Entire Minimal Parabolic Trajectories: The Planar Anisotropic Kepler Problem

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Abstract

We continue the variational approach to parabolic trajectories introduced in our previous paper (Barutello et al., Entire parabolic trajectories as minimal phase transitions. arXiv:1105.3358v1, 2011), which sees parabolic orbits as minimal phase transitions. We deepen and complete the analysis in the planar case for homogeneous singular potentials. We characterize all parabolic orbits connecting two minimal central configurations as free-time Morse minimizers (in a given homotopy class of paths). These may occur for at most one value of the homogeneity exponent. In addition, we link this threshold of existence of parabolic trajectories with the absence of collisions for all the minimizers of fixed-end problems, and also with the existence of action minimizing periodic trajectories with nontrivial homotopy type.

1. Introduction and Main Results

For a positive, singular potential $V \in \mathscr{C}^2(\mathbb{R}^d \setminus \{0\})$, vanishing at infinity, we study the Newtonian system

$$\ddot{x}(t) = \nabla V(x(t)), \tag{1}$$

searching for *parabolic* solutions, that is, entire solutions satisfying the zero-energy relation

$$\frac{1}{2}|\dot{x}(t)|^2 = V(x(t)), \quad \text{for every } t \in \mathbb{R}. \tag{2}$$

In the Kepler problem ($V(x) = 1/|x|$) all global zero-energy trajectories are, indeed, parabolas. In this paper we are concerned with $(-\alpha)$-homogeneous potentials, with $\alpha \in (0, 2)$. Within this class of potentials, parabolic trajectories are homoclinic to infinity, which represents the minimum of the potential.

In celestial mechanics, and more generally in the theory of singular Hamiltonian systems, parabolic trajectories play a central role and are known to carry precious information on the behavior of general solutions near collisions. On the other hand, parabolic trajectories are structurally unstable and therefore are usually considered beyond the range of application of variational or other global methods. In spite of this, in our previous paper [5], we introduced a new variational approach to their existence as minimal phase transitions.

The purpose of the present paper is to deepen and complete the analysis in the planar case $d = 2$; we will succeed in characterizing all parabolic orbits connecting two minimal central configurations as free-time Morse minimizers (in a given homotopy class of paths). In addition, we shall link the threshold of existence of parabolic trajectories with the absence of collisions for all the minimizers of fixed-ends problems and with the existence of action minimizing periodic trajectories with nontrivial homotopy type.

In the plane $\mathbb{R}^2$ we use the polar coordinates $x = (r \cos \vartheta, r \sin \vartheta) = (r, \vartheta)$ (despite the ambiguous notation, it will always be clear from the context whether a pair denotes either Cartesian or polar coordinates). Under this notation any $(-\alpha)$-homogeneous potential V can be written as

$$V(x) = \frac{U(\vartheta)}{r^\alpha},$$

where

$$U(\vartheta) := V(\cos \vartheta, \sin \vartheta).$$

The potential V is then a generalization of the *anisotropic Kepler potential* (extensively studied, for instance, in [16,17,21–23]), which actually corresponds to the value 1 of the parameter α and a specific U. For such potentials, it is well known that parabolic trajectories admit ingoing and outgoing asymptotic directions which are necessarily critical points of $U(\vartheta)$; these are called *central configurations*. We are mostly interested in parabolic trajectories connecting two *minimal* central configurations. To be more precise, given

$$0 \leqq \vartheta_1 \leqq \vartheta_2 < 2\pi,$$

we define the sets of potentials

$$\mathcal{U} = \mathcal{U}_{\vartheta_1 \vartheta_2} := \left\{ U \in \mathscr{C}^2(\mathbb{R}) : \begin{array}{l} \text{for every } \vartheta \in \mathbb{R} \text{ and } i = 1, 2 \\ U(\vartheta + 2\pi) = U(\vartheta) \\ U(\vartheta) \geqq U(\vartheta_1) = U(\vartheta_2) > 0 \\ U''(\vartheta_i) > 0 \end{array} \right\},$$

and, with a slight abuse of notation,

$$\begin{aligned} \mathcal{V} :&= \{V = (U, \alpha) : U \in \mathcal{U} \text{ and } \alpha \in (0, 2)\} \\ &= \left\{ V \in \mathscr{C}^2(\mathbb{R}^2 \setminus \{0\}) : V(x) = \frac{U(\vartheta)}{r^\alpha}, U \in \mathcal{U} \text{ and } \alpha \in (0, 2) \right\}. \end{aligned}$$

For a given $V \in \mathscr{V}$, we introduce the action functional

$$\mathscr{A}(x) = \mathscr{A}([a, b]; x) := \int_a^b \frac{1}{2}|\dot{x}(t)|^2 + V(x(t)) \, dt.$$

In our previous paper [5], we introduced the set of Morse parabolic minimizers associated to $\mathscr{A}$ and having asymptotic directions $\xi^- = (\cos \vartheta_1, \sin \vartheta_1)$ and $\xi^+ = (\cos \vartheta_2, \sin \vartheta_2)$. Nonetheless, since $\mathbb{R}^2 \setminus \{0\}$ is not simply connected, as a peculiar fact in the planar case one can also impose a topological constraint in the form of a homotopy class for the minimizer, for example imposing $h \in \mathbb{Z}$ counterclockwise rotations around the origin. Lifting such a trajectory to the universal covering of $\mathbb{R}^2 \setminus \{0\}$, this corresponds to joining ϑ_1 with $\vartheta_2 + 2h\pi$. Motivated by these considerations, we introduce the set

$$\Theta = \Theta_{\vartheta_1 \vartheta_2} := \{\vartheta \in \mathbb{R} : \vartheta = \vartheta_i + 2n\pi \text{ for some } n \in \mathbb{Z} \text{ and } i \in \{1, 2\}\}$$

and, given $\vartheta^- \neq \vartheta^+$ in $\Theta_{\vartheta_1 \vartheta_2}$ (or, more in general, $\vartheta^- \neq \vartheta^+$ central configurations), we define the following class of paths.

Definition 1. We say that $x = (r, \vartheta) \in H^1_{\text{loc}}(\mathbb{R})$ is a *parabolic trajectory* associated with ϑ^-, ϑ^+ and V, if it satisfies Equations (1), (2) and

- $\min_{t \in \mathbb{R}} r(t) > 0$;
- $r(t) \to +\infty$, $\vartheta(t) \to \vartheta^\pm$ as $t \to \pm\infty$.

We say that x is a (free time) *parabolic Morse minimizer* if, moreover, there holds

- for every $t_1 < t_2$, $t_1' < t_2'$, and $z = (\rho, \zeta) \in H^1(t_1', t_2')$, there holds

$$\rho(t_i') = r(t_i), \ \zeta(t_i') = \vartheta(t_i), \ i = 1, 2 \quad \implies \quad \mathscr{A}([t_1, t_2]; x) \leq \mathscr{A}([t_1', t_2']; z)$$

(this last property actually implies (1), (2)). A fixed time minimizer fulfills the above minimality condition only with $t_i' = t_i$.

Under the previous definition the following holds.

Theorem 1. *Let $U \in \mathscr{U}$ and $\vartheta^-, \vartheta^+ \in \Theta$, $\vartheta^- \neq \vartheta^+$ be fixed minimal central configurations; then*

- *there exists at most one $\bar{\alpha} = \bar{\alpha}(\vartheta^-, \vartheta^+, U) \in (0, 2)$ such that $V = (U, \alpha)$ admits a corresponding parabolic trajectory associated with $(\vartheta^-, \vartheta^+, U)$ if and only if $\alpha = \bar{\alpha}$;*
- *every parabolic trajectory associated with ϑ^-, ϑ^+ and U is a free time Morse minimizer;*
- *if $|\vartheta^+ - \vartheta^-| > \pi$, then there exists exactly one $\bar{\alpha}$ such that $V = (U, \alpha)$ admits a corresponding parabolic Morse minimizer if and only if $\alpha = \bar{\alpha}$.*

Let us point out that if $|\vartheta^+ - \vartheta^-| \leq \pi$, such a number $\bar{\alpha}(\vartheta^-, \vartheta^+, U)$ may or may not exist, depending on the properties of U. Some numerical approximations are contained in Table 1.

Table 1. Numerical approximations of the parabolic threshold $\bar{\alpha}(\vartheta^-, \vartheta^+, U)$ for the angular potential $U(\vartheta) = h - k \cos(l\vartheta)$

h	k	l	ϑ^-	ϑ^+	$\bar{\alpha}$
2	1	2	0	π	0.84
2	1	2	0	2π	1.46
2	1	2	0	3π	1.64
2	1	2	0	4π	1.73
2	1	3	0	$2\pi/3$	0.27
2	1	3	0	$4\pi/3$	1.19
2	1	3	0	2π	1.47
3	1	3	0	$2\pi/3$	$\not\exists\,(0)$
3	1	3	0	$4\pi/3$	1.01
3	1	3	0	2π	1.36
3	1.2	3	0	$2\pi/3$	0.03
3	1.2	3	0	$4\pi/3$	1.09
3	1.2	3	0	2π	1.40

The values are truncated to a precision of 2 decimal places, and are obtained by a bisection argument based on Proposition 2 and Lemmas 2, 3. Numerical integration of system (6) was performed by using the Dormand–Prince method

To proceed with the description of our results, let us extend the function $\bar{\alpha}$ to the whole of the possible triplets $(\vartheta^-, \vartheta^+, U) \in \Theta \times \mathcal{U}$ by setting its value to zero if there are no parabolic trajectories for any α. This exponent can be related to the presence or absence of collisions for both the fixed time and the free time Bolza problems within the sector defined by the angles ϑ^- and ϑ^+.

The problem of the exclusion of collisions for action minimizing trajectories has a long history, starting from the first elaborations in the late 1980s, for example [1,11,12,14,15,27,28], up to the extensive researches of the last decade, mostly motivated by the search of new symmetric collisionless periodic solutions to the n–body problem (for example [6,9,10,18]). Starting from the idea of averaged variation by Marchal [8,25], later made fully rigorous, extended and refined in [19], a rather complete analysis of the possible singularities of minimizing trajectories has been recently achieved in [3]. In the literature, minimal parabolic trajectories have been studied in connection with the absence of collisions for fixed-endpoint minimizers. More precisely, as remarked by Luz and Maderna [13], the property of being collisionless for all Bolza minimizers implies the absence of parabolic trajectories which are Morse minimal for the usual n–body problem with $\alpha = 1$. On the contrary, minimal parabolic arcs (that is, defined only on the half line) exist for every starting configuration, as proved by Maderna and Venturelli [24].

Special attention has been devoted to minimizers subject to topological constraints and to the existence of trajectories having a particular homotopy type (see for example [2,7,20,25,26,29]). For such constrained minimizers the averaged variation technique is not available, and other devices have to be designed to avoid the occurrence of collisions. Starting from [29], motivated by the search of periodic solutions having prism symmetry, a connection has been established between the apsidal angles of parabolic trajectories and the exclusion of collisions for minimizers with a given rotation angle. In fact, we can now draw a complete picture of the

role played by the parabolic orbits in the solution of the collision-free minimization problem with fixed ends.

