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Dimensionality of Local Minimizers of the Interaction Energy

D. Balagué, J. A. Carrillo, T. Laurent & G. Raoul

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Abstract

In this work we consider local minimizers (in the topology of transport distances) of the interaction energy associated with a repulsive–attractive potential. We show how the dimensionality of the support of local minimizers is related to the repulsive strength of the potential at the origin.

1. Introduction

Given a Borel measurable function $W : \mathbb{R}^N \to (-\infty, +\infty]$ that is bounded from below, the interaction energy of the Borel probability measure μ is given by

$$E_W[\mu] := \tfrac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y). \tag{1}$$

Our main goal will be to analyze the qualitative properties of local minimizers of the energy E_W in the set of Borel probability measures with the topology induced by transport distances. More specifically, we will show that the Hausdorff dimension of the support of local minimizers is directly related to the behavior at the origin of ΔW.

The interaction energy E_W arises in many contexts. In physical, biological, and material sciences it is used to model the effects of particles or individuals on each other via pairwise interactions. Given n particles located at $X_1, \ldots, X_n \in \mathbb{R}^N$, their discrete interaction energy is given by

$$E_W^n[X_1, \ldots, X_n] := \frac{1}{2n^2} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} W(X_i - X_j). \tag{2}$$

Formally, for a large number of particles the discrete energy (2) is well approximated by the continuum energy (1), where $\mathrm{d}\mu(x)$ is a general distribution of particles at location $x \in \mathbb{R}^N$. In fact, the continuum energy (1) of the discrete distribution $\frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}$ reduces to (2).

In models arising in material sciences [15,21,32,36,38], particles, nano-particles, or molecules self-assemble in a way to minimize energies similar to E_W^n. Analogously in applications to biological sciences [4,14,27,28,34], individuals in a social aggregate (for example, swarm, flock, school, or herd) self-organize in order to minimize similar types of energies. In these applications the potential W is typically repulsive in the short range so that particles/individuals do not collide, and attractive in the long range so that the particles/individuals gather to form a group or a structure. Therefore, one is often led to consider radially symmetric interaction potentials of the form $W(x) = w(|x|)$, where $w : [0, +\infty) \to (-\infty, +\infty]$ is decreasing on some interval $[0, r_0)$ and increasing on $(r_0, +\infty)$. The function w may or may not have a singularity at $r = 0$. We will refer to such potentials as being **repulsive–attractive**. Since w has a global minimum at r_0, it is obvious that if we consider only two particles, X_1 and X_2, in order to minimize $E_W^2[X_1, X_2]$ the two particles must be located at a distance r_0 from one another. While the situation is simple with two particles, it becomes very complicated for a large number of particles. Recent works [2,17–19,22,23,31,33,36,37] have shown that such repulsive–attractive potentials lead to the emergence of surprisingly rich geometric structures. The goal of the present paper is to understand how the dimensionality of these structures depends on the singularity of ΔW at the origin.

Let us describe the main results. Consider a repulsive–attractive potential $W(x) = w(|x|)$. Typically, the Laplacian of such a potential will be negative in a neighborhood of the origin. We show that if

$$\Delta W(x) \sim -\frac{1}{|x|^\beta} \quad \text{as } x \to 0 \tag{3}$$

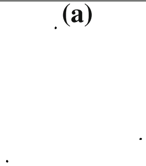
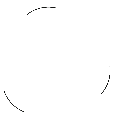
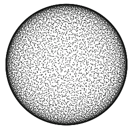
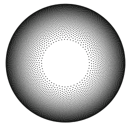
for some $0 < \beta < N$, then the support of local minimizers of E_W has Hausdorff dimension greater than or equal to β. The precise hypotheses needed on W for this result to be true, as well as the precise meaning of (3), can be found in the statement of Theorem 1. The exponent β appearing in (3) quantifies how repulsive the potential is at the origin. Therefore, our result can be intuitively understood as follows: the more repulsive the potential is at the origin, the higher the dimension of local minimizers will be.

Potentials satisfying (3) have a singular Laplacian at 0 and we refer to them as **strongly repulsive at the origin**. The second main result is devoted to potentials which are **mildly repulsive at the origin**, that is, potentials whose Laplacian does not blow up at the origin. To be more precise, we show that if

$$W(x) \sim -|x|^\alpha \quad \text{as } x \to 0 \quad \text{for some } \alpha > 2, \tag{4}$$

then a local minimizer of the interaction energy cannot be concentrated on smooth manifolds of any dimension except 0-dimensional sets. The exact hypotheses on W, as well as the precise meaning of (4), can be found in Theorem 2. Note that this result suggests that local minimizers of the interaction energy of mildly repulsive potentials have zero Hausdorff dimension—however we are currently unable to prove this stronger result.

Table 1. Local minimizers of the interaction energy E_W^n for various potentials $W(x)$

	Dim = 0	Dim = 1	Dim = 2
$\alpha = 2.5$	**(a)** 		
$\alpha = 1.5$		**(b)** 	**(c)** 
$\alpha = 0.5$			**(d)** 

In these computations $n = 10{,}000$. When ΔW does not blow-up at the origin (Case a) the Hausdorff dimension of the the support of minimizers is zero. When $\Delta W \sim -1/|x|^\beta$ as $x \to 0$, $0 < \beta < N$ (Cases b, c, d) the Hausdorff dimension of the the support of minimizers is greater than or equal to β

Summarizing, in this paper we show that if the Laplacian of the potential behaves like $-1/|x|^\beta$ around the origin, with $0 < \beta < N$, then the dimension of minimizers is at least β and if the Laplacian does not blow up at the origin, then the dimension is zero, see the precise statement in Theorems 1 and 2. This is illustrated in the case of two dimensions ($N = 2$) in Table 1, where we show some local minimizers of E_W with interaction potentials of the form

$$W(x) = -\frac{|x|^\alpha}{\alpha} + \frac{|x|^\gamma}{\gamma} \quad \alpha < \gamma, \tag{5}$$

so that $W(x) \sim -\frac{|x|^\alpha}{\alpha}$ and $\Delta W(x) \sim -\frac{1}{|x|^\beta}$ with $\beta = 2 - \alpha$ as $x \to 0$.

- Subfigure (a): $\alpha = 2.5$ and $\gamma = 15$. The support of the minimizer has zero Hausdorff dimension in agreement with Theorem 2. Actually, in this particular case it is supported on just three points.
- Subfigure (b) and (c): we consider two examples where the potentials have the same behavior at the origin, $\alpha = 1.5$, but different attractive long range behavior ($\gamma = 7$ and 2, respectively). Theorem 1 shows that the Hausdorff dimension of the support must be greater than or equal to $\beta = 2 - \alpha = 0.5$. Indeed, the minimizer for the first example has a one-dimensional support on three curves, whereas the minimizer for the second example has a two-dimensional support.

- Subfigure (d): $\alpha = 0.5$ and $\gamma = 5$. Theorem 1 proves that the Hausdorff dimension of the support must be greater than $\beta = 2 - \alpha = 1.5$. The numerical simulation demonstrates that it has dimension two.

In our extensive numerical experiments using gradient descent methods we never observed minimizers with a support that might be of non-integer Hausdorff dimension.

In most of this paper, we will consider local minimizers for the topology induced by the transport distance d_∞ (see Section 2 for a definition of d_∞). This topology is, indeed, the natural one to consider. In particular, gradient descent numerical methods based on particles typically lead to local minimizers for the d_∞-topology. Moreover, the topology induced by d_∞ is the finest topology among the ones induced by d_p, $1 \leqq p \leqq \infty$ (see Section 2 for a definition of d_p). As a consequence, local minimizers in the d_p-topology are automatically local minimizers in the d_∞-topology, and thus they are also covered by our study. In Section 5 we will discuss these questions in more detail.

Finally, let us mention that the gradient flow of the energy E_W in the Wasserstein sense d_2 [1,12,13] has been extensively studied in recent years [2,3,5–11,17,18, 24,31]. It leads to the nonlocal interaction equation

$$\frac{\partial \mu}{\partial t} + \operatorname{div}(\mu v) = 0, \quad v = -\nabla W * \mu, \tag{6}$$

where $\mu(t,x) = \mu_t(x)$ is the probability or mass density of particles at time t and at location $x \in \mathbb{R}^N$, and $v(t,x)$ is the velocity of the particles. Stability properties of steady states for (6) with repulsive–attractive potentials have only been analyzed very recently. In [2] we gave conditions for radial stability/instability of particular local minimizers. We should also mention that the one-dimensional case was analyzed in detail in [17,18]. Well-posedness theories for these repulsive–attractive potentials in various functional settings have been provided in [1,2,8,9, 11,24]. Stable steady states of (6) under certain set of perturbations are expected to be local minimizers of the energy functional (1) in a topology to be specified. Actually, this topology should determine the set of admissible perturbations. As already mentioned, the d_∞-stability is the one typically studied by performing equal mass particles simulations.

Finally, we can now interpret our dimensionality result in terms of the nonlocal evolution equation (6). The heuristic idea behind the implication, that (3) with $0 < \beta < N$ implies dimensionality larger than β of the support of local minimizers of E_W, can be understood in terms of the divergence of the velocity field in (6). In fact, it is straightforward to check that the divergence of the velocity field generated by a uniform density localized over a smooth manifold of dimension k is $+\infty$ on the manifold if and only if $k < \beta$ (this is equivalent to non-integrability of $-\Delta W$ on manifolds of dimension k). Heuristically, if $\operatorname{div} v = -\Delta W * \mu$ associated to μ diverges on its support, the density has a strong tendency to spread, the configuration is not stable and then μ is not a local minimizer. Therefore, we can reinterpret our result in Theorem 1 as follows: local minimizers of (1) have to be supported on manifolds where the divergence of their generated velocity field is not $+\infty$.

The plan of the paper is as follows. Section 2 will be devoted to the necessary background in optimal transport theory and notation. Strongly repulsive potentials are treated in Section 3, while mildly repulsive potentials are analyzed in Section 4. In Section 5, for the smaller subset of local minimizers in the d_2-topology, we show that we can use an Euler–Lagrange approach in the spirit of [4] to derive some properties of these minimizers. Extensive numerical tests as well as details of the algorithm used in order to minimize E_W^n are reported in Section 6.

2. Preliminaries in Transport Distances

We denote by $\mathcal{B}(\mathbb{R}^N)$ the family of Borel subsets of $\mathbb{R}^N$. Given a set $A \in \mathcal{B}(\mathbb{R}^N)$, its Lebesgue measure is denoted by $|A|$. We denote by $\mathcal{M}(\mathbb{R}^N)$ the set of (nonnegative) Borel measures on $\mathbb{R}^N$ and by $\mathcal{P}(\mathbb{R}^N)$ the set of Borel probability measures on $\mathbb{R}^N$. The support of $\mu \in \mathcal{M}(\mathbb{R}^N)$, denoted by $\mathrm{supp}(\mu)$, is the closed set defined by

$$\mathrm{supp}(\mu) := \{x \in \mathbb{R}^N : \mu(B(x,\varepsilon)) > 0 \quad \text{for all } \varepsilon > 0\}.$$

A measure $\rho \in \mathcal{M}(\mathbb{R}^N)$ is said to be a part of μ if $\rho(A) \leqq \mu(A)$ for all $A \in \mathcal{B}(\mathbb{R}^N)$ and it is not identically zero. This terminology is justified by the fact that if ρ is a part of μ, then μ can be written $\mu = \rho + \nu$ for some $\nu \in \mathcal{M}(\mathbb{R}^N)$ ($\nu = \mu - \rho$, to be more precise). We will say that a probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ can be decomposed as a convex combination of $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^N)$ if there exist $0 \leqq m_0, m_1 \leqq 1$ with $m_0 + m_1 = 1$ such that $\mu = m_0\mu_0 + m_1\mu_1$.

Let us introduce some notation related to the interaction potential energy. We denote by $B_W : \mathcal{P}(\mathbb{R}^N) \times \mathcal{P}(\mathbb{R}^N) \to (0, +\infty]$ the bilinear form defined by

$$B_W[\mu_1, \mu_2] := \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)\, \mathrm{d}\mu_1(x)\, \mathrm{d}\mu_2(y). \tag{7}$$

Obviously, we have that $E_W[\mu] = B_W[\mu, \mu]$. Let us define the shortcut notation

$$T_W[\mu_1, \mu_2] := E_W[\mu_1] - 2B_W[\mu_1, \mu_2] + E_W[\mu_2],$$

which will occur in several computations. For notational simplicity, we will drop the subscript for E_W, B_W, and T_W in detailed proofs while keeping it in the main statements.

