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On Formation of a Locally Self-Similar Collapse in the Incompressible Euler Equations

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Abstract

The paper addresses the question of the existence of a locally self-similar blow-up for the incompressible Euler equations. Several exclusion results are proved based on the L^p-condition for velocity or vorticity and for a range of scaling exponents. In particular, in N dimensions if in self-similar variables $u \in L^p$ and $u \sim \frac{1}{t^{\alpha/(1+\alpha)}}$, then the blow-up does not occur, provided $\alpha > N/2$ or $-1 < \alpha \leq N/p$. This includes the L^3 case natural for the Navier–Stokes equations. For $\alpha = N/2$ we exclude profiles with asymptotic power bounds of the form $|y|^{-N-1+\delta} \lesssim |u(y)| \lesssim |y|^{1-\delta}$. Solutions homogeneous near infinity are eliminated, as well, except when homogeneity is scaling invariant.

1. Introduction

In the theory of weak solutions to the Navier–Stokes equation, one of the cornerstone results is non-existence of self-similar blow-up for velocities in L^3 proved by Necas et al. [17], and further extended to the case of L^p, $p > 3$, by Tsai [22]. This was followed by the celebrated $L^{3,\infty}$-regularity criterion of Escauriaza et al. [11]. For its inviscid counterpart, the Euler equation, given by

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla p &= 0 \\ \nabla \cdot u &= 0, \end{aligned} \tag{1}$$

the self-similar blow-up has not yet been explored systematically in mathematical literature, despite an abundance of numerical data based on (1) pointing to such a

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possibility. BRACHET et al. [3] observe a pancake-like formation of vortex structures from Taylor-Green initial conditions. Simulations of KERR [14] present strong evidence of a singularity corresponding to scaling $u \sim \frac{1}{\sqrt{T-t}}$, the same as for the Navier–Stokes. BORATAV and PELZ [2] conducted tests on Kida's high-symmetry flows that revealed self-similar evolution of a focusing vortex dodecapole, again in the same scaling. A similar collapse was further observed in vortex filament models of PELZ [19], KIMURA [15], NG and BHATTACHARJEE [18], and others.

To describe the mathematical setup, let us assume that the fluid domain is $\mathbb{R}^N$, $N \geq 2$, although other choices are possible. Suppose that near some point $x^* \in \mathbb{R}^N$ a solution, initially starting from smooth data, organizes into a locally self-similar blowup. In other words, there is $\alpha > -1$, a $\rho_0 > 0$ and time $T > 0$ such that

$$
\begin{aligned}
u(x,t) &= \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}} v\left(\frac{x-x^*}{(T-t)^{\frac{1}{1+\alpha}}}\right) \\
p(x,t) &= \frac{1}{(T-t)^{\frac{2\alpha}{1+\alpha}}} q\left(\frac{x-x^*}{(T-t)^{\frac{1}{1+\alpha}}}\right),
\end{aligned}
\tag{2}
$$

for all $|x - x^*| < \rho_0$, and $t < T$ near T. For finite α the collapse is dynamically focusing, while for $\alpha = \infty$, the solution (2) becomes

$$
u(x,t) = (T-t)^{-1} v(x - x^*), \quad p(x,t) = (T-t)^{-2} q(x - x^*), \tag{3}
$$

which exhibits globally inflating characteristics. We normally assume in this case that $\rho_0 = \infty$.

Observe that the vorticity near the singularity scales like $\omega = \operatorname{curl} v \sim \frac{1}{T-t}$, making it a borderline case for the Beal–Kato–Majda criterion [1]. The Lipschitz constant of the vorticity direction field $\xi = \frac{\omega}{|\omega|}$ scales like $(T-t)^{-\frac{1}{1+\alpha}}$, again in no contradiction with Constantin and Fefferman's criterion for three-dimensional fluids [9,10]. In [12,13] Xinyu He shows existence of solutions to self-similar equations (7) on three-dimensional bounded and exterior domains with $\alpha = 1$. On exterior domains solutions exhibit the power-like decay similar to vortex models, $|v| \sim |y|^{-1}$, $|\nabla v| \sim |y|^{-2}$ under the same scaling. Although these solutions belong to different settings, interestingly, their decay rate appears critical for our results below. One can observe that $\alpha = N/2$ is the only scaling consistent with energy conservation for globally self-similar solutions if the helicity is not zero ([6], see also [20] for 'pseudo self-similar solutions'). A study of self-similar blow-up in the settings adopted here was undertaken by the first author in a series of works [4–7]. The two main results obtained were the following. First, if $v \in L^p(\mathbb{R}^3)$, $p \geq \frac{9}{2}$, $\alpha = \infty$, and the ansatz is (3) is global, that is, $\rho_0 = \infty$, then $v = 0$. Second, if $\|\nabla v\|_\infty < \infty$ and the vorticity belongs to $\cap_{0<p<p_0} L^p(\mathbb{R}^3)$, for some p_0, while $\alpha > -1$ is arbitrary, then v is irrotational, with $\omega = 0$ throughout.

In this paper we develop a new set of criteria that exclude locally self-similar collapse in physically relevant scalings. Let us observe that if the total energy of

u is finite, then by rescaling the energy in the ball $|x - x^*| \le \rho_0$, we have the bound

$$\int_{|y|<L\rho_0} |v(y)|^2 \lesssim L^{N-2\alpha}, \quad \text{for all } L > L_0. \tag{4}$$

Therefore, the case $\alpha > \frac{N}{2}$ is automatically excluded, while in the range $\alpha < \frac{N}{2}$ the energy of v may be unbounded. In all our results we avoid using the assumption of finiteness of total energy, keeping in mind, for instance, the three-dimensional vortex filament models, where the energy is naturally unbounded. We therefore examine the full range of $\alpha > -1$ and integrability conditions $v \in L^p$ for a possible collapse. If $v \in L^p,\ p > 2$, there are two special values of α to consider: $\alpha = \frac{N}{p}$ for the fact that $\|u\|_p$ is conserved under the self-similar evolution on the open space, and $\alpha = N/2$ as the boundary between local energy inflation and deflation regimes (see (4)). We will see that the cases $-1 < \alpha \le \frac{N}{p},\ \frac{N}{p} < \alpha \le \frac{N}{2}$, and $\alpha > \frac{N}{2}$ are, in fact, different in character, and we exclude solutions under the following conditions:

(i) $v \in L^p \cap C^1_{\text{loc}},\ p \ge 3$, and $-1 < \alpha \le \frac{N}{p}$ or $\alpha > \frac{N}{2}$;

