Minimization of the k-th eigenvalue of the Dirichlet Laplacian

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Abstract

For every $k \in \mathbb{N}$, we prove the existence of a quasi-open set minimizing the k-th eigenvalue of the Dirichlet Laplacian among all sets of prescribed Lebesgue measure. Moreover, we prove that every minimizer is bounded and has a finite perimeter. The key point is the observation that such quasi-open sets are shape subsolutions for an energy minimizing free boundary problem.

1. Introduction

The question of minimizing the first Dirichlet eigenvalue of the Laplace operator among all open sets of prescribed measure is an old one. Rayleigh conjectured in 1877 that the solution is the ball, and this was positively proved by Faber and Krahn in the 1920s. It was also noticed that the minimizer for the second eigenvalue is the union of two equal disjoint balls. For $k \geq 3$, not only is the optimal shape unknown (some conjectures based on numerical evidence are formulated in [12]), but the existence of an open and smooth optimal shape is also an unsolved question. For $k = 3$, the existence of a quasi-open set was proved in [6], while for $k \geq 4$ it was proved only that a minimizer exists, provided that for every lower index a bounded minimizer exists.

The key general existence result is due to Buttazzo and Dal Maso [7] where, in particular, it is proved that for every bounded open set $D \subseteq \mathbb{R}^N$, the problem

$$\min\{\lambda_k(A)|A \subseteq D, |A| = c, A \text{ quasi-open}\}$$

has a solution. Here, $\lambda_k(A)$ stands for the k-th eigenvalue of the Dirichlet Laplacian on the set A, multiplicities being counted. To replace D with $\mathbb{R}^N$ is not a trivial matter (see the note at the end of the paper concerning the recent result of Mazzoleni and Pratelli). The main challenge of [3] was to give a concentration-compactness result for the resolvent operators, which could explain the behaviour of a minimizing sequence of domains in $\mathbb{R}^N$. The missing step for proving the existence of a solution for λ_{k+1} was related to the regularity of the optimal shapes for $\lambda_1, \dots, \lambda_k$.

The main purpose of this paper is to introduce the analysis of the *shape subsolutions* for an energy minimizing free boundary problem as a tool to handle shape optimization problems associated to min-max functionals. As a direct consequence, one can prove not only the existence of a solution for the problem

$$\min\{\lambda_k(A)|A \subseteq \mathbb{R}^N, |A| = c\} \tag{1}$$

in the family of quasi-open sets, but can also extract some qualitative information, such as proving that *every* solution is a bounded set with finite perimeter.

In the literature, the regularity of the free boundaries is well understood only for energy like minimizers (see [1,2,8,11]). If for $k = 1$ or $k = 2$, problem (1) can be seen as a "classical" free boundary problem and thus regularity can be extracted using the ideas introduced by ALT AND CAFFARELLI [1], for $k \geq 3$, the k-th eigenvalue is a critical point and the regularity of the free boundary problems associated with this kind of functional requires a new approach.

The key tool we introduce in order to deal with this kind of problem relies on the analysis of the *shape subsolutions* for the energy. We observe that every solution A of (1) is a shape subsolution for an energy minimization problem, that is, there exists $\Lambda > 0$ (small enough) such that

$$\forall \tilde{A} \subseteq A \quad \min_{u \in H_0^1(A)} \frac{1}{2} \int |\nabla u|^2 \, \mathrm{d}x - \int u \, \mathrm{d}x + \Lambda|A|$$

$$\leq \min_{u \in H_0^1(\tilde{A})} \frac{1}{2} \int |\nabla u|^2 \, \mathrm{d}x - \int u \, \mathrm{d}x + \Lambda|\tilde{A}|. \tag{2}$$

The minimization of the (torsion) energy

$$E(A) = \min_{u \in H_0^1(A)} \frac{1}{2} \int |\nabla u|^2 \, \mathrm{d}x - \int u \, \mathrm{d}x \tag{3}$$

among all (quasi-open) sets A of prescribed measure, possibly satisfying some inclusion constraints, is a classical free boundary problem (see [2]) for which existence of a solution and primary regularity results are quite well understood.

