \end{aligned} \tag{34}$$

By the definition of the Liouville operator given by (3),

$$L_N^{\tau(t)}\boldsymbol{\zeta}_i = \mathbf{J}_{i-1,i} - \mathbf{J}_{i,i+1}.$$

Finally let us define

$$\mathbf{v}_j := (0, p_j, -p_j^2/2)^{\mathrm{T}}.$$

Hence with the definition of the symmetric operator given by (5),

$$\begin{aligned} S_N(\boldsymbol{\zeta}_j) &= -2\mathbf{v}_j + \mathbf{v}_{j+1} + \mathbf{v}_{j-1}, \quad j = 2, \dots, N-1 \\ S_N(\boldsymbol{\zeta}_N) &= -\mathbf{v}_N + \mathbf{v}_{N-1}, \quad S_N(\boldsymbol{\zeta}_1) = -\mathbf{v}_1 + \mathbf{v}_2. \end{aligned}$$

Lemma 2.

$$N L_N^{\tau(t)} \log g_t^N = \sum_{i=1}^N \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) \cdot \mathbf{J}_{i-1,i} + N\lambda_2(1,t)\tau(t) + a_N(t) \tag{35}$$

where $a_N(t)$ is such that

$$\lim_{N\to\infty} \frac{1}{N}\int_0^t \int a_N(s) f_s^N \,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{r}\,\mathrm{d}s = 0$$

Proof.

$$N L_N^{\tau} \log g_t^N(\mathbf{r},\mathbf{p}) = N\sum_{i=1}^N \boldsymbol{\lambda}\left(\frac{i}{N}, t\right)\cdot\left(\mathbf{J}_{i-1,i} - \mathbf{J}_{i,i+1}\right)$$

$$= N\sum_{i=1}^N \left(\boldsymbol{\lambda}\left(\frac{i}{N}, t\right) - \boldsymbol{\lambda}\left(\frac{i-1}{N}, t\right)\right)\cdot \mathbf{J}_{i-1,i} - \boldsymbol{\lambda}(0,t)\cdot\mathbf{J}_{0,1} + \boldsymbol{\lambda}(1,t)\cdot\mathbf{J}_{N.N+1}$$

Taking into account the boundary conditions on $\boldsymbol{\lambda}$ we have

$$\boldsymbol{\lambda}(0,t)\cdot\mathbf{J}_{0,1} = \lambda_2(0,t)V'(r_1) = 0 \tag{36}$$

and

$$\boldsymbol{\lambda}(1,t)\cdot\mathbf{J}_{N.N+1} = -p_N\lambda_1(1,t) - \lambda_2(1,t)\tau(t) + \lambda_3(1,t)\tau(t)p_N = -\lambda_2(1,t)\tau(t) \tag{37}$$

because $\tau(t)\lambda_3(1,t) = \lambda_1(1,t)$. Since $\boldsymbol{\lambda}$ is a C^2-function, we obtain (35) with

$$|a_N(t)| = \frac{C}{N}\sum_{i=1}^{N-1} \|\mathbf{J}_i\|.$$

It remains to show, that $\lim_{N\to\infty}\int_0^t\int \frac{a_N(s)}{N}\mathrm{d}\mu_s^N\mathrm{d}s = 0$. This will be an easy consequence of Lemma 10.

Lemma 3.

$$\partial_t \log g_t^N = \sum_{i=1}^N (D\tilde{\mathbf{J}})^{\mathrm{T}}\left(\tilde{\mathrm{u}}\left(\frac{i}{N}, t\right)\right)\partial_x\boldsymbol{\lambda}\left(\frac{i}{N}, t\right)\cdot\left(\boldsymbol{\zeta}_i - \tilde{\mathrm{u}}\left(\frac{i}{N}, t\right)\right)$$

Proof.

$$\frac{\partial}{\partial t}\log g_t^N = \frac{\partial}{\partial t}\sum_{i=1}^N\left(\boldsymbol{\lambda}\left(\frac{i}{N}, t\right)\cdot\boldsymbol{\zeta}_i - \Theta\left(\boldsymbol{\lambda}\left(\frac{i}{N}, t\right)\right)\right)$$

$$= \sum_{i=1}^N \partial_t\boldsymbol{\lambda}\left(\frac{i}{N}, t\right)\cdot\left(\boldsymbol{\zeta}_i - D\Theta\left(\boldsymbol{\lambda}\left(\frac{i}{N}, t\right)\right)\right)$$

By (8), $D\Theta\left(\boldsymbol{\lambda}(\frac{i}{N}, t)\right) = \tilde{\mathrm{u}}(\frac{i}{N}, t)$, and (33) the result follows.

Lemma 4. *Recall the definition of the symmetric operator given by (5).*

$$-\lim_{N\to\infty}\frac{1}{N}\int_0^t\int\left(NS_N\log g_s^N\right)f_s^N\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{r}\,\mathrm{d}s \leqq \lim_{N\to\infty}\frac{1}{\sigma N}\int_0^t H_N(s)\,\mathrm{d}s,$$

where σ is a constant independent of N with $0 < 2\sigma < \inf_{x,s}\lambda_3(x,s)$.

Observe that the smoothness of the solution of (11) guarantees $\inf_{x,s}\lambda_3(x,s) > 0$.