Definition 2. Given a potential V, we say that $x = (r, \vartheta) \in H^1(t_1, t_2)$ is a *fixed-time Bolza minimizer* associated with the ends $x_1 = r_1 e^{i\varphi_1}$, $x_2 = r_2 e^{i\varphi_2}$, if

- $r(t_i) = r_i$ and $\vartheta(t_i) = \varphi_i, i = 1, 2$;
- for every $z = (\rho, \zeta) \in H^1(t_1, t_2)$, there holds

$$\rho(t_i) = r_i, \ \zeta(t_i) = \varphi_i, \ i = 1, 2 \implies \mathscr{A}([t_1, t_2]; x) \leqq \mathscr{A}([t_1, t_2]; z).$$

If $\min_{t \in [t_1, t_2]} r(t) > 0$ we say that the Bolza minimizer is collisionless.

Theorem 2. *Let $U \in \mathscr{U}, \vartheta^- \neq \vartheta^+ \in \Theta$, and consider a perturbed potential*
$V = \dfrac{U(\vartheta)}{r^\alpha} + W$, *with $V \in \mathcal{C}^1(\mathbb{R}^2 \setminus 0), \alpha > \alpha'$ and*

$$\lim_{r \to 0} r^{\alpha'}(W(x) + r|\nabla W(x)|) = 0 . \tag{3}$$

If $\alpha > \bar{\alpha}(U, \vartheta^-, \vartheta^+)$ then all fixed-time Bolza minimizers associated to $x_1 = (r_1, \varphi_1)$ and $x_2 = (r_2, \varphi_2)$ within the sector $[\vartheta^-, \vartheta^+]$ are collisionless.

It is worthwhile noticing that if, conversely, $\alpha \leqq \bar{\alpha}(U, \vartheta^-, \vartheta^+)$, then there are always some Bolza problems which admit only colliding minimizers. In addition, when $\alpha = \bar{\alpha}(U, \vartheta^-, \vartheta^+)$, the very same arguments imply the following statement, which is connected with Lambert's results on the existence of the direct and inverse arcs for the planar Kepler problem [25,30].

Proposition 1. *Let $U \in \mathscr{U}, \vartheta^- \neq \vartheta^+ \in \Theta$, and V be a perturbed potential as in the previous theorem, with $\alpha = \bar{\alpha}(U, \vartheta^-, \vartheta^+)$. Given any pair of points x_1 and x_2 in the sector $(\vartheta^-, \vartheta^+)$, all fixed-time Bolza minimizers associated with x_1, x_2 within the sector $[\vartheta^- + \varepsilon, \vartheta^+ - \varepsilon]$, for some $\varepsilon > 0$, are collisionless.*

Some further interesting conclusions can be drawn, in the special case when $\vartheta^+ = \vartheta^- + 2k\pi$, which connect the parabolic threshold with the existence of non-collision periodic orbits having a prescribed winding number (this is connected with the minimizing property of Kepler ellipses, see [20]).

Theorem 3. *Let $U \in \mathscr{U}$ be such that all its local minima are non-degenerate global ones, and consider the potential $V = \dfrac{U(\vartheta)}{r^\alpha}$. Given any integer $k \neq 0$ and period $T > 0$, if*

$$\alpha > \bar{\alpha}(U, \vartheta^*, \vartheta^* + 2k\pi) , \quad \text{for every minimum } \vartheta^* \text{ of } U, \tag{4}$$

then any action minimizer in the class of T-periodic trajectories winding k times around zero is collisionless.

The outline of the paper is as follows: in Section 2 we exploit some results due to
Devaney [16,17] in order to rewrite Equations (1), (2) in terms of an equivalent
planar first-order system; this allows us to develop a first phase-plane analysis of
the dynamical properties of parabolic trajectories. In Section 3 we turn to the var-
iational properties of zero-energy solutions. In Section 4 we prove Theorem 1 in
the particular case in which $\pi < \vartheta^+ - \vartheta^- \leqq 2\pi$. Finally, Sections 5 and 6 are
devoted to the completion of the proof of Theorem 1 and to the proofs of Theorems
2 and 3, respectively.

2. Phase Plane Analysis

Following Devaney [16,17], an appropriate change of variables makes the dif-
ferential problem (1), (2) equivalent to a planar first order system, for which a phase
plane analysis can be carried out. This allows a first investigation of its trajectories
from a dynamical (as opposed to variational) point of view.

Let $U \in \mathcal{U}_{\vartheta_1 \vartheta_2}$, and let us assume for simplicity that U is a Morse function,
even though the only important assumption is that ϑ_1, ϑ_2 are non-degenerate. Intro-
ducing the Cartesian coordinates $q_1 = r \cos \vartheta$, $q_2 = r \sin \vartheta$ and the momentum
vector $(p_1, p_2) = (\dot{q}_1, \dot{q}_2)$, we write Equations (1) and (2) as

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{q}_2 = p_2 \\ \dot{p}_1 = \partial_{q_1} \left(r^{-\alpha} U(\vartheta) \right) = r^{-\alpha-2} \left(-U'(\vartheta)q_2 - \alpha U(\vartheta)q_1 \right) \\ \dot{p}_2 = \partial_{q_2} \left(r^{-\alpha} U(\vartheta) \right) = r^{-\alpha-2} \left(U'(\vartheta)q_1 - \alpha U(\vartheta)q_2 \right), \end{cases}$$

and

$$\frac{1}{2} \left(p_1^2 + p_2^2 \right) = \frac{U(\vartheta)}{r^\alpha}.$$

Since $U(\vartheta) \geqq U(\vartheta_1) = U(\vartheta_2) =: U_{\min} > 0$, we have that $|p| \neq 0$. As a
consequence, for every solution of the previous dynamical system we can find
smooth functions $z > 0$ and $\varphi \in \mathbb{R}$ in such a way that $p_1 = r^{-\alpha/2}z \cos \varphi$, $p_2 = r^{-\alpha/2}z \sin \varphi$. These functions satisfy

$$z = \sqrt{2U(\vartheta)}$$

and

$$\begin{cases} \dot{r} = r^{-\alpha/2}z \left(\cos \vartheta \cos \varphi + \sin \vartheta \sin \varphi \right) = r^{-\alpha/2}z \cos(\varphi - \vartheta) \\ \dot{\vartheta} = r^{-1-\alpha/2}z \left(\cos \vartheta \sin \varphi - \sin \vartheta \cos \varphi \right) = r^{-1-\alpha/2}z \sin(\varphi - \vartheta) \\ \dot{z} = r^{-1-\alpha/2}U'(\vartheta) \sin(\varphi - \vartheta) \\ \dot{\varphi} = \frac{1}{z}r^{-1-\alpha/2} \left[U'(\vartheta) \cos(\varphi - \vartheta) + \alpha U(\vartheta) \sin(\varphi - \vartheta) \right]. \end{cases}$$

This system has a singularity at $r = 0$ that can be removed by a change of time
scale. Assuming $r > 0$, we introduce the new variable τ via

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = zr^{1+\alpha/2}$$

in order to rewrite the dynamical system as (here "′" denotes the derivative with respect to τ)

$$\begin{cases} r' = rz^2 \cos(\varphi - \vartheta) = 2rU(\vartheta)\cos(\varphi - \vartheta) \\ z' = zU'(\vartheta)\sin(\varphi - \vartheta) \\ \vartheta' = z^2 \sin(\varphi - \vartheta) = 2U(\vartheta)\sin(\varphi - \vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta), \end{cases} \tag{5}$$

which contains the independent planar system

$$\begin{cases} \vartheta' = 2U(\vartheta)\sin(\varphi - \vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta). \end{cases} \tag{6}$$

It is immediately clear that the systems above enjoy global existence, and that the stationary points of (6) are the points (ϑ^*, φ^*), where $U'(\vartheta^*) = 0$ and $\sin(\varphi^* - \vartheta^*) = 0$.

Theorem 4. (DEVANEY [17]) *The path* $x = x(t)$ *satisfies* (1), (2) *if and only if* (ϑ, φ) *satisfies* (6) *(and* (r, z) *satisfies* (5)). *The function*

$$v(\tau) = \sqrt{U(\vartheta(\tau))} \cos\left(\varphi(\tau) - \vartheta(\tau)\right),$$

is non-decreasing on the solutions of (6), *which correspond to*

- *saddle-type equilibria* $(\vartheta^*, \vartheta^* + h\pi)$, $U'(\vartheta^*) = 0$, $U''(\vartheta^*) > 0$ *and* $h \in \mathbb{Z}$;
- *sink/source-type equilibria* $(\vartheta^*, \vartheta^* + h\pi)$, *where* $U'(\vartheta^*) = 0$, $U''(\vartheta^*) < 0$ *and* $h \in \mathbb{Z}$;
- *heteroclinic trajectories connecting two of the previous equilibria.*

To every trajectory of (6) *there corresponds infinitely many trajectories of* (5), *all equivalent through a radial homotheticity.*

The corresponding solutions of (1), (2) *satisfy the following:*

- *if*

$$\cos(\varphi - \vartheta) \to \pm 1 \ as \ \tau \to \pm\infty, \tag{7}$$

  then x *is globally defined and unbounded in the future/past (in* t *);*
- *if* $\cos(\varphi - \vartheta) \to \mp 1$ *as* $\tau \to \pm\infty$, *then* $t(\tau) \to T_\pm \in \mathbb{R}$ *and* $x(t) \to 0$ *as* $t \to T_\pm$.

In Fig. 1 we describe the phase plane for the dynamical system (6) when U is *isotropic* and, in particular, for the Kepler problem. On the other hand, if we take into account an anisotropic potential U in the class $\mathscr{U}_{\vartheta_1 \vartheta_2}$ and a homogeneous extension (U, α), $\alpha \in (0, 2)$, then we can deduce the following result (by time reversibility, it is not restrictive to assume that $\vartheta^- < \vartheta^+$).