The main challenge in our case is that the minimizers of (1) are only subsolutions in the sense of (2), so that full information on the free boundary cannot, a priori, be obtained. Roughly speaking, the main point which allows us to pass

from (1) to (2) is that the variation of eigenvalues for inner local perturbations of the geometric domain can be controlled by the variation of the torsion energy. This is not the case for outer geometric perturbations. Briefly, if we replace the strong information about the minimality in (1) by the weak information given by (2), we are able to handle the saddle point character of the *k*-th eigenvalue in the free boundary problem, but we lose all the information related to the outer perturbations. This is a main stop in order to extract more regularity for the solutions of (1), as, for example, openness of the optimal shapes.

2. Analysis of Shape Subsolutions

Let A be a measurable set of finite measure. We denote

$$\tilde{H}_0^1(A) := \{u \in H^1(\mathbb{R}^N) : u = 0 \text{ almost everywhere on } \mathbb{R}^N \setminus A\}.$$

It is well known that there exists a quasi-open set $\omega_A \subseteq A$ such that

$$H_0^1(\omega_A) = \tilde{H}_0^1(A).$$

Above, H^1 and H_0^1 stand for the classical Sobolev spaces. With this observation, and based on the monotonicity properties of the eigenvalues with respect to inclusions, solving problem (1) in the family of quasi-open sets with the classical definition of the Sobolev space, or solving it in the family of measurable sets associated to $\tilde{H}_0^1$, are equivalent. This remark remains true for a larger class of shape optimization problems associated to decreasing functionals with respect to inclusions. We refer the reader to [7] or [5, Chapter 4] for a collection of results involving the Dirichlet spectrum of the Laplace operator on quasi-open sets and γ-convergence.

As a consequence of the identity above, one can endow the family of measurable sets with a distance issued from the γ-convergence. Precisely, for two measurable sets A and B of finite measure, we introduce

$$d_\gamma(A, B) = \int |u_A - u_B| \, \mathrm{d}x,$$

where $u_A \in H_0^1(A)$ denotes the minimizer in (3) extended by zero to an element of $H^1(\mathbb{R}^N)$.

Definition 1. We say that a measurable set A of finite measure is a local shape subsolution for the energy problem if there exists $\delta > 0$ and $\Lambda > 0$ such that, for every measurable set $\tilde{A}, |\tilde{A} \setminus A| = 0, d_\gamma(A, \tilde{A}) \leqq \delta$ we have

$$E(A) + \Lambda|A| \leqq E(\tilde{A}) + \Lambda|\tilde{A}|. \tag{4}$$

The locality in the definition above is expressed through the γ-distance.

Here is the main result of this section.

Theorem 1. *Assume A is a local shape subsolution for the energy. Then A is bounded, has finite perimeter and its fine interior has the same measure as A (i.e., $|A \setminus \omega_A| = 0$).*

Proof. Fine interior. Using the subsolution property with $\tilde{A} = \omega_A$ and the fact that $E(A) = E(\omega_A)$ we get $|A| \leq |\omega_A|$, so that $|A \setminus \omega_A| = 0$.

Finite perimeter. Let us simply denote $u := u_A$ the unique minimizer of the torsion energy $E(A)$ in (3). In order to prove that A has finite perimeter we consider, for every $\varepsilon > 0$, the test function $u_\varepsilon = (u - \varepsilon)^+$ and write from (4)

$$\frac{1}{2} \int |\nabla u|^2 \, dx - \int u \, dx + \Lambda |\{u > 0\}| \leq \frac{1}{2} \int |\nabla u_\varepsilon|^2 \, dx - \int u_\varepsilon \, dx + \Lambda |\{u_\varepsilon > 0\}|.$$

This perturbation is valid, since $\lim_{\varepsilon \to 0} d_\gamma(A, \{u_\varepsilon > 0\}) = 0$.