Proof.

$$\begin{aligned}
&S_N\log g_s^N\\
&=\sum_{i=2}^{N-1}\boldsymbol{\lambda}\left(\frac{i}{N},t\right)\cdot(\mathbf{v}_{i-1}-2\mathbf{v}_i+\mathbf{v}_{i+1})+\boldsymbol{\lambda}\left(\frac{1}{N},t\right)\cdot(-\mathbf{v}_1+\mathbf{v}_2)+\boldsymbol{\lambda}(1,t)\\
&\quad\cdot(\mathbf{v}_{N-1}-\mathbf{v}_N)\\
&=\sum_{i=2}^{N-1}\left(\boldsymbol{\lambda}\left(\frac{i-1}{N},t\right)-2\boldsymbol{\lambda}\left(\frac{i}{N},t\right)+\boldsymbol{\lambda}\left(\frac{i+1}{N},t\right)\right)\cdot\mathbf{v}_i\\
&\quad+\left(\boldsymbol{\lambda}\left(\frac{2}{N},t\right)-\boldsymbol{\lambda}\left(\frac{1}{N},t\right)\right)\cdot\mathbf{v}_1+\left(\boldsymbol{\lambda}\left(\frac{N-1}{N},t\right)-\boldsymbol{\lambda}(1,t)\right)\cdot\mathbf{v}_N
\end{aligned}$$

In Lemma 10 we will show that the expectation of $\frac{1}{N}\sum_i\|\mathbf{v}_i\|$ is uniformly bounded for all N and hence, since $\boldsymbol{\lambda}$ is in C^2, the first term vanishes in the limit as $N\to\infty$.

Recall that by the entropy inequality (23), for any $\sigma > 0$ we have for $k\in\{1,\ldots,N\}$:

$$\frac{1}{N}\int p_k^2 f_s^N\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{r} \leqq \frac{1}{N\sigma}\log\int e^{\sigma p_k^2}\nu^N_{\boldsymbol{\lambda}(\cdot,s)}+\frac{1}{N\sigma}H(s).$$

Since this inequality is true for any $\sigma > 0$, the integral on the right hand side of the inequality is bounded as long as $\sigma < \inf_x\frac{1}{2}\lambda_3(x,s)$ and hence the first term vanishes as $N\to\infty$. The expected value of p_k can be controlled in a similar way.

So far we have from Lemmas 2, 3 and 4

$$\begin{aligned}
H_N(t)\leqq&\int_0^t\int\sum_{i=1}^N\partial_x\boldsymbol{\lambda}\left(\frac{i}{N},s\right)\\
&\times\left[\mathbf{J}_{i-1,i}-(D\tilde{\mathbf{J}})^{\mathrm{T}}\left(\tilde{\mathrm{u}}\left(\frac{i}{N},s\right)\right)\left(\boldsymbol{\zeta}_i-\tilde{\mathrm{u}}\left(\frac{i}{N},s\right)\right)\right]f_s^N\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{r}\,\mathrm{d}s\\
&-\int_0^t N\tau(s)\lambda_2(1,s)\,\mathrm{d}s+\frac{1}{\sigma}\int_0^t H_N(s)\,\mathrm{d}s+\int_0^t R_N(s)\,\mathrm{d}s
\end{aligned}\qquad(38)$$

where $R_N(t)$ is such that (27) holds.

By (30) we have

$$\int_0^1\partial_x\boldsymbol{\lambda}(x,t)\cdot\tilde{\mathbf{J}}(\tilde{\mathrm{u}}(x,t))\,\mathrm{d}x=\int\frac{\partial}{\partial x}\left(\frac{\lambda_1(x,t)\lambda_2(x,t)}{\lambda_3(x,t)}\right)\,\mathrm{d}x=\tau(t)\lambda_2(1,t),$$

and consequently we can replace $-N\tau(t)\lambda_2(1,t)$ by

$$-\sum_{i=1}^{N} \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) \cdot \tilde{\mathbf{J}}\left(\tilde{\mathrm{u}}\left(\frac{i}{N}, t\right)\right)$$

with an error uniformly bounded in N. It follows that from (38) we have

$$\begin{aligned} H_N(t) \leqq & \int_0^t \int \sum_{i=1}^{N} \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, s\right) \\ & \times \left[\mathbf{J}_{i-1,i} - \tilde{\mathbf{J}}\left(\tilde{\mathrm{u}}\left(\frac{i}{N}, s\right)\right) - (D\tilde{\mathbf{J}})^{\mathrm{T}}\left(\tilde{\mathrm{u}}\left(\frac{i}{N}, s\right)\right)\right. \\ & \left.\times \left(\boldsymbol{\zeta}_i - \tilde{\mathrm{u}}\left(\frac{i}{N}, s\right)\right)\right] f_s^N \,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{r}\,\mathrm{d}s \\ & + \frac{1}{\sigma}\int_0^t H_N(s)\,\mathrm{d}s + \int_0^t R_N(s)\,\mathrm{d}s. \end{aligned} \tag{39}$$

Our next goal is to prove a weak form of local equilibrium. In view of this we introduce microscopic averages over blocks of size $2k+1$. In what follows, for any vector field $\mathbf{Y}_i := (Y_{1,i}, Y_{2,i}, Y_{3,i})^{\mathrm{T}} : (\mathbb{R}^2)^3 \to \mathbb{R}^3$ we denote by $\mathbf{Y}_i^k := (Y_{1,i}^k, Y_{2,i}^k, Y_{3,i}^k)^{\mathrm{T}}$, block averages over blocks of length $2k+1$, where $k > 0$ is independent of N. For example

$$\boldsymbol{\zeta}_i^k = (\zeta_{1,i}^k, \zeta_{2,i}^k, \zeta_{3,i}^k)^{\mathrm{T}} := (r_i^k, p_i^k, -e_i^k)^{\mathrm{T}} := \frac{1}{2k+1}\sum_{|i-l|\leqq k} \boldsymbol{\zeta}_l. \tag{40}$$

These blocks are microscopically large but on the macroscopic scale they are small, thus N goes to infinity first and then k goes to infinity. We also need to introduce another small parameter ℓ and consider small macroscopic blocs of length ℓN at the boundaries.