(ii) $v \in L^2 \cap C^1_{\text{loc}},\ \alpha = \frac{N}{2}$, and for some $\delta > 0$ and $|y|$ large, one has

$$\frac{c}{|y|^{N+1-\delta}} \le |v(y)| \le C|y|^{1-\delta}. \tag{5}$$

The local C^1-condition is needed only for the local energy equality to hold, and is natural since we view T as the first time of regularity loss. The local energy equality will be our starting point in most arguments, although somewhat unusually for a self-similar problem, we will employ the full time-dependent version of it to be able to make a non-self-similar choice for a test function. As a result the local energy equality takes the form

$$\frac{1}{L^{N-2\alpha}} \int_{|y|\le L} |v|^2 \mathrm{d}y \lesssim \frac{1}{l^{N-2\alpha}} \int_{|y|\le l} |v|^2 \mathrm{d}y + \int_{l\le|y|\le L} \frac{|v|^3 + |v||q|}{|y|^{N+1-2\alpha}} \mathrm{d}y. \tag{6}$$

As we remarked above, the asymptotic character of terms in (6) depends on the range of α considered. Nonetheless, (6) allows us to control the growth of the energy either by the L^p-norm of v on the large scales in case (i) or through the use of power bounds on v as in (ii). This gives an improved bound on the trilinear integral in (6) by interpolation. The general strategy will then be to bootstrap between the growth of L^2 and L^3 norms of v over large balls $|y| < L$ via a repeated use of (6), until eventually the energy over $|y| < L$ displays a decay as $L \to \infty$, implying $v = 0$. It is precisely for $\frac{N}{p} < \alpha \le \frac{N}{2}$ when this algorithm fails to bootstrap. However, as a byproduct of the argument, we obtain

(iii) if $v \in L^p \cap C^1_{\text{loc}},\ p \ge 3,$ and $\frac{N}{p} < \alpha \le \frac{N}{2},$ then (4) holds.

So, the energy growth bound (4) is a natural internal feature of the blow-up, independent of the total energy assumption. In particular, if $v \in L^p,\ p \ge 3$, and $\alpha = \frac{N}{2}$, then automatically $v \in L^2$.

Coming back to the vortex models or He's solutions, notice that in those cases $v \in L^p$ for $p > 3$ (even if only at infinity) while $\alpha = 1$. Thus, they appear to be critical for the scope of (i).

We present several explicit homogeneous examples of solution pairs (v, q), see (10), (11), (12), (13), which although lacking sufficient local regularity to be fully qualified as counterexamples, serve as indicators that our arguments may be sharp. In Theorem 4.2 we demonstrate, however, that locally smooth homogenous at infinity solutions are trivial unless the homogeneity is consistent with the scaling, and even then the case $\alpha = N/2$ is excluded.

A criterion dimensionally equivalent to (i), but in terms of vorticity, is established using the self-similar equations in vorticity form, generalizing the results obtained by the first author. We have

(iv) Suppose $\alpha > -1$, $\omega \in L^p$, for some $0 < p < \frac{N}{1+\alpha}$, and the strain tensor $|\partial v + \partial^\top v| = o(1)$ as $|y| \to \infty$. Then v is a constant vector.

2. Technical Preliminaries

2.1. Self-Similar Equations and Pressure

If (u, p) is a distributional solution to (1), then the pair (v, q) satisfies

$$\frac{1}{1+\alpha} y \cdot \nabla v + \frac{\alpha}{1+\alpha} v = v \cdot \nabla v + \nabla q, \tag{7}$$

and the pressure necessarily satisfies the Poisson equation

$$\Delta q = -\operatorname{div}\operatorname{div}(v \otimes v) = -\partial_i \partial_j (v_i v_j). \tag{8}$$

If $v \in L^p$, $2 < p < \infty$ (respectively, L^∞) and $q \in L^{p/2}$ (respectively, BMO), then there is only one solution to (8), given by

$$q(y) = -\frac{|v|^2}{N} + P.V. \int_{\mathbb{R}^N} K_{ij}(y - z) v_i(z) v_j(z) \, \mathrm{d}z, \tag{9}$$

where the kernel is given by

$$K_{ij}(y) = \frac{N y_i y_j - \delta_{i,j} |y|^2}{N \omega_N |y|^{N+2}},$$

and $\omega_N = 2\pi^{N/2}(N\Gamma(N/2))^{-1}$ is the volume of the unit ball in $\mathbb{R}^N$. The pressure given by (9) is referred to as the associated pressure. Unless stated otherwise we will always assume that the pressure is associated, however not for every pair (v, q) solving (7), is q given by (9). Indeed, let

$$v = \langle 1, 0 \rangle, \quad q = \frac{\alpha}{1+\alpha} y_1. \tag{10}$$

This is a self-similar solution for any $\alpha > -1$. Clearly, (9) does not hold (see [16] for the role of such examples in uniqueness of solutions of the Navier–Stokes equation).

The equation in self-similar coordinates (7) has its own intrinsic scaling—if v is a solution to (7), then

$$v_\lambda(y) = \lambda v(y/\lambda), \quad q_\lambda(y) = \lambda^2 q(y/\lambda)$$

is also a solution to the same equation. This suggests that there may exist non-trivial examples of a 1-homogeneous solution. Indeed,

$$\begin{aligned} &v(y) = My, \quad q(y) = \frac{1}{2}\langle (M - M^2)y, y\rangle, \\ &\operatorname{Tr} M = 0, \quad M - M^2 \in \mathrm{Sym}_N. \end{aligned} \tag{11}$$

Another example is the following two-dimensional parallel shear flow

$$v(y) = \langle y_2^\alpha, 0\rangle, \quad q(y) = \frac{2\alpha}{(1+\alpha)^2} y_2^{\alpha+1}, \tag{12}$$

which in the case $\alpha = 1$ reduces to the natural homogeneity. A singular example of a solution of special interest to us is the α-point vortex

$$v(y) = \frac{y^\perp}{|y|^{\alpha+1}}, \quad q(y) = 0. \tag{13}$$

The equation for vorticity tensor $\omega = \frac{1}{2}\{\partial_i v_j - \partial_j v_i\}_{i,j=1}^N$ in self-similar variables reads

$$\omega + \frac{1}{1+\alpha} y \cdot \nabla\omega = v \cdot \nabla\omega - \omega\varsigma - \varsigma\omega, \tag{14}$$

where $\varsigma = \frac{1}{2}\{\partial_i v_j + \partial_j v_i\}_{i,j=1}^N$ is the strain tensor.

2.2. *Local Energy Equality*

All our results below hold under the presumption that the solution (u, p) is regular enough to satisfy the local energy equality, at least in the region of self-similarity:

$$\begin{aligned} &\int_{\mathbb{R}^N} |u(t_2, x)|^2 \sigma(t_2, x)\, \mathrm{d}x - \int_{\mathbb{R}^N} |u(t_1, x)|^2 \sigma(t_1, x)\, \mathrm{d}x \\ &\quad = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u(t, x)|^2 \partial_t \sigma(t, x)\, \mathrm{d}x\, \mathrm{d}t + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (|u|^2 + 2p) u \cdot \nabla\sigma\, \mathrm{d}x\, \mathrm{d}t, \end{aligned} \tag{15}$$

where $\sigma \in C_0^\infty((0, T) \times \mathbb{R}^N)$, and $0 < t_1 < t_2 < T$. This holds trivially for locally smooth solutions, $u, p \in C^1_{\mathrm{loc}}((0, T) \times \mathbb{R}^N)$. The weakest known regularity condition under which (15) still holds is a Besov-type regularity of smoothness $1/3$ (see [8,21]). It is not our goal, however, to pursue the sharpest local condition.

