Arch. Rational Mech. Anal. 208 (2013) 447–480
Digital Object Identifier (DOI) 10.1007/s00205-012-0600-x

Generalized Traveling Waves in Disordered Media: Existence, Uniqueness, and Stability

Andrej Zlatoš

Communicated by P. Rabinowitz

Abstract

We prove existence, uniqueness, and stability of transition fronts (generalized traveling waves) for reaction-diffusion equations in cylindrical domains with general inhomogeneous ignition reactions. We also show uniform convergence of solutions with exponentially decaying initial data to time translations of the front. In the case of stationary ergodic reactions, the fronts are proved to propagate with a deterministic positive speed. Our results extend to reaction-advection-diffusion equations with periodic advection and diffusion.

1. Introduction and Results

In this paper we study time-global solutions, called transition fronts or generalized traveling waves, of reaction-diffusion equations on infinite cylinders. We consider the PDE

$$u_t = \Delta u + f(x, u) \tag{1.1}$$

which is used in modeling of processes such as autocatalytic chemical reactions, propagation of advantageous genes in a population, and combustion. The function $u(t, x) \in [0, 1]$ is the (normalized) concentration of a reactant or allele, or the temperature of a combusting solid or gaseous medium. The non-negative reaction term f accounts for an increase of concentration/temperature due to a chemical reaction or burning and satisfies $f(x, 0) = f(x, 1) = 0$. We will be particularly interested in *ignition reactions*, which vanish for u smaller than some *ignition temperature* $\theta(x) > 0$ and are used in the modeling of combustion, but we will also treat general non-negative reactions. The function f will satisfy some uniform bounds but will otherwise be an arbitrary *non-periodic* function of x.

We will also consider the more general equation

$$u_t + q(x) \cdot \nabla u = \mathrm{div}(A(x)\nabla u) + f(x, u), \tag{1.2}$$

with an incompressible mean-zero vector field q representing advection and a uniformly elliptic diffusion operator $\operatorname{div}(A\nabla)$ representing inhomogeneous diffusion. Unlike f, both q and A will be assumed to be periodic (with the same period).

Our main goal is the proof of existence and uniqueness of transition fronts under very general conditions on f. Moreover, we also want to prove uniform convergence of arbitrary solutions of (1.1)/(1.2) with exponentially decaying initial data to these fronts, thus describing the behavior of very general solutions of the PDEs. We will do this for (1.1)/(1.2) with ignition reactions, and also prove existence of fronts for some non-ignition reactions, on the cylindrical domain $D \equiv \mathbb{R} \times \mathbb{T}^{d-1}$ (that is, $\mathbb{R} \times [0, 1]^{d-1}$ with periodic boundary conditions). However, all our results can be extended to open connected domains $D \subseteq \mathbb{R}^d$ with a smooth boundary which are periodic in the first variable and bounded in the others, with either periodic or Neumann boundary conditions on ∂D (the latter being $\nu \cdot \nabla u = 0$ for (1.1) or $\nu \cdot A\nabla u = 0$ and $q \cdot \nu = 0$ for (1.2), with ν the outward unit normal to ∂D). These include cylinders with periodically undulating boundaries and a periodic array of holes.

Such domains, unbounded in arbitrarily many variables, have been considered in [1], where transition fronts for *periodic* f (as well as q and A) were studied. We restrict ourselves here to domains unbounded in only one variable because this is essentially the only case when transition fronts for ignition reactions can be unique, even in homogeneous media (one moving right and one moving left). Moreover, we will show elsewhere [23] that there are examples of ignition reactions on $D = \mathbb{R}^2$ where no transition fronts exist! Nevertheless, some questions about solutions of (1.1)/(1.2) on domains unbounded in several variables can also be treated by our methods [23].

The following is the definition of a transition front from [2], adapted to our domain D.

Definition 1.1. A *transition front (moving to the right)* is a solution $w : \mathbb{R} \times D \to [0, 1]$ of (1.1) or (1.2) that is global in time and satisfies, for each $t \in \mathbb{R}$,

$$\lim_{x_1 \to -\infty} w(t, x) = 1 \qquad \text{and} \qquad \lim_{x_1 \to +\infty} w(t, x) = 0, \tag{1.3}$$

uniformly in $(x_2, \dots, x_d) \in \mathbb{T}^{d-1}$. In addition, the front must have a *bounded width* (uniformly in time). That is, if $I_\varepsilon(t) \subset \mathbb{R}$ is the smallest interval such that $w(t, x) \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$ for $x \in D \setminus (I_\varepsilon(t) \times \mathbb{T}^{d-1})$, then $L_{w,\varepsilon} \equiv \sup_{t \in \mathbb{R}} |I_\varepsilon(t)| < \infty$ for each $\varepsilon > 0$. The domain $I_{\varepsilon_0}(t) \times \mathbb{T}^{d-1}$ for some small $\varepsilon_0 > 0$ will be referred to as the *reaction zone*.

We define a *transition front moving to the left* as above, but with (1.3) replaced by

$$\lim_{x_1 \to -\infty} w(t, x) = 0 \qquad \text{and} \qquad \lim_{x_1 \to +\infty} w(t, x) = 1.$$

Remark. In cylindrical domains fronts moving both right and left can exist. Since the transformation $x_1 \mapsto -x_1$ interchanges the two directions, we will mostly consider only fronts moving to the right.

The simplest case of transition fronts is a *traveling front* whose shape is time-independent. The study of traveling fronts goes back to the works of Kolmogorov et al. [11] and Fisher [8] in 1937. Those researchers considered (1.1) in one spatial dimension $D = \mathbb{R}$ and with x-independent *KPP reaction* $f(u)$ (such that $0 < f(u) \leqq f'(0)u$ for $u \in (0, 1)$). In this case, traveling fronts of the form $w(t, x) = W(x - ct)$ exist precisely when the *front speed* $c \geqq c^*$, with the minimal speed being $c^* = 2\sqrt{f'(0)}$. The *front profile* W is time independent and satisfies the ODE $W_{xx} + cW_x + f(W) = 0$ with $W(\infty) = 0$ and $W(-\infty) = 1$. The situation is the same for general *positive reactions* (such that $f(u) > 0$ for $u \in (0, 1)$) but the formula for $c^* > 0$ is more complicated. In contrast, the front and its speed $c^* > 0$ are unique for *ignition reactions* (such that $f(u) = 0$ for $u \in [0, \theta]$ and $f(u) > 0$ for $u \in (\theta, 1)$, with $\theta > 0$).

The ansatz $w(t, x) = W(x - cte_1)$ also works in more dimensions when q, A and f are independent of x_1, and the answers are the same as above. In particular, the case of mean-zero shear flows q and x-independent A, f has been treated by Berestycki et al. [5], and Berestycki and Nirenberg [6]. On the other hand, for periodic q, A, f (with the same period p), the front profile can only be expected to be time-periodic in a moving frame in the sense $w(t + p/c, x + pe_1) = w(t, x)$ for some speed $c > 0$. Such *pulsating fronts* are of the form $w(t, x) = W(x_1 - ct, x)$ with the profile W decreasing in the first variable, periodic in the second, and satisfying $\lim_{s\to-\infty} W(s, x) = 1$ and $\lim_{s\to+\infty} W(s, x) = 0$uniformly in $x \in D$. Existence and uniqueness of pulsating fronts (with mean-zero q) was proved by Xin [20] for x-independent ignition reactions in $\mathbb{R}^d$, and by Berestycki and Hamel [1] for x-periodic ignition reactions (such that if $\tilde{\theta} \equiv \inf_x \theta(x)$, with $\theta(x) \equiv \inf\{u > 0 \mid f(x, u) > 0\}$ the *ignition temperature*, then $\tilde{\theta} > 0$ and $\sup_{x\in D} f(x, u) > 0$ for each $u \in (\tilde{\theta}, 1)$) in general periodic domains. [1] also treats x-periodic positive reactions (as above but with $\tilde{\theta} = 0$) and again proves existence of fronts with precisely the speeds $c \geqq c^*$ for some $c^* > 0$.

The situation is different for disordered media, when no such ansatz exists and one has to work directly with the original PDE. Constant or periodic front profiles cannot be expected and fronts need not have a well defined speed. The definition of a transition front, above, has been given by Berestycki and Hamel [2] in a more general setting and on arbitrary domains. An alternative definition has been given by Shen [18], who studied fronts in time-random one-dimensional media and established some sufficient conditions on their existence. This formalizes an earlier definition by Matano, which essentially requires the profile of the front to be a continuous function of the medium near the reaction zone.

Due to the above difficulties, existence and uniqueness of transition fronts in general disordered media has so far only been proved for (1.1) in one dimension, for some ignition reactions with x-independent ignition temperatures. Specifically, Nolen and Ryzhik [15], and independently Mellet et al. [14], have proved that such fronts exist on $D = \mathbb{R}$ when the reaction satisfies $b_0 f_0(u) \leqq f(x, u) \leqq b_1 f_0(u)$ and $f_0'(1) < 0$. Here $0 < b_0 \leqq b_1 < \infty$ and f_0 is of ignition type with $f_0(u) > 0$ if and only if $u \in (\theta, 1), \theta > 0$. Moreover, Mellet et al. [13] proved that if $f(x, u) = b(x) f_0(u)$ with $b(x) \in [b_0, b_1]$, then the (right-moving) front is unique and exponentially stable with respect to exponentially in space decaying perturbations. In addition, Vakulenko and Volpert [19] proved existence of fronts for (1.1) in one-dimensional with f some small perturbation of an x-independent *bistable reaction* $f_0(u)$ (such that $f_0'(0), f_0'(1) < 0$, $f_0(u) < 0$ for $u \in (0, \theta)$, and $f_0(u) > 0$ for $u \in (\theta, 1)$). We also mention here results for x-independent bistable reactions in some special domains—non-existence of moving fronts (while stationary fronts exist) in quickly widening cylinders by Chapuisat and Grenier [7], and later also Berestycki et al. [3], and existence of fronts in $\mathbb{R}^d$ with compact star-shaped obstacles [3].

The usage of one-dimensional techniques plays an important role in [13–15], as does the requirement of an x-independent ignition temperature θ.We present here a new method that can handle *more general reactions*, works *in several dimensions*, and *in the presence of (periodic) q and A*. In particular, we prove existence of a unique transition front when f lies between two arbitrary x-independent ignition reactions with different ignition temperatures. We also prove existence of fronts in the more general case when the upper bound is a positive reaction, only requiring a bound on its derivative at zero (these fronts are not unique in general, as is the case for homogeneous media). We note that the requirement of a bound of this type is necessary to guarantee existence and cannot be improved, except possibly by a constant (see Remark 1 after Theorem 1.3).

Let us now state our main results. We will start with the special case (1.1). We will assume the following hypotheses on f.

(**H1**) The reaction f is uniformly Lipschitz with constant $K \geqq 1$ and lies between two x-independent reactions, one of ignition type and the other positive or ignition. More specifically, there are Lipshitz functions f_0, f_1, decreasing on $[1-\varepsilon, 1]$ for some $\varepsilon > 0$, such that $f_0(u) \leqq f(x,u) \leqq f_1(u)$ for $(x,u) \in D \times [0,1]$. In addition, $f_0(0) = f_0(1) = f_1(0) = f_1(1) = 0$, there is $\theta \in (0,1)$ such that $f_0(u) = 0$ for $u \in [0,\theta]$ and $f_0(u) > 0$ for $u \in (\theta, 1)$, and there is $\theta' \in [0,1)$, such that $f_1(u) = 0$ for $u \in [0,\theta']$ and $f_1(u) > 0$ for $u \in (\theta', 1)$.

Assume that $c_0 > 0$ is the speed of the unique (right-moving) traveling front for (1.1) with f replaced by the x-independent reaction f_0 (this front is of the form $w(t,x) = W(x_1 - c_0 t)$). We then obtain existence of a transition front for f, provided $f_1'(0) < c_0^2/4$. Note that this condition is automatically satisfied when $f_1'(0) = 0$ (for example, if f_1 is ignition). For instance, a front exists in the case $f(x,u) = b(x) f_1(u)$ with $f_1'(0) = 0$ and b Lipschitz and uniformly bounded away from 0 and ∞. Moreover, if f_1 is ignition, then we also prove uniqueness of the (right-moving) front and that this unique front is a global attractor of general exponentially decaying initial data.

Our reaction f can have an x-dependent ignition temperature $\theta(x) \in [\theta', \theta]$ and we do not require $\sup_{x\in D} f(x,u) > 0$ for each $u \in (\inf_{x\in D}\theta(x), 1)$. Nevertheless, we will need to impose a similar but weaker natural hypothesis (which is automatically satisfied when $\theta' = \theta$) that if $f(x,u)$ is large enough (rather than just positive) for some (x,u) and $v \in (u,1)$, then $f(\cdot, v)$ cannot become uniformly arbitrarily small on some large neighborhood of $x \in D$.

Definition 1.2. Given any $\zeta > 0$, define

$$\alpha_f(x) \equiv \inf\big(\{u \in (0,1) \mid f(x,u) \geqq \zeta u\} \cup \{1\}\big). \tag{1.4}$$

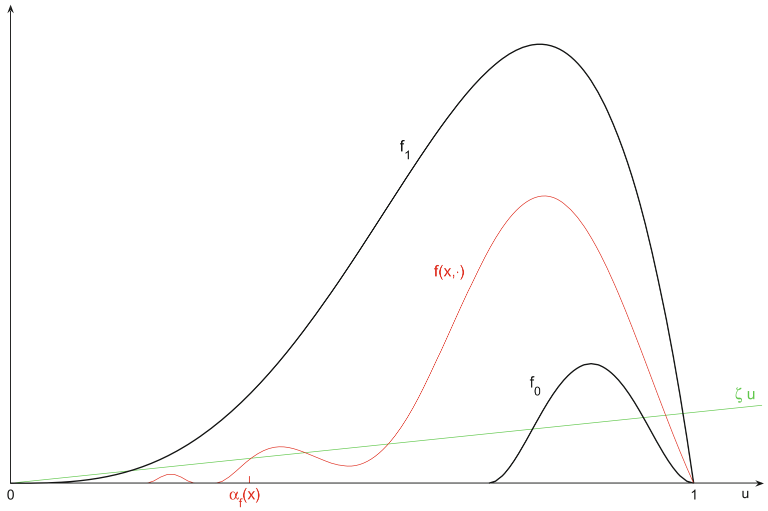

Fig. 1. Example of a reaction (at a fixed $x \in D$) satisfying our hypotheses

Assume $g \in C([0,1])$ with $g(0) = g(1) = 0$ and $g(u) > 0$ for $u \in (0,1)$. If for all $x \in D$ and all $u \in [\alpha_f(x), 1]$ we have $\sup_{|\tilde{x}_1 - x_1| \leqq \|g\|_\infty^{-1}} f(\tilde{x}, u) \geqq g(u)$, then we say that f *ζ-majorizes* g.

Remark. 1. If $\zeta > f_1'(0)$, then $\alpha_f(x) = \min(\{u \in (0,1) \mid f(x,u) = \zeta u\} \cup \{1\}) > \theta'$.

2. We could replace $\|g\|_\infty^{-1}$ by some large M, but then the notation would have to include it, thus being too cumbersome. Of course, we will consider f ζ-majorizing some small g (with a large $\|g\|_\infty^{-1}$), so this simplification can be made without loss.

The Fig. 1 gives an example of such an f at a fixed x (satisfying $f(x,u) > 0$ for $u \in [\alpha_f(x), 1]$, while the definition requires non-vanishing of f only at such u and *some* u-dependent point $\tilde{x}$ whose distance from x is uniformly bounded).

Here is our main result for (1.1).

Theorem 1.3. *Let f satisfy the hypotheses (H1). Assume that f ζ-majorizes g for some $\zeta < c_0^2/4$ and some g as in Definition 1.2.*

(i) *If $f_1'(0) < c_0^2/4$, then there exists a transition front w for* (1.1) *moving to the right with $w_t > 0$* (*and another moving to the left*).

(ii) *If f_1 is an ignition reaction (that is, $\theta' > 0$) and f is non-increasing in u on $[\theta'', 1]$ for some $\theta'' < 1$, then there is a unique* (*up to time shifts*) *transition front w_+ for* (1.1) *moving to the right* (*and another w_- moving to the left*).