Consequently,

$$\frac{1}{2} \int_{0 \leq u \leq \varepsilon} |\nabla u|^2 \, dx + \Lambda |\{0 \leq u \leq \varepsilon\}| \leq \int (u - u_\varepsilon) \, dx = \int_{0 \leq u \leq \varepsilon} u + \varepsilon |\{u > \varepsilon\}| \leq \varepsilon |A|.$$

Using the Cauchy–Schwarz inequality we have

$$\left(\int_{0 \leq u \leq \varepsilon} |\nabla u| \, dx \right)^2 \leq \int_{0 \leq u \leq \varepsilon} |\nabla u|^2 \, dx |\{0 \leq u \leq \varepsilon\}| \leq \frac{2}{\Lambda} \varepsilon^2 |A|^2,$$

so that

$$\int_{0 \leq u \leq \varepsilon} |\nabla u| \, dx \leq \sqrt{2/\Lambda} \varepsilon |A|.$$

Using the co-area formula, we find $\delta_n > 0, \delta_n \to 0$ such that

$$\mathscr{H}^{N-1}(\partial^* \{u > \delta_n\}) \leq \sqrt{2/\Lambda} |A|.$$

Passing to the limit, we get

$$\mathscr{H}^{N-1}(\partial^* \{u > 0\}) \leq \sqrt{2/\Lambda} |A|.$$

Since $\omega_A = \{u > 0\}$ and since $\omega_A = A$ almost everywhere, we get that the perimeter of A is less than $\sqrt{2/\Lambda} |A|$.

Boundedness. The proof of boundedness involves an estimate of Alt-Caffarelli type (see [1]) adapted to the energy minimization problem. First, we notice that

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^N} d_\gamma(A, A \setminus \overline{B}_r(x)) \leq \lim_{r \to 0} \sup(|u_A|_\infty^2 + |u_A|_\infty) \operatorname{cap}(B_r) = 0,$$

so that for r smaller than some $r_0 = r_0(|A|, \delta)$ and for every $x \in \mathbb{R}^N$ we have $d_\gamma(A, A \setminus \overline{B}_r(x)) < \delta$. $\quad\square$

We give the following result, which is adapted from [1, Lemma 3.4].

Lemma 1. *Let $\tau \in (0, 1)$. There exists $r_0 > 0, C_0 > 0$ such that for every $x_0 \in \mathbb{R}^N$ and $r \in (0, r_0)$*

$$\sup_{x \in B_{\sqrt{\tau}r}(x_0)} u(x) \leq C_0 r \implies u = 0 \text{ on } B_{\tau r}(x_0). \tag{5}$$

Proof of the Lemma. For the convenience of the reader, we give the proof which follows a perturbation argument similar to [1, Lemma 3.4] adapted to the energy E. In the following computations we assume that $x_0 = 0$ in order to simplify notation. For $\varepsilon > 0$, we introduce the following function:

$$\begin{cases} -\Delta v_r = 1 & \text{in } B_{\sqrt{\tau}r} \setminus \overline{B}_{\tau r} \\ v_r = 0 & \text{on } \partial B_{\tau r} \\ v_r = \varepsilon & \text{on } \partial B_{\sqrt{\tau}r} \end{cases}.$$

Rescaling v_r by $\psi_r(x) = v_r(rx)$, we get

$$\begin{cases} -\Delta \psi_r = r^2 & \text{in } B_{\sqrt{\tau}} \setminus \overline{B}_{\tau} \\ \psi_r = 0 & \text{on } \partial B_{\tau} \\ \psi_r = \varepsilon & \text{on } \partial B_{\sqrt{\tau}} \end{cases}.$$

Let $\varepsilon = \sup_{x \in B_{\sqrt{\tau}r}} u(x)$ (which will be controlled by $C_0 r$). Since $u \in L^\infty(\mathbb{R}^N)$, with norm controlled by $|A|$ (as a consequence of the isoperimetric inequalitiy for the L^∞ norm of the torsion function), we have $\varepsilon < +\infty$.

