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Convergence of Ginzburg–Landau Functionals in Three-Dimensional Superconductivity

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Abstract

In this paper we consider the asymptotic behavior of the Ginzburg–Landau model for superconductivity in three dimensions, in various energy regimes. Through an analysis via Γ-convergence, we rigorously derive a reduced model for the vortex density and deduce a curvature equation for the vortex lines. In the companion paper (Baldo et al. Commun. Math. Phys. 2012, to appear) we describe further applications to superconductivity and superfluidity, such as general expressions for the first critical magnetic field H_{c_1}, and the critical angular velocity of rotating Bose–Einstein condensates.

1. Introduction

In this paper we investigate the asymptotic behavior as $\epsilon \to 0$ of the functionals

$$E_\epsilon(u) \equiv E_\epsilon(u;\Omega) = \int_\Omega e_\epsilon(u)\,\mathrm{d}x = \int_\Omega \frac{1}{2}|Du|^2 + \frac{1}{\epsilon^2}W(u)\,\mathrm{d}x,$$

where $\epsilon > 0$, Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, $u = u^1 + iu^2 \in H^1(\Omega;\mathbb{C})$, $W : \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{R}$ is nonnegative and continuous, $W(u) = 0 \iff |u| = 1$, and is assumed to satisfy some growth condition at infinity and around its zero set (see hypothesis (H_q) below).

In the case $W(u) = \frac{(1-|u|^2)^2}{4}$, one usually refers to E_ϵ as the Ginzburg–Landau functional. This model is relevant to a variety of phenomena in quantum physics; in fact, as corollaries of its asymptotic analysis, we will derive, here and in the companion paper [2], reduced models for density of vortex lines (or curves) in three-dimensional superconductivity and Bose–Einstein condensation. In these physical applications, ϵ represents a (small) characteristic length, u corresponds to a wave function, $|u|^2$ to the density of superconducting or superfluid material contained in Ω. Moreover, the *momentum*, defined as the 1-form

$$ju \equiv (iu, du) \equiv u^1 du^2 - u^2 du^1,$$

represents the superconducting (resp. superfluid) current, and hence it is natural to interpret the Jacobian $Ju \equiv du^1 \wedge du^2$ as the *vorticity*, since $2Ju = d(ju)$. We refer the reader to the Appendix for notation used throughout this paper and background on differential forms and related material.

In the two-dimensional case, it has been recognized since [5] that for minimizers u_ϵ of E_ϵ (subject to appropriate boundary conditions), as $\epsilon \to 0$, the energy typically scales like $|\log \epsilon|$. In addition, there are a finite number of singular points, called *vortices*, where the energy density $e_\epsilon(u_\epsilon)\mathrm{d}x$ and the vorticity Ju_ϵ concentrate. Moreover, the rescaled energy $\frac{E_\epsilon(u_\epsilon)}{|\log \epsilon|}$ controls the total vorticity. These phenomena are robust, in the sense that analogous results hold in higher dimensions (see [6,24], where the limiting vorticity is supported in a codimension 2 minimal surface) and under weaker assumptions on u_ϵ, as stated in the following Γ-convergence result:

Theorem 1. ([1,22]) *Let $K > 0$, $n \geqq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and the potential W satisfy the growth condition*[1]

$$\liminf_{|u|\to\infty} \frac{W(u)}{|u|^q} > 0, \qquad \liminf_{|u|\to 1} \frac{W(u)}{(1-|u|)^2} > 0, \tag{H_q}$$

for some $q \geqq 2$. Then the following statements hold:

(i) Compactness and lower bound inequality. *For any sequence $u_\epsilon \in H^1(\Omega, \mathbb{C})$ such that*

$$E_\epsilon(u_\epsilon) \leqq K|\log \epsilon|, \tag{H_0}$$

we have, up to a subsequence, $Ju_\epsilon \to J$ in $W^{-1,p}$ for every $p < \frac{n}{n-1}$, where J is an exact measure-valued 2-form in Ω with finite mass $||J|| \equiv |J|(\Omega)$, and J has the structure of an $(n-2)$-rectifiable boundary with multiplicities in $\pi \cdot \mathbb{Z}$. Moreover,

$$\liminf_{\epsilon\to 0} \frac{E_\epsilon(u_\epsilon)}{|\log \epsilon|} \geqq ||J||. \tag{1.1}$$