Let us give a brief self-contained summary of the main concepts related to distances between measures in optimal transport theory; we refer to [20,26,35] for further details. A probability measure π on the product space $\mathbb{R}^N \times \mathbb{R}^N$ is said to be a transference plan between $\mu \in \mathcal{P}(\mathbb{R}^N)$ and $\nu \in \mathcal{P}(\mathbb{R}^N)$ if

$$\pi(A \times \mathbb{R}^N) = \mu(A) \quad \text{and} \quad \pi(\mathbb{R}^N \times A) = \nu(A) \tag{8}$$

for all $A \in \mathcal{B}(\mathbb{R}^N)$. If $\mu, \nu \in \mathcal{P}(\mathbb{R}^N)$, then

$$\Pi(\mu, \nu) := \{\pi \in \mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N) : (8) \text{ holds for all } A \in \mathcal{B}(\mathbb{R}^N)\}$$

denotes the set of admissible transference plans between μ and ν. Informally, if $\pi \in \Pi(\mu, \nu)$ then $d\pi(x, y)$ measures the amount of mass transferred from location x to location y. With this interpretation in mind, note that $\sup_{(x,y)\in\operatorname{supp}(\pi)} |x - y|$ represents the maximum distance that an infinitesimal element of mass from μ is moved by the transference plan π. We will work with the ∞-Wasserstein distance d_∞ between two probability measures μ, ν defined by

$$d_\infty(\mu, \nu) = \inf_{\pi\in\Pi(\mu,\nu)} \sup_{(x,y)\in\operatorname{supp}(\pi)} |x - y| , \tag{9}$$

which can take infinite values, though it is obviously finite for compactly supported measures. This distance induces a complete metric structure restricted to the set of probability measures with finite moments of all orders, $\mathcal{P}_\infty(\mathbb{R}^N)$, as proven in [20].

We recall that for $1 \leqq p < \infty$ the distance d_p between two measures μ and ν is defined by

$$d_p^p(\mu, \nu) = \inf_{\pi\in\Pi(\mu,\nu)} \left\{ \iint_{\mathbb{R}^N\times\mathbb{R}^N} |x - y|^p \, \mathrm{d}\pi(x, y) \right\} .$$

Note that $d_p(\mu, \nu) < \infty$ for $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^N)$, the set of probability measures with finite moments of order p. Since $d_p(\mu, \nu)$ is increasing as a function of $1 \leqq p < \infty$, one can show that it converges to $d_\infty(\mu, \nu)$ as $p \to \infty$. Since the distances are ordered with respect to p, it is obvious that the topologies are also ordered. More precisely, open sets for d_p are always open sets for d_∞, and thus, d_∞ induces the finest topology among d_p, $1 \leqq p \leqq \infty$. More properties of the distance d_∞ can be seen in [26].

Given $\mathcal{T} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ measurable, we say that ν is the push-forward of μ through $\mathcal{T}$, $\nu = \mathcal{T}\#\mu$, if $\nu[A] := \mu[\mathcal{T}^{-1}(A)]$ for all measurable sets $A \subset \mathbb{R}^N$, equivalently

$$\int_{\mathbb{R}^N} \varphi(x) \, \mathrm{d}\nu(x) = \int_{\mathbb{R}^N} \varphi(\mathcal{T}(x)) \, \mathrm{d}\mu(x)$$

for all $\varphi \in C_b(\mathbb{R}^N)$. In case there is a map $\mathcal{T} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ transporting μ onto ν, that is $\mathcal{T}\#\mu = \nu$, we immediately obtain

$$d_\infty(\mu, \nu) \leqq \sup_{y\in\operatorname{supp}(\mu)} |y - \mathcal{T}(y)| .$$

This comes from (9), by using the transference plan $\pi_{\mathcal{T}} = (\mathbb{1}_{\mathbb{R}^N} \times \mathcal{T})\#\mu$.

Lemma 1. *Assume that $\mu, \tilde{\mu} \in \mathcal{P}(\mathbb{R}^N)$ are two convex combinations: $\mu = m_0\mu_0 + m_1\mu_1$ and $\tilde{\mu} = m_0\tilde{\mu}_0 + m_1\mu_1$, where μ_0 and $\tilde{\mu}_0$ are supported in $B(x_0, \varepsilon)$ for some $x_0 \in \mathbb{R}^N$ and $\varepsilon > 0$. Then $d_\infty(\mu, \tilde{\mu}) \leqq 2\varepsilon$.*

Proof. Let $\pi_1 \in \Pi(\mu_1, \mu_1)$ be the transport plan induced by the identity map, that is,

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \phi(x, y) \, \mathrm{d}\pi_1(x, y) = \int_{\mathbb{R}^N} \phi(x, x) \, \mathrm{d}\mu_1(x),$$

and let $\pi_0 \in \Pi(\mu_0, \tilde{\mu}_0)$ be any transport plan between μ_0 and $\tilde{\mu}_0$. Note that the transference plan $\pi = m_0\pi_0 + m_1\pi_1 \in \Pi(\mu, \tilde{\mu})$ and $\mathrm{supp}(\pi) = \mathrm{supp}(\pi_0) \cup \mathrm{supp}(\pi_1)$. Since π_1 is supported on the diagonal, we have $\sup_{(x,y)\in\mathrm{supp}(\pi_1)} |x-y| = 0$. On the other hand, $\mathrm{supp}(\pi_0) \subset \mathrm{supp}(\mu_0) \times \mathrm{supp}(\tilde{\mu}_0) \subset B(x_0, \varepsilon) \times B(x_0, \varepsilon)$ and therefore $\sup_{(x,y)\in\mathrm{supp}(\pi_0)} |x-y| \leqq 2\varepsilon$. We conclude that $\sup_{(x,y)\in\mathrm{supp}(\pi)} |x-y| \leqq 2\varepsilon$ which implies $\inf_{\pi\in\Pi(\nu,\rho)} \sup_{(x,y)\in\mathrm{supp}(\pi)} |x-y| \leqq 2\varepsilon$.

3. Lower Bound on the Hausdorff Dimension of the Support

In this section we consider potentials which are strongly repulsive at the origin and we prove that if $\Delta W \sim -1/|x|^\beta$ as $x \to 0$, $0 < \beta < N$, then the Hausdorff dimension of the support of local minimizers of the interaction energy is greater than or equal to β. Actually, our result is slightly stronger: we prove that if μ is a local minimizer, then the support of any part of μ has Hausdorff dimension greater than or equal to β. Let us illustrate the importance of controlling not only the dimension of μ, but also the dimensions of the parts of μ. Suppose, for example, that $\Delta W \sim -1/|x|$ as $x \to 0$, then our result implies that any part of μ has Hausdorff dimension greater than or equal to 1. As a consequence, μ cannot have an atomic part. If $\Delta W \sim -1/|x|^{1.5}$ as $x \to 0$, then μ cannot have a part concentrated on a curve, and so on.

3.1. Hypotheses and Statement of the Main Result

In this section, we will assume that the potential $W : \mathbb{R}^N \to (-\infty, +\infty]$ satisfies the following hypotheses:

(H1) W is bounded from below.
(H2) W is lower semicontinuous.
(H3) W is uniformly locally integrable: $\exists M > 0$ such that $\int_{B(x,1)} W(y)\,\mathrm{d}y \leqq M$ for all $x \in \mathbb{R}^N$.

In order to state the main results of this section we will also need the following two definitions:

Definition 1. (*Generalized Laplacian*) Suppose $W : \mathbb{R}^N \to (-\infty, +\infty]$ is locally integrable. The approximate Laplacian of W is defined by

$$-\Delta^\varepsilon W(x) := \frac{2(N+2)}{\varepsilon^2}\left(W(x) - \fint_{B(x,\varepsilon)} W(y)\,\mathrm{d}y\right),$$

where $\fint_{B(x_0,r)} f(x)\,\mathrm{d}x$ stands for the average of f over the ball of radius r centered at x_0, and the generalized Laplacian of W is defined by

$$-\Delta^0 W(x) := \liminf_{n\to\infty} \left\{-\Delta^{(1/n)} W(x)\right\}.$$

Let us emphasize the fact that the function W in the above definition is defined pointwise in $\mathbb{R}^N$ (it possibly takes the value $+\infty$), and is Borel measurable. As a consequence, the functions $-\Delta^\varepsilon W$ and $-\Delta^0 W$ are also Borel measurable, defined pointwise in $\mathbb{R}^N$, and take values in $(-\infty, +\infty]$ and $[-\infty, +\infty]$, respectively.

Definition 2. (*β-repulsive potential*) Suppose $W : \mathbb{R}^N \to (-\infty, +\infty]$ is locally integrable. W is said to be β-repulsive at the origin if there exists $\varepsilon > 0$ and $C > 0$ such that

$$-\Delta^0 W(x) \geqq \frac{C}{|x|^\beta} \quad \text{for all } 0 < |x| < \varepsilon \tag{10}$$

$$-\Delta^0 W(0) = +\infty. \tag{11}$$

By doing a Taylor expansion one can easily check that $\Delta^0 W(x) = \Delta W(x)$ wherever W is twice differentiable. In particular, if W is twice differentiable away from the origin, as is often the case for potentials of interest, then (10) simply means that $-\Delta W(x) \geqq C/|x|^\beta$ for all $0 < |x| < \varepsilon$. The terminology "β-repulsive" is justified by the fact that the rate at which $\Delta^0 W(x)$ goes to $-\infty$ as x approaches the origin quantifies the repulsive strength of the potential at the origin, therefore the greater β is, the more repulsive the potential is around the origin. This is the rigorous mathematical formulation of what we meant in (3). Additionally to hypotheses (H1)–(H3), we will need the following technical assumption on the potential W:

(H4) There exists $C^* > 0$ such that

$$\Delta^\varepsilon W(x) < C^* \quad \forall x \in \mathbb{R}^N \text{ and } \forall \varepsilon \in (0, 1).$$

We are now ready to state the main theorems of this section:

Theorem 1. *Suppose W satisfies* (H1)–(H4) *and let μ be a local minimizer of the interaction energy with respect to the topology induced by d_∞. If W is β-repulsive at the origin, $0 < \beta < N$, then the Hausdorff dimension of the support of any part of μ is greater than or equal to β.*

Remark 1. Observe that (H3) and (H4) are conditions that restrict the growth of the potential and its derivatives at ∞. For instance, a potential growing algebraically at ∞ does not satisfy those assumptions. However, if we are only interested in the dimensionality of the support for compactly supported local minimizers, Theorem 1 holds under weaker assumptions not restricting the growth of the potential at ∞. That is, (H3) and (H4) can be substituted by (H3-loc) and (H4-loc):

(H3-loc) W is locally integrable.
(H4-loc) For every compact subset K of $\mathbb{R}^N$ there exists $C_K^* > 0$ such that

$$\Delta^\varepsilon W(x) < C_K^* \quad \forall x \in K \text{ and } \forall \varepsilon \in (0, 1),$$

with obvious changes in the proof.

Remark 2. In Theorem 1 (resp. Remark 1) potential W is assumed to be β-repulsive at the origin and to satisfy hypotheses (H1)–(H4) (resp. (H1)–(H2)–(H3-loc)–(H4-loc)). Whereas hypotheses (H1)–(H3) (resp. (H1)–(H2)–(H3-loc)) are easily verified for a given potential, hypotheses (H4) or (H4-loc), and the β-repulsivity are not as transparent. To clarify the meaning of these more technical assumptions let us consider the case where W is smooth away from the origin and satisfies

$$-\Delta W(x) \geqq \frac{C}{|x|^\beta} \quad \text{for all } 0 < |x| < \varepsilon \tag{12}$$

for some $0 < \beta < N$. Such a potential satisfies (10) as pointed out in the comment after Definition 2. Moreover, most potentials of interest satisfying (12) will also satisfy (11) and either (H4) or (H4-loc), but of course this need to be checked case by case. In Subsection 3.3 we consider some typical repulsive–attractive potentials satisfying (12) and we show that they satisfy (11) and either (H4) or (H4-loc), depending on their behavior at infinity.

3.2. Proof of Theorem 1

First note that without loss of generality we can replace hypothesis $(H1)$ by

(H1$'$) W is nonnegative,

since adding a constant to the potential W does not affect the local minimizers of E_W. The following lemma is classical:

Lemma 2. *Suppose W satisfies* (H1$'$) *and* (H2) *and let $\mu \in \mathcal{P}(\mathbb{R}^N)$. Then the function $V_\mu : \mathbb{R}^N \to [0, +\infty]$ defined by*

$$V_\mu(x) = (W * \mu)(x) = \int_{\mathbb{R}^N} W(x-y)\, d\mu(y)$$

is lower semicontinuous.

Proof. Suppose $x_n \to x$, then by Fatou's lemma we have

$$\begin{aligned} V_\mu(x) = \int_{\mathbb{R}^N} W(x-y)\,\mathrm{d}\mu(y) &\leqq \int_{\mathbb{R}^N} \liminf_n W(x_n-y)\,\mathrm{d}\mu(y) \\ &\leqq \liminf_n \int_{\mathbb{R}^N} W(x_n-y)\,\mathrm{d}\mu(y) = \liminf_n V_\mu(x_n) \end{aligned}$$

as desired.