For any smooth and bounded function $F : [0,1] \to \mathbb{R}$ and any bounded function $\psi : \mathbb{R}^2 \to \mathbb{R}$, we obtain the following summation by parts formula

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{N} F\left(\frac{i}{N}\right)\psi(r_i, p_i) = & \frac{1}{N}\sum_{i=[N\ell]}^{N-[N\ell]} F\left(\frac{i}{N}\right)\frac{1}{2k+1}\sum_{|j-i|\leqq k}\psi(r_j, p_j) \\ & + \mathcal{O}\left(\frac{k+N\ell}{N}\right). \end{aligned} \tag{41}$$

Here we first restricted the sum to configurations over $\{[N\ell], \ldots, N-[N\ell]\}$, for some small $\ell > 0$, such that $\ell \to 0$ after $N \to \infty$ and $\ell N >> k$. In this way, we avoid touching the boundary when we introduce the block averages. The error we made will vanish in the limit since $\ell \to 0$.

We also need to do some cut off in order to have only bounded variables.

Let $\mathcal{C}_{i,b} := \{e_{i-1}, e_i \leqq b\}$, and define

$$\mathbf{J}_{i-1,i}^b := \mathbf{J}_{i-1,i}\mathbf{1}_{\mathcal{C}_{i,b}} \quad \text{and} \quad \boldsymbol{\zeta}_i^b := \boldsymbol{\zeta}_i\mathbf{1}_{\mathcal{C}_{i,b}},$$

then these functions are bounded. Also denote as $\tilde{\mathbf{J}}^b(\tilde{\mathfrak{u}})$ the corresponding expectation with respect to the Gibbs measure of parameters $\boldsymbol{\lambda}(\tilde{\mathfrak{u}})$, that converges to $\tilde{\mathbf{J}}(\tilde{\mathfrak{u}})$ as $b \to \infty$.

Assumptions (1) on the potential assert that by the entropy inequality (23) with reference measure $\mathrm{d}\nu^N_{\boldsymbol{\lambda}(\cdot,t)}$, the error we make by the replacement of $\mathbf{J}_{i-1,i}$ and $\boldsymbol{\zeta}_i$ by $\mathbf{J}^b_{i-1,i}$ and $\boldsymbol{\zeta}^b_i$ respectively, is small in N if we can show that $\frac{1}{N}H_N(s) \to 0$ as $N \to \infty$.

For any $\sigma > 0$ small enough

$$
\begin{aligned}
&\int \sum_{i=1}^N \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, s\right) \mathbf{J}_{i-1,i} \mathbf{1}_{\mathcal{C}^c_{i,b}}\, \mathrm{d}\mu^N_s \\
&\leqq \frac{1}{\sigma} \sum_{i=1}^N \log\left(\int e^{\sigma \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, s\right) \mathbf{J}_{i-1,i} \mathbf{1}_{\mathcal{C}^c_{i,b}}}\, \mathrm{d}\nu_{\boldsymbol{\lambda}(\cdot,t)} \right) + \frac{H_N(s)}{\sigma} \\
&\leqq \frac{1}{\sigma} \sum_{i=1}^N \log\left(1 + \int_{\mathcal{C}^c_{i,b}} e^{\sigma \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, s\right) \mathbf{J}_{i-1,i}}\, \mathrm{d}\nu_{\boldsymbol{\lambda}(\cdot,t)} \right) + \frac{H_N(s)}{\sigma} \\
&= \frac{NC(b,\sigma)}{\sigma} + \frac{H_N(s)}{\sigma}
\end{aligned}
\tag{42}
$$

where $\lim_{b\to\infty} C(b,\sigma) = 0$ for any $\sigma > 0$.

Using that $\boldsymbol{\lambda}$ and $\mathfrak{u}$ are in C^2 and formula (41), we arrive at

$$
\begin{aligned}
&\sum_{i=1}^N \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, s\right) \Bigg[\mathbf{J}^b_{i-1,i} - \tilde{\mathbf{J}}^b\left(\tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right) \\
&- (D\tilde{\mathbf{J}})^{\mathrm{T}}\left(\tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right)\left(\boldsymbol{\zeta}^b_i - \tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right)\Bigg] \\
&\quad = \sum_{i=[N\ell]}^{N-[N\ell]} \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, s\right) \\
&\qquad \times \Bigg[\frac{1}{2k+1} \sum_{|l-i|\leqq k} \mathbf{J}^b_{l-1,l} - \tilde{\mathbf{J}}^b\left(\tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right) \\
&\qquad - (D\tilde{\mathbf{J}})^{\mathrm{T}}\left(\tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right)\left(\boldsymbol{\zeta}^{b,k}_i - \tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right)\Bigg] \\
&\qquad + \mathcal{O}(k + N\ell).
\end{aligned}
$$

The following theorem will be proved in Section 3.3.

