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Global Compensated Compactness Theorem for General Differential Operators of First Order

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Abstract

Let $A_1(x, D)$ and $A_2(x, D)$ be differential operators of the first order acting on l-vector functions $u = (u_1, \dots, u_l)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ with the smooth boundary $\partial\Omega$. We assume that the H^1-norm $\|u\|_{H^1(\Omega)}$ is equivalent to $\sum_{i=1}^2 \|A_i u\|_{L^2(\Omega)} + \|B_1 u\|_{H^{\frac{1}{2}}(\partial\Omega)}$ and $\sum_{i=1}^2 \|A_i u\|_{L^2(\Omega)} + \|B_2 u\|_{H^{\frac{1}{2}}(\partial\Omega)}$, where $B_i = B_i(x, \nu)$ is the trace operator onto $\partial\Omega$ associated with $A_i(x, D)$ for $i = 1, 2$ which is determined by the Stokes integral formula (ν: unit outer normal to $\partial\Omega$). Furthermore, we impose on A_1 and A_2 a cancellation property such as $A_1 A_2' = 0$ and $A_2 A_1' = 0$, where A_i' is the formal adjoint differential operator of $A_i (i = 1, 2)$. Suppose that $\{u_m\}_{m=1}^\infty$ and $\{v_m\}_{m=1}^\infty$ converge to u and v weakly in $L^2(\Omega)$, respectively. Assume also that $\{A_1 u_m\}_{m=1}^\infty$ and $\{A_2 v_m\}_{m=1}^\infty$ are bounded in $L^2(\Omega)$. If either $\{B_1 u_m\}_{m=1}^\infty$ or $\{B_2 v_m\}_{m=1}^\infty$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)$, then it holds that $\int_\Omega u_m \cdot v_m \, dx \to \int_\Omega u \cdot v \, dx$. We also discuss a corresponding result on compact Riemannian manifolds with boundary.

1. Introduction

The purpose of this paper is to establish a compensated compactness theorem for general differential operators of the first order. The convergence is proved not only in the sense of distributions in open sets in $\mathbb{R}^n$ but also in bounded domains Ω up to the boundary $\partial\Omega$. Let $A_1 = A_1(x, D)$ and $A_2 = A_2(x, D)$ be two differential operators in a domain Ω in $\mathbb{R}^n$ acting on l-vector functions $u = {}^t(u_1, \dots, u_l) \in L^2(\Omega)^l$ to $H^{-1}(\Omega)^{d_1}$ and to $H^{-1}(\Omega)^{d_2}$, respectively, where $D = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. For every point $x \in \Omega$, we consider a quadratic form $Q(\eta, \zeta) = Q(\eta, \zeta)(x) \equiv \sum_{j,k=1}^l q_{jk}(x)\eta_j\zeta_k$ for $\eta = {}^t(\eta_1, \dots, \eta_l)$, $\zeta = {}^t(\zeta_1, \dots, \zeta_l) \in \mathbb{R}^l$, where $q_{jk} \in C^\infty(\bar{\Omega})$, $j, k = 1, \dots, l$. The compensated compactness theorem states that under the following hypotheses (i) and (ii)

(i) $u_m \rightharpoonup u,\ v_m \rightharpoonup v$ weakly in $L^2(\Omega)^l$ as $m \to \infty$;
(ii) $\{A_1 u_m\}_{m=1}^{\infty}$ is bounded in $L^2(\Omega)^{d_1}$ and $\{A_2 v_m\}_{m=1}^{\infty}$ is bounded in $L^2(\Omega)^{d_2}$,

it holds that

$$Q(u_m, v_m) \to Q(u, v) \quad \text{in the sense of distributions in } \Omega \text{ as } m \to \infty. \tag{1.1}$$

A typical example of the compensated compactness theorem is so called *Div–Curl lemma*, where we may take $A_1 = \mathrm{div}$, $A_2 = \mathrm{rot}$ and $Q(\eta, \zeta) = \sum_{j=1}^{3} \eta_j \zeta_j$ with $l = n = 3$, $d_1 = 1$ and $d_2 = 3$. Roughly speaking, in the compensated compactness theorem, we need to investigate special structures of the quadratic form $Q(\eta, \zeta)$ in connection with the differential operators A_1 and A_2 which yields the convergence (1.1).

In the case when $A_1 = A_1(D)$ and $A_2 = A_2(D)$ are differential operators with constant coefficients of the homogeneous degree 1 as well as the quadratic form Q with the constant coefficients $\{q_{jk}\}_{j,k=1,\dots,l}$ in Ω, TARTAR [18] introduced an algebraic cancellation property

$$Q(\lambda, \lambda) = 0 \tag{1.2}$$

for all $\lambda =^t (\lambda_1, \dots, \lambda_l) \in \mathbb{R}^l$ such that $A_\alpha(\xi)\lambda = 0, \alpha = 1, 2$ for some $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ with $\xi \neq 0$, and proved (1.1). On the other hand, it seems to be important to handle the general differential operators $A_1 = A_1(x, D)$ and $A_2 = A_2(x, D)$ with *variable* coefficients for the standard scalar product $Q(\eta, \zeta) = \sum_{j=1}^{l} \eta_j \zeta_j$ in $\mathbb{R}^l$. In this direction, KAZHIKHOV [7] made use of the closed range theorem for A_1 and A_2 which yields necessarily orthogonal decompositions

$$L^2(\Omega)^l = Ker(A_\alpha) \oplus R(A_\alpha^*), \quad \alpha = 1, 2, \tag{1.3}$$

where $Ker(A_\alpha)$ and $R(A_\alpha^*)$ denote the kernel of A_α and the range of the adjoint operator A_α^* of A_α, respectively. In comparison with the case of differential operators $A_1 = A_1(D)$ and $A_2 = A_2(D)$ with constant coefficients, the inclusion relation

$$Ker(A_\alpha) \subset R(A_\beta^*), \quad \alpha \neq \beta \tag{1.4}$$

plays a substitutive role for the cancellation property (1.2). In any case, the main difficulty to prove (1.1) stems from treatment of $Ker(A_\alpha)$ for $\alpha = 1, 2$. More precisely, since A_α is invertible on $R(A_\alpha^*)$, the proof of (1.1) can be reduced to show that

$$Q(P_1 u_m, P_2 v_m) \to Q(P_1 u, P_2 v) \quad \text{in the sense of distributions in } \Omega \text{ as } m \to \infty, \tag{1.5}$$

where $P_\alpha : L^2(\Omega)^l \to Ker(A_\alpha)$, $\alpha = 1, 2$ is the orthogonal projection along (1.3). It should be noted that both (1.2) and (1.4) are sufficient conditions for (1.5).

In the present paper, we shall first make clear a special structure of the operator $A_\alpha = A_\alpha(x, D)$ so that $Ker(A_\alpha)$ is a finite dimensional subspace in $L^2(\Omega)^l$. Once $Ker(A_\alpha)$ is reduced to the finite dimensional space, it is easy to see that even weak convergence in $L^2(\Omega)^l$ of $\{u_m\}_{m=1}^{\infty}$ and $\{v_m\}_{m=1}^{\infty}$ yields (1.5). For such purpose, we need to introduce an appropriate boundary operator $B_\alpha(x, \nu)$ on $\partial\Omega$ and regard A_α as an unbounded operator in $L^2(\Omega)^l$ with the domain $D(A_\alpha) = \{u \in L^2(\Omega)^l;\ A_\alpha u \in L^2(\Omega)^{d_\alpha},\ B_\alpha(x, \nu)u|_{\partial\Omega} = 0\}$. In the next step, by assuming the corresponding cancellation property for (1.2) and (1.4) such as

$$A_\alpha A_\beta^* = 0, \quad \alpha \neq \beta, \tag{1.6}$$

we deal with the convergence on the subspace $R(A_\alpha^*)$. Since we control behavior of $\{u_m\}_{m=1}^\infty$ and $\{v_m\}_{m=1}^\infty$ on the boundary $\partial\Omega$, we establish a stronger convergence in the whole domain Ω such as

$$\int_\Omega Q(u_m, v_m)\,\mathrm{d}x \to \int_\Omega Q(u, v)\,\mathrm{d}x \quad \text{as } m \to \infty, \tag{1.7}$$

which includes (1.1).

As an application of our result, we prove Murat–Tartar's classical Div–Curl lemma [12–14,18] with additional lower order terms with variable coefficients. We also establish a generalized Div–Curl lemma for arbitrary differential l-forms via the exterior derivative d and its co-differential operator δ on compact Riemannian manifolds $(\bar{\Omega}, g)$ with boundary $\partial\Omega$. To this end, we introduce the tangential part τu and the normal part νu on $\partial\Omega$ for the differential l-form $u = \sum_{i_1<\cdots<i_l} u_{i_1\cdots i_l}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge x^{i_l}$ on $\bar{\Omega}$. A similar investigation in L^r-spaces can be seen in our previous papers [9] and [10].

There is a huge literature of generalization for variable coefficients of the Murat–Tartar's classical Div–Curl lemma. Making use of the technique of pseudo-differential operators, Gérard [6] established systematic treatments of micro local defect measures and their connection to orthogonality of two sequences $\{u_m\}_{m=1}^\infty$ and $\{v_m\}_{m=1}^\infty$ in $L^2(\Omega)^l$ which yields the convergence like (1.7). He applied his generalization to the problem of homogenization for the first order scalar differential operators with oscillating coefficients. Another generalization had been carried out by Tartar [19] who introduced a notion of H-measures independently of [6]. Indeed, compensated compactness can be obtained as a consequence of the localization principle of the support of the H-measure. He applied several properties of H-measures to propagation of both oscillation and concentration effects in the nonlinear partial differential equations arising from continuum mechanics and physics. A more generalized summary on compensated compactness was demonstrated by [20]. However, all of these convergences have been discussed in the sense of distributions in Ω. Although our result might be well-known so far as local convergence in the interior of Ω is concerned, we shall prove the *global* convergence such as (1.7) in the whole Ω in terms of the relation between the differential operators $A_\alpha(x, D)$ and the boundary operators $B_\alpha(x, \nu)$ for $\alpha = 1, 2$.

This paper is organized as follows. In Section 2, after precise definition of the differential operator $A_\alpha(x, D)$ together with the boundary operator $B_\alpha(x, \nu)$, we shall state our main theorem. Section 3 is devoted to the orthogonal decomposition (1.3) and the cancellation property (1.6). In particular, we need to pay attention to a certain vanishing property on the boundary value of the special forms which makes it easy to handle the convergence on $R(A_\alpha^*)$. Then the proof of our main theorem is established in Section 4. Finally in Section 5, some examples such as the generalized Div–Curl lemma on compact Riemannian manifolds are considered.