(iii) *In the setting of* (ii) *we have convergence of solutions with exponentially decaying initial data to time shifts of $w_\pm$ in the sense of Definition 1.4, below. Moreover, this convergence is uniform in f, a, u—the s_ε in the definition depends only on $f_0, f_1, \zeta, g, K, \theta'', Y, \mu, \nu, \varepsilon$ and the L_ν on f_0, ν.*

Definition 1.4. Let $w_\pm$ be some right- and left- moving transition fronts for (1.1) (or (1.2)) on D. We say that *solutions with exponentially decaying initial data converge to time shifts of* $w_\pm$ if the following hold for any $Y, \mu, \nu > 0$ and $a \in \mathbb{R}$:

(a) If u solves (1.1) (or (1.2)) with initial datum

$$(\theta + \nu)\chi_{(-\infty,a]}(x_1) \leqq u_0(x) \leqq \mathrm{e}^{-\mu(x_1-a-Y)},$$

then there is τ_u such that for every $\varepsilon > 0$ there is $s_\varepsilon > 0$, such that for each $t \geqq s_\varepsilon$,

$$\|u(t,x) - w_+(t+\tau_u, x)\|_{L_x^\infty} < \varepsilon \tag{1.5}$$

(and similarly for solutions exponentially decaying as $x_1 \to -\infty$ and for w_-).

(b) There is $L_\nu < \infty$ such that if $L \geqq L_\nu$ and u solves (1.1) (or (1.2)) with initial datum

$$(\theta + \nu)\chi_{[a-L,a+L]}(x_1) \leqq u_0(x) \leqq \min\{\mathrm{e}^{-\mu(x_1-a-L-Y)}, \mathrm{e}^{\mu(x_1-a+L+Y)}\},$$

then there are $\tau_{u,\pm}$ such that for every $\varepsilon > 0$ there is $s_\varepsilon > 0$, such that for each $t \geqq s_\varepsilon$,

$$\|u(t,x) - w_+(t+\tau_{u,+}, x) - w_-(t+\tau_{u,-}, x) + 1\|_{L_x^\infty} < \varepsilon. \tag{1.6}$$

As mentioned before, transition fronts need not exist in general (and are typically not unique) when the domain D is unbounded in more than one variable, even for ignition reactions. We next make several remarks which illuminate the necessity of the assumptions in Theorem 1.3 on cylindrical domains D, and thus show that our result is *qualitatively sharp*.

Remark. 1. The main condition here is $f_1'(0) < c_0^2/4$. Some condition of this type is necessary for the existence of fronts, as can be seen from a result of Nolen et al. [16], which shows that there are examples with $D = \mathbb{R}$ and $f_1'(0)$ arbitrarily close to c_0^2 where no transition fronts exist! In these examples, in fact, each global in time solution $0 < u < 1$ is a spatially extended pulse with $\|u(t,x)\|_{L_x^\infty} \to 0$ as $t \to -\infty$.

2. The condition $f_1'(0) < c_0^2/4$ is equivalent to $2\sqrt{f_1'(0)} < c_0$, meaning that minimal-speed fronts for KPP reactions $f(u)$ with $f'(0) = f_1'(0)$ are slower than the front for $f_0(u)$. It then follows that the graph of $f_1'(0)u$ must intersect that of $f_0(u)$ at some $u > 0$ because the minimal front speed is monotone with respect to the reaction. Thus we cannot treat the case when f_1 as a KPP reaction, which is not surprising in the light of the previous remark.

3. Some condition of non-vanishing of f after it has become large (such as in Definition 1.2) is also needed, otherwise a transition front connecting 0 and 1 might not exist. An example of such a situation can be obtained by taking $f(x,u) = f(u)$ with $f(u+\frac{1}{2}) \geqq f(u)$ for $u \in [0, \frac{1}{2}]$ (so $f(\frac{1}{2}) = 0$) with a strict inequality somewhere. It can be then shown that if there is a front connecting 0 and $a \in [0,1]$, then $a \leqq \frac{1}{2}$.

4. As mentioned earlier, in the general positive reaction case (i) transition fronts are not unique even in homogeneous media.
5. We also note that some decay assumption on u_0 in (iii) needs to be made. Indeed, it is not hard to show that if $f(x,u) = f_0(u)$ and u_0 decays slowly enough, then u will "overtake" all time shifts of w_+.
6. $L_\nu < \infty$ in Definition 1.4(b) is such that solutions of (1.1) with f_0 in place of f and $u_0(x) \geqq (\theta + \nu)\chi_{[-L_\nu, L_\nu]}$ are guaranteed to spread, that is, $u(t,x) \to 1$ locally uniformly in $x \in D$ as $t \to \infty$. This can be taken to be the L_ν from Lemma 5.1.

Theorem 1.3 extends to the more general case of (1.2) with periodic q and A which satisfy the following hypotheses.

(**H2**) The flow $q \in C^\eta(D)$ (for some $\eta > 0$) is incompressible $\nabla \cdot q \equiv 0$, p-periodic in x_1, and with mean-zero first coordinate $\int_{\mathcal{C}} q_1(x)\,\mathrm{d}x = 0$ (where $\mathcal{C} = [0,p] \times \mathbb{T}^{d-1}$). The matrix $A \in C^{1,\eta}(D)$ is symmetric, p-periodic, and with $\underline{A}I \leqq A(x) \leqq \bar{A}I$ for some $0 < \underline{A} \leqq \bar{A} < \infty$ and all $x \in D$.

Again, let $c_0 > 0$ be the speed of the unique (right-moving) pulsating front for (1.2), with f replaced by f_0 [1]. We also let $\zeta_0 > 0$ be such that the minimal pulsating front speed for (1.2) with f replaced by $\zeta_0 u(1-u)$ is c_0 [4]. Equivalently, ζ_0 is the unique positive number such that the right-hand side of (2.9) below with ζ replaced by ζ_0 equals c_0. For the left-moving front we have a possibly different speed $c_0^- > 0$, and we define $\zeta_0^- > 0$ accordingly, with e_1, q_1 replaced by $-e_1, -q_1$ in (2.10). The condition $f_1'(0) < c_0^2/4$ is now replaced by $f_1'(0) < \zeta_0$ and our main result for (1.2) is as follows.

Theorem 1.5. *Let q, A, f satisfy the hypotheses (H1), (H2).*
Assume that f ζ-majorizes g for some $\zeta < \zeta_0$ and some g as in Definition 1.2.

- (i) *If $f_1'(0) < \zeta_0$, then there exists a transition front w for (1.2) moving to the right with $w_t > 0$ (and another moving to the left when ζ_0 is replaced by ζ_0^- in the hypotheses).*
- (ii) *If f_1 is an ignition reaction (that is, $\theta' > 0$) and f is non-increasing in u on $[\theta'', 1]$ for some $\theta'' < 1$, then there is a unique (up to time shifts) transition front w_+ for (1.2) moving to the right (and another w_- moving to the left).*
- (iii) *In the setting of (ii) we have convergence of solutions with exponentially decaying initial data to time shifts of $w_\pm$ in the sense of Definition 1.4. Moreover, this convergence is uniform in f, a, u—the s_ε in the definition depends only on $q, A, f_0, f_1, \zeta, g, K, \theta'', Y, \mu, \nu, \varepsilon$ and the L_ν on q, A, f_0, ν.*

Remark. 1. Although the front speed is not well defined in general disordered media, it is easy to see from our proof that the reaction zone of $w_\pm$ moves with speed $\geqq c_0$ resp. $\geqq c_0^-$. We also prove an f-independent bound on the width of $w_\pm$ (see the remark after the proof of Lemma 3.1 and the last paragraph of Section 3). In addition, w_+ decays at an exponential rate $\geqq \lambda_\zeta$ as $x_1 \to \infty$ (and similarly w_- as $x_1 \to -\infty$) determined from (2.9) (see (3.1)). In fact, λ_ζ can be replaced here by any λ such that the fraction in (2.9) is smaller than c_0.

2. The proof of Theorem 1.5 can, in fact, be made independent of previous results on transition fronts. In particular, we can prove that a unique pulsating front

with a unique speed c_0 exists for (1.2) with the x-independent reaction f_0 (which we need in order to state Theorem 1.5). This is done at the end of the proof of Lemma 5.1. The only result we will need is convexity of the function $\kappa(\lambda)$ in (2.10) and $\kappa'(0) = 0$, from [1, Proposition 5.7(iii)].

We next note that (at least for fixed periodic A, q) the ignition fronts in (ii) satisfy the earlier mentioned definition of Matano, where the shape of the front depends continuously on the reaction f in the neighborhood of the reaction zone. This follows from the uniform in time convergence of general solutions to the front in (iii). We provide an application of this principle to periodic and random media.

Corollary 1.6. *Assume the hypotheses of Theorem 1.5*(ii) *and that f is also p-periodic in x_1. Then there is $c_\pm > 0$ such that the unique transition front $w_\pm$ from that theorem satisfies $w_\pm(t + p/c_\pm, x \pm pe_1) = w_\pm(t, x)$ (that is, $w_\pm$ is time-periodic in the frame moving right/left at the speed $c_\pm$, which is thus the speed of $w_\pm$).*

Remark. Existence, uniqueness, and periodicity of the transition front in this setting also follow from results in [1,2], but only within the class of transition fronts which have a *constant mean speed* (that is, there is $c \geqq 0$ such that for $\tilde{X}_w$ as below, $\sup_{(t,s)} \big| |\tilde{X}_w(t) - \tilde{X}_w(s)| - c|t - s| \big| < \infty$), with the extra assumption $\sup_{x \in D} f(x, u) > 0$ for each $u \in (\inf_{x \in D} \theta(x), 1)$, and with somewhat stronger regularity assumptions on q, A, f (the last two in [1]). Hence, our result of existence, uniqueness, and periodicity of pulsating fronts in periodic media with ignition reactions is new in this generality.

Proof. Consider only w_+. Let u, u' be solutions of (1.2) with initial data $u_0(x)$, $u_0(x + p)$, where u_0 is as in the first part of (iii). Then u, u' are p-translates of each other (in x_1), so the same is true about $w_+(t + \tau_u, x)$ and $w_+(t + \tau_{u'}, x)$. The result follows with $c_+ \equiv p/(\tau_u - \tau_{u'})$. □

Although fronts in disordered media do not have constant (mean) speeds in general, our results can be used to show that there is a deterministic asymptotic speed of fronts for (1.1)/(1.2) when f is random (stationary and ergodic with respect to translations in x_1). The *asymptotic front speed* of a transition front w is defined as

$$c \equiv \lim_{|t| \to \infty} \frac{|\tilde{X}_w(t)|}{|t|}, \tag{1.7}$$

provided the limit exists. Here $\tilde{X}_w(t)$ is the first coordinate of some point in the reaction zone of the front; for instance, one could take $\tilde{X}_w(t)$ such that $w(t, x) = \frac{1}{2}$ for some $x = (\tilde{X}_w(t), x_2, \dots, x_d)$. Such $\tilde{X}_w(t)$ may not be unique, but the requirement of a bounded width of the front w shows that the limit in (1.7) is independent of the choice.

Corollary 1.7. *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that a measurable function $f : \Omega \to L^\infty_{\mathrm{loc}}(D \times [0, 1])$ satisfies the hypotheses of Theorem 1.5*(ii) *(with ω-independent q, A) uniformly in $\omega \in \Omega$. In addition, assume that f is stationary and ergodic. That is, there is a group $\{\pi_k\}_{k \in \mathbb{Z}}$ of measure preserving transformations acting ergodically on Ω such that $f(\pi_k \omega; x, u) = f(\omega; x - kpe_1, u)$.*

Then there are constants $c_\pm > 0$ *such that the* ω*-dependent transition front* $w_{\pm,\omega}$ *from that theorem has asymptotic speed* $c_\pm$ *for almost all* $\omega \in \Omega$.

Remark. 1. Theorem 1.5(iii) then shows that solutions with (large enough) exponentially decaying initial data almost surely spread with asymptotic speed c_+ to the right and c_- to the left.
2. Corollary 1.7 was proved in [15] for (1.1) in the one-dimensional setting $D = \mathbb{R}$ and with the random reaction function $f(\omega; x, u) = b(x, \omega) f_0(u)$, where b is bounded below and above by positive constants and f_0 is of the ignition type.

Proof. Consider only w_+ and the case of p-periodic q, A. Let $v(x)$ be the function from Lemma 2.1 below with $\tilde{\theta}$ chosen as at the beginning of Section 3. Let u_m solve (1.2) with initial condition $u_m(0, x) = v(x - mpe_1)$ (so that $(u_m)_t > 0$). For integers $n \geqq m$, define

$$\tau_{m,n}(\omega) \equiv \inf \left\{ t \geqq 0 \,\middle|\, u_m(t, x) \geqq v(x - npe_1) \quad \text{for all } x \in D \right\}.$$

Then the proof of Theorem 1.5 (more precisely, (3.6) and (3.9), below) shows that $\tau_{m,n}(\omega) \in [C_0(n-m), C_1(n-m)]$ for some $0 < C_0 < C_1 < \infty$ and all ω. Moreover, $\tau_{m,n}$ is measurable because $f : \Omega \to L^\infty_{\text{loc}}(D \times [0, 1])$ is measurable and $\tau_{m,n} : L^\infty_{\text{loc}}(D \times [0, 1]) \to \mathbb{R}$ is a lower semi-continuous function of the reaction due to continuity of solutions of (1.2) with respect to $f \in L^\infty_{\text{loc}}(D \times [0, 1])$ and the properties of v.

The comparison principle shows that $\tau_{m,n}(\omega) \leqq \tau_{m,k}(\omega) + \tau_{k,n}(\omega)$ when $m \leqq k \leqq n$. We also have $\tau_{m+k,n+k}(\pi_k \omega) = \tau_{m,n}(\omega)$ for $k \in \mathbb{Z}$. Since the group $\{\pi_k\}_{k \in \mathbb{Z}}$ acts ergodically on Ω, the subadditive ergodic theorem [10, 12] shows that there is $\tau_+ \in [C_0, C_1]$ such that

$$\tau_+ = \lim_{n \to \infty} \frac{\tau_{0,n}(\omega)}{n} = \lim_{n \to \infty} \frac{\tau_{-n,0}(\omega)}{n}$$

for almost all ω. Uniform convergence (in ω) of the solution u_0 to the front $w_{+,\omega}$ and the proof of Theorem 1.5 (more precisely, (3.1) below) then show that $c_+ = p/\tau_+$ is the asymptotic speed of $w_{+,\omega}$ for almost all ω. □

Let us finish this introduction with a brief description of the proof of Theorem 1.5. In Section 2 we construct the front as a limit of a (sub)sequence of special solutions u_n of (1.2), increasing in time and initially (at a sequence of times $\tau_n \to -\infty$) supported increasingly farther to the left. The sequence τ_n is chosen so that the reaction zone for u_n arrives at the origin at $t = 0$. The main issue is to show that the u_n have a *uniformly (in n) bounded width* in the sense of Definition 1.1. This will boil down to showing that the leftmost point $Y_n(t)$, such that to the right of it u_n decays no slower than at a fixed exponential rate, cannot escape too far to the right from the leftmost point $X_n(t)$, such that to the right of it $u_n(t, x) < \alpha_f(x)$. When going from left to right, the former point can be thought of as the beginning of the region where u_n is 'small' (for obvious reasons), while the latter can be thought of as the end of the region where u_n is 'large enough' (because the end of the region where u_n is close to 1 will be within a uniform distance from $X_n(t)$, thanks to f ζ-majorizing g and thus u_n growing close to 1 within a uniformly bounded time after it becomes at least $\alpha_f(x)$ — see Lemma 2.6(i)). The uniform boundedness of $Y_n(t) - X_n(t)$ will be proved in Lemma 2.5 using the crucial hypothesis $f_1'(0) < \zeta_0$, which guarantees that the reaction is so small when $u_n(t,x) < \alpha_f(x)$ that it can make $Y_n(t)$ propagate to the right only with a speed strictly smaller than c_0 when $Y_n(t) - X_n(t)$ is large. At the same time, $X_n(t)$ always propagates to the right with a speed of at least c_0, because that is the case for the region where u_n is close to 1 (see Lemma 2.6(i)). Thus $Y_n(t) - X_n(t)$ cannot grow unbounded, as desired.

We note that this argument requires q and A to be periodic because the upper bound on the speed of propagation of $Y_n(t)$ is the speed of propagation of an exponentially decreasing super-solution (at small u) of (1.2), which is a solution if $f(x,u)$ is replaced by ζu. We were able to construct this solution when the latter PDE is x-periodic by solving the *cell problem* (2.10) below, but not for general non-periodic q, A. Similarly, the lower bound c_0 on the speed of propagation of $X_n(t)$ is the minimal front speed for the x-periodic PDE with $f(x,u)$ replaced by $f_0(u)$. It is possible that if one could show that these bounds depend on some local norms of q, A, but not on q, A themselves, and primarily not on the period p, then one could obtain a front for non-periodic q, A as a limit of fronts for a sequence of periodic q_k, A_k with growing periods. We leave this as an open problem.