Corollary 1. *Let* $\vartheta^- < \vartheta^+$ *belong to* $\Theta_{\vartheta_1 \vartheta_2}$ *and* $x = rs$ *be an associated parabolic Morse minimizer for* (U, α). *Then (a suitable choice of) the corresponding* (ϑ, φ) *is an heteroclinic connection between the saddles*

$$(\vartheta^-, \vartheta^- + \pi) \ and \ (\vartheta^+, \vartheta^+).$$

Moreover ϑ *is strictly increasing between* ϑ^- *and* ϑ^+.
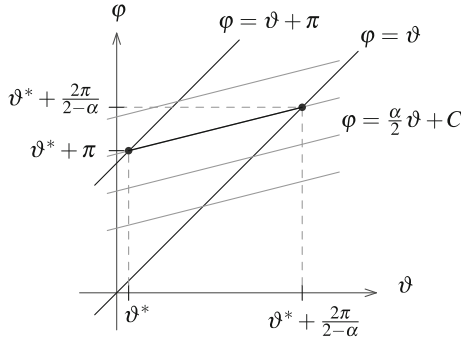
Fig. 1. The figure sketches the phase portrait of (6) when $U(\vartheta) \equiv 1$. The dynamical system reads $\varphi' = (\alpha/2)\,\vartheta' = \alpha\sin(\varphi - \vartheta)$, which critical points satisfy $\varphi = \vartheta + k\pi$, $k \in \mathbb{Z}$. Trajectories lie on the bundle $\varphi = (\alpha/2)\vartheta + C$, $C \in \mathbb{R}$, and, recalling condition (7), we deduce that parabolic solutions coincide with heteroclinic connections departing from points on $\varphi = \vartheta + (2k+1)\pi$ and ending on $\varphi = \vartheta + 2k\pi$, for some $k \in \mathbb{Z}$. For instance, when $k = 0$, we obtain heteroclinics connecting $(\vartheta^*, \vartheta^* + \pi)$ to $(2\pi/(2 - \alpha) + \vartheta^*, 2\pi/(2 - \alpha) + \vartheta^*)$, for some $\vartheta^* \in \mathbb{R}$. Going back to the original dynamical system, this implies that parabolic motions exist only when the angle between the ingoing and outgoing asymptotic directions is $2\pi/(2-\alpha)$; let us emphasize that such an angle is always greater than π. When $\alpha = 1$, that is, in the classical Kepler problem, this angle is 2π: the heteroclinic between $(\vartheta^*, \vartheta^* + \pi)$ and $(2\pi + \vartheta^*, 2\pi + \vartheta^*)$ actually describes a parabola whose axis forms an angle ϑ^* with the horizontal line

Proof. Since $\vartheta^{\pm}$ are minima for U we have that (ϑ, φ) connects the two saddles, say,

$$(\vartheta^-, \vartheta^- + h_1\pi) \text{ and } (\vartheta^+, \vartheta^+ + h_2\pi),$$

in such a way that

$$\lim_{\tau \to -\infty} [\varphi(\tau) - \vartheta(\tau)] = h_1\pi, \qquad \lim_{\tau \to +\infty} [\varphi(\tau) - \vartheta(\tau)] = h_2\pi.$$

Since x is globally defined, condition (7) holds, implying that $\cos(h_1\pi) = -1$ and $\cos(h_2\pi) = 1$, that is, h_1 is odd while h_2 is even. Since v is non-decreasing, we have that

$$-\sqrt{U_{\min}} = v(-\infty) < v(\tau) < v(+\infty) = \sqrt{U_{\min}}.$$

Now we observe that

$$v' = (2 - \alpha)\,[U(\vartheta)]^{3/2}\sin^2(\varphi - \vartheta) = (2 - \alpha)\sqrt{U(\vartheta)}\Big[U(\vartheta) - v^2\Big], \quad (8)$$

Hence v strictly increases. Then $\sin(\varphi - \vartheta) \neq 0$, therefore also ϑ is strictly monotone. Since $\vartheta^- < \vartheta^+$ we obtain that ϑ increases. But this finally implies that $\sin(\varphi - \vartheta) > 0$, for every τ. Summing up all the information we deduce that

$$h_1 = h_2 + 1. \quad \square$$

Motivated by the previous result, we devote the rest of the section to studying the properties of the stable and unstable trajectories associated with the saddle points of (6), in dependence of the parameter α. To start with, using Equation (8), we provide a necessary condition for the existence of saddle–saddle connections.

Lemma 1. *Let us assume that for some* $\alpha \in (0, 2)$ *there exists a saddle–saddle connection for* (6) *between* $(\vartheta^-, \vartheta^- + \pi)$ *and* $(\vartheta^+, \vartheta^+)$. *Then*

$$2 - \frac{2\pi}{\vartheta^+ - \vartheta^-} \leqq \alpha \leqq 2 - \frac{4}{\vartheta^+ - \vartheta^-} \arcsin \sqrt{\frac{U_{\min}}{U_{\max}}},$$

where $U_{\min} \leqq U(\vartheta) \leqq U_{\max}$, *for every* ϑ.

Proof. Let (ϑ, φ) be such a heteroclinic. Reasoning as in the proof of the previous corollary, one can deduce that both v and ϑ are (strictly) monotone in τ. It is then possible to write $v = v(\tau(\vartheta)) =: \hat{v}(\vartheta)$, obtaining that

$$\lim_{\vartheta \to \vartheta^\pm} \hat{v}(\vartheta) = \pm\sqrt{U_{\min}}.$$

With this notation we can write

$$\frac{d\hat{v}}{d\vartheta} = v'(\tau)\frac{d\tau}{d\vartheta} = \frac{2-\alpha}{2} \frac{\sqrt{U(\vartheta)}}{U(\vartheta)} \frac{U(\vartheta) - v^2}{\sin(\varphi - \vartheta)} = \frac{2-\alpha}{2}\sqrt{U(\vartheta) - \hat{v}^2}.$$

Integrating on $\vartheta \in [\vartheta^-, \vartheta^+]$, we obtain on one hand

$$\vartheta^+ - \vartheta^- \leqq \frac{2}{2-\alpha}\int_{-\sqrt{U_{\min}}}^{\sqrt{U_{\min}}} \frac{dv}{\sqrt{U_{\min} - v^2}} = \frac{2\pi}{2-\alpha}, \tag{9}$$

and on the other hand

$$\vartheta^+ - \vartheta^- \geqq \frac{2}{2-\alpha}\int_{-\sqrt{U_{\min}}}^{\sqrt{U_{\min}}} \frac{dv}{\sqrt{U_{\max} - v^2}} = \frac{4}{2-\alpha}\arcsin\sqrt{\frac{U_{\min}}{U_{\max}}}. \qquad \square \tag{10}$$

Using the previous arguments, together with standard results in structural stability, it is already possible, for appropriate values of α, to show the existence of saddle–saddle heteroclinic connections (see Fig. 2). In any case, if in principle saddle–saddle connections occur only for particular values of α, on the other hand, whenever $\vartheta^\pm$ are minima for U, *for every* α they correspond to saddle points. The above techniques allow us to study the dependence of their stable and unstable manifolds on α.

Lemma 2. *Let* (ϑ, φ) *denote the* (*unique, apart from time translations*) *unstable trajectory emanating from* $(\vartheta^-, \vartheta^- + \pi)$ *with increasing* ϑ. *Then it intersects the line* $\varphi = \vartheta + \pi/2$ *in a unique point with first coordinate* $\hat{\vartheta}^- = \hat{\vartheta}^-(\alpha)$. *Moreover,* ϑ *is strictly increasing on* $(\vartheta^-, \hat{\vartheta}^-]$ *and on the same interval* $\varphi = \varphi_\alpha(\vartheta)$ *can be expressed as a function of* ϑ. *Finally,*

$$\alpha_1 < \alpha_2 \quad implies \quad \hat{\vartheta}^-(\alpha_1) < \hat{\vartheta}^-(\alpha_2)$$

and $\varphi_{\alpha_1}(\vartheta) < \varphi_{\alpha_2}(\vartheta)$ *on* $(\vartheta^-, \hat{\vartheta}^-(\alpha_1)]$ (*see also Fig. 3*).
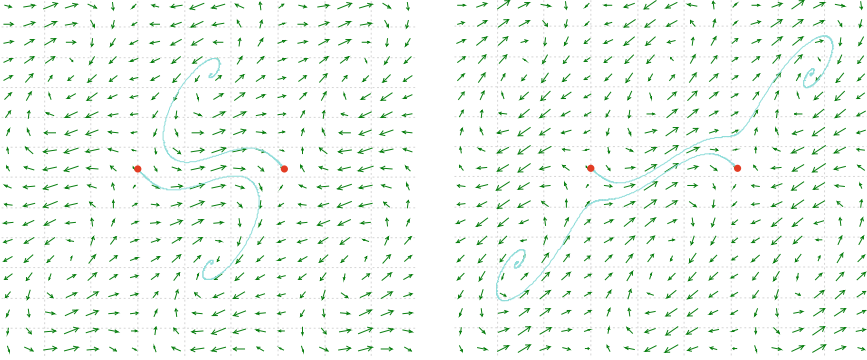
Fig. 2. The two pictures represent the phase portrait of the dynamical system (6) with $U(\vartheta) = 2 - \cos(2\vartheta)$, when $\alpha = 0.5$ (at left) or $\alpha = 1$ (at right). We focus our attention on the saddles $(0, \pi)$ and (π, π) (which satisfy condition (7)). From the mutual positions of the heteroclinic departing from $(0, \pi)$ and the one ending in (π, π) we deduce that the two vector fields are not topologically equivalent. By structural stability we infer the existence, for some $\bar{\alpha} \in (0.5, 1)$, of a saddle connection between $(0, \pi)$ and (π, π)
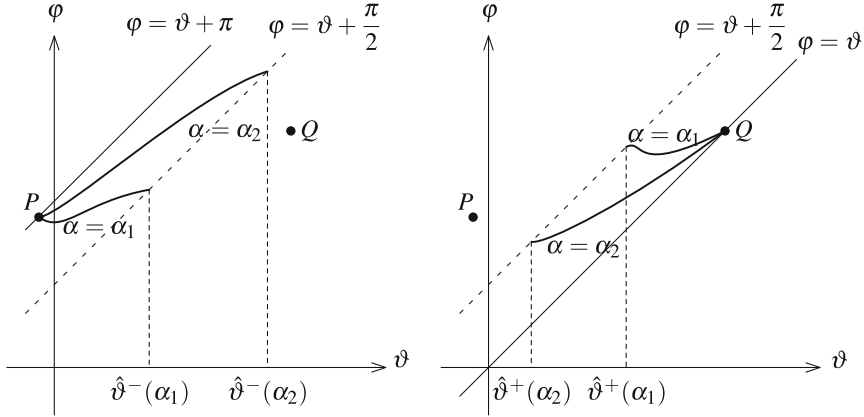

Fig. 3. The unstable (resp. stable) manifold emanating from $P = (\vartheta^-, \vartheta^- + \pi)$ (resp. entering in $Q = (\vartheta^+, \vartheta^+))$, and its dependence on α, according to Lemma 2 (resp. Lemma 3). Here $\alpha_1 < \alpha_2$

Proof. To start with we observe that, for any $\alpha \in (0, 2)$, the linearized matrix for (6) at $(\vartheta^-, \vartheta^- + \pi)$ is

$$J^- = U_{\min} \begin{pmatrix} 2 & -2 \\ \alpha - \mu^- & -\alpha \end{pmatrix},$$

where $\mu^- = U''(\vartheta^-)/U_{\min}$. The eigendirection correspondent to the heteroclinic emanating from $(\vartheta^-, \vartheta^- + \pi)$ is $v^- = (1, v_2^-) = (1, 1 - \lambda_+^-/2)$, where λ_+^- is the positive eigenvalue of J^-. A straight calculation provides

$$v_2^- = v_2^-(\alpha) = \frac{1}{2} + \frac{\alpha}{4} - \frac{1}{4}\sqrt{(2 - \alpha)^2 + 8\mu^-}.$$

On one hand, we have that

$$\frac{d}{d\alpha}v_2^-(\alpha) = \frac{1}{4} + \frac{2-\alpha}{4\sqrt{(2-\alpha)^2 + 8\mu^-}} > 0,$$

implying that, for different values of α, the corresponding unstable trajectories are ordered as claimed near $(\vartheta^-, \vartheta^- + \pi)$. On the other hand, since $v_2^- < 1$, we have that the trajectory is contained in the strip $\pi/2 < \varphi - \vartheta < \pi$ for large negative times.