By the construction of v_r we have that $u \wedge v_r \in H_0^1(A \setminus \overline{B}_{\tau r})$, so we can use it as a test function. We have

$$\frac{1}{2} \int_{B_{\sqrt{\tau}r}} |\nabla u|^2 \, dx - \int_{B_{\sqrt{\tau}r}} u \, dx + \Lambda |B_{\tau r} \cap A|$$

$$\leq \frac{1}{2} \int_{B_{\sqrt{\tau}r} \setminus \overline{B}_{\tau r}} |\nabla(u \wedge v_r)|^2 \, dx - \int_{B_{\sqrt{\tau}r} \setminus \overline{B}_{\tau r}} u \wedge v_r \, dx.$$

Using the Cauchy–Schwarz inequality and the equation satisfied by v_r,

$$\frac{1}{2} \int_{B_{\tau r}} |\nabla u|^2 \, dx - \int_{B_{\tau r}} u \, dx + \Lambda |B_{\tau r} \cap A|$$

$$\leq \frac{1}{2} \int_{(B_{\sqrt{\tau}r} \setminus \overline{B}_{\tau r}) \cap \{u > v\}} |\nabla v_r|^2 - |\nabla u|^2 \, dx + \int_{B_{\sqrt{\tau}r} \setminus \overline{B}_{\tau r}} (u - v)^+ \, dx$$

$$\leq \int_{(B_{\sqrt{\tau}r} \setminus \overline{B}_{\tau r}) \cap \{u > v\}} |\nabla v_r|^2 - \nabla u \nabla v_r \, dx + \int_{B_{\sqrt{\tau}r} \setminus \overline{B}_{\tau r}} (u - v)^+ \, dx$$

$$= -\int_{B_{\sqrt{\tau}r} \setminus \overline{B}_{\tau r}} \nabla(u - v_r)^+ \nabla v \, dx + \int_{B_{\sqrt{\tau}r} \setminus \overline{B}_{\tau r}} (u - v_r)^+ \, dx$$

$$= -\int_{\partial B_{\tau r}} \frac{\partial v_r}{\partial n} (u - v_r)^+ \, d\mathcal{H}^{N-1}$$

$$= |v_r'(\tau r)| \int_{\partial B_{\tau r}} (u - v_r)^+ \, d\mathcal{H}^{N-1} = |v_r'(\tau r)| \int_{\partial B_{\tau r}} u \, d\mathcal{H}^{N-1}.$$

Finally,

$$\frac{1}{2} \int_{B_{\tau r}} |\nabla u|^2 \, dx - \int_{B_{\tau r}} u \, dx + \Lambda |B_{\tau r} \cap A| \leq |v_r'(\tau r)| \int_{\partial B_{\tau r}} u \, d\mathcal{H}^{N-1}. \quad (6)$$

Let us assume $\varepsilon \leqq \frac{\Lambda}{2}$, which is controlled by the choice of C_0 and r_0. Then (6) gives

$$\frac{1}{2} \int_{B_{\tau r}} |\nabla u|^2 \, dx + \frac{1}{2}\Lambda|B_{\tau r} \cap A| \leqq |v'_r(\tau r)| \int_{\partial B_{\tau r}} u \, d\mathcal{H}^{N-1}. \qquad (7)$$

On the other hand, we have, by rescaling the boundary trace theorem in $W^{1,1}(B_1)$,

$$\int_{\partial B_{\tau r}} u \, d\mathcal{H}^{N-1} \leqq C(N) \left[\frac{1}{\tau r} \int_{B_{\tau r}} u \, dx + \int_{B_{\tau r}} |\nabla u| \, dx \right].$$

So using hypothesis (5) we get

$$\int_{\partial B_{\tau r}} u \, d\mathcal{H}^{N-1} \leqq C(N) \left[\frac{1}{\tau r} C_0 r |B_{\tau r} \cap A| + \int_{B_{\tau r}} |\nabla u| \, dx \right]$$

$$= C(N) \left[\frac{1}{\tau} C_0 |B_{\tau r} \cap A| + \int_{B_{\tau r}} |\nabla u| \, dx \right]$$

$$\leqq C(N) \left[\left(\frac{1}{\tau} C_0 + \frac{1}{2} \right) |B_{\tau r} \cap A| + \frac{1}{2} \int_{B_{\tau r}} |\nabla u|^2 \, dx \right].$$