2. Result

Let Ω be a bounded domain in $\mathbb{R}^n$ with the smooth boundary $\partial\Omega$. We consider a system $A(x, D) : C^\infty(\bar{\Omega})^l \mapsto C^\infty(\bar{\Omega})^d$ of differential operators of the first order defined by

$$A(x, D)u = {}^t\left(\sum_{j=1}^{l} A_{1j}(x, D)u_j, \dots, \sum_{j=1}^{l} A_{dj}(x, D)u_j\right)$$
$$\text{for } u = {}^t\ (u_1, \dots, u_l) \in C^\infty(\bar{\Omega})^l,$$

where

$$A_{ij}(x, D) = \sum_{k=1}^{n} a_{ijk}(x)\frac{\partial}{\partial x_k} + b_{ij}(x), \quad x \in \bar{\Omega}, D = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \tag{2.1}$$

with $a_{ijk}, b_{ij} \in C^\infty(\bar{\Omega})$ for $i = 1, \dots, d$, $j = 1, \dots, l$, $k = 1, \dots, n$. For simplicity, we assume that a_{ijk}, b_{ij} are real valued smooth coefficients of $A(x, D)$. Then the formal adjoint $A'(x, D) : C^\infty(\bar{\Omega})^d \mapsto C^\infty(\bar{\Omega})^l$ of $A(x, D)$ is defined by the relation

$$(A(\cdot, D)u, \varphi) = (u, A'(\cdot, D)\varphi), \quad u \in C_0^\infty(\Omega)^l, \varphi \in C_0^\infty(\Omega)^d,$$

where $(\cdot, \cdot)$ denotes the usual L^2-inner product on Ω. Indeed, for $A(x, D)$ defined by (2.1), we have the expression of $A'(x, D)$ as

$$A'(x, D)\varphi = {}^t\left(\sum_{i=1}^{d} A'_{1i}(x, D)\varphi_i, \dots, \sum_{i=1}^{d} A'_{li}(x, D)\varphi_i\right)$$
$$\text{for } \varphi = {}^t\ (\varphi_1, \dots, \varphi_d) \in C^\infty(\bar{\Omega})^d,$$

where

$$A'_{ji}(x, D) = -\sum_{k=1}^{n} a_{ijk}(x)\frac{\partial}{\partial x_k} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k}a_{ijk}(x) + b_{ij}(x), \quad j = 1, \dots, l, i = 1, \dots, d. \tag{2.2}$$

Then there exist operators $B(x, \nu) : C^\infty(\bar{\Omega})^l \mapsto C^\infty(\partial\Omega)^d$ and $B'(x, \nu) : C^\infty(\bar{\Omega})^d \mapsto C^\infty(\partial\Omega)^l$ such that the Stokes integral formula

$$(A(\cdot, D)u, \varphi) - (u, A'(\cdot, D)\varphi) = \langle B(\cdot, \nu)u, \varphi\rangle_{\partial\Omega}, \tag{2.3}$$
$$(A(\cdot, D)u, \varphi) - (u, A'(\cdot, D)\varphi) = \langle u, B'(\cdot, \nu)\varphi\rangle_{\partial\Omega} \tag{2.4}$$

holds for all $u \in C^\infty(\bar{\Omega})^l$ and all $\varphi \in C^\infty(\bar{\Omega})^d$, where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outer normal to $\partial\Omega$ and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the standard L^2-inner product on $\partial\Omega$. It is easy to see that $B(x, \nu)$ and $B'(x, \nu)$ are expressed as

$$B(x,\nu)u = \left(\sum_{j=1}^{l} B_{1j}(x,\nu)u_j, \ldots, \sum_{j=1}^{l} B_{dj}(x,\nu)u_j \right),$$
$$u =^t (u_1, \ldots, u_l) \in C^\infty(\bar{\Omega})^l \text{ with}$$
$$B_{ij}(x,\nu) = \sum_{k=1}^{n} a_{ijk}(x)\nu_k, \quad i = 1, \ldots, d,\ j = 1, \ldots, l, \tag{2.5}$$

and

$$B'(x,\nu)\varphi = \left(\sum_{i=1}^{d} B'_{1i}(x,\nu)\varphi_i, \ldots, \sum_{i=1}^{d} B'_{li}(x,\nu)\varphi_i \right),$$
$$\varphi =^t (\varphi_1, \ldots, \varphi_d) \in C^\infty(\bar{\Omega})^d \quad \text{with}$$
$$B'_{ji}(x,\nu) = \sum_{k=1}^{n} a_{ijk}(x)\nu_k, \quad j = 1, \ldots, l,\ i = 1, \ldots, d,$$

respectively.

Remark. By (2.3), the boundary operator $B(x, \nu)$ can be extended to the functions $u \in L^2(\Omega)^l$ with $A(x, D)u \in L^2(\Omega)^d$ so that $B(x, \nu)u \in H^{-\frac{1}{2}}(\partial\Omega)^d \equiv (H^{\frac{1}{2}}(\partial\Omega)^d)^*$, and the generalized Stokes formula holds

$$(A(\cdot, D)u, \varphi) - (u, A'(\cdot, D)\varphi) = \langle B(\cdot, \nu)u, \gamma\varphi \rangle_{\partial\Omega} \quad \text{for all } \varphi \in H^1(\Omega)^d, \tag{2.6}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)^d$ and $H^{\frac{1}{2}}(\partial\Omega)^d$, and γ is the usual trace operator from $H^1(\Omega)^d$ onto $H^{\frac{1}{2}}(\partial\Omega)^d$.

Similarly, by (2.4), for every $\varphi \in L^2(\Omega)^d$ with $A'(x, D)\varphi \in L^2(\Omega)^l$, we can define $B'(x, \nu)\varphi \in H^{-\frac{1}{2}}(\partial\Omega)^l$ with the generalized Stokes formula

$$(A(\cdot, D)u, \varphi) - (u, A'(\cdot, D)\varphi) = \langle \gamma u, B'(\cdot, \nu)\varphi \rangle_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega)^l. \tag{2.7}$$

In what follows, we shall regard the boundary operators $B(x, \nu)$ and $B'(x, \nu)$ as those in the generalized sense satisfying (2.6) and (2.7), respectively.

Let us consider two pairs $\{A_\alpha(x, D), A'_\alpha(x, D), B_\alpha(x, \nu), B'_\alpha(x, \nu)\}$ for $\alpha = 1, 2$ with $l_1 = l_2 = l$, that is,

$$A_1(x, D) : H^1(\Omega)^l \mapsto L^2(\Omega)^{d_1}, \quad A_2(x, D) : H^1(\Omega)^l \mapsto L^2(\Omega)^{d_2}$$

which satisfy (2.6) and (2.7) with $A = A_1$ and $A = A_2$. Throughout this paper, we impose the following assumption on A_1 and A_2.

Assumption. There is a constant $C = C(\Omega)$ such that

$$\|\nabla u\| \leqq C(\|A_1 u\| + \|A_2 u\| + \|u\| + \|B_1 u\|_{H^{\frac{1}{2}}(\partial\Omega)}), \tag{2.8}$$

$$\|\nabla u\| \leqq C(\|A_1 u\| + \|A_2 u\| + \|u\| + \|B_2 u\|_{H^{\frac{1}{2}}(\partial\Omega)}) \tag{2.9}$$

holds for all $u \in H^1(\Omega)^l$. Here and in what follows, $\|\cdot\|$ denotes the usual L^2-norm on Ω.

Our main theorem now reads:

Theorem 1. *Let two pairs* $\{A_\alpha(x, D), A'_\alpha(x, D), B_\alpha(x, \nu), B'_\alpha(x, \nu)\}, \alpha = 1, 2$ *satisfy* (2.6) *and* (2.7) *with* $A = A_1$ *and* $A = A_2$*. Let the Assumption hold. We assume the cancellation property*

$$A_2 A'_1 = 0, \quad A_1 A'_2 = 0. \tag{2.10}$$

Suppose that $\{u_m\}_{m=1}^\infty$ *and* $\{v_m\}_{m=1}^\infty$ *are sequences in* $L^2(\Omega)^l$ *satisfying the following conditions* (i), (ii) *and* (iii).

(i) $u_m \rightharpoonup u, v_m \rightharpoonup v$ *weakly in* $L^2(\Omega)^l$*;*
(ii) $\{A_1 u_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^{d_1}$ *and* $\{A_2 v_m\}_{m=1}^\infty$ *is bounded in* $L^2(\Omega)^{d_2}$*;*
(iii) *Either* $\{B_1 u_m\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^{d_1}$ *or* $\{B_2 v_m\}_{m=1}^\infty$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^{d_2}$.

Then it holds that

$$\int_\Omega u_m(x) \cdot v_m(x)\,\mathrm{d}x \to \int_\Omega u(x) \cdot v(x)\,\mathrm{d}x \quad \text{as } m \to \infty, \tag{2.11}$$

where $u(x) \cdot v(x) = \sum_{j=1}^l u_j(x) v_j(x)$ *is the standard scalar product in* $\mathbb{R}^l$ *at each point* $x \in \Omega$.

Remark 1. If we express A_1 and A_2 as in the form like (2.1), that is,

$$A_\alpha(x, D)u = {}^t\left(\sum_{j=1}^l A_{1j}^{(\alpha)}(x, D)u_j, \ldots, \sum_{j=1}^l A_{dj}^{(\alpha)}(x, D)u_j\right)$$
$$\text{for } u = {}^t\,(u_1, \ldots, u_l) \in H^1(\Omega)^l$$

with

$$A_{ij}^{(\alpha)}(x, D) = \sum_{k=1}^n a_{ijk}^{(\alpha)}(x) \frac{\partial}{\partial x_k} + b_{ij}^{(\alpha)}(x), \quad i = 1, \ldots, d_\alpha, \quad j = 1, \ldots, l, \quad \alpha = 1, 2,$$

then the cancellation property (2.10) can be written as

$$\sum_{j=1}^{l}(a_{rjs}^{(\alpha)}a_{ijk}^{(\beta)}+a_{rjk}^{(\alpha)}a_{ijs}^{(\beta)})=0,$$

$$\alpha,\beta=1,2,\quad \alpha\neq\beta,\quad 1\leqq\forall s,\forall k\leqq n,\quad r=1,\dots,d_\alpha, i=1,\dots d_\beta, \tag{2.12}$$

$$\sum_{j=1}^{l}\left(\sum_{\mu=1}^{n}a_{rj\mu}^{(\alpha)}\frac{\partial a_{ijk}^{(\beta)}}{\partial x_\mu}+a_{rjk}^{(\alpha)}\sum_{\mu=1}^{n}\frac{\partial a_{ij\mu}^{(\beta)}}{\partial x_\mu}-a_{rjk}^{(\alpha)}b_{ij}^{(\beta)}-a_{ijk}^{(\beta)}b_{rj}^{(\alpha)}\right)=0,$$

$$\alpha,\beta=1,2,\quad \alpha\neq\beta,\quad 1\leqq\forall k\leqq n,\quad r=1,\dots,d_\alpha, i=1,\dots d_\beta, \tag{2.13}$$

$$\sum_{j=1}^{l}\left(\sum_{\mu=1}^{n}a_{rj\mu}^{(\alpha)}(-\sum_{\sigma=1}^{n}\frac{\partial^2 a_{ij\sigma}^{(\beta)}}{\partial x_\mu\partial x_\sigma}+\frac{\partial b_{ij}^{(\beta)}}{\partial x_\mu})+b_{rj}^{(\alpha)}(-\sum_{\mu=1}^{n}\frac{\partial a_{ij\mu}^{(\beta)}}{\partial x_\mu}+b_{ij}^{(\beta)})\right)=0,$$

$$\alpha,\beta=1,2,\quad \alpha\neq\beta,\quad r=1,\dots,d_\alpha, i=1,\dots d_\beta. \tag{2.14}$$

Remark 2. Our proof is based on the orthogonal decomposition (1.3). A more precise argument will be discussed in (4.1). Such a method is closely related to the de Rham–Hodge–Kodaira decomposition for differential forms on Riemannian manifolds. Indeed, ROBBIN et al. [15] made use of it for showing weak continuity of the scalar product $u_m\cdot v_m$ as $m\to\infty$ by means of the exterior derivatives and their formal adjoints. A similar approach to the convergence up to the boundary was established by our previous paper [9]. It is known that application of the theory of the Hardy space is also useful to the proof of the div–curl lemma. See for example, COIFMAN et al. [2] and DAFNI [3]. It seems to be an interesting problem to investigate the relation between cancellation property (2.10) and functions in the Hardy space.

