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Serrin-Type Blowup Criterion for Full Compressible Navier–Stokes System

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Abstract

The authors establish a Serrin-type blowup criterion for the Cauchy problem of the three-dimensional full compressible Navier–Stokes system, which states that a strong or smooth solution exists globally, provided that the velocity satisfies Serrin's condition and that the temporal integral of the maximum norm of the divergence of the velocity is bounded. In particular, this criterion extends the well-known Serrin's blowup criterion for the three-dimensional incompressible Navier–Stokes equations to the three-dimensional full compressible system and is just the same as that of the barotropic case.

1. Introduction

The motion of a compressible viscous, heat-conductive, ideal polytropic fluid occupying a spatial domain $\Omega \subset \mathbb{R}^3$ is governed by the following full compressible Navier–Stokes system:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla \mathrm{div} u + \nabla P = 0, \\ c_v[(\rho\theta)_t + \mathrm{div}(\rho u \theta)] - \kappa \Delta \theta + P \mathrm{div} u = 2\mu|\mathfrak{D}(u)|^2 + \lambda(\mathrm{div} u)^2. \end{cases} \tag{1.1}$$

Here $t \geqq 0$ is time, $x \in \Omega$ is the spatial coordinate, and $\rho, u = \left(u^1, u^2, u^3\right)^{\mathrm{tr}}, \theta$, and $P = R\rho\theta$ $(R > 0)$ represent, respectively, the fluid density, velocity, absolute temperature, and pressure. In addition, $\mathfrak{D}(u)$ is the deformation tensor

$$\mathfrak{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{tr}}).$$

The constant viscosity coefficients μ and λ satisfy the physical restrictions

$$\mu > 0, \quad 2\mu + 3\lambda \geqq 0. \tag{1.2}$$

Positive constants c_v and κ are, respectively, the heat capacity and the ratio of the heat conductivity coefficient over the heat capacity.

Let $\Omega = \mathbb{R}^3$. For constants $\tilde{\rho} \geqq 0$ and $\tilde{\theta} \geqq 0$, we consider the Cauchy problem to (1.1) with the far field behavior:

$$(\rho, u, \theta)(x, t) \to (\tilde{\rho}, 0, \tilde{\theta}) \text{ as } |x| \to \infty, \tag{1.3}$$

and initial data

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0), \quad x \in \mathbb{R}^3. \tag{1.4}$$

There is a considerable body of literature on the large time existence and behavior of solutions to (1.1). The one-dimensional problem with strictly positive initial density and temperature has been studied extensively by many people, see [15,16] and the references therein. For the multi-dimensional case, the local existence and uniqueness of classical solutions are shown in [19,22] in the absence of vacuum. Recently, Cho and Kim [2] obtained the local existence and uniqueness of strong solutions for the case in which the initial density need not be positive and may vanish in open sets. The global classical solutions were first obtained by Matsumura and Nishida [18] for initial data close to a non-vacuum equilibrium in some Sobolev space H^s. Later, Hoff [6,7] studied the global weak solutions with strictly positive initial density and temperature for discontinuous initial data.

On the other hand, in the presence of a vacuum, this issue becomes much more complicated. Concerning viscous compressible fluids in a barotropic regime, where the state of these fluids at each instant $t > 0$ is completely determined by the density $\rho = \rho(x, t)$ and the velocity $u = u(x, t)$, the pressure P being an explicit function of the density, the major breakthrough is due to Lions [17] (see also Feireisl [4,5]), who obtained global existence of weak solutions, defined as solutions with finite energy, when the pressure $P(\rho) = a\rho^\gamma (a > 0, \gamma > 1)$ with suitably large γ. The main restriction on initial data is that the initial energy is finite. Recently, Huang et al. [13] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier–Stokes equations in three-dimensional space with smooth initial data which are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish, and even has compact support. This result can be regarded as a uniqueness and regularity theory of Lions–Feireisl's weak solutions in [4,5,17] with small initial energy. More recently, for $\tilde{\rho} > 0$ and $\tilde{\theta} > 0$, Huang and Li [10] obtained both the global classical solutions (which are unique) and the global weak ones to the Cauchy problem (1.1)–(1.4) which may have large oscillations, provided the initial energy is suitably small; in particular, the initial density may contain vacuum states. For $\tilde{\rho} = 0$ and $\tilde{\theta} = 0$, Xin [26] first showed that in the case where the initial density has compact support, any smooth solution to the Cauchy problem of the full compressible Navier–Stokes system without heat conduction blows up in finite time. See also the recent generalizations to the cases for non-compact but rapidly decreasing at far field initial densities [21].