Suppose now that W satisfies (H1$'$)–(H4). Note that hypothesis (H4) implies that $-\Delta^0 W \geqq -C^*$ and recall that by definition $-\Delta^0 W$ is a Borel measurable function that is defined pointwise in $\mathbb{R}^N$. As a consequence, for any $\mu \in \mathcal{P}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, the integral

$$\begin{aligned} (-\Delta^0 W * \mu)(x) &:= \int_{\mathbb{R}^N} (-\Delta^0 W)(x-y)\,\mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}^N} \left[(-\Delta^0 W)(x-y) + C^*\right] \mathrm{d}\mu(y) - C^* \end{aligned}$$

is well defined and belongs to $[-C^*, +\infty]$. The function $-\Delta^0 W * \mu$ is therefore defined pointwise in $\mathbb{R}^N$ and takes values in $[-C^*, +\infty]$. Note that the integral against $\mu \in \mathcal{P}(\mathbb{R}^N)$ makes sense, since $-\Delta^0 W$ is Borel measurable. Indeed, the measurability of W and $-\Delta^0 W$ also follows from (H2), since any lower semicontinuous function can be seen as the decreasing limit of continuous functions.

Lemma 3. *Suppose that W satisfies* (H1′)–(H4) *and let $\mu \in \mathcal{P}(\mathbb{R}^N)$. If x_0 is a local min of $V_\mu = W * \mu$, in the sense that there exists $\varepsilon_0 > 0$ such that*

$$V_\mu(x_0) \leqq V_\mu(x) \quad \textit{for almost every } x \in B(x_0, \varepsilon_0), \tag{13}$$

then $(\Delta^0 W * \mu)(x_0) \geqq 0$.

Proof. Assume that x_0 satisfies (13). Note first that $V_\mu(x_0) < +\infty$, since

$$\begin{aligned} V_\mu(x_0) &\leqq \operatorname{ess\,inf}_{B(x_0,\varepsilon_0)} V_\mu \leqq \frac{1}{|B(x_0, \varepsilon_0)|} \int_{B(x_0,\varepsilon_0)} V_\mu(x)\, \mathrm{d}x \\ &\leqq \frac{1}{|B(x_0, \varepsilon_0)|} \int_{\mathbb{R}^N} \int_{B(x_0-y,1)} W(z)\, \mathrm{d}z\, \mathrm{d}\mu(y) \\ &\leqq \frac{M}{|B(x_0, \varepsilon_0)|} < +\infty, \end{aligned}$$

where we have used Fubini's Theorem and Hypothesis (H3). Now, for $\varepsilon \leqq \varepsilon_0$ we have

$$\begin{aligned} 0 &\leqq \frac{2(N+2)}{\varepsilon^2} \left(\fint_{B(0,\varepsilon)} V_\mu(x_0 + x)\, \mathrm{d}x - V_\mu(x_0) \right) \\ &= \frac{2(N+2)}{\varepsilon^2} \left(\int_{\mathbb{R}^N} \fint_{B(0,\varepsilon)} W(x_0+x-y)\, \mathrm{d}x\, \mathrm{d}\mu(y) - \int_{\mathbb{R}^N} W(x_0 - y)\, \mathrm{d}\mu(y) \right). \end{aligned} \tag{14}$$

Note that hypothesis (H4) is equivalent to

$$\fint_{B(0,\varepsilon)} W(x_0 + x - y)\, \mathrm{d}x \leqq W(x_0 - y) + \frac{C^* \varepsilon^2}{2(N+2)}.$$

Since $V_\mu(x_0) < +\infty$, the functions $y \mapsto W(x_0 - y)$ and $y \mapsto \fint_{B(0,\varepsilon)} W(x_0 + x - y)\, \mathrm{d}x$ are μ-integrable and the difference of the integrals in (14) is equal to the integral of the difference. Therefore, we have:

$$\begin{aligned} 0 &\leqq \int_{\mathbb{R}^N} \frac{2(N+2)}{\varepsilon^2} \left(\fint_{B(0,\varepsilon)} W(x_0 - y + x)\, \mathrm{d}x - W(x_0 - y) \right) \mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}^N} \Delta^\varepsilon W(x_0 - y)\, \mathrm{d}\mu(y). \end{aligned} \tag{15}$$

Because of hypothesis (H4), we have that $-\Delta^\varepsilon W + C^* \geqq 0$ for all $\varepsilon \in (0, 1)$. Therefore, using Fatou's Lemma and (15):

$$\begin{aligned} &\int_{\mathbb{R}^N} \liminf_{n\to\infty} \left\{ -\Delta^{(1/n)} W(x_0 - y) + C^* \right\} \mathrm{d}\mu(y) \\ &\qquad \leqq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[-\Delta^{(1/n)} W(x_0 - y) + C^* \right] \mathrm{d}\mu(y) \leqq C^*, \end{aligned}$$

that is, $(\Delta^0 W * \mu)(x_0) \geqq 0$.

Proposition 1. *Suppose that W satisfies* (H1′)-(H2)-(H3). *Let μ be a local minimizer of the interaction energy with respect to d_∞ and assume that $E[\mu] < +\infty$. Then any point $x_0 \in supp(\mu)$ is a local minimizer of V_μ, in the sense that there exists $\varepsilon_0 > 0$ such that*

$$V_\mu(x_0) \leqq V_\mu(x) \quad \textit{for almost every} \quad x \in B(x_0, \varepsilon_0).$$

Proof. We argue by contradiction. Assume that there exists $x_0 \in \mathrm{supp}(\mu)$ which is not a local minimum of V_μ. Fix $\varepsilon > 0$. Then there exists a set $A \subset B(x_0, \varepsilon)$ of positive Lebesgue measure, such that for $x \in A$, $V_\mu(x) < V_\mu(x_0)$. The set A can be written as follows:

$$A = \cup_{n=1}^{\infty} \{x \in A;\ V_\mu(x) \leqq V_\mu(x_0) - 1/n\},$$

that is, A is an increasing union of measurable sets. Thanks to the continuity from below of the Lebesgue measure, it implies that

$$0 < |A| = \lim_{n\to\infty} |\{x \in A;\ V_\mu(x) \leqq V_\mu(x_0) - 1/n\}|,$$

and there exists n_0 such that $\tilde{A} := \{x \in A;\ V_\mu(x) \leqq V_\mu(x_0) - 1/n_0\}$ is of positive Lebesgue measure. Thanks to the lower semicontinuity of V_μ, there exists $\eta \in (0, \varepsilon)$ such that

$$\inf_{B(x_0,\eta)} V_\mu \geqq V_\mu(x_0) - \frac{1}{2n_0} \geqq \sup_{\tilde{A}} V_\mu + \frac{1}{2n_0}. \tag{16}$$

Notice that $x_0 \in \mathrm{supp}(\mu)$ implies $\mu(B(x_0, \eta)) > 0$. We can therefore define the probability measures μ_0, $\mu_{\tilde{A}}$ by

$$\mu_0(B) = \frac{1}{m_0}\mu(B \cap B(x_0, \eta)), \quad \mu_{\tilde{A}}(B) = \frac{1}{|\tilde{A}|}|B \cap \tilde{A}|$$

for any Borel set $B \in \mathcal{B}(\mathbb{R}^N)$, where $m_0 := \mu(B(x_0, \eta))$. Let us now write μ as a convex combination $\mu = m_0\mu_0 + m_1\mu_1$, and define the curve of measures

$$\begin{aligned}\mu_t &= (m_0 - t)\mu_0 + t\mu_{\tilde{A}} + m_1\mu_1 \\ &= m_0\left[\left(1 - \frac{t}{m_0}\right)\mu_0 + \frac{t}{m_0}\tilde{\mu}_{\tilde{A}}\right] + m_1\mu_1.\end{aligned}$$

It is clear by construction that $\mu_t \in \mathcal{P}(\mathbb{R}^N)$ for $t \in [0, m_0]$, and since $\tilde{\mu}_0 := \left(1 - \frac{t}{m_0}\right)\mu_0 + \frac{t}{m_0}\tilde{\mu}_{\tilde{A}}$ is supported in $B(x_0, \eta)$, Lemma 1 implies that $d_\infty(\mu, \mu_t) \leqq 2\varepsilon$. Inequality (16) shows that the function V_μ is greater on the region $B(x_0, \eta)$ than on the region $\tilde{A}$, therefore one would expect that transporting mass from one region to the other will decrease the interaction energy. Indeed we will show that $E[\mu_t] < E[\mu]$ for t small enough. Since ε was arbitrary, this will imply that we can always find a probability measure arbitrarily close to μ (in the sense of the d_∞) with strictly smaller energy. This is a contradiction concluding the proof.

We are left to show that $E[\mu_t] < E[\mu]$ for t small enough. Since $0 \leqq E[\mu] < \infty$ and it is given by

$$E[\mu] = m_0^2 E[\mu_0] + 2m_0 m_1 B[\mu_0, \mu_1] + m_1^2 E[\mu_1]$$

the three terms $E[\mu_0]$, $B[\mu_0, \mu_1]$ and $E[\mu_1]$ are all positive and finite. Note that $E[\mu_{\tilde{A}}]$ is also finite: indeed, since W is locally integrable by (H3), we have

$$\begin{aligned} E[\mu_{\tilde{A}}] &= \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)\, \mathrm{d}\mu_{\tilde{A}}(x)\, \mathrm{d}\mu_{\tilde{A}}(y) \\ &\leqq \frac{1}{|\tilde{A}|^2} \iint_{B(x_0,\varepsilon)\times B(x_0,\varepsilon)} W(x-y)\, \mathrm{d}x\, \mathrm{d}y < +\infty. \end{aligned}$$

From (16) and the fact that $B[\mu, \mu_0] \leqq \frac{1}{m_0} E[\mu] < +\infty$, we also have that

$$B[\mu, \mu_{\tilde{A}}] + \frac{1}{2n_0} \leqq B[\mu, \mu_0] < +\infty. \tag{17}$$

Using all these, we can show that all combinations of the bilinear form $B[\cdot, \cdot]$ for the measures μ_0, μ_1, and $\mu_{\tilde{A}}$ are finite:

$$E[\mu_0] < +\infty, \quad E[\mu_1] < +\infty, \quad E[\mu_{\tilde{A}}] < +\infty, \quad B[\mu_1, \mu_0] < +\infty, \tag{18}$$

$$B[\mu_{\tilde{A}}, \mu_0] \leqq \frac{1}{m_0} B[\mu_{\tilde{A}}, \mu] < +\infty, \quad B[\mu_{\tilde{A}}, \mu_1] \leqq \frac{1}{m_1} B[\mu_{\tilde{A}}, \mu] < +\infty, \tag{19}$$

where we have used (17) in order to obtain (19). Note that in (19) we have assumed $m_1 \neq 0$. If $m_1 = 0$, then μ_1 can be chosen to be zero and therefore $B[\mu_{\tilde{A}}, \mu_1] < +\infty$ trivially holds. Using (18)–(19), we are allowed to expand $E[\mu_t]$ as:

$$\begin{aligned} E[\mu_t] &= E[(m_0 - t)\mu_0 + m_1\mu_1 + t\mu_{\tilde{A}}] \\ &= (m_0 - t)^2 E[\mu_0] + m_1^2 E[\mu_1] + t^2 E[\mu_{\tilde{A}}] \\ &\quad + 2(m_0 - t) m_1 B[\mu_0, \mu_1] + 2(m_0 - t) t B[\mu_0, \mu_{\tilde{A}}] + 2 m_1 t B[\mu_1, \mu_{\tilde{A}}] \\ &= m_0^2 E[\mu_0] + 2 m_0 m_1 B[\mu_0, \mu_1] + m_1^2 E[\mu_1] \\ &\quad + 2t(m_0 B[m_0, \mu_{\tilde{A}}] + m_1 B[\mu_1, \mu_{\tilde{A}}]) - 2t(m_0 B[\mu_0, \mu_0] + m_1 B[\mu_0, \mu_1]) \\ &\quad + t^2 E[\mu_0] + t^2 E[\mu_{\tilde{A}}] - 2t^2 B[\mu_0, \mu_{\tilde{A}}] \\ &= E[\mu] + 2t(B[\mu_{\tilde{A}}, \mu] - B[\mu_0, \mu]) + t^2 T[\mu_0, \mu_{\tilde{A}}]. \end{aligned} \tag{20}$$

Note that in the above computations we have only used the bilinearity of $B[\cdot, \cdot]$ on the space of positive measures. However, a formal computation using the bilinearity of $B[\cdot, \cdot]$ on the space of signed measures leads to the same result in a much simpler way:

$$E[\mu_t] = E[\mu - t\mu_0 + t\mu_{\tilde{A}}] = E[\mu] + 2t(B[\mu_{\tilde{A}}, \mu] - B[\mu_0, \mu]) + t^2 T[\mu_0, \mu_{\tilde{A}}].$$

To conclude the proof, note that because of (17) the term $B[\mu_{\tilde{A}}, \mu] - B[\mu_0, \mu]$ appearing in (20) is strictly negative and, since the term $T[\mu_0, \mu_{\tilde{A}}] = E[\mu_0] - 2B[\mu_0, \mu_{\tilde{A}}] + E[\mu_{\tilde{A}}]$ is finite, we can choose t small enough so that $E[\mu_t] < E[\mu]$. This concludes the proof.