(ii) Upper bound (in) equality. *For any exact measure-valued 2-form J having the structure of an $(n-2)$-rectifiable boundary in Ω with multiplicities in $\pi \cdot \mathbb{Z}$, there exist $u_\epsilon \in H^1(\Omega, \mathbb{C})$ such that $Ju_\epsilon \to J$ in $W^{-1,p}$ for every $p < \frac{n}{n-1}$, and*

$$\lim_{\epsilon\to 0} \frac{E_\epsilon(u_\epsilon)}{|\log \epsilon|} = ||J||. \tag{1.2}$$

[1] See condition (2.2) in [1].

Other energy regimes arise naturally for E_ϵ and are interesting for applications. In particular the energy regime $E_\epsilon(u_\epsilon) \approx |\log\epsilon|^2$ corresponds to the onset of the mixed phase in type-II superconductors, and to the appearance of vortices in Bose–Einstein condensates. These situations have been extensively studied in the two-dimensional case, especially by Sandier and Serfaty in the case of superconductivity (see [30] and references therein). In this energy regime, the number of vortices is of order $|\log\epsilon|$, hence unbounded as $\epsilon \to 0$. Another feature is that the contribution of the vortices to the energy is of the same order as the contribution of the momentum, so that the limiting behavior can be described in terms of this last quantity, suitably normalized. A Γ-convergence result for $\frac{1}{g_\epsilon}E_\epsilon$ for general energy regimes $E_\epsilon(u_\epsilon) \lesssim g_\epsilon \ll \epsilon^{-2}$ has been proved, in the two-dimensional case, in [23], see also [30].

1.1. Main Results

A first result of this paper extends the asymptotic analysis of [23] to the three-dimensional case. We write $f_\epsilon \ll h_\epsilon$ (or $h_\epsilon \gg f_\epsilon$) to express $f_\epsilon = o(h_\epsilon)$ as $\epsilon \to 0$. We will use the notation

$$\mathcal{A}_0 := \{(J, v) : J \text{ is an exact measure-valued 2-form in } \Omega,\ v \in L^2(\Lambda^1\Omega)\}. \tag{1.3}$$

Measure-valued k-forms are discussed in the Appendix, see in particular Sections 5.1.1 and 5.1.2. Our conventions imply that a measure-value form J has finite mass, so that $\|J\| := |J|(\Omega) < \infty$, where $|J|$ denotes the total variation measure associated with J. We say that a measure-valued k-form J is *exact* if $J = dw$ in the sense of distributions for some measure-valued $k-1$-form w. We show in Lemma 12 that a measure-valued $(n-1)$-form J on a smooth bounded open $\Omega \subset \mathbb{R}^n$ is exact if and only if $dJ = 0$ and the associated flux through each component of the boundary $\partial\Omega$ vanishes. The latter condition follows automatically from the former if $\partial\Omega$ is connected.

Theorem 2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, $W(u)$ satisfy (H_q) for some $q \geqq 2$, and $|\log\epsilon| \ll g_\epsilon \ll \epsilon^{-2}$. Then the following statements hold:*

(i) Compactness and lower bound inequality. *For any sequence $u_\epsilon \in H^1(\Omega, \mathbb{C})$ such that*

$$\textit{for some } K > 0, \qquad E_\epsilon(u_\epsilon) \leqq K g_\epsilon, \tag{H_g}$$

there exist $(J, v) \in \mathcal{A}_0$ such that, after passing to a subsequence if necessary,

$$|u_\epsilon| \to 1 \quad \textit{in } L^q(\Omega), \qquad \frac{ju_\epsilon}{|u_\epsilon|\sqrt{g_\epsilon}} \rightharpoonup v \quad \textit{weakly in } L^2(\Lambda^1\Omega), \tag{1.4}$$

$$\frac{ju_\epsilon}{\sqrt{g_\epsilon}} \rightharpoonup v \quad \textit{weakly in } L^{\frac{2q}{q+2}}(\Lambda^1\Omega). \tag{1.5}$$