Theorem 3. (The one-block estimate) *For any ℓ, b:*

$$
\lim_{k\to\infty} \lim_{N\to\infty} \frac{1}{N} \sum_{i=[N\ell]}^{N-[N\ell]} \int_0^t \int \left| \frac{1}{2k+1} \sum_{|l-i|\leqq k} \mathbf{J}^b_{l-1,l} - \tilde{\mathbf{J}}^b\left(\boldsymbol{\zeta}^k_i\right) \right| f^N_s\, \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{r}\, \mathrm{d}s = 0.
\tag{43}
$$

With this theorem we obtain:

$$\frac{H_N(t)}{N} \leqq \frac{1}{N} \sum_{i=[N\ell]}^{N-[N\ell]} \int_0^t \int \boldsymbol{\Omega}\left(\boldsymbol{\zeta}_i^k, \tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right) f_s^N \, \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{r} \, \mathrm{d}s$$
$$+ \int_0^t \frac{R_{N,k,\ell,b}(s)}{N} \, \mathrm{d}s + \int_0^t \frac{H_N(s)}{N\sigma} \, \mathrm{d}s \tag{44}$$

for some $\sigma > 0. R_{N,k,\ell,b}$ is such that

$$\lim_{b\to\infty} \lim_{\ell\to 0} \lim_{k\to\infty} \lim_{N\to\infty} \int_0^t \frac{R_{N,k,\ell,b}(s)}{N} \, \mathrm{d}s = 0,$$

and we used (33) to define

$$\boldsymbol{\Omega}(\mathfrak{z}, \tilde{\mathfrak{u}}) := \partial_x \boldsymbol{\lambda} \cdot \left(\tilde{\mathbf{J}}(\mathfrak{z}) - \tilde{\mathbf{J}}(\tilde{\mathfrak{u}})\right) - \partial_t \boldsymbol{\lambda} \cdot (\mathfrak{z} - \tilde{\mathfrak{u}}).$$

Hence

$$D_{\mathfrak{z}} \boldsymbol{\Omega}(\mathfrak{z}, \tilde{\mathfrak{u}}) = \left((D\tilde{\mathbf{J}})^{\mathrm{T}}(\mathfrak{z}) \cdot \partial_x \boldsymbol{\lambda} - \partial_t \boldsymbol{\lambda}\right) \tag{45}$$

is equal to zero if $\mathfrak{z}$ is a solution of (33) and consequently:

$$\boldsymbol{\Omega}(\tilde{\mathfrak{u}}, \tilde{\mathfrak{u}}) = 0, \quad D_{\mathfrak{z}} \boldsymbol{\Omega}(\tilde{\mathfrak{u}}, \tilde{\mathfrak{u}}) = 0.$$

Applying the entropy inequality (23) on the sum in (44), we obtain that for some $\sigma > 0$ it is bounded above by

$$\frac{1}{N\sigma} \int_0^t \log \int \exp\left\{\sigma \sum_{i=[N\ell]}^{N-[N\ell]} \boldsymbol{\Omega}\left(\boldsymbol{\zeta}_i^k, \mathfrak{u}\left(\frac{i}{N}, s\right)\right)\right\} g_s^N \, \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{r} \, \mathrm{d}s$$
$$+ \frac{1}{N\sigma} \int_0^t H_N(s) \, \mathrm{d}s. \tag{46}$$

Hence it remains to prove that the first term of this expression is of order $\mathcal{O}(\frac{1}{N})$. This will be done using the following special case of Varadhan's lemma:

Theorem 4. (Varadhan's lemma) *Let $\nu_{\boldsymbol{\lambda}}^n$ be the product homogeneous measure with marginals $\nu_{\boldsymbol{\lambda}}$ given by (10) and with rate function $I : \mathbb{R}^2 \times \mathbb{R}_- \to \mathbb{R}$ defined by*

$$I(\mathbf{x}) := \Phi(\mathbf{x}) - \mathbf{x} \cdot \boldsymbol{\lambda} + \Theta(\boldsymbol{\lambda}).$$

Then for any bounded continuous function F on $\mathbb{R}^2 \times \mathbb{R}_-$

$$\lim_{n\to\infty} \frac{1}{n} \log \int e^{nF(\zeta)} \, \mathrm{d}\nu_{\boldsymbol{\lambda}}^n = \sup_{\mathbf{x}} \{F(\mathbf{x}) - I(\mathbf{x})\}.$$

Proof. A proof of this theorem can be adapted from [1,5,10].

In order to apply this Theorem we arrange the sum in (46) as sums over disjoint blocks and then take advantage of the fact that the local Gibbs measures are product measures. Assume without loss of generality that $2k+1$ divides $N-2[N\ell]$, then

$$\sum_{i=[N\ell]}^{N-[N\ell]} \boldsymbol{\Omega}\left(\boldsymbol{\zeta}_i^k, \tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right) = \sum_{j\in\{-k,\ldots,k\}} \sum_{i\in B_{[N\ell]}^{N-2[N\ell],k}} \tau_j \boldsymbol{\Omega}\left(\boldsymbol{\zeta}_i^k, \tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right)$$

where $B_{[N\ell]}^{N-2[N\ell],k} := \left\{r(2k+1)+[N\ell]+k;\, r\in\left\{0,\ldots,\frac{N-2[N\ell]-2k}{2k+1}\right\}\right\}$. In this way, for any fixed j, the terms in the sum over $i\in B_{[N\ell]}^{N-2[N\ell],k}$ depend on configurations in disjoint blocks. Thus the random variables

$$\tau_j \boldsymbol{\Omega}\left(\boldsymbol{\zeta}_i^k, \tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right)$$

are independent under $\nu_{\boldsymbol{\lambda}(\cdot,s)}^N$.

