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Analysis of Nematic Liquid Crystals with Disclination Lines

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Abstract

We investigate the structure of nematic liquid crystal thin films described by the Landau–de Gennes tensor-valued order parameter model with Dirichlet boundary conditions on the sides of nonzero degree. We prove that as the elasticity constant goes to zero in the energy, a limiting uniaxial nematic texture forms with a finite number of defects, all of degree $\frac{1}{2}$ or all of degree $-\frac{1}{2}$, corresponding to vertical disclination lines at those locations. We also state a result on the limiting behavior of minimizers of the Chern–Simons–Higgs model without magnetic field that follows from a similar proof.

1. Introduction

We investigate disclination line defects in a thin nematic liquid crystal by using a tensor-valued order parameter description based on the Landau–de Gennes theory. The unknown field Q in this theory is $\mathscr{S}$-valued such that $Q = Q(x, y)$, where $\mathscr{S}$ is the space of 3×3, real symmetric, traceless matrices, and (x, y) varies in a bounded domain Ω in $\mathbb{R}^2$. For simplicity, we assume that Ω is a simply connected bounded domain with a C^3 boundary in the plane, representing the reference configuration of a very thin liquid crystal material.

The Landau–de Gennes model is based on a phenomenological theory in which stable states of the liquid material correspond to minimizers (or stable states) of an energy formulated in terms of Q on Ω. The matrix $Q(\mathbf{x})$ models the second moments of the orientations of the rod-like liquid crystal molecules near $\mathbf{x}$. Its

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values describe the average orientation and phase of the liquid crystals near $\mathbf{x}$, measured through its eigenvectors and eigenvalues. (See Section 1.1 for more detail on this structure.) As such, Q is a measure of the microscopic anisotropy of their relative positions. In this paper, we consider fields $Q \in W^{1,2}(\Omega; \mathscr{S})$ with fixed uniaxial nematic boundary conditions of the form $Q = Q_0$ on $\partial\Omega$ (in the sense of trace). We assume throughout the paper that $(Q_0)_{ij} \in C^3(\partial\Omega)$ for all $1 \leqq i, j \leqq 3$, and that

$$Q_0(x, y) = s\left(\mathbf{n}_0(x, y) \otimes \mathbf{n}_0(x, y) - \frac{1}{3} I\right) \quad \text{for } (x, y) \in \partial\Omega, \tag{1.1}$$

where I is the 3×3 identity matrix, s is an arbitrary fixed nonzero real number, and $\mathbf{n}_0$ is a fixed vector field defined on $\partial\Omega$ satisfying $\mathbf{n}_0 = \langle n_1, n_2, 0\rangle$, $|\mathbf{n}_0| = 1$, and (1.1) on $\partial\Omega$. Note that Q_0 is invariant under changes in direction: $\mathbf{n}_0(x, y) \to -\mathbf{n}_0(x, y)$ at any point (x, y) in $\partial\Omega$, which allows boundary conditions of degree one-half, or integer multiples of one-half, for Q_0. Nonzero boundary conditions of this type are observed on the sides of thin liquid crystal materials exhibiting defects along curves, known as "disclination lines" in the material, whose intersections with horizontal cross-sections are isolated points of degree $\frac{1}{2}$ or $-\frac{1}{2}$. (See [3,20].) We analyze a class of equilibria for the Landau–de Gennes energy

$$F_\varepsilon(Q) = \int_\Omega [f_e(Q) + \varepsilon^{-2} f_b(Q)],$$

where $\varepsilon > 0$, defined for all $Q \in W^{1,2}(\Omega; \mathscr{S})$. Here, f_e is the elastic energy density in Ω given by

$$f_e(Q) = \frac{L_1}{2}\, Q_{ij,k}\, Q_{ij,k} + \frac{L_2}{2}\, Q_{ij,j}\, Q_{ik,k} + \frac{L_3}{2}\, Q_{ij,k}\, Q_{ik,j},$$

where each term above is summed over all i, j, k from 1 to 3. Here L_1, L_2, L_3 are constants, $Q_{ij,\alpha}$ denotes $\frac{\partial Q_{ij}}{\partial x_\alpha}$, and $(x_1, x_2, x_3) = (x, y, z)$. The above formula is valid in two- or three-dimensional reference domains. Since here we are considering a two-dimensional reference domain, Ω, we identify $Q(x, y)$ with $Q(x, y, 0)$ above, so that $Q_{ij,3} = 0$ for all $1 \leqq i, j \leqq 3$. We assume throughout the paper that

$$L_1 > 0 \quad \text{and} \quad L_1 + L_2 + L_3 > 0. \tag{1.2}$$

The term f_b is the bulk energy density given by a smooth real-valued function which depends on temperature as well as on Q. We assume that temperature is fixed and $f_b = f_b(Q)$ is a nonnegative C^∞ function defined on $\mathscr{S}$ such that $f_b(Q) = 0$ if and only if $Q \in \Lambda_s = \{Q \in \mathscr{S} : Q = s(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3} I) \text{ for some } \mathbf{m} \in \mathbb{S}^2\}$, where s is the fixed nonzero constant in the definition of Q_0. From our definitions in the next subsection, we shall see that the energy well, Λ_s, corresponds to a set of uniaxial states. Liquid crystals satisfy the principle of frame indifference and are macroscopically isotropic. As a consequence, f_b is assumed to be invariant with respect to orthogonal transformations, that is, we require

$$f_b(RQR^t) = f_b(Q) \quad \text{for all } R \in O(3) \quad \text{and} \quad Q \in \mathscr{S}. \tag{1.3}$$

Set

$$\mathscr{S}_0 = \{Q \in \mathscr{S} : Q_{i3} = Q_{3i} = 0 \quad \text{for } i = 1, 2\},$$
$$\mathscr{A}_0 = \{Q(x, y) \in W^{1,2}(\Omega; \mathscr{S}_0) : \quad Q = Q_0 \text{ on } \partial\Omega\},$$

and

$$\mathscr{A} = \{Q \in W^{1,2}(\Omega; \mathscr{S}) : \quad Q = Q_0 \text{ on } \partial\Omega\}.$$

Our goal in this paper is to investigate minimizers for F_ε in $\mathscr{A}_0$, and to analyze their behavior in the vanishing elastic energy limit, $\varepsilon \to 0$. The relevance for doing this is that due to the symmetries described above, these minimizers are critical points (equilibria) for the energy F_ε over the larger space $\mathscr{A}$, and thus satisfy the full set of Euler–Lagrange equations with respect to variations in $\mathscr{A}$. (We prove this in Lemma 2.1.) In addition, each $Q \in \mathscr{S}$ is described in terms of an orthonormal set of eigenvectors. (See (1.8).) For $Q \in \mathscr{S}_0$, we have

$$Q = s_1 \mathbf{m} \otimes \mathbf{m} + s_2 \mathbf{m}^\perp \otimes \mathbf{m}^\perp - \frac{1}{3}(s_1 + s_2) I \tag{1.4}$$

for some real numbers s_1 and s_2, and Q has an orthonormal basis of eigenvectors of the form

$$\{\mathbf{m}, \mathbf{m}^\perp, \mathbf{e}_3\} \text{ where } \quad |\mathbf{m}| = 1, \ \mathbf{m} = \langle m_1, m_2, 0\rangle,$$
$$\text{and} \quad \mathbf{m}^\perp = \langle -m_2, m_1, 0\rangle, \tag{1.5}$$

with eigenvalues

$$\lambda_1 = \frac{1}{3}(2s_1 - s_2), \quad \lambda_2 = \frac{1}{3}(2s_2 - s_1), \quad \lambda_3 = -\frac{1}{3}(s_1 + s_2). \tag{1.6}$$

(See [17].) Thus the minimization problem of F_ε over $\mathscr{A}_0$ models the behavior of a thin liquid crystal material occupying $\Omega \times (-\eta, \eta)$, with its top and bottom surfaces treated so as to fix $\mathbf{e}_3$ as a principal axis (eigenvector of Q) of the liquid crystal molecules throughout the body, with the other two principal axes (eigenvectors) in $\mathbb{R}^2 \times \{0\}$ and boundary values on its sides given by $Q = Q_0(x, y)$. The above problem includes a classic example from the liquid crystal literature, in which

$$f_b(Q) = f_b^0(Q) = \mathfrak{a}\, tr(Q^2) - \frac{2\mathfrak{b}}{3}\, tr(Q^3) + \frac{\mathfrak{c}}{2}(tr(Q^2))^2 + \mathfrak{d}$$
$$= \mathfrak{a}\left(\sum_{i=1}^{3} \lambda_i^2\right) - \frac{2\mathfrak{b}}{3}\left(\sum_{i=1}^{3} \lambda_i^3\right) + \frac{\mathfrak{c}}{2}\left(\sum_{i=1}^{3} \lambda_i^2\right)^2 + \mathfrak{d}. \tag{1.7}$$

Indeed, taking $\mathfrak{b}, \mathfrak{c} > 0$, $\mathfrak{a} < \frac{\mathfrak{b}^2}{27\mathfrak{c}}$, and an appropriate choice of $\mathfrak{d}$, we have $f_b^0 \geqq 0$ and $f_b^0(Q) = 0$ if and only if $Q \in \Lambda_s$ where $s = \frac{1}{4\mathfrak{c}}(\mathfrak{b} + \sqrt{\mathfrak{b}^2 - 24\mathfrak{a}\mathfrak{c}})$. (See [17].)

1.1. Definitions and Structural Assumptions

Our results require some structural assumptions on the bulk energy density f_b. In this section, we state these assumptions, along with some definitions and a change of variables in $\mathscr{A}_0$, that will be needed to state our main results.

It is well known (see [17]) that each $Q \in \mathscr{S}$ has an orthonormal set of eigenvectors and can be written as

$$Q = s_1\mathbf{n}\otimes\mathbf{n} + s_2\mathbf{k}\otimes\mathbf{k} - \frac{1}{3}(s_1+s_2)I, \tag{1.8}$$

where $\mathbf{n}$ and $\mathbf{k}$ are orthogonal unit vectors in $\mathbb{R}^3$; moreover, the eigenvalues of Q are given by the formula in (1.6).

Definition 1. Let $Q \in \mathscr{S}$. We say that Q is *isotropic* if all its eigenvalues are equal. (In this case, the structure of Q is that of a "normal" liquid.)

We say that Q is *uniaxial* if exactly two of its eigenvalues are equal. (In this case, Q has an axis of symmetry and its structure is "rod-like" or "disk-like".)

We say that Q is *biaxial* if all its eigenvalues are distinct. (In this case, there is no axis of complete rotational symmetry for Q and its structure is "board-like".)

By formula (1.6) for the eigenvalues of $Q \in \mathscr{S}$, it follows that Q is isotropic if and only if $s_1 = s_2 = 0$ (and hence all eigenvalues are zero); Q is uniaxial if and only if one of the following three conditions hold: $s_1 = 0$ and $s_2 \neq 0$, $s_2 = 0$ and $s_1 \neq 0$, or $s_1 = s_2 \neq 0$ (and hence all eigenvalues are nonzero and exactly two of the eigenvalues are equal). Finally, Q is biaxial for all other values of s_1 and s_2.

The above definition, when applied to a minimizer $Q_\varepsilon(\mathbf{x})$ of F_ε in $\mathscr{A}$ or $\mathscr{A}_0$, allows one to identify subregions of Ω in which the liquid crystal material is in the isotropic, uniaxial, or biaxial phase; nonempty subregions of any of these phases are possible for functions in $\mathscr{A}$ or $\mathscr{A}_0$. Note that $\Lambda_s \cap \mathscr{S}_0$ is a disconnected set of uniaxial states in $\mathscr{S}_0$ with two connected components: $\Lambda_s \cap \mathscr{S}_0 = \Lambda_s' \cup \{s(\mathbf{e}_3\otimes\mathbf{e}_3 - \frac{1}{3}I)\}$, where $\Lambda_s' = \{s(\mathbf{m}\otimes\mathbf{m} - \frac{1}{3}I) \colon \mathbf{m} = \langle m_1, m_2, 0\rangle,\ |\mathbf{m}| = 1\}$; also, the boundary values $Q_0(x,y)$ are valued in Λ_s'.

Definition 2. Let $\gamma \colon [0,1] \to \partial\Omega$ be a C^3 positively oriented parameterization of $\partial\Omega$ such that γ is one-to-one on $[0,1)$. For Q_0 as assumed above, choose a unit vector field $\tilde{\mathbf{n}}_0(\mathbf{x}) = \langle \tilde{n}_1(\mathbf{x}), \tilde{n}_2(\mathbf{x}), 0\rangle$ defined on $\partial\Omega$ satisfying (1.1) such that $\tilde{\mathbf{n}}_0(\gamma(\cdot)) \in C^1([0,1))$. We define the *degree* of Q_0 on $\partial\Omega$ by

$$\frac{1}{2\pi}\int_0^1 \tilde{\mathbf{n}}_0(\gamma(t))^\perp \cdot \frac{\mathrm{d}\tilde{\mathbf{n}}_0(\gamma(t))}{\mathrm{d}t}\,\mathrm{d}t \colon = \deg Q_0.$$

Since $\lim_{t\uparrow 1}\mathbf{n}_0(\gamma(t)) = \pm\mathbf{n}_0(\gamma(0))$ by (1.1) and the continuity of Q_0, it follows that $\deg Q_0 = \frac{k}{2}$ for some $k \in \mathbb{Z}$. Since we are interested in boundary conditions that correspond to a thin liquid crystal material with disclination-line type defects, we assume that k is nonzero, and thus without loss of generality, we shall assume throughout the paper that $k > 0$. As $\varepsilon \downarrow 0$ the effect of the bulk energy density f_b becomes more pronounced and minimizers tend to have their values located in a neighborhood of $\Lambda_s \bigcap \mathscr{S}_0$. We prove this rigorously in Lemma 3.2 and Corollary 3.3. Due to the boundary conditions, however, this cannot happen throughout Ω. We prove that the regions in which minimizers, $Q_\varepsilon(x)$, of F_ε take values outside a neighborhood of Λ_s' concentrate and quantize into k small subdomains. Moreover, for a subsequence as $\varepsilon_j \to 0$, these subdomains tend to k distinct points $\{a_1, \dots, a_k\}$ representing the cross sections of the limiting disclination lines.

In [20], Schopohl and Sluckin carried out a numerical investigation of equilibria for F_ε in $\mathscr{A}$, where $f_b = f_b^0$ is as in (1.7), with parameters so that the energy well is a set of uniaxial states Λ_s. They gave numerical evidence that uniaxial boundary conditions with nonzero degree give rise to equilibria that are nearly uniaxial away from topologically induced defects, about which the solutions are strongly biaxial. They pointed out that there is a subclass of equilibria which is contained in $\mathscr{A}_0$, and they developed simulations for the equilibria in this subclass. This is the class of solutions that we are studying here.