Now recall that, as above, $v(-\infty) = -\sqrt{U_{\min}}$ and that both ϑ and v are strictly increasing whenever v is smaller than $\sqrt{U_{\min}}$. We deduce that there exists exactly one $\hat{\tau}$ such that

$$v(\hat{\tau}) = 0, \text{ or equivalently } \varphi(\hat{\tau}) = \vartheta(\hat{\tau}) + \frac{\pi}{2}.$$

As a consequence, the value $\hat{\vartheta}^- = \vartheta(\hat{\tau})$ is well defined and, reasoning as in Corollary 1 and in Lemma 1, we can invert $\vartheta = \vartheta(\tau)$ on $(-\infty, \hat{\tau}]$. We deduce that we can write

$$\varphi_\alpha(\vartheta) := \varphi(\tau(\vartheta)), \text{ where } \frac{d\varphi_\alpha}{d\vartheta} = \frac{\alpha}{2} + \frac{U'(\vartheta)}{2U(\vartheta)}\cotan(\varphi_\alpha - \vartheta) \text{ on } (\vartheta^-, \hat{\vartheta}^-].$$
(11)

To conclude the proof we have to show that, if $\alpha_1 < \alpha_2$, then $\varphi_{\alpha_1}(\vartheta) < \varphi_{\alpha_2}(\vartheta)$ where they are defined. To this aim, let by contradiction $\vartheta^* > \vartheta^-$ be such that $\varphi_{\alpha_1}(\vartheta) < \varphi_{\alpha_2}(\vartheta)$ on $(\vartheta^-, \vartheta^*)$, and $\varphi_{\alpha_1}(\vartheta^*) = \varphi_{\alpha_2}(\vartheta*)$. But the above differential equation implies

$$\frac{d(\varphi_{\alpha_2} - \varphi_{\alpha_1})}{d\vartheta}(\vartheta^*) = \frac{\alpha_2 - \alpha_1}{2} > 0,$$

a contradiction. □

Arguing exactly as above, one can prove analogous properties for the stable manifolds.

Lemma 3. *Let (ϑ, φ) denote the (unique, apart from time translations) stable trajectory entering in $(\vartheta^+, \vartheta^+)$ with increasing ϑ. Then it intersects the line $\varphi = \vartheta + \pi/2$ in a unique point with first coordinate $\hat{\vartheta}^+ = \hat{\vartheta}^+(\alpha)$. Moreover, ϑ is strictly increasing on $[\hat{\vartheta}^+, \vartheta^-)$ and, on the same interval, $\varphi = \varphi_\alpha(\vartheta)$ can be expressed as a function of ϑ. Finally,*

$$\alpha_1 < \alpha_2 \text{ implies } \hat{\vartheta}^+(\alpha_1) > \hat{\vartheta}^+(\alpha_2),$$

and $\varphi_{\alpha_1}(\vartheta) > \varphi_{\alpha_2}(\vartheta)$ on $[\hat{\vartheta}^+(\alpha_1), \vartheta^+)$.

By uniqueness, the above unstable/stable trajectories cannot be crossed by any other orbit. To be more precise, we have the following.

Corollary 2. *Let*

$$\vartheta^* \in [\vartheta^-, \vartheta^+] \text{ such that } U'(\vartheta^*) = 0$$

be any *central configuration, and γ be a trajectory of system (6) emanating from $(\vartheta^*, \vartheta^* + \pi)$ and intersecting the set*

$$\Sigma := \left\{ (\vartheta, \varphi) : \ \vartheta^- \leqq \vartheta \leqq \vartheta^+, \ \vartheta + \frac{\pi}{2} \leqq \varphi \leqq \vartheta + \frac{3\pi}{2} \right\}.$$

Then, if γ exits from Σ, it must cross either the union of the segments

$$\left\{ \hat{\vartheta}^-(\alpha) \leqq \vartheta \leqq \vartheta^+, \ \varphi = \vartheta + \frac{\pi}{2} \right\}, \quad \left\{ \vartheta^- \leqq \vartheta \leqq \hat{\vartheta}^+(\alpha), \ \varphi = \vartheta + \frac{3\pi}{2} \right\},$$

or the vertical lines $\vartheta = \vartheta^\pm$. Analogously, for a trajectory asymptotic (in the future) to $(\vartheta^*, \vartheta^*)$, the entering set in

$$\Sigma' := \left\{ (\vartheta, \varphi) : \ \vartheta^- \leqq \vartheta \leqq \vartheta^+, \ \vartheta - \frac{\pi}{2} \leqq \varphi \leqq \vartheta + \frac{\pi}{2} \right\}$$

is the union of the segments

$$\left\{ \hat{\vartheta}^+(\alpha) \leqq \vartheta \leqq \vartheta^+, \ \varphi = \vartheta - \frac{\pi}{2} \right\}, \quad \left\{ \vartheta^- \leqq \vartheta \leqq \hat{\vartheta}^-(\alpha), \ \varphi = \vartheta + \frac{\pi}{2} \right\},$$

and of the vertical lines $\vartheta = \vartheta^\pm$.

Proof. We prove only the first part. If $\vartheta^* = \vartheta^-$ then $\gamma \equiv \gamma_1$, the unique unstable trajectory emanating from the corresponding saddle point with ϑ increasing considered in Lemma 2; but then it exits from Σ through the point $(\hat{\vartheta}^-(\alpha), \hat{\vartheta}^-(\alpha) + \pi/2)$. In the same way, if $\vartheta^* = \vartheta^+$ then $\gamma \equiv \gamma_2$, the unique unstable trajectory emanating from the corresponding saddle point with ϑ decreasing (recall that, if $(\vartheta(\tau), \varphi(\tau))$ solves (6), then $(\vartheta(-\tau), \varphi(-\tau) + \pi)$ also does); in such a case the exit point is $(\hat{\vartheta}^+(\alpha), \hat{\vartheta}^+(\alpha) + 3\pi/2)$. Finally, if $\vartheta^- < \vartheta^* < \vartheta^+$, then γ must lie above γ_1 and below γ_2, and the assertion follows. $\square$

The angles $\hat{\vartheta}^\pm(\alpha)$ defined above represent the (oriented) parabolic *apsidal angles* swept by the parabolic arc from the infinity up to the pericenter. As a consequence of the previous arguments, the appearance of a parabolic trajectory associated with the asymptotic directions $(\vartheta^-, \vartheta^+)$, or, equivalently, the existence of a heteroclinic connection between $(\vartheta^-, \vartheta^- + \pi)$ and $(\vartheta^+, \vartheta^+)$ can be expressed in terms of the corresponding apsidal angles. Summing up, we have proved the following.

Proposition 2. *Let $U \in \mathcal{U}$, $\vartheta^- < \vartheta^+ \in \Theta$, and the monotone functions $\hat{\vartheta}^-(\alpha)$, $\hat{\vartheta}^+(\alpha)$ be defined as in Lemmata 2, 3, respectively. Then system (6) admits a heteroclinic connection between $(\vartheta^-, \vartheta^- + \pi)$ and $(\vartheta^+, \vartheta^+)$ for some value $\alpha = \bar{\alpha} \in (0, 2)$ if and only if*

$$\hat{\vartheta}^-(\bar{\alpha}) = \hat{\vartheta}^+(\bar{\alpha}).$$

In particular, if such a value exists, then it is unique.

The function $\bar{\alpha}$ can be extended to all the triplets $U \in \mathcal{U}_{\vartheta_1 \vartheta_2}$ and ϑ^-, $\vartheta^+ \in \Theta_{\vartheta_1 \vartheta_2}$ as follows:

Definition 3. For any triplet $U \in \mathcal{U}$, $\vartheta^- < \vartheta^+ \in \Theta$, we define the function

$$\bar{\alpha}(\vartheta^-, \vartheta^+, U) = \inf \left\{ \alpha \in (0, 2) : \hat{\vartheta}^-(\alpha) > \hat{\vartheta}^+(\alpha) \right\}$$

If $\vartheta^- > \vartheta^+$ we define $\bar{\alpha}(\vartheta^-, \vartheta^+, U) = \bar{\alpha}(\vartheta^+, \vartheta^-, U)$.

In this way, the previous proposition proves the first point of Theorem 1.

As a final remark, let us notice that the apsidal angles defined above, and the corresponding stable and unstable trajectories, act as a "barrier" for any heteroclinic traveling in the strip $\vartheta^- \leq \vartheta \leq \vartheta^+$ and corresponding to a (not necessarily minimal) parabolic trajectory. Such arguments will turn out to be useful in the proof of Theorems 2 and 3.

Proposition 3. *Let $U \in \mathcal{U}$, ϑ^-, $\vartheta^+ \in \Theta$, and let us assume that*

$$\alpha > \bar{\alpha}(\vartheta^-, \vartheta^+, U).$$

Then (U, α) does not admit any (not necessarily minimal) parabolic trajectory completely contained in the sector $[\vartheta^-, \vartheta^+]$.

Proof. By Theorem 4 (and, in particular, condition (7)) such a parabolic trajectory $x = x(t)$ would correspond to a heteroclinic connection for system (6), joining an equilibrium (say) $(\vartheta^*, \vartheta^* + \pi)$ to another one $(\vartheta^{**} + 2h\pi, \vartheta^{**} + 2h\pi)$, with h being an integer. We want to prove that such a trajectory, under the above assumptions, cannot be completely contained in the strip $[\vartheta^-, \vartheta^+] \times \mathbb{R}$.