Using Hölder inequality, the first term in (46) is bounded above by

$$\begin{aligned}
&\frac{1}{N\sigma}\int_0^t \log\int \prod_{j\in\{-k,\ldots,k\}} \exp\left\{\sigma \sum_{i\in B_{[N\ell]}^{N-2[N\ell],k}} \tau_j \boldsymbol{\Omega}\left(\boldsymbol{\zeta}_i^k, \mathfrak{u}\left(\frac{i}{N}, s\right)\right)\right\} g_s^N\, \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{r}\, \mathrm{d}s\\
&\leqq \frac{1}{N\sigma(2k+1)}\int_0^t\\
&\quad \sum_{j\in\{-k,\ldots,k\}} \log\int \exp\left\{\sigma(2k+1) \sum_{i\in B_{[N\ell]}^{N-2[N\ell],k}} \tau_j \boldsymbol{\Omega}\left(\boldsymbol{\zeta}_i^k, \tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right)\right\} g_s^N\, \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{r}\, \mathrm{d}s\\
&= \frac{1}{N\sigma(2k+1)} \sum_{i=[N\ell]}^{N-[N\ell]} \int_0^t \log\int \exp\left\{\sigma(2k+1)\boldsymbol{\Omega}\left(\boldsymbol{\zeta}_i^k, \tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right)\right\} g_s^N\, \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{r}\, \mathrm{d}s.
\end{aligned}$$

Then, since all the functions in this expression are smooth and the family of local Gibbs measures converges weakly, we obtain that

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{N\to\infty} \frac{1}{(2k+1)N\sigma} \sum_{i=[N\ell]}^{N-[N\ell]} \int_0^t \log\int \exp\\
&\times\left\{\sigma(2k+1)\boldsymbol{\Omega}\left(\boldsymbol{\zeta}_i^k, \tilde{\mathfrak{u}}\left(\frac{i}{N}, s\right)\right)\right\} g_s^N\, \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{r}\, \mathrm{d}s\\
&= \lim_{k\to\infty} \frac{1}{\sigma(2k+1)} \int_0^t\int_0^1 \log\int \exp\left\{(2k+1)\sigma\boldsymbol{\Omega}(\boldsymbol{\zeta}_i^k, \tilde{\mathfrak{u}}(x,s))\right\}\, \mathrm{d}\nu_{\boldsymbol{\lambda}(x,s)}\, \mathrm{d}x\, \mathrm{d}s.
\end{aligned}$$

So now for each $x\in[0,1]$, the distribution of the particles in a box of size k is given by the invariant Gibbs measure with average $\mathfrak{u}(x,s)$ such that we can apply Theorem 4 on this product measure to obtain that the last expression is equal to

$$\frac{1}{\sigma}\int_0^t\int_0^1 \sup_{\mathfrak{z}}\{\sigma\boldsymbol{\Omega}\left(\mathfrak{z}, \tilde{\mathfrak{u}}(x,s)\right) - I(\mathfrak{z})\}\, \mathrm{d}x. \tag{47}$$

To conclude Theorem 2 it thus remains to show that this is equal to zero. Since I and $\boldsymbol{\Omega}$ are both convex, and both functions and their derivatives are vanishing at $\mathfrak{z} = \tilde{\mathfrak{u}}$, it follows from assumption (1) on the potential that $\sigma \boldsymbol{\Omega}(\mathfrak{z}, \tilde{\mathfrak{u}}) \leqq I(\mathfrak{z})$ for σ small enough. Hence there exists a σ such that the last expression is equal to zero.

This concludes the proof of Theorem 2.

Since

$$H_N(t) \leqq C \int_0^t H_N(s)\, \mathrm{d}s + \int_0^t R_{N,k,\ell,b}(s)\, \mathrm{d}s,$$

for some uniform constant C, it follows by Gronwall inequality that

$$\begin{aligned} H_N(t) &\leqq H_N(0) e^{Ct} + \int_0^t R_{N,k,\ell,b}(s) e^{C(t-s)}\, \mathrm{d}s \\ &\leqq e^{Ct} \left(H_N(0) + \int_0^t R_{N,k,\ell,b}(s)\, \mathrm{d}s \right). \end{aligned}$$

Hence the claim follows, since

$$\lim_{b\to\infty} \lim_{\ell\to 0} \lim_{k\to\infty} \lim_{N\to\infty} \int_0^t \frac{R_{N,k,\ell,b}(s)}{N}\, \mathrm{d}s = 0.$$

3.3. The One Block Estimate (Theorem 3)

We define the space-time average of the distribution

$$\bar{f}_t^{N,\ell,k} = \frac{1}{t} \int_0^t \frac{1}{[N(1-2\ell)]} \sum_{i=N\ell}^{N(1-\ell)} f_{s,i}^{N,k}\,(r_{-k}, p_{-k}, \ldots, r_k, p_k)\, \mathrm{d}s \tag{48}$$

where we defined the projections

$$\begin{aligned} & f_{s,i}^{N,k}(\tilde{r}_{-k}, \tilde{p}_{-k} \ldots, \tilde{r}_k, \tilde{p}_k) \\ &\quad = \int f_s^N(r_1, p_1, \ldots, r_{i-k-1}, p_{i-k-1}, \tilde{r}_{-k}, \tilde{p}_{-k}, \ldots, \tilde{r}_k, \tilde{p}_k, r_{i+k+1}, \\ &\qquad p_{i+k+1}, \ldots, r_N, p_N) \prod_{|i-l|>k} \mathrm{d}r_l\, \mathrm{d}p_l. \end{aligned} \tag{49}$$

and we denote

$$\mathrm{d}\bar{\mu}_t^{N,\ell,k} := \bar{f}_t^{N,\ell,k} \prod_{|l|\leqq k} \mathrm{d}r_l\, \mathrm{d}p_l \tag{50}$$

3.3.1. Tightness We have the following:

Lemma 5. (Tightness) *For each k fixed , the sequence* $(\bar{\mu}_t^{N,\ell,k})_{N \geqq 1}$ *of probability measures is tight.*