To state our main results, we use the following linear change of variables for the coefficients of each $Q \in \mathscr{A}_0$ in terms of unique functions $\mathbf{p} = (p_1, p_2)$ and r:

$$Q = Q(\mathbf{p}, r) = \begin{bmatrix} p_1 + \frac{r}{2} & p_2 & 0 \\ p_2 & \frac{r}{2} - p_1 & 0 \\ 0 & 0 & -r \end{bmatrix}. \tag{1.9}$$

By (1.1) and (1.4) each $Q \in \mathscr{A}_0$ corresponds to a unique $(\mathbf{p}, r) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{1,2}(\Omega)$ satisfying $\mathbf{p}|_{\partial\Omega} = \mathbf{p}_0$, $r|_{\partial\Omega} = r_0$, where $|\mathbf{p}_0| = \frac{|s|}{2}$, $r_0 = \frac{s}{3}$, and $\deg\,(\frac{\mathbf{p}_0}{|\mathbf{p}_0|}, \partial\Omega) = k = 2 \deg Q_0$. This can be seen by writing (since $\mathbf{n}_0 \otimes \mathbf{n}_0 = (-\mathbf{n}_0) \otimes (-\mathbf{n}_0)$)

$$\mathbf{n}_0(\gamma(t)) = \pm\langle \cos\alpha(t), \sin\alpha(t), 0\rangle$$

for each t in $[0, 1)$, where $\langle \cos\alpha(t), \sin\alpha(t), 0\rangle = \tilde{\mathbf{n}}_0(\gamma(t))$ and $\alpha \in C^1([0, 1))$. Then, using (1.1) and(1.4), we observe that

$$\mathbf{p}_0(\gamma(t)) = \frac{s}{2}\langle \cos 2\alpha(t), \sin 2\alpha(t)\rangle.$$

The representation (1.9) was motivated by the setting used for the simulations in [20]. We may then recast our minimum problem by considering the set

$$A_0 = \left\{(\mathbf{p}, r) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{1,2}(\Omega) \colon \mathbf{p} = \mathbf{p}_0 \text{ and } r = \frac{s}{3} \text{ on } \partial\Omega\right\}.$$

The mapping $Q = Q(\mathbf{p}, r) : A_0 \to \mathscr{A}_0$ is one-to-one and onto, and the eigenvalues for $Q(\mathbf{p}, r)$ are $\lambda_1 = \frac{r}{2} + |\mathbf{p}|$, $\lambda_2 = \frac{r}{2} - |\mathbf{p}|$, $\lambda_3 = -r$. By (1.3), f_b depends only on the invariants of Q; since $tr\,Q = 0$, these are $\det Q = (|\mathbf{p}|^2 - \frac{r^2}{4})r$ and $|Q|^2 = 2|\mathbf{p}|^2 + \frac{3}{2}\,r^2$. Thus, $f_b(Q) = g_b(|\mathbf{p}|^2, r)$ for some function g_b. We prove in Section 2 that minimizing $F_\varepsilon(Q)$ over $\mathscr{A}_0$ is equivalent to minimizing

$$G_\varepsilon(\mathbf{p}, r) = \int_\Omega [g_e(\nabla\mathbf{p}, \nabla r) + \varepsilon^{-2} g_b(|\mathbf{p}|^2, r)] \quad \text{for } (\mathbf{p}, r) \in A_0, \tag{1.10}$$

where $g_e(\nabla \mathbf{p}, \nabla r)$ is defined by

$$\begin{aligned} g_e = &\left(L_1 + \frac{(L_2+L_3)}{2}\right) |\nabla \mathbf{p}|^2 + \left(\frac{3L_1}{4} + \frac{(L_2+L_3)}{8}\right) |\nabla r|^2 \\ &+ \frac{(L_2+L_3)}{2} (p_{1x}r_x - p_{1y}r_y + r_x p_{2y} + r_y p_{2x}) \\ &+ |L_2+L_3|(p_{1x}p_{2y} - p_{1y}p_{2x}). \end{aligned} \tag{1.11}$$

This can be rewritten as

$$\begin{aligned} g_e = &L_1\left(|\nabla \mathbf{p}|^2 + \frac{3}{4}|\nabla r|^2\right) \\ &+ \frac{(L_2+L_3)}{2}\left(\left(p_{1x} + \frac{r_x}{2} + p_{2y}\right)^2 + \left(p_{2x} - p_{1y} + \frac{r_y}{2}\right)^2\right) \\ &\text{if } L_2 + L_3 \geqq 0, \end{aligned} \tag{1.12}$$

$$\begin{aligned} g_e = &(L_1+L_2+L_3)(|\nabla \mathbf{p}|^2 + \frac{3}{4}|\nabla r|^2) \\ &- \frac{(L_2+L_3)}{2}\left(\left(\frac{r_x}{2} - p_{1x} - p_{2y}\right)^2 + \left(p_{2x} - p_{1y} - \frac{r_y}{2}\right)^2 + |\nabla r|^2\right) \\ &\text{if } 0 > L_2 + L_3. \end{aligned} \tag{1.13}$$

The following structural conditions are assumed for $g_b(\mathfrak{p}, \mathfrak{r}) = g_b(|\mathbf{p}|^2, r)$:

$$\begin{cases} \text{i) } g_b \in C^\infty([0,\infty)\times\mathbb{R}),\ g_b \geqq 0 \text{ and } g_b(\frac{s^2}{4}, \frac{s}{3}) = 0, \\ \text{ii) For some } m_1, m_2, m_3 > 0 \\ \quad |g_{b,\mathfrak{p}}(|\mathbf{p}|^2, r)||\mathbf{p}| + |g_{b,\mathfrak{r}}(|\mathbf{p}|^2, r)| \leqq m_1(|\mathbf{p}|^3 + |r|^3) + m_2, \\ \qquad m_3(|\mathbf{p}|^4 + |r|^4) - 1 \leqq g_b(|\mathbf{p}|^2, r), \\ \text{iii) For some } \delta, m_4 > 0 \\ \quad m_4((|\mathbf{p}|^2 - \frac{s^2}{4})^2 + |r - \frac{s}{3}|^2) \leqq g_b(|\mathbf{p}|^2, r) \\ \qquad \text{for } ||\mathbf{p}| - \frac{|s|}{2}| + |r - \frac{s}{3}| < \delta. \end{cases} \tag{1.14}$$

Since $f_b(Q) = g_b(|\mathbf{p}|^2, r) = g_b(\mathfrak{p}, \mathfrak{r})$ under the change of variables (1.9), these are additional assumptions on f_b. From (1.2), (1.12), and (1.13) we see that g_e is a positive definite quadratic. Thus G_ε is strongly elliptic. It follows that minimizers for G_ε in A_0 exist and that the Euler–Lagrange equation is a semi–linear elliptic system for which minimizers are classical solutions ($C^\infty(\Omega) \bigcap C^2(\overline{\Omega})$ in our case). (See Theorem 2.2.)

In general the bulk energy well for $g_b(|\mathbf{p}|^2, r)$ corresponds to $\{(\mathbf{p}, r)\colon g_b(|\mathbf{p}|^2, r) = 0\}$. From our assumptions on f_b, the bulk energy well for f_b restricted to $\mathscr{S}_0$ is $\Lambda_s' \cup \{s(\mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{3}I)\}$. By the change of variables $Q \to (\mathbf{p}, r)$, Λ_s' corresponds to $\Gamma_s := \{(\mathbf{p}, r)\colon |\mathbf{p}|^2 = \frac{s^2}{4}, r = \frac{s}{3}\}$, and $\{s(\mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{3}I)\}$ corresponds to $\{(\mathbf{p}, r) = (\mathbf{0}, -\frac{2s}{3})\}$. We note that the structural conditions (1.14) require only that

$\{g_b = 0\}$ contains Γ_s as in (i), that it is bounded as in (ii), and that g_b has quadratic growth away from Γ_s as in (iii).

For the classic example, $f_b = f_b^0$ from the liquid crystal literature (with coefficients $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ as described above (see (1.7)), f_b^0 minimizes precisely on the uniaxial well Λ_s,

$$f_b^0(Q) = \mathfrak{a}\left(2|\mathbf{p}|^2 + \frac{3}{2}r^2\right) - 2\mathfrak{b}r\left(|\mathbf{p}|^2 - \frac{r^2}{4}\right) + \frac{\mathfrak{c}}{2}\left(2|\mathbf{p}|^2 + \frac{3}{2}r^2\right)^2 + \mathfrak{d} =: g_b^0(|\mathbf{p}|^2, r)$$

for $Q \in \mathscr{S}_0$ and $Q = Q(\mathbf{p}, r)$, and one can easily show that the structural assumptions (1.14) are satisfied for this example of g_b.

1.2. Main Results

In this section we state our main results on the structure of minimizers of the energy functional $F_\varepsilon(Q)$ over $\mathscr{A}_0$, using the fact that Q is a minimizer of F_ε in $\mathscr{A}_0$ if and only if $(\mathbf{p}, r)$ is a minimizer of G_ε in A_0 and $Q = Q(\mathbf{p}, r)$.

Theorem A. *Let $\{(\mathbf{p}_j, r_j)\}$ be a sequence of minimizers for $\{G_{\varepsilon_j}\}$, respectively over A_0, such that $\varepsilon_j \downarrow 0$. For ease of notation we consider $\mathbf{p}_j$ as a complex-valued function by identifying $\mathbb{R}^2$ and $\mathbb{C}$. Then for a subsequence $\{(\mathbf{p}_{j'}, r_{j'})\}$ there exists a harmonic function $h \in C^2(\overline{\Omega})$ and k points $\{a_1, \ldots, a_k\} \subset \Omega$ such that*

$$(|\mathbf{p}_{j'}(\mathbf{x})|, r_{j'}(\mathbf{x})) \to \left(\frac{|s|}{2}, \frac{s}{3}\right) \text{ in } C_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_k\}), \quad \text{and}$$

$$(\mathbf{p}_{j'}(\mathbf{x}), r_{j'}(\mathbf{x})) \to (\mathbf{p}^*(\mathbf{x}), r^*(\mathbf{x})) = \left(\frac{|s|}{2}\, e^{i(h(\mathbf{x}) + \sum_{\ell=1}^k \theta_\ell(\mathbf{x}))}, \frac{s}{3}\right) \tag{1.15}$$

in $W^{1,2}_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_k\}) \cap C_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_k\})$ and in $C^m_{\text{loc}}(\Omega\backslash\{a_1, \ldots, a_k\})$ for all $m > 0$, where $\theta_\ell = \theta_\ell(\mathbf{x})$ denotes the polar angle of $\mathbf{x}$ with respect to the center a_ℓ. In particular, for each sufficiently small $\rho > 0$, if j' is sufficiently large, setting $\Omega_\rho = \Omega\backslash \bigcup_{\ell=1}^k B_\rho(a_\ell)$, we have

$$\mathbf{p}_{j'}(\mathbf{x}) = |\mathbf{p}_{j'}(\mathbf{x})| e^{i(h_{j'}(\mathbf{x}) + \sum_{\ell=1}^k \theta_\ell(\mathbf{x}))} \text{ in } \overline{\Omega}_\rho \tag{1.16}$$

where $h_{j'}(\mathbf{x})$ is a function in $C^2(\overline{\Omega}_\rho)$ so that $e^{ih_{j'}(\mathbf{x})}$ has degree zero on $\partial\Omega$, and $\mathbf{p}_{j'}$ has degree 1 about each of the k defects $\{a_1, \ldots, a_k\}$.

From Theorem A and the change of variables between A_0 and $\mathscr{A}_0$, we obtain:

Corollary A. *Let $\{Q_j\}$ be a sequence of minimizers of $\{F_{\varepsilon_j}\}$, respectively over $\mathscr{A}_0$ such that $\varepsilon_j \downarrow 0$. Then for a subsequence of minimizers, we have $Q_{j'} = Q(\mathbf{p}_{j'}, r_{j'})$ where $\{\mathbf{p}_{j'}, r_{j'}\} \subset A_0$ satisfies Theorem A, and hence for each sufficiently small $\rho > 0$, if j' is sufficiently large, we have:*

$$Q_{j'}(\mathbf{x}) = s_{j'1}(\mathbf{x})(\mathbf{m}_{j'}(\mathbf{x}) \otimes \mathbf{m}_{j'}(\mathbf{x})) + s_{j'2}(\mathbf{x})(\mathbf{m}^\perp_{j'}(\mathbf{x}) \otimes \mathbf{m}^\perp_{j'}(\mathbf{x})) - \frac{1}{3}(s_{j'1}(\mathbf{x}) + s_{j'2}(\mathbf{x}))I \quad \text{in } \overline{\Omega}_\rho,$$

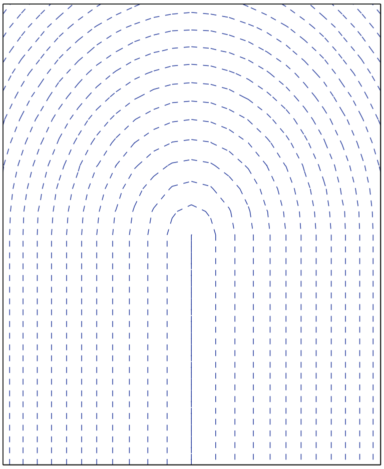

Fig. 1. $\frac{1}{2}$ degree defect in a nematic texture

where

$$\mathbf{m}_{j'}(\mathbf{x}) = \left\langle \cos\left(\frac{1}{2}(h_{j'}(\mathbf{x}) + \sum_{\ell=1}^{k}\theta_\ell(\mathbf{x}))\right), \sin\left(\frac{1}{2}\left(h_{j'}(\mathbf{x}) + \sum_{\ell=1}^{k}\theta_\ell(\mathbf{x})\right)\right), 0\right\rangle,$$

$$s_{j'1}(\mathbf{x}) = |\mathbf{p}_{j'}(\mathbf{x})| + \frac{3}{2}r_{j'}(\mathbf{x}), \quad s_{j'2}(\mathbf{x}) = \frac{3}{2}r_{j'}(\mathbf{x}) - |\mathbf{p}_{j'}(\mathbf{x})|,$$

and $Q_{j'}$ has degree $\frac{1}{2}$ about each a_ℓ. (See Fig. 1.)

In particular, $Q_{j'}(\mathbf{x})$ converges to a uniaxial field $Q^*(\mathbf{x})$ in $W^{1,2}_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \dots, a_k\}) \cap C_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \dots, a_k\})$ and in $C^m_{\text{loc}}(\Omega\backslash\{a_1, \dots, a_k\})$ for all $m > 0$ as $j' \to \infty$, where

$$Q^*(\mathbf{x}) = s\left(\mathbf{m}(\mathbf{x}) \otimes \mathbf{m}(\mathbf{x}) - \frac{1}{3}I\right) \quad \text{in } \Omega\backslash\{a_1, \dots, a_k\} \quad \text{when } s > 0,$$

and

$$Q^*(\mathbf{x}) = s(\mathbf{m}^\perp(\mathbf{x}) \otimes \mathbf{m}^\perp(\mathbf{x}) - \frac{1}{3}I) \quad \text{in } \Omega\backslash\{a_1, \dots, a_k\} \quad \text{when } s < 0.$$

Here,

$$\mathbf{m}(\mathbf{x}) = \left\langle \cos\left(\frac{1}{2}\left(h(\mathbf{x}) + \sum_{\ell=1}^{k}\theta_\ell(\mathbf{x})\right)\right), \sin\left(\frac{1}{2}\left(h(\mathbf{x}) + \sum_{i=1}^{k}\theta_\ell(\mathbf{x})\right)\right), 0\right\rangle \tag{1.17}$$

for all $\mathbf{x}$ in $\Omega\backslash\{a_1, \dots, a_k\}$. Note that $\mathbf{m}_{j'}$ and $\mathbf{m}$ are discontinuous while $Q_{j'}$ and Q are continuous on $\overline{\Omega}_\rho$.

The points $\{a_1, \dots, a_k\}$ represent the cross sections of the limiting disclination lines perpendicular to Ω. We prove that this set of points minimizes a renormalized energy $W(\mathbf{b})$ defined for $\mathbf{b} = (b_1, \ldots, b_k) \in \Omega^k$, which was introduced by Brezis et al. [1] in connection with their analysis of minimizing sequences $\{\mathbf{v}_\varepsilon\}$ for the Ginzburg–Landau energy

$$E_\varepsilon(\mathbf{v}) = \frac{1}{2}\int_\Omega \left[|\nabla \mathbf{v}|^2 + \frac{1}{2\varepsilon^2}(1 - |\mathbf{v}|^2)^2 \right] \tag{1.18}$$

for $\mathbf{v} \in \{\mathbf{w} \in W^{1,2}(\Omega; \mathbb{R}^2) \colon \mathbf{w} = \mathbf{p}_0/|\mathbf{p}_0| \text{ on } \partial\Omega\}$. (The renormalized energy $W(\mathbf{b})$ is defined by equation (3.24).) On the other hand, we analyze the limiting behavior of the energies, $\{F_{\varepsilon_j}(Q_j)\}$, as $j \to \infty$ and show that it depends strongly on the coefficients L_1, L_2, and L_3 as well as the parameters s and the degree $\frac{k}{2}$ of the boundary data Q_0. More precisely, we have:

Theorem B. *Let $\{(\mathbf{p}_j, r_j)\}$ be a sequence of minimizers for $\{G_{\varepsilon_j}\}$, respectively over A_0 (or equivalently, let $\{Q_j\}$ be a sequence of minimizers for $\{F_{\varepsilon_j}(Q_j)\}$, respectively over $\mathscr{A}_0$), for which $(a_1, \ldots, a_k)$ is a limiting configuration of defects as $\varepsilon_j \downarrow 0$ as described in Theorem A. Then*

$$F_{\varepsilon_j}(Q_j) = G_{\varepsilon_j}(\mathbf{p}_j, r_j) - (L_3 - L_2 + |L_3 + L_2|)\frac{s^2\pi k}{4}.$$

Furthermore, the renormalized energy $W(\mathbf{b})$ for the limiting problem minimizes at $\mathbf{a}$ and we have

$$\lim_{j\to\infty} \left[G_{\varepsilon_j}(\mathbf{p}_j, r_j) - \frac{(2L_1 + L_2 + L_3)s^2\pi k}{4} \ln \frac{1}{\varepsilon_j} \right] = (2L_1 + L_2 + L_3)\frac{s^2}{4} W(\mathbf{a}) + k\gamma.$$

Here γ is a fixed constant associated to the energy of each defect core.

