Arch. Rational Mech. Anal. 213 (2014) 269–286
Digital Object Identifier (DOI) 10.1007/s00205-014-0734-0

A General Class of Free Boundary Problems for Fully Nonlinear Elliptic Equations

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Communicated by F. Lin

Abstract

In this paper we study the fully nonlinear free boundary problem

$$\begin{cases} F(D^2u) = 1 & \text{almost everywhere in } B_1 \cap \Omega \\ |D^2u| \leqq K & \text{almost everywhere in } B_1 \setminus \Omega, \end{cases}$$

where $K > 0$, and Ω is an unknown open set. Our main result is the optimal regularity for solutions to this problem: namely, we prove that $W^{2,n}$ solutions are locally $C^{1,1}$ inside B_1. Under the extra condition that $\Omega \supset \{Du \neq 0\}$ and a uniform thickness assumption on the coincidence set $\{Du = 0\}$, we also show local regularity for the free boundary $\partial\Omega \cap B_1$.

1. Introduction and Main Result

1.1. Background

Since the seminal work of Caffarelli [2] on the analysis of free boundaries in the obstacle problem, many new techniques and tools have been developed to treat similar types of free boundary problems. The linear theory, that is, when the operator is the Laplacian, has been completely resolved in [7,16] for the Lipschitz right hand side f and when the equation is satisfied outside the set where u vanishes (this corresponds to the obstacle problem):

$$\Delta u = f\chi_{\{u \neq 0\}} \quad \text{in } B_1. \tag{1.1}$$

A. Figalli was partially supported by NSF Grant DMS-0969962. H. Shahgholian was partially supported by Swedish Research Council. A. Figalli acknowledges the Göran Gustafsson Foundation for his visiting appointment to KTH.

Passing below the Lipschitz threshold was a challenging task, as the previous techniques were using monotonicity formulas which failed when $f \in C^\alpha$. The main difficulty has been to prove the $C^{1,1}$-regularity of solutions. On the other hand, the regularity of the free boundary for the Laplacian case was still feasible (even in low-regularity cases) due to the fact that after blow-up the right hand side becomes a constant, and hence the monotonicity tool applies again. (We refer to the above reference for more details.)

A generalization of the problem towards a fully nonlinear operator $F(D^2u) = \chi_{\{u\neq 0\}}$ for the signed-problem (that is, $u \geqq 0$) was completely done by LEE [13] and later partial results were obtained by LEE–SHAHGHOLIAN [14] in the case of the no-sign obstacle problem. Here, two challenging problems were left: (i) $C^{1,1}$-regularity of u; (ii) Classification of global solutions.

Recently, using the harmonic analysis technique, ANDERSSON–LINDGREN–SHAHGHOLIAN [1] could prove a complete result for the Laplacian case, with f satisfying a Dini-condition. Actually their argument shows that if the elliptic equation $\Delta v = f$ admits a $C^{1,1}$-solution in B_1, then the corresponding free boundary problem also admits a $C^{1,1}$-solution. From here, the free boundary regularity follows as in the classical case. The heart of the matter in [1] lies in their Proposition 1 (due to John Andersson) which is a dichotomy between the growth of the solution and the decay of the volume of the coincidence set. Indeed, one can show that if (close to a free boundary point) the growth of the solution is not quadratic, then the volume of the complement set $B_r(x^0) \setminus \Omega$ decays fast enough to make the potential of this set twice differentiable at the origin. From this fact, they can then achieve the optimal growth.

In [1] the authors strongly relied on the linearity of the equation to consider projections of the solution onto the space of second order harmonic polynomials. Also, the linearity of the equation plays a crucial role in several of their estimates. Here, we introduce a suitable "fully nonlinear version" of this projection operation, and we are able to circumvent the difficulties coming from the nonlinear structure of the problem to prove $C^{1,1}$ regularity of the solution. Using this result, we can also show C^1-regularity of the free boundary under uniform thickness assumptions on the "coincidence set", which proves, in particular, that Lipschitz free boundaries are smooth. Nevertheless, a complete regularity of the free boundary still remains open due to a lack of new techniques to classify global solutions.

1.2. Setting of the Problem

Our aim here is to provide an optimal regularity result for solutions to a very general class of free boundary problems which include both the obstacle problem (that is, the right hand side is given by $\chi_{\{u\neq 0\}}$) and the more general free boundary problems studied in [8] (where the right hand side is of the form $\chi_{\{\nabla u\neq 0\}}$).

To include these examples in a unique general framework, we make the weakest possible assumption on the structure of the equation: we suppose that u solves a fully nonlinear equation inside an open set Ω, and in the complement of Ω we only assume that D^2u is bounded.

Notice that, in the above mentioned problems, the first step in the regularity theory is to show that viscosity solutions are $W^{2,p}$ for any $p < \infty$ (this is a relatively "soft" part), and then one wants to prove that actually solutions are $C^{1,1}$.

Since the first step is already pretty well understood [8,10,15], here we focus on the second one. Hence, we assume that $u : B_1 \to \mathbb{R}$ is a $W^{2,n}$ function satisfying

$$\begin{cases} F(D^2u) = 1 & \text{almost everywhere in } B_1 \cap \Omega \\ |D^2u| \leqq K & \text{almost everywhere in } B_1 \setminus \Omega, \end{cases} \tag{1.2}$$

where $K > 0$, and $\Omega \subset \mathbb{R}^n$ is some unknown open set. Since D^2u is bounded in the complement of Ω, we see that $F(D^2u)$ is bounded inside the whole B_1, therefore u is a so-called "strong L^n solution" to a fully nonlinear equation with bounded right hand side [5]. We refer to [4] as a basic reference to fully nonlinear equations and viscosity methods, and to [8,10,15] for several existence results for strong solutions to free boundary type problems.

Let us observe that, if $u \in W^{2,n}$, then $D^2u = 0$ almost everywhere inside both sets $\{u = 0\}$ and $\{\nabla u = 0\}$, so (1.2) includes as special cases both $F(D^2u) = \chi_{\{u\neq 0\}}$ and $F(D^2u) = \chi_{\{\nabla u\neq 0\}}$.

We assume that:

(H0) $F(0) = 0$.
(H1) F is uniformly elliptic with ellipticity constants $\lambda_0, \lambda_1 > 0$, that is

$$\mathscr{P}^-(Q-P) \leqq F(Q) - F(P) \leqq \mathscr{P}^+(Q-P)$$

for any P, Q symmetric, where $\mathscr{P}^-$ and $\mathscr{P}^+$ are the extremal Pucci operators:

$$\mathscr{P}^-(M) := \inf_{\lambda_0 \,\mathrm{Id} \leqq N \leqq \lambda_1 \,\mathrm{Id}} \operatorname{trace}(NM), \quad \mathscr{P}^+(M) := \sup_{\lambda_0 \,\mathrm{Id} \leqq N \leqq \lambda_1 \,\mathrm{Id}} \operatorname{trace}(NM).$$

(H2) F is either convex or concave.