If $g_\epsilon \leqq |\log \epsilon|^2$, *then in addition,*

$$\frac{|\log \epsilon|}{g_\epsilon} J u_\epsilon = \frac{|\log \epsilon|}{2 g_\epsilon} d(j u_\epsilon) \to J \qquad \text{in } W^{-1,p}(\Lambda^2 \Omega) \quad \forall\, p < 3/2. \tag{1.6}$$

The convergences in (1.5) *and* (1.6) *yield, in different scaling regimes,*

$$\text{if} \quad |\log \epsilon| \ll g_\epsilon \ll |\log \epsilon|^2 \text{ then } (J, v) \in \mathcal{A}_1 := \{(J, v) \in \mathcal{A}_0 : dv = 0\}, \tag{S_1}$$

$$\text{if} \quad g_\epsilon = |\log \epsilon|^2 \text{ then } (J, v) \in \mathcal{A}_2 := \{(J, v) \in \mathcal{A}_0 : J = \frac{1}{2} dv \in H^{-1}(\Lambda^2 \Omega)\}, \tag{S_2}$$

$$\text{if} \quad |\log \epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2} \text{ then } (J, v) \in \mathcal{A}_3 := \{(J, v) \in \mathcal{A}_0 : J = 0\}. \tag{S_3}$$

and in every case,

$$\liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon)}{g_\epsilon} \geqq ||J|| + \frac{1}{2} ||v||^2_{L^2(\Lambda^1 \Omega)}. \tag{1.7}$$

(ii) Upper bound (in)equality. *Assume that* $(g_\epsilon)_{\epsilon > 0}$ *satisfies one of the scaling conditions* (S_k), $k \in \{1, 2, 3\}$, *identified above, and that* $(J, v) \in \mathcal{A}_k$. *Then* $\exists\, U_\epsilon \in H^1(\Omega; \mathbb{C})$ *such that* (1.4), (1.5), (1.6) *hold, and*

$$\lim_{\epsilon \to 0} \frac{E_\epsilon(U_\epsilon)}{g_\epsilon} = ||J|| + \frac{1}{2} ||v||^2_{L^2(\Lambda^1 \Omega)}. \tag{1.8}$$

The compactness and lower bound assertions are either very easy, already known (see for example [31]) or are proved almost exactly as in the two-dimensional case. The upper bound (1.8) is the main new part of the theorem, and constitutes the most difficult part of the theorem.

Remark 1. Assume that $(g_\epsilon)_{\epsilon > 0}$ satisfies one of the scaling conditions (S_k), $k \in \{1, 2, 3\}$, identified above, and for $(J, v) \in \mathcal{A}_0$, set

$$E(J, v) := ||J|| + \frac{1}{2} ||v||^2_{L^2(\Lambda^1 \Omega)} \qquad \text{if } (J, v) \in \mathcal{A}_k, \tag{1.9}$$

and $E(J, v) := +\infty$ if $(J, v) \notin \mathcal{A}_k$. We express the Γ-convergence result of Theorem 2 using the notation

$$\frac{E_\epsilon(u_\epsilon)}{g_\epsilon} \xrightarrow{\Gamma} E(J, v), \tag{1.10}$$

where the Γ-limit is intended with respect to the convergences (1.4), (1.5), and (1.6). Notice that the contributions of vorticity and momentum are decoupled in the Γ-limit, due to the different scaling factors in (1.5), (1.6), except for the critical regime $g_\epsilon = |\log \epsilon|^2$, where the scalings of $J u_\epsilon$ and $j u_\epsilon$ coincide, and the limits satisfy $2J = dv$ (see Section 1.2 below). In particular, Theorem 2 expresses the fact that for regimes $g_\epsilon \ll |\log \epsilon|^2$, the contribution to the energy is given by the vorticity and the curl-free part of the momentum, while for $g_\epsilon \gg |\log \epsilon|^2$ the contribution of the vorticity vanishes asymptotically.