It is thus important to study the mechanism of blowup and the structure of possible singularities of strong (or smooth) solutions to the full compressible Navier–Stokes system (1.1). The pioneering work can be traced to Serrin's criterion [23] on the Leray–Hopf weak solutions to the three-dimensional incompressible Navier–Stokes equations, which states that if a weak solution u satisfies $u \in L^s(0, T; L^r)$, with

$$\frac{2}{s} + \frac{3}{r} \leqq 1, \quad 3 < r \leqq \infty, \tag{1.5}$$

then it is regular. Recently, this criterion was extended to the barotropic compressible Navier–Stokes equations by Huang et al. [11], who showed that if $T^* < \infty$ is the maximal time of existence of a strong (or classical) solution (ρ, u), then

$$\lim_{T \to T^*} \left(\|\mathrm{div} u\|_{L^1(0,T;L^\infty)} + \|\rho^{1/2} u\|_{L^s(0,T;L^r)} \right) = \infty,$$

with r and s as in (1.5). For more information on the blowup criteria of barotropic compressible flows, we refer to [8,11,12,14,24] and the references therein. In particular, Huang [8] and Huang–Xin [14] first established a blowup criterion, analogous to the Beale–Kato–Majda criterion for the ideal incompressible flows, for the strong and classical solutions to the viscous compressible barotropic flows

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty} \, \mathrm{d}t = \infty,$$

provided

$$7\mu > \lambda. \tag{1.6}$$

Later Fan et al. [3] extended the results of [8,14] to the full compressible Navier–Stokes system (1.1), that is, under the condition (1.6), if $T^* < \infty$ is the maximal time of existence of a strong (or classical) solution (ρ, u, θ), then

$$\lim_{T \to T^*} (\|\theta\|_{L^\infty(0,T;L^\infty)} + \|\nabla u\|_{L^1(0,T;L^\infty)}) = \infty.$$

Recently, under just the physical restrictions (1.2), Huang and Li [9] and Huang et al. [12] succeeded in removing the crucial condition (1.6) of [3,8,14] and in establishing the following blowup criterion:

$$\lim_{T \to T^*} \left(\|\theta\|_{L^2(0,T;L^\infty)} + \|\mathfrak{D}(u)\|_{L^1(0,T;L^\infty)} \right) = \infty,$$

where $\mathfrak{D}(u)$ is the deformation tensor. More recently, in the absence of a vacuum, Sun et al. [25] showed that

$$\lim_{T \to T^*} \left(\|\theta\|_{L^\infty(0,T;L^\infty)} + \left\| \left(\rho, \rho^{-1}\right) \right\|_{L^\infty(0,T;L^\infty)} \right) = \infty,$$

provided that (1.6) holds.

The aim of this paper is to improve all the previous blowup criterion results for the full compressible Navier–Stokes system (1.1) by removing the stringent condition (1.6), by allowing initial vacuum states, and furthermore, by describing the blowup mechanism only in terms of a Serrin-type criterion. Before stating our main result, we first explain the notations and conventions used throughout this paper. We denote

$$\int f \, \mathrm{d}x = \int_{\mathbb{R}^3} f \, \mathrm{d}x.$$

For $1 \leqq p \leqq \infty$ and integer $k \geqq 0$, the standard homogeneous and inhomogeneous Sobolev spaces are denoted by:

$$\begin{cases} L^p = L^p(\mathbb{R}^3), \quad W^{k,p} = W^{k,p}(\mathbb{R}^3), \quad H^k = W^{k,2}, \\ D^1 = \left\{ u \in L^6 \,\middle|\, \|\nabla u\|_{L^2} < \infty \right\}, \ D^{k,p} = \left\{ u \in L^1_{loc}(\mathbb{R}^3) \,\middle|\, \|\nabla^k u\|_{L^p} < \infty \right\}. \end{cases}$$

Then, the strong solutions to the Cauchy problem, (1.1)–(1.4), are defined as follows.

Definition 1. (*Strong Solutions*) For $\tilde{\rho} \geqq 0$ and $\tilde{\theta} = 0$, (ρ, u, θ) is called a strong solution to (1.1) in $\mathbb{R}^3 \times (0, T)$, if for some $q_0 > 3$,

$$\begin{cases} \rho \geqq 0, \quad \rho - \tilde{\rho} \in C([0, T]; W^{1,q_0}), \quad \rho_t \in C([0, T]; L^{q_0}), \\ (u, \theta) \in C([0, T]; D^1 \cap D^{2,2}) \cap L^2(0, T; D^{2,q_0}), \\ (u_t, \theta_t) \in L^2(0, T; D^1), \quad (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t) \in L^\infty(0, T; L^2) \end{cases}$$

and (ρ, u, θ) satisfies (1.1) almost everywhere in $\mathbb{R}^3 \times (0, T)$.