Combining

$$\int_{\partial B_{\tau r}} u \, d\mathcal{H}^{N-1} \leqq C(N) \left[\left(\frac{1}{\tau} C_0 + \frac{1}{2} \right) |B_{\tau r} \cap A| + \frac{1}{2} \int_{B_{\tau r}} |\nabla u|^2 \, dx \right]$$

with (7) we get

$$\min \left\{ \frac{1}{2}, \Lambda \right\} \leqq |v'_r(\tau r)| C(N) \left(\left(\frac{1}{\tau} C_0 + \frac{1}{2} \right) + \Lambda \right),$$

as soon as $\int_{B_{\tau r}} |\nabla u|^2 \, dx + |B_{\tau r} \cap A| > 0$. This leads to a contradiction, as soon as $|v'_r(\tau r)|$ is small enough. But $|v'_r(\tau r)| = \frac{\psi'_r(\tau)}{r} \to 0$ when r and C_0 are vanishing, hence for some r_0, C_0 small enough and all $r \leqq r_0$ we get a contradiction. Finally, we should have $\int_{B_{\tau r}} |\nabla u|^2 \, dx + |B_{\tau r} \cap A| = 0$, which gives us the conclusion of the Lemma. □

Proof of Theorem 1 (continuation). Assume for contradiction that the set A is unbounded. There exists a sequence of points $x_n \in A$ such that $|x_n| \to +\infty$ and the distance between any two of them is greater than $3r_0$. We can also assume that $u(x_n) \neq 0$, in the measure theoretic sense.

By the choice of x_n, using Lemma 1, we know that $\sup_{B_{\tau r_0}(x_n)} u \geqq C_0 r_0 \wedge \frac{1}{2}$, so there exists $y_n \in B_{\tau r_0}(x_n)$ such that $u(y_n) \geqq C_0 r_0 \wedge \frac{1}{2}$. Since the function

$$u(x) + \frac{|x - y_n|^2}{2N}$$

is subharmonic in $\mathbb{R}^N$, we get for every $r > 0$ that

$$C_0 r_0 \wedge \frac{1}{2} \leqq u(y_n) \leqq C(N) \frac{1}{r^N} \int_{B_r(y_n)} \left[u(x) + \frac{|x - y_n|^2}{2N} \right] dx.$$

For a small r there exists a constant C independent of n such that

$$\int_{B_r(y_n)} u(x)\,\mathrm{d}x \geqq Cr^{N+1},$$

which gives that $\int u(x)\,\mathrm{d}x = +\infty$, which is a contradiction with $u \in L^1(A)$. $\quad\square$

Remark 1. The construction of the constants C_0 and r_0 in Lemma 1 leads to a control of the diameter of A in terms of $|A|$, Λ and δ.

3. Minimization of the k-th Eigenvalue of the Dirichlet Laplacian

Let $A \subseteq \mathbb{R}^N$ be a quasi-open set of finite measure. Then the injection $H_0^1(A) \subseteq L^2(A)$ is compact and the spectrum of the Dirichlet Laplacian consists only of eigenvalues which can be denoted (counting the multiplicity)

$$0 < \lambda_1(A) \leqq \lambda_2(A) \leqq \cdots \leqq \lambda_k(A) \leqq \cdots \to +\infty.$$

For every $k \in \mathbb{N}$, there exists $u_k \in H_0^1(A) \setminus \{0\}$ such that $-\Delta u_k = \lambda_k(A)u_k$, this equality being understood in the sense

$$\forall \varphi \in H_0^1(A), \quad \int_A \nabla u_k \nabla \varphi \,\mathrm{d}x = \lambda_k(A) \int_A u_k \varphi \,\mathrm{d}x.$$

We consider the minimization problem

$$A \mapsto \lambda_k(A) + |A| \tag{8}$$

in the family of all quasi-open sets of $\mathbb{R}^N$. We notice that problems (1) and (8) are equivalent, and have the same solutions up to a homothety. In a first step, we assume that A is a solution to problem (8); the existence will be proved in a second step by an induction argument over k. We shall first prove only that, if it exists, A is also a shape subsolution of the energy problem, so Theorem 1 applies.