Remark 2. As observed in [1,22], replacing $W(u)$ by $\sigma \cdot W(u)$, $\sigma > 0$, and letting $\sigma \to 0$, the lower bound (1.7) can be sharpened to

$$\liminf_{\epsilon \to 0} \int_\Omega \frac{|\nabla u_\epsilon|^2}{2g_\epsilon} \geqq ||J|| + \frac{1}{2}||v||^2_{L^2(\Lambda^1\Omega)}. \tag{1.11}$$

Moreover, for a sequence u_ϵ satisfying (1.8), the potential part of the energy is a lower order term, that is,

$$\int_\Omega \frac{W(u_\epsilon)}{\epsilon^2} = o(g_\epsilon) \qquad \text{as } \epsilon \to 0. \tag{1.12}$$

Inequality (1.11) is also proved in [31].

Remark 3. In the two-dimensional case the Γ-convergence result of [23] is formulated exactly as Theorem 2 above, except for the convergence of the normalized Jacobians $\frac{|\log\epsilon|}{g_\epsilon} Ju_\epsilon$, which takes place in $W^{1,p}$ for any $p < 2$.

Remark 4. By localization, Theorem 2 implies the following: for any u_ϵ satisfying (H_g), the rescaled energy densities $\frac{e_\epsilon(u_\epsilon)\mathrm{d}x}{g_\epsilon}$ converge weakly as measures in Ω, upon passing to a subsequence, to a limiting measure μ, with $|J| + \frac{v^2}{2}\mathrm{d}x \leqq \mu$. It then follows that $\mu = |J| + \frac{v^2}{2}\mathrm{d}x$ for any sequence (u_ϵ) such that the convergences (1.4), (1.5), (1.6) and the upper bound equality (1.8) hold.

Remark 5. The final compactness assertion (1.6) is proved by establishing convergence in $W^{-1,1}$, and then interpolating, using the easy estimate $\|Ju_\epsilon\|_{L^1} \leqq \|Du\|^2_{L^2}$. For $|\log\epsilon| \ll g_\epsilon \ll \epsilon^{-2}$, (1.5) already implies that $\frac{|\log\epsilon|}{g_\epsilon} Ju_\epsilon \to 0$ in $W^{-1,\frac{2q}{q+2}}$. This can also be improved by interpolating with L^1 estimates (which imply $W^{-1,3/2}$ estimates) if $\frac{2q}{q+2} < \frac{3}{2}$.

Remark 6. The convergences (1.4), (1.5) and (1.6) have been already established in the analysis of [1,22,23]. In particular, for a domain $\Omega \subset \mathbb{R}^n$ with $n \geqq 4$, (1.4) and (1.5) still hold true, while the normalized Jacobians converge to J in $W^{-1,p}$ for any $p < \frac{n}{n-1}$. Moreover, assuming $g_\epsilon \leqq \epsilon^{-\gamma}$ for some $0 < \gamma < 2$, the convergence in (1.5) can be improved according to γ, see [23]. In [8], following [10], the convergence in (1.6) also has been proved to hold in $W^{1,\frac{n}{n-1}}$ (as well as in fractional spaces $W^{s,p}$ with $sp = n/(n-1)$) for $n \geqq 4$, and even in the case $n = 3$, assuming the condition $u \in L^q(\Omega)$ for $q > 6$ (see [8], Theorem 1.3 and Remark 1.6).