Our main result can be stated as follows:

Theorem 1. *Let $\tilde{\rho} \geqq 0$ and $\tilde{\theta} = 0$. For $\tilde{q} \in (3, 6]$, assume that the initial data $(\rho_0 \geqq 0, u_0, \theta_0 \geqq 0)$ satisfy*

$$\rho_0 - \tilde{\rho} \in H^1 \cap W^{1,\tilde{q}}, \quad (u_0, \theta_0) \in D^1 \cap D^{2,2}, \quad \rho_0 \theta_0^2 \in L^1, \tag{1.7}$$

and the compatibility conditions

$$-\mu \Delta u_0 - (\mu + \lambda)\nabla \mathrm{div} u_0 + R\nabla(\rho_0\theta_0) = \sqrt{\rho_0} g_1, \tag{1.8}$$

$$\kappa \Delta\theta_0 + \frac{\mu}{2}|\nabla u_0 + (\nabla u_0)^{tr}|^2 + \lambda(\mathrm{div} u_0)^2 = \sqrt{\rho_0} g_2, \tag{1.9}$$

with $g_1, g_2 \in L^2$. Let (ρ, u, θ) be a strong solution to the Cauchy problem (1.1)–(1.4). If $T^* < \infty$ is the maximal time of existence, then

$$\lim_{T \to T^*} \left(\|\mathrm{div}\, u\|_{L^1(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)} \right) = \infty, \tag{1.10}$$

with r and s as in (1.5).

A few remarks are in order:

Remark 1. The criterion (1.10) consists of two parts: the compressibility of the fluid and Serrin's criterion for incompressible Navier–Stokes equations. Therefore, Theorem 1 can be regarded as the Serrin-type blowup criterion for the three-dimensional full compressible Navier–Stokes system.

Remark 2. It is worth noting that the conclusion in Theorem 1 is somewhat surprising, since the criterion (1.10) is just the same as that of barotropic case [11]; in particular, it is independent of the temperature. In fact, as indicated by Sun et al. [25], it seems that the nonlinearity of the highly nonlinear terms $|\mathfrak{D}(u)|^2$ and $(\mathrm{div}u)^2$ in the temperature equation is stronger than that of $\mathrm{div}(\rho u \otimes u)$ in the momentum ones. However, (1.10) shows that the nonlinear term $|\nabla u|^2$ can be controlled, provided one can control $\mathrm{div}(\rho u \otimes u)$.

Remark 3. Theorem 1 also holds for classical solutions to the three-dimensional full compressible Navier–Stokes system.

We now comment on the analysis of this paper. Let (ρ, u, θ) be a strong solution described in Theorem 1. Suppose that (1.10) were false, that is,

$$\lim_{T \to T^*} \left(\|\mathrm{div} u\|_{L^1(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)} \right) \leqq M_0 < +\infty. \tag{1.11}$$

We want to show that indeed

$$\sup_{0 \leqq t \leqq T^*} \left(\|\rho - \tilde{\rho}\|_{H^1 \cap W^{1,\tilde{q}}} + \|\nabla u\|_{H^1} + \|\nabla \theta\|_{H^1} \right) \leqq C < +\infty.$$

Since the methods in all previous works [3,9,25] depend crucially on the $L_t^\infty L_x^\infty$-norm or $L_t^2 L_x^\infty$-norm of the temperature θ, some new ideas are needed to recover all the a priori estimates under only the assumption (1.11) without any a priori bounds on the temperature. In fact, we prove (see Lemma 6) that a control of the Serrin norm of the velocity and the $L_t^1 L_x^\infty$ of the divergence of the velocity u implies a control on the $L_t^\infty L_x^2$ norm of ∇u. The main idea, in order to obtain this control, is that good bounds on the temperature θ can be obtained by multiplying the equation for the temperature θ and those for the momentum ρu by θ and $u\theta$, respectively (see (3.4) and (3.5)). This is the key to the proof, and once that is obtained, the proof follows in the same way as in our previous paper [10].

The rest of the paper is organized as follows: in the next section, we collect some elementary facts and inequalities that will be needed later. The main result, Theorem 1, is proved in Section 3.

2. Preliminaries

In this section, we recall some known facts and elementary inequalities that will be used later. We begin with the local existence and uniqueness of strong solutions when the initial density might not be positive and may vanish in an open set.

Lemma 1. ([2]) *Assume that the initial data* $(\rho_0 \geqq 0, u_0, \theta_0 \geqq 0)$ *satisfy* (1.7)–(1.9). *Then there exists a positive time* $T_1 \in (0, \infty)$ *and a unique strong solution* (ρ, u, θ) *to the Cauchy problem* (1.1)–(1.4) *on* $\mathbb{R}^3 \times (0, T_1]$.