In Section 3 we show that if w is any transition front, then for each u_n one obtains L_x^∞-convergence of u_n to a time-shift of w as $t \to \infty$, and the rate of this convergence is *uniform in n*. This will show that any two fronts must be time shifts of each other. The argument starts with proving local stability of u_n via construction of sub- and super-solutions near u_n,which will trap w between small perturbations of two time-shifts of u_n. One then uses the strong maximum principle to obtain convergence of a fixed time shift of w to u_n. We note that a similar strategy was employed in the proof of uniqueness of one-dimensional ignition fronts [13]. However, our argument applies in a more general setting, and its second part is different from, as well as considerably simpler and shorter than, that in [13]. This is in part thanks to the use of the ideas from Section 2 described above, and in part because, unlike [13], we do not establish a rate of the convergence of u_n to a time shift of w.

We recycle this argument in Section 4 to show L_x^∞-convergence of u_n to a time shift of any solution u as in Theorem 1.5(iii), again at a *uniform rate (in u)*. Since u_n also converges uniformly to a time shift of w, the same will be true for u.

Of course, this all also proves the special case, Theorem 1.3. Finally, in Appendix A we show how to make our proof independent of previous results by using our arguments to obtain a slight improvement of Lemma 2.6(ii), below, which is from [21].

The author thanks Jean-Michel Roquejoffre for useful discussions. Partial support by NSF grants DMS-1113017 and DMS-1056327 and an Alfred P. Sloan Research Fellowship is also acknowledged.

2. Existence of Fronts for General Reactions

In this section we will prove Theorem 1.5(i) by finding a front moving to the right. Let us assume that the period of q, A in x_1 is $p = 1$, and thus the unit cell of periodicity is $\mathcal{C} = \mathbb{T}^d$ (the general case is treated identically). We will assume without loss of generality that $\zeta > f_1'(0)$ and that there is $\sigma \in (0, \zeta - f_1'(0))$ such that f ζ'-majorizes g for each $\zeta' > \zeta - \sigma$. This can be done because if $\zeta' > \zeta$, then f also ζ'-majorizes g, so we only need to change ζ to $(\max\{\zeta, f_1'(0)\} + \zeta_0)/2$ and pick $\sigma \equiv (\zeta_0 - \max\{\zeta, f_1'(0)\})/4$. The first assumption implies that inf can be replaced by min and $\geqq$ by $=$ in (1.4), and $\alpha_f(x)$ is uniformly bounded away from 0 and 1 (see (2.8)). The second guarantees that if a subsequence of reactions f_n which satisfy (H1) converges locally uniformly to f (each such sequence has a locally uniformly convergent subsequence) and each f_n ζ'-majorizes g for each $\zeta' > \zeta - \sigma$, then this f not only satisfies (H1) but also ζ'-majorizes g for each $\zeta' > \zeta - \sigma$. This claim would not be true with $\geqq$ in place of $>$. We note that all constants in this section will depend on $q, A, f_0, f_1, \zeta, g, K$ (also on ζ_0, σ, c_0 which already depend only on q, A, f_0, ζ) *but not on* f.

As in the one-dimensional case [14,15,18], we will look for the front as a limit of solutions with initial data specified at increasingly negative times and supported further and further to the left. These solutions will be monotonically increasing in time. We will therefore need

Lemma 2.1. *For each* $\tilde\theta \in (\theta, 1)$ *there exists a function* v *supported in* $(-\infty, 0) \times \mathbb{T}^{d-1} \subseteq D$ *with* $v(x) \in [0, \tilde\theta$ *for all* $x \in D$ *and* $v(x) = \tilde\theta$ *for all* x_1 *small enough such that*

$$- \operatorname{div}(A(x)\nabla v) + q(x) \cdot \nabla v \leqq f_0(v) \tag{2.1}$$

in the sense of distributions.

Proof. Take a nondecreasing function $\rho \in C(\mathbb{R}) \cup C^2(\mathbb{R}^+)$ with $\rho(v) = 0$ for $v \leqq 0$, $\rho(v) = v$ for $v \in [0, \frac{\theta+\tilde\theta}{2}]$, $\rho''(v) \leqq 0$ for $v \in [\frac{\theta+\tilde\theta}{2}, 1]$, and $\rho(v) = \tilde\theta$ for $v \geqq 1$.

Let $\tilde v < 0$ be a C^2 solution of $-\operatorname{div}(A\nabla\tilde v) + q \cdot \nabla\tilde v = q_1 - \operatorname{div}(Ae_1)$ on $\mathbb{T}^d$, periodically continued to D (here $e_1 = (1, 0, \ldots, 0)$ and $q_1 = q \cdot e_1$). Such $\tilde v$ exists because the integral of the right-hand side is zero, and because the left-hand side annihilates constants. Then we let $v_\varepsilon(x) \equiv \varepsilon(\tilde v(x) - x_1)$ and $v(x) \equiv \rho(v_\varepsilon(x))$ for $\varepsilon > 0$, so that v is supported in $(-\infty, 0)\times\mathbb{T}^{d-1}$. We have $-\operatorname{div}(A\nabla v_\varepsilon) + q\cdot\nabla v_\varepsilon = 0$ and so, for some distribution $T \geqq 0$ supported on the set $D_\varepsilon \equiv \{x \in D \mid v_\varepsilon(x) = 0\}$,

$$- \operatorname{div}(A\nabla v) + q \cdot \nabla v = -\varepsilon^2 \chi_{D\setminus D_\varepsilon}\rho''(v_\varepsilon)(\nabla\tilde v - e_1) \cdot A(\nabla\tilde v - e_1) - T.$$

If $\rho''(v_\varepsilon(x)) < 0$, then $v(x) = \rho(v_\varepsilon(x)) \in [\frac{\theta+\tilde\theta}{2}, \tilde\theta]$. Since f_0 is uniformly positive on this interval and A is a positive matrix, (2.1) follows, provided $\varepsilon > 0$ is small enough. □

We now fix $\tilde\theta < 1$ to be close to 1 so that $\tilde\theta > \theta_0$ from (2.8) below (in particular, $\tilde\theta > \theta$) and consider the corresponding v along with the functions $v_n(x) \equiv v(x + ne_1)$. For all $n \in \mathbb{N}$ let u_n solve (1.2) for $t > \tau_n$ with initial condition $u_n(\tau_n, x) = v_n(x)$, where $\tau_n \to -\infty$ will be chosen shortly. We then have

Lemma 2.2. *The functions u_n satisfy for all $t > \tau_n$ and $x \in D$*

$$(u_n)_t(t, x) > 0, \tag{2.2}$$

as well as

$$\lim_{t\to\infty} u_n(t, x) = 1 \tag{2.3}$$

locally uniformly in $x \in D$.

Proof. The time derivative $\dot{u}_n \equiv (u_n)_t$ satisfies $\dot{u}_n(\tau_n, x) \geqq 0$ due to (2.1) and periodicity of q, A, and it is not identically 0. Since $\dot{u}_n$ satisfies $(\dot{u}_n)_t + q \cdot \nabla \dot{u}_n = \operatorname{div}(A\nabla\dot{u}_n) + \frac{\partial f}{\partial u}(x, u_n)\dot{u}_n$ and $\frac{\partial f}{\partial u}$ is bounded, the strong maximum principle shows (2.2).

This means that for each n, the function $\tilde{u}(x) \equiv \lim_{t\to\infty} u_n(t, x)$ is well defined and satisfies $\tilde{u}(x) \in (0, 1]$ and

$$-\operatorname{div}(A\nabla\tilde{u}) + q \cdot \nabla\tilde{u} = f(x, \tilde{u}).$$

Lemma 2.3(ii) below now shows that $\tilde{u}$ is a constant, which is then 1 due to $\|\tilde{u}\|_\infty \geqq \tilde{\theta} > \theta$. Parabolic regularity then shows the limit to be locally uniform in D. □

We now choose $\tau_n < 0$ to be the unique time such that

$$u_n(0, 0) = \theta. \tag{2.4}$$

Note that $\tau_n \to -\infty$ (see the remark after Lemma 2.4).

Despite its apparent simplicity, we were not able to locate the following Liouville-type result in the literature.

Lemma 2.3. (i) *Let q, A be as in* (H2) *but without the assumptions of periodicity and q_1 being mean-zero. If the function u is bounded on D and satisfies*

$$-\operatorname{div}(A(x)\nabla u) + q(x) \cdot \nabla u = 0, \tag{2.5}$$

then u is constant.

(ii) *Let q, A be as in* (H2) *but without the assumption of periodicity, and let r be a bounded non-negative measurable function on D. If u is bounded and non-negative on D and satisfies*

$$-\operatorname{div}(A(x)\nabla u) + q(x) \cdot \nabla u = r(x)u, \tag{2.6}$$

then u is constant.

Remark. Note that if q is incompressible on D, then its mean $\bar{q} \equiv \int_{\mathbb{T}^{d-1}} q(x_1, x')\,\mathrm{d}x'$ is independent of x_1. Thus $\int_{[0,p]\times\mathbb{T}^{d-1}} q_1(x)\,\mathrm{d}x = 0$ is satisfied in (ii) for either all $p > 0$ or none.

Proof. Let u satisfy (2.5) or (2.6). Then $\tilde{u}(x_1) \equiv \min_{x' \in \mathbb{T}^{d-1}} u(x_1, x')$ cannot have a local minimum by the maximum principle unless it is constant. In either case the limits

$$\lim_{x_1 \to -\infty} \tilde{u}(x_1) = l_1 \qquad \text{and} \qquad \lim_{x_1 \to \infty} \tilde{u}(x_1) = l_2$$

exist. The Harnack inequality for the domains $(y, y+1) \times \mathbb{T}^{d-1} \subset (y-1, y+2) \times \mathbb{T}^{d-1}$ with $y \to \pm\infty$ (see, for example, [9, p. 199]) now shows that uniformly in $x' = (x_2, \dots, x_d)$,

$$\lim_{x_1 \to -\infty} u(x) = l_1 \qquad \text{and} \qquad \lim_{x_1 \to \infty} u(x) = l_2.$$

Parabolic regularity shows that $\lim_{x_1 \to \pm\infty} \nabla u(x) = 0$ uniformly in x'.

If $\bar{q}_1 = 0$, integrate (2.6) over D to get

$$0 = \int_D \operatorname{div}(A\nabla u) = \int_D \operatorname{div}(qu) - \int_D ru = (l_2 - l_1)\bar{q}_1 - \int_D ru = -\int_D ru.$$

Thus $ru \equiv 0$ and (2.6) becomes (2.5). We multiply (2.5) by u and integrate over D to get

$$-\int_D \nabla u \cdot A\nabla u = \int_D u \operatorname{div}(A\nabla u) = \frac{1}{2}\int_D \operatorname{div}(qu^2) = \frac{1}{2}(l_2^2 - l_1^2)\bar{q}_1 = 0. \tag{2.7}$$

Thus u must be constant, proving (ii) and the case $\bar{q}_1 = 0$ in (i).

If $\bar{q}_1 \neq 0$ in (i), integrate (2.5) over D to get

$$0 = \int_D \operatorname{div}(A\nabla u) = \int_D \operatorname{div}(qu) = (l_2 - l_1)\bar{q}_1.$$

Thus $l_2 = l_1$ and (2.7) finishes the proof of this case. □

We will now recover the transition front w as a limit of u_n along a subsequence as $n \to \infty$. Such a limit always exists by parabolic regularity and satisfies $w(0, 0) = \theta$ due to (2.4), but the main issue is to show that it is indeed a transition front for (1.2). The following four lemmas will ensure this fact.

Let us take ζ from the statement of Theorem 1.5 and let θ_j $(j = 0, 1)$ be the smallest positive number such that $f_j(\theta_j) = \zeta\theta_j$. Since $\zeta > f_1'(0)$, we have $0 < \theta_1 \leqq \theta_0 < 1, \theta < \theta_0$ and, for each $x \in D$,

$$\alpha_f(x) \in [\theta_1, \theta_0]. \tag{2.8}$$

We now let

$$c_\zeta \equiv \min_{\lambda > 0} \frac{\zeta + \kappa(\lambda)}{\lambda} \tag{2.9}$$

with $\kappa(\lambda)$ and $\gamma(x; \lambda) > 0$ the principal eigenvalue and eigenfunction for

$$\operatorname{div}(A\nabla\gamma) - (q + 2\lambda Ae_1) \cdot \nabla\gamma + (\lambda^2 e_1^T Ae_1 - \lambda \operatorname{div}(Ae_1) + \lambda q_1)\gamma = \kappa(\lambda)\gamma \tag{2.10}$$

on $\mathbb{T}^d$, normalized by $\sup_{x\in\mathbb{T}^d} \gamma(x;\lambda) = 1$ (for (1.1) we have $\gamma(x;\lambda) \equiv 1$ and $\kappa(\lambda) = \lambda^2$). We note that the minimum is achieved at some $\lambda_\zeta > 0$ because κ is a continuous function of λ and the fraction in (2.9) diverges to ∞ as $\lambda \to 0, \infty$. The latter follows from $\zeta > 0$ and

$$\kappa(\lambda) \geqq \underline{A}\lambda^2,$$

which is obtained after dividing (2.10) by γ and integrating over $\mathbb{T}^d$:

$$\kappa(\lambda) = \int_{\mathbb{T}^d} (\nabla \log \gamma - \lambda e_1) \cdot A(\nabla \log \gamma - \lambda e_1) \geqq \underline{A} \int_{\mathbb{T}^d} |\nabla \log \gamma|^2 + \lambda^2 \geqq \underline{A}\lambda^2.$$

We note that $c_\zeta < c_0$ because $\zeta < \zeta_0$ and ζ_0 was defined so that the right-hand side of (2.9) with ζ_0 in place of ζ equals c_0.

We continue $\gamma(x;\lambda_\zeta)$ periodically on D, and define

$$\Psi(s,x) \equiv \left[\inf_D \gamma(x;\lambda_\zeta)\right]^{-1} \mathrm{e}^{-\lambda_\zeta s} \gamma(x;\lambda_\zeta) > 0$$

(for (1.1) this is $\Psi(s,x) = \mathrm{e}^{-\lambda_\zeta s}$).Notice that $\Psi(0,x) \geqq 1$, and $\psi(t,x) \equiv \Psi(x_1 - c_\zeta t, x)$ is an (exponentially growing as $x_1 \to -\infty$) pulsating front with speed c_ζ for (1.2) with f replaced by ζu. In fact, [4] shows that c_ζ is also the minimal speed of a true pulsating front for (1.2) with any x-independent KPP reaction $\tilde{f}$ satisfying $\tilde{f}'(0) = \zeta$. This is why $c_\zeta < c_0$ will be a crucial component of our argument (and, in fact, any λ such that the fraction in (2.9) is smaller than c_0 would do in place of λ_ζ).

We now let for each $n \in \mathbb{N}$ and $t \geqq \tau_n$,

$$X_n(t) \equiv \sup\{x_1 \,|\, u_n(t,x) \geqq \alpha_f(x) \quad \text{for some } x = (x_1, x')\}$$

and

$$Y_n(t) \equiv \inf\{y \,|\, u_n(t,x) \leqq \Psi(x_1 - y, x) \quad \text{for all } x \in D\}.$$

Both these functions are non-decreasing because u_n is increasing and Ψ decreasing in their respective first variables. Continuity of $u_n(t,x)$, lower semi-continuity of $\alpha_f(x)$, and compactness of $\mathbb{T}^{d-1}$ imply that X_n is continuous from the right.

Lemma 2.4. *Let* $\xi \equiv \sup_{u\in(0,1)} f_1(u)/u \geqq \zeta$ *and* $c_\xi \equiv (\xi + \kappa(\lambda_\zeta))/\lambda_\zeta$. *Then for any* n *and* $t \geqq \tau \geqq \tau_n$ *we have*

$$Y_n(t) - Y_n(\tau) \leqq c_\xi(t - \tau). \tag{2.11}$$

Remark. Taking $\tau = \tau_n$ and $t = 0$, this and (2.4) give $\tau_n \to -\infty$.