3. Preliminary

For the proof of Theorem 1, let us introduce two operators S and T defined by $S,T: L^2(\Omega)^l\mapsto L^2(\Omega)^{d_1+d_2}$,

$$D(S)=\{u\in H^1(\Omega)^l;\, B_1u=0\quad\text{on }\partial\Omega\},\quad Su\equiv{}^t(A_1u,A_2u)\quad\text{for } u\in D(S),$$
$$D(T)=\{u\in H^1(\Omega)^l;\, B_2u=0\quad\text{on }\partial\Omega\},\quad Tu\equiv{}^t(A_1u,A_2u)\quad\text{for } u\in D(T).$$

It should be noted that $D(S)$ and $D(T)$ are dense in $L^2(\Omega)^l$ (see for example, DUVAUT and LIONS [4, Chapter 7, Lemmata 4.1, 6.1] and GEORGESGUE [5, Theorem 4.1.1]), and, hence, we may define the adjoint operators S^* and T^* of S and T from $L^2(\Omega)^{d_1+d_2}$ to $L^2(\Omega)^l$, respectively. By (2.6) and (2.7) it holds that

$$\begin{aligned} D(S^*)&=\{{}^t(p,w)\in L^2(\Omega)^{d_1}\times L^2(\Omega)^{d_2};\, A_1'p\in L^2(\Omega)^l, A_2'w\in L^2(\Omega)^l,\\ &\qquad B_2'w=0\quad\text{on }\partial\Omega\},\\ S^*({}^t(p,w))&=A_1'p+A_2'w\quad\text{for }{}^t(p,w)\in D(S^*), \end{aligned}\tag{3.1}$$

$$\begin{aligned} D(T^*)&=\{{}^t(p,w)\in L^2(\Omega)^{d_1}\times L^2(\Omega)^{d_2};\, A_1'p\in L^2(\Omega)^l, A_2'w\in L^2(\Omega)^l,\\ &\qquad B_1'p=0\quad\text{on }\partial\Omega\},\\ T^*({}^t(p,w))&=A_1'p+A_2'w\quad\text{for }{}^t(p,w)\in D(T^*). \end{aligned}\tag{3.2}$$

Furthermore, we have the following lemma.

Lemma 3.1. 1. *The kernels $Ker(S)$ and $Ker(T)$ of S and T are both finite dimensional subspaces of $L^2(\Omega)^l$.*
2. *The ranges $R(S)$ and $R(T)$ of S and T are both closed subspaces of $L^2(\Omega)^{d_1+d_2}$.*

Proof. The proofs for S and T are based on the the estimates (2.8) and (2.9) in the Assumption, respectively. So, we may only show the assertion on S.

1. By (2.8) we see that the unit ball in $Ker(S)$ is a bounded set in $H^1(\Omega)^l$, and, hence, the Rellich theorem states that it is a compact set in $L^2(\Omega)^l$. This implies that $Ker(S)$ is a finite dimensional subspace in $L^2(\Omega)^l$.
2. We make use of an auxiliary estimate; there exists a constant $\delta > 0$ such that

$$\|Sw\| \geqq \delta\|w\| \tag{3.3}$$

holds for all $w \in D(S) \cap Ker(S)^\perp$.

For the moment, let us assume (3.3). Suppose that $\{u_m\}_{m=1}^\infty \subset D(S)$ satisfies

$$Su_m \to f \quad \text{in } L^2(\Omega)^{d_1+d_2} \quad \text{as } m \to \infty.$$

By the orthogonal decomposition, u_m is expressed as

$$u_m = v_m + w_m, \quad v_m \in Ker(S), \quad w_m \in Ker(S)^\perp, \quad m = 1, 2, \ldots.$$

Since it follows from (3.3) that

$$\|Su_m - Su_l\| = \|S(w_m - w_l)\| \geqq \delta\|w_m - w_l\|, \quad m, l = 1, 2, \ldots,$$

we have that $w_m \to w$ in $L^2(\Omega)^l$ for some $w \in Ker(S)^\perp$. Since $Sw_m = Su_m \to f$ in $L^2(\Omega)^{d_1+d_2}$ and since S is a closed operator from $L^2(\Omega)^l$ to $L^2(\Omega)^{d_1+d_2}$, it holds that $w \in D(S)$ with $Sw = f$, which means that $f \in R(S)$. Hence, $R(S)$ is a closed subspace of $L^2(\Omega)^{d_1+d_2}$.

Now it remains to prove (3.3). We make use of a contradiction argument. Suppose the contrary. Then there is a sequence $\{w_m\}_{m=1}^\infty$ in $D(S) \cap Ker(S)^\perp$ with $\|w_m\| \equiv 1$ such that

$$\|Sw_m\| = \|A_1 w_m\| + \|A_2 w_m\| \leqq 1/m \quad \text{for all } m = 1, 2, \ldots.$$

By (2.8), we see that $\{w_m\}_{m=1}^\infty$ is a bounded sequence in $H^1(\Omega)^1$, and, hence, there is a subsequence of $\{w_m\}_{m=1}^\infty$, which we denote by $\{w_m\}_{m=1}^\infty$ itself, for simplicity, and a function $w \in Ker(S)^\perp$ such that $w_m \to w$ in $L^2(\Omega)^l$. Since $Sw_m \to 0$ in $L^2(\Omega)^{d_1+d_2}$, again by closedness of S it holds that $w \in D(S)$ with $Sw = 0$, that is, $w \in Ker(S)$. Since $w \in Ker(S)^\perp$, we have $w = 0$, which contradicts the property that $\|w_m\| \equiv 1$ for all $m = 1, \ldots$. This proves Lemma 3.1. □

Lemma 3.2. *Let* (2.10) *hold.*

1. *If* $w \in L^2(\Omega)^{d_2}$ *with* $A_2' w \in L^2(\Omega)^l$ *satisfies* $B_2' w = 0$ *in* $H^{-\frac{1}{2}}(\partial\Omega)^{d_2}$, *then we have*

$$B_1(x, \nu)(A_2' w) = 0 \quad \text{on } \partial\Omega \tag{3.4}$$

with the identity

$$(A_1' p, A_2' w) = 0 \quad \text{for all } p \in L^2(\Omega)^{d_1} \text{ with } A_1' p \in L^2(\Omega)^l. \tag{3.5}$$

2. *If* $p \in L^2(\Omega)^{d_1}$ *with* $A_1' p \in L^2(\Omega)^l$ *satisfies* $B_1' p = 0$ *in* $H^{-\frac{1}{2}}(\partial\Omega)^{d_1}$, *then we have*

$$B_2(x, \nu)(A_1' p) = 0 \quad \text{on } \partial\Omega \tag{3.6}$$

with the identity

$$(A_1' p, A_2' w) = 0 \quad \text{for all } w \in L^2(\Omega)^{d_2} \quad \text{with } A_2' w \in L^2(\Omega)^l. \tag{3.7}$$

Proof. 1. For every $q \in H^2(\Omega)^{d_2}$ we have by (2.6), (2.7) and (2.10) that

$$\begin{aligned}\langle B_1(\cdot, \nu) A_2' w, q\rangle_{\partial\Omega} &= (A_1(A_2' w), q) - (A_2' w, A_1' q)\\ &= -(A_2' w, A_1' q)\\ &= -(w, A_2(A_1' q)) + \langle B_2' w, A_1' q\rangle_{\partial\Omega}\\ &= 0,\end{aligned}$$

which implies $B_1(x, \nu) A_2' w = 0$ on $\partial\Omega$.

It is known ,that for every $p \in L^2(\Omega)^{d_1}$ with $A_1' p \in L^2(\Omega)^l$ there is a sequence $\{p_m\}_{m=1}^{\infty} \in C^\infty(\bar{\Omega})^{d_1}$ such that $p_m \to p$ in $L^2(\Omega)^{d_1}$ and $A_1' p_m \to A_1' p$ in $L^2(\Omega)^l$ (see for example, Georgesgue [5, Theorem 4.1.1]). Hence by passage to the limit, we may prove (3.5) for all $p \in C^\infty(\bar{\Omega})^{d_1}$. Since $B_2' w = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)^{d_2}$, it follows from (2.7) and (2.10) that

$$(A_1' p, A_2' w) = (A_2(A_1' p), w) - \langle A_1' p, B_2' w\rangle_{\partial\Omega} = 0 \quad \text{for all } p \in C^\infty(\bar{\Omega})^{d_1},$$

which yields (3.5).

2. Similarly, for every $\varphi \in H^2(\Omega)^{d_2}$ we have by (2.6), (2.7) and (2.10) that

$$\begin{aligned}\langle B_2(\cdot, \nu) A_1' p, \varphi\rangle_{\partial\Omega} &= (A_2 A_1' p, \varphi) - (A_1' p, A_2' \varphi)\\ &= -(A_1' p, A_2' \varphi)\\ &= -(p, A_1 A_2' \varphi) + \langle B_1' p, A_2' \varphi\rangle_{\partial\Omega}\\ &= 0,\end{aligned}$$

which implies $B_2(x, \nu)(A_1' p) = 0$ on $\partial\Omega$.

It is also known that for every $w \in L^2(\Omega)^{d_2}$ with $A_2' w \in L^2(\Omega)^l$ there is a sequence $\{w_m\}_{m=1}^{\infty} \in C^\infty(\bar{\Omega})^{d_2}$ such that $w_m \to w$ in $L^2(\bar{\Omega})^{d_2}$ and $A_2' w_m \to A_2' w$ in $L^2(\Omega)^l$. Hence, by passage to the limit, we may prove (3.7) for all $w \in C^\infty(\bar{\Omega})^{d_2}$. Since $B_1' p = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)^{d_1}$, it follows from (2.7) and (2.10) that

$$(A_1' p, A_2' w) = (p, A_1(A_2' w)) - \langle B_1' p, A_2' w\rangle_{\partial\Omega} = 0 \quad \text{for all } w \in C^\infty(\bar{\Omega})^{d_2},$$

which yields (3.7). This proves Lemma 3.2 □

4. Proof of Theorem

Case 1. Let us first consider the case when $\{B_1 u_m\}_{m=1}^{\infty}$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)^{d_1}$. In this case, we make use of the operator S. It follows from Lemma 3.1 (2) and the closed range theorem that

$$L^2(\Omega)^l = Ker(S) \oplus R(S^*) \quad \text{(orthogonal decomposition)}. \tag{4.1}$$

Let P and Q be orthogonal projections from $L^2(\Omega)^l$ onto $Ker(S)$ and $R(S^*)$ along (4.1), respectively. Then it holds

$$\begin{aligned} &u = Pu + Qu, \quad v = Pv + Qv, \\ &u_m = Pu_m + Qu_m, \quad v_m = Pv_m + Qv_m, \quad m = 1, 2, \dots, \end{aligned} \tag{4.2}$$

and we have

$$(u_m, v_m) = (Pu_m, Pv_m) + (Qu_m, Qv_m), \quad m = 1, 2, \dots. \tag{4.3}$$

Since $R(P) = Ker(S)$, we see by Lemma 3.2 (1) that P is a finite rank operator, so in particular, a compact operator. Hence by (i) it holds that

$$Pu_m \to Pu, \quad Pv_m \to Pv \quad \text{strongly in } L^2(\Omega)^{d_1} \quad \text{as } m \to \infty,$$

which yields

$$(Pu_m, Pv_m) \to (Pu, Pv) \quad \text{as } m \to \infty. \tag{4.4}$$

We next show that

$$(Qu_m, Qv_m) \to (Qu, Qv) \quad \text{as } m \to \infty. \tag{4.5}$$

Since Q is the projection operator from $L^2(\Omega)^l$ onto $R(S^*)$, there exist functions $p, \tilde{p}, p_m, \tilde{p}_m \in L^2(\Omega)^{d_1}$ with $A_1' p, A_1' \tilde{p}, A_1' p_m, A_1' \tilde{p}_m \in L^2(\Omega)^l$, and functions $w, \tilde{w}, w_m, \tilde{w}_m \in L^2(\Omega)^{d_2}$ with $A_2' w, A_2' \tilde{w}, A_2' w_m, A_2' \tilde{w}_m \in L^2(\Omega)^l$ and $B_2' w = B_2' \tilde{w} = B_2' w_m = B_2' \tilde{w}_m = 0$ on $\partial\Omega$ such that

$$Qu = A_1' p + A_2' w, \quad Qv = A_1' \tilde{p} + A_2' \tilde{w} \tag{4.6}$$

$$Qu_m = A_1' p_m + A_2' w_m, \quad Qv_m = A_1' \tilde{p}_m + A_2' \tilde{w}_m, \quad m = 1, 2, \dots. \tag{4.7}$$