Next, we recall the well-known Sobolev inequality, which will be frequently used later (see [20]).

Lemma 2. *For $p \in (1, \infty)$ and $q \in (3, \infty)$, there exists a generic constant $C > 0$, which depends only on p and q, such that for $f \in D^1$ and $g \in L^p \cap D^{1,q}$, we have*

$$\|f\|_{L^6} \leqq C\|\nabla f\|_{L^2}, \quad \|g\|_{L^\infty} \leqq C\|g\|_{L^p} + C\|\nabla g\|_{L^q}. \tag{2.1}$$

We now state an elementary estimate which follows directly from the standard L^p-estimate for the following elliptic system derived from the momentum equations, $(1.1)_2$:

$$\Delta G = \operatorname{div}(\rho\dot{u}), \qquad \mu\Delta\omega = \nabla \times (\rho\dot{u}),$$

where

$$\dot{f} \triangleq f_t + u \cdot \nabla f, G \triangleq (2\mu + \lambda)\operatorname{div}u - P, \omega \triangleq \nabla \times u, \tag{2.2}$$

are the material derivative of f, the effective viscous flux, and the vorticity, respectively.

Lemma 3. *Let (ρ, u, θ) be a smooth solution to (1.1) (1.3). Then there exists a generic positive constant C depending only on μ such that for any $p \in [2, 6]$,*

$$\|\nabla G\|_{L^p} + \|\nabla\omega\|_{L^p} \leqq C\|\rho\dot{u}\|_{L^p}. \tag{2.3}$$

Finally, the following Beale–Kato–Majda-type inequality, which can be found in [11] and was first proved in [1] when $\operatorname{div}u = 0$, will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla\rho\|_{L^2\cap L^q}$.

Lemma 4. *For $3 < q < \infty$, there exists a constant $C(q) > 0$ such that the following estimate holds for all $\nabla u \in L^2 \cap D^{1,q}$:*

$$\begin{aligned}\|\nabla u\|_{L^\infty} \leqq & C(\|\operatorname{div}u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^q}) \\ & + C\|\nabla u\|_{L^2} + C.\end{aligned} \tag{2.4}$$

3. Proof of Theorem 1

Before proving Theorem 1, we state some a priori estimates under the condition (1.11). First, the following upper bound of the density follows immediately from both (1.11) and (1.1) (see [11, Lemma 3.4]).

Lemma 5. *Assume that (1.11) holds. Then it holds that for $0 \leqq T < T^*$,*

$$\sup_{0\leqq t\leqq T} \|\rho\|_{L^\infty} \leqq C, \tag{3.1}$$

where (and in what follows) C, C_1, and C_2 denote generic constants depending only on M_0, μ, λ, R, κ, c_v, T^*, and the initial data.

Then, we derive the following key estimate on the $L^\infty(0,T;L^2)$-norm of ∇u.

Lemma 6. *Under the condition* (1.11)*, it holds that for* $0 \leqq T < T^*$,

$$\sup_{0\leqq t\leqq T}\int\left((\rho-\tilde\rho)^2+\rho\theta^2+|\nabla u|^2\right)\mathrm{d}x + \int_0^T\int\left(|\nabla\theta|^2+\rho|\dot u|^2\right)\mathrm{d}x\,\mathrm{d}t \leqq C. \tag{3.2}$$

Proof. First, applying the standard maximum principle to $(1.1)_3$, together with $\theta_0 \geqq 0$ (see [3,4]), shows that

$$\inf_{\mathbb{R}^3\times[0,T]}\theta(x,t)\geqq 0. \tag{3.3}$$

Multiplying $(1.1)_3$ by θ and integrating the resulting equation over $\mathbb{R}^3$ lead to

$$c_v\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\theta^2\,\mathrm{d}x+2\kappa\|\nabla\theta\|_{L^2}^2\leqq C\|\mathrm{div}u\|_{L^\infty}\int\rho\theta^2\,\mathrm{d}x+C\int\theta|\nabla u|^2\,\mathrm{d}x. \tag{3.4}$$