Proof. This is immediate from the definition of $Y_n(\tau)$ and the fact that the function $\phi(t,x) \equiv \Psi(x_1 - Y_n(\tau) - c_\xi(t-\tau), x)$ is a supersolution of (1.2) for $t > \tau$ (it solves (1.2) with ξu in place of f). □

This and $Y_n(\tau_n) < \infty$ show that $Y_n(t) < \infty$ (then also $X_n(t) < \infty$ by (2.8)) and that Y_n is continuous because it is non-decreasing. The following is a crucial step in our proof of the existence of fronts.

Lemma 2.5. *There is $C_1 < \infty$ such that for all $n \in \mathbb{N}$ and all $t \geqq \tau_n$,*

$$|Y_n(t) - X_n(t)| \leqq C_1. \tag{2.12}$$

Proof. The bound $X_n(t) - Y_n(t) \leqq C_1$ for a large enough C_1 is obvious from the definitions of X_n, Y_n, and (2.8), so let us show $Y_n(t) - X_n(t) \leqq C_1$. We let C_1 (to be chosen later) be larger than $C_0 \equiv \max\{Y_n(\tau_n) - X_n(\tau_n), 1\}$, the latter independent of n and finite due to v_n being supported in a half-strip, $\tilde{\theta} > \theta_0$, and (2.8). Let us assume that $Y_n(t_1) - X_n(t_1) > C_1$ for some n and $t_1 \geqq \tau_n$, and let $t_0 \equiv \max\{t < t_1 \,|\, Y_n(t) - X_n(t) \leqq C_0\} \geqq \tau_n$. The maximum exists and $Y_n(t_0) - X_n(t_0) = C_0$ because Y_n is continuous and X_n non-decreasing.

Let $\tilde{Y}_n(t) \equiv Y_n(t_0) + c_\zeta(t - t_0)$ and if $X_n(t) \geqq \tilde{Y}_n(t)$ for some $t \in [t_0, t_1]$, let t_2 be the first such time (recall that X_n is continuous from the right). Then $\psi(t, x) \equiv \Psi(x_1 - \tilde{Y}_n(t), x)$ is a solution and $u_n(t, x)$ a subsolution in $\tilde{D} \equiv \{(t, x) \in (t_0, t_2) \times D \,|\, x_1 > \tilde{Y}_n(t)\}$ of (1.2) with ζu in place of f. Moreover, $\psi(t_0, x) \geqq u_n(t_0, x)$ for $x_1 > \tilde{Y}_n(t_0)$ and $\psi(t, x) \geqq u_n(t, x)$ for $t \in [t_0, t_2]$ and $x_1 = \tilde{Y}_n(t)$ (because $\Psi(0, x) \geqq 1$). Thus $\psi(t, x) \geqq u_n(t, x)$ in $\tilde{D}$ by the comparison principle, meaning that $Y_n(t) \leqq \tilde{Y}_n(t)$ for $t \in [t_0, t_2]$. But then $Y_n(t_2) - X_n(t_2) \leqq 0 < C_0$, which is a contradiction with the choice of t_0. Hence, no such time $t_2 \in [t_0, t_1]$ exists and the above argument gives $Y_n(t) \leqq \tilde{Y}_n(t)$ for $t \in [t_0, t_1]$.

That is, Y_n increases with an average speed of at most c_ζ on $[t_0, t_1]$. On the other hand, the next lemma shows that X_n increases with an average speed of at least $c_0 - \varepsilon$ (for any $\varepsilon > 0$) after an initial time delay t_ε (in the sense of (2.13) below with $\tau = t_0$). Since $c_\zeta < c_0$, we can pick $\varepsilon \equiv (c_0 - c_\zeta)/2 > 0$ and obtain

$$\begin{aligned} Y_n(t) - X_n(t) &\leqq Y_n(t_0) + c_\zeta(t - t_0) - X_n(t_0) - (c_0 - \varepsilon)(t - t_0 - t_\varepsilon) \\ &= C_0 + (c_0 - \varepsilon)t_\varepsilon - (c_0 - \varepsilon - c_\zeta)(t - t_0) \end{aligned}$$

for $t \in [t_0, t_1]$. If we now let $C_1 \equiv C_0 + c_0 t_\varepsilon$, then it follows that $Y_n(t_1) - X_n(t_1) < C_1$, a contradiction. Thus $Y_n(t) - X_n(t) \leqq C_1$ for a large enough C_1 and all n and $t \geqq \tau_n$. □

Recall that $c_0^- > 0$ is the speed of the unique left-moving front for (1.2) with reaction f_0.

Lemma 2.6. (i) *For every $\varepsilon > 0$ there is $t_\varepsilon < \infty$ such that if $u : [0, \infty) \times D \to [0, 1]$ solves (1.2) with $u_t \geqq 0$ and $u(0, \tilde{x}) \geqq \alpha_f(\tilde{x})$ for some $\tilde{x} \in D$, then for each $t \geqq 0$ we have*

$$\inf\left\{u(t + t_\varepsilon, x) \,\middle|\, x_1 - \tilde{x}_1 \in [-(c_0^- - \varepsilon)t, (c_0 - \varepsilon)t]\right\} \geqq 1 - \varepsilon.$$

(ii) *There is $L' > 0$ such that for every $\varepsilon > 0$ there is $t'_\varepsilon < \infty$ satisfying the following. If $u : [0, \infty) \times D \to [0, 1]$ solves (1.2) and $\inf\left\{u(0, x) \,\middle|\, |x_1 - \tilde{x}_1| \leqq L'\right\} \geqq (1 + \theta)/2$ for some $\tilde{x} \in D$, then for each $t \geqq 0$ we have*

$$\inf\left\{u(t + t'_\varepsilon, x) \,\middle|\, x_1 - \tilde{x}_1 \in [-(c_0^- - \varepsilon)t, (c_0 - \varepsilon)t]\right\} \geqq 1 - \varepsilon.$$

Remark. 1. Part (i) also shows that for $t \geqq \tau \geqq \tau_n$,

$$X_n(t) - X_n(\tau) \geqq (c_0 - \varepsilon)(t - \tau - t_\varepsilon). \tag{2.13}$$

2. Of course, the constants $t_\varepsilon, t'_\varepsilon$ are independent of f.

Proof. (ii)This is an immediate consequence of Proposition 3.4 in [21], which proves the same result for $f(x,u) = f_0(u)$. In Appendix A we will provide an alternative proof of this result (in fact, with $c_0 - \varepsilon$, $c_0^- - \varepsilon$ replaced by c_0, c_0^-), thus making our proof independent of [21]. □

(i) Consider σ from the beginning of this section and let

$$\alpha'_f(x) \equiv \inf\{u \in (0,1] \mid f(x,u) \geqq (\zeta - \sigma/2)u\}.$$

Then $\alpha'_f(x) \leqq \alpha_f(\tilde{x}) - \sigma\theta_1/4K$ for $x \in B_{\sigma\theta_1/4K}(\tilde{x})$, since by the definition of $\alpha_f(\tilde{x})$ and (2.8),

$$\begin{aligned} f\left(x, \alpha_f(\tilde{x}) - \frac{\sigma\theta_1}{4K}\right) &\geqq \zeta\alpha_f(\tilde{x}) - \frac{\sigma\theta_1}{2} \geqq \left(\zeta - \frac{\sigma}{2}\right)\alpha_f(\tilde{x}) \\ &\geqq \left(\zeta - \frac{\sigma}{2}\right)\left(\alpha_f(\tilde{x}) - \frac{\sigma\theta_1}{4K}\right) \end{aligned}$$

for those values of x. (Also, $\zeta - \sigma/2 > f_1'(0)$ shows that $\alpha'_f(x)$ is the minimum of the set in its definition, as well as uniformly bounded away from 0.) So $u(0,\tilde{x}) \geqq \alpha_f(\tilde{x})$, $u_t \geqq 0$, and parabolic regularity show that there is $\delta > 0$ (independent of $f, \tilde{x}$) such that

$$u(1,x) \geqq \alpha'_f(x) \qquad \text{for all } x \in B_\delta(\tilde{x}). \tag{2.14}$$

Assume that (i) is false. Taking L' from (ii) and using (ii), this means that for each $n \in \mathbb{N}$, there is a solution w_n of (1.2) with some reaction f_n satisfying all the hypotheses (in particular, f_n ζ'-majorizes g for each $\zeta' > \zeta - \sigma$), such that $w_n(0,\tilde{x}^n) \geqq \alpha_{f_n}(\tilde{x}^n)$, $(w_n)_t \geqq 0$, and $w_n(n,x^n) < (1+\theta)/2$ for some x^n with $|x_1^n - \tilde{x}_1^n| \leqq L'$. After possible translation in x_1 it is sufficient to consider $\tilde{x}^n \in \mathbb{T}^d$, so we can assume $\tilde{x}^n \to \tilde{x}$ (otherwise we choose a subsequence). Then (2.14) gives $w_n(1,\tilde{x}) \geqq \alpha'_{f_n}(\tilde{x})$ for large n.

By parabolic regularity, the functions w_n are uniformly bounded in $C^{1,\eta;2,\eta}([1,\infty)\times D)$ for some $\eta > 0$. Thus there is a subsequence (which we again denote w_n) converging in $C^{1;2}_{\text{loc}}([1,\infty)\times D)$ to a solution $\tilde{w} \geqq 0$ of (1.2) on $(1,\infty)\times D$ with some reaction f (locally uniform limit of f_n), satisfying $\tilde{w}_t \geqq 0$, but then $w(x) \equiv \lim_{t\to\infty} \tilde{w}(t,x)$ exists and satisfies

$$q \cdot \nabla w = \operatorname{div}(A\nabla w) + f(x,w).$$

Lemma 2.3 and boundedness of $f(x,u)/u$ show that w is a constant and $f(x,w(x)) \equiv 0$. This constant is then 1 because $w(\tilde{x}) \geqq \liminf_n \alpha'_{f_n}(\tilde{x}) \geqq \alpha'_f(\tilde{x})$ and f $(\zeta - \sigma/2)$-majorizes g (being locally uniform limit of f_n).

But $w_n(n,x^n) < (1+\theta)/2$ and $(w_n)_t \geqq 0$ show for all $t \geqq 1$,

$$\|1 - \tilde{w}(t,\cdot)\|_{L^\infty([\tilde{x}_1 - L', \tilde{x}_1 + L'])} \geqq \frac{1-\theta}{2} > 0.$$

Parabolic regularity again shows that this contradicts $w \equiv 1$, thus finishing the proof. □

For $\varepsilon > 0$ we define

$$Z_{n,\varepsilon}^{-}(t) \equiv \sup\left\{y \,\middle|\, u_n(t,x) \geqq 1-\varepsilon \quad \text{when } x_1 \leqq y\right\},$$
$$Z_{n,\varepsilon}^{+}(t) \equiv \inf\left\{y \,\middle|\, u_n(t,x) \leqq \varepsilon \quad \text{when } x_1 \geqq y\right\}.$$

Clearly both are finite. The following will ensure a bounded width of the constructed transition front.

Lemma 2.7. *For any $\varepsilon \in (0, \min\{c_0, c_0^-\})$ let $\tilde{t}_\varepsilon \equiv t_\varepsilon + C_1(\min\{c_0, c_0^-\} - \varepsilon)^{-1}$ with t_ε from Lemma 2.6*(i) *and C_1 from Lemma 2.5. Then there is $L_\varepsilon < \infty$ such that for all n and $t \geqq \tau_n + \tilde{t}_\varepsilon$,*

$$Z_{n,\varepsilon}^{+}(t) - Z_{n,\varepsilon}^{-}(t) \leqq L_\varepsilon.$$

Proof. Notice that Lemma 2.5 and continuity of Y_n show that if X_n has jumps, they cannot be larger than $2C_1$. This and $(u_n)_t > 0$ mean that for each $t \geqq \tau_n + \tilde{t}_\varepsilon$ and any closed subinterval I of $(-\infty, X_n(t - \tilde{t}_\varepsilon)]$ of length $2C_1$, there is $x_1 \in I$ and $x' \in \mathbb{T}^{d-1}$ with $u_n(t - \tilde{t}_\varepsilon, x_1, x') \geqq \alpha_f(x_1, x')$. Then Lemma 2.6(i) shows that $u_n(t,x) \geqq 1-\varepsilon$ whenever $x_1 \leqq X_n(t-\tilde{t}_\varepsilon) + C_1$. On the other hand, Lemmas 2.4 and 2.5 show that $u_n(t,x) \leqq \varepsilon$ whenever

$$x_1 \geqq X_n(t - \tilde{t}_\varepsilon) + C_1 + c_\xi \tilde{t}_\varepsilon + l_\varepsilon,$$

where l_ε is such that $\Psi(l_\varepsilon, x) \leqq \varepsilon$ for all $x \in D$. Thus $L_\varepsilon \equiv c_\xi \tilde{t}_\varepsilon + l_\varepsilon$ works. □

Having Lemma 2.7, the proof of Theorem 1.5 is now standard. Parabolic regularity shows that the functions u_n are uniformly bounded in $C^{1,\eta;2,\eta}([\tau_n+1,\infty)\times D)$, so we can find a subsequence converging in $C^{1;2}_{\text{loc}}(\mathbb{R}\times D)$ to a function w on $\mathbb{R}\times D$, which then is also a solution of (1.2). Moreover, (2.4) gives $w(0,0) = \theta$, which together with Lemma 2.7, (2.11), and (2.12) ensures (1.3) as well as a bounded width of w. Thus w is a transition front in the sense of Definition 1.1. The claim $w_t > 0$ is immediate from (2.2) and the strong maximum principle for w_t. The exponential decay in Remark 1 follows from Lemma 2.5.

3. Uniqueness of Fronts for Ignition Reactions

We will now prove Theorem 1.5(ii), again assuming that the period of q in x_1 is $p = 1$. Since now $f_1'(0) = 0$, we have automatically $\zeta > f_1'(0)$, and we again assume that f ζ'-majorizes g for each $\zeta' > \zeta - \sigma$. All constants in this section will depend on $q, A, f_0, f_1, \zeta, g, K, \theta''$ *but not on* f. Without loss of generality we only need to consider fronts moving to the right, which we will denote by w.

We can assume $\theta'' \geqq \theta_0$ (otherwise we change θ'' to θ_0) and let

$$\varepsilon_0 \equiv \frac{1}{2}\min\{\theta', 1-\theta''\} \qquad \text{and} \qquad \tilde{\theta} \equiv 1 - \frac{\varepsilon_0}{2},$$

thus fixing v from Lemma 2.1.

We let w be an arbitrary front for (1.2) (in particular, we do not assume $w_t > 0$) and u the solution of (1.2) with the fixed initial condition v. Our strategy is as follows. First we will show that w has to decrease exponentially as $x \to \infty$, in the same way as u_n in the last section. Then we will use this to show that u has to converge to some time shift of w in L^∞ as $t \to \infty$ (thus any two fronts for f must approach each other up to a time shift as $t \to \infty$). Finally, we will show that the rate of this convergence is uniform for all f and all fronts w with uniformly bounded width. This gives a uniform convergence of solutions u_n with initial data $v_n(x) \equiv v(x+n)$ to time shifts of any front w, and the uniqueness of the front follows.

Let w be an arbitrary transition front for (1.2). Consider Ψ from the last section and let

$$\begin{aligned} X_w(t) &\equiv \sup\{x_1 \mid w(t,x) \geqq \alpha_f(x) \quad \text{for some } x=(x_1,x')\}, \\ Y_w(t) &\equiv \inf\{y \mid w(t,x) \leqq \Psi(x_1-y,x) \quad \text{for all } x \in D\}, \\ Z^-_{w,\varepsilon}(t) &\equiv \sup\left\{y \,\middle|\, w(t,x) \geqq 1-\varepsilon \quad \text{when } x_1 \leqq y\right\}, \\ Z^+_{w,\varepsilon}(t) &\equiv \inf\left\{y \,\middle|\, w(t,x) \leqq \varepsilon \quad \text{when } x_1 \geqq y\right\}, \end{aligned}$$

as well as

$$L_{w,\varepsilon} \equiv \sup_{t\in\mathbb{R}}\{Z^+_{w,\varepsilon}(t) - Z^-_{w,\varepsilon}(t)\}, \qquad Z_w(t) \equiv Z^-_{w,\varepsilon_0}(t), \qquad \text{and} \qquad L_w \equiv L_{w,\varepsilon_0}.$$

All of these, except possibly $Y_w(t)$, are finite because w is a transition front, and we have $X_w(t) \in [Z^-_{w,\varepsilon}(t), Z^+_{w,\varepsilon}(t)]$ for $\varepsilon \leqq \varepsilon_0 (\leqq \min\{\theta_1, 1-\theta_0\})$. The next lemma shows $Y_w(t) < \infty$.