Under the additional assumption that W is not singular at the origin, we can obtain a slightly stronger version of Proposition 1 which will be needed in Section 4.

Proposition 2. *Assume that W and μ satisfy the same hypotheses as Proposition 1. Assume, moreover, that $W(0) < +\infty$. Then any point $x_0 \in \operatorname{supp}(\mu)$ is a local minimizer of V_μ in the classical sense and V_μ is constant on any connected compact set $K \subset \operatorname{supp}(\mu)$.*

Proof. The proof of the first statement is similar to the proof of Proposition 1. We argue by contradiction: assume that $\mu \in \mathcal{P}(\mathbb{R}^N)$ is a local minimizer of $E[\cdot]$ and that there exists $x_0 \in \operatorname{supp}(\mu)$ which is not a (classical) local minimum of V_μ. Fix $\varepsilon > 0$, then there exists $x_a \in B(x_0, \varepsilon)$ such that $V_\mu(x_a) < V_\mu(x_0)$. But since V_μ is lower semicontinuous, there exists $0 < \eta < \varepsilon$ such that

$$V_\mu(x_a) < V_\mu(x_0) \leqq V_\mu(x) + \frac{V_\mu(x_0) - V_\mu(x_a)}{2} \quad \text{for all } x \in B(x_0, \eta). \tag{21}$$

We then define μ_0 and μ_1 as in the proof of Proposition 1. Following a different idea, now, we send mass from μ_0 to a Dirac Delta at the location x_a instead of distributing it evenly over a set $\tilde{A}$ of nonzero Lebesgue measure: instead of letting $\mu_t = (m_0 - t)\mu_0 + t\mu_{\tilde{A}} + m_1\mu_1$ as before, we now define $\mu_t = (m_0 - t)\mu_0 + t\delta_{x_a} + m_1\mu_1$. The same expansion leads to

$$E[\mu_t] = E[\mu - t\mu_0 + t\delta_{x_a}] = E[\mu] + 2t(B[\delta_{x_a}, \mu] - B[\mu_0, \mu]) + t^2 T[\mu_0, \delta_{x_a}].$$

From (21) we obtain that the term $B[\delta_{x_a}, \mu] - B[\mu_0, \mu]$ is strictly negative. In order to conclude the argument we need the term $T[\mu_0, \delta_{x_a}] = E[\mu_0] - 2B[\mu_0, \delta_{x_a}] + E[\delta_{x_a}]$ to be finite. Note that $E[\delta_{x_a}] = W(0)/2$, therefore it is necessary for $W(0)$ to be finite in order to conclude the proof.

We now prove the second statement. We follow classical arguments from potential theory, see [29, Proposition 0.4], for instance. Let K be a connected compact set contained in $\operatorname{supp}(\mu)$ and consider the sets $A = \{x \in K : V_\mu(x) > \inf_K V_\mu\}$ and $B = \{x \in K : V_\mu(x) = \inf_K V_\mu\}$. Since V_μ is lower semicontinuous, the set A is open relative to K. Let us show that B is also open relative to K. We argue by contradiction. Suppose there exists $x_a \in B$ such that for every $\varepsilon > 0$ there exists $x_{0,\varepsilon} \in K$ with $|x_a - x_{0,\varepsilon}| < \varepsilon$ and $V_\mu(x_a) < V_\mu(x_{0,\varepsilon})$. Then, following the exact same steps as in the proof of the first statement, we can construct a probability measure with lower energy than μ and whose d_∞ distance to μ is smaller than ε, therefore leading to a contradiction and proving that B is open relative to K. Since K is connected, then either A or B must be empty. But since V_μ is lower semicontinuous, it has to reach its minimum on compact sets, and therefore $A = \emptyset$ and $B = K$.

Remark 3. Since $\operatorname{supp}(\mu)$ is closed, the connected components of $\operatorname{supp}(\mu)$ are also closed. So the second statement of Proposition 2 implies that V_μ is constant on any bounded connected component of $\operatorname{supp}(\mu)$. In particular, if μ is compactly supported, then V_μ is constant on any connected component of $\operatorname{supp}(\mu)$.

Combining Lemma 3 and Proposition 1 we obtain:

Corollary 1. *Suppose that W satisfies* (H1′)–(H4). *If μ is a local minimizer of the interaction energy with respect to d_∞ and $E[\mu] < +\infty$, then $(\Delta^0 W * \mu)(x) \geqq 0$ for all $x \in \operatorname{supp}(\mu)$.*

We recall the following result from [16, Theorem 4.13], see also [25, Chapter 8].

Proposition 3. *Let A be a Borel subset of $\mathbb{R}^N$, and $s \geqq 0$. If there exists a probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ supported on A such that*

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{|x-y|^s} < \infty,$$

then $\dim_H A \geqq s$, with $\dim_H$ being the Hausdorff dimension of A.

We are now ready to prove the main theorem.

Proof of Theorem 1. Let ρ be a nonzero part of μ, that is, $\mu = \rho + \nu$ for some nonnegative measure ν. Let $A = \operatorname{supp}(\rho)$ and let us show that $\dim_H A \geqq \beta$. Choose ε small enough so that (10) holds, choose $x_0 \in A$ and define the measure

$$\mu_0(B) = \rho(B \cap B(x_0, \varepsilon/2)).$$

Clearly, μ can be written $\mu = \mu_0 + \mu_1$, where μ_0 and μ_1 are two (nonnegative) measures of mass $m_0 > 0$, and $m_1 \geqq 0$ and where μ_0 is supported in $A \cap B(x_0, \varepsilon/2)$. Then from (10) we get:

$$\begin{aligned} &C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\mathrm{d}\mu_0(x)\,\mathrm{d}\mu_0(y)}{|x-y|^\beta} \leqq \iint_{\mathbb{R}^N \times \mathbb{R}^N} -\Delta^0 W(x-y)\,\mathrm{d}\mu_0(x)\,\mathrm{d}\mu_0(y) \\ &= \iint_{\mathbb{R}^N \times \mathbb{R}^N} -\Delta^0 W(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu_0(y) - \iint_{\mathbb{R}^N \times \mathbb{R}^N} -\Delta^0 W(x-y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_0(y) \\ &= -\int_{\mathbb{R}^N \times \mathbb{R}^N} (\Delta^0 W * \mu)(y)\,\mathrm{d}\mu_0(y) + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Delta^0 W(x-y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_0(y) \\ &\leqq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Delta^0 W(x-y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_0(y) \leqq C^* m_1 m_0 < +\infty. \end{aligned}$$

We have used the fact that $\Delta^0 W * \mu$ is nonnegative on the support of μ from Corollary 1 and that $\Delta^0 W(x) < C^*$ by hypothesis (H4). We then apply Proposition 3 to the probability measure μ_0/m_0, which is supported on A, to obtain $\dim_H A \geqq \beta$.

3.3. Example of Potentials Satisfying the Hypotheses of Theorem 1

In this subsection we consider the class of potentials

$$W_\alpha(x) = c\, h_\alpha(x) + \psi(x), \tag{22}$$

where $\psi \in C^3(\mathbb{R}^N)$ is bounded from below, $c > 0$ and $h_\alpha : \mathbb{R}^N \to (-\infty, \infty]$ is the power-law function:

$$h_\alpha(x) = -|x|^\alpha/\alpha,$$

for $x \neq 0$ and $\alpha \in \mathbb{R}$ with the convention $h_0(x) = -\log |x|$. We define $h_\alpha(0) = 0$ if $\alpha > 0$, or $h_\alpha(0) = +\infty$ if $\alpha \leqq 0$. The potentials W_α are typical examples of repulsive–attractive potentials behaving like $-|x|^\alpha/\alpha$ around the origin. It is trivial to check that W_α satisfies (H1)–(H2)–(H3-loc) for any $\alpha > -N$ (in the case $\alpha \geqq 0$ the function ψ needs to grow fast enough at infinity for hypothesis (H1) to hold). Note, also, that for $x \neq 0$ we have

$$-\Delta W_\alpha(x) = c\frac{(\alpha + N - 2)}{|x|^{2-\alpha}} - \Delta\psi(x), \tag{23}$$

and therefore if $\alpha + N - 2 > 0$, then W_α satisfies (10) from the definition of β-repulsivity with $\beta = 2 - \alpha$. The goal of this subsection is to show that W_α also satisfies (11) and (H4-loc).

We start by checking (11). An explicit computation gives

$$\begin{aligned} -\Delta^\varepsilon h_\alpha(0) &= \frac{2(N+2)}{\varepsilon^2}\left(h_\alpha(0) - \fint_{B(0,\varepsilon)} h_\alpha(y)\,\mathrm{d}y\right) \\ &= \begin{cases} 2(N+2)\frac{N}{N+\alpha}\frac{\varepsilon^{\alpha-2}}{\alpha} & \text{if } \alpha > 0 \\ +\infty & \text{if } 2 - N \leqq \alpha \leqq 0, \end{cases} \end{aligned}$$

where we have used the fact that $h_\alpha(0) = 0$ for $\alpha > 0$ and $h_\alpha(0) = +\infty$ for $\alpha \leqq 0$. Letting $\varepsilon \to 0$ and using the fact that $\Delta\psi(0)$ is finite we obtain

$$-\Delta^0 W(0) = +\infty \quad \text{for all } \alpha < 2. \tag{24}$$

Combining (23) and (24) we see that for $2 - N < \alpha < 2$ the potential W_α is β-repulsive with $\beta = 2 - \alpha \in (0, N)$.

We now show that for $\alpha > 2 - N$ the potentials W_α satisfy hypothesis (H4-loc). The key point is that the functions h_α are superharmonic for $\alpha > 2 - N$. Let us recall the definition of superharmonicity:

Definition 3. A lower semicontinuous function $h : \mathbb{R}^N \to (-\infty, \infty]$ is said to be superharmonic on the connected open set Ω if it is not identically equal to $+\infty$ on Ω and if

$$h(x) \geqq \fint_{B(x,r)} h(y)\,\mathrm{d}y$$

for all $x \in \Omega$ and $r > 0$ such that $B(x, r) \subset \Omega$.

We also recall that if $h \in C^2(\Omega)$, then h is superharmonic on Ω if and only if $\Delta h(x) \leqq 0$ for all $x \in \Omega$. To see that the functions h_α are superharmonic for $\alpha > 2 - N$, first note that for $x \neq 0$

$$\Delta h_\alpha(x) = -\frac{(\alpha + N - 2)}{|x|^{2-\alpha}} \leqq 0.$$

Therefore h_α is superharmonic on $\mathbb{R}^N\backslash\{0\}$ and it can be easily checked that it satisfies the super-mean value property at the origin [29, Definition 2.1]. Both together imply that it is actually superharmonic on the full space $\mathbb{R}^N$, [29, Corollary 2.1]. As a consequence, we directly obtain from the definition of the approximate Laplacian that $\Delta^\varepsilon h_\alpha(x) \leqq 0$ and therefore

$$\Delta^\varepsilon W_\alpha = c\Delta^\varepsilon h_\alpha + \Delta^\varepsilon \psi \leqq \Delta^\varepsilon \psi.$$

To conclude, we note that since $\psi \in C^3(\mathbb{R}^N)$, a simple Taylor expansion shows that $\Delta^\varepsilon \psi$ converges uniformly to $\Delta\psi$ on compact sets. Indeed, the expansion gives

$$\begin{aligned}
\Delta^\varepsilon \psi(x) &= \frac{2(N+2)}{\varepsilon^2} \fint_{B(0,\varepsilon)} \psi(x+y) - \psi(x)\,\mathrm{d}y \\
&= \frac{2(N+2)}{\varepsilon^2} \fint_{B(0,\varepsilon)} y^T \nabla\psi(x) + y^T H\psi(x) y + O(\varepsilon^3)\,\mathrm{d}y \qquad (25)\\
&= \frac{2(N+2)}{\varepsilon^2} \left(\Delta\psi(x) \fint_{B(0,\varepsilon)} y_1^2\,\mathrm{d}y + O(\varepsilon^3) \right) \qquad (26)\\
&= \Delta\psi(x) + O(\varepsilon).
\end{aligned}$$

To go from (25) to (26) we have used the fact that most of the terms in the Taylor expansion are equal to zero after integrating on spheres, due to symmetry. The only remaining terms are the ones involved in the Laplacian. Note that since the partial derivatives of ψ of order 3 are bounded on compact subsets of $\mathbb{R}^N$, then the error term is uniform for x in compact sets. Since $\Delta\psi$ is bounded on compact sets, we conclude that for $\alpha > 2 - N$ the potential W_α satisfies (H4-loc). We summarize this discussion in the following proposition:

Proposition 4. *If $2-N < \alpha < 2$ and if $\psi \in C^3(\mathbb{R}^N)$, then W_α is $(2-\alpha)$-repulsive around the origin and satisfies* (H4-loc).