To estimate the last term on the right-hand side of (3.4), we multiply $(1.1)_2$ by $u\theta$ and integrate the resulting equation over $\mathbb{R}^3$ to obtain

$$\begin{aligned}\mu\int\theta|\nabla u|^2\,\mathrm{d}x\leqq{}&\eta\int\rho|\dot u|^2\,\mathrm{d}x+C\varepsilon\|\nabla\theta\|_{L^2}^2+C\|\mathrm{div}u\|_{L^\infty}\int\rho\theta^2\,\mathrm{d}x\\&+C(\varepsilon)\int|u|^2|\nabla u|^2\,\mathrm{d}x+C(\varepsilon,\eta)\int\rho\theta^2|u|^2\,\mathrm{d}x,\end{aligned} \tag{3.5}$$

where we have used

$$\int|\rho\dot u\cdot u\theta|\,\mathrm{d}x\leqq\eta\int\rho|\dot u|^2\,\mathrm{d}x+C(\eta)\int\rho\theta^2|u|^2\,\mathrm{d}x,$$

$$\begin{aligned}\left|\int\nabla Pu\theta\,\mathrm{d}x\right|&=\left|\int P\theta\mathrm{div}u\,\mathrm{d}x+\int Pu\cdot\nabla\theta\,\mathrm{d}x\right|\\&\leqq\varepsilon\|\nabla\theta\|_{L^2}^2+C\|\mathrm{div}u\|_{L^\infty}\int\rho\theta^2\,\mathrm{d}x+C(\varepsilon)\int\rho\theta^2|u|^2\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}&\int(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u)u\theta\,\mathrm{d}x\\&\leqq-\int\left(\mu\theta|\nabla u|^2+(\mu+\lambda)\theta(\mathrm{div}u)^2\right)\mathrm{d}x+C\int|\nabla\theta||u||\nabla u|\,\mathrm{d}x\\&\leqq-\int\mu\theta|\nabla u|^2\,\mathrm{d}x+\varepsilon\|\nabla\theta\|_{L^2}^2+C(\varepsilon)\int|u|^2|\nabla u|^2\,\mathrm{d}x.\end{aligned}$$

Combining (3.5) with (3.4), we obtain, after choosing ε suitably small, that

$$\begin{aligned}&c_v\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\theta^2\,\mathrm{d}x+\int\left(\theta|\nabla u|^2+\kappa|\nabla\theta|^2\right)\mathrm{d}x\\&\leqq C\eta\int\rho|\dot u|^2\,\mathrm{d}x+C\|\mathrm{div}u\|_{L^\infty}\int\rho\theta^2\,\mathrm{d}x\\&\quad+C(\eta)\int\left(\rho\theta^2|u|^2+|u|^2|\nabla u|^2\right)\mathrm{d}x.\end{aligned} \tag{3.6}$$

Next, multiplying (1.1)$_2$ by u_t and integrating the resulting equation over $\mathbb{R}^3$ show that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2\right)\mathrm{d}x+\int\rho|\dot{u}|^2\,\mathrm{d}x\\
&=\int\rho\dot{u}\cdot(u\cdot\nabla)u\,\mathrm{d}x+\int P\mathrm{div}u_t\,\mathrm{d}x\\
&\leqq\frac{1}{4}\int\rho|\dot{u}|^2\,\mathrm{d}x+C\int|u|^2|\nabla u|^2\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int P\mathrm{div}u\,\mathrm{d}x-\int P_t\mathrm{div}u\,\mathrm{d}x\\
&=\frac{1}{4}\int\rho|\dot{u}|^2\,\mathrm{d}x+C\int|u|^2|\nabla u|^2\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int P\mathrm{div}u\,\mathrm{d}x\\
&\quad-\frac{1}{2(2\mu+\lambda)}\frac{\mathrm{d}}{\mathrm{d}t}\int P^2\,\mathrm{d}x-\frac{1}{2\mu+\lambda}\int P_tG\,\mathrm{d}x. \qquad (3.7)
\end{aligned}$$

Noticing that (1.1)$_3$ and (2.3) lead to

$$\begin{aligned}
&\left|\int P_tG\,\mathrm{d}x\right|\\
&=\frac{R}{c_v}\left|\int\left(-c_v\mathrm{div}(\rho\theta u)+\kappa\Delta\theta-P\mathrm{div}u+2\mu|\mathfrak{D}(u)|^2+\lambda(\mathrm{div}u)^2\right)G\,\mathrm{d}x\right|\\
&\leqq C\int|\nabla G|\left(\rho\theta|u|+|\nabla\theta|\right)\,\mathrm{d}x+C\|\mathrm{div}u\|_{L^\infty}\left(\|\nabla u\|_{L^2}^2+\int\rho\theta^2\,\mathrm{d}x\right)\\
&\quad+C\int\theta|\nabla u|^2\,\mathrm{d}x\\
&\leqq\delta\int\rho|\dot{u}|^2\,\mathrm{d}x+C(\delta)\|\nabla\theta\|_{L^2}^2+C(\delta)\int\rho^2|u|^2\theta^2\,\mathrm{d}x+C\int\theta|\nabla u|^2\,\mathrm{d}x\\
&\quad+C\|\mathrm{div}u\|_{L^\infty}\left(\|\nabla u\|_{L^2}^2+\int\rho\theta^2\,\mathrm{d}x\right),
\end{aligned}$$