We will now work out a special form of the local energy equality (15) in the case of self-similar solutions. First, we take one preliminary step by assuming, without loss of generality, that the center of the blow-up is the origin, $x^* = 0$, the radius of the ball where (2) holds is $\rho_0 = 1$, and since the Euler equations are time reversible we can assume that $T = 0$ and the self-similar blow-up occurs backward in time for $0 < t < 1$, that is,

$$u(x,t) = t^{-\frac{\alpha}{1+\alpha}} v\left(xt^{-\frac{1}{1+\alpha}}\right), \quad p(x,t) = t^{-\frac{2\alpha}{1+\alpha}} p\left(xt^{-\frac{1}{1+\alpha}}\right).$$

Let us fix a radial test function σ, that is, $\sigma(x) = \sigma(|x|)$, such that $\sigma \geq 0$, $\sigma(r) = 1$, for $0 \leq r \leq \frac{1}{2}$, and $\sigma(r) = 0$, for $r > 1$. Using σ, (15) takes the form

$$\|u(t_2)\sigma\|_2^2 = \|u(t_1)\sigma\|_2^2 + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (|u|^2 + 2p) u \cdot \nabla\sigma(x) \mathrm{d}x \mathrm{d}t. \tag{16}$$

In self-similar variables, the above translates into the following. For $-1 < \alpha < \infty$,

$$\begin{aligned} t_2^{\frac{N-2\alpha}{1+\alpha}} &\int_{|y| \leq t_2^{-\frac{1}{1+\alpha}}} |v(y)|^2 \sigma\left(yt_2^{\frac{1}{1+\alpha}}\right) \mathrm{d}y \\ &= t_1^{\frac{N-2\alpha}{1+\alpha}} \int_{|y| \leq t_1^{-\frac{1}{1+\alpha}}} |v(y)|^2 \sigma\left(yt_1^{\frac{1}{1+\alpha}}\right) \mathrm{d}y \\ &\quad + \int_{t_1}^{t_2} t^{\frac{N-3\alpha}{1+\alpha}} \int_{\mathbb{R}^N} (|v|^2 + 2q) v \cdot \nabla\sigma\left(yt^{\frac{1}{1+\alpha}}\right) \mathrm{d}y \, \mathrm{d}t. \end{aligned} \tag{17}$$

Let us change the order of integration in the last integral, noting that in view of the definition of σ, $\frac{1}{2}t_2^{-\frac{1}{1+\alpha}} \leq |y| \leq t_1^{-\frac{1}{1+\alpha}}$,

$$\int_{\frac{1}{2}t_2^{-\frac{1}{1+\alpha}} \leq |y| \leq t_1^{-\frac{1}{1+\alpha}}} (|v|^2 + 2q) v \cdot k(y) \, \mathrm{d}y \, \mathrm{d}t,$$

where

$$k(y) = \int_{t_1}^{t_2} t^{\frac{N-3\alpha}{1+\alpha}} \nabla\sigma(yt^{\frac{1}{1+\alpha}}) \, \mathrm{d}t.$$

Noting, again, that in view of the definition of σ the interval of integration is, in fact, restricted to $\{|y|^{-1-\alpha}/2 \leq t \leq |y|^{-1-\alpha}\}$, we obtain

$$|k(y)| \leq \int_{|y|^{-1-\alpha}/2}^{|y|^{-1-\alpha}} t^{\frac{N-3\alpha}{1+\alpha}} \, \mathrm{d}t \lesssim \frac{1}{|y|^{N+1-2\alpha}}.$$

Using this estimate in (17) and changing the notation in the first two integrals with $l_1 = t_2^{-1/(1+\alpha)}$ and $l_2 = t_1^{-1/(1+\alpha)}$, we obtain the inequality

$$\begin{aligned} &\left| \frac{1}{l_2^{N-2\alpha}} \int_{|y| \leq l_2} |v(y)|^2 \sigma(y/l_2) \, \mathrm{d}y - \frac{1}{l_1^{N-2\alpha}} \int_{|y| \leq l_1} |v(y)|^2 \sigma(y/l_1) \, \mathrm{d}y \right| \\ &\quad \leq C \int_{l_1/2 \leq |y| \leq l_2} \frac{|v|^3 + |q||v|}{|y|^{N+1-2\alpha}} \, \mathrm{d}y, \end{aligned} \tag{18}$$

for all $1 < l_1 < l_2$, and it is valid if $-1 < \alpha < \infty$. This inequality will be our starting point in much of what follows.

2.3. Global Energy Equality

The global energy equality holds under additional L^3-integrability conditions at infinity.

Theorem 2.1. *Let* $u \in C_t^w L_x^2 \cap L_t^3 L_x^3 \cap C^1_{\mathrm{loc}}$ *be a weak solution to the Euler equations on* $\mathbb{R}^N$. *Then* u *conserves energy on* $[0, T]$.

Proof. Let $\sigma_R(x) = \sigma(x/R)$. By the local energy equality, we have

$$\|u(t_2)\sigma_R\|_2^2 - \|u(t_1)\sigma_R\|_2^2 = \frac{1}{R}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(|u|^2 + 2p)u \cdot \nabla\sigma(x/R)\mathrm{d}x\mathrm{d}t. \tag{19}$$

Since $u \in L^3_{t,x}$, then $p \in L^{3/2}_{t,x}$ and hence $(|u|^2 + 2p)u \in L^1_{t,x}$. Then, clearly, the integral on the right-hand side tends to zero as $R \to \infty$. □

So, if (2) is a part of a solution satisfying the assumptions of the above theorem, then at least the energy in the ball of self-similarity should remain bounded, which immediately translates into the bound (4). This clearly implies that $v = 0$ if α satisfies $\alpha > N/2$. We thus obtain the following conclusion.

Corollary 2.2. *Suppose* $u \in C_t^w L_x^2 \cap L_t^3 L_x^3 \cap C^1_{\mathrm{loc}}$ *is a weak solution to the Euler equations on* $\mathbb{R}^N$ *with a locally self-similar collapse. If* $\alpha > \frac{N}{2}$, *then the collapse does not occur. Otherwise,* (4) *holds.*

As a by-product of our proofs below we show that the conclusions of this corollary hold only under L^p-integrability assumptions on the self-similar profile v. In other words, a self-similar solution, even if viewed independently from the ambient flow, still behaves as if it was embedded in a global in space finite energy solution.