Finally, let us give examples of repulsive–attractive potentials $W(x) = w(|x|)$ that satisfy (H1)–(H4) rather than (H1)–(H2)–(H3-loc)–(H4-loc). In order to provide these examples we will use the already constructed potential W_α and ensure that they behave well as $|x| \to \infty$.

Since $W(x) = w(|x|)$ is assumed to be repulsive–attractive, the function w is increasing for r greater than some $r_0 > 0$. Condition (H3) therefore implies that w is bounded on $(r_0, +\infty)$. On the other hand, condition (H4) implies some bound on the derivatives of W. If $2 - N < \alpha < 0$, then $h_\alpha(x) \to 0$ as $|x| \to \infty$. It can then be easily checked that if the function ψ in (22) and its partial derivatives up to order three are bounded in $\mathbb{R}^N$, then W_α satisfies (H1)–(H4). If $0 \leqq \alpha < 2$, one can construct a family of potentials $\tilde{W}_\alpha$ that satisfies (H1)–(H4) by letting $\tilde{W}_\alpha = W_\alpha$ in some ball $B(0, R)$, and by extending $\tilde{W}_\alpha$ outside of this ball in such a way that:

1. $\tilde{W}_\alpha$ is three times continuously differentiable away from the origin,
2. $\tilde{W}_\alpha$ and its derivative up to order three are bounded in $\{x \in \mathbb{R}^N : |x| > R/2\}$.

4. Mild Repulsion Implies 0-Dimensionality

In this section, we will show that if the potential is mildly repulsive, meaning that it behaves locally near zero like $-|x|^\alpha$ with $\alpha > 2$, then the support of the measure cannot contain measures concentrated on smooth manifolds of any dimension except 0-dimensional sets.

Definition 4. Let $1 \leqq k \leqq N$. A probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ is said to have a regular k-dimensional part if it can be written

$$\mu = \mu_1 + \mu_2,$$

where μ_1 is a nonnegative measure on $\mathbb{R}^N$ and μ_2 is defined by

$$\int_{\mathbb{R}^N} \psi(x)\, \mathrm{d}\mu_2(x) = \int_{\mathcal{M}} \psi(x) f(x)\, \mathrm{d}\sigma(x) \quad \forall \psi \in C_0(\mathbb{R}^N),$$

for some C^2 manifold $\mathcal{M}$ of dimension k embedded in $\mathbb{R}^N$ and a non-identically zero nonnegative function $f : \mathcal{M} \to (0, +\infty]$ integrable with respect to the surface measure $d\sigma(x)$ on $\mathcal{M}$. Moreover, to avoid pathological situations, we assume that there exists $x_0 \in \mathcal{M}$, $c, \kappa > 0$ such that

$$f(x) \geqq c \quad \forall x \in \mathcal{M} \cap B(x_0, \kappa). \tag{27}$$

We now state the main result of this section:

Theorem 2. *Let $W \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which is equal to $-|x|^\alpha/\alpha$ in a neighborhood of the origin. If $\alpha > 2$ then a local minimizer of the interaction energy with respect to d_∞ cannot have a k-dimensional component for any $1 \leqq k \leqq N$.*

For the above theorem to be true it is not necessary for the potential to be exactly equal to a power law $-|x|^\alpha$, $\alpha > 2$, around the origin. It is enough for the potential to behave like $-|x|^\alpha$, $\alpha > 2$, at the origin in a precise convexity sense, see Theorem 3.

4.1. Preliminaries on Convexity

To prove Theorem 2, we need some convex analysis concepts, see [1,13] and the references therein. The term *modulus of convexity* refers to any convex function ϕ on the positive reals satisfying

(ϕ_0) $\phi : [0, \infty) \longrightarrow \mathbb{R}$ is continuous, $\phi(0) = 0$, and $\phi(x) \neq 0$ for $x > 0$.

(ϕ_1) $\phi(x) \geqq -kx$ for some $k < \infty$.

Now, we can quantify the convexity of certain functions in terms of a modulus of convexity.

Definition 5. A function $h : [0, +\infty) \to \mathbb{R}$ is $\phi-$uniformly convex on (a, b) if there exists a modulus of convexity ϕ such that

$$h\left(\frac{r_1 + r_2}{2}\right) \leqq \frac{1}{2}\,(h(r_1) + h(r_2)) - \frac{1}{4}\int_0^{|r_1 - r_2|} \phi(t)\, \mathrm{d}t, \tag{28}$$

for all $r_1, r_2 \in (a, b)$.

A function $h : [0, +\infty) \to \mathbb{R}$ is $\lambda-$convex on (a, b) if it is ϕ-uniformly convex with $\phi(s) = \lambda s$ and $\lambda \in \mathbb{R}$.

Note that if h is $\lambda-$convex, then (28) reads

$$h\left(\frac{r_1+r_2}{2}\right) \leqq \frac{1}{2}h(r_1) + \frac{1}{2}h(r_2) - \frac{\lambda}{8}(r_1-r_2)^2, \tag{29}$$

for all $r_1, r_2 \in (a,b)$. It is equivalent to assume that the function $h(r) - \frac{\lambda}{2}r^2$ is convex on (a,b). The following proposition can be easily proven:

Proposition 5. (Convexity properties of power laws)

(i) *If* $q \in (1,2]$*, then* $h(r) = r^q$ *is* $\lambda-$*convex on* $[0,R]$ *for* $\lambda = \inf_{(0,R)} h'' = q(q-1)R^{q-2} > 0$*, and thus, uniformly convex on* $[0,R]$.
(ii) *If* $q > 2$*, then* $h(r) = r^q$ *is* $\phi-$*uniformly convex on* $\mathbb{R}_+$*, with* $\phi(t) = 2^{2-q}t^{q-1}/q$*. That is*

$$h\left(\frac{r_1+r_2}{2}\right) \leqq \frac{1}{2}(h(r_1)+h(r_2)) - 2^{-q}|r_1-r_2|^q, \tag{30}$$

for $r_1, r_2 \geqq 0$.

4.2. Proof of Theorem 2

In this subsection we prove the following generalization of Theorem 2.

Theorem 3. *Let* $W(x) = w(|x|)$ *be continuously differentiable, bounded from below, and decreasing as a function of* $|x|$ *in a neighborhood of the origin. Assume, moreover that* W *behaves like the power law* $-|x|^\alpha$, $\alpha > 2$*, near the origin, in the sense that for some* $C^* > 0$ *and* $R > 0$ *small enough,* $w(r) = -h(r^2)$ *satisfies:*

- *If* $\alpha \in (2,4]$*,* h *is* $\lambda-$*convex on* $[0,R]$ *with* $\lambda = C^* R^{\alpha/2-2}$.
- *If* $\alpha \in (4,\infty)$*,* h *is* $\phi-$*uniformly convex on* $[0,R]$*, with* $\phi(t) = C^* t^{\alpha/2-1}$,

and $C^*|w'(r)| \leqq r^{\alpha-1}$ *on* $[0,R]$*. Then a local minimizer of the interaction energy with respect to* d_∞ *cannot have a k-dimensional part for any* $1 \leqq k \leqq N$.

Theorem 2 is a direct consequence of Theorem 3, thanks to Proposition 5.

We first provide an explicit formula for how the energy changes when perturbing a local minimizer:

Lemma 4. *Suppose that* $W : \mathbb{R}^N \to (-\infty,+\infty]$ *is symmetric, lower semicontinuous and bounded from below with* $W(0) < +\infty$*. Let* $\mu \in \mathcal{P}(\mathbb{R}^N)$ *be a local minimizer of the interaction energy with respect to* d_∞ *and* $E[\mu] < +\infty$*. Given a connected domain* $\Omega \subseteq \operatorname{supp}(\mu)$*, a Borel map* $\pi : \Omega \to \Omega$ *and a convex decomposition* $\mu = m_1\mu_1 + m_2\mu_2$ *with* $\operatorname{supp}(\mu_1) \subset \Omega$*, we have that*

$$E[m_1(\pi\#\mu_1) + m_2\mu_2] - E[m_1\mu_1 + m_2\mu_2] = m_1^2\, T[\pi\#\mu_1, \mu_1]. \tag{31}$$

Proof of Theorem 1. Since B in (7) is a bilinear form,

$$\begin{aligned} &E[m_1(\pi\#\mu_1)+m_2\mu_2]-E[m_1\mu_1+m_2\mu_2]\\ &\quad=m_1^2B[\pi\#\mu_1,\pi\#\mu_1]+2m_1m_2B[\pi\#\mu_1,\mu_2]\\ &\qquad-m_1^2B[\mu_1,\mu_1]-2m_1m_2B[\mu_1,\mu_2]. \end{aligned} \tag{32}$$

We now use the fact that μ is a local minimizer to express the terms involving μ_2 as terms involving only μ_1. Proposition 2 implies that the function $V_\mu(x)$ is constant on the connected domain Ω. Since $\pi(\Omega)\subset\Omega$, we have:

$$\int_{\mathbb{R}^N} W(\pi(x)-y)\,\mathrm{d}\mu(y)=\int_{\mathbb{R}^N} W(x-y)\,\mathrm{d}\mu(y)\quad\forall x\in\Omega,$$

and therefore, since $\mu=m_1\mu_1+m_2\mu_2$,

$$\begin{aligned} &m_1\int_{\mathbb{R}^N} W(\pi(x)-y)\,\mathrm{d}\mu_1(y)+m_2\int_{\mathbb{R}^N} W(\pi(x)-y)\,\mathrm{d}\mu_2(y)\\ &\quad=m_1\int_{\mathbb{R}^N} W(x-y)\,\mathrm{d}\mu_1(y)+m_2\int_{\mathbb{R}^N} W(x-y)\,\mathrm{d}\mu_2(y) \end{aligned}$$

for all $x\in\Omega$. Since $\mathrm{supp}(\mu_1)\subset\Omega$, we can integrate both sides with respect to $d\mu_1(x)$ and obtain, after multiplication by m_1,

$$\begin{aligned} &m_1^2\int_{\mathbb{R}^N\times\mathbb{R}^N} W(\pi(x)-y)\,\mathrm{d}\mu_1(y)\mathrm{d}\mu_1(x)+m_1m_2\int_{\mathbb{R}^N\times\mathbb{R}^N} W(\pi(x)-y)\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(x)\\ &\quad=m_1^2\int_{\mathbb{R}^N\times\mathbb{R}^N} W(x-y)\,\mathrm{d}\mu_1(y)\,\mathrm{d}\mu_1(x)+m_1m_2\int_{\mathbb{R}^N\times\mathbb{R}^N} W(x-y)\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(x), \end{aligned}$$

or equivalently, using the B-notation,

$$2m_1^2B[\mu_1,\pi\#\mu_1]+2m_1m_2B[\mu_2,\pi\#\mu_1]=2m_1^2B[\mu_1,\mu_1]+2m_1m_2B[\mu_2,\mu_1],$$

and therefore rearranging the terms, we can express the terms involving μ_2 in terms of the ones involving only μ_1:

$$2m_1m_2[B[\pi\#\mu_1,\mu_2]-B[\mu_1,\mu_2]]=2m_1^2[B[\mu_1,\mu_1]-B[\mu_1,\pi\#\mu_1]].$$

The desired identity (31) is readily obtained by plugging the last equality into (32), recalling the definition of $T[\mu,\nu]$ in Section 2.

Definition 6. Let $1\leqq k\leqq N$. We denote by D_ε^k the the k-dimensional disk of radius ε:

$$D_\varepsilon^k=\{(x_1,\ldots,x_N):\ x_1^2+\cdots+x_k^2\leqq\varepsilon^2 \text{ and } x_{k+1}=\cdots=x_N=0\},$$

and by $\nu_{\varepsilon,k}\in\mathcal{P}(\mathbb{R}^N)$ the uniform probability distribution on D_ε^k, that is, the probability measure defined as

$$\int_{\mathbb{R}^N}\psi(x)\,\mathrm{d}\nu_{\varepsilon,k}(x)=\frac{1}{|D_\varepsilon^k|}\int_{x_1^2+\cdots+x_k^2\leqq\varepsilon^2}\psi(x_1,\ldots,x_k,0,\ldots,0)\,\mathrm{d}x_1\cdots\mathrm{d}x_k$$

for all $\psi\in C^0(\mathbb{R}^N)$, where $|D_\varepsilon^k|$ is the Lebesgue measure of dimension k of D_ε^k, that is, $|D_\varepsilon^k|=\sigma_k\varepsilon^k$ with σ_k being the area of the unit k-dimensional ball.