Under assumptions (H0)–(H2) above, strong L^n solutions are also viscosity solutions [5], so classical regularity results for fully nonlinear equations [3] show that $u \in W^{2,p}_{\mathrm{loc}}(B_1)$ for all $p < \infty$. In addition, by [6], D^2u belongs to BMO.

Our primary aim here is to prove uniform optimal $C^{1,1}$-regularity for u. This is a key step in order to be able to perform an analysis of the free boundary.

Remark 1.1. In order to keep the presentation simple and to highlight the main ideas in the proof, we decided to restrict ourselves to the "clean" case $F(D^2u) = 1$ inside Ω. However, under suitable regularity assumptions on F and f, we expect our arguments to work for the general class of equations $F(x, u, \nabla u, D^2u) = f$ inside Ω.

1.3. Main Results

Our main result in this paper concerns optimal regularity of solutions to (1.2). In order to simplify the notation and avoid dependence of constants on $\|u\|_{L^\infty(B_1)}$, we call a constant *universal* if it depends on the dimension, K, the ellipticity constants of F, and $\|u\|_{L^\infty(B_1)}$ only.

Theorem 1.2. (Interior $C^{1,1}$ regularity) *Let* $u : B_1 \to \mathbb{R}$ *be a* $W^{2,n}$ *solution of* (1.2), *and assume that* F *satisfies* (H0)–(H2). *Then there exists a universal constant* $\bar{C} > 0$ *such that*

$$|D^2 u| \leqq \bar{C}, \quad in\ B_{1/2}.$$

In order to investigate the regularity of the free boundary, we need to restrict ourselves to a more specific situation than the one in (1.2). Indeed, as discussed in Section 3, even if we assume that $D^2u = 0$ outside Ω, non-degeneracy of solutions (a key ingredient to study the regularity of the free boundary) may fail. As we will see, a sufficient condition to show non-degeneracy of solutions is to assume that $\Omega \supset \{\nabla u \neq 0\}$. Still, once non-degeneracy is proved, the lack of strong tools (available in the Laplacian case) such as monotonicity formulas makes the regularity of the free boundary a very challenging issue.

To state our result we need to introduce the concept of "thickness". Set $\Lambda := B_1 \setminus \Omega$, and for any set E let $\mathrm{MD}(E)$ denote the smallest possible distance between two parallel hyperplanes containing E. Then, we define the *thickness* of the set Λ in $B_r(x)$ as

$$\delta_r(u, x) := \frac{\mathrm{MD}(\Lambda \cap B_r(x))}{r}.$$

We notice that δ_r enjoys the scaling property $\delta_1(u_r, 0) = \delta_r(u, x)$, where $u_r(y) = u(x + ry)/r^2$.

Our result provides regularity for the free boundary under a uniform thickness condition. As a corollary of our result, we deduce that Lipschitz free boundaries are C^1, and hence smooth [11].

Theorem 1.3. (Free boundary regularity) *Let* $u : B_1 \to \mathbb{R}$ *be a* $W^{2,n}$ *solution of* (1.2). *Assume that* F *is convex and satisfies* (H0)–(H1), *and that one of the following conditions holds:*

- $\Omega \supset \{\nabla u \neq 0\}$ *and* F *is of class* C^1;
- $\Omega \supset \{u \neq 0\}$.

Suppose further that there exists $\varepsilon > 0$ *such that*

$$\delta_r(u, z) > \varepsilon \quad \forall r < 1/4,\ z \in \partial\Omega \cap B_r(0).$$

Then $\partial\Omega \cap B_{r_0}(0)$ *is a* C^1*-graph, where* r_0 *depends only on* ε *and the data.*

The important difference between this theorem and previous results of this form is that here we assume thickness of Λ in a uniform neighborhood of the origin rather than at the origin only. The reason for this fact is that this allows us to classify global solutions arising as blow-ups around such "thick points". Once this is done, then local regularity follows in pretty standard way.

The paper is organized as follows: In Section 2 we prove Theorem 1.2. Then in Section 3 we investigate the non-degeneracy of solutions, and classify global solutions under a suitable thickness assumption. In Section 4 we show directional monotonicity for local solutions, that gives a Lipschitz regularity for the free boundary. This Lipschitz regularity can then be improved to C^1. The details of such an analysis are by-now classical and only indicated briefly in Section 5.

2. Proof of Theorem 1.2

2.1. Technical Preliminaries

In this section we shall gather some technical tools that are interesting in their own right, and may even be applied to other problems. Throughout all of the section, we assume that F satisfies (H0)–(H2).

With no loss of generality, here we will perform all our estimates at the origin, and later on we will apply such estimates at all points where u is twice differentiable, showing that D^2u is universally bounded at all such points. This will give a complete optimal regularity for u; see Section 2.2.

For all $r < 1/4$, we define

$$A_r := \{x : \; rx \in B_r \setminus \Omega\} = \frac{B_r \setminus \Omega}{r} \subset B_1. \tag{2.1}$$

We recall that, by [6, Theorem A] (see also [9, Appendix] for a simpler proof of this estimate in the more general context of parabolic equations),

$$\|D^2u\|_{BMO(B_{3/4})} \leqq C$$

for some universal constant C, which implies in particular that

$$\sup_{r\in(0,1/4)} \fint_{B_r(0)} |D^2u(y) - (D^2u)_{r,0}|^2 \,\mathrm{d}y \leqq C, \quad (D^2u)_{r,0} := \fint_{B_r(0)} D^2u(y)\,\mathrm{d}y. \tag{2.2}$$

Here we first show that in (2.2) we can replace $(D^2u)_{r,0}$ with a matrix in $F^{-1}(1)$ (a direct proof of this result is also given in [9, Appendix]).