after choosing δ suitably small, we obtain from (3.7) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int H(x,t)\,\mathrm{d}x+\int\rho|\dot{u}|^2\,\mathrm{d}x\\
&\leqq C_1\int\left(\theta|\nabla u|^2+\kappa|\nabla\theta|^2\right)\,\mathrm{d}x+C\int\rho^2|u|^2\theta^2\,\mathrm{d}x\\
&\quad+C\int|u|^2|\nabla u|^2\,\mathrm{d}x+C\|\mathrm{div}u\|_{L^\infty}\left(\|\nabla u\|_{L^2}^2+\int\rho\theta^2\,\mathrm{d}x\right), \qquad (3.8)
\end{aligned}$$

where

$$H(x,t)\triangleq\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2-2P\mathrm{div}u+\frac{P^2}{2\mu+\lambda}.$$

Then, choosing constant $C_2\geqq C_1+1$ suitably large such that

$$\mu|\nabla u|^2-2P\mathrm{div}u+C_2c_v\rho\theta^2\geqq\frac{\mu}{2}|\nabla u|^2+\rho\theta^2, \qquad (3.9)$$

and adding (3.6) multiplied by C_2 to (3.8), we have, after choosing η suitably small, that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(H+C_2c_v\rho\theta^2\right)\,\mathrm{d}x\\&+\frac{1}{2}\int\rho|\dot{u}|^2\,\mathrm{d}x+\int\left(\theta|\nabla u|^2+\kappa|\nabla\theta|^2\right)\,\mathrm{d}x\\&\leqq C\|\mathrm{div}u\|_{L^\infty}\int\left(|\nabla u|^2+\rho\theta^2\right)\,\mathrm{d}x\\&+C\int\left(\rho\theta^2|u|^2+|u|^2|\nabla u|^2\right)\,\mathrm{d}x.\end{aligned}\tag{3.10}$$

Hölder's inequality and (2.3) yield that for r, s as in (1.5),

$$\begin{aligned}&\int\left(\rho\theta^2|u|^2+|u|^2|\nabla u|^2\right)\,\mathrm{d}x\\&\leqq C\|u\|_{L^r}^2\left(\|\rho^{1/2}\theta\|_{L^2}^{2(1-3/r)}\|\theta\|_{L^6}^{6/r}+\|\nabla u\|_{L^2}^{2(1-3/r)}\|\nabla u\|_{L^6}^{6/r}\right)\\&\leqq C(\delta)(1+\|u\|_{L^r}^s)\left(\|\nabla u\|_{L^2}^2+\|\rho^{1/2}\theta\|_{L^2}^2\right)\\&\quad+\delta\left(\|\rho^{1/2}\dot{u}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right),\end{aligned}\tag{3.11}$$

where in the second inequality, we have used the following simple fact:

$$\begin{aligned}\|\nabla u\|_{L^6}&\leqq C(\|G\|_{L^6}+\|\omega\|_{L^6}+\|P\|_{L^6})\\&\leqq C(\|\nabla G\|_{L^2}+\|\nabla\omega\|_{L^2}+\|\theta\|_{L^6})\\&\leqq C\|\rho^{1/2}\dot{u}\|_{L^2}+C\|\nabla\theta\|_{L^2},\end{aligned}\tag{3.12}$$

due to (2.1) and (2.3). Combining (3.10) with (3.11), we obtain, after choosing δ suitably small and using Gronwall's inequality, (3.9), and (1.11) that

$$\sup_{0\leqq t\leqq T}\int\left(\rho\theta^2+|\nabla u|^2\right)\,\mathrm{d}x+\int_0^T\int\left(|\nabla\theta|^2+\rho|\dot{u}|^2\right)\,\mathrm{d}x\,\mathrm{d}t\leqq C.\tag{3.13}$$

Finally, since $\rho-\tilde{\rho}$ satisfies

$$(\rho-\tilde{\rho})_t+\mathrm{div}((\rho-\tilde{\rho})u)+\tilde{\rho}\mathrm{div}u=0.\tag{3.14}$$

Multiplying (3.14) by $\rho-\tilde{\rho}$ and integrating the resulting equation over $\mathbb{R}^3$, then using (3.1), we obtain

$$(\|\rho-\tilde{\rho}\|_{L^2}^2)'(t)\leqq C\|\rho-\tilde{\rho}\|_{L^2}^2+C\|\nabla u\|_{L^2}^2,$$

which together with (3.13) yields (3.2). The proof of Lemma 6 is completed.

Finally, the following Lemma 7 will deal with the higher order estimates of the solutions which are needed to guarantee the extension of a local strong solution to a global one under the conditions (1.7)–(1.9) and (1.11).