Remark 7. In the scaling $g_\epsilon = |\log\epsilon|$ studied in Theorem 1, arguments in the proof of Theorem 2 can easily be adapted to show that $\frac{E_\epsilon(u_\epsilon)}{g_\epsilon} \xrightarrow{\Gamma} E(J, v)$, where the Γ-limit is again intended with respect to the convergences (1.4), (1.5) and (1.6), and where $E(J, v)$ is defined exactly as in (1.9), except that $E(J, v)$ is set equal to $+\infty$ unless $dv = 0$ *and* J has the structure of a rectifiable boundary. This is an improvement over Theorem 1 (see analogous results in [7] for critical points of E_ϵ, and in [4] for minimizers with local energy bounds), and in fact is valid in $\mathbb{R}^n$ for any $n \geqq 3$.

Remark 8. The validity of (1.7), (1.8) in dimension $n \geqq 4$ remains an open issue for energy regimes $g_\epsilon \gg |\log \epsilon|$. A major difficulty is to determine the correct generalization of the total variation term $\|J\|$ in (1.9). Different candidates include the total variation with respect to the comass norm, the Euclidean norm, and the mass norm, see [16]. For measure-valued 2-forms in $\mathbb{R}^3$, all of these coincide.

The most reasonable conjecture is that the mass norm is the suitable one for the higher-dimensional generalization of Theorem 2, but this seems difficult to prove. The arguments we give to prove (1.7) are, in fact, presented in $\mathbb{R}^n$, and for $n \geqq 4$ they prove that (1.7) holds with $\|J\|$ replaced by the *comass* of J, which in general is strictly less than the mass of J. Lower bounds involving the comass norm in $\mathbb{R}^n$, $n \geqq 4$, are also proved in [31].

By way of illustration, for the (constant) measure-valued 2-form $J = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ on an open set $\Omega \subset \mathbb{R}^4$, one has comass$(J) = |\Omega|$, the Euclidean total variation of J is $\sqrt{2}|\Omega|$, and mass$(J) = 2|\Omega|$.

For $|\log \epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$, the total variation term does not appear in the limiting functional, so the issue of mass versus comass does not arise, and the proof of the lower bound (1.7) is straightforward; in fact it follows from arguments we give here. The upper bound (1.8) is probably also easier in this case than for $|\log \epsilon| \ll g_\epsilon \leqq |\log \epsilon|^2$.

Replacing assumption (H_q) for $W(u)$ with the following (verified in particular for sequences of minimizers)

$$\exists\, C > 1 \quad \text{such that } |u_\epsilon| \leqq C \qquad \forall \epsilon < 1, \tag{H_∞}$$

and taking into account Remark 6, a variant of Theorem 2 can be formulated as follows:

Theorem 3. *In the hypotheses of Theorem 2, we have*

(i) Compactness. *For any sequence $u_\epsilon \in H^1(\Omega, \mathbb{C})$ verifying (H_g) and (H_∞) we have, up to a subsequence,*

$$\frac{ju_\epsilon}{\sqrt{g_\epsilon}} \rightharpoonup v \text{ weakly in } L^2(\Lambda^1\Omega), \quad \frac{|\log \epsilon|}{g_\epsilon} Ju_\epsilon \to J \text{ in } W^{-1,3/2}(\Lambda^2\Omega), \tag{1. 13}$$

where J is an exact measure-valued 2-form in Ω, with finite mass $||J|| \equiv |J|(\Omega)$.

(ii) Γ-convergence. *Assuming that g_ϵ respects one of the scaling conditions S_k from Theorem 2, we have*

$$\frac{E_\epsilon(u_\epsilon)}{g_\epsilon} \xrightarrow{\Gamma} E(J, v), \tag{1. 14}$$

with respect to the convergence (1. 13), where $E(J, v)$ is defined in (1.9), taking into account the relevant scaling regime.