Lemma 2.1. *There exists* $C > 0$ *universal such that*

$$\min_{F(P)=1} \fint_{B_r(0)} |D^2u(y) - P|^2 \,dy \leqq C \quad \forall r \in (0, 1/4). \tag{2.3}$$

Proof. Set $Q_r := (D^2u)_{r,0}$. Since $F(D^2u)$ is bounded inside B_1 and F is λ_1-Lipschitz (this is a consequence of (H1)), using (2.2) we get

$$\begin{aligned}|F(Q_r)| &= \left| \fint_{B_r(0)} F\left(Q_r - D^2u(y) + D^2u(y)\right) \mathrm{d}y \right| \\ &\leqq \fint_{B_r(0)} \left(\left| F(D^2u(y)) \right| + \lambda_1 \left| Q_r - D^2u(y) \right| \right) \mathrm{d}y \\ &\leqq C\left(1 + \sqrt{\fint_{B_r(0)} |D^2u(y) - (D^2u)_{r,0}|^2 \,\mathrm{d}y}\right) \leqq C.\end{aligned}$$

Thus we have proved that $F(Q_r)$ is universally bounded. By ellipticity and continuity [see (H1)] we easily deduce that there exists a universally bounded constant $\beta \in \mathbb{R}$ such that $F(Q_r + \beta\,\mathrm{Id}) = 1$. Since

$$\fint_{B_r(0)} |D^2u(y) - (Q_r + \beta\,\mathrm{Id})|^2 \,\mathrm{d}y \leqq 2\fint_{B_r(0)} |D^2u(y) - Q_r|^2 \,\mathrm{d}y + 2\beta^2,$$

this proves the result. □

For any $r \in (0, 1/4)$, let $P_r \in F^{-1}(1)$ denote a minimizer in (2.3) (although P_r may not be unique, we just choose one).

We first show that P_r cannot change too much on a dyadic scale:

Lemma 2.2. *There exists a universal constant C_0 such that*

$$|P_{2r} - P_r| \leqq C_0 \quad \forall\, r \in (0, 1/8).$$

Proof. By the estimate

$$\fint_{B_r(0)} |D^2u(y) - P_r|^2 \,\mathrm{d}y + \fint_{B_{2r}(0)} |D^2u(y) - P_{2r}|^2 \,\mathrm{d}y \leqq C$$

[see (2.3)], we obtain

$$\begin{aligned}|P_{2r} - P_r|^2 &\leqq 2\fint_{B_r(0)} |D^2u(y) - P_r|^2 \,\mathrm{d}y + 2\fint_{B_r(0)} |D^2u(y) - P_{2r}|^2 \,\mathrm{d}y \\ &\leqq 2\fint_{B_r(0)} |D^2u(y) - P_r|^2 \,\mathrm{d}y + 2^{n+1}\fint_{B_{2r}(0)} |D^2u(y) - P_{2r}|^2 \,\mathrm{d}y \leqq C,\end{aligned}$$

which proves the result. □

The following result shows that if P_r is bounded, then (up to a linear function) so is $|u|/r^2$ inside B_r.

Lemma 2.3. *Assume that $u(0) = \nabla u(0) = 0$. Then there exists a universal constant C_1 such that*

$$\sup_{B_r(0)} \left| u - \frac{1}{2}\langle P_r y, y\rangle \right| \leqq C_1 r^2 \quad \forall\, r \in (0, 1/8). \tag{2.4}$$

In particular

$$\sup_{B_r(0)} |u| \leqq (C_1 + |P_r|) r^2. \tag{2.5}$$

Proof. By Lemma 2.1 we know that

$$\left\| D^2 \frac{u(ry)}{r^2} - P_r \right\|_{L^2(B_1)} \leqq C,$$

that is the function $\bar{u}_r(y) := u(ry)/r^2 - \frac{1}{2}\langle P_r y, y\rangle$ satisfies

$$\left\| D^2 \bar{u}_r \right\|_{L^2(B_1)} \leqq C.$$

By Poincaré inequality, this implies that there exists a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|\bar{u}_r - \ell\|_{L^2(B_{5/6})} \leqq C.$$

Let us define $\hat{u} := \bar{u}_r - \ell$. Since $F(P_r + D^2\hat{u}(y)) = F(D^2 u(ry)) \in L^\infty(B_1)$ and $F(P_r) = 1$, by [4, Theorem 4.8(2)] applied to the subsolutions $\hat{u}_+$ and $\hat{u}_-$ of the elliptic operators $Q \mapsto F(P_r + Q) - 1$ and $Q \mapsto 1 - F(P_r - Q)$ respectively, we obtain that

$$\|\hat{u}\|_{L^\infty(3/4)} \leqq C.$$

Then, by interior $C^{1,\alpha}$ estimates (see for instance [4, Chapter 5.3] and [3, Theorem 2]) we deduce that

$$\|\hat{u}\|_{C^{1,\alpha}(B_{1/2})} \leqq C,$$

so in particular (by the definition of $\hat{u}$)

$$|\bar{u}_r(0) - \ell(0)| + |\nabla \bar{u}_r(0) - \nabla \ell(0)| \leqq C.$$

Since by assumption $\bar{u}_r(0) = \nabla \bar{u}_r(0) = 0$, this implies that the linear function ℓ is uniformly bounded inside $B_{1/2}$, hence

$$\sup_{B_{r/2}(0)} \left| \frac{u - \frac{1}{2}\langle P_r y, y\rangle}{r^2} \right| = \|\bar{u}_r\|_{L^\infty(B_{1/2})} \leqq \|\hat{u}\|_{L^\infty(B_{1/2})} + \|\ell\|_{L^\infty(B_{1/2})} \leqq C. \tag{2.6}$$

To prove that we can actually replace $r/2$ with r in the equation above [see (2.4)], we first apply (2.6) with $2r$ in place of r to get

$$\sup_{B_r(0)} \left| \frac{u - \frac{1}{2}\langle P_{2r} y, y\rangle}{(2r)^2} \right| \leqq C,$$

and then we conclude by Lemma 2.2. □

We now prove that if $|P_r|$ is sufficiently large then the measure of A_r [see (2.1)] has to decay in a geometric fashion.

Proposition 2.4. *There exists $M > 0$ universal such that, for any $r \in (0, 1/8)$, if $|P_r| \geqq M$ then*

$$|A_{r/2}| \leqq \frac{|A_r|}{2^n}.$$

Proof. Set $u_r(y) := u(ry)/r^2$, and let

$$u_r(y) = \frac{1}{2}\langle P_r y, y\rangle + v_r(y) + w_r(y), \tag{2.7}$$

where v_r is defined as the solution of

$$\begin{cases} F(P_r + D^2 v_r) - 1 = 0 & \text{in } B_1, \\ v_r = u_r(y) - \frac{1}{2}\langle P_r y, y\rangle & \text{on } \partial B_1, \end{cases} \tag{2.8}$$

and by definition $w_r := u_r - \frac{1}{2}\langle P_r y, y\rangle - v_r$.