Then it holds that

$$(Qu, Qv) = (A_1' p, A_1' \tilde{p}) + (A_2' w, A_2' \tilde{w}), \tag{4.8}$$

$$(Qu_m, Qv_m) = (A_1' p_m, A_1' \tilde{p}_m) + (A_2' w_m, A_2' \tilde{w}_m), \tag{4.9}$$

$$\|Qu_m\|^2 = \|A_1' p_m\|^2 + \|A_2' w_m\|^2, \quad \|Qv_m\|^2 = \|A_1' \tilde{p}_m\|^2 + \|A_2' \tilde{w}_m\|^2, \tag{4.10}$$

$$\|A_1 u_m\| = \|A_1 Qu_m\| = \|A_1 A_1' p_m\|, \quad \|A_2 v_m\| = \|A_2 Qv_m\| = \|A_2 A_2' \tilde{w}_m\| \tag{4.11}$$

for all $m = 1, 2, \dots$. Indeed, (4.8), (4.9) and (4.10) are a consequence of (3.5). Since $Pu_m, Pv_m \in Ker(S)$, we have $A_1 Pu_m = 0$ and $A_2 Pv_m = 0$, and, hence, it follows from (4.2), (4.7) and (2.10) that

$$A_1 u_m = A_1 Q u_m = A_1 A_1' p_m, \quad A_2 v_m = A_2 Q v_m = A_2 A_2' \tilde{w}_m,$$

which yields (4.11). Furthermore, we have that

$$A_1' p_m \rightharpoonup A_1' p, \quad A_2' w_m \rightharpoonup A_2' w \quad \text{weakly in } L^2(\Omega)^l, \tag{4.12}$$

$$A_1' \tilde{p}_m \rightharpoonup A_1' \tilde{p}, \quad A_2' \tilde{w}_m \rightharpoonup A_2' \tilde{w} \quad \text{weakly in } L^2(\Omega)^l, \tag{4.13}$$

as $m \to \infty$. In fact, by (i) it is easy to see that $Qu_m \rightharpoonup Qu$ weakly in $L^2(\Omega)^l$. For every $\varphi \in L^2(\Omega)^l$, there exist $q \in L^2(\Omega)^{d_1}$ with $A_1' q \in L^2(\Omega)^l$, and $\eta \in L^2(\Omega)^{d_2}$ with $A_2' \eta \in L^2(\Omega)^l$ and $B_2' \eta = 0$ on $\partial\Omega$ such that

$$Q\varphi = A_1' q + A_2' \eta.$$

Since $A_1' p_m, A_1' p \in R(S^*) = R(Q)$, it follows from (2.10), (3.5), (4.6) and (4.7) that

$$\begin{aligned}
(A_1' p_m - A_1' p, \varphi) &= (A_1' p_m - A_1' p, Q\varphi) \\
&= (A_1' p_m, A_1' q) - (A_1' p, A_1' q) \\
&= (Qu_m, A_1' q) - (A_1' p, A_1' q) \\
&\to (Qu, A_1' q) - (A_1' p, A_1' q) \\
&= (A_1' p, A_1' q) - (A_1' p, A_1' q) = 0, \\
(A_2' w_m - A_2' w, \varphi) &= (A_2' w_m - A_2' w, Q\varphi) \\
&= (A_2' w_m, A_2' \eta) - (A_2' w, A_2' \eta) \\
&= (Qu_m, A_2' \eta) - (A_2' w, A_2' \eta) \\
&\to (Qu, A_2' \eta) - (A_2' w, A_2' \eta) \\
&= (A_2' w, A_2' \eta) - (A_2' w, A_2' \eta) = 0,
\end{aligned}$$

which implies (4.12). The validity of (4.13) can be shown quite similarly as above.

To prove (4.5) we need the following proposition.

Proposition 4.1. 1. *The sequence* $\{A_1' p_m\}_{m=1}^{\infty}$ *is bounded in* $H^1(\Omega)^l$.
2. *The sequence* $\{A_2' \tilde{w}_m\}_{m=1}^{\infty}$ *is bounded in* $H^1(\Omega)^l$.

For a moment, let us assume Proposition 4.1. Then we have by (4.12), (4.13) and the Rellich compactness theorem that

$$A_1' p_m \to A_1' p, \quad A_2' \tilde{w}_m \to A_2' \tilde{w} \quad \text{strongly in } L^2(\Omega)^l,$$

which yields again by virtue of (4.12), (4.13) that

$$(A_1' p_m, A_1' \tilde{p}_m) \to (A_1' p, A_1' \tilde{p}), \quad (A_2' w_m, A_2' \tilde{w}_m) \to (A_2' w, A_2' \tilde{w}) \tag{4.14}$$

as $m \to \infty$. Now from (4.8), (4.9) and (4.14), we obtain (4.5).

Finally, it remains to prove Proposition 4.1.

Proof of Proposition 4.1 1. By (3.4), it holds $B_1(x,\nu)A_2'w_m = 0$ on $\partial\Omega$. Since $Pu_m \in D(S)$, we have $B_1(x,\nu)Pu_m = 0$ on $\partial\Omega$, which yields by virtue of (4.2) and (4.7) that

$$B_1(A_1'p_m) = B_1(Qu_m - A_2'w_m) = B_1(Qu_m) = B_1u_m, \quad m = 1, 2, \ldots.$$

Hence, it follows from (2.10), (4.10), (4.11) and the Assumption that

$$\begin{aligned}\|\nabla(A_1'p_m)\| &\leqq C(\|A_1A_1'p_m\| + \|A_1'p_m\| + \|B_1(A_1'p_m)\|_{H^{\frac{1}{2}}(\partial\Omega)})\\ &\leqq C(\|A_1u_m\| + \|Qu_m\| + \|B_1u_m\|_{H^{\frac{1}{2}}(\partial\Omega)})\\ &\leqq C(\|A_1u_m\| + \|u_m\| + \|B_1u_m\|_{H^{\frac{1}{2}}(\partial\Omega)}).\end{aligned}$$

Then by the hypotheses (i), (ii) and (iii), we have that

$$\sup_{m=1,\ldots}(\|\nabla(A_1'p_m)\| + \|A_1'p_m\|) < \infty,$$

which implies (1).

2. By (3.4) we have $B_1(x,\nu)A_2'w_m = 0$ on $\partial\Omega$. Hence, it follows from (2.10), (4.10), (4.11) and the Assumption that

$$\begin{aligned}\|\nabla(A_2'\tilde{w}_m)\| &\leqq C(\|A_2A_2'\tilde{w}_m\| + \|A_2'\tilde{w}_m\| + \|B_1(A_2'\tilde{w}_m)\|_{H^{\frac{1}{2}}(\partial\Omega)})\\ &\leqq C(\|A_2v_m\| + \|Qv_m\|)\\ &\leqq C(\|A_2v_m\| + \|v_m\|).\end{aligned}$$

Then by the hypotheses (i) and (ii), we have that

$$\sup_{m=1,\ldots}(\|\nabla(A_2'\tilde{w}_m)\| + \|A_2'\tilde{w}_m\|) < \infty,$$

which implies (2). This proves Proposition 4.1.

Case 2. We next consider the case when $\{B_2v_m\}_{m=1}^{\infty}$ is bounded in $H^{\frac{1}{2}}(\Omega)^{d_2}$. In such a case, we make use of the operator T. The proof is quite similar to that of the Case 1. However, for the reader's convenience, we give the complete proof. By Lemma 3.1 and the closed range theorem, we have an orthogonal decomposition

$$L^2(\Omega)^l = Ker(T) \oplus R(T^*), \tag{4.15}$$

where $\dim Ker(T) < \infty$. In the same way as in (4.2), let us denote by $\hat{P}$ and $\hat{Q}$ the orthogonal projections from $L^2(\Omega)^l$ onto $Ker(T)$ and $R(T^*)$ along (4.15), respectively. Since $\hat{P}$ is a finite rank operator, similarly to (4.5), it suffices to show that

$$(\hat{Q}u_m, \hat{Q}v_m) \to (\hat{Q}u, \hat{Q}v) \quad \text{as } m \to \infty. \tag{4.16}$$

By (3.2), there are functions $q, \tilde{q}, q_m, \tilde{q}_m$ with $A_1'q, A_1'\tilde{q}, A_1'q_m, A_1'\tilde{q}_m \in L^2(\Omega)^l$ and $B_1'q = B_1'\tilde{q} = B_1'q_m = B_1'\tilde{q}_m = 0$ on $\partial\Omega$, and functions $s, \tilde{s}, s_m, \tilde{s}_m \in L^2(\Omega)^{d_2}$ with $A_2's, A_2'\tilde{s}, A_2's_m, A_2'\tilde{s}_m \in L^2(\Omega)^l$ such that

$$\hat{Q}u = A_1'q + A_2's, \quad \hat{Q}v = A_1'\tilde{q} + A_2'\tilde{s},$$
$$\hat{Q}u_m = A_1'q_m + A_2's_m, \quad \hat{Q}v_m = A_1'\tilde{q}_m + A_2'\tilde{s}_m, \quad m = 1, 2, \ldots. \tag{4.17}$$

Then in the same way as in (4.8)–(4.11), we have that

$$(\hat{Q}u, \hat{Q}v) = (A_1'q, A_1'\tilde{q}) + (A_2's, A_2'\tilde{s}), \tag{4.18}$$
$$(\hat{Q}u_m, \hat{Q}v_m) = (A_1'q_m, A_1'\tilde{q}_m) + (A_2's_m, A_2'\tilde{s}_m), \tag{4.19}$$
$$\|\hat{Q}u_m\|^2 = \|A_1'q_m\|^2 + \|A_2's_m\|^2, \quad \|\hat{Q}v_m\|^2 = \|A_1'\tilde{q}_m\|^2 + \|A_2'\tilde{s}_m\|^2, \tag{4.20}$$
$$\|A_1u_m\| = \|A_1\hat{Q}u_m\| = \|A_1A_1'q_m\|, \quad \|A_2v_m\| = \|A_2\hat{Q}v_m\| = \|A_2A_2'\tilde{s}_m\| \tag{4.21}$$

for all $m = 1, 2, \ldots$. Indeed, (4.18), (4.19) and (4.20) are a consequence of (3.7). Since $u_m = \hat{P}u_m + \hat{Q}u_m$, $v_m = \hat{P}v_m + \hat{Q}v_m$ and since $\hat{P}u_m, \hat{P}v_m \in Ker(T)$, we have $A_1\hat{P}u_m = 0$ and $A_2\hat{P}v_m = 0$, and, hence, it follows from (4.17) and (2.10) that

$$A_1u_m = A_1\hat{Q}u_m = A_1A_1'q_m, \quad A_2v_m = A_2\hat{Q}v_m = A_2A_2'\tilde{s}_m,$$

which yields (4.21). In comparison with (4.12) and (4.13), we next show that

$$A_1'q_m \rightharpoonup A_1'q, \quad A_2's_m \rightharpoonup A_2's \quad \text{weakly in } L^2(\Omega)^l, \tag{4.22}$$
$$A_1'\tilde{q}_m \rightharpoonup A_1'\tilde{q}, \quad A_2'\tilde{s}_m \rightharpoonup A_2'\tilde{s} \quad \text{weakly in } L^2(\Omega)^l, \tag{4.23}$$

as $m \to \infty$. In fact, by (i) it is easy to see that $\hat{Q}u_m \rightharpoonup \hat{Q}u$ weakly in $L^2(\Omega)^l$. For every $\varphi \in L^2(\Omega)^l$, there exist $\psi \in L^2(\Omega)^{d_1}$ with $A_1'\psi \in L^2(\Omega)^l$ and $B_1'\psi = 0$ on $\partial\Omega$, and $\eta \in L^2(\Omega)^{d_2}$ with $A_2'\eta \in L^2(\Omega)^l$, such that

$$\hat{Q}\varphi = A_1'\psi + A_2'\eta.$$

Since $A_1'q_m, A_1'q \in R(T^*) = R(\hat{Q})$, it follows from (2.10), (3.7) and (4.17) that

$$\begin{aligned}
(A_1'q_m - A_1'q, \varphi) &= (A_1'q_m - A_1'q, \hat{Q}\varphi) \\
&= (A_1'q_m, A_1'\psi) - (A_1'q, A_1'\psi) \\
&= (\hat{Q}u_m, A_1'\psi) - (A_1'q, A_1'\psi) \\
&\to (\hat{Q}u, A_1'\psi) - (A_1'q, A_1'\psi) \\
&= (A_1'q, A_1'\psi) - (A_1'q, A_1'\psi) = 0, \\
(A_2's_m - A_2's, \varphi) &= (A_2's_m - A_2's, \hat{Q}\varphi) \\
&= (A_2's_m, A_2'\eta) - (A_2's, A_2'\eta) \\
&= (\hat{Q}u_m, A_2'\eta) - (A_2's, A_2'\eta) \\
&\to (\hat{Q}u, A_2'\eta) - (A_2's, A_2'\eta) \\
&= (A_2's, A_2'\eta) - (A_2's, A_2'\eta) = 0,
\end{aligned}$$

which implies (4.22). The proof of (4.23) can be done in the same way as above.