3. Exclusions Based on Velocity

3.1. The Energy Conservative Scaling $\alpha = \frac{N}{2}$

As outlined in the introduction, the case of $\alpha = \frac{N}{2}$ is special since it is the only scaling compatible with the energy conservation law if (2) was defined globally in space. What distinguishes it from a pure technical point of view is the absence of weights in front of energy integrals in the energy balance relation (18). Our main result for this case is the exclusion of solutions with a power spread.

Theorem 3.1. *Suppose* $\alpha = \frac{N}{2}$, *and suppose* $v \in L^2(\mathbb{R}^N) \cap C^1_{\mathrm{loc}}$ *and the pressure* q *are given by* (9). *Suppose there exists a* $\delta > 0$ *and* $C, c > 0$ *such that*

$$\frac{c}{|y|^{N+1-\delta}} \le |v(y)| \le C|y|^{1-\delta}, \tag{20}$$

for all sufficiently large y. *Then* $v = 0$.

A few comments are in order. Example (11) shows relevance of the upper bound to the natural scaling of the equations, although of course it has infinite energy. The lower bound may seem to be artificial, especially given Theorem 4.2, below, where homogeneous profiles with decay $|v| \sim |y|^{-\beta}$ are excluded for any $\beta \geq N/2$. However, as we will see from the proof, it is essentially a way of dealing with the non-locality of the pressure.

Proof. We start with the basic energy equality (18). Using that $\alpha = \frac{N}{2}$, the factors in front of the energies disappear and we obtain

$$\int_{|y|\leq l_2/2} |v|^2 \, dy \leq \int_{|y|\leq l_1} |v|^2 \, dy + C \int_{l_1/2\leq|y|\leq l_2} \frac{|v|^3 + |q||v|}{|y|} \, dy. \tag{21}$$

Taking $l_1 = L = l_2/4$, we obtain

$$\int_{L\leq|y|\leq 2L} |v|^2 \, dy \leq C \int_{\frac{1}{2}L\leq|y|\leq 4L} \frac{|v|^3 + |q||v|}{|y|} \, dy. \tag{22}$$

The proof will now proceed by showing the following claim: for all $M \in \mathbb{N}$ there exists a $C_M > 0$ such that

$$\int_{L\leq|y|\leq 2L} |v|^2 dy \leq \frac{C_M}{L^M},$$

for all L sufficiently large. This immediately runs into contradiction with the lower bound of (20). The exact value of the power $N + 1 - \delta$ is not important at this point, but it will be crucial in the course of proving the claim.

Using our assumption (20) and the energy bound (22), we have

$$\int_{L\leq|y|\leq 2L} |v|^2 \, dy \lesssim \frac{1}{L^\delta} \int_{\frac{1}{2}L\leq|y|\leq 4L} |v|^2 \, dy + \frac{1}{L} \int_{\frac{1}{2}L\leq|y|\leq 4L} |v||q| \, dy. \tag{23}$$

Now our goal is to find suitable bounds on the pressure and the last integral in (23). Notice that

$$\int_{\mathbb{S}^{N-1}} K_{ij}(\theta) d\sigma(\theta) = 0, \tag{24}$$

for all i, j. Let us split the pressure as follows:

$$q = q_0 + q_1 + q_2 + q_3,$$

where q_0 is the local part of (9), and

$$\begin{aligned} q_1(y) &= \int_{|z|\leq L/4} K_{ij}(y-z) v_i(z) v_j(z) \, dz, \\ q_2(y) &= \int_{L/4\leq|z|\leq 8L} K_{ij}(y-z) v_i(z) v_j(z) \, dz, \\ q_3(y) &= \int_{|z|\geq 8L} K_{ij}(y-z) v_i(z) v_j(z) \, dz. \end{aligned}$$

Clearly, only estimates on the non-local quantities q_i are necessary. Since $|y-z| \sim L$ for all $|z| \le L/4$ and $\frac{1}{2}L \le |y| \le 4L$, we have

$$|q_1(y)| \lesssim \frac{1}{L^N} \int_{|z| \le L/4} |v|^2 \, dz \le \frac{\|v\|_2^2}{L^N}.$$

Thus, in view of (20),

$$\begin{aligned}\frac{1}{L} \int_{\frac{1}{2}L \le |y| \le 4L} |v||q_1| \, dy &\lesssim \frac{1}{L^{N+1}} \int_{\frac{1}{2}L \le |y| \le 4L} |v|^2 |v|^{-1} \, dy \\ &\lesssim \frac{1}{L^\delta} \int_{\frac{1}{2}L \le |y| \le 4L} |v|^2 \, dy.\end{aligned}$$

As to q_2, we have

$$\begin{aligned}\frac{1}{L} \int_{\frac{1}{2}L \le |y| \le 4L} |v||q_2| \, dy &\le \frac{1}{L} \left(\int_{\frac{1}{2}L \le |y| \le 4L} |v|^2 dy \right)^{1/2} \left(\int_{\mathbb{R}^N} |q_2|^2 dy \right)^{1/2} \\ &\lesssim \frac{1}{L} \left(\int_{\frac{1}{2}L \le |y| \le 4L} |v|^2 dy \right)^{1/2} \left(\int_{L/4 \le |y| \le 8L} |v|^4 dy \right)^{1/2} \\ &\lesssim \frac{1}{L^\delta} \left(\int_{\frac{1}{2}L \le |y| \le 4L} |v|^2 dy \right)^{1/2} \left(\int_{L/4 \le |y| \le 8L} |v|^2 dy \right)^{1/2} \\ &\lesssim \frac{1}{L^\delta} \int_{L/4 \le |y| \le 8L} |v|^2 dy.\end{aligned}$$

And as to q_3, we trivially have $|q_3(y)| \lesssim \frac{1}{L^N} \|v\|_2^2$. Thus,

$$\begin{aligned}\frac{1}{L} \int_{\frac{1}{2}L \le |y| \le 4L} |v||q_3| \, dy &\lesssim \frac{1}{L^{N+1}} \int_{\frac{1}{2}L \le |y| \le 4L} |v| \, dy \lesssim \frac{1}{L^\delta} \int_{\frac{1}{2}L \le |y| \le 4L} |v|^2 \, dy \\ &= \frac{1}{L^\delta} \sum_{k=-1}^{1} \int_{2^k L \le |y| \le 2^{k+1} L} |v|^2 \, dy.\end{aligned}$$

Putting the obtained estimates together into (23), we conclude that there exists a constant $C > 0$ such that, for all L large enough,

$$\int_{L \le |y| \le 2L} |v|^2 \, dy \le \frac{C}{L^\delta} \sum_{k=-2}^{2} \int_{2^k L \le |y| \le 2^{k+1} L} |v|^2 \, dy. \tag{25}$$

Let us now iterate the estimate above m times, applying it to each integral in the sum

$$\begin{aligned}\int_{L \le |y| \le 2L} |v|^2 dy &\le \frac{C^m}{L^{m\delta}} \sum_{k_1, \ldots, k_m = -2}^{2} \int_{2^{k_1 + \ldots + k_m} L \le |y| \le 2^{k_1 + \ldots + k_m + 1} L} |v|^2 \, dy \\ &\lesssim \frac{C_m}{L^{m\delta}}.\end{aligned}$$

Since m can be arbitrary, the claim is proved. □

3.2. The Energy Non-Conservative Scaling $\alpha \neq \frac{N}{2}$

As we mentioned earlier, some cases of non-conservative scaling appear physically relevant. Additionally, in the range $-1 < \alpha < \frac{N}{2}$, a possibly infinite energy of the self-similar profile v is not in contradiction with the finiteness of the global energy, as long as (4) holds. Our main result in the energy non-conservative scaling is the following.