Investigations from the physics literature of nematic textures in thin flat or curved surfaces (thin shells) can be found in [3,6,16,18], and [23]. In [5] Fatkullin and Slastikov proposed and investigated a model for two-dimensional nematics (assuming that $L_1 > 0$, $L_2 = L_3 = 0$, and Q is a two-dimensional tensor, that is, $Q = Q(\mathbf{p}, r)$ is in $\mathscr{A}_0$ with $r(\mathbf{x}) \equiv 0$) combining Onsager–Maier–Saupe and Landau–de Gennes theories. This led them to analyze a variational problem closely related to the Ginzburg–Landau energy (1.18).

Our final theorem describes how our results in this paper relate to earlier investigations of complex Ginzburg–Landau type functionals that have multiply-connected energy wells. The closest study in this respect is [8] by Han and Kim, in which they analyzed the asymptotic behavior for sequences of minimizers to the Chern–Simons–Higgs (CSH) and the Maxwell–Chern–Simons–Higgs (MCSH) energies used to model aspects of superconductivity.

For the (CSH) model one seeks (using our notation) minimizers $\mathbf{p}_\varepsilon$ to

$$C_\varepsilon(\mathbf{p}) = \int_\Omega \left[\frac{1}{2}|\nabla \mathbf{p}|^2 + \varepsilon^{-2}|\mathbf{p}|^2(1 - |\mathbf{p}|^2)^2 \right] \tag{1.19}$$

for $\mathbf{p} \in B_0 = \{\mathbf{v} \in W^{1,2}(\Omega; \mathbb{R}^2) \colon \mathbf{v} = \mathbf{p}_0 \text{ on } \partial\Omega\}$. Here $\mathbf{p}_0 \in C^3(\partial\Omega)$, $|\mathbf{p}_0| = 1$, and $\deg(\mathbf{p}_0, \partial\Omega) = k > 0$ with $k \in \mathbb{N}$.

For the (MCSH) model one seeks minimizers $(\mathbf{p}_{\varepsilon,q}, r_{\varepsilon,q})$ to

$$C_{\varepsilon,q}(\mathbf{p}, r) = \int_\Omega [\frac{1}{2}|\nabla \mathbf{p}|^2 + q^{-2}|\nabla r|^2 + |\mathbf{p}|^2 r^2 + q^2(\varepsilon^{-1}(|\mathbf{p}|^2 - 1) + r)^2] \tag{1.20}$$

for $(\mathbf{p}, r) \in B_0 \times W_0^{1,2}(\Omega)$. The following two results are from [8]:

i) For fixed $\varepsilon > 0$, from any sequence of minimizers for (1.20) with $q \to \infty$ one can find a subsequence $\{(\mathbf{p}_{\varepsilon,q_\ell}, r_{\varepsilon,q_\ell})\}$ and a minimizer $\mathbf{p}_\varepsilon$ to (1.19) for which

$$\mathbf{p}_{\varepsilon,q_\ell} \rightharpoonup \mathbf{p}_\varepsilon \quad \text{and} \quad C_{\varepsilon,q_\ell}(\mathbf{p}_{\varepsilon,q_\ell}, r_{\varepsilon,q_\ell}) \to C_\varepsilon(\mathbf{p}_\varepsilon) \text{ as } q_\ell \to \infty.$$

ii) For fixed $q > 0$, from any sequence of minimizers for (1.20) with $\varepsilon \to 0$ there exists a subsequence $\{(\mathbf{p}_{\varepsilon_\ell,q}, r_{\varepsilon_\ell,q})\}$, a point $\mathbf{a}^q = (a_1^q, \dots, a_k^q) \in \Omega^k$, and a function $\mathbf{p}_q^*$ as in (1.15) so that $\mathbf{p}_{\varepsilon_\ell,q} \to \mathbf{p}_q^*$ in the sense of Theorem A as $\varepsilon_\ell \to 0$.

The functionals (1.10) and (1.20) are quite different. The bulk energy well for $C_{\varepsilon,q}$ is $\mathbb{S}^1 \times \{0\} \bigcup \{(\mathbf{0}, \varepsilon^{-\mathbf{1}})\}$ and the second component is eventually outside of any bounded set as $\varepsilon \to 0$. This is in contrast to the bulk energy well for G_ε, which does not vary with ε. The analysis in [8] is based on this feature and cannot be applied to (1.10). Furthermore, the bounds in the estimates used to prove ii) diverge as $q \to \infty$, so they cannot be used to determine $\lim_{\varepsilon\to 0}(\lim_{q_\ell\to\infty} C_{\varepsilon,q_\ell}(\mathbf{p}_{\varepsilon,q_\ell}, r_{\varepsilon,q_\ell})) = \lim_{\varepsilon\to 0} C_\varepsilon(\mathbf{p}_\varepsilon)$ or the limiting behavior of minimizers of C_ε, and this was left open. Our analysis, however, applies to these issues directly. The same arguments we use to prove Theorems A and B give the following result:

Theorem C. *Let $\{\mathbf{p}_\varepsilon\}$ be a sequence of minimizers for* (1.19) *such that $\varepsilon \to 0$. Then there exists a subsequence $\{\mathbf{p}_{\varepsilon_\ell}\}$, a point $\mathbf{a} = (a_1, \dots, a_k) \in \Omega^k$, and a function $\mathbf{p}^*$ as in* (1.15) *for which $\mathbf{p}_{\varepsilon_\ell} \to \mathbf{p}^*$ in the sense of Theorem* A. *Moreover $W(\cdot)$ minimizes at $\mathbf{a}$ and*

$$\lim_{\ell\to\infty} \left[C_{\varepsilon_\ell}(\mathbf{p}_{\varepsilon_\ell}) - \pi k \ln \frac{1}{\varepsilon_\ell} \right] = W(\mathbf{a}) + k\gamma$$

for a fixed constant γ.

Other related work is given in the papers [11,12], and [21] in which the authors develop asymptotic properties for the (CSH) energy using Γ– convergence techniques. This approach gives less detailed information than in our setting. However, it is not restricted to sequences of minimizers as in our case, and the authors apply it to more general energies and scalings.

Our paper is organized as follows. In Section 2 we prove regularity of minimizers and show that minimizers for G_ε in A_0 correspond to a family of equilibria for F_ε in $\mathscr{A}$. In Section 3 we prove a number of a priori estimates for minimizers of G_ε and use them to prove Theorems A and B, developing the qualitative features of minimizers for G_ε. Our results in this section expand on investigations of minimizers for the Ginzburg–Landau energy E_ε (1.18) done by Brezis–Bethuel–Hélein, Fanghua Lin, and Struwe. (See [1,13,14], and [22]). The energies E_ε and G_ε differ in two main respects. First, the elastic term in the energy density for E_ε is the Dirichlet energy density, whereas for G_ε it is a coupled quadratic in ∇Q. Second, the energy well for the bulk energy density for E_ε is $\mathbb{S}^1$, while the energy well for G_ε is a bounded disconnected set consisting of the circle Γ_s and the point $(0, -\frac{2s}{3})$. In Corollary 3.3, we prove that for ε small, minimizers of G_ε take their values near Γ_s outside an exceptional set whose measure is $O(\varepsilon^2)$. As a result, Γ_s plays a role similar to the energy well for E_ε, and we prove that this exceptional set is contained in a neighborhood of k defects (vortices). The results in Section 3 are proved assuming the a priori estimate

$$\varepsilon^{-2}\int_\Omega g_b(|\mathbf{p}_\varepsilon|^2, r_\varepsilon) \leqq M \tag{1.21}$$

for some constant $M < \infty$, for the family of equilibria $\{(\mathbf{p}_\varepsilon, r_\varepsilon)\colon 0 < \varepsilon < \varepsilon_1\}$ that are considered. In Section 4 we prove, using a Pohozaev identity, that (1.21) is always satisfied if Ω is a disk and $0 < \varepsilon < 1$. We then use this result to establish (1.21) for the case in which Ω is a C^3 bounded simply connected domain and $\{(\mathbf{p}_\varepsilon, r_\varepsilon)\}$ are minimizers, where ε_1 depends on s, L_1, L_2, L_3, Ω, k, and the constants in (1.14), and M depends on these terms and in addition on $\|\mathbf{p}_0\|_{W^{1,2}(\partial\Omega)}$. Our approach for this part is similar to one used by del Pino and Felmer [4] in which they established the analogue of (1.21) for the simpler energy (1.18).

2. The Landau–de Gennes Energy

By definition of f_e, we have

$$\begin{aligned} f_e(Q) = \frac{L_1}{2}|\nabla Q|^2 + \frac{(L_2+L_3)}{2}|\text{div } Q|^2 \\ + \frac{L_3}{2}(Q_{ij,k}Q_{ik,j} - Q_{ij,j}Q_{ik,k}), \end{aligned}$$

where div Q is the column vector whose ith entry is the divergence of the ith row of Q, $Q_{ij,j}$. The last term in f_e is a null–Lagrangian; its integral over Ω is constant on

$$\mathscr{M} = \{Q \in W^{1,2}(\Omega; \mathbb{R}^{3\times 3})\colon Q = Q_0 \text{ on } \partial\Omega\}$$

and its first variation at any element of $\mathscr{M}$ is zero. Set

$$f_e'(Q) = \frac{L_1}{2}|\nabla Q|^2 + \frac{(L_2+L_3)}{2}|\text{div} Q|^2.$$

We say that F_ε and $F_\varepsilon' = \int_\Omega [f_e' + \varepsilon^{-2} f_b]$ are *equivalent* since their first variations on $\mathscr{M}$ agree, $\delta_V F_\varepsilon(Q) = \delta_V F_\varepsilon'(Q)$, where

$$\begin{aligned} &\delta_V F(Q) = DF(Q)[V] := \partial_t F(Q + tV) \quad \text{at} \quad t = 0 \\ &\text{for all} \quad Q \in \mathscr{M} \quad \text{and} \quad V \in W_0^{1,2}(\Omega; \mathbb{R}^{3\times 3}). \end{aligned}$$

For $Q \in \mathscr{A}$ we write

$$Q(\mathbf{x}) = \begin{bmatrix} z_1(\mathbf{x}) & z_2(\mathbf{x}) & z_4(\mathbf{x}) \\ z_2(\mathbf{x}) & z_3(\mathbf{x}) & z_5(\mathbf{x}) \\ z_4(\mathbf{x}) & z_5(\mathbf{x}) & -z_1(\mathbf{x}) - z_3(\mathbf{x}) \end{bmatrix}$$

and for $Q \in \mathscr{A}_0$ we additionally have $z_4(\mathbf{x}) = z_5(\mathbf{x}) = 0$. The Euler–Lagrange equations for F'_ε derived by variations in $\mathscr{A}$ ($\mathscr{A}_0$) consist of the five (three) equations $\delta_{z_\ell} F'_\varepsilon = 0$ for $\ell = 1, \dots, 5$ ($\ell = 1, 2, 3$).

We can now show that an equilibrium with respect to variations in $\mathscr{A}_0$ is also an equilibrium with respect to variations in $\mathscr{A}$. We have:

Lemma 2.1. *Let $Q \in \mathscr{A}_0$ solve $\delta_{z_\ell} F_\varepsilon(Q) = 0$ for $\ell = 1, 2, 3$, then $\delta_{z_4} F_\varepsilon(Q) = \delta_{z_5} F_\varepsilon(Q) = 0$, as well.*

Proof. Since $f_b = \tilde{f}_b(\det Q, |Q|^2)$ it is easy to see that $\partial_{z_4} f_b(\hat{Q}) = \partial_{z_5} f_b(\hat{Q}) = 0$ for $\hat{Q} \in \mathscr{S}_0$. It follows directly that $\delta_{z_4} F'_\varepsilon(Q) = \delta_{z_5} F'_\varepsilon(Q) = 0$ for any $Q(\mathbf{x}) \in \mathscr{A}_0$. □

Theorem 2.2. *For each $\varepsilon > 0$, minimizers for $F_\varepsilon(Q)$ in $\mathscr{A}_0$ exist and are of class $C^\infty(\Omega) \bigcap C^2(\overline{\Omega})$.*

Proof. Recall that by (1.2), $L_1 > 0$ and $L_1 + L_2 + L_3 > 0$. We consider two cases.

i) $L_2 + L_3 \geqq 0$. From the discussion above we can work with the energy F'_ε instead of F_ε. Its energy density is the sum of nonnegative terms and f'_e is a positive definite quadratic in $\nabla \mathbf{z}$, $\mathbf{z} = (z_1, z_2, z_3)$. The first variation of F'_ε in $\mathscr{A}_0$ results in a semilinear elliptic system of three equations in three unknowns. From standard elliptic theory (see [7]) minimizers for F'_ε in $\mathscr{A}_0$ exist, they are weak solutions to the resulting elliptic system and they are classical ($C^\infty(\Omega) \bigcap C^2(\overline{\Omega})$).

ii) $0 > L_2 + L_3$. Let curl Q denote the matrix-valued function whose ith row is the curl of the ith row of Q. Then $|\nabla Q|^2 - |\text{div } Q|^2 - |\text{curl } Q|^2 = (Q_{ij,k} Q_{ik,j} - Q_{ij,j} Q_{ik,k})$ is a null Lagrangian. As a result, if we set

$$f''_e(Q) = \frac{(L_1 + L_2 + L_3)}{2} |\nabla Q|^2 - \frac{(L_2 + L_3)}{2} |\text{curl } Q|^2,$$

then $f_e - f''_e$ is a null Lagrangian, f''_e is a positive definite quadratic in $\nabla \mathbf{z}$, and we can argue as in the previous case. □

Setting $p_1 = (z_1 - z_3)/2$, $p_2 = z_2$, and $r = z_1 + z_3$ then $Q \in \mathscr{A}_0$ is given in terms of $(\mathbf{p}, r) \in A_0$ by (1.9). The minimum problem for F_ε in $\mathscr{A}_0$ is recast as the minimum problem for G_ε as defined in (1.10) in A_0, where g_e as expressed in (1.12) and (1.13) directly corresponds to f'_e and f''_e in cases i) and ii) above, respectively. Moreover, we have

$$g_e(\mathbf{p}, r) = f_e(Q) - \frac{(L_3 - L_2 + |L_3 + L_2|)}{4} (Q_{ij,k} Q_{ik,j} - Q_{ij,j} Q_{ik,k}).$$

Corollary 2.3. *If* $(\mathbf{p}, r) \in A_0$ *and* $Q = Q(\mathbf{p}, r)$ *then*

$$G_\varepsilon(\mathbf{p}, r) = F_\varepsilon(Q) + (L_3 - L_2 + |L_3 + L_2|)\frac{s^2 \pi k}{4}.$$

Proof. It suffices to evaluate $\int_\Omega (Q_{ij,k} Q_{ik,j} - Q_{ij,j} Q_{ik,k})$. As this is a null-Lagrangian we are free to choose $(\mathbf{p}, r) \in A_0$, and we set $r = \frac{s}{3}$. It follows from (1.9) that

$$\begin{aligned}\int_\Omega (Q_{ij,k} Q_{ik,j} - Q_{ij,j} Q_{ik,k}) &= -4 \int_\Omega (p_{1x} p_{2y} - p_{1y} p_{2x}) \\ &= -4k|B_{\frac{|s|}{2}}(0)| = -s^2 \pi k.\end{aligned}$$

□

Corollary 2.4. *Minimizers* $(\mathbf{p}_\varepsilon, r_\varepsilon)$ *for* G_ε *in* A_0 *exist, they are of class* $C^\infty(\Omega) \bigcap C^2(\overline{\Omega})$*, and they correspond to minimizers for* F_ε *in* $\mathscr{A}_0$ *by the relation* (1.9)*.*

3. The Asymptotic Problem

By Theorem 2.2, equations (1.12)–(1.13), and our assumptions on Q_0, it follows that minimizers $(\mathbf{p}_\varepsilon, r_\varepsilon)$ for G_ε in A_0 are classical solutions to the boundary value problem

$$\begin{cases} \mathscr{L}_1(\mathbf{p}, r) := -2L_1 \Delta p_1 - (L_2 + L_3)[\Delta p_1 + \frac{1}{2}(r_{xx} - r_{yy})] = -\frac{2p_1}{\varepsilon^2} g_{b,\mathfrak{p}} \\ \mathscr{L}_2(\mathbf{p}, r) := -2L_1 \Delta p_2 - (L_2 + L_3)[\Delta p_2 + r_{xy}] = -\frac{2p_2}{\varepsilon^2} g_{b,\mathfrak{p}} \\ \mathscr{L}_3(\mathbf{p}, r) := -\frac{3}{2} L_1 \Delta r - \frac{(L_2+L_3)}{2}[p_{1xx} - p_{1yy} + 2p_{2xy} + \frac{1}{2}\Delta r] = -\frac{1}{\varepsilon^2} g_{b,\mathfrak{r}} \end{cases} \tag{3.1}$$

in Ω,

$$\text{and } r = \frac{s}{3}, \qquad \mathbf{p} = \mathbf{p}_0 \quad \text{on } \partial\Omega, \tag{3.2}$$

with $|\mathbf{p}_0| = \frac{|s|}{2}$ on $\partial\Omega$ and $\deg\left(\frac{\mathbf{p}_0}{|\mathbf{p}_0|}, \partial\Omega\right) = k > 0$.