Set $f_r := F(D^2 u_r) \in L^\infty(B_1)$ [recall that $|D^2 u_r| \leqq K$ almost everywhere inside A_r, see (1.2)]. Notice that, since $f_r = 1$ outside A_r,

$$F(D^2 u_r) - F(P_r + D^2 v_r) = (f_r - 1)\chi_{A_r},$$

so it follows by (H1) that w_r solves

$$\begin{cases} \mathscr{P}^-(D^2 w_r) \leqq (f_r - 1)\chi_{A_r} \leqq \mathscr{P}^+(D^2 w_r) & \text{in } B_1, \\ w_r = 0 & \text{on } \partial B_1. \end{cases} \tag{2.9}$$

Hence, since f_r is universally bounded, we can apply the ABP estimate [4, Chapter 3] to deduce that

$$\sup_{B_1} |w_r| \leqq C\|\chi_{A_r}\|_{L^n(B_1(0))} = C|A_r|^{1/n}. \tag{2.10}$$

Also, since $F(P_r) = 1$ and v_r is universally bounded on ∂B_1 [see (2.4)], by Evans–Krylov's theorem [4, Chapter 6] applied to (2.8) we have

$$\|D^2 v_r\|_{C^{0,\alpha}(B_{3/4}(0))} \leqq C. \tag{2.11}$$

This implies that w_r solves the fully nonlinear equation with Hölder coefficients

$$G(x, D^2 w_r) = (f_r - 1)\chi_{A_r} \quad \text{in } B_{3/4}, \quad G(x, Q) := F(P_r + D^2 v_r(x) + Q) - 1.$$

Since $G(x, 0) = 0$, we can apply [3, Theorem 1] with $p = 2n$, and using (2.10) we obtain

$$\int_{B_{1/2}(0)} |D^2 w_r|^{2n} \leqq C\left(\|w_r\|_{L^\infty(B_{3/4})} + \|\chi_{A_r}\|_{L^{2n}(B_{3/4}(0))}\right)^{2n} \leqq C\,|A_r| \tag{2.12}$$

(recall that $|A_r| \leqq |B_1|$).

We are now ready to conclude the proof: since $|D^2 u_r| \leqq K$ almost everywhere inside A_r (by (1.2)), recalling (2.7) we have

$$\int_{A_r \cap B_{1/2}(0)} |D^2 v_r + D^2 w_r + P_r|^{2n} = \int_{A_r \cap B_{1/2}(0)} |D^2 u_r|^{2n} \leqq K^{2n}|A_r|.$$

Therefore, by (2.11) and (2.12),

$$\begin{aligned}|A_r \cap B_{1/2}(0)|\,|P_r|^{2n} &= \int_{A_r\cap B_{1/2}(0)} |P_r|^{2n} \\ &\leqq 3^{2n}\left(\int_{A_r\cap B_{1/2}(0)} |D^2 v_r|^{2n} + \int_{A_r\cap B_{1/2}(0)} |D^2 w_r|^{2n} + K^{2n}|A_r|\right) \\ &\leqq 3^{2n}\left(|A_r \cap B_{1/2}(0)|\,\|D^2 v_r\|^{2n}_{L^\infty(B_{1/2}(0))} + \int_{B_{1/2}(0)} |D^2 w_r|^{2n} + K^{2n}|A_r|\right) \\ &\leqq C\,|A_r \cap B_{1/2}(0)| + C\,|A_r|.\end{aligned}$$

Hence, if $|P_r|$ is sufficiently large we obtain

$$|A_r \cap B_{1/2}(0)|\,|P_r|^{2n} \leqq C|A_r| \leqq \frac{1}{4^n}|P_r|^{2n}|A_r|.$$

Since $|A_{r/2}| = 2^n|A_r \cap B_{1/2}(0)|$, this gives the desired result. □

2.2. *Proof of Theorem 1.2*

Since by assumption $|D^2u| \leqq K$ almost everywhere outside Ω, it suffices to prove that $|D^2u(x^0)| \leqq C$ for almost everywhere $x^0 \in \bar{\Omega} \cap B_{1/2}$, for some $C > 0$ universal.

Fix $x^0 \in \bar{\Omega} \cap B_{1/2}$ such that u is twice differentiable at x^0, and x^0 a Lebesgue point for D^2u (these properties hold at almost every point). With no loss of generality we can assume that $x^0 = 0$ and that $u(0) = \nabla u(0) = 0$.

Let $M > 0$ as in Proposition 2.4. We distinguish two cases:

(i) $\liminf_{k\to\infty} |P_{2^{-k}}| \leqq 3M$.
(ii) $\liminf_{k\to 0} |P_{2^{-k}}| \geqq 3M$.

Using (2.5) and the fact that u is twice differentiable at 0, in case (i) we immediately obtain

$$|D^2u(0)| \leqq \liminf_{k\to\infty} \sup_{B_{2^{-k}}(0)} \frac{2|u|}{2^{-2k}} \leqq 2(C_1 + 3M).$$

In case (ii), let us define

$$k_0 := \inf\left\{k \geqq 2 \,:\, |P_{2^{-j}}| \geqq 2M \quad \forall\, j \geqq k\right\}.$$

By the assumption that $\liminf_{k\to 0} |P_{2^{-k}}| \geqq 3M$, we see that $k_0 < \infty$. In addition, since $P_{1/4}$ is universally bounded, to enlarge M we can assume that $k_0 \geqq 3$.

Let us observe that, since by definition $|P_{2^{-k_0-1}}| \leqq 2M$, by Lemma 2.2 we obtain

$$|P_{2^{-k_0}}| \leqq 2M + C_0. \tag{2.13}$$

We now define the function $\bar{u}_0 := 4^{k^0} u(2^{-k_0}x) - \frac{1}{2}\langle P_{2^{-k_0}}x, x\rangle$. Observe that $\bar{u}_0$ is a solution of the fully nonlinear equation

$$G(D^2\bar{u}_0) = (f_{2^{-k_0}} - 1)\chi_{A_{2^{-k_0}}} \quad \text{in } B_1, \tag{2.14}$$

where $G(Q) := F(P_{2^{-k_0}} + Q) - 1$ and $f_{2^{-k_0}}(x) := F(D^2u(2^{-k_0}x))$ is universally bounded. In addition, since $|P_{2^{-k}}| \geqq 2M$ for all $k \geqq k_0$, Proposition 2.4 gives

$$|A_{2^{-k_0+j}}| \leqq 2^{-jn}|A_{2^{-k_0}}| \leqq 2^{-jn}|B_1| \quad \forall j \geqq 0,$$

from which we deduce that $(f_{2^{-k_0}} - 1)\chi_{A_{2^{-k_0}}}$ decays in L^n geometrically fast:

$$⨍_{B_r} \left|(f_{2^{-k_0}} - 1)\chi_{A_{2^{-k_0}}}\right|^n \leqq C ⨍_{B_r} |\chi_{A_{2^{-k_0}}}| \leqq Cr^n \quad \forall r \in (0,1).$$

Hence, since $G(0) = 0$, we can apply [3, Theorem 3] to deduce that $\bar{u}_0$ is $C^{2,\alpha}$ at the origin, with universal bounds. In particular this implies

$$|D^2\bar{u}_0(0)| \leqq C.$$

Since $D^2u(0) = D^2\bar{u}_0(0) + P_{2^{-k_0}}$ and $P_{2^{-k_0}}$ is universally bounded [see (2.13)], this concludes the proof.