Lemma 7. *Under the condition* (1.11)*, it holds that for* $0 \leqq T < T^*$*,*

$$\sup_{0\leqq t\leqq T} (\|\rho-\tilde{\rho}\|_{H^1\cap W^{1,\tilde{q}}} + \|\nabla u\|_{H^1} + \|\nabla\theta\|_{H^1}) \leqq C. \tag{3.15}$$

To prove Lemma 7, we need the following estimates on both $\dot{u}$ and $\dot{\theta}$, whose proofs are omitted here since they are similar to those of Lemma 4.1 and (4.28) in [10].

Lemma 8. *Under the condition* (1.11)*, it holds that for* $0 \leqq T < T^*$*,*

$$\sup_{0\leqq t\leqq T} \int \left(|\nabla\theta|^2+\rho|\dot{u}|^2\right) \mathrm{d}x + \int_0^T \int \left(\rho\dot{\theta}^2+|\nabla\dot{u}|^2\right) \mathrm{d}x\, \mathrm{d}t \leqq C, \tag{3.16}$$

$$\sup_{0\leqq t\leqq T} \|\rho^{1/2}\dot{\theta}\|_{L^2} + \int_0^T \|\nabla\dot{\theta}\|_{L^2}^2\, \mathrm{d}t \leqq C. \tag{3.17}$$

Proof of Lemma 7 Following [10,11], we will prove Lemma 7. First, it follows from (3.2) and (3.16) that for any $\delta \in (0, 1]$,

$$\int \theta^2|\nabla u|^2\, \mathrm{d}x \leqq C\|\theta\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 \leqq \delta\|\nabla^2\theta\|_{L^2}^2 + C(\delta).$$

This, along with the standard L^2-estimate of $(1.1)_3$, gives

$$\begin{aligned}\|\nabla^2\theta\|_{L^2}^2 &\leqq C\int \rho\dot{\theta}^2\, \mathrm{d}x + C\int \rho^2\theta^2|\nabla u|^2\, \mathrm{d}x + C\int |\nabla u|^4\, \mathrm{d}x\\ &\leqq C\int \rho\dot{\theta}^2\, \mathrm{d}x + C\delta\|\nabla^2\theta\|_{L^2}^2 + C(\delta) + C\int |\nabla u|^4\, \mathrm{d}x,\end{aligned}$$

which together with (3.16), (3.17), and (3.12) shows

$$\sup_{0\leqq t\leqq T} \|\nabla\theta\|_{H^1} \leqq C. \tag{3.18}$$

It thus follows from (2.1)–(2.3) and (3.18) that

$$\begin{aligned}&\int_0^T \left(\|\mathrm{div}u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2\right) \mathrm{d}t\\ &\leqq C\int_0^T (\|G\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2 + \|P\|_{L^\infty}^2)\, \mathrm{d}t + C\\ &\leqq C\int_0^T (\|G\|_{L^2}^2 + \|\nabla G\|_{L^6}^2 + \|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^6}^2)\, \mathrm{d}t\\ &\quad +C\int_0^T \|\theta\|_{L^\infty}^2\, \mathrm{d}t + C\\ &\leqq C\int_0^T \|\rho\dot{u}\|_{L^6}^2\, \mathrm{d}t + C\\ &\leqq C.\end{aligned} \tag{3.19}$$

Next, for $2 \leqq p \leqq \tilde{q}$, $|\nabla\rho|^p$ satisfies

$$\begin{aligned}&(|\nabla\rho|^p)_t + \operatorname{div}(|\nabla\rho|^p u) + (p-1)|\nabla\rho|^p \operatorname{div}u\\&\quad + p|\nabla\rho|^{p-2}(\nabla\rho)^{tr}\nabla u(\nabla\rho) + p\rho|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\operatorname{div}u = 0.\end{aligned}$$

This yields that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^p} &\leqq C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p} + C\|\nabla^2 u\|_{L^p}\\&\leqq C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p} + C\|\nabla\dot{u}\|_{L^2} + C,\end{aligned}\tag{3.20}$$

where we have used

$$\begin{aligned}\|\nabla^2 u\|_{L^p} &\leqq C\left(\|\rho\dot{u}\|_{L^p} + \|\nabla P\|_{L^p}\right)\\&\leqq C\left(\|\rho\dot{u}\|_{L^2} + \|\nabla\dot{u}\|_{L^2} + \|\nabla\rho\|_{L^p}\right) + C\\&\leqq C\left(1 + \|\nabla\dot{u}\|_{L^2} + \|\nabla\rho\|_{L^p}\right),\end{aligned}\tag{3.21}$$

due to (3.16) and (3.18). It thus follows from (2.4), (3.2), and (3.21) that

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\leqq C + C(\|\operatorname{div}u\|_{L^\infty} + \|\omega\|_{L^\infty})\ln(e + \|\nabla^2 u\|_{L^{\tilde{q}}})\\&\leqq C + C(\|\operatorname{div}u\|_{L^\infty} + \|\omega\|_{L^\infty})\ln(e + \|\nabla\rho\|_{L^{\tilde{q}}})\\&\quad + C(\|\operatorname{div}u\|_{L^\infty} + \|\omega\|_{L^\infty})\ln(e + \|\nabla\dot{u}\|_{L^2}).\end{aligned}\tag{3.22}$$