Similarly to Proposition 4.1, we need

Proposition 4.2. 1. *The sequence* $\{A_1' q_m\}_{m=1}^{\infty}$ *is bounded in* $H^1(\Omega)^l$.
2. *The sequence* $\{A_2' \tilde{s}_m\}_{m=1}^{\infty}$ *is bounded in* $H^1(\Omega)^l$.

For a moment, let us assume Proposition 4.2. Then we have by (4.22), (4.23) and the Rellich compactness theorem that

$$A_1' q_m \to A_1' q, \quad A_2' \tilde{s}_m \to A_2' \tilde{s} \quad \text{strongly in } L^2(\Omega)^l,$$

which yields again by virtue of (4.22), (4.23) that

$$(A_1' q_m, A_1' \tilde{q}_m) \to (A_1' q, A_1' \tilde{q}), \quad (A_2' s_m, A_2' \tilde{s}_m) \to (A_2' s, A_2' \tilde{s}) \tag{4.24}$$

as $m \to \infty$. Now from (4.18), (4.19) and (4.24), we obtain (4.16).

Finally, it remains to prove Proposition 4.2.

Proof of Proposition 4.2 1. By (3.6) we have $B_2(x, \nu) A_2' q_m = 0$ on $\partial\Omega$. Hence, it follows from (2.10), (4.20), (4.21) and the Assumption that

$$\begin{aligned}
\|\nabla(A_1' q_m)\| &\leqq C(\|A_1 A_1' q_m\| + \|A_1' q_m\| + \|B_2(A_1' q_m)\|_{H^{\frac{1}{2}}(\partial\Omega)}) \\
&\leqq C(\|A_1 u_m\| + \|\hat{Q} u_m\|) \\
&\leqq C(\|A_1 u_m\| + \|u_m\|).
\end{aligned}$$

Then by the hypotheses (i) and (ii), we have that

$$\sup_{m=1,\ldots} (\|\nabla(A_1' q_m)\| + \|A_1' q_m\|) < \infty,$$

which implies (1).

2. By (3.6), it holds that $B_2(x, \nu) A_1' \tilde{q}_m = 0$ on $\partial\Omega$. Since $\hat{P} v_m \in D(T)$, we have $B_2(x, \nu) \hat{P} v_m = 0$ on $\partial\Omega$. Hence, by (4.17) and the expression v_m as $v_m = \hat{P} v_m + \hat{Q} v_m$, it holds that

$$B_2(A_1' \tilde{s}_m) = B_2(\hat{Q} v_m - A_1' \tilde{q}_m) = B_2(\hat{Q} v_m) = B_2 v_m, \quad m = 1, 2, \ldots.$$

Hence it follows from (2.10), (4.20), (4.21) and the Assumption that

$$\begin{aligned}
\|\nabla(A_2' \tilde{s}_m)\| &\leqq C(\|A_2 A_2' \tilde{s}_m\| + \|A_2' \tilde{s}_m\| + \|B_2(A_2' \tilde{s}_m)\|_{H^{\frac{1}{2}}(\partial\Omega)}) \\
&\leqq C(\|A_2 v_m\| + \|\hat{Q} v_m\| + \|B_2 v_m\|_{H^{\frac{1}{2}}(\partial\Omega)}) \\
&\leqq C(\|A_2 v_m\| + \|v_m\| + \|B_2 v_m\|_{H^{\frac{1}{2}}(\partial\Omega)}).
\end{aligned}$$

Then by the hypotheses (i), (ii) and (iii), we have that

$$\sup_{m=1,\ldots} (\|\nabla(A_2' \tilde{s}_m)\| + \|A_2' \tilde{s}_m\|) < \infty,$$

which implies (2).

This proves Proposition 4.2 and the proof of Theorem 1 is now complete.

5. Applications

5.1. Global Div–Curl Lemma in Bounded Domains

The classical Div–Curl lemma deals with the convergence in the sense of distributions (see for example, TARTAR [18]). On the other hand, our global version makes it possible to treat the convergence in the whole domain up to the boundary. First, we consider the global Div–Curl lemma on 3-dimensional vector fields.

Corollary 5.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Suppose that $\{u_m\}_{m=1}^{\infty}$ and $\{v_m\}_{m=1}^{\infty}$ are sequences of* 3*-dimensional vector fields in Ω satisfying the following conditions* (i), (ii) *and* (iii).

(i) $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ *weakly in* $L^2(\Omega)^3$*;*
(ii) $\{\mathrm{div}\, u_m\}_{m=1}^{\infty}$ *is bounded in* $L^2(\Omega)$*, and* $\{\mathrm{rot}\, v_m\}_{m=1}^{\infty}$ *is bounded in* $L^2(\Omega)^3$*;*
(iii) *Either* $\{u_m \cdot \nu\}_{m=1}^{\infty}$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)$*, or* $\{v_m \times \nu\}_{m=1}^{\infty}$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)^3$*,*

where $v_m \times \nu$ denotes the standard vector product in $\mathbb{R}^3$. Then it holds that

$$\int_{\Omega} u_m(x) \cdot v_m(x)\, \mathrm{d}x \to \int_{\Omega} u(x) \cdot v(x)\, \mathrm{d}x \quad \text{as } m \to \infty,$$

where $u(x) \cdot v(x) = \sum_{j=1}^{3} u_j(x) v_j(x)$ is the standard scalar product in $\mathbb{R}^3$ at each point $x \in \Omega$.

Remark. In the hypothesis (iii), we do not need to assume both bounds of $\{u_m \cdot \nu\}_{m=1}^{\infty}$ and $\{v_m \times \nu\}_{m=1}^{\infty}$ on $\partial\Omega$. It is sufficient to assume that one of them is bounded. A more precise result in $L^r(\Omega)^3$ for $1 < r < \infty$ was established in our previous paper [9].

Proof of Corollary 5.1. Let us define differential operators A_1 and A_2 with the expression as in (2.1). For A_1, we take $l = n = 3$, $d_1 = 1$ and set

$$A_1 u \equiv \mathrm{div}\, u = \sum_{j=1}^{3} A_{1j}^{(1)}(x, D) u_j \quad \text{for } u = {}^t(u_1, u_2, u_3) \in H^1(\Omega)^3,$$

where $A_{1j}^{(1)}(x, D) \equiv \sum_{k=1}^{3} a_{1jk}^{(1)}(x) \frac{\partial}{\partial x_k} + b_{1j}^{(1)}(x)$ with $a_{1jk}^{(1)}(x) = \delta_{jk}, b_{1j}(x) = 0$, $j, k = 1, 2, 3$. Concerning A_2, we take $l = n = 3$, $d_2 = 3$ and set

$$A_2 v \equiv \mathrm{rot}\, v = {}^t\left(\sum_{j=1}^{3} A_{1j}^{(2)}(x, D) v_j, \sum_{j=1}^{3} A_{2j}^{(2)}(x, D) v_j, \sum_{j=1}^{3} A_{3j}^{(2)}(x, D) v_j\right)$$
$$\text{for } v = {}^t(v_1, v_2, v_3) \in H^1(\Omega)^3,$$

where $A_{ij}^{(2)}(x, D) \equiv \sum_{k=1}^3 a_{ijk}^{(2)}(x)\frac{\partial}{\partial x_k} + b_{ij}^{(2)}(x)$, $i, j = 1, 2, 3$ with

$$(a_{1jk}^{(2)})_{k\to 1,2,3}^{j\downarrow 1,2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (a_{2jk}^{(2)})_{k\to 1,2,3}^{j\downarrow 1,2,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$
$$(a_{3jk}^{(2)})_{k\to 1,2,3}^{j\downarrow 1,2,3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_{ij}^{(2)}(x) = 0, \quad i, j = 1, 2, 3. \tag{5.1}$$

Then it follows from (2.5) that

$$B_1 u = \sum_{j=1}^3 \nu_j u_j = \nu \cdot u, \quad u = {}^t (u_1, u_2, u_3), \tag{5.2}$$

$$B_2 v = {}^t(\nu_2 v_3 - \nu_3 v_2, \nu_3 v_1 - \nu_1 v_3, \nu_1 v_2 - \nu_2 v_1) = \nu \times v, \quad v = {}^t(v_1, v_2, v_3). \tag{5.3}$$

By DUVAUT and LIONS [4, Chapter VII Theorem 6.1] and [8, Theorem 2], we have

$$\|\nabla u\| \leqq C(\|\text{div } u\| + \|\text{rot } u\| + \|u\| + \|\nu \cdot u\|_{H^{\frac{1}{2}}(\Omega)}) \quad \text{for all } u \in H^1(\Omega)^3, \tag{5.4}$$

$$\|\nabla v\| \leqq C(\|\text{div } v\| + \|\text{rot } v\| + \|v\| + \|\nu \times v\|_{H^{\frac{1}{2}}(\Omega)}) \quad \text{for all } v \in H^1(\Omega)^3, \tag{5.5}$$

which implies that the estimates (2.8) and (2.9) in the Assumption hold.

We next show the cancellation property (2.10). For that purpose, we may prove (2.12). For $\alpha = 1, \beta = 2$, we have by (5.1) that

$$\sum_{j=1}^3 (a_{1js}^{(1)} a_{ijk}^{(2)} + a_{1jk}^{(1)} a_{ijs}^{(2)}) = \sum_{j=1}^3 (\delta_{js} a_{ijk}^{(2)} + \delta_{jk} a_{ijs}^{(2)})$$
$$= a_{isk}^{(2)} + a_{iks}^{(2)} = 0, \quad i, k, s = 1, 2, 3. \tag{5.6}$$

The case for $\alpha = 2, \beta = 1$ of (2.12) can be handled in the same way, so we obtain (2.12). Now the desired convergence is a consequence of Theorem 1. This proves Corollary 5.1.

The global version of Div–Curl lemma as in Corollary 5.1 can be generalized for the operators A_1 and A_2 with lower order terms. The cancellation property (2.10) plays an essential role for such generalization.

Corollary 5.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let $b = {}^t (b_1, b_2, b_3) \in C^1(\bar{\Omega})^3$ be an irrotational vector field in $\bar{\Omega}$, that is,* rot $b = 0$. *Suppose that $\{u_m\}_{m=1}^\infty$ and $\{v_m\}_{m=1}^\infty$ are sequences of* 3*-dimensional vector fields in Ω satisfying the following conditions* (i), (ii) *and* (iii).