3. Non-degeneracy and Global Solutions

3.1. Local Non-degeneracy

Non-degeneracy is a corner-stone for proving smoothness of the free boundary. This property says that the function grows quadratically (and not slower) away from the free boundary points, that is, $\sup_{B_r(x^0)} |u - u(x^0) - (x - x^0)\cdot\nabla u(x^0)| \gtrsim r^2$ for any $x^0 \in \overline{\Omega}$. However, while in the case $\Delta u = \chi_{\{u\neq 0\}}$ or $\Delta u = \chi_{\{\nabla u\neq 0\}}$ non-degeneracy is known to hold true, in the case $\Delta u = \chi_{\{D^2u\neq 0\}}$ non-degeneracy may fail.

To see this, one can consider the one dimensional problem $u'' = \chi_{\{u''\neq 0\}}$. Every solution is obtained by linear functions and quadratic polynomial glued together in a $C^{1,1}$ way. In particular, if $\{I_j\}_{j\ in\mathbb{N}}$ is a countable family of disjoint intervals, the function

$$u(t) := \int_0^t \int_0^s \chi_\Omega(\tau)\, \mathrm{d}\tau\, \mathrm{d}s, \quad \Omega := \cup_j I_j$$

satisfies $u'' = \chi_\Omega = \chi_{\{u''\neq 0\}}$, and if we choose I_j such that

$$\frac{|\Omega \cap (-r,r)|}{2r} \to 0 \quad \text{as } r \to 0,$$

then it is easy to check that $u(r) = o(r^2)$ as $r \to 0$.

A possible way to rule out the above counterexample may be to consider only points in $\overline{\Omega}$ such that Ω has a uniform density inside $B_r(x^0)$. We will not investigate this direction here. Instead, we show that non-degeneracy holds under the additional assumption that $\Omega \supset \{\nabla u \neq 0\}$ (which is sufficient to include into our analysis the cases $F(D^2u) = \chi_{\{u\neq 0\}}$ and $F(D^2u) = \chi_{\{\nabla u\neq 0\}}$).

Lemma 3.1. *Let $u : B_1 \to \mathbb{R}$ be a $W^{2,n}$ solution of* (1.2), *assume that F satisfies* (H0)–(H2), *and that $\Omega \supset \{\nabla u \neq 0\}$. Then, for any $x^0 \in \overline{\Omega} \cap B_{1/2}$,*

$$\max_{\partial B_r(x^0)} u \geqq u(x^0) + \frac{r^2}{2n\lambda_1} \quad \forall r \in (0, 1/4).$$

Proof. By approximation, it suffices to prove the estimate for $x^0 \in \Omega$. In addition, since $D^2u = 0$ almost everywhere inside the set $\{\nabla u = 0\}$, $F(D^2u) = 1$ in $\Omega \cap B_1$, and $F(0) = 0$ [by (H0)], we see that $\{\nabla u = 0\}$ has measure zero inside $\Omega \cap B_1$. This implies that the set $\Omega \cap \{\nabla u \neq 0\} \cap B_1$ is dense inside $\overline{\Omega \cap B_1}$, and so we only need to prove the result when $x^0 \in \Omega \cap \{\nabla u \neq 0\} \cap B_1$.

Let us define the $C^{1,1}$ function (recall that $u \in C^{1,1}$ because of Theorem 1.2)

$$v(x) := u(x) - \frac{|x - x^0|^2}{2n\lambda_1}.$$

By (H1) we see that

$$F(D^2v) = F\big(D^2u - \mathrm{Id}/(n\lambda_1)\big) \geqq F(D^2u) - \mathscr{P}^+\big(\mathrm{Id}/(n\lambda_1)\big) \geqq 0 \quad \text{in } \Omega \cap B_1. \tag{3.1}$$

We claim that

$$\max_{\partial B_r(x^0)} v = \sup_{B_r(x^0)} v. \tag{3.2}$$

Indeed, if there exists an interior maximum point $y \in B_r(x^0)$, then

$$0 = \nabla v(y) = \nabla u(y) - \frac{y - x^0}{n\lambda_1}. \tag{3.3}$$

Since by assumption $x^0 \in \{\nabla u \neq 0\}$ we have $\nabla u(x^0) \neq 0$, so by (3.3) $y \neq x^0$. In particular $\nabla u(y) = \frac{y - x^0}{n\lambda_1} \neq 0$, and thus $y \in \Omega$. Recalling that v is a subsolution for F inside $\Omega \cap B_1$ [see (H0) and (3.1)], by the strong maximum principle v is constant in a neighborhood of y. Thus, the set of maxima of v is both relatively open and closed in $B_r(x_0)$, which implies that v is constant there and (3.2) is trivially satisfied.

Thanks to the claim we obtain

$$\max_{\partial B_r(x^0)} u - \frac{r^2}{2n\lambda_1} = \max_{\partial B_r(x^0)} v \geqq v(x^0) = u(x^0),$$

which proves the result. □

3.2. Classification of Global Solutions

Now that non-degeneracy is proven, we can start considering blow-up solutions and try to classify them. We shall treat the case $\Omega \supset \{\nabla u \neq 0\}$. Our results would work also for the case $\Omega \supset \{D^2u \neq 0\}$ if the assumptions are strengthened in a way that solutions stay stable/invariant in a blow-up regime.

Since we will use the thickness to measure sets, we need some facts about its stability properties: Let us first recall the definition for $\delta_r(u,x)$:

$$\delta_r(u,x) := \frac{\mathrm{MD}(\Lambda \cap B_r(x))}{r}, \quad \Lambda := B_1 \setminus \Omega.$$

We remark that, for polynomial global solutions $P_2 = \sum a_j x_j^2$ (with a_j such that $F(D^2P_2) = 1$), one has

$$\delta_r(P_2, 0) = 0. \tag{3.4}$$

Indeed, the zeros of the gradient of a second degree homogeneous polynomial P_2 always lie on a hyperplane.