To start with, we observe that h must be equal to either 0 or 1. Indeed, the function $v(\tau)$ is non-decreasing along any trajectory, and $v = 0$ whenever $\varphi = \vartheta + \pi/2 + k\pi$, k integer. Without loss of generality, we can assume $h = 0$, so that the trajectory we are considering joins $(\vartheta^*, \vartheta^* + \pi)$ to $(\vartheta^{**}, \vartheta^{**})$. Let us assume by contradiction that it is completely contained in the strip $[\vartheta^-, \vartheta^+] \times \mathbb{R}$; but then, using the notations of Corollary 2, it must both exit Σ and enter Σ' across a single point belonging to the line $\varphi = \vartheta + \pi/2$ and the strip. This immediately provides a contradiction with the selfsame corollary, since

$$\alpha > \bar{\alpha} \quad \Longrightarrow \quad \hat{\vartheta}^-(\bar{\alpha}) > \hat{\vartheta}^+(\bar{\alpha}). \quad \square$$

3. Minimality Properties Near Equilibria

The purpose of this section is to develop a first investigation into the minimality properties of zero energy solutions of (1) with respect to the Maupertuis' functional

$$J(x) = J([a, b]; x) := \int_a^b \frac{1}{2} |\dot{x}(t)|^2 \, dt \cdot \int_a^b V(x(t)) \, dt,$$

where $V = (U, \alpha) \in \mathcal{V}$. Indeed, let us recall that

$$\min \left\{ \mathscr{A}([a', b']; y) : a' < b', \, y \in H^1(a', b') + \text{further conditions} \right\}$$
$$= \min \left\{ \sqrt{2J([a, b]; x)} : x \in H^1(a, b) + \text{same conditions} \right\}$$

for every (fixed) $a < b$, indeed J is invariant under reparameterizations (see [1]). As a consequence, every parabolic trajectory is a critical point of J, at least when restricted on suitably small bounded intervals.

In particular, we want to evaluate the second differential of J along zero-energy critical points. In order to do this, we first perform a change of time-scale essentially equivalent to that of Devaney, which we exploited in Section 2. In polar coordinates J reads as

$$J(r, \vartheta) = \int_a^b \frac{1}{2} \left[\dot{r}^2(t) + r^2(t)\dot{\vartheta}^2(t) \right] \, dt \cdot \int_a^b \frac{U(\vartheta(t))}{r^\alpha(t)} \, dt;$$

introducing the time-variable

$$\tau = \tau(t) = \int_a^t r^{-(2+\alpha)/2}(\xi) \, d\xi$$

we obtain (noting with a prime " $'$ " the derivative with respect to τ)

$$J(r, \vartheta) = \int_a^{\tau^*} \frac{1}{2} \left[\left(r^{-(2+\alpha)/4} r' \right)^2 + \left(r^{(2-\alpha)/4} \vartheta' \right)^2 \right] \, d\tau \cdot \int_a^{\tau^*} r^{(2-\alpha)/2} U(\vartheta) \, d\tau,$$

where r and ϑ depend now on τ, and

$$\tau^* = \int_a^b r^{-\frac{2+\alpha}{2}} \, dt.$$

We introduce the change of variables

$$\rho = r^{\frac{2-\alpha}{4}}, \quad \rho' = \frac{2-\alpha}{4} r^{-\frac{2+\alpha}{4}} r'$$

in order to obtain the Maupertuis' functional depending on (ρ, ϑ), that is,

$$J(\rho, \vartheta) = F(\rho, \vartheta) \, G(\rho, \vartheta),$$

where

$$F(\rho, \vartheta) = \int_0^{\tau^*} \frac{8}{(2-\alpha)^2} (\rho')^2 + \frac{1}{2} (\rho \vartheta')^2 \, d\tau, \quad G(\rho, \vartheta) = \int_0^{\tau^*} \rho^2 U(\vartheta) \, d\tau.$$

The energy relation (2) written in terms of τ, ρ and ϑ yields $F(\rho, \vartheta) = G(\rho, \vartheta)$. Now let (ρ, ϑ) be a critical point of J; then

$$dJ(\rho, \vartheta) = dF(\rho, \vartheta) \, G(\rho, \vartheta) + F(\rho, \vartheta) \, dG(\rho, \vartheta) = 0.$$

From the energy relation we then deduce that if (ρ, ϑ) is a zero-energy critical point for J, then $dF(\rho, \vartheta) = -dG(\rho, \vartheta)$. As a consequence we can write

$$d^2 J(\rho, \vartheta) = G(\rho, \vartheta) \left[d^2 F(\rho, \vartheta) + d^2 G(\rho, \vartheta) \right] - 2 \left[dG(\rho, \vartheta) \right]^2.$$

More explicitly, given a compactly supported variation (λ, ξ) and a zero-energy critical point for J, (ρ, ϑ), we have that

$$d^2 F(\rho, \vartheta)[(\lambda, \xi), (\lambda, \xi)] = \int_0^{\tau^*} \frac{16(\lambda')^2}{(2-\alpha)^2} + (\rho\xi')^2 + 4\vartheta'\xi'\rho\lambda + (\vartheta')^2\lambda^2 \, d\tau,$$

$$dG(\rho, \vartheta)(\lambda, \xi) = \int_0^{\tau^*} 2\rho\lambda U(\vartheta) + \rho^2 U'(\vartheta)\xi \, d\tau,$$

$$d^2 G(\rho, \vartheta)[(\lambda, \xi), (\lambda, \xi)] = \int_0^{\tau^*} 2\lambda^2 U(\vartheta) + 4\lambda\rho U'(\vartheta)\xi + \rho^2 U''(\vartheta)\xi^2 \, d\tau.$$

In the rest of the paper we will prove that trajectories asymptotic to minimal central configurations are, indeed, at least locally, minimizers for J. The main result of this section concerns the non-minimality of trajectories which are asymptotic to "sufficiently" non-minimal central configurations.

Proposition 4. *Let $\bar{\vartheta}$ be such that $U'(\bar{\vartheta}) = 0$, and let (ρ, ϑ) be any critical point of J, defined for $\tau \in [0, +\infty)$, such that $\vartheta(\tau) \to \bar{\vartheta}$ as $\tau \to +\infty$. Finally, let α be such that*

$$U''(\bar{\vartheta}) < -\frac{(2-\alpha)^2}{8} U(\bar{\vartheta}). \tag{12}$$

Then, for $a' < b'$ sufficiently large, (ρ, ϑ) restricted to (a', b') is neither a minimum for $\mathscr{A}$, nor for J.

Proof. We prove the result for the Maupertuis' functional J; indeed, the computations for the action are similar but simpler (recall that, under the above notations, $\mathscr{A}(\rho, \vartheta) = F(\rho, \vartheta) + G(\rho, \vartheta)$). More precisely, we are going to provide a compactly supported variation $(0, \xi)$ along which $d^2 J(\rho, \vartheta)$ will result negative. By the above calculations we have

$$d^2 J(\rho, \vartheta)[(0, \xi), (0, \xi)]$$
$$= \int_{a'}^{b'} \rho^2 U(\vartheta) \, d\tau \cdot \int_{a'}^{b'} \rho^2 \left[(\xi')^2 + U''(\vartheta)\xi^2 \right] \, d\tau - 2 \left(\int_{a'}^{b'} \rho^2 U'(\vartheta)\xi \, d\tau, \right)^2$$
$$\leqq C \int_{a'}^{b'} \rho^2 \left[(\xi')^2 + (\mu + \varepsilon)\xi^2 \right] \, d\tau,$$

where $C > 0$, $\varepsilon > 0$ is small, $a' < b'$ are large and $\mu := U''(\bar{\vartheta})$.

Now, we claim that the solutions of the linear equation

$$(\rho^2\xi')' = (\mu + 2\varepsilon)\rho^2\xi \tag{13}$$

have infinitely many zeroes for τ large; as a consequence, choosing a', b' to be two of such zeroes, testing with ξ and integrating by parts, one would obtain

$$\int_{a'}^{b'} \rho^2 \left[(\xi')^2 + (\mu + \varepsilon)\xi^2 \right] d\tau = -\varepsilon \int_{a'}^{b'} \rho^2 \xi^2 d\tau < 0,$$

providing the desired result.

In order to establish the oscillatory nature of Equation (13) we will apply Sturm comparison principle. First of all, by combining the Euler–Lagrange equation for ρ

$$\frac{16}{(2-\alpha)^2} \rho'' = (\vartheta')^2 \rho + 2\rho U(\vartheta),$$

and the zero-energy relation

$$\frac{8}{(2-\alpha)^2} (\rho')^2 + \frac{1}{2}(\vartheta')^2 \rho^2 = \rho^2 U(\vartheta),$$

we have that the function

$$p(\tau) := \frac{\rho'(\tau)}{\rho(\tau)} \quad \text{satisfies} \quad p' = -2p^2 + \frac{(2-\alpha)^2}{4} U(\vartheta)$$

on $[0, +\infty)$. But then, since $\vartheta(\tau) \to \bar{\vartheta}$ as $\tau \to +\infty$, by elementary comparison we easily obtain

$$\lim_{\tau \to +\infty} \frac{\rho'(\tau)}{\rho(\tau)} = \sqrt{\frac{(2-\alpha)^2}{8} U(\bar{\vartheta})} =: \gamma.$$

We finally infer that, for some constant k, and for τ large, there holds $\rho(\tau) < ke^{(\gamma+\varepsilon)\tau}$. But then Sturm comparison principle applies to (13) and to

$$(k^2 e^{2(\gamma+\varepsilon)\tau} \xi')' = (\mu + 2\varepsilon)k^2 e^{2(\gamma+\varepsilon)\tau} \xi,$$

yielding that every nodal interval of the second equation contains (at least) one zero of the first one; to conclude we observe that this last equation writes

$$\xi'' + 2(\gamma + \varepsilon)\xi' - (\mu + 2\varepsilon)\xi = 0,$$

which is oscillatory if and only if, for some $\varepsilon > 0$, there holds $(\gamma+\varepsilon)^2 + (\mu+2\varepsilon) < 0$, that is, if and only if

$$\mu < -\gamma^2. \quad \square$$

Corollary 3. *Let $\bar{\vartheta}$ be such that $U'(\bar{\vartheta}) = 0$, and let x, defined for $t \in [0, +\infty)$, be a solution of (1), (2), such that $x(t)/|x(t)| \to (\cos \bar{\vartheta}, \sin \bar{\vartheta})$ as $t \to +\infty$. Finally let α satisfy condition (12). Then, x can neither be a free-time Morse minimizer, nor a fixed-time one.*

Let us mention that this result completely agrees with the one proved, in the complementary case of collision trajectories, in [4]; on the other hand, quite surprisingly, it is not clear wether trajectories corresponding to "not too-strict" maxima for U (that is, maxima such that $-\gamma^2 < U''(\bar{\vartheta}) < 0$) may be minimizers for J.

4. Constrained Minimizers

In this section we prove Theorem 1 in the case in which $\vartheta^-, \vartheta^+ \in \Theta_{\vartheta_1 \vartheta_2}$ are such that

$$\pi < \vartheta^+ - \vartheta^- \leqq 2\pi.$$

By time reversibility, the case $-2\pi \leqq \vartheta^+ - \vartheta^- < -\pi$ will also follow. In such situation the results in [5] apply almost straightforwardly; we summarize them here, making explicit the minor changes we need in the present situation.

The main idea is that, since parabolic minimizers exist only for special values of α, one first introduces more general objects which, on the contrary, exist for every α.