Choose a finite covering $\mathscr{U}$ of the C^3 manifold $\overline{\Omega}$ by coordinate neighborhoods with uniformly bounded C^3 structure, and a constant ε_0 in (0, 1) (depending only on Ω and $\mathscr{U}$) such that for all $x_0 \in \overline{\Omega}$, $\overline{B}_{2\varepsilon_0}(x_0)$ is contained in a set in $\mathscr{U}$. Throughout this section we assume (1.21) holds for all minimizers $\mathbf{z}_\varepsilon = (\mathbf{p}_\varepsilon, r_\varepsilon)$ for G_ε in A_0 for all $0 < \varepsilon < \varepsilon_1 \leqq \varepsilon_0$, where ε_1 depends only on s, L_1, L_2, L_3, Ω, k, and the constants in (1.14), and M depends on these terms and in addition on $\|\mathbf{p}_0\|_{W^{1,2}(\partial\Omega)}$. This will be proved in Section 4.

We begin this section by proving several a priori estimates, namely Lemma 3.1 to Lemma 3.6, for solutions to (3.1) and (3.2) that satisfy (1.21) for the above M and $0 < \varepsilon < \varepsilon_1$. These and Proposition 3.7 to Corollary 3.13 will be applied to minimizers of G_ε to prove Theorems A and B at the end of this section.

In this section, unless otherwise stated, we denote by C and C_j, positive constants depending at most on $\mathbf{p}_0$, s, L_1, L_2, L_3, Ω, and the constants in (1.14). Additional dependence, for example on M, will be denoted by $C(M)$.

Lemma 3.1. *Let* $\mathbf{z}_\varepsilon = (\mathbf{p}_\varepsilon, r_\varepsilon)$ *satisfy* (1.21), (3.1), *and* (3.2) *for* $0 < \varepsilon < \varepsilon_1$. *Then* $|\mathbf{z}_\varepsilon|$ *and* $|\varepsilon\nabla\mathbf{z}_\varepsilon|$ *are uniformly bounded in* $\overline{\Omega}$ *by a constant* $C(M)$ *independent of* ε *for all* $0 < \varepsilon < \varepsilon_1$.

Proof. Let $\overline{\mathbf{x}} \in \overline{\Omega}$ and let $\varepsilon \in (0, \varepsilon_1)$. Set

$$\tilde{\mathbf{z}}(\mathbf{y}) = \mathbf{z}_\varepsilon(\varepsilon\mathbf{y} + \overline{\mathbf{x}}) \quad \text{for } \mathbf{y} \in \overline{\tilde{\Omega}} = \{\mathbf{y} \colon \varepsilon\mathbf{y} + \overline{\mathbf{x}} \in \overline{\Omega}\}.$$

Then in $\tilde{\Omega}$, $\tilde{\mathbf{z}}$ satisfies the system obtained by setting $\varepsilon = 1$ in (3.1). Let $\tilde{B}_r = B_r(0) \cap \overline{\tilde{\Omega}}$. From (1.21) and the growth estimate (1.14) on g_b, we have

$$\|\tilde{\mathbf{z}}\|_{L^4(\tilde{B}_1)} \leqq C(M) \qquad \text{for } 0 < \varepsilon < \varepsilon_1.$$

Write (3.1) as $\mathscr{L}\mathbf{z} = \varepsilon^{-2}\mathbf{f}(\mathbf{z})$, where $\mathbf{f}(\mathbf{z}) = [-2p_1 g_{b,\mathfrak{p}}, -2p_2 g_{b,\mathfrak{p}}, -g_{b,\mathfrak{r}}]^t$ and $\mathscr{L}$ is the second order elliptic operator with constant coefficients. From (1.21) and the L^4 estimate, we have

$$\int_{\tilde{B}_1} |\mathbf{f}(\tilde{\mathbf{z}}) \cdot \tilde{\mathbf{z}}| \leqq C(M) \qquad \text{for } 0 < \varepsilon < \varepsilon_1.$$

In addition, we have $\|\tilde{\mathbf{z}}|_{C^\ell(\partial\tilde{\Omega})} \leqq c_\ell$ for $0 < \varepsilon < \varepsilon_1$ and $\ell \leqq 3$, where c_ℓ depends only on Ω and $\mathbf{p}_0$.

We use $\varphi^2(\tilde{\mathbf{z}} - \psi)$ as a test function in (3.1), where φ is a cutoff function vanishing near $|\mathbf{y}| = 1$, such that $\varphi = 1$ on $\tilde{B}_{3/4}$, and ψ is a smooth function equal to $\tilde{\mathbf{z}}$ on $\partial\tilde{\Omega}$. The above inequalities and elliptic estimates give $\|\tilde{\mathbf{z}}\|_{1,2;\tilde{B}_{3/4}} \leqq C(M)$. This implies that $\mathbf{f}(\tilde{\mathbf{z}}) \in L^2(\tilde{B}_{3/4})$ and we see that $\|\tilde{\mathbf{z}}\|_{2,2;\tilde{B}_{5/8}} \leqq C(M)$. Elliptic estimates imply that $\tilde{\mathbf{z}} \in W^{3,2}(\tilde{B}_{9/16})$, and by differentiating the equation we obtain $\|\tilde{\mathbf{z}}\|_{3,2;\tilde{B}_{9/16}} \leqq C(M)$. It follows that $\|\tilde{\mathbf{z}}\|_{C^1(\overline{\tilde{B}}_{1/2})} \leqq C(M)$ uniformly for $0 < \varepsilon < \varepsilon_1$. The assertions then follow by scaling back to $\mathbf{z}_\varepsilon(\mathbf{x})$. □

Set $\mathscr{O}_\mu \colon = \{(\mathbf{p}, r) \colon ||\mathbf{p}| - \frac{|s|}{2}| + |r - \frac{s}{3}| \leqq \mu\}$. Note that $\mathscr{O}_0 = \Gamma_s$. Below, $\mathscr{H}^n(E)$ denotes the n-dimensional Hausdorff measure of E.

Lemma 3.2. *Let* $\mathbf{z}_\varepsilon$ *satisfy* (1.21), (3.1), *and* (3.2). *Set* $\mathscr{B}(\varepsilon, \mu) = \{\mathbf{x} \in \Omega \colon \mathbf{z}_\varepsilon(\mathbf{x}) \notin \mathscr{O}_\mu\}$, $P_1(x, y) = x$, *and* $P_2(x, y) = y$. *Let* $0 < \mu < \delta$, *where* δ *is given in* (1.14). *Then*

$$\mathscr{H}^1(P_1(\mathscr{B}(\varepsilon, \mu)) \leqq C(\mu, M)\varepsilon \quad \textit{and} \quad \mathscr{H}^1(P_2(\mathscr{B}(\varepsilon, \mu)) \leqq C(\mu, M)\varepsilon$$

for all $0 < \varepsilon < \varepsilon_1$.

Proof. Note that $\mathbf{z}_\varepsilon(\mathbf{x}) \in \mathscr{O}_0$ for each $\mathbf{x} \in \partial\Omega$. Let $(x', y') \in \mathscr{B}(\varepsilon, \mu)$, and set $\ell_{x'} = \{(x', y) \colon y \in \mathbb{R}\}$. Since this line intersects $\partial\Omega$, there must exist $(x', y'') \in \ell_{x'}$ so that $\mathbf{z}(x', y'') \in \partial\mathscr{O}_{\mu/2}$. It follows from Lemma 3.1 that there is a $C_1(\mu, M) > 0$ so that

$$\mathbf{z}(x', y) \in \mathscr{O}_{3\mu/4} \backslash \mathscr{O}_{\mu/4} \quad \text{for } |y - y''| < C_1\varepsilon.$$

From (1.14), then, we see that there exists $C_2(\mu, M) > 0$ so that $C_2\varepsilon \leqq \int_{\ell_{x'}} g_b(|\mathbf{p}_\varepsilon|^2, r_\varepsilon) d\mathscr{H}^1(y)$. Thus

$$C_2\varepsilon\mathscr{H}^1(P_1(\mathscr{B}(\varepsilon,\mu)) \leqq \int_\Omega g_b(|\mathbf{p}_\varepsilon|^2, r_\varepsilon) \leqq \varepsilon^2 M.$$

The estimate for $P_2(\mathscr{B}(\varepsilon,\mu))$ follows in the same manner. □

Since $\mathscr{B}(\varepsilon,\mu) \subset P_1(\mathscr{B}(\varepsilon,\mu)) \times P_2(\mathscr{B}(\varepsilon,\mu))$ for $\mu > 0$ we have the following.

Corollary 3.3. *Let* $\mathbf{z}_\varepsilon$ *satisfy* (1.21), (3.1), *and* (3.2). *For any* $\mu \in (0,\delta)$ *if* $0 < \varepsilon < \varepsilon_1$ *then* $\mathscr{H}^2(\mathscr{B}(\varepsilon,\mu)) \leqq C(\mu, M)\varepsilon^2$.

This estimate leads to a statement for all $\mathbf{x} \in \Omega$. We use the fact that $(\mathbf{p}_\varepsilon, r_\varepsilon)$ is bounded together with Corollary 3.3 for $\mathbf{x} \in \mathscr{B}(\varepsilon,\mu)$, and the growth estimate (1.14) for $\mathbf{x} \in \Omega\backslash\mathscr{B}(\varepsilon,\mu)$ to get

Corollary 3.4. *Let* $\mathbf{z}_\varepsilon$ *satisfy* (1.21), (3.1), *and* (3.2). *If* $0 < \varepsilon < \varepsilon_1$ *then*

$$\varepsilon^{-2}\int_\Omega \left(\left(r_\varepsilon(\mathbf{x}) - \frac{s}{3}\right)^2 + \left(|\mathbf{p}_\varepsilon(\mathbf{x})|^2 - \frac{s^2}{4}\right)^2\right) \leqq C(M). \tag{3.3}$$

Lemma 3.5. *Let* $\mathbf{z}_\varepsilon$ *satisfy* (1.21), (3.1), *and* (3.2). *If* $0 < \varepsilon < \varepsilon_1$ *then*

$$\int_\Omega |\nabla r_\varepsilon|^2 \leqq C(M).$$

Proof. We first record an energy estimate for linear elliptic systems applied to (3.1) and (3.2),

$$\|\mathbf{p}_\varepsilon\|^2_{2,2;\Omega} + \|r_\varepsilon\|^2_{2,2;\Omega} \leqq c_1\left(\varepsilon^{-4}\left(\|\mathbf{p}_\varepsilon g_{b,\mathfrak{p}}\|^2_{2;\Omega} + \|g_{b,\mathfrak{r}}\|^2_{2;\Omega}\right) + \|\mathbf{p}_0\|^2_{2,2;\partial\Omega}\right),$$

where c_1 depends on L_1, L_2, L_3 and Ω. Since g_b minimizes on $\mathscr{O}_0$ we have

$$\begin{aligned} &|g_{b,\mathfrak{p}}(|\mathbf{p}_\varepsilon(\mathbf{x})|^2, r_\varepsilon(\mathbf{x}))|^2 + |g_{b,\mathfrak{r}}(|\mathbf{p}_\varepsilon(\mathbf{x})|^2, r_\varepsilon(\mathbf{x}))|^2 \\ &\leqq C\left(\left(|\mathbf{p}_\varepsilon(\mathbf{x})|^2 - \frac{s^2}{4}\right)^2 + \left(r_\varepsilon(\mathbf{x}) - \frac{s}{3}\right)^2\right). \end{aligned} \tag{3.4}$$

Thus, using (3.3) we find

$$\|r_\varepsilon\|^2_{2,2;\Omega} \leqq C(\varepsilon^{-2} + 1).$$

It then follows from this inequality and (3.3) that

$$\begin{aligned} \int_\Omega |\nabla r_\varepsilon|^2 = -\int_\Omega \left(r_\varepsilon - \frac{s}{3}\right)\Delta r_\varepsilon &\leqq \varepsilon^{-1}\|r_\varepsilon - \frac{s}{3}\|_{2;\Omega}\varepsilon\|r_\varepsilon\|_{2,2;\Omega} \\ &\leqq C(M). \end{aligned}$$

□

Lemma 3.6. *There is a constant* $\varepsilon_2 \in (0, \varepsilon_1]$ *depending only on* Ω *and* $k = \deg(\frac{\mathbf{p}_0}{|\mathbf{p}_0|}, \partial\Omega)$*, and a constant* $C(M)$ *independent of* ε *so that if* $(\mathbf{p}_\varepsilon, r_\varepsilon)$ *is a minimizer for* G_ε *in* A_0 *and* $0 < \varepsilon < \varepsilon_2$ *then*

$$\int_\Omega |\nabla \mathbf{p}_\varepsilon|^2 \leqq \frac{s^2}{4}\, 2\pi\, k \ln \frac{1}{\varepsilon} + C(M).$$

Proof. We first construct a comparison function for the energy in (1.18). Choose a set of distinct points $\{b_1, \dots, b_k\} \subset \Omega$, depending only on Ω and k such that

$$\min\{|b_n - b_\ell|, \quad \operatorname{dist}(b_n, \partial\Omega);\ 1 \leqq n, \ell \leqq k, n \neq \ell\} = \overline{\varepsilon}$$

is maximal. Define

$$\mathbf{w}_\varepsilon(\mathbf{x}) = \prod_{\ell=1}^{k} \zeta\left(\frac{|\mathbf{x} - b_\ell|}{\varepsilon}\right) \frac{(\mathbf{x} - b_\ell)}{|\mathbf{x} - b_\ell|}\, e^{i j_\varepsilon(\mathbf{x})}$$

where $\zeta(t) \in C^2(\mathbb{R})$ such that $\zeta(t) = 0$ for $t \leqq \frac{1}{2}$, $\zeta(t) = 1$ for $1 \leqq t$, and $j_\varepsilon(\cdot)$ is harmonic in Ω such that $\mathbf{w}_\varepsilon = \frac{\mathbf{p}_0}{|\mathbf{p}_0|}$ on $\partial\Omega$ for $\varepsilon < \overline{\varepsilon}$. Then, one has $E_\varepsilon(\mathbf{w}_\varepsilon) \leqq \pi k \ln(\frac{1}{\varepsilon}) + c_0$ for $0 < \varepsilon < \overline{\varepsilon}$, where E_ε is given in (1.18) and c_0 depends only on Ω and $\mathbf{p}_0$. We next set $(\mathbf{w}', r') = (\frac{|s|}{2}\mathbf{w}_\varepsilon, \frac{s}{3}) \in A_0$ and use this as our comparison function for G_ε. Set $\varepsilon_2 = \min\{\overline{\varepsilon}, \varepsilon_1\}$. Then, for $\varepsilon \in (0, \varepsilon_2]$, using (1.11) and (1.14) we find that

$$\begin{aligned} G_\varepsilon(\mathbf{w}', r') \leqq{}& (L_1 + \frac{L_2 + L_3}{2}) \int_\Omega |\nabla \mathbf{w}'|^2 \\ &+ |L_2 + L_3| \int_\Omega (w'_{1,x} w'_{2,y} - w'_{1,y} w'_{2,x}) + C_1. \end{aligned}$$

The second integral on the right depends only on $\mathbf{w}'|_{\partial\Omega}$. Thus we get

$$G_\varepsilon(\mathbf{w}', r') \leqq \left(L_1 + \frac{L_2 + L_3}{2}\right) \frac{s^2}{4}\, 2\pi k \ln \frac{1}{\varepsilon} + C_1.$$

Next, we use

$$\int_\Omega g_e(\nabla \mathbf{p}_\varepsilon, \nabla r_\varepsilon) \leqq G_\varepsilon(\mathbf{p}_\varepsilon, r_\varepsilon) \leqq G_\varepsilon(\mathbf{w}', r').$$

From (1.11) and suppressing the subscript ε we see

$$\begin{aligned} &\left(L_1 + \frac{(L_2 + L_3)}{2}\right) \int_\Omega |\nabla \mathbf{p}|^2 + \frac{(L_2 + L_3)}{2} \int_\Omega (p_{1x} r_x - p_{1y} r_y + r_x p_{2y} + r_y p_{2x}) \\ &\quad + |L_2 + L_3| \int_\Omega (p_{1x} p_{2y} - p_{1y} p_{2x}) \leqq (2L_1 + L_2 + L_3)\, \frac{s^2}{4} \pi k \ln\left(\frac{1}{\varepsilon}\right) + C_1. \end{aligned}$$