Proof. From the definition of $\bar{\mu}_t^{N,\ell,k}$ we have

$$\int \left(\frac{1}{2k+1} \sum_{|l| \leqq k} e_l \right) \bar{f}_t^{N,\ell,k} \prod_{|l| \leqq k} \mathrm{d}r_l \, \mathrm{d}p_l$$

$$= \frac{1}{t} \int_0^t \frac{1}{[N(1-2\ell)]} \sum_{i=N\ell}^{N(1-\ell)}$$

$$\times \left(\int \left(\frac{1}{2k+1} \sum_{|l| \leqq k} e_l \right) f_{s,i}^{N,k} \, \mathrm{d}r_{-k} \, \mathrm{d}p_{-k} \ldots \mathrm{d}r_k \, \mathrm{d}p_k \right) \mathrm{d}s$$

$$\leqq \frac{1}{t} \int_0^t \int \left(\frac{1}{N(1-2\ell)} \sum_{l=1}^{N} e_l \right) f_s^N \prod_{l \in \mathbb{Z}} \mathrm{d}r_l \, \mathrm{d}p_l \, \mathrm{d}s \leqq C$$

by Lemma 10, and this implies the tightness.

Lemma 5 asserts that for each fixed k there exists a limit point $\mu_t^{\ell,k}$ of the sequence $(\bar{\mu}_t^{N,\ell,k})_{N \geqq 1}$. On the other hand, since the sequence $(\mu_t^{\ell,k})_{k \geqq 1}$ forms a consistent family of measures, by Kolmogorov's theorem, for $k \to \infty$, there exists a unique probability measure μ on the configuration space $\{(r_i, p_i)_{i \in \mathbb{Z}} \in (\mathbb{R}^2)^\infty\}$, such that the restriction of μ on $\{(r_j, p_j)_{j \in \{-k,\ldots,+k\}} \in (\mathbb{R}^2)^{2k+1}\}$ is $\mu_t^{\ell,k}$.

3.3.2. Proof of the One-Block-Estimate Let us define the formal generator $\mathcal{G}$ of the infinite dynamics by

$$\mathcal{G} := \mathcal{L} + \gamma \mathcal{S}, \tag{51}$$

with the antisymmetric part

$$\mathcal{L} := \sum_{j \in \mathbb{Z}} \left\{ p_j \left(\frac{\partial}{\partial r_j} - \frac{\partial}{\partial r_{j+1}} \right) + \left(V'(r_{j+1}) - V'(r_j) \right) \frac{\partial}{\partial p_j} \right\} \tag{52}$$

and the symmetric part

$$\mathcal{S} := \sum_{i \in \mathbb{Z}} \left(f \left(\mathbf{r}, \mathbf{p}^{j,j+1} \right) - f(\mathbf{r}, \mathbf{p}) \right). \tag{53}$$

In Section 3.3.3 we will prove the following proposition:

Proposition 1. *Any limit point* μ *of* $\bar{\mu}_t^{N,\ell,k}$, *for* $N \to \infty$ *and then* $k \to \infty$, *satisfies the following properties:*

(i) *it has finite entropy density: there exists a constant* $C > 0$ *such that for all subsets* $\Lambda \subset \mathbb{Z}$

$$H\left(\mu|_{\Lambda}\middle|\nu^{|\Lambda|}_{(\tau\beta,0,\beta)}\right) \leqq C|\Lambda|,$$

(i) *it is translation invariant: for any local function F and any $j \in \mathbb{Z}$,*

$$\int F \,\mathrm{d}\mu = \int (\tau_j F)\,\mathrm{d}\mu$$

where τ_j denotes the spatial shift by j on the configurations.

(i) *it is stationary with respect to the operator $\mathcal{G}$: for any smooth bounded local function F*

$$\int (\mathcal{G}F)\,\mathrm{d}\mu = 0.$$

With this proposition, we can apply the ergodic theorem from [2].

Theorem 5. (Ergodicity) *Any limit point μ of $\bar{\mu}_t^{N,\ell,k}(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p})$ is a convex combination of Gibbs measures, that is,*

$$\mu(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) = \prod_{i\in\mathbb{Z}} g_{\boldsymbol{\lambda}}(r_i, p_i)\mathrm{d}r_i \mathrm{d}p_i.$$

The proof of Theorem 5 is contained in [2], see also [1] for more details. The idea of the proof is the following: by Proposition 1 one can prove that μ is separately stationary for $\mathcal{L}$ and $\mathcal{S}$. This implies that the distribution of momenta conditioned on position $\mu(\mathrm{d}\mathbf{p}|\mathbf{r})$ is exchangeable. This is the only point where we need the noise in the dynamics.

Proof of Theorem 3. Recall that we need to prove (43). By Lemma 5 and (ii), (iii) of Proposition 1, it is enough to show that for each b and ℓ

$$\limsup_{k\to\infty} \sup_{\mu\in\mathcal{G}} \int \left| \frac{1}{2k+1} \sum_{l=-k+1}^{k} \mathbf{J}^b_{l-1,l} - \tilde{\mathbf{J}}^b\left(\frac{1}{2k+1}\sum_{l=-k+1}^{k} \boldsymbol{\zeta}_l\right)\right| \mathrm{d}\mu = 0$$

where $\mathcal{G}$ is the set of Gibbs measures. But this is just the law of large numbers and holds in the limit as $k \to \infty$. □

3.3.3. Proof of Proposition 1

Lemma 6. *Any limit probability μ of $\bar{\mu}_t^{N,\ell,k}$, for $N \to \infty$ and then $k \to \infty$, is translation invariant.*

Proof. Let F be a bounded, local function depending on configurations only through $-m, \ldots, m$ for some $m \geqq 0$. Then there exists for each $z \in \mathbb{Z}$ an integer k such that $|m + z| \leqq k$. Since $(\bar{f}_t^{N,\ell,k})_N$ is tight, it suffices to prove that for each z

$$\lim_{k\to\infty} \lim_{N\to\infty} \int (F - \tau_z F)\bar{f}_t^{N,\ell,k} \prod_{|l|\leqq k} \mathrm{d}r_l\, \mathrm{d}p_l = 0 \tag{54}$$

that follows easily from the definition (50).