Theorem 3.2. *Suppose $v \in L^p \cap C^1_{\mathrm{loc}}$ for some $3 \le p \le \infty$, and the pressure q is given by (9). If $-1 < \alpha \le \frac{N}{p}$ or $\frac{N}{2} < \alpha < \infty$, then $v = 0$. If $\alpha = \infty$, and, additionally, the self-similar solution (3) is global, then $v = 0$.*

The scaling $\alpha = N/p$ is notable for the fact that the L^p-norm of the solution is conserved. If $\alpha < N/p$, it deflates as $t \to 0$, and if $\alpha > N/p$, it inflates. The sharpness of this scaling is suggested by the α-point vortex (13). Even though it fails to satisfy the required regularity near the origin, it does belong to L^p near infinity precisely when $2/p < \alpha$. He's solutions in exterior domains with asymptotic decay $|v(y)| \sim \frac{1}{|y|}$, hence in L^3_{weak}, are suggestive of the criticality of $\alpha = N/p$ as well.

In the following we consider only the case when $p < \infty$, postponing the technicalities of the case $p = \infty$ to Section 3.2.5.

3.2.1. Proof in the range $-1 < \alpha \le \frac{N}{p}$ In this range we can eliminate the l_2-integral from (18). Our claim is

$$\frac{1}{l_2^{N-2\alpha}} \int_{|y|\le l_2} |v(y)|^2 \sigma(y/l_2)\, \mathrm{d}y \to 0,$$

as $l_2 \to \infty$. Indeed, for a fixed large $M > 0$ and $l_2 > M$, we have, by the Hölder inequality,

$$\frac{1}{l_2^{N-2\alpha}} \int_{|y|\le l_2} |v(y)|^2 \sigma(y/l_2)\, \mathrm{d}y \le \frac{1}{l_2^{N-2\alpha}} \int_{|y|\le M} |v(y)|^2\, \mathrm{d}y + l_2^{2\alpha-2N/p} \left(\int_{M\le|y|\le l_2} |v|^p\, \mathrm{d}y \right)^{2/p}.$$

Letting $l_2 \to 0$, the first integral disappears, and we have

$$\le \left(\int_{M\le|y|} |v|^p\, \mathrm{d}y \right)^{2/p} \to 0,$$

as $M \to \infty$. So, (18) takes the form (using that $\sigma = 1$ on $|y| < 1/2$, and replacing $l_1/2$ with L)

$$\frac{1}{L^{N-2\alpha}} \int_{|y|\le L} |v|^2\, \mathrm{d}y \le C \int_{L\le|y|} \frac{|v|^3 + |q||v|}{|y|^{N+1-2\alpha}}\, \mathrm{d}y. \tag{26}$$

By the Hölder inequality we obtain

$$\frac{1}{L^{N-2\alpha}} \int_{|y|\le L} |v|^2\, \mathrm{d}y \le C L^{2\alpha-1-3N/p} \left(\int_{L\le|y|} (|v|^3 + |q||v|)^{p/3}\, \mathrm{d}y \right)^{3/p},$$

and hence,

$$\int_{|y|\le L} |v|^2 \, \mathrm{d}y \le L^{\beta_p}, \text{ where } \beta_p = N - 1 - \frac{3N}{p}. \tag{27}$$

If $\beta_p < 0$, then the proof is finished by sending $L \to \infty$. Otherwise, by interpolation, we have

$$\int_{|y|\le L} |v|^3 \mathrm{d}y \le CL^{\beta_p \alpha_p}, \text{ where } \alpha_p = \frac{p-3}{p-2}. \tag{28}$$

Coming back repeatedly to the inequality (26), we will be able to bootstrap on the growth of energy, now based on a better estimate for the L^3-norms (28), but first we have to establish the corresponding estimates on the growth of the pressure.

Lemma 3.3. *Let*

$$\int_{|y|\le L} |v|^2 \mathrm{d}y \le CL^{a_2} \tag{29}$$

and

$$\int_{|y|\le L} |v|^3 \mathrm{d}y \le CL^{a_3} \tag{30}$$

hold for all large L, and $a_2 < N$, $\frac{3a_2 - N}{2} \le a_3$. *Then*

$$\int_{|y|\le L} |q|^{3/2} \mathrm{d}y \le CL^{a_3}. \tag{31}$$

In order not to verify the assumptions on the exponents every time, we simply note that they are verified for any couple a_2, a_3 with

$$a_2 \le N - \frac{2N}{p}, \quad a_3 = a_2 \alpha_p. \tag{32}$$

Clearly, $a_2 = \beta_p$, $a_3 = \beta_p \alpha_p$ is such a couple.

Proof. As before, let $q = q_0 + \tilde{q}$, where q_0 is the local and $\tilde{q}$ is the non-local part of the pressure. We can split

$$\int_{|y|\le L} |\tilde{q}|^{3/2} \mathrm{d}y \le \int_{|y|\le L} \left| \int_{|z|\le 2L} K_{ij}(y-z) v_i(z) v_j(z) \mathrm{d}z \right|^{3/2} \mathrm{d}y$$
$$+ \int_{|y|\le L} \left| \int_{|z|\ge 2L} K_{ij}(y-z) v_i(z) v_j(z) \mathrm{d}z \right|^{3/2} \mathrm{d}y = A + B.$$

By standard boundedness,

$$A \le C \int_{|z|\le 2L} |v|^3 \mathrm{d}z \le CL^{a_3},$$

as required. As to B, we use a dyadic decomposition,

$$B \leq \int_{|y|\leq L} \left(\sum_{k=1}^{\infty} \int_{2^k L \leq |z| \leq 2^{k+1} L} \frac{1}{|y-z|^N} |v(z)|^2 \mathrm{d}z \right)^{3/2} \mathrm{d}y.$$