Again, the third integral is a constant depending on $\mathbf{p}_0$. The lemma will follow once we show that we can bound the second integral appropriately. To do this we multiply the third equation in (3.1) by $(r - \frac{s}{3})$ and integrate over Ω. We get using Lemma 3.5 that for $0 < \varepsilon < \varepsilon_2$:

$$\left|\frac{(L_2 + L_3)}{2} \int_\Omega (p_{1x} r_x - p_{1y} r_y + p_{2x} r_y + p_{2y} r_x)\right|$$
$$\leqq \varepsilon^{-2} \int_\Omega |g_{b,\mathfrak{r}}| \cdot |r - \frac{s}{3}| + C_2(M)$$
$$\leqq \varepsilon^{-2} \int_\Omega \left(|g_{b,\mathfrak{r}}|^2 + |r - \frac{s}{3}|^2\right) + C_2(M).$$

Finally using (3.3) and (3.4) we see that the last integral is bounded by a constant $C(M)$ independent of ε for $0 < \varepsilon < \varepsilon_2$. □

We are in a position to apply Lin's Structure Proposition; see [15]. Significant parts of the proposition were also proved by Jerrard [10] and Sandier [19]. Define

$$J_\varepsilon(\mathbf{v}) = \int_\Omega j_\varepsilon(\mathbf{v}), \text{ where}$$
$$j_\varepsilon(\mathbf{v}) = \frac{1}{2}\left[|\nabla \mathbf{v}|^2 + \frac{1}{2\varepsilon^2}\left(\frac{s^2}{4} - |\mathbf{v}|^2\right)^2\right].$$

Proposition 3.7. *For fixed $s \neq 0$ and a constant K suppose that*

$$\mathbf{p}_\varepsilon \in \{\mathbf{v} \in W^{1,2}(\Omega; \mathbb{R}^2) \colon \mathbf{v} = \mathbf{p}_0 \textit{ on } \partial\Omega\} \textit{ such that}$$
$$\mathbf{p}_0 \in C^3(\partial\Omega),\ |\mathbf{p}_0| = \frac{s}{2},\ \textit{deg}\left(\frac{\mathbf{p}_0}{|\mathbf{p}_0|}, \partial\Omega\right) = k > 0,$$
$$J_\varepsilon(\mathbf{p}_\varepsilon) \leqq \pi\, \frac{s^2}{4}\, k \ln\, \frac{1}{\varepsilon} + K,$$

where $0 < \varepsilon < \eta$. Fix $0 < \alpha_0 < \frac{1}{8}$. There are positive constants $\eta_0 \in (0, \eta)$ and $\rho_0 > 0$ depending on $K, \Omega, \mathbf{p}_0$, and α_0 so that if $\varepsilon < \eta_0$ then for each $\mathbf{p}_\varepsilon$, there are points $\{a_1^\varepsilon, \ldots, a_k^\varepsilon\}$ satisfying

$$\min\{|a_n^\varepsilon - a_\ell^\varepsilon|,\ \textit{dist}(a_n^\varepsilon, \partial\Omega);\ 1 \leqq n, \ell \leqq k, n \neq \ell\} \geqq \rho_0,$$

and constants $\alpha_m(\varepsilon)$, $\alpha_0 \leqq \alpha_m \leqq 2\alpha_0$ for $1 \leqq m \leqq k$ so that $\varepsilon^{\alpha_m} \int_{\partial B_m} j_\varepsilon(\mathbf{p}_\varepsilon) \leqq C$ for a fixed constant $C(s)$, $|\mathbf{p}_\varepsilon| \geqq \frac{|s|}{4}$ on ∂B_m, and deg $(\frac{\mathbf{p}_\varepsilon}{|\mathbf{p}_\varepsilon|}, \partial B_m) = 1$ where $B_m := B_{\varepsilon^{\alpha_m}}(a_m^\varepsilon)$. Furthermore, for any sequence $\{\mathbf{p}_{\varepsilon_\ell}\}$ with $\varepsilon_\ell \downarrow 0$, there exists a subsequence $\{\varepsilon_{\ell(q)}\}$, points $\{a_1, \ldots, a_k\}$ and a function $h(\mathbf{x})$ so that

$$a_m^{\varepsilon_{\ell(q)}} \to a_m \quad \textit{and} \quad \mathbf{p}_{\varepsilon_{\ell(q)}} \to \mathbf{p}* = \prod_{m=1}^{k} \frac{(\mathbf{x} - a_m)}{|\mathbf{x} - a_m|}\, e^{ih(\mathbf{x})}\, \frac{|s|}{2}$$

as $q \to \infty$, where the convergence is strongly in $L^2(\Omega)$, weakly in

$$W^{1,2}_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_k\}), \quad \textit{and} \quad \|h\|_{W^{1,2}(\Omega)} \leqq C_1$$

for some constant $C_1 = C_1(K, \Omega, \mathbf{p}_0)$.

We take into account (1.21), Corollary 3.4, Lemma 3.5, Lemma 3.6 and apply the Proposition to a sequence of minimizers.

Lemma 3.8. *Let* $\{(\mathbf{p}_\varepsilon, r_\varepsilon)\}$ *be a sequence of minimizers for* $\{G_\varepsilon\}$ *in* A_0 *such that* $\varepsilon \downarrow 0$. *Then for a subsequence* $\{(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell})\}$ *we have* $\mathbf{p}_{\varepsilon_\ell} \to \mathbf{p}^*$ *as in Proposition 3.7 and*

$$r_{\varepsilon_\ell} \rightharpoonup \frac{s}{3} \text{ in } W^{1,2}(\Omega).$$

The next two lemmas strengthen the notion of convergence using the fact that we are working with a sequence of minimizers. Set $\Omega_\rho = \Omega \backslash \bigcup_{j=1}^{k} \overline{B}_\rho(a_j)$.

Lemma 3.9. *Let* $\{(\mathbf{p}_\ell, r_\ell)\}$ *be a sequence of minimizers for* $\{G_{\varepsilon_\ell}\}$ *in* A_0 *such that* $\varepsilon_\ell \downarrow 0$, *converging to* $(\mathbf{p}^*, \frac{s}{3})$, *in* $L^2(\Omega)$, *where* $\mathbf{p}^*(\mathbf{x}) = \frac{|s|}{2} \prod_{j=1}^{k} \frac{(\mathbf{x}-a_j)}{|\mathbf{x}-a_j|} e^{ih(\mathbf{x})}$. *Then for each* $0 < \rho < \frac{\rho_0}{2}$,

$$(\mathbf{p}_\ell, r_\ell) \to \left(\mathbf{p}^*, \frac{s}{3}\right) \text{ in } W^{1,2}(\Omega_\rho) \quad \text{and} \quad \lim_{\varepsilon_\ell \to 0} \varepsilon_\ell^{-2} \int_{\Omega_\rho} g_b(|\mathbf{p}_\ell|^2, r_\ell) = 0.$$

Moreover $\Delta h = 0$ *in* Ω.

Proof. Applying Lemma 3.8 and using a diagonal argument, we have $(\mathbf{p}_\ell, r_\ell) \rightharpoonup (\mathbf{p}^*, \frac{s}{3})$ in $W^{1,2}(\Omega_\rho)$ for each $\rho > 0$, as above. Furthermore, $(\mathbf{p}_\ell, r_\ell)$ is a local minimizer for

$$\int_{\Omega_\rho} [g_e(\nabla \mathbf{p}, \nabla r) + \varepsilon_\ell^{-2} g_b(|\mathbf{p}|^2, r)].$$

To prove strong convergence it is enough to show that for each $\overline{\mathbf{x}} \in \overline{\Omega} \backslash \{a_1, \dots, a_k\}$ there exists a neighborhood $\mathscr{U}_{\overline{\mathbf{x}}}$ of $\overline{\mathbf{x}}$, on which $(\mathbf{p}_\ell, r_\ell) \to (\mathbf{p}^*, \frac{s}{3})$ in $W^{1,2}(\mathscr{U}_{\overline{\mathbf{x}}} \cap \Omega)$. We first consider the case $\overline{\mathbf{x}} \notin \partial\Omega$ and take $\overline{d} = \overline{d}(\overline{\mathbf{x}}) > 0$ such that $\overline{B}_{3\overline{d}} = \overline{B}_{3\overline{d}}(\overline{\mathbf{x}}) \subset \Omega \backslash \{a_1, \dots, a_k\}$. Then $\sum_{j=1}^{k} \theta_j(\mathbf{x}) + h(\mathbf{x})$ is single-valued here and we write $\mathbf{p}^* = \frac{|s|}{2} e^{i\omega(\mathbf{x})}$ on $B_{2\overline{d}}$. From Lemma 3.8 and (3.3), for each $\rho > 0$, there exists $C_0(\rho, M) < \infty$ independent of ℓ so that

$$\int_{\Omega_\rho} \left[|\nabla \mathbf{p}_\ell|^2 + |\nabla r_\ell|^2 + \varepsilon_\ell^{-2}((|\mathbf{p}_\ell|^2 - \frac{s^2}{4})^2 + (r_\ell - \frac{s}{3})^2)\right] \leqq C_0(\rho, M). \tag{3.5}$$

Take $\rho < \overline{d}$. Then for any subsequence $\{(\mathbf{p}_{\ell_j}, r_{\ell_j})\}$ of $\{(\mathbf{p}_\ell, r_\ell)\}$ (possibly after passing to a further subsequence that we do not relabel) d can be chosen, $\overline{d} \leqq d \leqq 2\overline{d}$ so that

$$\int_{\partial B_d} [|\partial_\tau \mathbf{p}_{\ell_j}|^2 + |\partial_\tau r_{\ell_j}|^2 + \varepsilon_{\ell_j}^{-2}((|\mathbf{p}_{\ell_j}|^2 - \frac{s^2}{4})^2 + (r_{\ell_j} - \frac{s}{3})^2)] \leqq C_1(\overline{\mathbf{x}}, M), \tag{3.6}$$

where ∂_τ denotes the tangential derivative. Thus $(|\mathbf{p}_{\ell_j}|, r_{\ell_j}) \to (\frac{|s|}{2}, \frac{s}{3})$ uniformly on ∂B_d and $(\mathbf{p}_{\ell_j}, r_{\ell_j}) \rightharpoonup (\mathbf{p}^*, \frac{s}{3})$ in $W^{1,2}(\partial B_d)$. Since deg $(\frac{\mathbf{p}^*}{|\mathbf{p}^*|}, \partial B_d) = 0$, it follows that deg $(\frac{\mathbf{p}_{\ell_j}}{|\mathbf{p}_{\ell_j}|}, \partial B_d) = 0$ for j sufficiently large, and we can write $\mathbf{p}_{\ell_j}(\mathbf{x}) = |\mathbf{p}_{\ell_j}(\mathbf{x})|e^{i\omega_{\ell_j}(\mathbf{x})}$ for $\mathbf{x} \in \partial B_d$. We define $\tilde{\omega}_{\ell_j}(\mathbf{x})$ and $\tilde{\omega}(\mathbf{x})$ on B_d as the harmonic extensions of $\omega_{\ell_j}|_{\partial B_d}$ and $\omega|_{\partial B_d}$, respectively. It follows that

$$\tilde{\omega}_{\ell_j} \rightharpoonup \omega \text{ in } W^{1,2}(\partial B_d) \quad \text{and} \quad \tilde{\omega}_{\ell_j} \to \tilde{\omega} \text{ in } W^{1,2}(B_d). \tag{3.7}$$

The first limit follows from [9] and the second follows from elliptic regularity theory. We next construct comparison functions

$$(\tilde{\mathbf{p}}_{\ell_j}, \tilde{r}_{\ell_j}) := (|\tilde{\mathbf{p}}_{\ell_j}|e^{i\tilde{\omega}_{\ell_j}}, \tilde{r}_{\ell_j}) \quad \text{on } B_d,$$

such that $(\tilde{\mathbf{p}}_{\ell_j}, \tilde{r}_{\ell_j}) = (\mathbf{p}_{\ell_j}, r_{\ell_j})$ on ∂B_d.

This is done by setting

$$(|\tilde{\mathbf{p}}_{\ell_j}|, \tilde{r}_{\ell_j}) = \left(\frac{|s|}{2}, \frac{s}{3}\right) \quad \text{on } B_{d-\varepsilon_{\ell_j}},$$

and for each θ define $(|\tilde{\mathbf{p}}_{\ell_j}|, \tilde{r}_{\ell_j})(|\mathbf{x}|, \theta)$ to be linear for $d - \varepsilon_{\ell_j} \leqq |\mathbf{x}| \leqq d$. Then based on (3.6) and (3.7) it follows that $(\tilde{\mathbf{p}}_{\ell_j}, \tilde{r}_{\ell_j}) \to (\tilde{\mathbf{p}}, \tilde{r}) = (\frac{|s|}{2}\, e^{i\tilde{\omega}}, \frac{s}{3})$ in $W^{1,2}(B_d)$. Moreover,

$$\int_{B_d} g_e(\nabla\tilde{\mathbf{p}}, 0) = \lim_{j\to\infty} \int_{B_d} [g_e(\nabla\tilde{\mathbf{p}}_{\ell_j}, \nabla\tilde{r}_{\ell_j}) + \varepsilon_{\ell_j}^{-2} g_b(|\tilde{\mathbf{p}}_{\ell_j}|^2, \tilde{r}_{\ell_j})].$$

From the minimality of $(\mathbf{p}_{\ell_j}, r_{\ell_j})$ and the weak lower semicontinuity of $\int_{B_d} g_e$ we have

$$\begin{aligned} \int_{B_d} g_e(\nabla\mathbf{p}^*, 0) &\leqq \limsup_{j\to\infty} \int_{B_d} [g_e(\nabla\mathbf{p}_{\ell_j}, \nabla r_{\ell_j}) + \varepsilon_{\ell_j}^{-2} g_b(|\mathbf{p}_{\ell_j}|^2, r_{\ell_j})] \\ &\leqq \int_{B_d} g_e(\nabla\tilde{\mathbf{p}}, 0). \end{aligned} \tag{3.8}$$

From (1.11) it follows that $\int_{B_d} g_e(\nabla\mathbf{p}, 0)$ minimizes in the set $\{\mathbf{p} = \frac{|s|}{2}\, e^{if} \in W^{1,2}(B_d) \colon f = \omega \text{ on } \partial B_d\}$ if and only if $\Delta f = 0$ in B_d. Thus $\tilde{\mathbf{p}}$ is the unique minimizer and $\tilde{\mathbf{p}} = \mathbf{p}^*$ on B_d; see [2]. From (1.12) and (1.13) we see that $\int_{B_d} g_e(\nabla\mathbf{p}, \nabla r)$ is the sum of weakly lower semi-continuous integrals. We have shown that the sum is weakly continuous on the sequence $\{(\mathbf{p}_{\ell_j}, r_{\ell_j})\}$. It follows that each of its terms is weakly continuous on this sequence, as well. Thus $\int_{B_d} |\nabla\mathbf{p}_{\ell_j}|^2 \to \int_{B_d} |\nabla\mathbf{p}^*|^2$ and $\int_{B_d} |\nabla r_{\ell_j}|^2 \to 0$ as $j \to \infty$. Thus $(\mathbf{p}_{\ell_j}, r_{\ell_j}) \to (\mathbf{p}^*, \frac{s}{3})$ in $W^{1,2}(B_d)$ and, as a result, the full sequence $(\mathbf{p}_\ell, r_\ell) \to (\mathbf{p}^*, \frac{s}{3})$ in $W^{1,2}(B_{\overline{d}})$. A further consequence is that

$$\lim_{\ell\to\infty} \varepsilon_\ell^{-2} \int_{B_{\overline{d}}} g_b(|\mathbf{p}_\ell|^2, r_\ell) = 0.$$

Moreover, we have shown that $\mathbf{p}^* = \frac{|s|}{2} e^{i(\sum_{j=1}^k \theta_j + h(\mathbf{x}))}$, where $\Delta h = 0$ in $\Omega\backslash\{a_1, \dots, a_k\}$. From Proposition 3.7 we have that $h \in W^{1,2}(\Omega)$; this implies that the singularities are removable.