(i) *$u_m \rightharpoonup u$, $v_m \rightharpoonup v$ weakly in $L^2(\Omega)^3$;*
(ii) $\{\text{div } u_m + b \cdot u_m\}_{m=1}^\infty$ *is bounded in $L^2(\Omega)$, and* $\{\text{rot } v_m + b \times v_m\}_{m=1}^\infty$ *is bounded in $L^2(\Omega)^3$;*
(iii) *Either $\{u_m \cdot \nu\}_{m=1}^\infty$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)$, or $\{v_m \times \nu\}_{m=1}^\infty$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)^3$.*

Then it holds that

$$\int_\Omega u_m(x)\cdot v_m(x)\,\mathrm{d}x \to \int_\Omega u(x)\cdot v(x)\,\mathrm{d}x \quad \textit{as } m\to\infty.$$

Proof. Let us take the differential operators A_1 and A_2 as

$$A_1(x,D)u \equiv \operatorname{div} u + b\cdot u,\quad A_2(x,D)v \equiv \operatorname{rot} v + b\times v \quad \text{for } u, v\in H^1(\Omega)^3.$$

Then the coefficients $a^{(1)}_{1jk}$ and $a^{(2)}_{ijk}$, $i, j, k = 1, 2, 3$, are the same as in the proof of Corollary 5.1. As for the coefficients $b^{(1)}_{1j}$ and $b^{(2)}_{ij}$, we may take

$$b^{(1)}_{1j} = b_j(x),\quad j=1,2,3 \quad (b^{(2)}_{ij})^{i\downarrow 1,2,3}_{j\to 1,2,3} = \begin{pmatrix} 0 & -b_3 & b_2\\ b_3 & 0 & -b_1\\ -b_2 & b_1 & 0\end{pmatrix}. \tag{5.7}$$

Obviously by (2.5) the trace operators B_1 and B_2 are the same as (5.2) and (5.3), respectively. Since $b\in C^1(\bar\Omega)^3$, the estimates (2.8) and (2.9) in the Assumption follow from (5.4) and (5.5), respectively.

We next show the cancellation property (2.10) which is equivalent to (2.12), (2.13) and (2.14). Indeed, we have seen that (2.12) is a consequence of (5.6). Since the coefficients $a^{(1)}_{1jk}$ and $a^{(2)}_{ijk}$ are constants in $\bar\Omega$ for all $i, j, k = 1, 2, 3$, we see that the left hand side of (2.13) for $\alpha = 1$, $\beta = 2$ can be reduced to

$$\sum_{j=1}^3 (a^{(1)}_{1jk} b^{(2)}_{ij} + a^{(2)}_{ijk} b^{(1)}_{1j}) = \sum_{j=1}^3 (\delta_{jk} b^{(2)}_{ij} + a^{(2)}_{ijk} b_j) = b^{(2)}_{ik} + \sum_{j=1}^3 a^{(2)}_{ijk} b_j,\quad i,k=1,2,3.$$

Hence by virtue of (5.1) and (5.7), it holds (2.13) for $\alpha = 1$, $\beta = 2$. The case for $\alpha = 2$, $\beta = 1$ can be handled in the same way, so we obtain (2.13). Concerning (2.14), we have by (5.1), (5.7) and the hypothesis $\operatorname{rot} b = 0$ that

$$\begin{aligned}
\sum_{j=1}^3\left(\sum_{\mu=1}^3 a^{(1)}_{1j\mu}\frac{\partial b^{(2)}_{ij}}{\partial x_\mu} + b^{(1)}_{1j} b^{(2)}_{ij}\right)_{i\downarrow 1,2,3} &= \sum_{j=1}^3\left(\sum_{\mu=1}^3 \delta_{j\mu}\frac{\partial b^{(2)}_{ij}}{\partial x_\mu} + b^{(2)}_{ij} b_j\right)_{i\downarrow 1,2,3}\\
&= -\operatorname{rot} b + b\times b\\
&= 0,\\
\sum_{j=1}^3\left(\sum_{\mu=1}^3 a^{(2)}_{rj\mu}\frac{\partial b^{(1)}_{1j}}{\partial x_\mu} + b^{(2)}_{rj} b^{(1)}_{1j}\right)_{r\downarrow 1,2,3} &= \sum_{j=1}^3\left(\sum_{\mu=1}^3 a^{(2)}_{rj\mu}\frac{\partial b_j}{\partial x_\mu} + b^{(2)}_{rj} b_j\right)_{r\downarrow 1,2,3}\\
&= \operatorname{rot} b + b\times b\\
&= 0,
\end{aligned}$$

which implies (2.14). Now the desired convergence is a consequence of Theorem 1. This proves Corollary 5.2.

Moreover, Corollary 5.2 can be generalized in n-dimensional vector fields. Indeed, we have

Corollary 5.3. . *Let $n \geqq 2$ and let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Assume that $b = {}^t(b_1, b_2, \ldots, b_n) \in C^1(\bar{\Omega})^n$ is an irrotational vector field in $\bar{\Omega}$, that is, $\partial b_j/\partial x_i - \partial b_i/\partial x_j = 0$ for all $1 \leqq i < j \leqq n$. Suppose that $\{u_m\}_{m=1}^{\infty}$ and $\{v_m\}_{m=1}^{\infty}$ are sequences of n-dimensional vector fields in Ω satisfying the following conditions* (i), (ii) *and* (iii).

(i) $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ *weakly in* $L^2(\Omega)^n$;

(ii) $\{\mathrm{div}\, u_m + b \cdot u_m\}_{m=1}^{\infty}$ *is bounded in* $L^2(\Omega)$, *and* $\{\frac{\partial v_{m,i}}{\partial x_j} - \frac{\partial v_{m,j}}{\partial x_i} + v_{m,i}b_j - v_{m,j}b_i\}_{m=1}^{\infty}$ *is bounded in* $L^2(\Omega)$ *for all* $1 \leqq i < j \leqq n$;

(iii) *Either* $\{u_m \cdot \nu\}_{m=1}^{\infty}$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)$, *or* $\{v_{m,i}\nu_j - v_{m,j}\nu_i\}_{m=1}^{\infty}$ *is bounded in* $H^{\frac{1}{2}}(\partial\Omega)$ *for all* $1 \leqq i < j \leqq n$.

Then it holds that

$$\int_{\Omega} u_m(x) \cdot v_m(x)\, \mathrm{d}x \to \int_{\Omega} u(x) \cdot v(x)\, \mathrm{d}x \quad \text{as } m \to \infty,$$

where $u(x) \cdot v(x) = \sum_{j=1}^{n} u_j(x) v_j(x)$ *is the standard scalar product in* $\mathbb{R}^n$ *at each point* $x \in \Omega$.

Proof. Let us define differential operators A_1 and A_2 with the expression as in (2.1). For A_1, we take $l = n$, $d_1 = 1$, and set

$$A_1 u = \mathrm{div}\, u + b \cdot u = \sum_{j=1}^{n} A_{1j}^{(1)}(x, D) u_j \quad \text{for } u = {}^t(u_1, \ldots, u_n) \in H^1(\Omega)^n,$$

where $A_{1j}^{(1)}(x, D) = \sum_{k=1}^{n} a_{1jk}^{(1)}(x) \frac{\partial}{\partial x_k} + b_{1j}^{(1)}(x)$, $j = 1, \ldots, n$ with

$$a_{1jk}^{(1)}(x) = \delta_{jk}, \quad b_{1j}^{(1)}(x) = b_j(x), \quad j, k = 1, \ldots, n. \tag{5.8}$$

Concerning A_2, we take $l = n$, $d_2 = n(n-1)/2$, and set

$$\begin{aligned} A_2(x, D)v &= \left(\frac{\partial v_j}{\partial x_k} - \frac{\partial v_k}{\partial x_j} + v_j b_k - b_j v_k\right)_{1 \leqq j < k \leqq n} \\ &= {}^t\left(\sum_{j=1}^{n} A_{1j}^{(2)}(x, D) v_j, \ldots, \sum_{j=1}^{n} A_{\frac{n(n-1)}{2}j}^{(2)}(x, D) v_j\right) \\ &\qquad \text{for } v = {}^t(v_1, \ldots, v_n) \in H^1(\Omega)^n, \end{aligned}$$

where $A_{ij}^{(2)}(x, D) = \sum_{k=1}^{n} a_{ijk}^{(2)}(x) \frac{\partial}{\partial x_k} + b_{ij}^{(2)}(x)$, $i = 1, \ldots, n(n-1)/2$, $j = 1, \ldots, n$ with the following expression. Let us define a positive integer $\sigma(n, l)$ by

$$\sigma(n, l) \equiv \frac{(l-1)(2n-l)}{2} \quad \text{for } l = 1, \ldots, n-1.$$

For $i = \sigma(n,l)+1, \ldots, \sigma(n,l)+n-l$ with $l = 1, 2, \ldots, n-1$, it holds

$$a_{ijk}^{(2)}(x) = \begin{cases} 1, & j = l, \quad k = i - \sigma(n,l) + l, \\ -1, & j = i - \sigma(n,l) + l, \quad k = l, \\ 0, & \text{otherwise}, \end{cases} \tag{5.9}$$

$$b_{ij}^{(2)}(x) = \begin{cases} b_{i-\sigma(n,l)+l}(x), & j = l, \\ -b_l(x), & j = i - \sigma(n,l) + l, \\ 0, & \text{otherwise}. \end{cases} \tag{5.10}$$

By (2.5) we see that

$$\begin{aligned} B_1 u &= \sum_{j=1}^{n} \nu_j u_j, \\ B_2 v &= \left(v_l \nu_{i-\sigma(n,l)+l} - v_{i-\sigma(n,l)+l} \nu_l\right)_{i=\sigma(n,l)+1,\ldots,\sigma(n,l)+n-l,\, l=1,\ldots,n-1} \\ &= (v_1\nu_2 - v_2\nu_1, v_1\nu_3 - v_3\nu_1, \ldots, v_{n-1}\nu_n - v_n\nu_{n-1}) \end{aligned}$$

for $u =^t (u_1, \ldots, u_n), v =^t (v_1, \ldots v_n) \in H^1(\Omega)^n$. Then it follows from Georgescu [5, Corollary 4.2.3] that the estimates (2.8) and (2.9) in the Assumption are fulfilled.

We next show (2.12), (2.13) and (2.14). Concerning (2.12) for $\alpha = 1, \beta = 2$, we have by (5.8), (5.9) and (5.10) that

$$\sum_{j=1}^{n} (a_{1js}^{(1)} a_{ijk}^{(2)} + a_{1jk}^{(1)} a_{ijs}^{(2)}) = \sum_{j=1}^{n} (\delta_{js} a_{ijk}^{(2)} + \delta_{jk} a_{ijs}^{(2)}) = a_{isk}^{(2)} + a_{iks}^{(2)} = 0$$

for all $s, k = 1, \ldots, n$, $i = 1, 2, \ldots, n(n-1)/2$. The case for $\alpha = 2, \beta = 1$ can be handled in the same way. As for (2.13) for $\alpha = 1, \beta = 2$, we have similarly to the above that

$$\begin{aligned} \sum_{j=1}^{n} (a_{1jk}^{(1)} b_{ij}^{(2)} + a_{ijk}^{(2)} b_{1j}^{(1)}) &= \sum_{j=1}^{n} (\delta_{jk} b_{ij}^{(2)} + a_{ijk}^{(2)} b_j) \\ &= b_{ik}^{(2)} + \sum_{j=1}^{n} a_{ijk}^{(2)} b_j \\ &= \begin{cases} 0, \quad k = 1, \ldots, l-1, \\ b_{i-\sigma(n,l)+l} - b_{i-\sigma(n,l)+l}, \quad k = l, \\ 0, \quad k = l+1, \ldots, i - \sigma(n,l) + l - 1, \\ -b_l + b_l, \quad k = i - \sigma(n,l) + l, \\ 0, \quad k = i - \sigma(n,l) + l + 1, \ldots, n \end{cases} \\ &= 0 \end{aligned}$$

for all $i = \sigma(n,l)+1, \ldots, \sigma(n,l)+n-l$ with $l = 1, \ldots, n-1$. Since $l = 1, \ldots, n-1$ is arbitrarily taken, this implies (2.13) for $\alpha = 1, \beta = 2$. The case $\alpha = 2, \beta = 1$ can be handled in the same way, so we obtain (2.13).