We thus deduce from (3.22) and (3.20) that

$$f'(t) \leqq Cg(t)f(t)\ln f(t),\tag{3.23}$$

with

$$\begin{cases} f(t) \triangleq e + \|\nabla\rho\|_{L^{\tilde{q}}},\\ g(t) \triangleq (1 + \|\nabla\dot{u}\|_{L^2} + \|\operatorname{div}u\|_{L^\infty} + \|\omega\|_{L^\infty})\ln(e + \|\nabla\dot{u}\|_{L^2}).\end{cases}$$

It follows from (3.23), (3.19), (3.16), and Gronwall's inequality that

$$\sup_{0\leqq t\leqq T}\|\nabla\rho\|_{L^{\tilde{q}}} \leqq C,\tag{3.24}$$

which combined with (3.22), (3.16), and (3.19) directly gives

$$\int_0^T \|\nabla u\|_{L^\infty}\,\mathrm{d}t \leqq C.\tag{3.25}$$

Taking $p = 2$ in (3.20), we get, by using (3.25), (3.16), and Gronwall's inequality, that

$$\sup_{0\leqq t\leqq T}\|\nabla\rho\|_{L^2} \leqq C,\tag{3.26}$$

which together with (3.21) and (3.18) yields

$$\sup_{0\leqq t\leqq T}\|\nabla^2 u\|_{L^2} \leqq C\sup_{0\leqq t\leqq T}\left(\|\rho\dot{u}\|_{L^2} + \|\nabla\rho\|_{L^2} + \|\nabla\theta\|_{L^2}\right) \leqq C.$$

This, along with (3.24), (3.26), and (3.2), finishes the proof of Lemma 7.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1 Suppose that (1.10) were false, that is, (1.11) holds. Since the generic constant C in Lemma 7 remains uniformly bounded for all $T < T^*$, the functions $(\rho, u, \theta)(x, T^*) \triangleq \lim_{t \to T^*}(\rho, u, \theta)(x, t)$ satisfy the conditions imposed on the initial data (1.7) at the time $t = T^*$. Furthermore, standard arguments yield that $\rho\dot{u}, \rho\dot{\theta} \in C([0, T]; L^2)$, which implies

$$(\rho\dot{u}, \rho\dot{\theta})(x, T^*) = \lim_{t \to T^*}(\rho\dot{u}, \rho\dot{\theta}) \in L^2.$$

Hence, we have

$$\begin{aligned}
&-\mu\Delta u - (\mu + \lambda)\nabla \mathrm{div} u + R\nabla(\rho\theta)|_{t=T^*} = \sqrt{\rho}(x, T^*)g_1(x),\\
&\kappa\Delta\theta + \frac{\mu}{2}|\nabla u + (\nabla u)^{tr}|^2 + \lambda(\mathrm{div} u)^2|_{t=T^*} = \sqrt{\rho}(x, T^*)g_2(x),
\end{aligned}$$

with

$$g_1(x) \triangleq \begin{cases} \rho^{-1/2}(x, T^*)(\rho\dot{u})(x, T^*), & \text{for } x \in \{x|\rho(x, T^*) > 0\},\\ 0, & \text{for } x \in \{x|\rho(x, T^*) = 0\}, \end{cases}$$

and

$$g_2(x) \triangleq \begin{cases} \rho^{-1/2}(x, T^*)[c_v\rho\dot{\theta} + R\rho\theta \mathrm{div} u](x, T^*), & \text{for } x \in \{x|\rho(x, T^*) > 0\},\\ 0, & \text{for } x \in \{x|\rho(x, T^*) = 0\}, \end{cases}$$

satisfying $g_1, g_2 \in L^2$ due to (3.16), (3.17), and (3.15). Thus, $(\rho, u, \theta)(x, T^*)$ also satisfies (1.8) and (1.9). Therefore, one can take $(\rho, u, \theta)(x, T^*)$ as the initial data and apply Lemma 1 to extend the local strong solution beyond T^*. This contradicts the assumption on T^*. The proof of Theorem 1 is finished.

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