Definition 4. We say that $x = (r, \vartheta) \in H^1_{loc}(\mathbb{R})$ is a *constrained Morse minimizer* if

– $\min_t r(t) = 1$;
– $r(t) \to +\infty, \vartheta(t) \to \vartheta^\pm$ as $t \to \pm\infty$;
– for every $t_1 < t_2, t_1' < t_2'$, and $z \in H^1(t_1', t_2')$, there holds

$$z(t_i') = x(t_i), \ i = 1, 2, \ \min_{[t_1', t_2']} |z| = \min_{[t_1, t_2]} r \implies \mathscr{A}([t_1, t_2]; x) \leqq \mathscr{A}([t_1', t_2']; z).$$

We denote with $\mathscr{M} = \mathscr{M}(U, \alpha)$ the set of constrained Morse minimizers.

As for Definition 1, the previous definition also makes sense for any pair of central configurations, not necessarily for minimal ones. From this point of view, Proposition 4 provides a necessary condition for $\mathscr{M}$ to be non-empty in the case of non-minimal central configurations. In any case, when not explicitly remarked, we will always refer to constrained minimizers between minimal asymptotic configurations.

The following two lemmas describe the main properties of constrained minimizers; they are a direct consequence of the theory developed in [5], Sections 5 and 6

Lemma 4. *For every $\alpha \in (0, 2)$ the set $\mathscr{M}$ is not empty. If $x = (r, \vartheta) \in \mathscr{M}$ then (up to a time translation) there exist $t_* \leqq 0 \leqq t_{**}$ such that:*

1. $r(t) = 1$ *if and only if $t \in [t_*, t_{**}], \dot{r}(t) < 0$ (resp. > 0) if and only if $t < t_*$ (resp. $t > t_{**}$);*
2. x *satisfies (1) for every $t \notin [t_*, t_{**}]$ and (2) for every t;*
3. *one of the following alternatives hold:*
   (a) $t_* < t_{**}, x$ *is $\mathscr{C}^1$ for every $t, \dot{r} \equiv 0$ in $[t_*, t_{**}]$;*
   (b) $t_* = t_{**} = 0$ *and x is $\mathscr{C}^1$ for every t;*
   (c) $t_* = t_{**} = 0$ *and $\dot{x}$ has a jump discontinuity at 0, with*

$$-\dot{r}(0^-) = \dot{r}(0^+) > 0, \qquad \dot{\vartheta}(0^-) = \dot{\vartheta}(0^+).$$

Definition 5. In view of the previous lemma, for any $x = (r, \vartheta) \in \mathcal{M}$, we define its (*angular*) *position and velocity jumps*, respectively, as

$$\Delta_{\text{pos}}(x) := |\vartheta(t_{**}) - \vartheta(t_*)|, \qquad \Delta_{\text{vel}}(x) := |\dot{r}(t_{**}^+) - \dot{r}(t_*^-)|$$

(in particular, they cannot both be different from 0, while they are both 0 if and only if alternative (b) above holds).

Lemma 5. *Let* $0 < \alpha_1 < \alpha_2 < 1$ *and let us assume that there exists* $x_i \in \mathcal{M}(U, \alpha_i), i = 1, 2,$ *such that*

$$\Delta_{\text{pos}}(x_1) > 0 \quad \text{and} \quad \Delta_{\text{vel}}(x_2) > 0.$$

Then there exist $\bar{\alpha} \in (\alpha_1, \alpha_2)$ *and* $\bar{x} \in \mathcal{M}(U, \bar{\alpha})$ *such that*

$$\Delta_{\text{pos}}(\bar{x}) = \Delta_{\text{vel}}(\bar{x}) = 0 \quad \text{and } \bar{x} \text{ is a corresponding free Morse minimizer.}$$

In the planar case, the general theory we have recalled above can be complemented using the results regarding Devaney's system, which we obtained in Section 2.

Remark 1. Let $x = (r, \vartheta) \in \mathcal{M}$ and $t_* \leqq 0 \leqq t_{**}$ be as in Lemma 4. Via the variable and time changes introduced in Section 2, we can define $\tau_* \leqq 0 \leqq \tau_{**}$ in order to obtain that $x|_{\{t < t_*\}}$ corresponds, in the phase plane of system (6), to a part of the unstable trajectory emanating from $(\vartheta^-, \vartheta^- + \pi)$, with ϑ increasing (and $\tau < \tau_*$). Moreover, $\vartheta^- < \vartheta < \hat{\vartheta}^-(\alpha)$ along the trajectory (both of these facts descend from the fact that, for $t < t_*$, $\dot{r}$ is negative, and thus, also, is r' for $\tau < \tau_*$). Analogously, $x|_{\{t > t_{**}\}}$ corresponds to a part of the stable trajectory entering in $(\vartheta^+, \vartheta^+)$ (with $\tau > \tau_{**}$), and $\hat{\vartheta}^+(\alpha) < \vartheta < \vartheta^+$.

Finally, for $\tau \in (\tau_*, \tau_{**})$ (whenever such an interval is nonempty), (ϑ, φ) lies in a 1-to-1 way on the line of equation $\varphi = \vartheta + \pi/2$. In particular, (ϑ, φ) is completely contained in the strip

$$\left\{ (\vartheta, \varphi) : \vartheta^- < \vartheta < \vartheta^+, \ \vartheta < \varphi < \vartheta + \pi \right\}.$$

Taking into account Lemmata 2 and 3, it is possible to give a full characterization of constrained Morse minimizers in terms of the functions $\hat{\vartheta}^\pm(\alpha)$ defined therein.

Proposition 5. *Let* $U \in \mathcal{U}, \alpha \in (0, 2), \vartheta^\pm$ *as above. Then the corresponding constrained Morse minimizer* x_α *is unique (up to time translations) and (see also Fig. 4)*

– $\Delta_{\text{pos}}(x_\alpha) > 0$ *if and only if* $\hat{\vartheta}^-(\alpha) < \hat{\vartheta}^+(\alpha)$;
– $\Delta_{\text{vel}}(x_\alpha) > 0$ *if and only if* $\hat{\vartheta}^-(\alpha) > \hat{\vartheta}^+(\alpha)$;
– $\Delta_{\text{pos}}(x_\alpha) = \Delta_{\text{vel}}(x_\alpha) = 0$ *if and only if* $\hat{\vartheta}^-(\alpha) = \hat{\vartheta}^+(\alpha)$.

Proof. Let x_α be *any* element of $\mathcal{M}(U, \alpha)$ and let us denote with (ϑ, φ) the corresponding arc in the Devaney plane. Moreover, let $\tau_* \leqq 0 \leqq \tau_{**}$ be the values of τ corresponding to t_*, t_{**}, respectively.
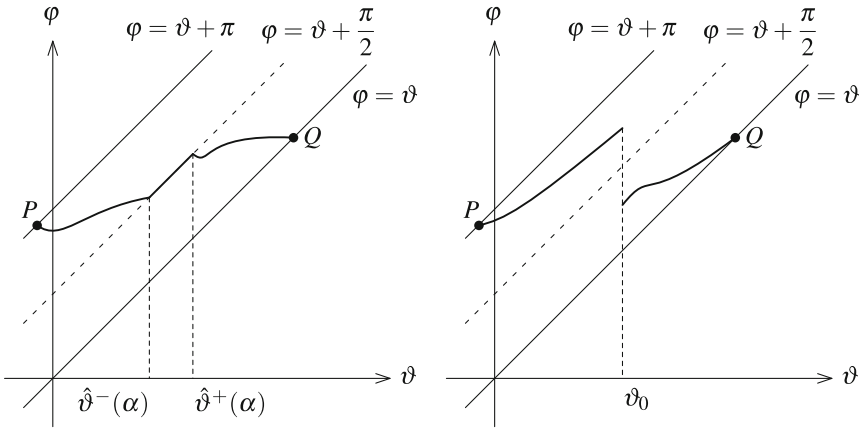
Fig. 4. On the left, a position-jumping constrained minimizer between $P = (\vartheta^-, \vartheta^- + \pi)$ and $Q = (\vartheta^+, \vartheta^+)$, with $\Delta_{\text{pos}}(x_\alpha) = \hat{\vartheta}^+(\alpha) - \hat{\vartheta}^+(\alpha)$. On the right, a velocity-jumping one; in such a case, the jump discontinuity is symmetric with respect to $(\vartheta_0, \vartheta_0 + \pi/2)$ (see Equation (14))

We start by assuming that $\Delta_{\text{vel}}(x_\alpha) > 0$. This means that $\dot{r}$ never vanishes, it is not defined in $t_* = t_{**} = 0$, and $-\dot{r}(0^-) = \dot{r}(0^+) > 0$. Since x_α, and hence ϑ, are continuous, we obtain that φ must be discontinuous. More precisely, letting $\vartheta_0 := \vartheta(0)$ and recalling system (5), we have that

$$-2U(\vartheta_0)\cos(\varphi(0^-) - \vartheta_0) = -r'(0^-) = r'(0^+) = 2U(\vartheta_0)\cos(\varphi(0^+) - \vartheta_0),$$

which implies (also recalling Remark 1)

$$\frac{\varphi(0^+) + \varphi(0^-)}{2} = \vartheta_0 + \frac{\pi}{2}. \tag{14}$$

On the other hand, for τ negative (resp. positive) we have that φ must be greater (resp. less) than $\vartheta + \pi/2$. Recalling Lemmas 2 and 3, we deduce that $\hat{\vartheta}^-(\alpha) > \hat{\vartheta}^+(\alpha)$. Let us now show that for every α satisfying this last condition, there exists exactly one $\vartheta_0 \in (\hat{\vartheta}^+(\alpha), \hat{\vartheta}^-(\alpha))$ such that condition (14) holds. This, together with the fact that $r(0) = 1$, will imply uniqueness for the velocity-jumping minimizer. Thanks to Lemmas 2 and 3 we have that, for $\vartheta \in (\hat{\vartheta}^+(\alpha), \hat{\vartheta}^-(\alpha))$, both the unstable manifold $\varphi = \varphi_-(\vartheta)$ and the stable one $\varphi = \varphi_+(\vartheta)$ are well defined as functions of ϑ, and that they both satisfy Equation (11), that is,

$$\frac{\mathrm{d}\varphi_\pm}{\mathrm{d}\vartheta} = \frac{\alpha}{2} + \frac{U'(\vartheta)}{2U(\vartheta)}\cotan(\varphi_\pm - \vartheta).$$

Let us define the (smooth) auxiliary function $\psi(\vartheta) := \varphi_+(\vartheta) + \varphi_-(\vartheta) - 2\vartheta - \pi$. Then condition (14) is equivalent to $\psi(\vartheta_0) = 0$. We easily obtain $\pm\psi(\hat{\vartheta}^\pm(\alpha)) > 0$ and

$$\frac{\mathrm{d}\psi}{\mathrm{d}\vartheta} = \alpha + \frac{U'(\vartheta)}{2U(\vartheta)}\left[\cotan(\varphi_+ - \vartheta) + \cotan(\varphi_- - \vartheta)\right] - 2$$

$$= \frac{U'(\vartheta)}{2U(\vartheta)}\frac{\sin(\psi + \pi)}{\sin(\varphi_+ - \vartheta) + \sin(\varphi_- - \vartheta)} - (2 - \alpha).$$

We deduce that $\psi(\vartheta_0) = 0$ implies $d\psi(\vartheta_0)/d\vartheta < 0$, so that ψ has exactly one zero as claimed.