It remains to show (2.14). For $\alpha = 1$, $\beta = 2$, we have by (5.8), (5.9) and (5.10) that

$$\sum_{j=1}^{n}\left(\sum_{\mu=1}^{n} a_{1j\mu}^{(1)}\frac{\partial b_{ij}^{(2)}}{\partial x_\mu} + b_{1j}^{(1)}b_{ij}^{(2)}\right) = \sum_{j=1}^{n}\left(\sum_{\mu=1}^{n}\delta_{j\mu}\frac{\partial b_{ij}^{(2)}}{\partial x_\mu} + b_{ij}^{(2)}b_j\right) = \sum_{j=1}^{n}\frac{\partial b_{ij}^{(2)}}{\partial x_j} + \sum_{j=1}^{n} b_{ij}^{(2)}b_j \tag{5.11}$$

for $i = 1, \ldots, n(n-1)/2$. For $i = \sigma(n,l)+1, \ldots, \sigma(n,l)+n-l$ with $l = 1, \ldots, n-1$ we have by (5.9) that

$$\sum_{j=1}^{n} b_{ij}^{(2)}b_j = b_{il}^{(2)}b_l + b_{ii-\sigma(n,l)+l}^{(2)}b_{i-\sigma(n,l)+l} = b_{i-\sigma(n,l)+l}b_l - b_l b_{i-\sigma(n,l)+l} = 0.$$

Since $l = 1, \ldots, n-1$ is arbitrarily taken, it holds that

$$\sum_{j=1}^{n} b_{ij}^{(2)}b_j = 0 \quad \text{for all } i = 1, \ldots, n(n-1)/2. \tag{5.12}$$

Since $\partial b_j/\partial x_k - \partial b_k/\partial x_j = 0$, $1 \leqq j < k \leqq n$, we have by (5.10) that

$$\sum_{j=1}^{n}\frac{\partial b_{ij}^{(2)}}{\partial x_j} = \frac{\partial b_{il}^{(2)}}{\partial x_l} + \frac{\partial b_{ii-\sigma(n,l)+l}}{\partial x_{i-\sigma(n,l)+l}} = \frac{\partial b_{i-\sigma(n,l)+l}}{\partial x_l} - \frac{\partial b_l}{\partial x_{i-\sigma(n,l)+l}} = 0$$

for $i = \sigma(n,l)+1, \ldots, \sigma(n,l)+n-l$ with $l = 1, \ldots, n-1$. Since $l = 1, \ldots, n-1$ is arbitrarily taken, this implies that

$$\sum_{j=1}^{n}\frac{\partial b_{ij}^{(2)}}{\partial x_j} = 0 \quad \text{for all } i = 1, \ldots, n(n-1)/2. \tag{5.13}$$

Hence, from (5.11), (5.12) and (5.13) we obtain (2.14) for $\alpha = 1$, $\beta = 2$. In the case for $\alpha = 2$, $\beta = 1$, we have by (5.12) that

$$\begin{aligned}\sum_{j=1}^{n}\left(\sum_{\mu=1}^{n} a_{rj\mu}^{(2)}\frac{\partial b_{1j}^{(1)}}{\partial x_\mu} + b_{rj}^{(2)}b_{1j}^{(1)}\right) &= \sum_{j,\mu=1}^{n} a_{rj\mu}^{(2)}\frac{\partial b_j}{\partial x_\mu} + \sum_{j=1}^{n} b_{rj}^{(2)}b_j \\ &= \sum_{j,\mu=1}^{n} a_{rj\mu}^{(2)}\frac{\partial b_j}{\partial x_\mu} \\ &= \frac{\partial b_l}{\partial x_{r-\sigma(n,l)+l}} - \frac{\partial b_{r-\sigma(n,l)+l}}{\partial x_l} \\ &= 0\end{aligned}$$

for $r = \sigma(n,l)+1, \ldots, \sigma(n,l)+n-l$ with $l = 1, \ldots, n-1$. Since $l = 1, \ldots, n-1$ is arbitrarily taken, this implies (2.14) for $\alpha = 2$, $\beta = 1$. Now the desired convergence follows from Theorem 1. This completes the proof of Corollary 5.3.

5.2. Global Div–Curl Lemma on Riemaniann Manifolds with Boundary

Let $(\bar{\Omega}, g)$ be a compact n-dimensional Riemannian manifold with smooth boundary $\partial\Omega$. We regard $\partial\Omega$ as a C^∞-sub-manifold of $\bar{\Omega}$. Then there is a canonical inclusion $\bigwedge T_x(\partial\Omega) \hookrightarrow \bigwedge T_x\bar{\Omega}$, where T_xM is the tangent space of the manifold M at $x \in M$, and, where $\bigwedge T_xM \equiv \oplus_{l=0}^{n} \bigwedge^l T_xM$. Notice that $\bigwedge^l T_xM$ is the l-exterior product of T_xM. For each $x \in \partial\Omega$, let us denote by ν_x the vector in $T_x\bar{\Omega}$ which is orthogonal to $T_x(\partial\Omega)$ and oriented toward the exterior of Ω, and which has the norm 1. For every l-form u on $\bar{\Omega}$, that is, $u \in \bigwedge^l(T\bar{\Omega})$, we define its tangential part τu and its normal part νu as

$$\tau u = \nu\rfloor(\nu \wedge u), \quad \nu u = \nu\rfloor u, \tag{5.14}$$

where $\nu\rfloor : \bigwedge^l(T\bar{\Omega}) \to \bigwedge^{l-1}(T\bar{\Omega}), l = 1, \ldots, n$, is the interior product defined by

$$(\nu\rfloor u)(X_1, \ldots, X_{l-1}) = u(X_1, \ldots, X_{l-1}, \nu) \quad \text{for } X_1, \ldots, X_{l-1} \in T\bar{\Omega}.$$

Then it holds the identity

$$u = \tau u + \nu \wedge (\nu u) \quad \text{for all } u \in \textstyle\bigwedge^l(T\bar{\Omega}).$$

Let us denote by $d : \bigwedge^l(T\bar{\Omega}) \to \bigwedge^{l+1}(T\bar{\Omega}), l = 0, 1, \ldots, n-1$, the exterior derivative and by $* : \bigwedge^l(T\bar{\Omega}) \to \bigwedge^{n-l}(T\bar{\Omega}), l = 0, 1, \ldots, n$, the Hodge star operator, respectively. We define the codifferential operator $\delta : \bigwedge^l(T\bar{\Omega}) \to \bigwedge^{l-1}(T\bar{\Omega}), l = 1, \ldots, n$, by $\delta = (-1)^{n+1} * d * \chi^n$, where $\chi u = (-1)^l u$ for $u \in \bigwedge^l(T\bar{\Omega})$. It is known that $\bigwedge^l(T\bar{\Omega}), l = 0, 1, \ldots, n$, has a Hilbert structure with the scalar product $(\cdot, \cdot)$ such as

$$(u, v) \equiv \int_\Omega u \wedge *v, \quad \text{for } u, v \in \textstyle\bigwedge^l(T\bar{\Omega}). \tag{5.15}$$

Based on this scalar product on $\bigwedge^l(T\bar{\Omega})$, we may define the Lebesgue space $L^2(\Omega)^l$ and the Sobolev space $H^1(\Omega)^l$. See, for example, Morrey [11].

We next consider the generalized Stokes formula on $(\bar{\Omega}, g)$ corresponding to (2.6) and (2.7). Let us introduce two spaces $H_d(\Omega)^{l-1}$ and $H_\delta(\Omega)^l$ for $l = 1, \ldots, n$, by

$$\begin{aligned} H_d(\Omega)^{l-1} &\equiv \{u \in L^2(\Omega)^{l-1};\, du \in L^2(\Omega)^l\}, \\ H_\delta(\Omega)^l &\equiv \{v \in L^2(\Omega)^l;\, \delta v \in L^2\Omega)^{l-1}\}. \end{aligned} \tag{5.16}$$

Then the boundary operators τ and ν defined by (5.14) can be extended uniquely as continuous linear operators

$$\begin{aligned} \tau : u \in H_d(\Omega)^{l-1}(\Omega) &\to \tau u \in H^{-\frac{1}{2}}(\partial\Omega)^{l-1}, \\ \nu : v \in H_\delta(\Omega)^l &\to \nu v \in H^{-\frac{1}{2}}(\partial\Omega)^{l-1}, \end{aligned} \tag{5.17}$$

where $H^{-\frac{1}{2}}(\partial\Omega)^{l-1}$ is the dual space of $H^{\frac{1}{2}}(\partial\Omega)^{l-1}$. Moreover, it holds by the generalized Stokes formula that

$$(du, v) - (u, \delta v) = \langle \tau u, \nu v\rangle_{\partial\Omega}, \quad l = 1, \ldots, n \tag{5.18}$$

for all $u \in H_d(\Omega)^{l-1}$ and $v \in H^1(\Omega)^l$, or for all $u \in H^1(\Omega)^{l-1}$ and $v \in H_\delta(\Omega)^l$, where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)^{l-1}$ and $H^{\frac{1}{2}}(\partial\Omega)^{l-1}$. For details, we refer to MORREY [11, Lemma 7.5.3] and GEORGESCU [5, Theorem 4.1.8].

An application of our theorem to the Div–Curl lemma now reads:

Corollary 5.4. *Let $(\bar{\Omega}, g)$ be an n-dimensional compact Riemannian manifold with smooth boundary $\partial\Omega$. Suppose that $\{u_m\}_{m=1}^\infty$ and $\{v_m\}_{m=1}^\infty$ are sequences of $L^2(\Omega)^l$ for $l = 1, \ldots, n-1$. We assume the following three hypotheses (i), (ii) and (iii).*

(i)

$$u_m \rightharpoonup u, \quad v_m \rightharpoonup v \quad \textit{weakly in } L^2(\Omega)^l;$$

(ii) *$\{du_m\}_{m=1}^\infty$ is bounded in $L^2(\Omega)^{l+1}$, and $\{\delta v_m\}_{m=1}^\infty$ is bounded in $L^2(\Omega)^{l-1}$;*

(iii) *Either $\{\tau u_m\}_{m=1}^\infty$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)^l$, or $\{\nu v_m\}_{m=1}^\infty$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)^{l-1}$.*

Then it holds that

$$(u_m, v_m) \to (u, v) \quad \textit{as } m \to \infty,$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)^l$ defined by (5.15).

Proof. Since $(\bar{\Omega}, g)$ is not the Euclidean space, but a compact Riemannian manifold with boundary $\partial\Omega$, we cannot apply Theorem 1 directly to Corollary 5.4. Indeed, although we take $A_1 = d$ and $A_2 = \delta$ in (2.1), it is impossible to define the boundary operators B_1 and B_2 so that the identity (2.3) holds. However, based on the generalized Stokes formula (5.18), we shall establish a proof of Corollary 5.4 with a certain modification of that of Theorem 1.

The boundary operators τ and ν in (5.17) play a substitutive role for B_1 and B_2 in (2.6). In fact, concerning the Assumption, it follows from GEROGESCU [5, Corollary 4.2.3] that

$$\|\nabla u\| \leqq C(\|du\| + \|\delta u\| + \|u\| + \|\tau u\|_{H^{\frac{1}{2}}(\partial\Omega)}), \tag{5.19}$$

$$\|\nabla u\| \leqq C(\|du\| + \|\delta u\| + \|u\| + \|\nu u\|_{H^{\frac{1}{2}}(\partial\Omega)}) \tag{5.20}$$

for all $u \in H^1(\Omega)^l$. Let us define two operators S and T by

$$D(S) = \{u \in H^1(\Omega)^l; \tau u = 0 \quad \text{on } \partial\Omega\}, \quad Su \equiv^t (du, \delta u) \quad \text{for } u \in D(S),$$
$$D(T) = \{u \in H^1(\Omega)^l; \nu u = 0 \quad \text{on } \partial\Omega\}, \quad Tu \equiv^t (du, \delta u) \quad \text{for } u \in D(T).$$

Similarly to (3.1) and (3.2), we have by the generalized Stokes formula (5.18), that

$$D(S^*) = \{{}^t(p, w) \in H_\delta(\Omega)^{l+1} \times H_d(\Omega)^{l-1}; \tau w = 0 \quad \text{on } \partial\Omega\},$$
$$S^*({}^t(p, w)) = \delta p + dw \quad \text{for } {}^t(p, w) \in D(S^*), \tag{5.21}$$
$$D(T^*) = \{{}^t(p, w) \in H_\delta(\Omega)^{l+1} \times H_d(\Omega)^{l-1}; \nu p = 0 \quad \text{on } \partial\Omega\},$$
$$T^*({}^t(p, w)) = \delta p + dw \quad \text{for } {}^t(p, w) \in D(T^*). \tag{5.22}$$

Then similarly to Lemma 3.1, we have by (5.19) and (5.20) the following proposition.