The following Lemma, combined with Lemma 4, shows that if a flat k-dimensional disk is contained in the support of a local minimizer, then the energy can be reduced by concentrating all the mass contained in the disk into a single point. As a consequence, the support of a local minimizer cannot contain a flat k-dimensional disk.

Lemma 5. *Suppose that $W(x) = -h(|x|^2)$ satisfies the assumptions of Theorem 3. Then, there exists $c_{k,\alpha} > 0$ such that for ε small enough,*

$$T[\delta_0, \nu_{\varepsilon,k}] = B[\delta_0, \delta_0] - 2B[\delta_0, \nu_{\varepsilon,k}] + B[\nu_{\varepsilon,k}, \nu_{\varepsilon,k}] \leqq -c_{k,\alpha}\varepsilon^{\alpha}.$$

Proof. Since W is bounded from below and $W(0) < +\infty$, we can assume without loss of generality that $W(0) = 0$ by adding a suitable constant to W. Then, the first term $B[\delta_0, \delta_0]$ is equal to zero. Symmetrizing the integral involved in the second term we obtain:

$$\begin{aligned} B[\delta_0, \nu_{\varepsilon,k}] = -\frac{1}{2}\int_{\mathbb{R}^N} h(|y|^2)\,\mathrm{d}\nu_{\varepsilon,k}(y) &= -\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} h(|y|^2)\,\mathrm{d}\nu_{\varepsilon,k}(x)\,\mathrm{d}\nu_{\varepsilon,k}(y) \\ &= -\frac{1}{4}\int_{\mathbb{R}^N\times\mathbb{R}^N} [h(|x|^2) + h(|y|^2)]\,\mathrm{d}\nu_{\varepsilon,k}(x)\,\mathrm{d}\nu_{\varepsilon,k}(y). \end{aligned}$$

Since the density of the measure $\nu_{\varepsilon,k}$ is symmetric by definition, we can also symmetrize the third term and obtain:

$$\begin{aligned} B[\nu_{\varepsilon,k}, \nu_{\varepsilon,k}] &= -\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} h(|x-y|^2)\,\mathrm{d}\nu_{\varepsilon,k}(x)\,\mathrm{d}\nu_{\varepsilon,k}(y) \\ &= -\frac{1}{4}\int_{\mathbb{R}^N\times\mathbb{R}^N} [h(|x-y|^2) + h(|x+y|^2)]\,\mathrm{d}\nu_{\varepsilon,k}(x)\,\mathrm{d}\nu_{\varepsilon,k}(y). \end{aligned}$$

Combining the three terms we find

$$T[\delta_0, \nu_{\varepsilon,k}] = \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} A(x,y)\,\mathrm{d}\nu_{\varepsilon,k}(x)\,\mathrm{d}\nu_{\varepsilon,k}(y), \tag{33}$$

with

$$A(x,y) := h(|x|^2) + h(|y|^2) - \frac{h(|x-y|^2) + h(|x+y|^2)}{2}.$$

Under the assumptions of Theorem 2, h is convex on $(0, 2\varepsilon^2)$; since $h(0) = 0$, we deduce

$$h(r_i^2) \leqq \frac{r_i^2}{r_1^2 + r_2^2} h(r_1^2 + r_2^2),$$

for $r_i \geqq 0,\ i = 1, 2$. Using the above inequalities for $i = 1, 2$ we get

$$h(|x|^2) + h(|y|^2) \leqq h(|x|^2 + |y|^2) = h\left(\frac{1}{2}|x+y|^2 + \frac{1}{2}|x-y|^2\right).$$

In the rest of this Lemma, C denotes some generic constant that will change from step to step. For $\alpha \in (2,4]$, h is $\lambda-$convex on $(0, 2\varepsilon^2)$ with $\lambda = C\varepsilon^{\alpha-4}$, so that plugging into (29), we obtain

$$h(|x|^2) + h(|y|^2) \leqq \frac{1}{2}h(|x+y|^2) + \frac{1}{2}h(|x-y|^2) - C\varepsilon^{\alpha-4}(x\cdot y)^2. \tag{34}$$

Combining (33) and (34), together with the change of variables $(x, y) = \varepsilon(\tilde{x}, \tilde{y})$, we get, dropping the tildes:

$$\begin{aligned} T[\delta_0, \nu_{\varepsilon,k}] &\leqq -C\varepsilon^{\alpha-4}\int_{\mathbb{R}^N\times\mathbb{R}^N}(x\cdot y)^2\,\mathrm{d}\nu_{\varepsilon,k}(x)\,\mathrm{d}\nu_{\varepsilon,k}(y)\\ &= -C\varepsilon^{\alpha}\int_{\mathbb{R}^N\times\mathbb{R}^N}(x\cdot y)^2\,\mathrm{d}\nu_{1,k}(x)\,\mathrm{d}\nu_{1,k}(y). \end{aligned}$$

For $\alpha \geqq 4$, h is $\phi-$convex with $\phi(t) = Ct^{\alpha/2-1}$, so that plugging into (30), we obtain

$$h(|x|^2) + h(|y|^2) \leqq \frac{1}{2}h(|x+y|^2) + \frac{1}{2}h(|x-y|^2) - C|x\cdot y|^{\alpha/2}. \tag{35}$$

Combining (33) and (35) with an ε-dilation change of variables, we similarly get:

$$\begin{aligned} T[\delta_0, \nu_{\varepsilon,k}] &\leqq -C\int_{\mathbb{R}^N\times\mathbb{R}^N}|x\cdot y|^{\alpha/2}\,\mathrm{d}\nu_{\varepsilon,k}(x)\,\mathrm{d}\nu_{\varepsilon,k}(y)\\ &= -C\varepsilon^{\alpha}\int_{\mathbb{R}^N\times\mathbb{R}^N}|x\cdot y|^{\alpha/2}\,\mathrm{d}\nu_{1,k}(x)\,\mathrm{d}\nu_{1,k}(y). \end{aligned}$$

The last Lemma combined to Lemma 4 shows that the support of a local minimizer cannot contain a flat k-dimensional disk of radius ε. To conclude the proof of Theorem 3, we need to introduce some differential geometry tools. Let $R > 0$, and $g : D_R^k \to \mathbb{R}^{N-k}$ a C^2-function such that $g(0) = 0$, $\nabla g(0) = 0$. We define the parameterization P_g of the graph of g as follows:

$$\begin{aligned} P_g : D_R^k &\longrightarrow \quad \mathbb{R}^N,\\ (x', 0) \quad &\mapsto \ (x', g(x')), \end{aligned} \tag{36}$$

where $x' = (x_1, \ldots, x_k)$ stands for the k first coordinates. Let us remark that classical differential geometry implies that any C^2-manifold can be locally parameterized by such graphs by conveniently choosing the axis and reordering of variables. Moreover, this can be done in such a way that the volume element of the graph J_g is as close as desired to the unit volume element of the flat tangent space by taking R small enough, see [30]. More precisely, there exists a constant C_g depending only on the second derivatives of g on D_R^k such that

$$\|J_g - 1\|_{L^\infty(D_\varepsilon^k)} \leqq C_g\varepsilon, \tag{37}$$

for $0 < \varepsilon < R$ small enough.

Lemma 6. *If W satisfies the assumptions of Theorem 3, and $g \in C^2(D_R^k, \mathbb{R}^{N-k})$ satisfies $g(0) = 0$, $\nabla g(0) = 0$, then for $\varepsilon > 0$ small enough,*

$$T[\delta_0, P_g \# \nu_{\varepsilon,k}] - T[\delta_0, \nu_{\varepsilon,k}] \leqq \frac{2^{\alpha-1}\varepsilon^\alpha}{C^*} \|\nabla g\|_{L^\infty(D_\varepsilon^k)}.$$

Proof. Note that by continuity for $\varepsilon > 0$ small enough, we have $\|\nabla g\|_{L^\infty(D_\varepsilon^k)} \leqq 1$. We first point out that

$$T[\delta_0, P_g \# \nu_{\varepsilon,k}] - T[\delta_0, \nu_{\varepsilon,k}] = \int_{\mathbb{R}^N \times \mathbb{R}^N} A(x, y) \, \mathrm{d}\nu_{\varepsilon,k}(x) \, \mathrm{d}\nu_{\varepsilon,k}(y),$$

with

$$A(x, y) = \frac{w(|P_g(x) - P_g(y)|) - w(|x - y|)}{2} - [w(|P_g(x)|) - w(|x|)].$$

Thanks to the definition of the parameterization P_g, $|P_g(x) - P_g(y)| \geqq |x - y|$. Moreover, since w is decreasing in a neighborhood of the 0, the first term in $A(x, y)$ is negative for $\max(|x|, |y|) < \varepsilon$ small enough. To estimate the second term, we use the mean value theorem for g around $x' = 0$, remembering that $g(0) = 0$:

$$|P_g(x) - (x', 0)| = |g(x') - g(0)| \leqq \varepsilon \|\nabla g\|_{L^\infty(D_\varepsilon^k)},$$

since $C^*|w'(r)| \leqq r^{\alpha-1}$, we conclude

$$w(|P_g(x)|) - w(|x|) \leqq \|w'\|_{L^\infty([0,2\varepsilon])} \varepsilon \|\nabla g\|_{L^\infty(D_\varepsilon^k)} \leqq \frac{2^{\alpha-1}\varepsilon^\alpha}{C^*} \|\nabla g\|_{L^\infty(D_\varepsilon^k)}.$$

Proof of the Theorem 2. Assume that μ is a local minimizer of E in d_∞ and that it has a regular k-dimensional part in the sense of Definition 4. Let $\mathcal{M}$ be the C^1-submanifold on which this component is supported, and f be the density on $\mathcal{M}$ of this component. Let $x_0 \in \mathcal{M}$, and $c, \kappa > 0$ satisfying (27). As discussed above and without loss of generality, we can assume that $x_0 = 0$ and that $\mathcal{M}$ is locally the graph of a C^2-function $g : D_R^k \to \mathbb{R}^{N-k}$, for some $\kappa > R > 0$, such that $g(0) = 0$, $\nabla g(0) = 0$.

Let P_g be the parameterization defined in (36). Note that for $\varepsilon \leqq R$, $\mu_1^\varepsilon := P_g \# \nu_{\varepsilon,k} \in \mathcal{P}(\mathbb{R}^N)$ is absolutely continuous with respect to the volume element on $\mathcal{M}$ with a density still denoted by μ_1^ε, and satisfying [by (37)]

$$\|\mu_1^\varepsilon\|_{L^\infty(\mathcal{M},d\sigma)} \leqq \frac{1}{|D_\varepsilon^k| \, I_\varepsilon} \leqq \frac{1}{|D_\varepsilon^k| \, (1 - C_g \varepsilon)}, \quad \text{with } I_\varepsilon = \inf_{x \in \bar{D}_\varepsilon^k} J_g(x).$$

Therefore, choosing $m_1 = \frac{c}{2} |D_\varepsilon^k| \, (1 - C_g \varepsilon)$, then $f(x) > m_1 \mu_1^\varepsilon$ on $x \in D_\varepsilon^k$, and we can decompose μ as a convex combination

$$\mu = m_1 \mu_1^\varepsilon + m_2 \mu_2^\varepsilon,$$

where $\mu_2^\varepsilon \in \mathcal{P}(\mathbb{R}^N)$.

Now, we are going to send all mass from μ_1^ε to a Dirac Delta at $x_0 = 0$. Let us define $\pi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ by $\pi \equiv 0$ and $\mu^\varepsilon := m_1\pi\#\mu_1^\varepsilon + m_2\mu_2^\varepsilon$ (note that $\pi\#\mu_1^\varepsilon = \delta_0$). μ^ε is then a small perturbation of μ in d_∞:

$$d_\infty(\mu, \mu^\varepsilon) \leqq \varepsilon(1 + \|\nabla g\|_{L^\infty(D_\varepsilon^k)}). \tag{38}$$

To check this just take a map $\mathcal{T}$ in Definition 9 such that $\mathcal{T}(x) = x$ for all $x \in \mathcal{M}/P_g(D_\varepsilon^k)$ and such that $\mathcal{T}(x) = 0$ for $x \in P_g(D_\varepsilon^k)$. Thus, the maximum displacement produced by the transport map $\mathcal{T}$ is bounded by the maximum of $|P_g(x)|$ for $x \in P_g(D_\varepsilon^k)$ leading to (38), using that $g(0) = 0$ and the mean value theorem.