Lemma 7. *Any limit measure μ is stationary in time with respect to the generator $\mathcal{G} = \mathcal{L} + \gamma\mathcal{S}$, that means for any bounded smooth local function $F(\mathbf{r}, \mathbf{p})$*

$$\int \mathcal{G}F \, \mathrm{d}\mu = 0. \tag{55}$$

Proof. We have to show that for some $k \geqq m$

$$\lim_{k\to\infty} \lim_{N\to\infty} \int \mathcal{G}F \; \bar{f}_t^{N,\ell,k} \prod_{|l|\leqq k} \mathrm{d}r_l \, \mathrm{d}p_l = 0.$$

With (50), the integral is equal to

$$\frac{1}{t}\int_0^t \frac{1}{[N(1-2\ell)]} \sum_{j=N\ell}^{N(1-\ell)} \int \mathcal{G}F f_{s,j}^{N,k} \prod_{|l|\leqq k} \mathrm{d}r_l \, \mathrm{d}p_l \, \mathrm{d}s. \tag{56}$$

Define the space average

$$\bar{F} := \frac{1}{[N(1-2\ell)]} \sum_{j=N\ell}^{N(1-\ell)} \tau_j F,$$

and observe that $\mathcal{G}_N^{\tau(t)} \bar{F} = \mathcal{G}\bar{F}$, then we can rewrite (56) as

$$\begin{aligned} \frac{1}{Nt}\int_0^t \int (N\mathcal{G}_N^{\tau(s)}\bar{F}) \; f_s^N \, \mathrm{d}\mathbf{r} \, \mathrm{d}\mathbf{p} \, \mathrm{d}s &= \frac{1}{tN}\int_0^t \int \bar{F}\partial_s f_s^N \, \mathrm{d}\mathbf{r} \, \mathrm{d}\mathbf{p} \, \mathrm{d}s \\ &= \frac{1}{tN}\left\{ \int \bar{F} f_t^N \, \mathrm{d}\mathbf{r} \, \mathrm{d}\mathbf{p} - \int \bar{F} f_0^N \, \mathrm{d}\mathbf{r} \, \mathrm{d}\mathbf{p} \right\}. \end{aligned}$$

This expression converges to 0 when $N \to \infty$, since $\bar{F}$ is a bounded function.

3.3.4. Entropy Density For some integer $n \geqq 1$, define by Λ^n a box of length $2n+1$ and by Λ_i^n a box of length $2n+1$ and centered at i. Furthermore, let

$$\nu_{(\tau\beta,0,\beta)}^{\infty}(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) := \prod_{i\in\mathbb{Z}} \nu_{(\tau\beta,0,\beta)}(\mathrm{d}r_i, \mathrm{d}p_i)$$

and

$$H_{\Lambda^k}(\mu|\nu_{(\tau\beta,0,\beta)}^{\infty}) := H(\mu|_{\Lambda^k}|\nu_{(\tau\beta,0,\beta)}^{k}).$$

We obtain the following lemma:

Lemma 8. *The limit point μ has finite entropy density, that means there exists a constant $C > 0$ such that for all subsets Λ^k*

$$H_{\Lambda^k}(\mu|\nu_{(\tau\beta,0,\beta)}^{\infty}) \leqq C|\Lambda^k|.$$

Proof. By convexity of the relative entropy, we have

$$\begin{aligned} & H\left(\bar{\mu}_t^{N,\ell,k} \,\middle|\, \nu^k_{(\tau\beta,0,\beta)}\right) \\ & \leqq \frac{1}{[N(1-2\ell)]} \sum_{j=N\ell}^{N(1-\ell)} H\left(\frac{1}{t}\int_0^t f^{N,k}_{s,j}\, \mathrm{d}r_{-k}\, \mathrm{d}p_{-k} \ldots \mathrm{d}r_k\, \mathrm{d}p_k\, \mathrm{d}s \,\middle|\, \nu^k_{(\tau\beta,0,\beta)}\right) \\ & = \frac{1}{[N(1-2\ell)]} \sum_{j=N\ell}^{N(1-\ell)} H_{\Lambda^k_j}\left(\bar{\mu}_t^N \,\middle|\, \nu^N_{(\tau\beta,0,\beta)}\right) \end{aligned} \tag{57}$$

where $\bar{\mu}_t^N := \bar{f}_t^N \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{p}$ with

$$\bar{f}_t^N := \frac{1}{t}\int_0^t f_s^N\, \mathrm{d}s. \tag{58}$$

Relative entropy is superadditive in the following sense (see for example [1]): let $(\Lambda_i)_{i\in I\subset\mathbb{N}}$ be a family of disjoint subsets of $\mathbb{Z}$. Then

$$H_{\bigcup_{i\in I}\Lambda_i}\left(\bar{\mu}_t^N \,\middle|\, \nu^N_{(\tau\beta,0,\beta)}\right) \geqq \sum_{i\in I} H_{\Lambda_i}\left(\bar{\mu}_t^N \,\middle|\, \nu^N_{(\tau\beta,0,\beta)}\right).$$