The next observation is the stability of $\delta_r(u,x)$ under scaling: more precisely, if $x \in \partial\Omega \cap B_1$ and we rescale u as $u_r(y) := \frac{u(x+ry)-u(x)}{r^2}$ (notice that $\nabla u(x) = 0$ for all $x \in \partial\Omega$), then

$$\delta_r(u,x) = \delta_1(u_r, 0) \tag{3.5}$$

which along with the fact that $\limsup_{r\to 0} \Lambda(u_r) \subset \Lambda(u_0)$ whenever u_r converges to some function u_0 (see [15, Proposition 3.17 (iv)]) gives

$$\limsup_{r\to 0} \delta_r(u, x^0) \leqq \delta_1(u_0, 0). \tag{3.6}$$

Since any limit of u_r will be a global solution of (1.2) [that is, it solves (1.2) in the whole $\mathbb{R}^n$], we are interested in classifying global solutions.

In the next proposition we classify global solution with a "thick free boundary".

Proposition 3.2. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a $W^{2,n}$ solution of* (1.2) *inside $\mathbb{R}^n$, assume that F is convex and satisfies* (H0)–(H1), *and that $\Omega \supset \{\nabla u \neq 0\}$. Assume that there exists $\varepsilon_0 > 0$ such that*

$$\delta_r(u, x^0) \geqq \varepsilon_0 \quad \forall r > 0,\ \forall x^0 \in \partial\Omega. \tag{3.7}$$

Then u is a half-space solution, that is, up to a rotation, $u(x) = \gamma[(x_1)_+]^2/2 + c$, where $\gamma \in (1/\lambda_1, 1/\lambda_0)$ is such that $F(\gamma e_1 \otimes e_1) = 1$ and $c \in \mathbb{R}$.

Proof. We first prove that u is convex. Suppose by contradiction that u is not, and set

$$m := \inf_{z\in\Omega,\, e\in\mathbb{S}^{n-1}} \partial_{ee}u(z) < 0.$$

Observe that, thanks to Theorem 1.2, u is globally $C^{1,1}$ in $\mathbb{R}^n$, so m is finite.

Let us consider sequences $y^j \in \Omega$ and $e^j \in \mathbb{S}^{n-1}$ such that

$$\partial_{e^je^j}u(y^j) \to m \quad \text{as } j \to \infty.$$

Rescale u at y^j with respect to $d_j := \operatorname{dist}(y^j, \partial\Omega)$, that is,

$$u_j(x) := \frac{u(d_j x + y^j) - u(y^j) - d_j \nabla u(y^j) \cdot x}{d_j^2}.$$

Notice that, since $\nabla u = 0$ on $\partial\Omega$, $\nabla u_j = \ell_j$ on $\partial\Omega_j$, where $\Omega_j := (\Omega - y^j)/d_j$ and $\ell_j := -\nabla u(y_j)/d_j \in \mathbb{R}^n$.

Also, $|\ell_j| \leqq C$ (by the $C^{1,1}$ regularity of u), up to a subsequence $\ell_j \to \ell_\infty$. By rotating the system of coordinates, we can assume that (again up to subsequences) $e^j \to e_1$. Then the functions u^j still satisfy (1.2) and they converge to another global solution u_∞ which satisfies $\partial_{11}u_\infty(0) = -m$. Let us observe that, by the convexity of F, $\partial_{11}u_\infty$ is a supersolution of the linear operator $F_{ij}(D^2u_\infty)\partial_{ij}$. Hence, since $\partial_{11}u_\infty(z) \geqq -m$ inside $B_1(0)$, by the strong maximum principle we deduce that $\partial_{11}u_\infty \equiv -m$ inside the connected component containing $B_1(0)$ (call it Ω_∞).

Notice that, by replacing $u_\infty(x)$ with $u_\infty(x) - \ell_\infty \cdot x$, we can assume that $\nabla u_\infty(x) = 0$ on $\partial\Omega_\infty$. Also, since $\partial_{ee}u_\infty(z) \geqq -m$ inside $B_1(0)$ for any $e \in \mathbb{S}^{n-1}$, it follows that e_1 is an eigenvector of D^2u at every point (which corresponds to the smallest eigenvalue). In particular, this implies that $\partial_{1j}u_\infty = 0$ for any $j = 2, \ldots, n$ inside Ω_∞. Hence, integrating u_∞ in the direction e_1 gives

$$u_\infty(x) = P(x) \quad \text{inside } \Omega_\infty, \tag{3.8}$$

where

$$P(x) := -mx_1^2/2 + ax_1 + b(x'), \quad x' = (x_2, \ldots, x_n).$$

We now observe that the set where $\partial_1 P$ vanishes corresponds to the hyperplane $\{x_1 = a/m\}$. Hence, since $\nabla u_\infty = 0$ (in particular $\partial_1 u_\infty = 0$) on $\partial\Omega_\infty$, we deduce that $\partial\Omega_\infty \subset \{x_1 = a/m\}$. We now distinguish two cases:

- If $\partial\Omega_\infty \neq \{x_1 = a/m\}$ then the set Ω_∞ contains $\mathbb{R}^n \setminus \{x_1 = a/m\}$ (since $\partial_1 u_\infty$ cannot vanish anywhere else), and so $F(D^2u_\infty) = 1$ almost everywhere in $\mathbb{R}^n$. Then we apply Evans–Krylov's Theorem [4, Chapter 6] to $u_\infty(Ry)/R^2$ inside B_1 (notice that these functions are uniformly bounded inside B_1 thanks to the global $C^{1,1}$ regularity) to deduce that

$$\sup_{x,z \in B_R} \frac{|D^2u_\infty(x) - D^2u_\infty(z)|}{|x - z|^\alpha} \leqq \frac{C}{R^\alpha}.$$

  Letting $R \to \infty$ we obtain that D^2u_∞ is constant, and so u_∞ is a second order polynomial.
- If $\partial\Omega_\infty = \{x_1 = a/m\}$, since $\nabla u_\infty = 0$ on $\partial\Omega_\infty$ we get that $\nabla_{x'}P = 0$ on the hyperplane $\{x_1 = a/m\}$. Hence b is constant and so

$$u_\infty = -mx_1^2/2 + ax_1 + b \quad \text{inside } \{x_1 > a/m\},$$

  which contradicts (H0) and (H1) (because $F(D^2u_\infty) = 1$ while $D^2u_\infty = -m\mathrm{Id}$ is negative definite).

In conclusion we have proved that if u is not convex, then u_∞ is a second order polynomial. Invoking the thickness assumption (3.7) and the stability properties (3.5)–(3.6) along with (3.4) (notice that the stability properties, although stated in a slightly different context, still hold in this situation), we conclude that u_∞ cannot be a second degree polynomial, and thus a contradiction.

Hence, we have proved that u is convex, which implies that $\{\nabla u = 0\}$ is a convex set (since for a convex function any critical point is a minimum, and the set of minima is convex). Recall that, since $F(D^2u) = 1$ in Ω, we have $|\Omega \setminus \{\nabla u \neq 0\}| = 0$, and by convexity of $\{\nabla u = 0\}$ and the thickness assumption it is easy to see that $\Omega = \{\nabla u \neq 0\}$ (notice that, since $u \in C^{1,1}$, the set $\{\nabla u \neq 0\}$ is open).