Lastly if $\bar{\mathbf{x}} \in \partial\Omega$, we take a neighborhood $\mathscr{U}_{\bar{\mathbf{x}}}$ and $d \in (0, \varepsilon_0)$ so that there exists a smooth diffeomorphism defined on B_d satisfying $\psi(\bar{\mathbf{x}}) = \bar{\mathbf{x}}$ and

$$\psi : B_d^+ = \{\mathbf{y} + \bar{\mathbf{x}} : y_1^2 + y_2^2 < d,\ y_2 \geqq 0\} \underset{onto}{\to} \mathscr{U}_{\bar{\mathbf{x}}}.$$

We can then carry out the radial construction of $(|\tilde{\mathbf{p}}_\ell|, \tilde{r}_\ell)$ in B_d^+, push this forward to $\mathscr{U}_{\bar{\mathbf{x}}}$, and then argue as in the previous case. □

We next prove that $\{(|\mathbf{p}_\ell|, r_\ell)\}$ converges uniformly to $(\frac{|s|}{2}, \frac{s}{3})$ outside of a neighborhood of $\{a_1, \dots, a_k\}$. The proof is similar to that in [13] Theorem A. This is possible since the density g_e can be expressed as the positive definite quadratic (1.12) or (1.13).

Lemma 3.10. *Let* $(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell}) = (\mathbf{p}_\ell, r_\ell)$ *be a convergent sequence of minimizers for* $\{G_{\varepsilon_\ell}\}$ *in* A_0 *as in Lemma* 3.9. *Given* $\rho \in (0, \frac{\rho_0}{2})$ *and* $\mu \in (0, \frac{\delta}{2})$*, there exists* ℓ_0 *so that*

$$(\mathbf{p}_\ell(\mathbf{x}), r_\ell(\mathbf{x})) \in \mathscr{O}_\mu \quad \textit{for all } \mathbf{x} \in \Omega_\rho \quad \textit{and} \quad \ell > \ell_0.$$

Proof. Assume there exists $\mathbf{x}_\ell \in \Omega_\rho$ such that

$$(\mathbf{p}_\ell(\mathbf{x}_\ell), r_\ell(\mathbf{x}_\ell)) \notin \mathscr{O}_\mu \quad \text{for} \quad \ell \in \mathbb{N}.$$

Since $\partial\Omega$, it follows that there exists $\mathbf{z}_\ell \in \Omega_\rho$ such that $(\mathbf{p}_\ell(\mathbf{z}_\ell), r_\ell(\mathbf{z}_\ell)) \in \partial\mathscr{O}_\mu$. Using Lemma 3.1 we see there is a $c(\mu) > 0$ so that

$$(\mathbf{p}_\ell(\mathbf{x}), r_\ell(\mathbf{x})) \in \mathscr{O}_{\frac{3\mu}{2}} \setminus \mathscr{O}_{\frac{\mu}{2}} \quad \text{for} \quad \mathbf{x} \in B_{c\varepsilon_\ell}(\mathbf{z}_\ell) \bigcap \Omega_\rho.$$

It follows from (1.14) that there is a constant $\beta(\mu) > 0$ so that

$$g_b(|\mathbf{p}_\ell(\mathbf{x})|^2, r_\ell(\mathbf{x})) \geqq \beta \quad \text{for} \quad \mathbf{x} \in B_{c\varepsilon_\ell}(\mathbf{z}_\ell) \bigcap \Omega_\rho.$$

Thus we conclude for ℓ sufficiently large that there is a constant $C_2 > 0$ so that

$$\varepsilon_\ell^{-2} \int_{B_{c\varepsilon_\ell}(\mathbf{z}_\ell) \bigcap \Omega_\rho} g_b(|\mathbf{p}_\ell|^2, r_\ell) \geqq C_2.$$

On the other hand, it follows from Lemma 3.9 that the left side tends to zero as $\ell \to \infty$. □

In the next two lemmas we prove that if a sequence of minimizers $\{(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell})\}$ converges in $W^{1,2}_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \dots, a_k\})$, then in fact it is bounded in $W^{j,2}_{\text{loc}}(\Omega\backslash\{a_1, \dots, a_k\})$ for all j. Our arguments are based on three features: first that $\{(|\mathbf{p}_{\varepsilon_\ell}|, r_{\varepsilon_\ell})\}$ converges uniformly to $(\frac{|s|}{2}, \frac{s}{3})$ on K for each $K \subset\subset \Omega\backslash\{a_1, \dots, a_k\}$, second that $(\frac{s^2}{4}, \frac{s}{3})$ is a nondegenerate minimum point for g_b, and third that g_e is strongly elliptic. A corresponding result is proved for minimizing sequences to the Ginzburg–Landau energy (1.18) in [1]. In that case the Euler–Lagrange equations are diagonal and the authors are able to apply estimates for elliptic equations. Here, our arguments rely only on L^2 estimates for elliptic systems.

Lemma 3.11. *Let $\{(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell})\}$ be a sequence of minimizers for $\{G_{\varepsilon_\ell}\}$ in A_0 converging in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1,\ldots,a_k\})$ as $\varepsilon_\ell \to 0$. Then for $K \subset\subset \overline{\Omega}\backslash\{a_1,\ldots,a_k\}$ there exist constants ℓ_0 and E so that if $\ell \geqq \ell_0$, then*

$$\|D^2(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell})\|_{2;K} \leqq E$$

Proof. It suffices to establish the estimate in a neighborhood of each point in $\overline{\Omega}\backslash\{a_1,\ldots,a_k\}$. We first consider the case of $x_0 \in \Omega\backslash\{a_1,\ldots,a_k\}$. Then $\overline{B_{2d}}(\mathbf{x}_0) \subset \Omega\backslash\{a_1,\ldots,a_k\}$ for some $d(\mathbf{x}_0) \in (0,\varepsilon_0)$. Fixing $\mathbf{x}_0$ and η, $0 < \eta < \frac{|s|}{6}$, we take d and ℓ_0 so that

$$\int_{B_{2d}(\mathbf{x}_0)} (|D\mathbf{p}_{\varepsilon_\ell}|^2 + |Dr_{\varepsilon_\ell}|^2) < \eta \tag{3.9}$$

and

$$\|\mathbf{p}_{\varepsilon_\ell}| - \frac{|s|}{2}| + |r_{\varepsilon_\ell} - \frac{s}{3}| < \eta \text{ on } B_{2d}(\mathbf{x}_0) \tag{3.10}$$

for all $\ell \geqq \ell_0$.

Let $\zeta \in C^2_c(B_{2d}(\mathbf{x}_0))$ be such that $\zeta = 1$ on $B_d(\mathbf{x}_0)$. We suppress the subscripts and write $(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell}) = (\mathbf{p}, r)$. Then multiplying (3.1) by $-\partial_{x_j}(\zeta^2\partial_{x_j}(\mathbf{p}, r))$, we get using the strong ellipticity of the system that there exists a constant $\Lambda(L_1, L_2, L_3) > 0$ for which

$$\begin{aligned}&\Lambda\|\zeta D\partial_{x_j}(\mathbf{p}, r)\|^2_{2;B_{2d}} + \varepsilon_\ell^{-2}\int_{B_{2d}} \zeta^2[\mathscr{D}^2 g_b](\partial_{x_j}(|\mathbf{p}|^2, r))\cdot(\partial_{x_j}(|\mathbf{p}|^2, r))\\ &\leqq C\||D\zeta|\partial_{x_j}(\mathbf{p}, r)\|^2_{2;B_{2d}} - \varepsilon_\ell^{-2}\int_{B_{2d}} 2g_{b,\mathfrak{p}}|\partial_{x_j}\mathbf{p}|^2\zeta^2.\end{aligned}$$

Here $\mathscr{D}g_b = (\partial_{\mathfrak{p}}g_b, \partial_{\mathfrak{r}}g_b)$ and $[\mathscr{D}^2 g_b]$ is the Hessian of g_b. Using (1.14), (3.10) and taking η sufficiently small, we have

$$\lambda\int_{B_{2d}} \zeta^2|\partial_{x_j}(|\mathbf{p}|^2, r)|^2 \leqq \int_{B_{2d}} \zeta^2[\mathscr{D}^2 g_b]\partial_{x_j}(|\mathbf{p}|^2, r)\cdot\partial_{x_j}(|\mathbf{p}|^2, r)$$

for some $\lambda > 0$.

From equations (3.1), using $|\mathbf{p}| \geqq \frac{|s|}{4}$ on B_{2d}, we get

$$\varepsilon_\ell^{-4}\int_{B_{2d}} \zeta^2(g^2_{b,\mathfrak{p}} + g^2_{b,\mathfrak{r}}) = \varepsilon_\ell^{-4}\int_{B_{2d}} \zeta^2|\mathscr{D}g_b|^2 \leqq C\int_{B_{2d}} \zeta^2|D^2(\mathbf{p}, r)|^2.$$

Thus we find

$$\begin{aligned}&\|\zeta D^2(\mathbf{p}, r)\|^2_{2;B_{2d}} + \varepsilon_\ell^{-4}\|\zeta\mathscr{D}g_b\|^2_{2;B_{2d}} + \varepsilon_\ell^{-2}\|\zeta D(|\mathbf{p}|^2, r)\|^2_{2;B_{2d}}\\ &\leqq C_0\int_{B_{2d}} \zeta^2|D\mathbf{p}|^4 + C_1\\ &\leqq C_2\int_{B_{2d}} \zeta^2|D^2\mathbf{p}|^2\cdot\int_{B_{2d}} |D\mathbf{p}|^2 + C_3.\end{aligned} \tag{3.11}$$

The last estimate follows by applying the Sobolev estimate

$$\left(\int_\Omega \varphi^2\right)^{1/2} \leqq c\int_\Omega (|D\varphi| + |\varphi|), \tag{3.12}$$

with $\varphi = \zeta|D\mathbf{p}|^2$ and $c = c(\Omega)$. Choosing η small in (3.9) the first term on the right of (3.11) can be absorbed into the left and the lemma is proved for the case of $K \subset\subset \Omega\backslash\{a_1, \dots, a_k\}$.

Assume next that $x_0 \in \partial\Omega$ and $d < \varepsilon_0$, so that $\overline{B_{2d}}(x_0)$ is contained in a coordinate patch in which we can locally flatten $\partial\Omega$ near x_0. We consider the special case where $\partial\Omega$ is already locally flat,

$$\begin{aligned} &B_{2d}(x_0) \cap (\Omega\backslash\{a_1, \dots, a_k\}) \\ &\quad = B^+_{2d}(x_0) = \{(x_1, x_2)\colon (x_1 - x_{01})^2 + (x_2 - x_{02})^2 < 4d^2 \text{ and } x_2 \geqq x_{02}\}. \end{aligned}$$

Let $\zeta \in C^\infty_c(B_{2d}(x_0))$ such that $\zeta = 1$ on $B_d(x_0)$. Let $(\tilde{\mathbf{p}}, \tilde{r}) \in W^{2,2}(\Omega)$ such that $(\tilde{\mathbf{p}}, \tilde{r}) = (\mathbf{p}_0, \frac{s}{3})$ on $\partial\Omega$. Again suppressing subscripts, we multiply (3.1) by $\partial_{x_1}(\zeta^2\partial_{x_1}(\mathbf{p} - \tilde{\mathbf{p}}, r - \tilde{r}))$ and integrate by parts. Then, for any $0 < \theta < 1$ we get

$$\begin{aligned} \Lambda\|\zeta D\partial_{x_1}(\mathbf{p}, r)\|^2_{2;B^+_{2d}} &\leqq C_1\||D\zeta||\partial_{x_1}(\mathbf{p}, r)|\|^2_{2;B^+_{2d}} \\ &+\theta\varepsilon_\ell^{-4}\|\zeta\mathscr{D}g_b\|^2_{2;B^+_{2d}} + \frac{1}{\theta}\left(\int_{B^+_{2d}} |\partial_{x_1}\mathbf{p}|^4\zeta^2 + C_2\right). \end{aligned} \tag{3.13}$$

We next multiply (3.1) by

$$-\partial_{x_2}(\zeta^2\partial_{x_2}(\mathbf{p}, r)) = -\zeta^2\partial^2_{x_2}(\mathbf{p}, r) - 2\zeta\,\partial_{x_2}\zeta\,\partial_{x_2}(\mathbf{p}, r).$$

Using the ellipticity of $\mathcal{L}$ we get

$$\begin{aligned} &\frac{L_1}{2}\|\zeta^2\partial^2_{x_2}(\mathbf{p}, r)\|^2_{2;B^+_{2d}} - \Lambda_1(\|\zeta^2 D\partial_{x_1}(\mathbf{p}, r)\|^2_{2;B^+_{2d}} \\ &\quad +\||D\zeta||D(\mathbf{p}, r)|\|^2_{2;B^+_{2d}}) \\ &\quad \leqq -\int_{B^+_{2d}} \mathcal{L}(\mathbf{p}, r)\cdot\partial_{x_2}(\zeta^2\partial_{x_2}(\mathbf{p}, r)) = I, \end{aligned} \tag{3.14}$$

where $\Lambda_1 = \Lambda_1(L_1, L_2, L_3)$.

From (3.1) we have

$$I = \int_{B^+_{2d}} [2p_1 g_{b,\mathfrak{p}}, 2p_2 g_{b,\mathfrak{p}}, g_{b,\mathfrak{r}}]^t\cdot\partial_{x_2}(\zeta^2\partial_{x_2}(\mathbf{p}, r)).$$

Here we integrate by parts. Since g_b minimizes at $(|\mathbf{p}|^2, r) = (s^2, \frac{s}{3})$, it follows that $g_{b,\mathfrak{p}} = g_{b,\mathfrak{r}} = 0$ on $\partial\Omega$. Thus the boundary term will vanish and we find that

$$\begin{aligned} I &= -\varepsilon_\ell^{-2}\int_{B^+_{2d}} \partial_{x_2}[2p_1 g_{b,\mathfrak{p}}, 2p_2 g_{b,\mathfrak{p}}, g_{b,\mathfrak{r}}]^t\zeta^2\partial_{x_2}(\mathbf{p}, r) \\ &\leqq 2\varepsilon_\ell^{-2}\int_{B^+_{2d}} |g_{b,\mathfrak{p}}||\partial_{x_2}\mathbf{p}|^2\zeta^2. \end{aligned} \tag{3.15}$$

Combining (3.13), (3.14), and (3.15) we see that there exists $\Lambda_2(L_1, L_2, L_3) > 0$ so that

$$\Lambda_2(\|\zeta^2 D^2(\mathbf{p}, r)\|^2_{2;B^+_{2d}} + \varepsilon_\ell^{-4}\|\zeta \mathscr{D} g_b\|^2_{2;B^+_{2d}}) \leqq C_2\||D\zeta||D(\mathbf{p}, r)|\|^2_{2;B^+_{2d}}$$
$$+\theta\varepsilon_\ell^{-4}\|\zeta\mathscr{D} g_b\|^2_{2;B^+_{2d}} + \frac{1}{\theta}\left(\int_{B^+_{2d}} |D\mathbf{p}|^4\zeta^2 + C_3\right).$$

From this point, the argument proceeds just as above. In the general case one first flattens the boundary and analyzes the system in local coordinates in the same manner. □

Lemma 3.12. *Let $\{(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell})\}$ be the sequence of minimizers for $\{G_{\varepsilon_\ell}\}$ from the previous lemma. For each integer $j > 2$ and set $K \subset\subset \Omega\backslash\{a_1, \ldots, a_k\}$ there are constants E_j so that*

$$\|(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell})\|_{j,2;K} \leqq E_j \textit{ for } \ell \geqq \ell_0.$$

Proof. Choose $\eta < \frac{|s|}{6}$ so that $[\mathscr{D}^2 g_b] \geqq \lambda I$ on $\mathscr{O}_\eta$. We suppress the subscript ε_ℓ and assume that $\ell \geqq \ell_0$, where ℓ_0 is from the previous lemma. We further assume that $d \in (0, \varepsilon_0)$ is sufficiently small so that $\overline{B_d}(\mathbf{x}_0) \subset \Omega\backslash\{a_1, \ldots, a_k\}$ and so that (3.10) holds. Assume that there exists a constant $E_q < \infty$ so that

$$\|(\mathbf{p}, r)\|^2_{q,2;B_d} + \varepsilon_\ell^{-2}\|\left(|\mathbf{p}|^2 - \frac{s^2}{4}, r - \frac{s}{3}\right)\|^2_{q-1,2;B_d}$$
$$+\varepsilon_\ell^{-4}\|(g_{b,\mathfrak{p}}, g_{b,r})\|^2_{q-2,2;B_d} \leqq E_q \tag{3.16}$$

holds for $q = j - 1$. We prove this estimate for $q = j$ where E_{j-1} is replaced by a possibly larger constant, E_j, and d by $d/2$. Note that we already have (3.16) for $q = 2$ from Lemma 3.11. Let ∂^γ be a derivative of order $j - 1$ and D^q be the collection of all partial derivatives of order q. Let $\zeta \in C_c^\infty(B_d)$ be such that $\zeta = 1$ on $B_{d/2}$. We use $(-1)^{j-1}\partial^\gamma(\zeta^2\partial^\gamma(\mathbf{p}, r))$ as a test function in (3.1) and find

$$\Lambda\|\zeta^2|D\partial^\gamma(\mathbf{p}, r)|\|^2_{2;\Omega} \leqq C\||D\zeta|\partial^\gamma(\mathbf{p}, r)\|^2_{2;\Omega}$$
$$-\varepsilon_\ell^{-2}\int_\Omega \zeta^2\partial^\gamma(g_{b,\mathfrak{p}}2\mathbf{p}, g_{b,r})\cdot\partial^\gamma(\mathbf{p}, r) = I - \Pi. \tag{3.17}$$