Lemma 3.1. *There is $\tilde{C}_2 < \infty$ (depending on L_w if $w_t \not\geqq 0$) such that for all t we have*

$$|Y_w(t) - Z_w(t)| \leqq \tilde{C}_2. \tag{3.1}$$

Proof. Again $Z_w(t) - Y_w(t) \leqq \tilde{C}_2$ (with a uniform bound) is immediate so we are left with proving $Y_w(t) - Z_w(t) \leqq \tilde{C}_2$. We fix any $\varepsilon \in (0, \varepsilon_0)$ and for $t \in \mathbb{R}$ define

$$\begin{aligned} \alpha_{f,\varepsilon}(x) &\equiv \inf\{u \in (\varepsilon, 1] \mid f(x,u) \geqq \zeta(u-\varepsilon)\} \uparrow \alpha_f(x) \quad \text{as } \varepsilon \to 0, \\ X_{w,\varepsilon}(t) &\equiv \sup\{x_1 \mid w(t,x) \geqq \alpha_{f,\varepsilon}(x) \text{ for some } x=(x_1,x')\} \downarrow X_w(t) \quad \text{as } \varepsilon \to 0, \\ Y_{w,\varepsilon}(t) &\equiv \inf\{y \mid w(t,x) \leqq \Psi(x_1-y,x)+\varepsilon \text{ for all } x \in D\} \uparrow Y_w(t) \quad \text{as } \varepsilon \to 0. \end{aligned}$$

The convergences hold because $\theta' > 0$, w and f are continuous, and $\mathbb{T}^{d-1}$ is compact. Note that $\alpha_{f,\varepsilon}(x) \in [\theta', \theta'']$ because $f(x,\cdot)$ decreases on $[\theta'', 1]$. Thus for any t and $\varepsilon \leqq \varepsilon_0$,

$$0 \leqq X_{w,\varepsilon}(t) - Z_w(t) \leqq L_w \tag{3.2}$$

by $\varepsilon_0 < \theta', 1-\theta''$. Hence it is sufficient to show

$$Y_{w,\varepsilon}(t) - X_{w,\varepsilon}(t) \leqq C'_2 \tag{3.3}$$

with C'_2 independent of ε (then use (3.2) and take $\varepsilon \to 0$ to obtain $Y_w(t) - Z_w(t) \leqq \tilde{C}_2 \equiv C'_2 + L_w$). We will prove (3.3) using the argument from Lemma 2.5. We do not have $w_t > 0$ here, but Lemma 2.6(i) will not be needed. Instead, Lemma 2.6(ii) will suffice thanks to (3.2).

Pick any $t_0 \in \mathbb{R}$ and notice that $\Psi(0, x) \geqq 1$ implies $Y_{w,\varepsilon}(t_0) \leqq Z^+_{w,\varepsilon}(t_0)$. We also have $X_{w,\varepsilon}(t_0) \geqq Z_w(t_0) \geqq Z^-_{w,\varepsilon}(t_0)$ by (3.2), so

$$Y_{w,\varepsilon}(t_0) - X_{w,\varepsilon}(t_0) \leqq L_{w,\varepsilon}.$$

As long as $X_{w,\varepsilon}(t) \leqq Y_{w,\varepsilon}(t)$ for $t \geqq t_0$, the argument in Lemma 2.5 shows that $Y_{w,\varepsilon}(t)$ increases with an average speed of at most c_ζ (that is, $Y_{w,\varepsilon}(t) \leqq Y_{w,\varepsilon}(t_0) + c_\zeta(t - t_0)$) because $\psi(t, x) \equiv \Psi(x_1 - Y_{w,\varepsilon}(t_0) - c_\zeta(t - t_0), x) + \varepsilon$ solves (1.2) with $\zeta(u-\varepsilon)$ in place of f. On the other hand, Lemma 2.6(ii) means that $Z_w(t)$ (and thus also $X_{w,\varepsilon}(t)$ due to (3.2)) increases with an average speed of at least $(c_0 + c_\zeta)/2 > c_\zeta$ after an initial time delay $t'_{(c_0-c_\zeta)/2}$ (independent of t_0). Thus the faster moving $X_{w,\varepsilon}(t)$ will catch up with $Y_{w,\varepsilon}(t)$ and we have $X_{w,\varepsilon}(t_1) \geqq Y_{w,\varepsilon}(t_1)$ for some $t_1 \in [t_0, t_0 + t''_\varepsilon]$. Here, t''_ε is independent of t_0 because the speed difference $\geqq (c_0 - c_\zeta)/2$ and initial distance $\leqq L_{w,\varepsilon}$ for all t_0. After time t_1, the argument of Lemma 2.5 shows again that $Y_{w,\varepsilon}(t) - X_{w,\varepsilon}(t)$ must stay uniformly bounded above (independently of ε). Indeed, $Y_{w,\varepsilon}(t)$ is again continuous (using $\xi \equiv \sup_{u\in(\varepsilon_0,1)} f_1(u)/(u - \varepsilon_0) \geqq \zeta$ in Lemma 2.4) and increases with an average speed of at most c_ζ when $X_{w,\varepsilon}(t) \leqq Y_{w,\varepsilon}(t)$. On the other hand, starting from any time $\tau \in \mathbb{R}$, $X_{w,\varepsilon}(t)$ increases with an average speed of at least $(c_0 + c_\zeta)/2$ after an initial time delay $t'_{(c_0-c_\zeta)/2}$ (independent of ε) due to (3.2) and Lemma 2.6(ii). This proves the existence of C'_2 (depending on L_w but independent of ε) such that (3.3) holds for all $t \geqq t_0 + t''_\varepsilon$. Since t_0 has been arbitrary, (3.3) holds for all t and all $\varepsilon \in (0, \varepsilon_0)$, and taking $\varepsilon \to 0$ gives (3.1).

In particular, $Y_w(t)$ is finite and, as in Lemma 2.4, we obtain for $t \geqq \tau$,

$$Y_w(t) - Y_w(\tau) \leqq c_\xi(t - \tau). \tag{3.4}$$

Finally, note that if $w_t \geqq 0$, then Lemma 2.6(i) applies to w. As a result, the proof of (3.3) for small enough ε is identical to that of (2.12) (using that $f(\zeta - \sigma/2)$-majorizes g). Hence we do not need (3.2) to show that $X_{w,\varepsilon}(t)$ increases with average speed larger than $(c_0+c_\zeta)/2$, and C'_2 becomes L_w-independent. Then $\varepsilon \to 0$ gives $Y_w(t) - X_w(t) \leqq C'_2$ and thus

$$|Y_w(t) - X_w(t)| \leqq C'_2. \tag{3.5}$$

This and L_w-independent upper bounds on $X_w(t - t_{\varepsilon_0}) - Z_w(t)$ (from Lemma 2.6(i)) and on $X_w(t) - X_w(t - t_{\varepsilon_0})$ (from (3.5) and (3.4)) show that $\tilde{C}_2$ in (3.1) is also L_w-independent. □

Remark. Notice that this result, together with the definition of Y_w and (3.8) below, shows that $L_{w,\varepsilon}$ depends only on L_w and ε.

Let u be the solution of (1.2) with initial condition $v(x)$ from Lemma 2.1 (with fixed $\tilde{\theta} \equiv 1 - \varepsilon_0$). Let us define

$$\begin{aligned} X_u(t) &\equiv \sup\{x_1 \mid u(t,x) \geqq \alpha_f(x) \quad \text{for some } x = (x_1, x')\}, \\ Y_u(t) &\equiv \inf\{y \mid u(t,x) \leqq \Psi(x_1 - y, x) \quad \text{for all } x \in D\}, \\ Z^-_{u,\varepsilon}(t) &\equiv \sup\left\{y \,\middle|\, u(t,x) \geqq 1 - \varepsilon \quad \text{when } x_1 \leqq y\right\}, \\ Z^+_{u,\varepsilon}(t) &\equiv \inf\left\{y \,\middle|\, u(t,x) \leqq \varepsilon \quad \text{when } x_1 \geqq y\right\}, \end{aligned}$$

as well as

$$L_{u,\varepsilon} \equiv \sup_{t \geqq t'_\varepsilon}\{Z^+_{u,\varepsilon}(t) - Z^-_{u,\varepsilon}(t)\}, \qquad Z_u(t) \equiv Z^-_{u,\varepsilon_0}(t) \qquad \text{and} \qquad L_u \equiv L_{u,\varepsilon_0},$$

with t'_ε from Lemma 2.6(ii). All these are finite as in Section 2 and again

$$Y_u(t) - Y_u(\tau) \leqq c_\xi(t - \tau). \tag{3.6}$$

Lemmas 2.5–2.7 and $u_t > 0$ again show

$$|Y_u(t) - Z_u(t)| \leqq C_2 \tag{3.7}$$

for some f-independent C_2. We also have that for each $\varepsilon > 0$ there are $C_\varepsilon, \tilde{C}_\varepsilon < \infty$ (the latter L_w-dependent if $w_t \ngeqq 0$) such that for any $t \geqq \tau$,

$$Z^-_{w,\varepsilon}(t) \geqq Z_w(\tau) + \frac{c_0 + c_\zeta}{2}(t - \tau) - \tilde{C}_\varepsilon, \tag{3.8}$$

$$Z^-_{u,\varepsilon}(t) \geqq Z_u(\tau) + \frac{c_0 + c_\zeta}{2}(t - \tau) - C_\varepsilon \quad (\tau \geqq t'_\varepsilon \text{ if } \varepsilon \leqq 1 - \tilde{\theta} = \tfrac{\varepsilon_0}{2}). \tag{3.9}$$

Here, (3.9) holds because Lemma 2.6(ii) shows that

$$Z^-_{u,\varepsilon}(t) \geqq Z_u(\tau - t'_\varepsilon) + \frac{c_0 + c_\zeta}{2}(t - \tau)$$

and $Z_u(\tau) - Z_u(\tau - t'_\varepsilon)$ is uniformly bounded in τ due to (3.6) and (3.7). The same argument works for w, but it uses (3.1) and so $\tilde{C}_\varepsilon$ depends on L_w via $\tilde{C}_2$ (unless $w_t \geqq 0$).

Before we can show that u converges to some time shift of w, we need to prove that once u is close to a time shift of w, it will not depart far from it.

Lemma 3.2. *For each $\varepsilon > 0$ there is a $\delta > 0$ (depending also on L_w if $w_t \ngeqq 0$) such that:*

(i) *If $w(t_1, x) \leqq u(t_0, x) + \delta$ for some $t_0 \geqq 1, t_1 \in \mathbb{R}$ and all $x \in D$, then $w(t + t_1 - t_0, x) \leqq u(t, x) + \varepsilon$ for all $t \geqq t_0$ and $x \in D$.*

(ii) *If $w(t_1, x) \geqq u(t_0, x) - \delta$ for some $t_0 \geqq 1, t_1 \in \mathbb{R}$ and all $x \in D$, then $w(t + t_1 - t_0, x) \geqq u(t, x) - \varepsilon$ for all $t \geqq t_0$ and $x \in D$.*

Proof. This is proved via construction of a suitable supersolution and subsolution. To do that, let $\kappa(\lambda_\zeta/2) \geqq 0$ and $\gamma(x; \lambda_\zeta/2) > 0$ be from (2.10) with $\lambda = \lambda_\zeta/2$. If we continue $\gamma(x; \lambda_\zeta/2)$ periodically on D and let

$$\Phi(s, x) \equiv \left[\inf_D \gamma(x; \lambda_\zeta) \inf_D \gamma(x; \lambda_\zeta/2)\right]^{-1} \mathrm{e}^{-\lambda_\zeta s/2} \gamma(x; \lambda_\zeta/2) > 0,$$

then (recall that $\sup_D \gamma(x; \lambda_\zeta) = 1$)

$$\Phi(s, x) \geqq \mathrm{e}^{\lambda_\zeta s/2} \Psi(s, x) \tag{3.10}$$

for $(s, x) \in \mathbb{R} \times D$. We also have from the convexity of $\kappa(\lambda)$ [1, Proposition 5.7(iii)] and $\kappa(0) = 0$ that

$$\kappa\left(\frac{\lambda_\zeta}{2}\right) \leqq \frac{\kappa(\lambda_\zeta)}{2} < \frac{c_\zeta \lambda_\zeta}{2}.$$

Thus, for each $y \in \mathbb{R}$ the function $\phi(t, x) \equiv \Phi(x_1 - y - c_\zeta t, x)$ satisfies

$$\phi_t + q \cdot \nabla\phi - \operatorname{div}(A\nabla\phi) = \left[\frac{c_\zeta \lambda_\zeta}{2} - \kappa\left(\frac{\lambda_\zeta}{2}\right)\right] \phi \geqq 0. \tag{3.11}$$

Next pick $\omega \leqq 1 \leqq \Omega$ so that, for each f as in the statement of Theorem 1.5 (with fixed $q, A, f_0, f_1, \zeta, g, K, \theta''$),

$$0 < \omega \leqq \inf\{u_t(t, x) \,|\, t \geqq 1 \quad \text{and } x_1 \in [Z_u(t), Z^+_{u,\varepsilon_0}(t)]\}, \tag{3.12}$$

$$\infty > \Omega \geqq \sup\{u_t(t, x) \,|\, t \geqq 1 \quad \text{and } x \in D\}. \tag{3.13}$$

The existence of such Ω follows from parabolic regularity and boundedness of u. The existence of ω is guaranteed by $u_t > 0$ and is proved as follows. Assume the contrary, that is, there are sequences f_n and $(t_n, x^n) \in [1, \infty) \times [Z_u(t_n), Z^+_{u,\varepsilon_0}(t_n)] \times \mathbb{T}^{d-1}$ such that $u_t(t_n, x^n) \to 0$.As at the end of Section 2, the functions $u_n(t, x) \equiv u(t + t_n, x + \lfloor x_1^n \rfloor e_1)$ contain a subsequence which converges in $C^{1;2}_{\mathrm{loc}}$ to a solution $\tilde{u}$ of (1.2) on $(-1, \infty) \times D$ (with the same q, A and some Lipschitz reaction $\tilde{f}(x, u) \in [f_0(u), f_1(u)]$ which is a locally uniform limit of a subsequence of $f_n(x + \lfloor x_1^n \rfloor e_1, u)$). But then $\tilde{u}_t(0, \tilde{x}) = 0$ for some $\tilde{x} \in \mathbb{T}^d$, and so $\tilde{u}_t \geqq 0$ and the strong maximum principle for $\tilde{u}_t$ show $\tilde{u}_t \equiv 0$. Since $Z^+_{u,\varepsilon_0}(t) - Z_u(t)$ is uniformly bounded (in t and f) due to (3.7), we have $\limsup_{x_1 \to \infty} \tilde{u}(t, x) \leqq \varepsilon_0$ and $\liminf_{x_1 \to -\infty} \tilde{u}(t, x) \geqq 1 - \varepsilon_0$. This contradicts $\tilde{u}_t \equiv 0$ because

$$Z_{\tilde{u}}(t) \equiv \sup\{y \,|\, \tilde{u}(t, x) \geqq 1 - \varepsilon_0 \quad \text{when } x_1 \leqq y\}$$

again grows with a positive average speed after an initial time delay, by Lemma 2.6(ii).

Finally, assume without loss of generality that $\varepsilon \leqq \varepsilon_0$ and $t_1 = t_0$ (otherwise we shift w in t by $t_1 - t_0$), and increase K so that

$$K \geqq \frac{\lambda_\zeta (c_0 - c_\zeta)}{4}. \tag{3.14}$$

(i) Notice that (3.7) and the hypothesis show that (after possibly increasing C_2 by a constant depending only on Ψ and thus not on f, u, w)

$$Z_u(t_0) \geqq Z_w(t_0) - C_2. \tag{3.15}$$

Let

$$\begin{aligned} C'_\varepsilon &\equiv C_2 + \tilde{C}_2 + C_\varepsilon + \tilde{C}_\varepsilon + 1, \\ b_\varepsilon &\equiv \frac{\varepsilon\lambda_\zeta(c_0 - c_\zeta)\mathrm{e}^{-\lambda_\zeta C'_\varepsilon/2}\omega}{4\Omega K \sup_D \Phi(0,x)}, \\ \beta(t) &\equiv \frac{\varepsilon}{\Omega}\left(1 - \mathrm{e}^{-\lambda_\zeta(c_0-c_\zeta)(t-t_0)/4}\right), \qquad (3.16) \\ \phi_+(t,x) &\equiv b_\varepsilon \Phi(x_1 - Y_w(t_0) - c_\zeta(t-t_0), x). \qquad (3.17) \end{aligned}$$

Then $b_\varepsilon \leqq \varepsilon$ by (3.14), $\omega \leqq \Omega$, and $\Phi(0,x) \geqq 1$. Also, (3.11) holds for ϕ_+, and we define

$$z_+(t,x) \equiv \tilde{u}_+(t,x) + \phi_+(t,x) \equiv u(t+\beta(t),x) + \phi_+(t,x)$$

for $t \geqq t_0$. Our aim is to show

$$z_+(t,x) \geqq w(t,x) \tag{3.18}$$

for all $x \in D$ and $t \geqq t_0$. This estimate might not appear very useful because ϕ_+ is unbounded but it will suffice. The reason is that at $t = t_0$, the function ϕ_+ is large only where both u, w are close to 1 and therefore also to each other. This setup will persist for all $t \geqq t_0$ because ϕ_+ travels with speed c_ζ, which is strictly smaller than the speeds of propagation of u and w. Thus, in fact, ϕ_+ decays near the reaction zones of u, w as t grows.