We now show that the set $\Lambda(u) = \{\nabla u = 0\}$ is a half-space. For simplicity we may assume the origin is on the free boundary. Consider a blow-down u_∞ obtained as a limit (up to a subsequence) of $u(Ry)/R^2$ as $R \to \infty$. It is not hard to realize that $\Lambda(u_\infty) = \{x \in \Lambda(u) : \ tx \in \Lambda(u)\ \forall t > 0\}$. In other words, the coincidence set for the blow-down is convex, and coincides with the largest cone (with vertex at the origin) in the coincidence set of the function u. Assume by contradiction that $\Lambda(u_\infty)$ is not a half-space. Then, in some suitable system of coordinates,

$$\Lambda(u_\infty) \subset \mathcal{C}_{\theta_0} := \left\{x \in \mathbb{R}^n : \ x = (\rho\cos\theta, \rho\sin\theta, x_3, \dots, x_n),\ \theta_0 \leqq |\theta| \leqq \pi\right\}$$

for some $\theta_0 > \pi/2$. Hence, if we choose $\theta_1 \in (\pi/2, \theta_0)$ and set $\alpha := \pi/\theta_1$, then it is easy to check that, for $\beta > 0$ sufficiently large (the largeness depending only on θ_1 and the ellipticity constants of F), the function

$$v = r^\alpha\left(e^{-\beta\sin(\alpha\theta)} - e^{-\beta}\right)$$

is a positive subsolution for the linear operator $F_{ij}(D^2u)\partial_{ij}$ inside $\mathbb{R}^n \setminus \mathcal{C}_1$ (see for instance [13]), and it vanishes on $\partial\mathcal{C}_{\theta_1}$. Hence, because $\partial_1 u_\infty > 0$ inside $\mathbb{R}^n \setminus \mathcal{C}_{\theta_0}$ (by convexity of u_∞) and $\theta_0 > \theta_1$, by the comparison principle we deduce that

$$v \leqq \partial_1 u_\infty.$$

However, since $\alpha < 1$, this contradicts the Lipschitz regularity of $\partial_1 u_\infty$ at the origin.

So $\Lambda(u_\infty)$ is a half space, and since $\Lambda(u_\infty) \subset \Lambda(u)$ and the latter set is convex, we deduce that $\Lambda(u)$ is a half-space as well.

Finally, to conclude the proof, we apply Krylov's boundary $C^{2,\alpha}$ estimates [12] (see also the recent results in [17]) inside the half-ball $B_1 \setminus \Lambda(u)$ to the uniformly bounded functions $u(Ry)/R^2$ to get

$$\sup_{x,z\in B_R\setminus\Lambda(u)} \frac{|D^2u(x) - D^2u(z)|}{|x-z|^\alpha} \leqq \frac{C}{R^\alpha}.$$

Letting $R \to \infty$ we obtain that D^2u is constant, and so u is a second order polynomial inside the half-space $\mathbb{R}^n \setminus \Lambda(u)$. Since $\nabla u = 0$ on the hyperplane $\partial\Lambda(u)$, it is immediate to check that u has to be a half-space solution. □

4. Local Solutions and Directional Monotonicity

In this section we shall prove a directional monotonicity for solutions to our equations. In the next section we will use Lemmas 4.1 and 4.2 below to show that, if u is close enough to a half-space solution $\gamma[(x_1)_+]^2$ in a ball B_r, then for any $e \in \mathbb{S}^{n-1}$ with $e \cdot e_1 \geqq s > 0$ we have $C_0\partial_e u - u \geqq 0$ inside $B_{r/2}$.

4.1. The Case $\Omega \supset \{u \neq 0\}$

Lemma 4.1. *Let* $u : B_1 \to \mathbb{R}$ *be a* $W^{2,n}$ *solution of* (1.2) *with* $\Omega \supset \{u \neq 0\}$. *Assume that* $C_0\partial_e u - u \geqq -\varepsilon_0$ *in* B_1 *for some* $C_0, \varepsilon_0 \geqq 0$, *and that* F *is convex and satisfies* (H0)–(H1). *Then* $C_0\partial_e u - u \geqq 0$ *in* $B_{1/2}$ *provided* $\varepsilon_0 \leqq 1/(8n\lambda_1)$.

Proof. Since F is convex, for any matrix M we can choose an element P^M inside $\partial F(M)$ (the subdifferential of F at M) in such a way that the map $M \mapsto P^M$ is measurable. Then, since that $u \in C^{2,\alpha}_{\text{loc}}(\Omega)$ (by Evans–Krylov's Theorem [4, Chapter 6]), we can define the measurable uniformly elliptic coefficients

$$a_{ij}(x) := (P^{D^2u(x)})_{ij} \in \partial F(D^2u(x)).$$

We now notice two useful facts: first of all, since $a_{ij} \in \partial F(D^2u)$, by convexity of F we deduce that, for any $x \in \Omega$ and $h > 0$ small such that $x + he \in \Omega$,

$$a_{ij}(x)\frac{\partial_{ij}u(x+he) - \partial_{ij}u(x)}{h} \leqq \frac{F(D^2u(x+he)) - F(D^2u(x))}{h} = 0,$$

so, by letting $h \to 0$,

$$a_{ij}\partial_{ij}\partial_e u \leqq 0 \quad \text{in } \Omega. \tag{4.1}$$

Also, again by the convexity of F and recalling that $F(0) = 0$, we have

$$a_{ij}\partial_{ij}u \geqq F(D^2u) - F(0) = 1 \quad \text{in } \Omega. \tag{4.2}$$

Now, let us assume by contradiction that there exists $y_0 \in B_{1/2}$ such that $C_0\partial_e u(y_0) - u(y_0) < 0$, and consider the function

$$w(x) := C_0\partial_e u(x) - u(x) + \frac{|x - y_0|^2}{2n\lambda_1}.$$

Thanks to (4.1), (4.2), and assumption (H1) (which implies that $\lambda_0 \,\mathrm{Id} \leqq a_{ij} \leqq \lambda_1 \,\mathrm{Id}$) we deduce that w is a supersolution of the linear operator $\mathscr{L} := a_{ij}\partial_{ij}$. Hence, by the maximum principle,

$$\min_{\partial(\Omega\cap B_1)} w = \min_{\Omega\cap B_1} w \leqq w(y_0) < 0,$$

where the first inequality follows from the fact that $y_0 \in \Omega \cap B_{1/2}$ (since $u = \nabla u = 0$ outside Ω).