Given that $|y-z| \sim |z|$, we continue

$$\begin{aligned} B &\leq L^N \left(\sum_{k=1}^{\infty} \frac{1}{2^{Nk} L^N} \int_{2^k L \leq |z| \leq 2^{k+1} L} |v(z)|^2 \mathrm{d}z \right)^{3/2} \\ &\leq \frac{C}{L^{N/2}} \left(\sum_{k=1}^{\infty} \frac{2^{ka_2} L^{a_2}}{2^{Nk}} \right)^{3/2} \leq C L^{\frac{3a_2 - N}{2}} \leq C L^{a_3}, \end{aligned}$$

where the latter holds due to imposed assumptions. □

Now, using the obtained estimates (28) and (31) in (26), we obtain

$$\begin{aligned} \frac{1}{L^{N-2\alpha}} \int_{|y|\leq L} |v|^2 \, \mathrm{d}y &\leq \frac{C}{L^{N+1-2\alpha}} \sum_{k=0}^{\infty} \frac{1}{2^{k(N+1-2\alpha)}} \int_{2^k L \leq |y| \leq 2^{k+1} L} (|v|^3 + |v||q|) \, \mathrm{d}y \\ &\leq L^{\beta_p \alpha_p - N - 1 + 2\alpha} \sum_{k=0}^{\infty} 2^{k(\beta_p \alpha_p - N - 1 + 2\alpha)}. \end{aligned}$$

Notice that in the range $\alpha \leq N/p$, the power in the series is negative. Hence,

$$\int_{|y|\leq L} |v|^2 \mathrm{d}y \leq C L^{\beta_p \alpha_p - 1} \quad \text{and} \quad \int_{|y|\leq L} |v|^3 \mathrm{d}y \leq C L^{\beta_p \alpha_p^2 - \alpha_p}. \tag{33}$$

Once again, the new exponents satisfy (32), hence

$$\int_{|y|\leq L} |q|^{3/2} \mathrm{d}y \leq C L^{\beta_p \alpha_p^2 - \alpha_p}. \tag{34}$$

Substituting this into (26), we obtain

$$\int_{|y|\leq L} |v|^2 \mathrm{d}y \leq C L^{\beta_p \alpha_p^2 - \alpha_p - 1}, \tag{35}$$

and so on. Noting that on each step the assumptions on the exponents are satisfied (even improved), we arrive at

$$\int_{|y|\leq L} |v|^2 \mathrm{d}y \leq C L^{\beta_p \alpha_p^n - \alpha_p^{n-1} - \cdots - 1}. \tag{36}$$

For n sufficiently large the power will become negative, implying that $v = 0$.

3.2.2. Proof in the range $\frac{N}{2} < \alpha < \infty$ Starting from the same energy equality (18), we obtain

$$\frac{1}{l_2^{N-2\alpha}} \int_{|y|\le l_2/2} |v|^2 \mathrm{d}y \lesssim \frac{1}{l_1^{N-2\alpha}} \int_{|y|\le l_1} |v|^2 \mathrm{d}y + \int_{l_1/2\le|y|\le l_2} \frac{|v|^3 + |q||v|}{|y|^{N+1-2\alpha}} \mathrm{d}y.$$

Let us fix $l_1 = 2$ and $l_2 = 2L >> 2$. Then

$$\int_{|y|\le L} |v|^2 \,\mathrm{d}y \lesssim L^{N-2\alpha} + L^{N-2\alpha} \int_{1\le|y|\le 2L} \frac{|v|^3 + |q||v|}{|y|^{N+1-2\alpha}} \,\mathrm{d}y, \tag{37}$$

and by the Hölder inequality,

$$\lesssim L^{N-2\alpha} + L^{N-2\alpha} \left(\int_{1<|y|<2L} \frac{1}{|y|^{(N+1-2\alpha)p/(p-3)}} \,\mathrm{d}y \right)^{(p-3)/p}.$$

Since $N - 2\alpha < 0$, the only case we have to consider is when $(N + 1 - 2\alpha)\, p/(p-3) < N$. In this case the estimate above gives

$$\int_{|y|\le L} |v|^2 \mathrm{d}y \lesssim L^{N-2\alpha} + L^{\beta_p}.$$

If $\beta_p < 0$ the proof is finished. Otherwise, we obtain

$$\int_{|y|\le L} |v|^2 \,\mathrm{d}y \lesssim L^{\beta_p}, \quad \text{and} \quad \int_{|y|\le L} |v|^3 \,\mathrm{d}y \lesssim L^{\beta_p \alpha_p}. \tag{38}$$

We are in a position to initiate the bootstrap argument as before, but with some modifications. Plugging (38) into (37), we find

$$\int_{|y|\le L} |v|^2 \,\mathrm{d}y \lesssim L^{N-2\alpha} + \frac{1}{L} \sum_{k=-1}^{[\log_2 L]} 2^{k(N+1-2\alpha)} \int_{L/2^{k+1}<|y|<L/2^k} (|v|^3 + |q||v|) \,\mathrm{d}y$$

$$\lesssim L^{N-2\alpha} + L^{\beta_p\alpha_p - 1} \sum_{k=-1}^{[\log_2 L]} 2^{k(N+1-2\alpha-\beta_p\alpha_p)}.$$

If the power $N + 1 - 2\alpha - \beta_p\alpha_p \ge 0$, we obtain

$$\lesssim L^{N-2\alpha} + L^{N-2\alpha} \log_2 L \to 0, \quad \text{as } L \to \infty.$$

In this case the proof is over. Otherwise, we obtain

$$\lesssim L^{N-2\alpha} + L^{\beta_p\alpha_p - 1}.$$

If $\beta_p\alpha_p - 1 < 0$, the proof is over. Otherwise,

$$\int_{|y|\le L} |v|^2 \,\mathrm{d}y \lesssim L^{\beta_p\alpha_p - 1}, \quad \text{and} \quad \int_{|y|\le L} |v|^3 \,\mathrm{d}y \lesssim L^{\beta_p\alpha_p^2 - \alpha_p}.$$

The iteration will certainly terminate at a step when the power

$$\beta_p\alpha_p^n - \alpha_p^{n-1} - \cdots - 1$$

becomes negative, or earlier.

3.2.3. Implications of the proof to the range $N/p < \alpha \le N/2$ The proof given in the previous section yields the following corollary.

Corollary 3.4. *Suppose* $\frac{N}{p} < \alpha \le \frac{N}{2}$*. Then one has*

$$\int_{|y|\le L} |v|^2 \, \mathrm{d}y \lesssim L^{N-2\alpha}. \tag{39}$$

Indeed, as the power $\beta_p \alpha_p^n - \alpha_p^{n-1} - \cdots - 1$ becomes negative, the term $L^{N-2\alpha}$ becomes dominant. There is only one place of the argument which needs extra attention. It happens if at some point we run into the logarithmic bound

$$\int_{|y|\le L} |v|^2 \, \mathrm{d}y \lesssim L^{N-2\alpha} \log_2 L.$$

Then, for any $\varepsilon > 0$ we have

$$\int_{|y|\le L} |v|^2 \, \mathrm{d}y \lesssim L^{N-2\alpha+\varepsilon}, \quad \text{and} \quad \int_{|y|\le L} |v|^3 \, \mathrm{d}y \lesssim L^{(N-2\alpha+\varepsilon)\alpha_p}.$$

The conditions (32) are still satisfied for small ε, so the pressure has the analogous growth bound. Returning to (37) and performing dyadic splitting of the integral as before, we obtain

$$\int_{|y|\le L} |v|^2 \mathrm{d}y \lesssim L^{N-2\alpha} + L^{(N-2\alpha+\varepsilon)\alpha_p - 1} \sum_{k=0}^{[\log_2 L]} 2^{k(N+1-2\alpha-(N-2\alpha+\varepsilon)\alpha_p)}.$$

The power in the sum is strictly positive. So, we obtain (39).