Let us come to the case in which $\Delta_{\mathrm{vel}}(x_\alpha) = 0$. Using Lemmas 2 and 3 again, we have that both the unstable trajectory and the stable one meet the line $\varphi = \vartheta + \pi/2$ in exactly one point. We deduce that, for some $\tau_* \leqq \tau_{**}$

$$\vartheta(\tau_*) = \hat\vartheta^-(\alpha), \qquad \vartheta(\tau_{**}) = \hat\vartheta^+(\alpha).$$

This, if also $\Delta_{\mathrm{pos}}(x_\alpha) = 0$, immediately yields $\hat\vartheta^-(\alpha) = \hat\vartheta^+(\alpha)$. On the other hand, let us assume that $\tau_* < \tau_{**}$. Then, by minimality, the corresponding segment on the line $\varphi = \vartheta + \pi/2$ must be traveled with ϑ monotone; since ϑ is $\mathscr{C}^1$ and $\vartheta'(\tau^*-) > 0$, we deduce that $\vartheta' > 0$ on $[\tau_*, \tau_{**}]$, that is, $\hat\vartheta^-(\alpha) < \hat\vartheta^+(\alpha)$. Again, in both cases, the uniqueness of x_α inside its category is due to the initial conditions $r(\tau_*) = r(\tau_{**}) = 1$.

Now the proof easily follows, indeed, in each of the two triplets of conditions, at least one instance must occur and each one excludes the others. □

We are ready to prove our main theorem in the present case.

Proof (Proof of Theorem 1, case $\pi < \vartheta^+ - \vartheta^- \leqq 2\pi$). As already mentioned, the first part of the theorem is a consequence of Proposition 2 and Definition 3, while the second easily follows by comparing Proposition 2 and the third instance of Proposition 5. To prove the last part we can use Lemma 5 in combination with Proposition 5. In this way, we are left to show the existence of two values α_1, α_2 such that the order between $\hat\vartheta^-(\alpha_i)$ and $\hat\vartheta^+(\alpha_i)$ is reversed by switching between $i = 1$ and $i = 2$. To this aim, reasoning exactly as in the proof of Lemma 1, one can prove the analogues of estimates (9), (10), that is,

$$\frac{2}{2-\alpha} \arcsin\sqrt{\frac{U_{\min}}{U_{\max}}} \leqq \hat\vartheta^-(\alpha) - \vartheta^- \leqq \frac{\pi}{2-\alpha},$$

$$\frac{2}{2-\alpha} \arcsin\sqrt{\frac{U_{\min}}{U_{\max}}} \leqq \vartheta^+ - \hat\vartheta^+(\alpha) \leqq \frac{\pi}{2-\alpha}.$$

Summing up and rearranging, we obtain

$$(\vartheta^+ - \vartheta^-) - \frac{2\pi}{2-\alpha} \leqq \hat\vartheta^+(\alpha) - \hat\vartheta^-(\alpha) \leqq (\vartheta^+ - \vartheta^-) - \frac{4}{2-\alpha} \arcsin\sqrt{\frac{U_{\min}}{U_{\max}}}.$$

It is now trivial, taking into account the limitations for $\vartheta^+ - \vartheta^-$, to verify that if α_1 is small then $\hat\vartheta^-(\alpha_1) < \hat\vartheta^+(\alpha_1)$, while if α_2 is near 2 then the opposite inequality holds. □

We conclude this section with a few words about the case $0 < \vartheta^+ - \vartheta^- \leqq \pi$.

Remark 2. If $0 < \vartheta^+ - \vartheta^- \leq \pi$, then explicit conditions can be provided to show that the number $\bar{\alpha}$, and hence parabolic minimizers, may or may not exist, depending on the properties of U. For instance, if U is a small perturbation of a constant (that is, V is an anisotropic small perturbation of an isotropic potential), then $\bar{\alpha}$ does not exist, recall Fig. 1. On the other hand, it is possible to construct angular potentials U with arbitrarily small $\vartheta^+ - \vartheta^-$, such that the corresponding $\bar{\alpha}$ exists: roughly speaking, this can be done by choosing U very larger than $U_{\min}$ on a compact subinterval of $(\vartheta^-, \vartheta^+)$, see Lemma 6.11 in [5].

5. General Winding Number

In the previous section we ruled out the case in which $\vartheta^+ - \vartheta^- \in (\pi, 2\pi]$. This section is devoted to reformulating the case

$$2h\pi < \vartheta^+ - \vartheta^- \leq 2(h+1)\pi, \quad h \geq 1$$

in terms of that previous case, completing the proof of Theorem 1 (again, the case $-2(h+1)\pi \leq \vartheta^+ - \vartheta^- < -2h\pi$ is easily treated using time reversibility). This can be done using the following conformal change of variables.

Lemma 6. *Let* $x = (r, \vartheta)$ *be defined for* $t \in [a, b]$, *with* $\min_t r > 0$ *and let* $y = (\rho, \varphi)$ *be defined for* $\tau \in [a', b']$, *with* $\min_\tau \rho > 0$. *Let us assume that, for some* $\beta > 0$, *there holds*

$$\tau = a' + \int_a^t r^{2(1-\beta)/\beta} \, dt, \quad r(t) = \rho^\beta(\tau), \quad \vartheta(t) = \beta\varphi(\tau),$$

$b' = a' + \int_a^b r^{2(1-\beta)/\beta} \, dt$. *Finally, let* U *be* 2π-*periodic and*

$$V(x) = \frac{U(\vartheta)}{r^\alpha}.$$

Then

$$\int_a^b \frac{1}{2}|\dot{x}|^2 + V(x) \, dt = \beta^2 \int_{a'}^{b'} \frac{1}{2}|y'|^2 + \tilde{V}(y) \, d\tau,$$

where

$$\tilde{V}(y) = \frac{\tilde{U}(\varphi)}{\rho^{\tilde{\alpha}}} \text{ with } \tilde{U}(\varphi) = \frac{U(\beta\varphi)}{\beta^2} \text{ and } \tilde{\alpha} = 2 - \beta(2 - \alpha).$$

Proof. By direct computation we have

$$V(x) = \frac{U(\vartheta)}{r^\alpha} = \frac{U(\beta\varphi)}{\rho^{\alpha\beta}}$$

and

$$|\dot{x}|^2 = \dot{r}^2 + r^2\dot{\vartheta}^2 = \beta^2\rho^{2\beta-2}\dot{\rho}^2 + \beta^2\rho^{2\beta}\dot{\varphi}^2$$

$$= \beta^2\rho^{2\beta-2}\left[(\rho')^2 + \rho^2(\vartheta')^2\right]\left(\frac{d\tau}{dt}\right)^2 = \beta^2\rho^{2(1-\beta)}|y'|^2.$$

Substituting the action, we obtain

$$\int_a^b \frac{1}{2}|\dot{x}|^2 + V(x)\,dt = \beta^2 \int_{a'}^{b'} \left(\rho^{2(1-\beta)} \frac{|y'|^2}{2} + \frac{U(\beta\varphi)/\beta^2}{\rho^{\alpha\beta}} \right) \cdot \rho^{-2(1-\beta)}\,d\tau. \qquad \square$$

Remark 3. It is immediate to show that if $U \in \mathscr{U}_{\vartheta_1\vartheta_2}$, $\vartheta^\pm \in \Theta_{\vartheta_1\vartheta_2}$, and $\tilde{U}$ is defined as in the previous lemma, then $\tilde{U} \in \mathscr{U}_{\frac{\vartheta_1}{\beta}\frac{\vartheta_2}{\beta}}$ and $\frac{\vartheta^\pm}{\beta} \in \Theta_{\frac{\vartheta_1}{\beta}\frac{\vartheta_2}{\beta}}$.

We are in a position to conclude the proof of Theorem 1. This is done through the following proposition.

Proposition 6. *Let* $2h\pi < \vartheta^+ - \vartheta^- \leq 2(h+1)\pi$ *for some* $h \geq 1$ *and let us define*

$$\tilde{\vartheta}^\pm = \frac{\vartheta^\pm}{h+1} \quad \text{and} \quad \tilde{U}(\vartheta) = \frac{U((h+1)\vartheta)}{(h+1)^2}.$$

Then $\pi < \tilde{\vartheta}^+ - \tilde{\vartheta}^- \leq 2\pi$ *and*

$$\bar{\alpha}(\vartheta^-, \vartheta^+, U) = 2 - \frac{2 - \bar{\alpha}(\tilde{\vartheta}^-, \tilde{\vartheta}^+, \tilde{U})}{h+1},$$

the latter being well defined by Section 4.

Proof. We have to show that (U, α), $\alpha \in (0, 2)$, admits a parabolic Morse minimizer if and only if α is equal to the right-hand side of the expression above. To start with, we observe that if

$$\alpha \leq 2 - \frac{1}{h},$$

then (U, α) cannot admit a parabolic Morse minimizer. Indeed, on the contrary, Lemma 1 would apply, yielding

$$2 - \frac{2\pi}{\vartheta^+ - \vartheta^-} \leq \alpha,$$

in contradiction with the fact that $\vartheta^+ - \vartheta^- > 2h\pi$. On the other hand, if $\alpha > 2 - 1/h$, we can apply Lemma 6 and Remark 3, obtaining that trajectories connecting $\vartheta^\pm$ with potential (U, α) correspond to trajectories connecting $\tilde{\vartheta}^\pm$ with potential $(\tilde{U}, \tilde{\alpha})$, with $\tilde{\alpha} = 2 - (h+1)(2-\alpha)$. As a consequence, in order to prove the proposition, we simply have to show that the results of Section 4 can be applied to this latter context. To this aim, the only non-immediate thing to check is that $\tilde{\alpha} \in (0, 2)$. This is easily proved by monotonicity, since

$$2 - \frac{1}{h} < \alpha < 2 \quad \Longrightarrow \quad 1 - \frac{1}{h} < \tilde{\alpha} < 2. \qquad \square$$

6. Proof of Theorems 2 and 3

The strategy in the proof of both theorems is the following. To start with, we assume by contradiction the existence of a colliding minimizer and we study a class of constrained minimization problems, restricting to the paths having distance from the origin of at least ε. Next, we let $\varepsilon \to 0$ and perform a blow-up procedure obtaining as a limit a global zero-energy path, which connects two central configurations at $r \to \infty$ and solves the equation outside the constraint. Finally, we obtain a contradiction to the existence of such a path by exploiting the results obtained in the previous sections. To this last aim, a crucial tool is given by the following lemma, which is a generalization of Proposition 3 to fixed-time constrained minimizers with $\Delta_{\mathrm{vel}} = 0$ connecting (not necessarily minimal) central configurations.