Since μ_1^ε has a connected support that contains $\pi(\text{supp}(\mu_1^\varepsilon)) = \{0\}$, we can apply Lemma 4 to get:

$$\begin{aligned} E[\mu^\varepsilon] - E[\mu] &= m_1^2\, T[\pi\#\mu_1^\varepsilon, \mu_1^\varepsilon] \\ &= m_1^2\, T[\pi\#\mu_1^\varepsilon, \nu_{\varepsilon,k}] + m_1^2 \left(T[\pi\#\mu_1^\varepsilon, \mu_1^\varepsilon] - T[\pi\#\mu_1^\varepsilon, \nu_{\varepsilon,k}]\right). \end{aligned}$$

Since $\pi\#\mu_1^\varepsilon = \delta_0$, we can use Lemma 5 to estimate the first term. Moreover, since $\mu_1^\varepsilon = P_g\#\nu_{\varepsilon,k}$, we can use Lemma 6 to estimate the last two terms, so that we finally conclude

$$E[\mu^\varepsilon] - E[\mu] \leqq m_1^2 \left[-c_{k,\alpha}\varepsilon^\alpha + C\varepsilon^\alpha \|\nabla g\|_{L^\infty(D_\varepsilon^k)}\right].$$

Note that $g \in C^1(D_R^k)$ and $\nabla g(0) = 0$ imply that $\|\nabla g\|_{L^\infty(D_\varepsilon^k)} \to 0$ as $\varepsilon \to 0$; thus, if $\varepsilon > 0$ is small enough, we get that $E[\mu^\varepsilon] - E[\mu] < 0$.

Thus, μ^ε is a better competitor in the minimization of E for ε arbitrarily small. This leads to a contradiction with the fact that μ is a local minimizer of E, showing Theorem 2.

5. Euler–Lagrange Approach to Study Local Minimizers in the d_2-Topology

So far, we have used transport plans to build perturbed measures. This enabled us to study local minimizers of the interaction energy with respect to the d_∞-topology. To study local minimizers with respect to the d_2-topology, it is actually possible to use a more classical Euler–Lagrange approach, as we will present in this section. The Euler–Lagrange conditions that we will derive were formally obtained in [4] by perturbing densities inside and outside their support. Here, we provide a rigorous proof in the case of probability measures endowed with the distance d_2.

Theorem 4. *Consider an interaction potential W satisfying* (H1)–(H2). *Let us consider $\mu \in \mathcal{P}_2(\mathbb{R}^N)$ a local minimizer of E with respect to d_2, such that $E[\mu] < \infty$. Then,*

(i) $(W * \mu)(x) = 2E[\mu]$ *μ-almost everywhere*
(ii) $(W * \mu)(x) \leqq 2E[\mu]$ *for all $x \in$* $\text{supp}(\mu)$.
(iii) $(W * \mu)(x) \geqq 2E[\mu]$ *for almost everywhere $x \in \mathbb{R}^N$.*

Proof. As usual, we assume that $W \geqq 0$ without loss of generality. Lemma 2 implies that $W * \mu$ is well defined, lower semicontinuous and non-negative.

In order to prove the first two items, let us choose $\varphi \in C_0^\infty(\mathbb{R}^N)$ to define

$$\nu = \left(\varphi - \int_{\mathbb{R}^N} \varphi \, \mathrm{d}\mu\right) \mu := a(x)\mu,$$

and $\mu_\varepsilon = \mu + \varepsilon\nu = (1 + \varepsilon a(x))\mu$ with $\varepsilon > 0$ to be specified. It is clear that $\mu_\varepsilon(\mathbb{R}^N) = 1$, since $a(x)$ has zero integral with respect to μ. Moreover, since $a(x) \geqq -2\|\varphi\|_{L^\infty}$, then $\mu_\varepsilon \geqq 0$ for $\varepsilon < \frac{1}{2\|\varphi\|_{L^\infty}} = \varepsilon_\varphi$. Therefore, $\mu_\varepsilon \in \mathcal{P}(\mathbb{R}^N)$ for all $\varepsilon < \varepsilon_\varphi$. It is easy to check that $\mu_\varepsilon \in \mathcal{P}_2(\mathbb{R}^N)$, that $\mu_\varepsilon \rightharpoonup \mu$ weakly-$*$ as measures, and

$$\int_{\mathbb{R}^N} |x|^2 \, \mathrm{d}\mu_\varepsilon \to \int_{\mathbb{R}^N} |x|^2 \, \mathrm{d}\mu.$$

Therefore, we have that

$$d_2(\mu_\varepsilon, \mu) \to 0 \quad \text{as } \varepsilon \to 0.$$

Note that it is not true in general that $d_\infty(\mu_\varepsilon, \mu) \to 0$ as $\varepsilon \to 0$. Consider, for instance, $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ and $\varphi|_{[0,1]} = x$, then $\mu_\varepsilon = (\frac{1}{2} - \varepsilon)\delta_0 + (\frac{1}{2} + \varepsilon)\delta_1$, and $d_\infty(\mu_\varepsilon, \mu) = 1$ for any $\varepsilon > 0$.

Now, since μ is a local minimizer in d_2, then $E[\mu_\varepsilon] \geqq E[\mu]$ for ε small enough. Moreover, since μ has finite energy, then $E[\mu_\varepsilon] < \infty$ and we can expand it as

$$\begin{aligned} \frac{E[\mu_\varepsilon] - E[\mu]}{\varepsilon} &= \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) \, \mathrm{d}\nu(x) \, \mathrm{d}\mu(y) \\ &+ \frac{\varepsilon}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) \, \mathrm{d}\nu(x) \, \mathrm{d}\nu(y) \geqq 0, \end{aligned}$$

with both integral terms well-defined. As $\varepsilon \to 0$, we easily get

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) \, \mathrm{d}\nu(x) \, \mathrm{d}\mu(y) \geqq 0,$$

or equivalently,

$$\int \varphi[(W * \mu)(x) - 2E[\mu]] \, \mathrm{d}\mu(x) \geqq 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since one can take either φ or $-\varphi$ as test functions, we deduce

$$\int \varphi[(W * \mu)(x) - 2E[\mu]] \, \mathrm{d}\mu(x) = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, and thus (i) is satisfied for almost everywhere μ.

Let us now prove (ii). Take $x \in \operatorname{supp}(\mu)$. Then there exists $\{x_n\}_{n \in \mathbb{N}} \to x$ with $x_n \in \operatorname{supp}(\mu)$, such that $(W * \mu)(x_n) = 2E[\mu]$. The existence of such a sequence is ensured since $\mu(B(x,\varepsilon)) > 0$ for all $\varepsilon > 0$ by definition of the support of μ. Then, by lower semicontinuity of $W * \mu$ we get

$$(W * \mu)(x) \leqq \liminf_{n\to\infty} (W * \mu)(x_n) = 2E[\mu],$$

and then (ii) is satisfied.

In order to show (iii), we consider different variations to the arguments constructed above. Take $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, $\psi \geqq 0$ and then take

$$\nu = \psi - \left(\int_{\mathbb{R}^N} \psi \, \mathrm{d}x\right)\mu.$$

Again, defining $\mu_\varepsilon = \mu + \varepsilon\nu$, then it verifies $\mu_\varepsilon(\mathbb{R}^N) = 1$ and if $\varepsilon < 1/\int \psi \, \mathrm{d}x$, then $\mu_\varepsilon \geqq 0$. As previously, it is easy to check that

$$d_2(\mu_\varepsilon, \mu) \to 0 \quad \text{as } \varepsilon \to 0.$$

Note, again, that it is not true in general that $d_\infty(\mu_\varepsilon, \mu) \to 0$ as $\varepsilon \to 0$.

Proceeding similarly to point (i), we get

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) \, \mathrm{d}\nu(y) \, \mathrm{d}\mu(x) \geqq 0,$$

taking $\varepsilon \to 0$ in $E[\mu_\varepsilon] \geqq E[\mu]$. Therefore, we conclude that

$$\int_{\mathbb{R}^N} ((W * \mu)(x) - 2E[\mu])\psi \, \mathrm{d}x \geqq 0,$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \geqq 0$. This readily implies (iii).

Remark 4. Note that putting together (i), (ii), and (iii) in the previous theorem, we conclude that

$$\begin{cases} (W * \mu)(x) = 2E[\mu] & \text{for almost everywhere } x \in \operatorname{supp}(\mu) \\ (W * \mu)(x) \geqq 2E[\mu] & \text{for almost everywhere } x \in \mathbb{R}^N \setminus \operatorname{supp}(\mu), \end{cases}$$

if μ is absolutely continuous with respect to the Lebesgue measure. These two properties are the Euler–Lagrange conditions that were found for densities in [4].

Remark 5. Let us now clarify the differences between local minimizers in the d_2-topology and local minimizers in the $d_\infty-$topology. Following [17], consider as an example the interaction potential $W(x) := -x^2 + \frac{x^4}{2}$ in one dimension. Then,

$$\rho_m = m\delta_0 + (1-m)\delta_1$$

is a critical point of the interaction energy for any $m \in [0, 1]$. Theorem 3.1 in [17] shows that the measure ρ_m is a local minimizer in the $d_\infty-$topology as soon as $m \in (1/3, 2/3)$. Indeed, what is proven is the stronger statement that ρ_m is locally asymptotically stable for the aggregation equation (6) with respect to any perturbation in the $d_\infty-$topology. However, $E(\rho_m) = \frac{1}{2}(m - \frac{1}{2})^2 - \frac{1}{8}$, so only one of them, namely $\rho_{1/2}$, can be a local minimizer of the energy in the d_2-topology (and one can prove that it actually is).

This shows that the set of local minimizers with respect to the d_2-topology is strictly contained in the set of local minimizers with respect to the d_∞-topology. Moreover, numerical simulations suggest that, for $m \in (1/3, 2/3)$, ρ_m is actually stable (although not asymptotically stable) with respect to small d_2-perturbations. As a consequence, when using a gradient flow approach to numerically compute minimizers of the interaction energy via particles, one obtains d_∞-local minimizers, which typically are not d_2-local minimizers (see for example Fig. 2 of [17]).

6. Numerical Experiments

In this section we conduct a numerical investigation of the local minimizers of the discrete interaction energy (2) with a high number of particles. The gradient flow of (2) is given by the system of ODEs:

$$\dot{X}_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \nabla W(X_i - X_j). \tag{39}$$

In order to efficiently find local minimizers of (2), we solve (39) by an explicit Euler scheme with an adaptive time step chosen as the largest possible, such that the discrete energy (2) decreases. This scheme is nothing other than a gradient descent method for the discrete energy (2). Although this method might not be accurate enough for the dynamics, it is efficient to find local minimizers of the discrete energy. In stiffer situations an explicit Runge–Kutta method is used instead. These methods are essentially the ones proposed in [36,37]. The results of these simulations in two dimensions with power-law potentials were presented in the introduction, see Table 1. In Subsection 6.1 we discuss similar computations in three dimensions. We also provide numerical experiments suggesting that for some potentials, there are local minimizers of the interaction energy with mixed dimensionality, that is, local minimizers that are the sum of measures whose supports have different Hausdorff dimensions.

In Subsection 6.2 we show how our numerical results can be further understood by using the results from [2,23,37], where a careful stability analysis of a ring solution (in two dimensions) and a spherical shell solution (in three dimensions) was conducted. We also show how this stability analysis connects to the analytical results presented in this paper.