3.2.4. Proof in the case $\alpha = \infty$ In this case, since we assume that the self-similar solution is global, we can start with (19), which implies

$$\int_{|y|\le L} |v|^2 \, \mathrm{d}y \lesssim \frac{1}{L} \int_{L/2 \le |y| \le 2L} (|v|^3 + |q||v|) \, \mathrm{d}y.$$

This, in turn, implies (27) by the same Hölder estimates as before. The rest of the proof follows the bootstrap scheme of Section 3.2.1.

3.2.5. Theorem 3.2 in the case $p = \infty$ Only a few minor modifications are needed to extend the above argument to the case $v \in L^\infty$, $q \in BMO$. In the case $\alpha \le 0$ we start from (17) and subtract from q the averages over dyadically divided time intervals. This, after changing the order as in (18), results in the following inequality (in place of (26)):

$$\frac{1}{L^{N-2\alpha}} \int_{|y|\le L} |v|^2 \, \mathrm{d}y \le C \sum_{k=1}^{\infty} \int_{2^k L \le |y| \le 2^{k+1} L} \frac{|v|^3 + |q - \bar{q}_k||v|}{|y|^{N+1-2\alpha}} \, \mathrm{d}y, \tag{40}$$

where $\bar{q}_k = \frac{1}{Vol(2^k L \le |z| \le 2^{k+1} L)} \int_{2^k L \le |z| \le 2^{k+1} L} q(z) \mathrm{d}z$. Using that

$$\int_{2^k L \le |y| \le 2^{k+1} L} |q(y) - \bar{q}_k| \mathrm{d}y \lesssim (2^k L)^N \|q\|_{BMO},$$

we immediately obtain (27), with $\beta_\infty = N - 1$ as expected. Note that again the constants β_∞ and $\alpha_\infty = 1$ satisfy the requirements of Lemma 3.3. From this point on the argument proceeds as before.

In the case $\infty > \alpha > 0$ a similar argument replaces (37) with

$$\int_{|y|\le L} |v|^2\,\mathrm{d}y \lesssim L^{N-2\alpha} + L^{N-2\alpha} \sum_{k=-1}^{[\log_2 L]} \int_{L/2^{k+1}<|y|<L/2^k} \frac{|v|^3 + |q-\bar{q}_k||v|}{|y|^{N+1-2\alpha}}\,\mathrm{d}y,$$

where $\bar{q}_k = \frac{1}{Vol(L/2^{k+1}\le|z|\le L/2^k)} \int_{L/2^{k+1}\le|z|\le L/2^k} q(z)\mathrm{d}z$. The rest of the argument goes as before. And finally, the modifications made in Section 3.2.4 for the case $\alpha = \infty$ carry over to these settings in a similar manner.

4. Exclusions Based on Vorticity

The condition in terms of vorticity that excludes a non-trivial blow-up stated and proved in [6] involves a requirement on decay at infinity in the sense that all L^p-norms for $0 < p < p_0$ are finite. In this section we will eliminate solutions under a much weaker condition. Recall that $\varsigma = \frac{1}{2}\{\partial_i v_j + \partial_j v_i\}_{i,j=1}^N$ denotes the strain tensor.

Theorem 4.1. *Suppose $v \in C^1_{\mathrm{loc}}(\mathbb{R}^N)$ is a solution of (7) with $-1 < \alpha < \infty$ satisfying the following conditions:*

(i) $|\varsigma(y)| = o(1)$ *as* $|y| \to \infty$,
(ii) $\omega \in L^p$, *for some* $0 < p < \frac{N}{1+\alpha}$.

Then, v is a constant vector field.

We note that He's examples [13], although in different settings, with $|\omega| \sim \frac{1}{|y|^2}$ in three dimensions and $\alpha = 1$ corresponding to $\omega \in L^p$ for all $p > \frac{N}{1+\alpha} = \frac{3}{2}$. It points to the sharpness of our condition (ii). Furthermore, the value of $p = \frac{N}{1+\alpha}$ appears naturally critical for the fact that the vorticity of the self-similar solution preserves this particular L^p-norm. Let us recall that for a similar reason the exponent $p^* = \frac{N}{\alpha}$ is critical for velocity in Theorem 3.2. The two are conjugate through the Sobolev embedding. Indeed, if $v \to 0$ at infinity, $-1 < \alpha \le N-1$, then $\omega \in L^p$ implies $v \in L^{p^*}$. This brings us back in agreement with the range of Theorem 3.2, although the end-point case cannot be excluded here.

Proof. From (i) by the Fundamental Theorem of Calculus, the radial component of velocity is

$$|v_r(y)| = o(|y|), \quad \text{as } |y| \to \infty. \tag{41}$$

Indeed, we have

$$v(y) = v(0) + \int_0^1 \nabla v(ty)\cdot y\,\mathrm{d}t.$$

Then

$$v_r(y) = v(y) \cdot \frac{y}{|y|} = v(0) \cdot \frac{y}{|y|} + \frac{1}{|y|} \int_0^1 y \cdot \varsigma(ty) \cdot y \, \mathrm{d}t,$$

and the claim follows. Observe

$$\infty > \|\omega\|_p^p = \int_0^\infty \int_{|y|=r} |w|^p \, \mathrm{d}S_r \, \mathrm{d}r.$$

Hence, there exists a sequence $R_j \uparrow \infty$ such that

$$R_j \int_{|y|=R_j} |w|^p \mathrm{d}S_{R_j} \to 0 \quad \text{as } j \to \infty.$$