Let $0 < \delta \leqq b_\varepsilon \inf_D \Phi(2|\ln b_\varepsilon|/\lambda_\zeta, x)$. Then

$$z_+(t_0,x) \geqq w(t_0,x) \tag{3.19}$$

for $x_1 \leqq Y_w(t_0) + 2|\ln b_\varepsilon|/\lambda_\zeta$ by the hypothesis $w(t_0,x) \leqq u(t_0,x) + \delta$ and our choice of δ, and for $x_1 \geqq Y_w(t_0) + 2|\ln b_\varepsilon|/\lambda_\zeta$ by the definition of $Y_w(t_0)$ and by $b_\varepsilon \Phi(2|\ln b_\varepsilon|/\lambda_\zeta + y, x) \geqq \Psi(2|\ln b_\varepsilon|/\lambda_\zeta + y, x)$ when $y \geqq 0$ (see (3.10)).

Moreover, we will prove that z_+ is a supersolution of (1.2) for $t \geqq t_0$, with $f(x,u) = 0$ when $u \geqq 1$. Since (3.11) gives

$$\begin{aligned} &(z_+)_t + q \cdot \nabla z_+ - \operatorname{div}(A\nabla z_+) \\ &\quad \geqq f(x,z_+) + \left[f(x,\tilde{u}_+) - f(x,z_+) + \beta'(t)u_t(t+\beta(t),x)\right], \end{aligned}$$

this will be established if we show that the square bracket is non-negative. This is clearly true for $x_1 \leqq Z_u(t+\beta(t))$, since then $\tilde{u}_+(t,x) \geqq \theta''$ and so $f(x,\tilde{u}_+) \geqq f(x,z_+)$. Next, (3.9), (3.15), (3.1), and $\beta(t) \geqq 0$ for $t \geqq t_0$ give

$$Z^-_{u,\varepsilon}(t+\beta(t)) - Y_w(t_0) - c_\zeta(t-t_0) \geqq \frac{c_0 - c_\zeta}{2}(t-t_0) - C'_\varepsilon, \tag{3.20}$$

so for $x_1 \geqq Z^-_{u,\varepsilon}(t+\beta(t))$ we find using (3.14),

$$\begin{aligned}\phi_+(t,x) &\leqq b_\varepsilon e^{-\lambda_\zeta[(c_0-c_\zeta)(t-t_0)/2-C'_\varepsilon]/2} \sup_D \Phi(0,x) \\ &= \frac{\beta'(t)\omega}{K} \leqq \varepsilon \frac{\lambda_\zeta(c_0-c_\zeta)}{4K} \leqq \varepsilon. \end{aligned} \tag{3.21}$$

This gives for $x_1 \geqq Z^-_{u,\varepsilon}(t+\beta(t))$,

$$|f(x,\tilde{u}_+) - f(x,z_+)| \leqq K\phi_+ \leqq \beta'(t)\omega.$$

Thus the square bracket is again non-negative for $Z_u(t+\beta(t)) \leqq x_1 \leqq Z^+_{u,\varepsilon_0}(t+\beta(t))$ due to (3.12) and $Z_u(t+\beta(t)) \geqq Z^-_{u,\varepsilon}(t+\beta(t))$. The same is true for $x_1 \geqq Z^+_{u,\varepsilon_0}(t+\beta(t))$ because then (3.21) implies $z_+(t,x) \leqq \varepsilon_0 + \varepsilon \leqq 2\varepsilon_0 \leqq \theta'$, yielding $f(x,z_+) = 0$.
Hence z_+ is a supersolution of (1.2) with (3.19), meaning that (3.18) holds. Thus for $x_1 \geqq Z^-_{u,\varepsilon}(t+\beta(t))$,

$$\begin{aligned} u(t,x) - w(t,x) &\geqq z_+(t,x) - w(t,x) - \phi_+(t,x) - \beta(t)\Omega \\ &\geqq 0 - \varepsilon - \varepsilon = -2\varepsilon \end{aligned}$$

using (3.21) and (3.16), and for $x_1 \leqq Z^-_{u,\varepsilon}(t+\beta(t))$,

$$u(t,x) - w(t,x) \geqq \tilde{u}_+(t,x) - \beta(t)\Omega - 1 \geqq \tilde{u}_+(t,x) - \varepsilon - 1 \geqq -2\varepsilon.$$

This proves (i) with 2ε in place of ε. Note that δ also depends on C'_ε, and thus on L_w when $w_t \ngeqq 0$.

(ii) Recall that we assume $\varepsilon \leqq \varepsilon_0$, $t_1 = t_0$, and (3.14), and let us also assume $\varepsilon \leqq c_0^{-1}$. This time (3.1) and the hypothesis give (after increasing $\tilde{C}_2$ by an f, u, w-independent constant)

$$Z_w(t_0) \geqq Z_u(t_0) - \tilde{C}_2. \tag{3.22}$$

Then the proof goes along the same lines as in (i) but using

$$\begin{aligned} \phi_-(t,x) &\equiv b_\varepsilon \Phi(x_1 - Y_u(t_0) - c_\zeta(t-t_0), x), \\ z_-(t,x) &\equiv \tilde{u}_-(t,x) - \phi_-(t,x) \equiv u(t-\beta(t),x) - \phi_-(t,x). \end{aligned} \tag{3.23}$$

This time

$$\begin{aligned} (z_-)_t + q\cdot\nabla z_- - \operatorname{div}(A\nabla z_-) \leqq f(x,z_-) - \big[&f(x,z_-) - f(x,\tilde{u}_-) \\ &+ \beta'(t)u_t(t-\beta(t),x)\big], \end{aligned}$$

with $f(x,u) = 0$ for $u \leqq 0$, and

$$z_-(t_0,x) \leqq w(t_0,x) \tag{3.24}$$

if δ is as in (i). We again need to show that the square bracket is non-negative.

For $x_1 \geqq Z^+_{u,\varepsilon_0}(t-\beta(t))$ we have $\tilde{u}_-(t,x) \leqq \varepsilon_0$, so $f(x,\tilde{u}_-)=0$ and the square bracket is non-negative. For $Z_u(t-\beta(t)) \leqq x_1 \leqq Z^+_{u,\varepsilon_0}(t-\beta(t))$ the same is true because

$$Z^-_{u,\varepsilon}(t-\beta(t)) - Y_u(t_0) - c_\zeta(t-t_0) \geqq \frac{c_0-c_\zeta}{2}(t-t_0) - C'_\varepsilon \tag{3.25}$$

(from (3.9), (3.7), and $\beta(t) \leqq \varepsilon \leqq c_0^{-1} \leqq 2/(c_0+c_\zeta)$) again gives for $x_1 \geqq Z^-_{u,\varepsilon}(t-\beta(t))$,

$$\phi_-(t,x) \leqq b_\varepsilon \mathrm{e}^{-\lambda_\zeta[(c_0-c_\zeta)(t-t_0)/2 - C'_\varepsilon]/2} \sup_D \Phi(0,x) = \frac{\beta'(t)\omega}{K} \leqq \varepsilon \frac{\lambda_\zeta(c_0-c_\zeta)}{4K} \leqq \varepsilon,$$
$$|f(x,\tilde{u}_-) - f(x,z_-)| \leqq K\phi_- \leqq \beta'(t)\omega. \tag{3.26}$$

For $x_1 \leqq Z_u(t-\beta(t))$ we have $\tilde{u}_-(t,x) \geqq 1-\varepsilon_0$, so the bracket is non-negative as long as $\phi_-(t,x) \leqq \varepsilon_0$ (because then $1-\theta'' \leqq z_-(t,x) \leqq \tilde{u}_-(t,x)$). This means that z_- is a subsolution of (1.2), where $\phi_-(t,x) \leqq \varepsilon_0$.

Since (3.8), (3.22), and (3.7) imply

$$Z^-_{w,\varepsilon}(t) - Y_u(t_0) - c_\zeta(t-t_0) \geqq \frac{c_0-c_\zeta}{2}(t-t_0) - C'_\varepsilon,$$

(3.26) also holds for $x_1 \geqq Z^-_{w,\varepsilon}(t)$. Thus z_- is a subsolution of (1.2) on the set where $\phi_-(t,x) \leqq \varepsilon_0$ while on the complement of that set we have $x_1 \leqq Z^-_{w,\varepsilon}(t)$ and so

$$w(t,x) \geqq 1-\varepsilon \geqq 1-\varepsilon_0 \geqq 1-\phi_-(t,x) \geqq z_-(t,x).$$

This together with (3.24) gives $z_-(t,x) \leqq w(t,x)$ for $t \geqq t_0$ and $x \in D$. The rest of the proof is analogous to (i), with $Z^-_{w,\varepsilon}(t)$ in place of $Z^-_{u,\varepsilon}(t+\beta(t))$. □

Lemma 3.3. *If*

$$\tau_w \equiv \inf\{\tau \mid \liminf_{t\to\infty} \inf_{x\in D} [w(t+\tau,x) - u(t,x)] \geqq 0\}, \tag{3.27}$$

then $-\infty < \tau_w < \infty$. *Moreover, the infimum is also a minimum and so*

$$\liminf_{t\to\infty} \inf_{x\in D} [w(t+\tau_w,x) - u(t,x)] \geqq 0. \tag{3.28}$$

Proof. The set in (3.27) is an interval (a,∞) for some $a \leqq \infty$ due to $u_t > 0$. Inequality (3.8) shows $\lim_{t\to\infty} Z_w(t) = \infty$, so $f_0 > 0$ on $[1-\varepsilon_0, 1)$ yields $w(\tau,x) \geqq \tilde{\theta}\chi_{(-\infty,0]}(x_1)$ for some $\tau < \infty$ and all $x \in D$. Thus $w(\tau,x) \geqq u(0,x)$ for all $x \in D$, and the comparison principle shows $w(t+\tau,x) \geqq u(t,x)$ for all $t \geqq 0$ and $x \in D$. Hence $\tau_w < \infty$.

In the opposite direction, notice that (3.9) gives $\lim_{t\to\infty} Z^-_{u,\delta}(t) = \infty$. Hence for each $\delta > 0$, the hypothesis of Lemma 3.2(i) is satisfied with $t_1 = 0$ and a large t_0. Then Lemma 3.2(i) and (3.12) prove for all $t \geqq t_0$ that $\inf_{x\in D}[w(t-t_0,x) - u(t+2\varepsilon/\omega,x)] \leqq -\varepsilon$, provided we choose $\varepsilon > 0$ small enough and then δ, t_0 according to Lemma 3.2(i). Thus $\tau_w > -t_0 - 2\varepsilon/\omega > -\infty$.

Hence τ_w is finite, so the infimum must be a minimum by (3.13). □

Lemma 3.4. *We have*

$$\lim_{t\to\infty} \|w(t+\tau_w, x) - u(t,x)\|_{L^\infty_x} = 0. \tag{3.29}$$

Proof. We can assume without loss of generality that $\tau_w = 0$ (otherwise we shift w in t). Then (3.28) reads

$$\liminf_{t\to\infty} \inf_{x\in D} [w(t,x) - u(t,x)] \geqq 0, \tag{3.30}$$

and we are left with proving

$$\limsup_{t\to\infty} \sup_{x\in D} [w(t,x) - u(t,x)] \leqq 0. \tag{3.31}$$

Assume this is not true. Then by Lemma 3.2(i), there is $\delta_0 > 0$ such that for all $t \geqq 1$,

$$\sup_{x\in D} [w(t,x) - u(t,x)] \geqq \delta_0. \tag{3.32}$$

Moreover, the definition of $\tau_w = 0$ and Lemma 3.2(ii) show that for each $\tau > 0$ there is $\delta_\tau > 0$ such that, for all $t \geqq 1$,

$$\inf_{x\in D} [w(t-\tau,x) - u(t,x)] \leqq -\delta_\tau. \tag{3.33}$$

Finally, we claim that $Z_w(t) - Z_u(t)$ stays bounded as $t \to \infty$. The lower bound follows from (3.30) and $L_u, L_w < \infty$. The upper bound follows from (3.33) for $\tau = 1$, $L_{u,\delta_1}, L_{w,\delta_1} < \infty$, and a uniform upper bound on $Z_w(t) - Z_w(t-1)$ (due to (3.1) and (3.4)).

As before, there is a sequence $t_n \to \infty$ such that the functions $w(t+t_n, x + \lfloor Z_w(t_n)\rfloor e_1)$ and $u(t+t_n, x+\lfloor Z_w(t_n)\rfloor e_1)$ converge in $C^{1;2}_{\text{loc}}(\mathbb{R}\times D)$ to two solutions $\tilde{w}, \tilde{u}$ of (1.2) with some reaction $\tilde{f}$ which has all the properties of f. Moreover, $\tilde{w}, \tilde{u}$ are both transition fronts because of the boundedness of $Z_w(t) - Z_u(t)$ and the properties of w, u (namely, (3.1), (3.4), (3.8), (3.6), (3.7), and (3.9)). We also have $\tilde{u}_t \geqq 0$ as well as

$$\begin{aligned} \tilde{w}(t,x) &\geqq \tilde{u}(t,x) && \text{for all } (t,x) \in \mathbb{R}\times D, && (3.34)\\ \sup_{x\in D}[\tilde{w}(t,x) - \tilde{u}(t,x)] &\geqq \delta_0 && \text{for all } t \in \mathbb{R}, && (3.35)\\ \inf_{x\in D}[\tilde{w}(t-\tau,x) - \tilde{u}(t,x)] &\leqq -\delta_\tau && \text{for all } t\in\mathbb{R}, \tau > 0. \end{aligned}$$

This is thanks to (3.30), (3.32), (3.33), $t_n \to \infty$, and the uniform boundedness in t of $\max\{Z^+_{w,\varepsilon}(t), Z^+_{u,\varepsilon}(t)\} - \min\{Z^-_{w,\varepsilon}(t), Z^-_{u,\varepsilon}(t)\}$ (for any $\varepsilon > 0$).