Proposition 5.1. 1. *The kernels $Ker(S)$ and $Ker(T)$ of S and T are both finite dimensional subspaces of $L^2(\Omega)^l$.*
2. *The ranges $R(S)$ and $R(T)$ of S and T are both closed subspaces of $L^2(\Omega)^{l+1} \times L^2(\Omega)^{l-1}$.*

As for cancellation property (2.10), we make use of the well-known fact that

$$d^2 = 0, \quad \delta^2 = 0. \tag{5.23}$$

Instead of Lemma 3.2, we have the following proposition.

Proposition 5.2. 1. *If $w \in H_d(\Omega)^{l-1}$ satisfies $\tau w = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)^{l-1}$, then it holds that*

$$\tau(dw) = 0 \quad \text{on } \partial\Omega$$

with the identity

$$(\delta p, dw) = 0 \quad \text{for all } p \in H_\delta(\Omega)^{l+1}. \tag{5.24}$$

2. *If $p \in H_\delta(\Omega)^{l+1}$ satisfies $\nu p = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)^l$, then it holds that*

$$\nu(\delta p) = 0 \quad \text{on } \partial\Omega$$

with the identity

$$(\delta p, dw) = 0 \quad \text{for all } w \in H_d(\Omega)^{l-1}. \tag{5.25}$$

In the case when $w \in C^1(\bar{\Omega})^{l-1}$ and $p \in C^1(\bar{\Omega})^{l+1}$, this proposition is shown by Morrey [11, Lemma 7.5.2].

Proof of Proposition 5.2 1. For every $q \in C^\infty(\partial\Omega)^l$, there is an $\omega \in C^\infty(\bar{\Omega})^{l+1}$ such that $q = \nu\omega$, and, hence, it follows from (5.18) and (5.23) that

$$\begin{aligned}\langle \tau(dw), q\rangle_{\partial\Omega} &= \langle \tau(dw), \nu\omega\rangle_{\partial\Omega}\\ &= (d(dw), \omega) - (dw, \delta\omega)\\ &= -(dw, \delta\omega)\\ &= -(w, \delta(\delta\omega)) - \langle \tau w, \nu(\delta\omega)\rangle_{\partial\Omega}\\ &= 0.\end{aligned}$$

Since $q \in C^\infty(\partial\Omega)^l$ is arbitrarily taken, and since $C^\infty(\partial\Omega)^l$ is dense in $H^{\frac{1}{2}}(\partial\Omega)^l$, we obtain from the above that $\tau(dw) = 0$ on $\partial\Omega$.

We next show (5.24). Since $\tau w = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)^{l-1}$, we have by (5.18) that

$$(\delta p, dw) = (\delta(\delta p), w) + \langle \nu(\delta p), \tau w\rangle_{\partial\Omega} = 0 \quad \text{for all } p \in C^\infty(\bar{\Omega})^{l+1}.$$

Since $C^\infty(\bar{\Omega})^{l+1}$ is dense in $H_\delta(\Omega)^{l+1}$ (see GEROGESCU [5, Lemma 4.1.7]), the above identity yields (5.24).

2. For every $\varphi \in C^\infty(\partial\Omega)^{l-1}$, there exists an $\eta \in C^\infty(\bar{\Omega})^{l-1}$ such that $\varphi = \tau\eta$ on $\partial\Omega$. Hence, it follows from (5.18) and (5.23) that

$$\begin{aligned}\langle \nu(\delta p), \varphi\rangle_{\partial\Omega} &= \langle \nu(\delta p), \tau\eta\rangle_{\partial\Omega}\\ &= -(\delta(\delta p), \eta) + (\delta p, d\eta)\\ &= (\delta p, d\eta)\\ &= (p, d(d\eta)) - \langle \nu p, \tau(d\eta)\rangle_{\partial\Omega}\\ &= 0.\end{aligned}$$

Since $\varphi \in C^\infty(\partial\Omega)^{l-1}$ is arbitrarily taken and since $C^\infty(\partial\Omega)^{l-1}$ is dense in $H^{\frac{1}{2}}(\partial\Omega)^{l-1}$, we obtain from the above that $\nu(\delta p) = 0$ on $\partial\Omega$.

We next show (5.25). Since $\nu p = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)^l$, we have by (5.18) that

$$(\delta p, dw) = (p, d(dw)) - \langle \nu p, \tau(dw)\rangle_{\partial\Omega} = 0 \quad \text{for all } w \in C^\infty(\bar{\Omega})^{l-1}.$$

Since $C^\infty(\bar{\Omega})^{l-1}$ is dense in $H_d(\Omega)^{l-1}$ (see also GEROGESCU [5, Lemma 4.1.7]), the above identity yields (5.25). This proves Proposition 5.2.

Completion of the proof of Corollary 5.4 Since Propositions 5.1 and 5.2 play a substitutive role for Lemmata 3.1 and 3.2, respectively, the argument in Section 4 is applicable to the proof of Corollary 5.4 for $A_1 = d$, $A_2 = \delta$ with B_1 and B_2 replaced by $B_1 u = \tau u$ and $B_2 v = \nu v$. However, for the reader's convenience, we shall give a complete proof. Let us consider first the case when $\{\tau u_m\}_{m=1}^\infty$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)^l$. By Proposition 5.1, we have the orthogonal decomposition (4.1) for S and S^* in (5.21). Since the projection $P : L^2(\Omega)^l \to Ker(S)$ is a finite rank operator, we have (4.4), and, hence, it remains to prove (4.5). Similarly to (4.6) and (4.7), there exist $p, \tilde{p}, p_m, \tilde{p}_m \in H_\delta(\Omega)^{l+1}$ and $w, \tilde{w}, w_m, \tilde{w}_m \in H_d(\Omega)^{l-1}$ with $\tau w = \tau\tilde{w} = \tau w_m = \tau\tilde{w}_m = 0$ on $\partial\Omega$ such that

$$Qu = \delta p + dw, \quad Qv = \delta\tilde{p} + d\tilde{w}, \tag{5.26}$$

$$Qu_m = \delta p_m + dw_m, \quad Qv_m = \delta\tilde{p}_m + d\tilde{w}_m, \quad m = 1, 2, \ldots, \tag{5.27}$$

where Q is the projection from $L^2(\Omega)^l$ onto $R(S^*)$. In the same way as in (4.8)–(4.13), we have by (5.23) and Proposition 5.2 (1) that

$$(Qu, Qv) = (\delta p, \delta \tilde{p}) + (dw, d\tilde{w}), \quad (Qu_m, Qv_m) = (\delta p_m, \delta \tilde{p}_m) + (dw_m, d\tilde{w}_m), \tag{5.28}$$

$$\|Qu_m\|^2 = \|\delta p_m\|^2 + \|dw_m\|^2, \quad \|Qv_m\|^2 = \|\delta \tilde{p}_m\|^2 + \|d\tilde{w}_m\|^2, \tag{5.29}$$

$$\|du_m\| = \|d\delta p_m\|, \quad \|\delta v_m\| = \|\delta d\tilde{w}_m\| \tag{5.30}$$

for all $m = 1, 2, \ldots$ and that

$$\delta p_m \rightharpoonup \delta p, \quad \delta \tilde{p}_m \rightharpoonup \delta \tilde{p}, \quad dw_m \rightharpoonup dw, \quad d\tilde{w}_m \rightharpoonup d\tilde{w} \quad \text{weakly in } L^2(\Omega)^l \tag{5.31}$$

as $m \to \infty$. Notice that $dP\alpha = 0$ and $\delta P\alpha = 0$ for all $\alpha \in L^2(\Omega)^l$.

Moreover, similarly to Proposition 4.1, we have

Proposition 5.3. 1. *The sequence* $\{\delta p_m\}_{m=1}^{\infty}$ *is bounded in* $H^1(\Omega)^l$.
2. *The sequence* $\{d\tilde{w}_m\}_{m=1}^{\infty}$ *is bounded in* $H^1(\Omega)^l$.

For a moment, let us assume this proposition. Then by (5.31) and the Rellich compactness theorem we have that

$$\delta p_m \to \delta p, \quad d\tilde{w}_m \to d\tilde{w} \quad \text{strongly in } L^2(\Omega)^l,$$

and, hence, again by (5.31) and (5.28) it holds that

$$(Qu_m, Qv_m) \to (Qu, Qv) \quad \text{as } m \to \infty,$$

which implies (4.5).

Now, it remains to prove Proposition 5.3.

Proof of Proposition 5.3. 1. Since $Pu_m \in Ker(S) \subset D(S)$, we have $\tau(Pu_m) = 0$ on $\partial\Omega$. Hence, it follows from (5.27) and Proposition 5.2 (1) that

$$\tau(\delta p_m) = \tau(Qu_m - dw_m) = \tau(Qu_m) = \tau u_m, \quad m = 1, 2, \ldots.$$

By (5.19), (5.23), (5.29) and (5.30) we have

$$\begin{aligned}
\|\nabla(\delta p_m)\| + \|\delta p_m\| &\leqq C(\|d(\delta p_m)\| + \|\delta(\delta p_m)\| + \|\delta p_m\| + \|\tau(\delta p_m)\|_{H^{\frac{1}{2}}(\partial\Omega)}) \\
&\leqq C(\|du_m\| + \|Qu_m\| + \|\tau u_m\|_{H^{\frac{1}{2}}(\partial\Omega)}) \\
&\leqq C(\|du_m\| + \|u_m\| + \|\tau u_m\|_{H^{\frac{1}{2}}(\partial\Omega)})
\end{aligned}$$

for all $m = 1, 2, \ldots$. Then by the hypotheses (i), (ii) and (iii), it holds that

$$\sup_{m=1,2,\ldots} (\|\nabla(\delta p_m)\| + \|\delta p_m\|) < \infty,$$

which implies the assertion (1).

2. By Proposition 5.2 (1), we have $\tau \tilde{w}_m = 0$ on $\partial\Omega$, and, hence, it follows from (5.19), (5.23), (5.29) and (5.30) that

$$\begin{aligned}\|\nabla(d\tilde{w}_m)\| + \|d\tilde{w}_m\| &\leqq C(\|d(d\tilde{w}_m)\| + \|\delta(d\tilde{w}_m)\| + \|d\tilde{w}_m\| + \|\tau(d\tilde{w}_m)\|_{H^{\frac{1}{2}}(\partial\Omega)}) \\ &\leqq C(\|\delta v_m\| + \|Qv_m\|) \\ &\leqq C(\|\delta v_m\| + \|v_m\|)\end{aligned}$$

for all $m = 1, 2, \ldots$. Then by the hypotheses (i) and (ii), it holds that

$$\sup_{m=1,2,\ldots} (\|\nabla(d\tilde{w}_m)\| + \|d\tilde{w}_m\|) < \infty,$$

which implies the assertion (2). This proves Proposition 5.3.

Now, we complete the proof of Corollary 5.4 under the hypothesis that $\{\tau u_m\}_{m=1}^{\infty}$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)^l$. In the case when $\{\nu v_m\}_{m=1}^{\infty}$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)^l$, instead of S and S^*, we make use of the operators T and T^* in (5.22) with the aid of Propositions 5.1 and 5.2 (2). Since the argument of the proof is quite parallel to that of the above case, we may omit it. This completes the proof of Corollary 5.4.

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