From (3.16) we have $I \leqq C_0(E_{j-1}, d)$. We write

$$\partial^\gamma(g_{b,\mathfrak{p}}2\mathbf{p}, g_{b,\mathfrak{r}})\cdot\partial^\gamma(\mathbf{p}, r) = \partial^\gamma(g_{b,\mathfrak{p}}, g_{b,\mathfrak{r}})\cdot(2\mathbf{p}\cdot\partial^\gamma\mathbf{p}, \partial^\gamma r)$$
$$+\sum_{\substack{|\alpha|\leqq j-2\\ \alpha+\beta=\gamma}} a_\alpha\partial^\alpha g_{b,\mathfrak{p}}\partial^\beta\mathbf{p}\cdot\partial^\gamma\mathbf{p}, \tag{3.18}$$

$$2\mathbf{p}\cdot\partial^\gamma\mathbf{p} = \partial^\gamma|\mathbf{p}|^2 + \sum_{\substack{\alpha+\beta=\gamma\\ 1\leqq|\alpha|\leqq j-2}} b_\alpha\partial^\alpha\mathbf{p}\cdot\partial^\beta\mathbf{p}, \tag{3.19}$$

and

$$\partial^\gamma(g_{b,\mathfrak{p}}, g_{b,\mathfrak{r}}) = [\mathscr{D}^2 g_b]\partial^\gamma(|\mathbf{p}|^2, r) + \sum_{\sum_{\alpha,\beta}(|\alpha|\ell_\alpha+|\beta|m_\beta)=j-1} c_{\alpha\beta} \prod_{|\alpha|\leqq j-2} (\partial^\alpha|\mathbf{p}|^2)^{\ell_\alpha} \cdot \prod_{|\beta|\leqq j-2} (\partial^\beta r)^{m_\beta}, \tag{3.20}$$

where a_α, b_α are constants, $\ell_0 = m_0 = 0$, and $c_{\alpha\beta}(\mathbf{x}) = (c^1_{\alpha\beta}(\mathbf{x}), c^2_{\alpha\beta}(\mathbf{x}))$ are bounded. Inserting (3.18), (3.19), and (3.20) into the right side of (3.17), we have for $B_d = B_d(\mathbf{x}_0)$:

$$\begin{aligned} II &= \varepsilon_\ell^{-2}\int_{B_d} \zeta^2[\mathscr{D}^2 g_b]\partial^\gamma(|\mathbf{p}|^2, r)\cdot\partial^\gamma(|\mathbf{p}|^2, r) \\ &\quad +\varepsilon_\ell^{-2}\int_{B_d}\zeta^2\sum c_{\alpha\beta}\Pi(\partial^\alpha|\mathbf{p}|^2)^{\ell_\alpha}(\Pi\partial^\beta r)^{m_\beta}\cdot\partial^\gamma(|\mathbf{p}|^2, r) \\ &\quad +\varepsilon_\ell^{-2}\int_{B_d}\zeta^2\partial^\gamma g_{b,\mathfrak{p}}\left(\sum b_\alpha\partial^\alpha\mathbf{p}\cdot\partial^\beta\mathbf{p}\right) \\ &\quad +\varepsilon_\ell^{-2}\int_{B_d}\zeta^2\left(\sum a_\alpha\partial^\alpha g_{b,\mathfrak{p}}\partial^\beta\mathbf{p}\cdot\partial^\gamma\mathbf{p}\right) \\ &= III + IV + V + VI. \end{aligned}$$

Just as in Lemma 3.11, we have

$$\lambda\varepsilon_\ell^{-2}\int_{B_d}\zeta^2|\partial^\gamma(|\mathbf{p}|^2, r)|^2 \leqq III.$$

From Sobolev's theorem, the derivatives in IV of order less than $j-2$ are bounded. It follows then for any $\theta > 0$ that

$$\begin{aligned} |IV| &\leqq C_1\varepsilon_\ell^{-2}\int_{B_d}\zeta^2\left(\sum_{t=1}^{j-2}|D^t(|\mathbf{p}|^2, r)|^2\right)|\partial^\gamma(|\mathbf{p}|^2, r)| \\ &\leqq \theta\varepsilon_\ell^{-4}\int_{B_d}\zeta^4|D^{j-2}(|\mathbf{p}|^2, r)|^4 + \frac{C_2(E_{j-1}, d)}{\theta}. \end{aligned}$$

Then, using (3.12) and (3.16) we see

$$|IV| \leqq \theta C_3(E_{j-1})\varepsilon_\ell^{-2}\int_{B_d}\zeta^2|D^{j-1}((|\mathbf{p}|^2, r)|^2 + \frac{C_4(E_{j-1}, d)}{\theta}.$$

To estimate $|V|$, we write $\partial^\gamma = \partial_{x'}\partial^{\gamma'}$ for some x' and integrate by parts to get

$$|V| \leqq \theta\varepsilon_\ell^{-4}\int_{B_d}\zeta^2|D^{j-2}g_{b,\mathfrak{p}}|^2 + \theta C_5(E_{j-1})\int_{B_d}\zeta^2|D^j\mathbf{p}|^2 + \frac{C_6(E_{j-1}, d)}{\theta^2}.$$

To bound $|VI|$, we first consider the terms with $\alpha \neq \mathbf{0}$. For these $|\beta| < j-1$ and we see we can bound these terms just as was done for V. The term with $\alpha = \mathbf{0}$ can be bounded by $\frac{C_7}{\theta}\frac{g^2_{b,\mathfrak{p}}}{\varepsilon_\ell^4} + \theta|\zeta D^{j-1}\mathbf{p}|^4$. The integral of the first term over B_d is bounded by applying (3.16) for $q = 2$ (which follows from Lemma 3.11), and the second by $\theta C_8(E_{j-1}) \int_{B_d} \zeta^2 |D^j \mathbf{p}|^2 + C_9(E_{j-1}, d)$. Thus

$$|VI| \leqq C_{10}(E_{j-1})\theta\Big(\varepsilon_\ell^{-4} \int_{B_d} \zeta^2 |D^{j-2} g_{b,\mathfrak{p}}|^2 + \int_{B_d} \zeta^2 |D^j \mathbf{p}|^2\Big) + \frac{C_{11}(E_{j-1}, d)}{\theta}.$$

Summing on $|\gamma| = j - 1$ and collecting the estimates for $III, \ldots, VI$ we find

$$\begin{aligned}
&\Lambda \int_{B_d} \zeta^2 |D^j(\mathbf{p}, r)|^2 + \lambda(\varepsilon_\ell^{-2} \int_{B_d} \zeta^2 |D^{j-1}(|\mathbf{p}|^2, r)|^2 \\
&\leqq \theta C_{12}(E_{j-1})\Big(\int_{B_d} \zeta^2 |D^j(\mathbf{p}, r)|^2 + \varepsilon_\ell^{-2} \int_{B_d} \zeta^2 |D^{j-1}(|\mathbf{p}|^2, r)|^2 \\
&\quad + \varepsilon_\ell^{-4} \int_{B_d} \zeta^2 |D^{j-2}(g_{b,\mathfrak{p}}, g_{b,\mathfrak{r}})|^2\Big) + \frac{C_{13}(E_{j-1}, d)}{\theta^2}.
\end{aligned} \tag{3.21}$$

From (3.1) we have $\varepsilon_\ell^{-2}(g_{b,\mathfrak{p}}, g_{b,\mathfrak{r}}) = -(\frac{\mathbf{p}}{|\mathbf{p}|^2} \cdot (\mathscr{L}_1, \mathscr{L}_2), \mathscr{L}_3)(\mathbf{p}, r)$. Using this, the estimate $|\mathbf{p}| \geqq \frac{|s|}{4}$, and Sobolev's theorem we get

$$\varepsilon_\ell^{-4} \int_{B_d} \zeta^2 |D^{j-2}(g_{b,\mathfrak{p}}, g_{b,\mathfrak{r}})|^2 \leqq C_{14}(E_{j-1}) \int_{B_d} \zeta^2 |D^j(\mathbf{p}, r)|^2 + C_{15}(E_{j-1}, d).$$

Inserting this estimate into (3.21) and choosing θ sufficiently small, we obtain (3.16) for $q = j$ and d replaced by $d/2$. □

Corollary 3.13. *Let $\{(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell})\}$ be a sequence of minimizers for $\{G_{\varepsilon_\ell}\}$ in A_0 converging to $(\mathbf{p}^*, r^*)$ in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_k\})$. Then for each integer m,*

$$(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell}) \to (\mathbf{p}^*, r^*) \text{ in } C_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_k\}),$$

and in $C^m_{\mathrm{loc}}(\Omega\backslash\{a_1, \ldots, a_k\})$ as $\ell \to \infty$.

Proof of Theorem A. Let $\{(\mathbf{p}_\varepsilon, r_\varepsilon)\}$ be a sequence of minimizers for $\{G_\varepsilon\}$ in A_0 for which (1.21) holds and such that $\varepsilon \downarrow 0$. Then by applying Lemma 3.8 it follows that there exists a subsequence $\{(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell})\}$ and points $\{a_1, \ldots, a_k\} \subset \Omega$ so that

$$(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell}) \rightharpoonup \left(\frac{|s|}{2} \prod_{j=1}^{k} \frac{\mathbf{x} - a_j}{|\mathbf{x} - a_1|} e^{ih(\mathbf{x})}, \frac{s}{3}\right) = (\mathbf{p}^*, \frac{s}{3})$$
$$\text{in } W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_k\}) \times W^{1,2}(\Omega).$$

By Lemma 3.10 for each $\rho \in (0, \varepsilon_0)$, $(|\mathbf{p}_{\varepsilon_\ell}|, r_{\varepsilon_\ell}) \to (\frac{|s|}{2}, \frac{s}{3})$ uniformly on $\overline{\Omega}_\rho = \overline{\Omega}\backslash \bigcup_{j=1}^k B_\rho(a_j)$, and from Lemma 3.9

$$(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell}) \to \left(\mathbf{p}^*, \frac{s}{3}\right) \text{ in } W^{1,2}(\Omega_\rho).$$

Moreover $h(\mathbf{x})$ is harmonic in Ω.

Finally, by applying Corollary 3.13 we see that

$$(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell}) \to (\mathbf{p}^*, \frac{s}{3}) \text{ in } C(\overline{\Omega}_\rho) \quad \text{and} \quad C^m_{\mathrm{loc}}(\Omega_\rho)$$

for each integer m. □

We need to establish several properties for the following minimum problem in order to prove Theorem B. Let $\beta \in \mathbb{C}$, $|\beta| = 1$ and define

$$L\left(\frac{\varepsilon}{\mu};\beta\right) := L\left(\frac{\varepsilon}{\mu},1;\beta\right) = L(\varepsilon,\mu;\beta)$$
$$= \inf_{(\mathbf{v},r)\in\mathfrak{A}_\beta} \int_{B_\mu} [g_e(\nabla\mathbf{v},\nabla r) + \varepsilon^{-2} g_b(|\mathbf{v}|^2, r)]$$
$$+(2L_1 + L_2 + L_3)\frac{|s|^2}{4}\pi \ln\left(\frac{\varepsilon}{\mu}\right), \tag{3.22}$$

where

$$\mathfrak{A}_\beta = \{(\mathbf{v},r) \in W^{1,2}(B_\mu) \colon \mathbf{v}(\mathbf{x}) = \frac{\beta|s|}{2}\frac{x}{|x|} \text{ and } r(\mathbf{x}) = \frac{s}{3} \text{ for } |x| = \mu\}.$$

Lemma 3.14. *$L(\tau;\beta)$ is independent of β for all $\beta \in \mathbb{C}$ with $|\beta| = 1$. Moreover $L(\tau) := L(\tau;\beta)$ is a nondecreasing function of τ for $\tau > 0$ such that $\gamma := \lim_{\tau\downarrow 0} L(\tau) > -\infty$.*

Proof. For any $T \in SO(2)$, consider the change of variables by rotation, $\mathbf{y} = T\mathbf{x}$ for $\mathbf{x} \in B_1$ and set

$$R = \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The energy density is frame indifferent and, as such, satisfies

$$f_e(\nabla_{\mathbf{y}}\tilde{Q}(\mathbf{y})) + \tau^{-2} f_b(\tilde{Q}(\mathbf{y})) = f_e(\nabla_{\mathbf{x}} Q(\mathbf{x})) + \tau^{-2} f_b(Q(\mathbf{x})),$$

where $\tilde{Q}(\mathbf{y}) = RQ(T^t\mathbf{y})R^t$. This translates into a statement of invariance for g_e and g_b,

$$g_e(\nabla_{\mathbf{y}}\tilde{\mathbf{p}}(\mathbf{y}), \nabla_{\mathbf{y}}\tilde{r}(\mathbf{y})) + \tau^{-2} g_b(|\tilde{\mathbf{p}}(\mathbf{y})|^2, \tilde{r}(\mathbf{y})) = g_e(\nabla_{\mathbf{x}}\mathbf{p}(\mathbf{x}), \nabla_{\mathbf{x}} r(\mathbf{x}))$$
$$+\tau^{-2} g_b(|\mathbf{p}(\mathbf{x})|^2, r(\mathbf{x})),$$

where $\tilde{\mathbf{p}}(\mathbf{y}) = T^2\mathbf{p}(T^t\mathbf{y})$ and $\tilde{r}(\mathbf{y}) = r(T^t\mathbf{y})$. Let $\beta = \beta_1 + i\beta_2$. Then the boundary condition for $\mathbf{p}(\mathbf{x})$ as a vector in $\mathbb{R}^2$ reads as $\mathbf{p}_0(\mathbf{x}) = \frac{|s|}{2}Kx$ for $|\mathbf{x}| = 1$, where

$$K = \begin{bmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{bmatrix}.$$

Given $T \in SO(2)$, the boundary condition for $\tilde{\mathbf{p}}(\mathbf{y})$ becomes $\tilde{\mathbf{p}}_0(\mathbf{y}) = \frac{|s|}{2}T^2KT^t\mathbf{y}$ for $|\mathbf{y}| = 1$. In particular, if we let $T = K^t$, we get $\tilde{\mathbf{p}}_0(\mathbf{y}) = \frac{|s|}{2}\mathbf{y}$ for $|\mathbf{y}| = 1$. Thus the mapping $(\mathbf{p},r) \in \mathfrak{A}_\beta \to (\tilde{\mathbf{p}},\tilde{r}) \in \mathfrak{A}_1$ is an isometry such that $G_\tau(\mathbf{p},r) = G_\tau(\tilde{\mathbf{p}},\tilde{r})$. In particular, we see that $L(\tau;\beta) = L(\tau;1) = L(\tau)$.