The sum in (57) can be rearranged in $2k+1$ sums of sums over disjoint blocks, then applying the superadditivity (57) is bounded by

$$\frac{(2k+1)}{[N(1-2\ell)]} H\left(\bar{\mu}_t^N \,\middle|\, \nu^N_{(\tau\beta,0,\beta)}\right).$$

We will prove in Lemma 9 that there exists a finite constant C independent of N, such that

$$H\left(\bar{\mu}_t^N \,\middle|\, \nu^N_{(\tau\beta,0,\beta)}\right) \leqq CN. \tag{59}$$

By Lemma 5 the sequence $(\bar{\mu}_t^{N,\ell,k})_N$ is tight. Since by Lemmata 6 and 7 each limit point μ of $(\bar{\mu}_t^{N,\ell,k})_N$ is translation invariant and stationary, we can conclude the proof by the lower semi continuity of the relative entropy.

To complete the proof of Lemma 8 it remains to show (59).

Lemma 9. *If*

$$H\left(f_0^N \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p} \mid \nu^N_{(\tau\beta,0,\beta)}\right) \leqq C_1 N$$

for some uniform constant $C_1 > 0$*, then for any* $N \in \mathbb{N}$ *there exists a constant* $C_2 > 0$ *such that*

$$H\left(\bar{\mu}_t^N \mid \nu^N_{(\tau\beta,0,\beta)}\right) \leqq C_2 N,$$

where $\bar{\mu}_t^N$ *is defined by* (58).

Proof. Recall from Lemma 1 that the relative entropy with respect to Lebesgue measure is nonincreasing in time since the Lebesgue measure is stationary with respect to the generator $\mathcal{G}_N^{\tau(t)}$ (for any t). Therefore, as in the proof of Lemma 1 we can write

$$H\left(f_t^N \mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p} \,|\, g^N_{(\tau\beta,0,\beta)}\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}\right) - H\left(f_0^N \mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p} \,|\, g^N_{(\tau\beta,0,\beta)}\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}\right)$$
$$\leqq -\int \log g^N_{(\tau\beta,0,\beta)} f_t^N \mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p} + \int \log g^N_{(\tau\beta,0,\beta)} f_0^N \mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}.$$

The last line is then equal to

$$= -\int_0^t \int f_s^N N G_N^{\tau(s)} \log g^N_{(\tau\beta,0,\beta)}\ \mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}\,\mathrm{d}s$$
$$= -\beta N \int_0^t \tau(s) \int f_s^N p_N\ \mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}\,\mathrm{d}s.$$

Since the last line is equal to the expectation of $\beta \sum_{j=1}^N (e_j(t) - e_j(0))$, by Lemma 10 it is bounded by CN for some constant C.

Hence, by convexity of $H(\cdot|\cdot)$,

$$H\left(\bar{\mu}_t^N \mid \nu^N_{(\tau\beta,0,\beta)}\right) \leqq (C_1 + C)N.$$

3.4. Energy Bound

We prove here a deterministic bound on the total energy inside the system, independent of the realizations of the noise of the dynamics.

Lemma 10. *If the initial configuration satisfy*

$$\sum_{j=1}^N e_j(0) \leqq CN$$

then there exists a constant $\tilde{C}(t)$ *independent of* N *such that*

$$\sum_{j=1}^N e_j(t) \leqq \tilde{C}(t)N \tag{60}$$

Proof. Define

$$F_N(t) = \sum_{j=1}^N e_j(t) - \tau(t)q_N(t) = \sum_{j=1}^N \left(e_j(t) - \tau(t)r_j(t)\right).$$

Computing the time evolution of this function we have

$$F_N(t) = F_N(0) - \int_0^t \tau'(s)q_N(s)\,\mathrm{d}s. \tag{61}$$

Consequently

$$\sum_{j=1}^{N} e_j(t) = \tau(t)q_N(t) - \tau(0)q_N(0) + \sum_{j=1}^{N} e_j(0) - \int_0^t \tau'(s)q_N(s)\,\mathrm{d}s. \quad (62)$$

By condition (1), we have that

$$\begin{aligned} |q_N| &\leqq \sum_j |r_j| \leqq \sqrt{N}\left(\sum_j |r_j|^2\right)^{1/2} \\ &\leqq C_-^{1/2}\sqrt{N}\left(\sum_j (1+V(r_j))\right)^{1/2} \\ &\leqq C_-^{1/2}\sqrt{N}\left(\sum_j (1+e_j)\right)^{1/2}. \end{aligned}$$

Then we can estimate

$$\begin{aligned} \left|\int_0^t \tau'(s)q_N(s)\,\mathrm{d}s\right| &\leqq \|\tau'\|_\infty \int_0^t |q_N(s)|\,\mathrm{d}s \\ &\leqq \|\tau'\|_\infty C_-^{1/2}\sqrt{N}\int_0^t \left(\sum_{j=1}^{N}(1+e_j(s))\right)^{1/2} \mathrm{d}s. \end{aligned}$$

Defining $\bar{e}_N(t) = \frac{1}{N}\sum_{j=1}^{N} e_j(t)$ we have then

$$\begin{aligned} \bar{e}_N(t) \leqq\ & C_-^{1/2}\|\tau\|_\infty\left(\sqrt{\bar{e}_N(t)+1} + \sqrt{\bar{e}_N(0)+1}\right) \\ &+\bar{e}_N(0) + \|\tau'\|_\infty C_-^{1/2}\int_0^t \sqrt{\bar{e}_N(s)+1}\,\mathrm{d}s, \end{aligned}$$

which implies (60).

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