We define $Z^\pm_{\tilde{w},\varepsilon}(t)$, $Z^\pm_{\tilde{u},\varepsilon}(t)$, $L_{\tilde{w},\varepsilon}$, $L_{\tilde{u},\varepsilon}$ analogously to $Z^\pm_{w,\varepsilon}(t)$, $L_{w,\varepsilon}$. Then $Z^\pm_{\tilde{w},\varepsilon}(t) \geqq Z^\pm_{\tilde{u},\varepsilon}(t)$ for any $\varepsilon > 0$ by (3.34), and $Z^+_{\tilde{w},\varepsilon}(t) - Z^-_{\tilde{u},\varepsilon}(t)$ is uniformly bounded in t, because $Z_w(t) - Z_u(t)$ stays bounded as $t \to \infty$. We let $Z^+(t) \equiv \max\{Z^+_{\tilde{w},\varepsilon_0}(t), Z^+_{\tilde{u},\varepsilon_0}(t+1)\}$ and $Z^-(t) \equiv Z^-_{\tilde{u},\varepsilon_0}(t)$ so that $Z^+(t) - Z^-(t)$ is also uniformly bounded in t. Inequality (3.35) shows that for each t, there is $x^t \in [Z^-_{\tilde{u},\delta_0}(t), Z^+_{\tilde{w},\delta_0}(t)] \times \mathbb{T}^{d-1}$ such that

$$\tilde{w}(t, x^t) - \tilde{u}(t, x^t) \geqq \delta_0.$$

Then (3.34) and the Harnack inequality give the existence of $\delta' > 0$ (which is t-independent by $\sup_{\mathbb{R}}(Z^+(t) - Z^-(t)) < \infty$) such that

$$\tilde{w}(t, x) - \tilde{u}(t, x) \geqq \delta' \quad \text{whenever } x_1 \in [Z^+(t), Z^-(t)],$$

and so (3.13) yields the existence of $\tau \in (0, 1)$ such that

$$\tilde{w}(t, x) \geqq \tilde{u}(t + \tau, x) \quad \text{whenever } x_1 \in [Z^+(t), Z^-(t)].$$

We finish the proof with an argument similar to [13]. We define $z(t, x) \equiv \tilde{w}(t, x) - \tilde{u}(t + \tau, x) \in C^{1,\eta;2,\eta}(\mathbb{R} \times D)$ and notice that z then satisfies

$$z_t + q \cdot \nabla z - \operatorname{div}(A\nabla z) = r(t, x)z$$

with $|r(t, x)| \leqq K$. We also have

$$z(t, x) \geqq 0 \qquad \text{when } x_1 \in [Z^+(t), Z^-(t)], \tag{3.36}$$

$$\inf_{x\in D} z(t, x) \leqq -\delta_\tau \qquad \text{for each } t \in \mathbb{R},$$

$$r(t, x) \leqq 0 \qquad \text{when } x_1 \notin [Z^+(t), Z^-(t)]. \tag{3.37}$$

The last inequality holds because $\tilde{f}$ is non-increasing outside $[\varepsilon_0, 1 - \varepsilon_0]$ and $\tilde{u}_t \geqq 0$. Moreover, $z(t, x) \to 0$ as $\operatorname{dist}(x_1, [Z^+(t), Z^-(t)]) \to \infty$ uniformly in t because $\tilde{w}, \tilde{u}$ are transition fronts and hence have bounded width. Let (t_n, x^{t_n}) be such that

$$\lim_{n\to\infty} z(t_n, x^{t_n}) = \delta'' \equiv \inf_{(t,x)\in\mathbb{R}\times D} z(t, x) < 0.$$

Notice that we then have a uniform bound on $\operatorname{dist}(x_1^{t_n}, [Z^+(t_n), Z^-(t_n)])$. Again, a subsequence of the sequence of functions $z(t + t_n, x + \lfloor x_1^{t_n} \rfloor e_1)$ converges in $C^{1;2}_{\mathrm{loc}}(\mathbb{R} \times D)$ to a function $\tilde{z}$ with

$$\tilde{z}(0, \tilde{x}) = \delta'' = \inf_{(t,x)\in\mathbb{R}\times D} \tilde{z}(t, x) < 0$$

for some $\tilde{x} \in \mathbb{T}^{d-1}$, and satisfying (due to (3.36) and (3.37))

$$\tilde{z}_t + q \cdot \nabla \tilde{z} - \operatorname{div}(A\nabla \tilde{z}) \geqq 0 \quad \text{where } \tilde{z}(t, x) \leqq 0.$$

The strong maximum principle then forces $\tilde{z}(t, x) = \delta'' < 0$ for $t < 0$, a contradiction with uniform boundedness of $\operatorname{dist}(x_1^{t_n}, [Z^+(t_n), Z^-(t_n)])$ and (3.36). This proves (3.31) and we are done. □

Our final ingredient is the claim that the convergence in (3.29) is uniform in f and w.

Lemma 3.5. *For any $C > 0$ and fixed $q, A, f_0, f_1, \zeta, g, K, \theta''$, the convergence in Lemma 3.4 is uniform in all f as above, and in all fronts w with $L_w \leqq C$.*

Remark. We will see at the end of this section that the hypothesis $L_w \leqq C$ is satisfied for some $C < \infty$ and all f, w. Thus the convergence is uniform in all f, w, as in Theorem 1.5(ii).

Proof. Assume the contrary. Thus for some $C, \varepsilon > 0$ and each $n \in \mathbb{N}$, there are w_n, u_n as in Lemma 3.4—solving (1.2) with reactions f_n (which satisfy the hypotheses of Theorem 1.5(ii) with uniform $q, A, f_0, f_1, \zeta, g, K, \theta''$) and with $\tau_{w_n} = 0$ after a translation of w_n in t—such that $L_{w_n} \leqq C$ and for some $t_n \to \infty$

$$\|w_n(t_n, x) - u_n(t_n, x)\|_{L_x^\infty} > \varepsilon. \tag{3.38}$$

We will obtain a contradiction by finding a subsequence of $\{(f_n, w_n, u_n)\}_n$ which converges locally uniformly to (f, w, u) such that $L_w \leqq C$, and (3.29) is violated.

By parabolic regularity, for some $\eta > 0$, the w_n are uniformly bounded in $C^{1,\eta;2,\eta}(\mathbb{R} \times D)$ and the u_n in $C^{1,\eta;2,\eta}([a, \infty) \times D)$ (for any $a > 0$). We can thus choose a subsequence (which we again index by n) such that $f_n \to f$ in $C_{\text{loc}}(D)$, $w_n \to w$ in $C^{1;2}_{\text{loc}}(\mathbb{R} \times D)$ and $u_n \to u$ in $C^{1;2}_{\text{loc}}((0, \infty) \times D)$. Therefore w, u solve (1.2) on $\mathbb{R} \times D$ and $(0, \infty) \times D$, respectively. Also, $u(0, x) = u_n(0, x) = v(x)$ holds because the f_n are uniformly bounded and v is continuous, so $\|u_n(t, x) - v(x)\|_{L_x^\infty} \to 0$ as $t \downarrow 0$, uniformly in n. We note that the limiting reaction f again satisfies all the hypotheses, including ζ'-majorization of g for $\zeta' > \zeta - \sigma$.

Next we show that w is a front. The $Z_{w_n}(0)$ must be uniformly bounded above, because otherwise $w_n(-1, x) \geqq v(x)$ for large n and all $x \in D$, meaning that $\tau_{w_n} \leqq -1$, a contradiction. Similarly, the $Y_{w_n}(0)$ are uniformly bounded below because of (3.4) for w_n and the fact that $Y_{w_n}(t'_\delta)$ are uniformly bounded below for each $\delta > 0$ (by the argument in the second part of the proof of Lemma 3.3 and $\tau_{w_n} = 0$). Then Lemma 3.1 and $L_{w_n} \leqq C$ show that $Z_{w_n}(0)$ and $Y_{w_n}(0)$ are uniformly bounded below and above, as are the average growth rates of $Z_{w_n}(t)$ and $Y_{w_n}(t)$ (due to (3.4), (3.8), and (3.1)). It follows from (3.8) and $L_{w_n} \leqq C$ that the locally uniform limit w is, indeed, a transition front with $L_w \leqq C$.

Thus Lemma 3.4 applies to this w and we have (3.29) for some τ_w. So for each $\delta > 0$ there is $s_\delta \geqq t'_\delta$ such that $\|w(s_\delta + \tau_w, x) - u(s_\delta, x)\|_{L_x^\infty} < \delta$. Hence for each $M, \delta > 0$ and all large enough n we have $\|w_n(s_\delta + \tau_w, x) - u_n(s_\delta, x)\|_{L_x^\infty(-M,M)} < 2\delta$. Since $Z^-_{w_n,\delta}(s_\delta + \tau_w)$, $Y_{w_n}(s_\delta + \tau_w)$, $Y_{u_n}(s_\delta)$ are uniformly bounded in n by the argument above and $s_\delta \geqq t'_\delta$, it follows that, in fact, $\|w_n(s_\delta + \tau_w, x) - u_n(s_\delta, x)\|_{L_x^\infty} < 2\delta$ for all large enough n. Then $\delta > 0$ being arbitrary and Lemma 3.2, together, show that for each $\varepsilon' > 0$ there are $N_{\varepsilon'}, r_{\varepsilon'}$ such that for all $n > N_{\varepsilon'}$ and $t > r_{\varepsilon'}$,

$$\|w_n(t + \tau_w, x) - u_n(t, x)\|_{L_x^\infty} < \varepsilon'. \tag{3.39}$$

Now (3.12) and $\varepsilon_0 \leqq \frac{1}{4}$ show for these n, t,

$$\|w_n(t + \tau_w, x) - u_n(t + \tau_w, x)\|_{L_x^\infty} \geqq \min\left\{\tau_w \omega, \frac{1}{2}\right\} - \varepsilon'.$$

If $\tau_w \neq 0$, then this contradicts $\tau_{w_n} = 0$ and (3.29) when we take ε' small enough. Therefore $\tau_w = 0$. But then after taking $\varepsilon' = \varepsilon$ and $n > N_\varepsilon$ such that $t_n > r_\varepsilon$, we obtain a contradiction between (3.38) and (3.39) with $t = t_n$. □

We can now proceed to prove Theorem 1.5(ii). Let w be a transition front for (1.2) and translate it in t so that $w(0, 0) = \theta$. This is possible by (3.8), although such translation may not be unique. Define $f_n(x, u) \equiv f(x - ne_1, u)$ and let u_n solve (1.2) with reaction f_n and $u_n(0, x) \equiv v(x)$. Pick t_n so that $u_n(t_n, ne_1) = \theta$ and consider the front $w_n(t, x) \equiv w(t - t_n, x - ne_1)$ for (1.2) with f_n.

We have $t_n \to \infty$ by (3.4) as well as $L_{w_n} = L_w$ for each n. Then Lemma 3.5 shows that for any $T \in \mathbb{R}$, uniformly in $t \geqq T$,

$$\begin{aligned} &\|w(t + \tau_{w_n}, x) - u_n(t + t_n, x + ne_1)\|_{L^\infty_x} \\ &\quad = \|w_n(t + t_n + \tau_{w_n}, x + ne_1) - u_n(t + t_n, x + ne_1)\|_{L^\infty_x} \to 0 \qquad (3.40) \end{aligned}$$

as $n \to \infty$. This and $u_n(t_n, ne_1) = \theta$ show $w(\tau_{w_n}, 0) \to \theta$ as $n \to \infty$. Then τ_{w_n} must be bounded in n by $w(0, 0) = \theta$, (3.8), and $L_w < \infty$. So there is a subsequence converging to some $\tau \in \mathbb{R}$. It follows from (3.13) and (3.40) that for each $t \in \mathbb{R}$,

$$\|w(t + \tau, x) - u_n(t + t_n, x + ne_1)\|_{L^\infty_x} \to 0$$

along this subsequence.

If now w_1, w_2 are two fronts for the same f, then we can choose the same subsequence for both, which gives the existence of τ_1, τ_2 such that $w_1(t + \tau_1, x) = w_2(t + \tau_2, x)$ for each t, x. Thus the two fronts are time shifts of each other, that is, each front for this f is a time shift of the front w constructed at the end of Section 2.

Since this front satisfies $w_t > 0$, the constants in this section do not depend on L_w. In particular, $\tilde{C}_2$ in (3.1) does not, which in turn gives a uniform in f bound on L_w. This proves the remark after Lemma 3.5.

4. Stability of Fronts for Ignition Reactions

We will now prove Theorem 1.5(iii). We make the same assumptions as at the beginning of the last section, and all constants will again depend on $q, A, f_0, f_1, \zeta, g, K, \theta''$ as well as on Y, μ, ν, *but not on* f. Let us denote by $w_\pm$ the unique right- and left-moving fronts from the last section.

Without loss of generality, we will assume $\mu \leqq \lambda_\zeta/2$ and (3.14). We can also assume $a = 0$, after possibly shifting the domain. It will be notationally convenient to let u be the solution of (1.2) with initial condition v (with $\tilde{\theta} \equiv 1 - \varepsilon_0/2$ as in the last section), and w the solution of (1.2) in question, with initial condition $w(0, x) \equiv w_0(x)$.

Let us prove claim (a) in Theorem 1.5(iii) (that is, Definition 1.4(a) with f, w_0-uniform constants). We have

$$w_0(x) \leqq \mathrm{e}^{-\mu(x_1 - Y)}, \qquad (4.1)$$

$$w_0(x) \geqq \tilde{\theta}\chi_{(-\infty,0)}(x_1), \qquad (4.2)$$

where we have also assumed that $\nu = \tilde{\theta} - \theta$. We can do this without loss of generality due to the following. Lemma 2.6(ii), in fact, holds with $\theta + \nu$ in place of $(1+\theta)/2$ for any $\nu > 0$ (see [21] or Lemma 5.1) but with ν-dependent t'_ε and L'. So if (4.2) holds with $\theta + \nu$ in place of $\tilde{\theta}$, then $w(\tau', x) \geqq \tilde{\theta}\chi_{(-\infty,0)}(x_1)$ for $\tau' \equiv t'_{1-\tilde{\theta}} + L'/(c_0 - \varepsilon_0)$. We also have that if

$$Y_w(t) \equiv \inf\{y \mid w(t,x) \leqq \Phi(x_1 - y, x) \quad \text{for all } x \in D\} \tag{4.3}$$

with

$$\Phi(s,x) \equiv 2\left[\inf_D \gamma(x;\lambda_\zeta) \inf_D \gamma(x;\mu)\right]^{-1} \mathrm{e}^{-\mu s}\gamma(x;\mu) > 0,$$

corresponding to $\lambda = \mu$ in (2.10), then (3.4) holds with $c_\xi = (\xi + \kappa(\mu))/\mu$ (and ξ from Lemma 2.4). In particular, $Y_w(t)$ is again finite because (4.1) and

$$\Phi(s,x) \geqq 2\mathrm{e}^{-\mu s} \tag{4.4}$$

imply $Y_w(0) \leqq Y$. Thus $w(\tau', x) \leqq \mathrm{e}^{-\mu(x_1 - Y - Y')}$ holds with the f, w-independent constant $Y' \equiv c_\xi \tau' + \mu^{-1} \sup_D \log \Phi(0,x)$. So (4.1) and (4.2) are satisfied for $w(\tau', x)$ in place of $w_0(x)$ and $Y + Y'$ in place of Y, with τ', Y' independent of f, w.

We define $X_w, Z^\pm_{w,\varepsilon}, L_{w,\varepsilon}, Z_w, L_w$ as before and $Y_w(t)$ by (4.3). The proof of claim (a) in Theorem 1.5(iii) will be essentially identical to the argument in Lemmas 3.2–3.5, after we have established the basic properties (3.1), (3.4), (3.8) for w. We will then show uniform convergence of u to a time shift of w. Since u also uniformly converges to a time shift of w_+, claim (a) in Theorem 1.5(iii) will thus be proved. The proof of claim (b) at the end of this section will be a slight variation on the same theme.

Lemma 4.1. *The estimates* (3.1)*,* (3.4)*, and* (3.8) *hold with f, u, w-independent constants.*

Proof. We have already proved (3.4) and, obviously, we also have

$$Y_w(t) \geqq Z_w(t) - \tilde{C}_2 \tag{4.5}$$

for some f, w-independent $\tilde{C}_2$.

Next, we note that (4.2) gives $w_0(x) \geqq v(x)$, thus $w(t,x) \geqq u(t,x)$, and so

$$Z^\pm_{w,\varepsilon}(t) \geqq Z^\pm_{u,\varepsilon}(t) \qquad \text{and} \qquad Z_w(t) \geqq Z_u(t) \tag{4.6}$$

for all $t \geqq 0$. It is therefore sufficient to show that there is an f, u, w-independent t_1 such that

$$Y_w(t) \leqq Y_u(t + t_1) \tag{4.7}$$

for all $t \geqq 0$, because then (3.1) and (3.8) (with new f, u, w-independent constants) follow from (4.5), (4.6) and (3.6), (3.7), (3.9).

We prove this by bounding w above by a uniformly bounded time shift of u plus a small perturbation. The argument is very similar to that in the proof of Lemma 3.2(i), with μ in place of $\lambda_\zeta/2$. We let

$$b_0 \equiv \frac{\varepsilon_0 \mu (c_0 - c_\zeta)\omega}{2\Omega K \sup_D \Phi(0,x)} \qquad (\leqq \varepsilon_0 \text{ due to (3.14) and } \mu \leqq \lambda_\zeta/2),$$
$$Y_0 \equiv Y + \frac{|\log b_0|}{\mu},$$

and choose t_0 so that

$$Z_u(t_0) \geqq Y_0 + C_{\varepsilon_0} + C_2. \tag{4.8}$$

This can be done uniformly in f, u, w thanks to (3.9) and $Z_u(0)$ depending only on v. Let

$$\beta(t) \equiv \frac{\varepsilon_0}{\Omega}\left(1 - \mathrm{e}^{-\mu(c_0 - c_\zeta)(t-t_0)/2}\right),$$
$$\phi(t,x) \equiv b_0 \Phi(x_1 - Y_0 - c_\zeta(t-t_0), x),$$
$$z(t,x) \equiv \tilde{u}(t,x) + \phi(t,x) \equiv u(t+\beta(t),x) + \phi(t,x),$$

so that (4.4) and (4.1) give

$$z(t_0,x) \geqq \phi(t_0,x) \geqq w_0(x) \tag{4.9}$$

for all $x \in D$. Convexity of $\kappa(\lambda)$, $\kappa(0)=0$, and $\mu < \lambda_\zeta$ yield

$$\kappa(\mu) \leqq \frac{\mu}{\lambda_\zeta}\kappa(\lambda_\zeta) < c_\zeta \mu,$$

so that, again, $\phi_t + q\cdot\nabla\phi - \operatorname{div}(A\nabla\phi) \geqq 0$.