6.1. Numerical Experiments in three dimensions

First, we numerically compute local minimizers of E_W^n, where W is the power-law potential defined by (5). Recall that $\Delta W(x) \sim -1/|x|^\beta$ with $\beta = 2 - \alpha$ as $x \to 0$. The computations are performed with $n = 2{,}500$ particles. The results are shown in Table 2 and are discussed below:

Table 2. Minimizers of E_W^n in $\mathbb{R}^3$ for various power-law potentials with $n = 2{,}500$

	Dim = 0	Dim = 1	Dim = 2	Dim = 3
$\alpha = 2.5$	**(a)** 			
$\alpha = 1.25$		?	**(b)** 	**(c)** 
$\alpha = 0.5$			**(d)** 	**(e)** 
$\alpha = -0.5$				**(f)** 

- Subfigure (a): $\alpha = 2.5$ and $\gamma = 5$. The support of the minimizer has zero Hausdorff dimension, in agreement with Theorem 2. Actually, in this particular case it is supported on four points, forming a tetrahedron.
- Subfigures (b) and (c): the two potentials have the same behavior at the origin, $\alpha = 1.25$, but different attractive long range behavior ($\gamma = 15$ and $\gamma = 1.4$ respectively). Theorem 1 shows that the Hausdorff dimension of the support must be greater than or equal to $\beta = 2 - \alpha = 0.75$. Numerically, we observe that the local minimizer for the first example has a two-dimensional support and the minimizer for the second example has a three-dimensional support. We did not choose the value $\alpha = 1.5$ because we were not able to obtain a change of dimensionality of the stable steady states varying $\gamma > \alpha$. Note that $\alpha = 1.5$ is always above the instability curve for radial perturbations, which meets line $\alpha = \gamma$ at the point $(\sqrt{2}, \sqrt{2})$. See Fig. 4 and Subsection 6.2 for more details.
- Subfigures (d) and (e): the two potentials have the same behavior at the origin, $\alpha = 0.5$, but different attractive long range behaviors ($\gamma = 23$ and $\gamma = 1.4$ respectively). Theorem 1 shows that the Hausdorff dimension of the support must be greater than or equal to $\beta = 2 - \alpha = 1.5$. Numerically, we observe that the local minimizer for the first example has a two-dimensional support and the local minimizer for the second example has a three-dimensional support.
- Subfigure (f): $\alpha = -0.5$ and $\gamma = 5$. Theorem 1 proves that the Hausdorff dimension of the support must be greater than $\beta = 2 - \alpha = 2.5$, which can also be observed numerically. In Fig. 1, we haverepresented the radius of particles to the

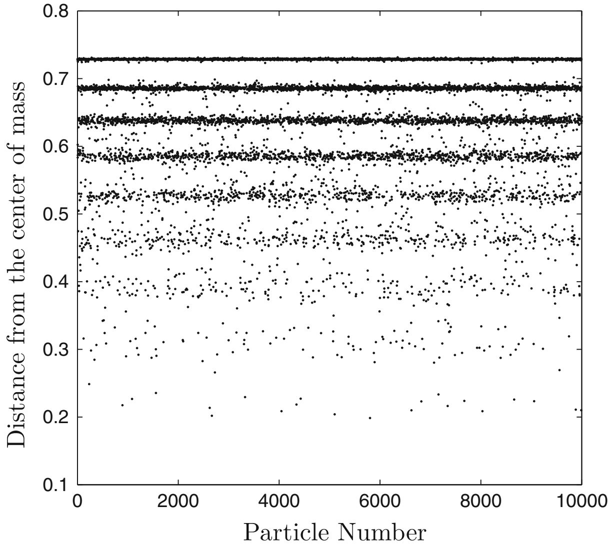


Fig. 1. Distances of the particles from the center of mass for the power law potential with $\alpha = -0.5$, $\gamma = 5$ in three dimensional. Case (f) in Subsection 6.1 in Table 2 with $n = 10{,}000$

center of mass. The particles seem to organize into successive two-dimensional layers. Such lattices were also observed in [22], and it is related to the finite number of particles used in the simulations.

Note that we were not able to find examples of interaction potentials leading numerically to a local minimizer with one dimensional support. We could, however, observe such situations with an additional asymmetric confining potential; thus we believe it should be possible to produce such cases.

A natural question following Tables 1 and 2 is whether it is possible to produce local minimizers that are the sums of two measures whose supports have different Hausdorff dimensions. A possible candidate was already observed in [37]. Here, we analyze the possibility more carefully with a much larger number of particles. From our simulations, it seems that the interaction potential $W(x) = w(|x|)$ with w defined by

$$-w'(r) = \tanh((1-r)a) + b, \quad a = 5, \ b = 0.5,$$

leads numerically to a local minimizer consisting of a ball (Hausdorff dimension three) inside a spherical shell (Hausdorff dimension two), see Fig. 2.

The distance of each particle to the center of mass is displayed on the right part of Fig. 2. The inner ball appears to be composed of five equally spaced layers of particles. This is most likely due to the fact that particles are organized into a lattice configuration, and therefore the distances between the particles and the origin do not form a continuum. It is instructive to compare the distribution of the radius of the particles in the right subplot of Fig. 2 with the one in Fig. 1 for the case of an approximated local minimizer with three-dimensional support, that is, Case (f) of Table 2. Although Theorem 1 guarantees that the support of the local minimizer corresponding to Fig. 1 has Hausdorff dimension greater than or equal

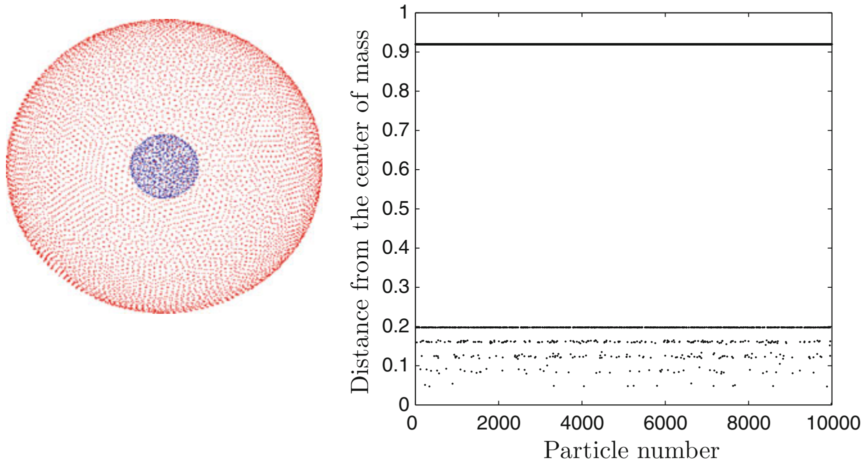


Fig. 2. *Left* Local minimizer in three-dimensional with $n = 10{,}000$. *Right* Distance of the particles from the center of mass

to 2.5, we can also observe that particles arrange themselves in layers. Notice that in dimension $N = 2$, such artifacts also appear in simulations using a finite number of particles, see Figure 4 in [22].

6.2. Relationship with Previous Works on Ring and Shell Solutions

An important characteristic of the analysis performed in the main theorems of this work is that we do not assume a specific shape on the local minimizers. If, on the contrary, one is interested by the special case of delta ring minimizers (in two dimensions), or spherical shell minimizers (in three dimensions), perturbative methods provide more detailed results.

In [23] the local stability of discrete ring solutions, made of N-particle equally distributed in a circle, was studied for the N-particle system (39). The authors considered the power law interaction potentials (5), and conducted a formal linear stability analysis for the continuum ring solution as steady state of the aggregation equation (6) by taking $N \to \infty$. Those predictions were then confirmed numerically. They could not obtain nonlinear stability of the ring solution, particularly because there is no spectral gap as $N \to \infty$, that is, the largest negative eigenvalue tends to 0 when $N \to \infty$. In [2], the nonlinear stability of the ring solutions was proved for radial perturbations, corroborating some of the formal results of [23], together with the instability due to fattening in the complementary set of parameters.

We have represented this set of parameters in Fig. 3, as well as all the parameters used in the two-dimensional numerical simulations of this article (Tables 1, 3). As the caricature presented in Table 3 shows, crossing the lower border of this set, curve $\alpha = \gamma/(\gamma - 1)$, leads to a "fattening" of the delta ring, that is, to minimizers with dimensionality 2, see [23,2]. On the other hand, crossing its upper border, given by the curve marked with 3, does not modify the dimensionality of the stable steady states as long as $\alpha < 2$ (they remain one-dimensional), but leads to a

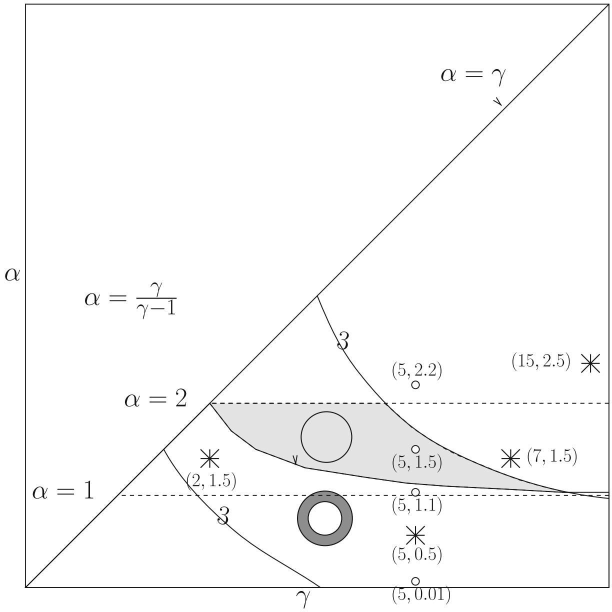
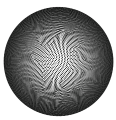
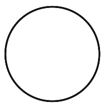
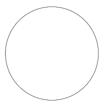
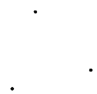


Fig. 3. Sketch of all the computed cases in dimension $N = 2$. The parameters (γ, α) used in Table 1 are marked with $*$, while those used in Table 3 are marked with $\circ$. Notice that $\alpha < \gamma$ is necessary for the interaction potential to be confining. In *dark gray* is represented the set of parameters such that a delta ring could be a local minimizer

Table 3. Evolution of local minimizers when $\alpha > 0$ increases, while $\gamma = 5$ remains constant

			
$\alpha = 0.01$	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 2.2$

The computations were done with $n = 10{,}000$ particles

"shape" instability towards a triangular configuration that breaks the ring into three connected one-dimensional components, as in case (b) of Table 1.

Finally, if $\alpha > 2$, local minimizers become of dimensionality 0, as predicted by Theorem 2, whereas if $\alpha < 1$, all the minimizers are of dimensionality 2, as shown by Theorem 1.

In three dimensions, a linear stability analysis of discrete spherical shell solutions is also possible, but it leads to more cumbersome instability curves, see [36,37]. Again, the results in [2] give the "fattening" instability curve dividing instability from stability under radial perturbations. In Fig. 4, we have represented only the set of parameters such that the spherical shells are not local minimizers for spherically symmetric perturbations, as well as all the parameters (γ, α) used for three-dimensional numerical simulations in this article in Table 2. Just as we have

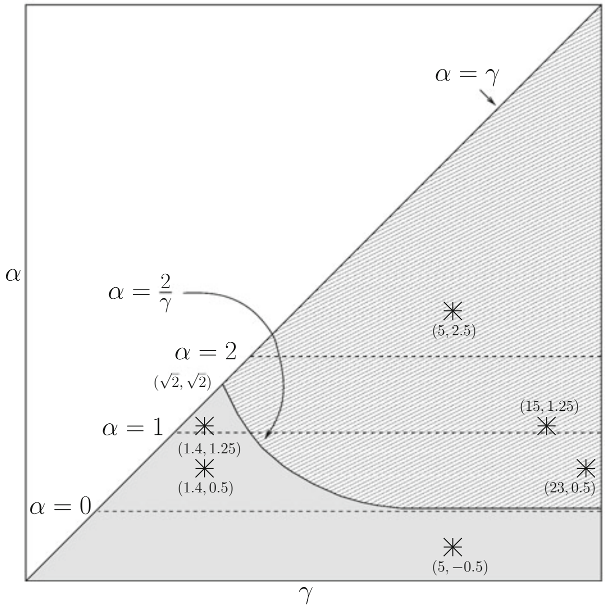
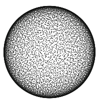
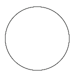
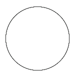


Fig. 4. Sketch of all the computed cases in dimension $N = 3$. The parameters (γ, α) used in Table 2 are marked with $*$. Notice that $\alpha < \gamma$ is necessary for the interaction potential to be confining. *The curve* is the limit between parameters leading to spherical shell solutions for radial perturbations (*above the curve*) and to minimizers of dimensionality 3 (*below the curve*)

Table 4. Local minimizers with the power-law potential (5) and the perturbed potential (40), $n = 10{,}000$

	Powers	$p = 3$	$p = 5$
$(\gamma, \alpha) = (2, 1.5)$			

observed in the two-dimensional case, crossing the lower border of this set leads to a "fattening" instability of the spherical shell.

Notice, finally, that it is also possible to modify the dimensionality of the local minimizers with other perturbations of power law potentials. As an example, in Table 4, we consider the following perturbations of the power law potential (5):

$$W(x) = -\frac{|x|^\alpha}{\alpha} + \frac{|x|^\gamma}{\gamma} + \frac{3}{2p}\cos(px), \quad \alpha < \gamma, \;\; p = 3, 5. \tag{40}$$

In Table 4 we have represented the power-law case in the first column, and the perturbations in the next two. For $(\gamma, \alpha) = (2, 1.5)$, the unperturbed power-law potential leads to a local minimizer with Hausdorff dimension two. When we add the perturbation $p = 3$, the dimension of the minimizer changes to one. Notice that the perturbation does not alter the local behavior of the potential at the origin or at infinity, suggesting that Theorem 1 is probably sharp, at least in terms of natural dimensions.

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Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
08193 Bellaterra,
Spain.
e-mail: dbalague@mat.uab.cat

and

Department of Mathematics,
Imperial College London,
South Kensington Campus,
London SW7 2AZ,
UK.
e-mail: carrillo@imperial.ac.uk

and

Department of Mathematics,
University of California-Riverside,
Riverside,
CA 92521,
USA.
e-mail: laurent@math.ucr.edu

and

Centre d'Ecologie Fonctionnelle et Evolutive, UMR 5175,
CNRS, 1919 Route de Mende,
34293 Montpellier Cedex 5, France.
e-mail: raoul@cefe.cnrs.fr

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