Since $w \geqq 0$ on $\partial\Omega$ and $|x - y_0|^2 \geqq 1/4$ on ∂B_1, it follows that

$$0 > \min_{\partial B_1} w \geqq -\varepsilon_0 + \frac{1}{8n\lambda_1},$$

a contradiction if $\varepsilon_0 < 1/(8n\lambda_1)$. □

4.2. The Case $\Omega \supset \{\nabla u \neq 0\}$

Lemma 4.2. *Let $u : B_1 \to \mathbb{R}$ be a $W^{2,n}$ solution of* (1.2) *with $\Omega \supset \{\nabla u \neq 0\}$. Assume that $C_0\partial_e u - |\nabla u|^2 \geqq -\varepsilon_0$ in B_1 for some $C_0, \varepsilon_0 \geqq 0$, and that F is convex, of class C^1, and satisfies* (H0)–(H1). *Then $C_0\partial_e u - |\nabla u|^2 \geqq 0$ in $B_{1/2}$ provided $\varepsilon_0 \leqq \lambda_0/(4n^2\lambda_1^3)$.*

Proof. By differentiating the equation $F(D^2u) = 1$ inside Ω, we deduce that

$$F_{ij}(D^2u)\partial_{ij}\nabla u = 0. \tag{4.3}$$

We now observe that, since $F_{ij} \in C^0$ (because $F \in C^1$) and $D^2u \in C^{2,\alpha}_{\text{loc}}(\Omega)$ (by Evans–Krylov's Theorem [4, Chapter 6]), ∇u solves a linear elliptic equation with continuous coefficients, so by standard elliptic theory $\nabla u \in W^{2,p}_{\text{loc}}(\Omega)$ for any $p < \infty$. Hence, we can apply the linear operator $F_{ij}(D^2u)\partial_{ij}$ to the $W^{2,p}_{\text{loc}}$ function $|\nabla u|^2$, and using (4.3) we obtain

$$\begin{aligned} F_{ij}(D^2u)\partial_{ij}|\nabla u|^2 &= 2\left(F_{ij}(D^2u)\partial_{ij}\partial_k u\right)\cdot\partial_k u + 2F_{ij}(D^2u)\partial_{ij}u\partial_{ik}u \\ &= 2F_{ij}(D^2u)\partial_{ij}u\partial_{ik}u. \end{aligned}$$

Now, if for every point $x \in \Omega$ we choose a system of coordinates so that D^2u is diagonal, since $F_{ii}(D^2u) \geqq \lambda_0$ for all $i = 1, \dots, n$ [by (H1)] we obtain

$$F_{ij}(D^2u(x))\partial_{ij}|\nabla u|^2(x) = 2F_{ii}(D^2u(x))\,(D_{ii}u(x))^2 \geqq 2\lambda_0|D^2u(x)|^2,$$

where $|D^2u(x)| := \sqrt{\sum_{ij}\left(D_{ij}u(x)\right)^2} = \sqrt{\sum_i (D_{ii}u(x))^2}$ (since $D^2u(x)$ is diagonal). Using (H1) again, we also have

$$1 = F(D^2u) - F(0) \leqq \sqrt{n}\lambda_1|D^2u| \quad \text{inside } \Omega,$$

so by combining the two estimates above we get

$$F_{ij}(D^2u))\partial_{ij}|\nabla u|^2 \geqq 2\lambda_0/(n\lambda_1^2). \tag{4.4}$$

Thanks to (4.3) and (4.4), we conclude exactly as before, considering now the function

$$w(x) := C_0\partial_e u(x) - |\nabla u|^2(x) + \frac{\lambda_0|x-y_0|^2}{n^2\lambda_1^3}.$$

□

5. Proof of Theorem 1.3

As already mentioned in the introduction, once we know that blow-up solutions around "thick points" are half-space solutions (Proposition 3.2) and we can improve almost directional monotonicity to full directional monotonicity (Lemmas 4.1 and 4.2), then the proof of Theorem 1.3 becomes standard. For convenience of the reader, we briefly sketch it here.

We consider only the case when $\Omega \supset \{u \neq 0\}$ (the other being analogous).

Take $x \in \partial\Omega \cap B_{1/8}$, and rescale the solution around x, that is, consider $u_r(y) := [u(x + ry) - u(x)]/r^2$. Because of the uniform $C^{1,1}$ estimate provided by Theorem 1.2, we can find a sequence $r_j \to 0$ such that u_{r_j} converges locally in C^1 to a global solution u_0 satisfying $u_0(0) = 0$. Moreover, by our thickness assumption on the free boundary of u and (3.6), it follows that the minimal diameter property holds for all $r > 0$ and all points on the free boundary $\partial\Omega(u_0)$. Then, by Proposition 3.2 we deduce that u_0 is of the form $u_0(y) = \gamma[(y \cdot e_x)_+]^2/2$ with $\gamma \in [1/\lambda_1, 1/\lambda_0]$ and $e_x \in \mathbb{S}^{n-1}$.

Notice now that, for any $s \in (0, 1)$, we can find a large constant C_s such that

$$C_s \partial_e u_0 - u_0 \geqq 0 \quad \text{inside } B_1$$

for all directions $e \in \mathbb{S}^{n-1}$ such that $e \cdot e_x \geqq s$. Since $u_{r_j} \to u_0$ in C^1_{loc}, we deduce that, for j sufficiently large (the largeness depending on s), the assumptions of Lemma 4.1 are satisfied with $u = \tilde{u}_{r_j}$. Hence

$$C_s \partial_e u_{r_j} - u_{r_j} \geqq 0 \quad \text{in } B_{1/2}, \tag{5.1}$$

and since $u_{r_j}(0) = 0$ a simple ODE argument shows that $u_{r_j} \geqq 0$ in $B_{1/4}$ (see the proof of [15, Lemmas 4.4 and 4.5]).

Using (5.1) again, this implies that $\partial_e u_{r_j} \geqq 0$ inside $B_{1/4}$, and so in terms of u we deduce that there exists $r = r(s) > 0$ such that

$$\partial_e u \geqq 0 \quad \text{inside } B_r(x)$$

for all $e \in \mathbb{S}^{n-1}$ such that $e \cdot e_x \geqq s$.

A simple compactness argument shows that r is independent of the point x, which implies that the free boundary is s-Lipschitz. Since s can be taken arbitrarily small (provided one reduces the size of r), this actually proves that the free boundary is C^1 (compare, for instance with [15, Theorem 4.10]). Higher regularity follows from the classical work of Kinderlehrer–Nirenberg [11].

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(*Received December 23, 2012 / Accepted February 19, 2014*)
Published online April 4, 2014 – © Springer-Verlag Berlin Heidelberg (2014)