The monotonicity property of $L(\tau)$ follows by the same argument for (1.18) given in [1], Chapter 3. A lower bound $\underline{m}$ for minimizers for the energy (1.18) with

$\Omega = B_1$ is proved in [1], Chapter 5. Let $\mathbf{u}_\varepsilon$ be such a minimizer with $\mathbf{u}_\varepsilon(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$ on ∂B_1. If $(\mathbf{v}_\tau, r_\tau)$ is a minimizer for (3.22) with $\mu = 1$ and $\varepsilon = \tau$, it follows that

$$E_\tau(\frac{2}{|s|}\mathbf{v}_\tau) \geqq E_\tau(\mathbf{u}_\tau) \geqq -\pi \ln(\tau) - \underline{m}.$$

Thus, using (3.3) we have

$$\frac{1}{2}\int_{B_1} |\nabla \mathbf{v}_\tau|^2 \geqq -\frac{s^2}{4}\pi \ln(\tau) - \underline{m}'.$$

The existence of a finite lower bound for $L(\tau)$ follows from this and the estimates in the proof of Lemma 3.6. □

Proof of Theorem B. The relation between F_ε and G_ε is proved in Corollary 2.3. We establish the asymptotic relation by arguing as in [1], Chapter 8. Let

$$\Upsilon = \{\mathbf{b} = (b, \dots, b_k) \in \Omega^k : b_i \neq b_j \text{ if } i \neq j\},$$

and for $\mathbf{b} \in \Upsilon$ set

$$\mathbf{q}_b(\mathbf{x}) = \frac{|s|}{2} \prod_{j=1}^{k} \frac{(\mathbf{x} - b_j)}{|\mathbf{x} - b_j|} \, e^{i\mathbf{h_b}(\mathbf{x})},$$

where $\mathbf{h_b}(\mathbf{x})$ is harmonic in Ω and is determined (mod 2π) by the condition $\mathbf{q_b} = \mathbf{p}_0$ on $\partial\Omega$. From [1], Chapter 8 we have

$$\frac{1}{2}\int_{\Omega\setminus \bigcup_{j=1}^{k} B_\rho(b_j)} |\nabla \mathbf{q_b}|^2 = \frac{s^2}{4}\left(\pi k \ln\frac{1}{\rho} + W(\mathbf{b})\right) + O(\rho) \quad \text{as} \quad \rho \to 0, \tag{3.23}$$

where $W(\mathbf{b})$ is the renormalized energy for (1.18) given in [1]. We express this using our notation. Set $R(\mathbf{x}) = \sum_{j=1}^{k} \ln|\mathbf{x} - b_j|$ and $\tau = \nu^\perp$, where ν is the exterior unit normal to $\partial\Omega$. Then

$$W(\mathbf{b}) = -\pi \sum_{\ell \neq j} \log|b_\ell - b_j| + \frac{1}{2}\int_{\partial\Omega} R\partial_\nu R - \int_{\partial\Omega} h_{\mathbf{b}}\partial_\tau R + \frac{1}{2}\int_\Omega |\nabla h_{\mathbf{b}}|^2. \tag{3.24}$$

Note that using (1.11), we have

$$g_e(\nabla \mathbf{q_b}, 0) = \left(L_1 + \frac{L_2 + L_3}{2}\right)|\nabla \mathbf{q_b}|^2 + |L_2 + L_3|(q_{\mathbf{b}1,x}q_{\mathbf{b}2,y} - q_{\mathbf{b}1,y}q_{\mathbf{b}2,x}), \tag{3.25}$$

and that $q_{\mathbf{b}1,x}q_{\mathbf{b}2,y} - q_{\mathbf{b}1,y}q_{\mathbf{b}2,x} = 0$ since $|\mathbf{q_b}| = \frac{|s|}{2}$.

We next construct a comparison function for (1.10). Let $\mathbf{b} \in \Upsilon$. Then for $0 < \varepsilon_\ell << \rho$ and for ρ sufficiently small (depending on Ω and $\mathbf{b}$), we define

$$(\tilde{\mathbf{p}}_{\varepsilon_\ell}, \tilde{r}_{\varepsilon_\ell}) = \begin{cases} (\mathbf{q_b}, s/3) & \text{for } \mathbf{x} \in \Omega \setminus \bigcup_{j=1}^{k} B_\rho(b_j), \\ (\mathbf{v}_0(\mathbf{x} - b_j), s/3) & \text{for } \rho/2 \leqq |\mathbf{x} - b_j| \leqq \rho, \\ (\mathbf{v}_j((\mathbf{x} - b_j)), r_j(\mathbf{x} - b_j)) & \text{for } \mathbf{x} \in B_{\rho/2}(b_j). \end{cases}$$

Here $(\mathbf{v}_j, r_j)$ minimizes $\int_{B_{\rho/2}(0)} [g_e + \varepsilon_\ell^{-2} g_b]$ with boundary conditions $(\frac{|s|}{2} \frac{\beta_j \mathbf{x}}{|\mathbf{x}|}, \frac{s}{3})$ on $\partial B_{\rho/2}(0)$ and $\beta_j = \prod_{\substack{\ell=1 \\ \ell \neq j}}^{k} \frac{(b_j - b_\ell)}{|b_j - b_\ell|} e^{ih_{\mathbf{b}}(b_j)}$. The function $\mathbf{v}_0$ is a minimal harmonic map valued in $\{|\mathbf{v}| = |\frac{s}{2}|\}$ such that $\tilde{\mathbf{p}}_{\varepsilon_\ell}$ is continuous. From Lemma 3.14 we have

$$\begin{aligned} \int_{B_{\rho/2}(0)} &[g_e(\nabla \mathbf{v}_j, \nabla r_j) + \varepsilon_\ell^{-2} g_b(|\mathbf{v}_j|^2, r_j)] \\ &= (2L_1 + L_2 + L_3) \frac{s^2 \pi}{4} \ln\left(\frac{\rho}{2\varepsilon_\ell}\right) + \gamma + o_\varepsilon(1) \end{aligned} \tag{3.26}$$

as $\varepsilon_\ell \to 0$. Then from (3.23), (3.25), and Lemma 3.14 we get

$$\begin{aligned} G_{\varepsilon_\ell}(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell}) &\leqq G(\tilde{\mathbf{p}}_{\varepsilon_\ell}, \tilde{r}_{\varepsilon_\ell}) \\ &= (2L_1 + L_2 + L_3) \frac{s^2}{4} \left(\pi k \ln\left(\frac{1}{\varepsilon_\ell}\right) + W(\mathbf{b}) \right) + k\gamma \\ &\quad + O(\rho) + o_\varepsilon(1). \end{aligned}$$

Let $\mathbf{a} \in \Upsilon$ be a limiting configuration as in Theorem A. Then from Lemma 3.9 and (3.23-26) we have

$$\begin{aligned} G_{\varepsilon_\ell}(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell}) &\geqq (2L_1 + L_2 + L_3) \frac{s^2}{4} \left(\pi k \ln\left(\frac{1}{\varepsilon_\ell}\right) + W(\mathbf{a}) \right) + k\gamma \\ &\quad + O(\rho) + o_\varepsilon(1). \end{aligned}$$

Just as in [1], choosing $\varepsilon_\ell = \varepsilon_\ell(\rho) << \rho$ with $\rho \to 0$, we arrive at our assertion. It follows from these two inequalities that W minimizes at $\mathbf{b} = \mathbf{a}$ and that the limit for $G_{\varepsilon_\ell}(\mathbf{p}_{\varepsilon_\ell}, r_{\varepsilon_\ell})$ as $\ell \to \infty$ is established. □

4. The Pohozaev Identity

In this section we show that (1.21) always holds for minimizers of G_ε in A_0 if Ω is simply connected and $0 < \varepsilon < \varepsilon_1$, where ε_1 depends on s, L_1, L_2, L_3, Ω, k, and the constants in (1.14), and M depends on these terms and $\|\mathbf{p}_0\|_{W^{1,2}(\partial\Omega)}$, as well. We first prove (1.21) for solutions to (3.1-2) in the case of a disk using the Pohozaev identity.

Lemma 4.1. *Let* $(\mathbf{p}, r) = (\mathbf{p}_\varepsilon, r_\varepsilon)$ *be a solution to* (3.1-2) *where* $\Omega = \Omega_R = B_R(0)$ *and* $0 < \varepsilon < 1$. *Then there is a constant* $C_0 = C_0(R, L_1, L_2, L_3, \|\mathbf{p}_0\|_{1,2;\partial B_R}, s)$ *so that*

$$\varepsilon^{-2}\int_{B_R} g_b(|\mathbf{p}|^2, r) \leqq C_0.$$

Proof. We multiply the system (3.1) by $-\nabla(p_1, p_2, r)\mathbf{x}$ and integrate over B_R. We find

$$\begin{aligned}
0 = &\int_{B_R}[(2L_1 + L_2 + L_3)(\Delta\mathbf{p}\cdot\nabla\mathbf{p}\cdot\mathbf{x}) + \left(\frac{3L_1}{2}+\frac{L_2+L_3}{4}\right)\Delta r\nabla r\cdot\mathbf{x}\\
&-\varepsilon^{-2}\nabla g\cdot\mathbf{x}]\\
&+\frac{(L_2+L_3)}{2}\int_{B_R}[2r_{xy}\nabla p_2\cdot\mathbf{x} + 2p_{2xy}\nabla r\cdot\mathbf{x}]\\
&+\frac{(L_2+L_3)}{2}\int_{B_r}[(r_{xx}-r_{yy})\nabla p_1\cdot\mathbf{x} + (p_{1xx}-p_{1yy})\nabla r\cdot\mathbf{x}]\\
=:&\ I + \frac{(L_2+L_3)}{2}II + \frac{(L_2+L_3)}{2}III.
\end{aligned} \tag{4.1}$$

We can calculate I as in [1], Chapter 3,

$$\begin{aligned}
I = &R\left(L_1 + \frac{(L_2+L_3)}{2}\right)\int_{\partial B_R}(|\mathbf{p}_\nu|^2 - |\mathbf{p}_\tau|^2)\\
&+R\left(\frac{3L_1}{4}+\frac{(L_2+L_3)}{8}\right)\int_{\partial B_R}(|r_\nu|^2-|r_\tau|^2) + 2\varepsilon^{-2}\int_{B_R} g_b.
\end{aligned} \tag{4.2}$$

Here $\mathbf{p}_\tau$ and r_τ are tangential derivatives. Note that $r_\tau = 0$ and $\mathbf{p}_\tau = \mathbf{p}_{0\tau}$ on ∂B_R. To calculate II, we write

$$\begin{aligned}
\int_{B_R} r_{xy}\nabla p_2\cdot\mathbf{x} &= \int_{B_R}(r_{xy}xp_{2x} + r_{xy}yp_{2y})\\
&= -\int_{B_R}(xr_x p_{2xy} + yr_y p_{2xy})\\
&\quad+\frac{1}{R}\int_{\partial B_R} xy(p_{2x}r_x + p_{2y}r_y).
\end{aligned}$$

Using this and the fact that $r_\tau = 0$ on ∂B_R, we get

$$II = \frac{2}{R}\int_{\partial B_R} xyp_{2\nu}r_\nu.$$

To calculate III, we change variables, $x' = (x-y)/\sqrt{2}$, $y' = (x+y)/\sqrt{2}$. Then

$$III = 2\int_{B_R}(r_{x'y'}\nabla p_1\cdot\mathbf{x} + p_{1x'y'}\nabla r\cdot\mathbf{x}) = \frac{2}{R}\int_{\partial B_R} x'y'p_{1\nu}r_\nu.$$

Writing $(x, y) = (R\cos\theta, R\sin\theta)$, then, it follows that $(x', y') = (R\cos(\theta + \frac{\pi}{4}), R\sin(\theta + \frac{\pi}{4}))$. Thus $II + III = R\int_{\partial B_R} r_\nu(\cos 2\theta, \sin 2\theta)\cdot \mathbf{p}_\nu$. Finally, we see that

$$\left|\left(\frac{L_2+L_3}{2}\right)(II+III)\right| \leqq R\,\frac{|L_2+L_3|}{2}\,\left(\int_{\partial B_R}\left(\frac{|r_\nu|^2}{4}+|\mathbf{p}_\nu|^2\right)\right). \quad (4.3)$$

Thus using (4.1), (4.2) and (4.3) with (1.2), we get

$$\begin{aligned}
&R\left(L_1+\frac{(L_2+L_3)}{2}\right)\int_{\partial B_R}|\mathbf{p}_{0_\tau}|^2\\
&\geqq R(L_1+\frac{L_2+L_3}{2}-\frac{|L_2+L_3|}{2})\int_{\partial B_R}|\mathbf{p}_\nu|^2\\
&\quad +R\left(\frac{3L_1}{4}+\frac{(L_2+L_3)}{8}-\frac{|L_2+L_3|}{8}\right)\int_{\partial B_R}|r_\nu|^2\\
&\quad +2\varepsilon^{-2}\int_{B_R} g_b \geqq 2\varepsilon^{-2}\int_{B_R} g_b.
\end{aligned}$$

□

Lemma 4.2. *Let Ω be a C^3 bounded simply connected domain in $\mathbb{R}^2$. There is a constant $0 < \varepsilon_1 \leqq \varepsilon_0$ such that if $(\mathbf{p}, r) = (\mathbf{p}_\varepsilon, r_\varepsilon)$ is a minimizer for G_ε in A_0 and $0 < \varepsilon < \varepsilon_1$, then*

$$\varepsilon^{-2}\int_\Omega g_b(|\mathbf{p}|^2, r) \leqq M.$$

Here ε_1 depends on $s, L_1, L_2, L_3, \Omega, k$, and the constants in (1.14) and M depends on these terms and $\|\mathbf{p}_0\|_{W^{1,2}(\partial\Omega)}$.

Proof. Set $R = 2(\text{diam}(\Omega))$ and assume that $0 \in \Omega$. We construct an extension of $\mathbf{p}$. Let $\hat{\mathbf{p}} \in W^{1,2}(B_R(0)\setminus\Omega)$ valued in $\{|\hat{\mathbf{p}}| = \frac{|s|}{2}\}$ and such that $\hat{\mathbf{p}}$ is a minimal harmonic map satisfying $\hat{\mathbf{p}} = \mathbf{p}_0$ on $\partial\Omega$ and $\hat{\mathbf{p}}(\mathbf{x}) = \frac{|s|}{2}(\frac{\mathbf{x}}{|\mathbf{x}|})^k$ on $\partial B_R(0)$. Note that $\|\hat{\mathbf{p}}\|_{1,2;B_R(0)\setminus\Omega} \leqq C\|\mathbf{p}_0\|_{1,2;\partial\Omega}$. Set

$$(\mathbf{p}', r') = \begin{cases} (\mathbf{p}, r) & \text{for } \mathbf{x} \in \Omega,\\ (\hat{\mathbf{p}}, \frac{s}{3}) & \text{for } \mathbf{x} \in B_R \setminus \Omega.\end{cases}$$

Let $\tilde{G}_\varepsilon = \int_{B_R}[g_e + \frac{1}{2\varepsilon^2}\, g_b]$, and let $(\tilde{\mathbf{p}}, \tilde{r})$ be a minimizer for $\tilde{G}_\varepsilon$ such that $(\tilde{\mathbf{p}}, \tilde{r}) = (\hat{\mathbf{p}}, \frac{s}{3})$ on ∂B_R. We can apply Lemma 4.1 (with ε replaced by $\sqrt{2}\varepsilon$) and the results from Section 3 to $\tilde{G}_\varepsilon$ and $(\tilde{\mathbf{p}}, \tilde{r})$ for the case of $\Omega = B_R$. In particular, from the proof of Theorem B, there are constants C_1 and $0 < \eta_1 < 1$, depending on $s, L_1, L_2, L_3, \Omega, k$, and the constants in (1.14) so that

$$(2L_1 + L_2 + L_3)\frac{s^2}{4}\,\pi k \ln\frac{1}{\varepsilon} - C_1 \leqq \tilde{G}_\varepsilon(\tilde{\mathbf{p}}, \tilde{r}) \leqq \tilde{G}_\varepsilon(\mathbf{p}', r')$$

for all $0 < \varepsilon < \eta_1$. Note that

$$\begin{aligned}\tilde{G}_\varepsilon(\mathbf{p}', r') &= \int_\Omega [g_e(\nabla \mathbf{p}, \nabla r) + \frac{1}{2\varepsilon^2}\, g_b(|\mathbf{p}|^2, r)] \\ &\quad + \int_{B_R \setminus \Omega} g_e(\nabla \hat{\mathbf{p}}, 0) \\ &= G_\varepsilon(\mathbf{p}, r) - \frac{1}{2\varepsilon^2} \int_\Omega g_b(|\mathbf{p}|^2, r) + C_2,\end{aligned}$$

where C_2 depends only on $\|\mathbf{p}_0\|_{1,2;\partial\Omega}$ and the constants in (1.14). Thus

$$(2L_1 + L_2 + L_3)\frac{s^2}{4}\pi k \ln \frac{1}{\varepsilon} + \frac{1}{2\varepsilon^2} \int_\Omega g_b(|\mathbf{p}|^2, r) \leqq G_\varepsilon(\mathbf{p}, r) + C_1 + C_2. \tag{4.4}$$

Next we consider the comparison map $(\mathbf{w}', r')$ constructed in Lemma 3.6 defined for $\varepsilon < \overline{\varepsilon} = \eta_2$. Since $(\mathbf{p}, r)$ is a minimizer for G_ε, we get

$$G_\varepsilon(\mathbf{p}, r) \leqq G_\varepsilon(\mathbf{w}', r') \leqq (2L_1 + L_2 + L_3)\frac{s^2}{4}\pi k \ln \frac{1}{\varepsilon} + C_3$$

for all $\varepsilon < \overline{\varepsilon} = \eta_1$, where C_3 depends only on $\mathbf{p}_0$, Ω, L_1, L_2, L_3, and the constants in (1.14). It follows from this and (4.4) that

$$\varepsilon^{-2} \int_\Omega g_b(|\mathbf{p}|^2, r) \leqq 2(C_1 + C_2 + C_3) =: M$$

for all $0 < \varepsilon < \varepsilon_1 = \min\{\eta_1, \eta_2, \varepsilon_0\}$. □

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