We multiply (14) by $\omega|\omega|^{p-2}$ and write it in the form

$$\begin{aligned} |\omega|^p + \frac{1}{p(\alpha+1)} \operatorname{div}(y|\omega|^p) - \frac{N}{p(\alpha+1)} |\omega|^p \\ = \frac{1}{p} \operatorname{div}(v|\omega|^p) - \hat{\varsigma}|\omega|^p, \end{aligned} \tag{42}$$

where $\hat{\varsigma} = (\omega\varsigma \cdot \omega + \varsigma\omega \cdot \omega)|\omega|^{-2}$. Let us fix an $R > 0$, integrate (42) over the annulus $\{R < |y| < R_j\}$, and apply the divergence theorem to have

$$\begin{aligned} &\left(\frac{N}{p(\alpha+1)} - 1\right) \int_{R<|y|<R_j} |\omega|^p \mathrm{d}y \\ &\quad + \frac{R}{p(\alpha+1)} \int_{|y|=R} |\omega|^p \mathrm{d}S_R - \frac{R_j}{p(\alpha+1)} \int_{|y|=R_j} |\omega|^p \mathrm{d}S_{R_j} \\ &= \int_{R<|y|<R_j} \hat{\varsigma}|\omega|^p \mathrm{d}y + \frac{1}{p} \int_{|y|=R} v_r |\omega|^p dS_R - \frac{1}{p} \int_{|y|=R_j} v_r |\omega|^p dS_{R_j}. \end{aligned}$$

Then, passing $j \to \infty$, one obtains

$$\begin{aligned} \left(\frac{N}{p(\alpha+1)} - 1\right) \int_{|y|>R} |\omega|^p \mathrm{d}y + \frac{R}{p(\alpha+1)} \int_{|y|=R} |\omega|^p \mathrm{d}S_R \\ = \int_{|y|>R} \hat{\varsigma}|\omega|^p \mathrm{d}y + \frac{1}{p} \int_{|y|=R} v_r |\omega|^p dS_R. \end{aligned}$$

Thus, by choosing R sufficiently large, and using (i) and (41), we can ensure

$$\leq \frac{1}{2} \left(\frac{N}{p(\alpha+1)} - 1\right) \int_{|y|>R} |\omega|^p \mathrm{d}y + \frac{R}{2p(\alpha+1)} \int_{|y|=R} |\omega|^p \mathrm{d}S_R.$$

Consequently,

$$\int_{|y|>R} |\omega|^p \mathrm{d}y = \int_{|y|=R} |\omega|^p \mathrm{d}S_R = 0,$$

and hence, $\omega = 0$ on $\{y \in \mathbb{R}^3 \mid |y| > R\}$. Now we apply the result of [6] to conclude $\omega = 0$ on $\mathbb{R}^N$. Then there exists a harmonic function h such that $v = \nabla h$. By (i), the Hessian matrix $\nabla\nabla h$ is bounded and vanishes at infinity. Since each entry is harmonic, by the Liouville Theorem, $\nabla\nabla h = 0$, and therefore h is a quadratic polynomial. But, then from the condition $|\nabla h| = o(|y|)$, ∇h is constant. □

4.1. Homogeneous Near Infinity Solutions

Given the plethora of two-dimensional homogeneous examples in Section 2.1, it is natural to ask whether one can find a locally smooth self-similar profile homogeneous near infinity. We say that a field $v \in C^1_{\text{loc}}(\mathbb{R}^N)$ is β-homogeneous near infinity if, for some $\beta \in \mathbb{R}$, and for large enough y, the profile v is given by

$$v(y) = \frac{V(y|y|^{-1})}{|y|^\beta}, \tag{43}$$

for some $V \in C^1(\mathbb{S}^{N-1}; \mathbb{R}^N)$. The message of the following theorem is to show that the homogeneity has to be consistent with the scaling of the self-similarity, and even this is excluded in the energy-conservative case.

Theorem 4.2. *Suppose v is a β-homogeneous near infinity solution and any of these conditions are satisfied*

(i) $0 < \beta < \alpha$,
(ii) $-1 < \alpha < \beta$,
(iii) $\alpha = \beta = \frac{N}{2}$.

Then $v = 0$, except in the case $\beta = 0$, which implies that v is constant.

Proof. In the case (i), since $\beta > 0$, $v \in L^p$ for all $p > p_0$. If, in addition $\alpha > N/2$, then an application of Theorem 3.2 concludes the proof. Otherwise, by Corollary 3.4, (39) holds. On the other hand,

$$\int_{L \le |y| \le 2L} |v|^2 \, \mathrm{d}y \sim L^{N-2\beta} \int_{\mathbb{S}^{N-1}} |V(\theta)|^2 \mathrm{d}S(\theta),$$

which necessitates $\beta \ge \alpha$, unless $V = 0$. If $V = 0$, however, then Theorem 4.1 or the result of [6] applies to find $v = \nabla h$ for some harmonic function h. Since $h = const$ near infinity, h is constant throughout by the Liouville Theorem, which implies $v = 0$.

In case (ii) we have $|\nabla v| \sim \frac{1}{|y|^{\beta+1}}$. Since $-1 < \alpha < \beta$, there exists a $p > 0$ with $\frac{N}{1+\beta} < p < \frac{N}{1+\alpha}$. For this p, $\omega \in L^p$, and Theorem 4.1 applies. Note that only in the case $\beta = 0$ may the constant velocity be different from zero.

In case (iii), Corollary 3.4 implies $v \in L^2$. However, for any $M > 0$,

$$\int_{L \le |y| \le ML} |v|^2 \, \mathrm{d}y = \log M \int_{\mathbb{S}^{N-1}} |V(\theta)|^2 \mathrm{d}S(\theta).$$

This implies $V = 0$ and the argument proceeds as before. □

Let us note that Theorem 4.2 could be extended to the range of $-1 < \beta \le 0$ and $\beta < \alpha$ if we postulate the corresponding asymptotic bound on the pressure:

$$|q(y)| \lesssim |y|^{-2\beta} \tag{44}$$

for large y. Example (10) demonstrates that (44) may, in fact, fail for some solutions. Now, assuming (44), the local energy inequality implies

$$\frac{1}{L^{N-2\alpha}}\int_{|y|\le L/2}|v|^2\,\mathrm{d}y \le \frac{1}{l^{N-2\alpha}}\int_{|y|\le l}|v|^2\,\mathrm{d}y + \int_{l/2\le|y|\le L}\frac{|v|^3+|v||q|}{|y|^{N+1-2\alpha}}\,\mathrm{d}y.$$

By a direct computation, with l fixed, and L large,

$$L^{N-2\beta}\|V\|^2_{L^2(\mathbb{S}^{N-1})} \lesssim \int_{|y|\le L/2}|v|^2\,\mathrm{d}y \lesssim L^{N-2\alpha} + L^{N-2\alpha}\int_{cl\le|y|\le L}\frac{1}{|y|^{N+1-2\alpha+3\beta}}\,\mathrm{d}y.$$

If $N+1-2\alpha+3\beta \ge N$, then the above implies

$$L^{N-2\beta}\|V\|^2_{L^2(\mathbb{S}^{N-1})} \lesssim L^{N-2\alpha}\log L,$$

and hence $V=0$. Otherwise,

$$L^{N-2\beta}\|V\|^2_{L^2(\mathbb{S}^{N-1})} \lesssim L^{N-2\alpha} + L^{N-1-3\beta},$$

implying again that $V=0$, since $\beta > -1$.

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