Theorem 2. *Assume A is a solution of problem* (8). *Then A is a local shape subsolution for the energy problem.*

Proof. This is a direct consequence of [4, Theorem 3.4] (see also Lemma 3 in the Appendix below) which asserts that there exists a constant $c_k(A)$ depending on A and k such that

$$\left| \frac{1}{\lambda_k(\tilde{A})} - \frac{1}{\lambda_k(A)} \right| \leqq c_k(A)d_\gamma(\tilde{A}, A). \tag{9}$$

Since $E(\tilde{A}) - E(A) = \frac{1}{2}d_\gamma(\tilde{A}, A)$, for δ small enough, such that $\delta \leqq \frac{1}{4c_k(A)}\frac{1}{\lambda_k(A)}$ and for every $\tilde{A}$ such that $\tilde{A} \subseteq A$ and $d_\gamma(\tilde{A}, A) \leqq \delta$, we get that

$$\lambda_k(\tilde{A}) - \lambda_k(A) \leqq c_k'(A)(E(\tilde{A}) - E(A)), \tag{10}$$

where $c_k'(A)$ depends on δ, k and A. Combining the optimality of A expressed in (8) by $|A| - |\tilde{A}| \leqq \lambda_k(\tilde{A}) - \lambda_k(A)$ together with (10), we obtain that A is a local shape subsolution for the energy problem, with constant $\Lambda = \frac{1}{c_k'(A)}$. $\quad\square$

We recall the following result from [6, Theorem 3.5] which is based on a con-centration—compactness argument for the resolvent operators (see [3, Theorem 2.2]).

Lemma 2. *Assume that a bounded minimizer exists for problem (1) for $k = 1, \dots, N$. Then, at least one minimizer exists for $k = N + 1$.*

We can now formulate the existence result and give a piece of qualitative infor-mation on the minimizers.

Theorem 3. *For every $k \in \mathbb{N}$, problem (1) has at least one solution. Moreover, every solution is bounded and has finite perimeter.*

Proof. The fact that every solution, if it exists, is bounded and has finite perimeter is a consequence of Theorems 1 and 2.

For $k = 1, 2$ the (unique) solution is a ball and two equal balls, respectively. The existence of a solution for an arbitrary k, follows from the previous lemma, provided that a bounded solution exists for $j = 1, 2, \dots, k - 1$. Since this is true for $k = 1, 2$, an induction argument based on Theorems 1 and 2 concludes the proof. $\square$

4. Appendix

For the convenience of the reader, we recall the proof of inequality (9) given in [4, Theorem 3.4], rephrased here in the context of quasi-open sets.

Lemma 3. *Let $A \subseteq \mathbb{R}^N$ be a quasi-open set of finite measure. For every $k \in \mathbb{N}$, there exists a constant $c_k(A)$ depending only on A such that for every $j \leqq k$ and for every quasi-open set $B \subseteq A$, we have*

$$\left| \frac{1}{\lambda_j(A)} - \frac{1}{\lambda_j(B)} \right| \leqq c_k(A) d_\gamma(A, B). \tag{11}$$

Proof. Let us fix $k \in \mathbb{N}$. We consider $V_k \subseteq L^2(\mathbb{R}^N)$ the linear space generated by the first k eigenfunctions of the Dirichlet–Laplacian on A. The space V_k is a finite dimensional subspace of $H_0^1(A)$. We denote by $R_A : L^2(A) \to L^2(A)$ the resolvent operator of the Dirichlet–Laplacian $R_A = (-\Delta)^{-1}$, defined by $R_A(f) = u_{A,f}$, where $u_{A,f} \in H_0^1(A)$ satisfies $-\Delta u_{A,f} = f$, in the sense that

$$\forall \varphi \in H_0^1(A), \quad \int_A \nabla u_{A,f} \nabla \varphi \, \mathrm{d}x = \int_A f \varphi \, \mathrm{d}x.$$

We set $P_k : L^2(A) \to V_k$ as the L^2-orthogonal projector on V_k and introduce the finite rank, positive, self adjoint operators

$$T_k^A = P_k \circ R_A \circ P_k,$$

$$T_k^B = P_k \circ R_B \circ P_k.$$

Denoting $\mu_j(T_k^A)$, $\mu_j(T_k^B)$ the jth eigenvalues of the operators T_k^A, T_k^B, respectively (multiplicites being counted), we have

$$\forall j = 1, \ldots k, \quad \mu_j(T_k^B) \leqq \frac{1}{\lambda_j(B)}, \quad \mu_j(T_k^A) = \frac{1}{\lambda_j(A)}. \tag{12}$$

Indeed, for every $j = 1, \ldots, k$ we have (see for instance [10, Corollaries 3 and 4, pp. 1089–1090])

$$\mu_j(T_k^B) = \mu_j(P_k \circ R_B \circ P_k) \leqq \mu_1(P_k)^2 \frac{1}{\lambda_j(B)}.$$

Since $\mu_1(P_k) = |P_k|_{\mathscr{L}(L^2(A))} = 1$ the first inequality in (12) comes immediately.