It then follows, as in the proof of Lemma (3.2)(i), that z is a supersolution of (1.2). The argument is identical, with $\varepsilon = \varepsilon_0$, μ in place of $\lambda_\zeta/2$, (3.20) replaced by

$$Z_u(t+\beta(t)) - Y_0 - c_\zeta(t-t_0) \geqq \frac{c_0 - c_\zeta}{2}(t-t_0) + C_2 \tag{4.10}$$

(which is immediate from (4.8) and (3.9)), and (3.21) by

$$\phi(t,x) \leqq b_0 \mathrm{e}^{-\mu[(c_0-c_\zeta)(t-t_0)/2 + C_2]} \sup_D \Phi(0,x)$$
$$= \frac{\beta'(t)\omega}{K}\mathrm{e}^{-\mu C_2} \leqq \varepsilon_0 \frac{\mu(c_0-c_\zeta)}{2K} \leqq \varepsilon_0, \tag{4.11}$$

for $x_1 \geqq Z_u(t+\beta(t))$.

Thus (4.9) yields $z(t+t_0,x) \geqq w(t,x)$ for all $t \geqq 0$ and $x \in D$.

However, $\Psi(s,x) \leqq \frac{1}{2}\Phi(s,x)$ for $s \geqq 0$ and $b_0 \leqq \frac{1}{2}$ give

$$w(t,x) \leqq \tilde{u}(t+t_0,x) + \phi(t+t_0,x) \leqq \frac{1}{2}\Phi\left(x_1 - Y_u\left(t+t_0+\beta(t)\right), x\right)$$
$$+\frac{1}{2}\Phi\left(x_1 - Y_0 - c_\zeta(t-t_0), x\right)$$

for $x_1 \geqq Y_u(t+t_0+\beta(t))$. Since $Y_u(t+t_0+\beta(t)) \geqq Y_0+c_\zeta(t-t_0)$ (by (4.10) and (3.7)) and $\Phi(s,x) \geqq 1$ for $s \leqq 0$, we obtain $w(t,x) \leqq \Phi(x_1 - Y_u(t+t_0+\beta(t)), x)$ for $t \geqq 0$ (and all x_1). This and $\beta(t) \leqq \varepsilon_0/\Omega$ now yield (4.7) with $t_1 \equiv t_0 + \varepsilon_0/\Omega$. □

Lemma 4.2. *Lemmas 3.2, 3.3, 3.4 hold for u, w as above, and the convergence in Lemma 3.4 is uniform in all f, u, w as above (with fixed q, A, f_0, f_1, ζ, g, K, θ'', Y, μ, ν).*

Proof. This is virtually identical to the proofs of Lemmas 3.2–3.5, with the following adjustments. In Lemma 3.2 one considers only $t_1 \geqq 1$ and λ_ζ is replaced by μ in the proof. In the proof of Lemma 3.4 we cannot assume $\tau_w = 0$, but this causes no change to the argument. This is also the case in the proof of Lemma 3.5, but uniformity in f of the argument in Lemma 3.3 shows that the τ_w are uniformly bounded (for fixed Y, μ, ν). Thus, in the proof of Lemma 3.5 we can simply consider a sequence w_n with convergent τ_{w_n} and the proof is unchanged. The condition $L_w \leqq C$ can be omitted because it is automatic from Lemma 4.1. Finally, the limit w obtained in the proof of Lemma 3.5 along a subsequence of $\{w_n\}_n$ is not a front, but satisfies (4.1) and (4.2), and thus the just-proved Lemma 3.4 holds for w, too. □

This and f-uniform convergence of u to a time shift of w_+ prove f, w-uniform convergence of w to a time shift of w_+ in L^∞_x, claim (a) in Theorem 1.5(iii).

The proof of claim (b) is virtually identical, with separate treatments of the two reaction zones of w (one on either side of $x_1 = a$) moving right and left. This requires the adjustment of the definition of $Z^-_{w,\varepsilon}(t)$ (for the right-moving reaction zone) to

$$Z^-_{w,\varepsilon}(t) \equiv \sup\{y \geqq a \,\big|\, w(t,x) \geqq 1-\varepsilon \quad \text{when } x_1 \in [a,y]\},$$

and a restriction of all the estimates to $x_1 \geqq a$. The rest of the proof is unchanged because our subsolution z_- in Lemma 3.2(ii) is, in fact, negative for $t \geqq t_0$ and $x_1 < a$, as long as t_0 is large enough (depending on ε) so that $\phi_-(t,x) \geqq 1$ for these t, x_1 (see (3.23)). Thus we still obtain $z_- \leqq w$ and ultimately prove f, w-uniform convergence in $L^\infty_x(D^+_a)$ of w to a time shift of u (and hence of w_+), with $D^+_a \equiv [a,\infty) \times \mathbb{T}^{d-1}$. A similar treatment of the left-moving reaction zone of w gives a f, w-uniform convergence in $L^\infty_x(D^-_a)$ of w to a time shift of w_-, with $D^-_a \equiv (-\infty, a] \times \mathbb{T}^{d-1}$. Since $w_\pm$ converge f-uniformly to 1 in $L^\infty_x(D^\mp_a)$ by Lemma 2.6, the claim follows.

Appendix: The Spreading Lemma and Transition Fronts for Homogeneous Ignition Reactions

We will now show how one can use our arguments to obtain a proof of Lemma 2.6(ii), which is from [21], without the use of [21]. In fact, we will prove a slightly stronger result. In the course of this proof we will also prove Theorem 1.5 for $f(x,u) = f_0(u)$, showing that c_0 in that theorem is well defined. Recall that

$c_0, c_0^- > 0$ are the speeds of the unique right- and left-moving fronts for (5.1) below.

Lemma 5.1. *Let* q, A, f_0 *be as in* (H1), (H2). *Then for each* $\nu > 0$ *there is* $L_\nu > 0$ *such that, for every* $\varepsilon > 0$, *there is* $t'_\varepsilon < \infty$ *satisfying the following. If* $u : [0, \infty) \times D \to [0, 1]$ *solves*

$$u_t + q(x) \cdot \nabla u = \operatorname{div}(A(x)\nabla u) + f_0(u), \tag{5.1}$$

and $\tilde{x} \in D$ *is such that* $\inf\left\{u(0, x) \,\middle|\, |x_1 - \tilde{x}_1| \leqq L_\nu\right\} \geqq \theta + \nu$, *then for each* $t \geqq 0$ *we have*

$$\inf\left\{u(t + t'_\varepsilon, x) \,\middle|\, x_1 - \tilde{x}_1 \in [-c_0^- t, c_0 t]\right\} \geqq 1 - \varepsilon. \tag{5.2}$$

Remark. The comparison principle then gives the same result for solutions of (1.2) with $f \geqq f_0$. This also gives Lemma 2.6(i) with c_0, c_0^- in place of $c_0 - \varepsilon, c_0^- - \varepsilon$. Thus $(c_0 + c_\zeta)/2$ can be replaced by c_0 in (3.8) and (3.9), proving the first claim in Remark 1 after Theorem 1.5.

Proof. Let $f(x, u) \equiv f_0(u)$ and let us re-prove Theorem 1.5 without relying on Lemma 2.6, as originally stated. We first let $\tilde{\theta} = \theta + \nu$ and construct a compactly supported initial datum $v(x) \leqq \tilde{\theta}$ that satisfies (2.1). This is done as in Lemma 2.1, but cutting off v on both sides. We let $\tilde{v}_+$ be as $\tilde{v}$ in that lemma and let $\tilde{v}_-$ be a C^2 solution of $-\operatorname{div}(A\nabla\tilde{v}_-) + q \cdot \nabla\tilde{v}_- = -q_1 + \operatorname{div}(Ae_1)$ on $\mathbb{T}^d$, periodically continued to D. Let $v_{\pm,\varepsilon(x)} \equiv \varepsilon(\tilde{v}_\pm(x) \mp x_1)$ and $v(x) \equiv \rho(\min\{v_{+,\varepsilon}(x), v_{-,\varepsilon}(x) + 4\})$ for $\varepsilon > 0$. So v is compactly supported and, if ε is small, then there is $a \in \mathbb{R}$ and $l > 0$ such that

$$v(x) = \begin{cases} \tilde{\theta} & x_1 \in [a - l, a + l], \\ \rho(v_{+,\varepsilon}(x)) & x_1 \geqq a, \\ \rho(v_{-,\varepsilon}(x) + 4) & x_1 \leqq a \end{cases}$$

(a is such that $v_{\pm,\varepsilon}(x) \approx \pm 2$ when $x_1 \approx a$). We also have $-\operatorname{div}(A\nabla v_{\pm,\varepsilon}) + q \cdot \nabla v_{\pm,\varepsilon} = 0$, so for some distribution $T \geqq 0$ supported on the set $D_\varepsilon \equiv \{x \in D \mid v_{+,\varepsilon}(x) = 0 \text{ or } v_{-,\varepsilon}(x) = 0\}$ and with $\tilde{v}$ standing for $\tilde{v}_\pm$ when $\pm(x_1 - a) \geqq 0$,

$$\begin{aligned} -\operatorname{div}(A\nabla v) + q \cdot \nabla v \; &= -\varepsilon^2 \chi_{D \setminus D_\varepsilon} \rho''(\min\{v_{+,\varepsilon}(x), v_{-,\varepsilon}(x) + 4\}) \\ &\qquad (\nabla\tilde{v} - e_1) \cdot A(\nabla\tilde{v} - e_1) - T, \end{aligned}$$

which is again less than or equal to $f_0(v)$ if ε is small enough.

Next, let $L_\nu < \infty$ be such that v is supported in $[-L_\nu, L_\nu] \times \mathbb{T}^{d-1}$. Let u solve (5.1) with $u(0, x) = v(x)$, so that Lemma 2.2 holds for u and $t > 0$. Thus, there is $\tau_\nu < \infty$ such that

$$u(\tau_\nu, x) \geqq \tilde{\theta} \chi_{[-L_\nu - p, L_\nu + p]}(x_1) \geqq \max\{v(x - pe_1), v(x), v(x + pe_1)\}.$$

This, (2.3) for u, and the comparison principle then prove Lemma 5.1 with $c'_0 \equiv p/\tau_\nu > 0$ in place of c_0, c_0^-. This, in turn, proves Lemma 2.6 with c'_0 in place of $c_0 - \varepsilon, c_0^- - \varepsilon$.

We now notice that f_0 being independent of x and positive on $(\theta, 1)$ shows that f_0 ζ-majorizes some (ζ-dependent) g as in Theorem 1.6 for each $\zeta > 0$. We therefore choose $\zeta > 0$ small enough (and a corresponding g) so that $c_\zeta < c_0'$ in (2.9) and f_0 ζ'-majorizes g for all $\zeta' > \zeta - \sigma$ (with $\sigma \equiv \zeta/2$). This can be done because $\kappa(0) = \kappa'(0) = 0$ [1, Proposition 5.7(iii)]. Now we can perform the rest of the proof of Theorem 1.5 for (5.1) using c_0' in place of c_0, c_0^- (and, in particular, $c_\zeta < c_0'$ in place of $c_\zeta < c_0$). This yields the existence of a unique right-moving transition front for (5.1) with some speed $c_0 > 0$ and, similarly, a left-moving transition front with speed $c_0^- > 0$, as well as convergence of general solutions to them as in Theorem 1.5(iii). So the solution u, above, converges in L_x^∞ to some time shifts of these fronts in the sense of (1.6). This and the comparison principle now prove (5.2). □

References

1. Berestycki, H., Hamel, F.: Front propagation in periodic excitable media. *Comm. Pure and Appl. Math.* **55**, 949–1032 (2002)
2. Berestycki, H., Hamel, F.: Generalized transition waves and their properties. *Comm. Pure Appl. Math.* **65**, 592–648 (2012)
3. Berestycki, H., Hamel, F., Matano, M.: Bistable travelling waves around an obstacle. *Comm. Pure Appl. Math.* **62**, 729–788 (2009)
4. Berestycki, H., Hamel, F., Nadirashvili, N.: The speed of propagation for KPP type problems, I—Periodic framework, J. Eur. Math. Soc. **7**, 173–213 (2005)
5. Berestycki, H., Larrouturou, B., Lions, P.L. Multi-dimensional travelling wave solutions of a flame propagation model. *Arch. Rational Mech. Anal.* **111**, 33–49 (1990)
6. Berestycki, H., Nirenberg, L. : Traveling fronts in cylinders. *Ann. Inst. H. Poincar Anal. Non Linaire* **9**, 497–572 (1992)
7. Chapuisat, G., Grenier, E. : Existence and nonexistence of traveling wave solutions for a bistable reaction-diffusion equation in an infinite cylinder whose diameter is suddenly increased. *Comm. Partial Differential Equations* **30**, 1805–1816 (2005)
8. Fisher, R.: The wave of advance of advantageous genes. *Ann. Eugenics* **7**, 355–369 (1937)
9. Gilbarg, D., Trudinger, N.: *Elliptic Partial Differential Equations Equations of Second Order*. Springer, Berlin, 1998
10. Kingman, J.F.C.:The ergodic theory of subadditive stochastic processes. *J. R. Stat. Soc. Sr. B* **30**, 499–510 (1968)
11. Kolmogorov, A.N., Petrovskii, I.G., Piskunov, N.S.: Étude de l'équation de la chaleur de matière et son application à un problème biologique. *Bull. Moskov. Gos. Univ. Mat. Mekh.* **1**, 1–25 (1937)
12. Liggett, T.M.: An improved subadditive ergodic theorem. *Ann. Probab.* **13**, 1279–1285 (1985)
13. Mellet, A., Nolen, J., Roquejoffre, J.M., Ryzhik, L.: Stability of generalized transition fronts. *Commun. PDE* **34**, 521–552 (2009)
14. Mellet, A., Roquejoffre, J.M., Sire, Y.: Generalized fronts for one-dimensional reaction-diffusion equations. *Discrete Contin. Dyn. Syst.* **26**, 303–312 (2010)
15. Nolen, J., Ryzhik, L.: Traveling waves in a one-dimensional heterogeneous medium. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**, 1021–1047 (2009)
16. Nolen, J., Roquejoffre, J.M., Ryzhik, L., Zlatoš, A.: Existence and non-existence of Fisher-KPP transition fronts. *Arch. Ration. Mech. Anal.* **203**, 217–246 (2012)
17. Ryzhik, L., Zlatoš, A.: KPP pulsating front speed-up by flows. *Commun. Math. Sci.* **5**, 575–593 (2007)

18. Shen, W.: Traveling waves in diffusive random media. *J. Dyn. Diff. Eqs.* **16**, 1011–1060 (2004)
19. Vakulenko, S., Volpert, V.: Generalized travelling waves for perturbed monotone reaction-diffusion systems. *Nonlinear Anal.* **46**, 757–776 (2001)
20. Xin, J.: Existence of planar flame fronts in convective-diffusive media. *Arch. Rational Mech. Anal.* **121**, 205–233 (1992)
21. Xin, J.: Existence and nonexistence of travelling waves and reaction-diffusion front propagation in periodic media. *J. Stat. Phys.* **73**, 893–926 (1993)
22. Zlatoš, A.: Transition fronts in inhomogeneous Fisher-KPP reaction-diffusion equations. *J. Math. Pures Appl.* **98**, 89–102 (2012)
23. Zlatoš, A.: Spreading of reaction and transition fronts in inhomogeneous media (preprint)

Department of Mathematics,
University of Wisconsin,
Madison, WI 53706, USA.
e-mail: zlatos@math.wisc.edu
e-mail: andrej@math.wisc.edu

(*Received July 18, 2012 / Accepted November 12, 2012*)
Published online January 8, 2013 –