Lemma 7. *Let (U, α) be fixed and $x = (r, \vartheta) \in H^1_{\mathrm{loc}}(\mathbb{R})$ be such that, for some $t_* \leqq 0 \leqq t_{**}$, it holds*

- *x is $\mathscr{C}^1$ and is minimal under fixed-time variations;*
- *$|x| \to \infty$ and $x/|x| \to \widetilde{\vartheta}^{\pm}$ as $t \to \pm\infty$;*
- *$r(t) \equiv 1$ if and only if $t \in [t_*, t_{**}]$, $\dot{r}(t) < 0$ (resp. $\dot{r}(t) > 0$) if and only if $t < t_*$ (resp. $t > t_{**}$);*
- *x solves (1) for $t \notin [t_*, t_{**}]$ and (2) for every t;*
- *there exist $\vartheta^{\pm}$ minimal central configurations such that $[\widetilde{\vartheta}^-, \widetilde{\vartheta}^+] \subset [\vartheta^-, \vartheta^+]$.*

Then $\alpha \leqq \bar{\alpha}(\vartheta^-, \vartheta^+, U)$.

Proof. Reasoning as in Remark 1, we can project x to Devaney's phase plane. As usual, the corresponding graph consists of the junction of three arcs in the strip: the part of an unstable trajectory emanating from (say) $(\widetilde{\vartheta}^-, \widetilde{\vartheta}^- + \pi)$ up to A, its crossing point with the straight line $\varphi = \vartheta + \pi/2$; the arc of the stable manifold entering in $(\widetilde{\vartheta}^+, \widetilde{\vartheta}^+)$ back to B, its crossing point with the same straight line; and, finally, a segment joining the two crossings, which is traveled monotonically in ϑ by minimality. Since ϑ must be $\mathscr{C}^1$ across the whole junction, and trajectories of (6) cross the line $\varphi = \vartheta + \pi/2$ with increasing ϑ, we infer that

$$\vartheta_A \leqq \vartheta_B.$$

On the other hand, since the whole junction is completely contained in the strip $[\vartheta^-, \vartheta^+]$, Corollary 2 implies that

$$\vartheta_A \geqq \hat{\vartheta}^-(\alpha), \qquad \vartheta_B \leqq \hat{\vartheta}^+(\alpha),$$

and the conclusion follows from the definition of $\bar{\alpha}$. ☐

Remark 4. In the previous lemma $\alpha = \bar{\alpha}$ forces $\vartheta_A = \vartheta_B$ and hence $\widetilde{\vartheta}^{\pm} = \vartheta^{\pm}$.

Proof (Proof of Theorem 2). Taking advantage of the conformal equivariance of the problem, arguing as in Section 4 we can reduce to the case $\vartheta^+ \leqq \vartheta^- + 2\pi$. We argue by contradiction, assuming that for some $x_1 = (r_1, \varphi_1)$, $x_2 = (r_2, \varphi_2)$ in the sector $[\vartheta^-, \vartheta^+]$ and $t_1 < t_2$ there exists a Bolza minimizer completely contained in the sector and traveling through the origin. As we did in Definition 4 for Morse

minimizers, we can introduce the notion of constrained Bolza ones. More precisely, let us consider the set of paths within the sector having the required endpoints:

$$\Gamma := \left\{ x = (r, \vartheta) : r(t_i) = r_i, \ \vartheta(t_i) = \varphi_i, \ \vartheta(t) \in [\vartheta^-, \vartheta^+] \text{ for } t \in [t_1, t_2] \right\}.$$

Next, we consider a small parameter $\varepsilon > 0$ and we compare the values of the two following constrained minimization problems: the one featuring equality constraint

$$c_\varepsilon^c := \min \left\{ \mathscr{A}(t_1, t_2; x) : x \in \Gamma \text{ and } \min_{[t_1, t_2]} r(t) = \varepsilon \right\}$$

with the obstacle-type problem

$$c_\varepsilon^c := \min \left\{ \mathscr{A}(t_1, t_2; x) : x \in \Gamma \text{ and } \min_{[t_1, t_2]} r(t) \geqq \varepsilon \right\}$$

(it is standard to prove that they are both achieved). Of course, c_ε is nondecreasing in ε and $c_\varepsilon \leqq c_\varepsilon^c, \forall \varepsilon > 0$. Arguing as in the proof of Theorem 18 in [29], if $c_\varepsilon < c_\varepsilon^c$ for every small positive ε, then we are done. Hence, we can reduce our analysis to the case of a vanishing sequence $\varepsilon_n \to 0$ with $c_{\varepsilon_n} = c_{\varepsilon_n}^c$ and such that the two constrained minimization problems share the same class of minimizers. Let us take a sequence x_n of such minimizers: they can interact with the constraints in essentially two ways. On one hand, they are $\mathscr{C}^1$ when they touch the lines $\vartheta = \varphi_i$. On the other hand, concerning the circular constraint as in Section 4, we may have either $\Delta_{\text{vel}}(x_n) > 0$ or $\Delta_{\text{vel}}(x_n) = 0$ (it can be shown that the classification in terms of position and velocity jumps also holds for fixed-time minimizers, at least for ε small, see [5, Proposition 3.6]). It is immediate to rule out the case $\Delta_{\text{vel}}(x_n) > 0$, because a local variation can be easily produced in contradiction with the fact that $c_{\varepsilon_n} = c_{\varepsilon_n}^c$. Following the argument of the proof of Proposition 20 of [29] again, one sees that the energies are uniformly bounded along the sequence. Defining the blow-up sequence

$$\hat{x}_n(t) = \frac{1}{\varepsilon_n} x_n(\varepsilon_n^{-\frac{2+\alpha}{2}} t),$$

we can argue as in [29] (pp. 486–488) to pass to the limit and find a zero-energy $\mathscr{C}^1$-path, minimal under fixed-time variations for the homogeneous potential (U, α). We observe that such paths cannot touch the lines $\vartheta = \varphi_i$: indeed, it would be a $\mathscr{C}^1$ junction, in contradiction to the uniqueness for Cauchy problems. As a consequence, the blow-up limit consists of a pair of parabolic arcs, connected by a circular arc, within the sector $(\vartheta^-, \vartheta^+)$. The two parabolic arcs have ingoing and outgoing asymptotic central configurations $\widetilde{\vartheta}^-, \widetilde{\vartheta}^+$ such that $\vartheta^- \leqq \widetilde{\vartheta}^- < \widetilde{\vartheta}^+ \leqq \vartheta^+$. Since $\alpha > \bar{\alpha}$ this contradicts Lemma 7. □

Remark 5. The previous proof, together with Remark 4, immediately provides Proposition 1. Moreover, it is possible to show that, if $\alpha = \bar{\alpha}$, then any Bolza minimizer within the sector is either collisionless or it collides with ingoing/outgoing directions precisely ϑ^- and ϑ^+.

Proof (Proof of Theorem 3). First of all, we can take advantage of the conformal invariance to reduce to the case $k = 1$. Next, we again set the constrained minimization problems over the set of loops winding one time around the origin:

$$c_\varepsilon^c(\alpha, U) = \min \left\{ \mathscr{A}(0, T; x) \; ; \; x(0) = x(T) \; , \; \deg(x, 0) = 1 \text{ and } \min_{[0,T]} r(t) = \varepsilon \right\}$$

(here $\deg(x, 0)$ denotes the topological degree of the map x). We also set

$$c^c = \liminf_{\varepsilon \to 0} c_\varepsilon^c.$$

It is easy to prove that a minimizing periodic trajectory in this class corresponds to a simple loop. We remark that, under the previous notation, our aim is to prove that there exists $\varepsilon > 0$ such that $c_\varepsilon^c < c^c$. This will be done in two steps.

Step 1: If every maximum of U satisfies condition (12) *then there exists $\varepsilon > 0$ such that $c_\varepsilon^c(\alpha, U) \leqq c^c$.* Indeed, if not, we would have $c_\varepsilon^c > c^c$ for all positive ε and hence, for every small $\varepsilon_2 > 0$, we can find a smaller ε_1 such that

$$c_{\varepsilon_1, \varepsilon_2} = \min \left\{ \mathscr{A}(0, T; x) \; ; \; x(0) = x(T) \; , \; \deg(x, 0) = 1 \text{ and } \min_{[0,T]} r(t) \in [\varepsilon_1, \varepsilon_2) \right\}$$

is achieved. In this way, we find the existence of a fixed-time constrained minimizing trajectory with $\Delta_{\mathrm{vel}} = 0$. Reasoning again as in [29], letting $\varepsilon_2 \to 0$ and going to a blow-up sequence, we find in the limit a parabolic fixed-time constrained minimizing trajectory with $\Delta_{\mathrm{vel}} = 0$. Now we look at its asymptotic central configurations and we go to the phase plane. We have to deal with the case when the corresponding trajectory connects a pair of stationary points $(\widetilde{\vartheta}^-, \widetilde{\vartheta}^- + \pi)$ and $(\widetilde{\vartheta}^+, \widetilde{\vartheta}^+)$ and, by the absence of self intersections, we infer $\widetilde{\vartheta}^+ \leqq \widetilde{\vartheta}^- + 2\pi$. Now, if $\widetilde{\vartheta}^-$ is a maximum for U, then thanks to Corollary 3, we reach a contradiction. On the other hand, if $\widetilde{\vartheta}^-$ is a minimum, we can apply Lemma 7 with $[\vartheta^-, \vartheta^+] := [\widetilde{\vartheta}^-, \widetilde{\vartheta}^- + 2\pi]$ and obtain a contradiction with the fact that $\alpha > \bar{\alpha}(\widetilde{\vartheta}^-, \widetilde{\vartheta}^- + 2\pi, U)$.

Step 2: if U and $\widetilde{U}$ share the same global minimizers, at the same level U_{min}, then $c^c(\alpha, U) = c^c(\alpha, \widetilde{U})$. Indeed let $(r(t), \vartheta(t))$ achieve $c^c(\alpha, U)$, then also $(r(t), \vartheta^*)$, for any ϑ^* minimal configuration for U, achieves the same level. On this last path the actions with potentials U and $\widetilde{U}$ coincide. Therefore $c^c(\alpha, \widetilde{U}) \leqq c^c(\alpha, U)$; the claim follows by exchanging the roles of U and $\widetilde{U}$.

Step 3: conclusion. Let U satisfy the assumptions of the theorem. We can always construct another Morse potential $\widetilde{U} \in \mathscr{U}$, still satisfying (4), such that $\min \widetilde{U} = \min U$, $\widetilde{U} \geqq U$, $\widetilde{U} \neq U$ and, last but not least, $\widetilde{U}$ satisfies (12). Now, by Step 1, there exists $\varepsilon > 0$ such that $c_\varepsilon^c(\alpha, \widetilde{U}) \leqq c^c(\alpha, \widetilde{U})$, the former being achieved by a collisionless loop $\widetilde{x}$. Evaluating the action relative to U along $\widetilde{x}$ we obtain

$$c_\varepsilon^c(\alpha, U) < c_\varepsilon^c(\alpha, \widetilde{U}) \leqq c^c(\alpha, U),$$

as was to be shown. $\square$

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