To prove the second inequality in (12), we notice in the same way that

$$\mu_j(T_k^A) \leqq \frac{1}{\lambda_j(A)}. \tag{13}$$

Moreover, for every $j = 1, \ldots, k$ if u_j is the jth eigenfunction of R_A associated with $\lambda_j(A)$ we have that

$$T_k^A u_j = P_k \circ R_A \circ P_k u_j = \mu_j(A) u_j, \tag{14}$$

since $P_k u_j = u_j$. Combining (13) and (14) we get the second inequality in (12).

Consequently, we have for every $j = 1, \ldots, k$

$$0 \leqq \frac{1}{\lambda_j(A)} - \frac{1}{\lambda_j(B)} \leqq \mu_j(T_k^A) - \mu_j(T_k^B)$$

$$\leqq |T_k^A - T_k^B|_{\mathscr{L}(L^2(A))} = |P_k \circ R_A \circ P_k - P_k \circ R_B \circ P_k|_{\mathscr{L}(L^2(A))}.$$

But

$$|P_k \circ R_A \circ P_k - P_k \circ R_B \circ P_k|_{\mathscr{L}(L^2(A))}$$
$$= \sup_{|u|_{L^2(A)} \leqq 1} \langle (P_k \circ R_A \circ P_k - P_k \circ R_B \circ P_k) u, u \rangle_{L^2(A) \times L^2(A)}$$
$$= \sup_{|u|_{L^2(A)} \leqq 1} \langle (R_A - R_B) P_k u, P_k u \rangle_{L^2(A) \times L^2(A)}.$$

Let us note that $Range(P_k) \subseteq L^\infty(A)$ and, moreover, that

$$P_k : L^2(A) \to L^\infty(A)$$

is bounded. Indeed, let $u \in L^2(A)$, $|u|_{L^2(A)} \leqq 1$ and $P_k u = \alpha_1 u_1 + \cdots + \alpha_k u_k$. Here, the eigenfunctions $u_1, \ldots, u_k$ of the Dirichlet Laplacian on A are supposed to be L^2-normalised. Since $|P_k u|_{L^2(A)} \leqq 1$, we get $\sum_{j=1}^k \alpha_j^2 \leqq 1$, hence $|\alpha_j| \leqq 1$ for every $j = 1, \ldots, k$. From [9, Example 2.1.8] (which holds as well for quasi-open sets of finite measure as a consequence of the density of smooth open sets in the family of quasi-open sets, for the γ-distance) we have that

$$|u_j|_{L^\infty(A)} \leqq C\lambda_j(A)^{N/4},$$

where the constant C depends only on the dimension of the space. Finally, we observe that

$$|P_k u|_{L^\infty(A)} \leqq C \sum_{j=1}^{k} |\alpha_j| \lambda_j(A)^{N/4} := C_k(A).$$

We have

$$\langle (R_A - R_B) P_k u, P_k u \rangle_{L^2(A) \times L^2(A)}$$

$$\leqq \int_A |R_A(P_k u) - R_B(P_k u)||P_k u| \, \mathrm{d}x$$

$$\leqq C_k(A) \int_A |R_A(P_k u) - R_B(P_k u)| \, \mathrm{d}x \leqq 2C_k(A) \int_A R_A(|P_k u|) - R_B(|P_k u|) \, \mathrm{d}x$$

$$\leqq 2C_k(A)^2 \int_A R_A(1) - R_B(1) \, \mathrm{d}x = 2C_k(A)^2 d_\gamma(A, B).$$

The last inequality is a consequence of the weak maximum principle. □

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