1.2. The Critical Regime $g_\epsilon = |\log \epsilon|^2$

Let us specialize the statements of Theorems 2 and 3 to the critical regime $g_\epsilon = |\log \epsilon|^2$, where the scaling factors in (1.4), (1.5) and (1.6) are equal, and hence the normalized vorticity is related to the momentum by the formula $2J = dv$. We then have

$$\frac{E_\epsilon(u_\epsilon)}{|\log \epsilon|^2} \xrightarrow{\Gamma} E(v), \tag{1. 15}$$

where, for $v \in L^2(\Lambda^1\Omega)$, we define

$$E(v) := E\Big(\frac{dv}{2}, v\Big) = \frac{1}{2}||dv|| + \frac{1}{2}||v||^2_{L^2(\Lambda^1\Omega)}, \tag{1. 16}$$

if the mass $||dv|| \equiv |dv|(\Omega)$ is finite, $E(v) = +\infty$ otherwise. The Γ-limit is intended with respect to the convergences (1.4), (1.5) and (1.6).

Clearly Theorem 3 yields the same conclusion (1. 15), this time with respect to the convergence (1. 13), which in this case reads

$$\frac{ju_\epsilon}{|\log \epsilon|} \rightharpoonup v \text{ weakly in } L^2(\Lambda^1\Omega), \quad \frac{2Ju_\epsilon}{|\log \epsilon|} \to dv \text{ in } W^{-1,3/2}(\Lambda^2\Omega). \tag{1. 17}$$

1.3. Applications to Superconductivity

As a first application of the above results in the energy regime $g_\epsilon = |\log \epsilon|^2$, we describe the asymptotic behavior of the Ginzburg–Landau functional for superconductivity

$$\mathcal{F}_\epsilon(u, A) = \int_\Omega \frac{|du - iAu|^2}{2} + \frac{1}{\epsilon^2}W(u)\,\mathrm{d}x + \int_{\mathbb{R}^3} \frac{|dA - h_{\rm ex}|^2}{2}\,\mathrm{d}x,$$

defined for $\Omega \subset \mathbb{R}^3$, where the 2-form $h_{\rm ex} \in L^2_{loc}(\Lambda^2\mathbb{R}^3)$ is an external applied magnetic field and the 1-form $A \in H^1(\Lambda^1 R^3)$ is the induced vector potential (gauge field). It does not change the problem to assume that $h_{\rm ex}$ has the form $h_{\rm ex} = dA_{\rm ex}$ for some $A_{\rm ex} \in H^1_{loc}(\Lambda^1\mathbb{R}^3)$, so we will always make this assumption.

Let $\dot H^1_*(\Lambda^1\mathbb{R}^3) := \{A \in \dot H^1(\Lambda^1\mathbb{R}^3) : d^*A = 0\}$, and define the inner product $(A, B)_{\dot H^1_*(\Lambda^1\mathbb{R}^3)} := (dA, dB)_{L^2(\Lambda^2\mathbb{R}^3)}$. This makes $\dot H^1_*(\Lambda^1\mathbb{R}^3)$ into a Hilbert space, satisfying in addition the Sobolev inequality

$$\|A\|_{L^6(\Lambda^1\mathbb{R}^3)} \leqq C\|A\|_{\dot H^1_*(\Lambda^1\mathbb{R}^3)}.$$

We will study $\mathcal{F}_\epsilon(v, A)$ for $(v, A) \in H^1(\Omega; \mathbb{C}) \times [A_{\rm ex} + \dot H^1_*(\Lambda^1\mathbb{R}^3)]$; this is reasonable in view of the gauge-invariance of $\mathcal{F}_\epsilon$, that is, the fact that

$$\mathcal{F}_\epsilon(u, A) = \mathcal{F}_\epsilon(u \cdot e^{i\phi}, A + d\phi) \qquad \forall \phi \in H^1(\mathbb{R}^3). \tag{1. 18}$$

It is useful to decompose $\mathcal{F}_\epsilon$ as follows (see for example [9]):

$$\mathcal{F}_\epsilon(u, A) = E_\epsilon(u) + \mathcal{I}(u, A) + \mathcal{M}(A, h_{\rm ex}) + \mathcal{R}(u, A), \tag{1. 19}$$

with

$$\mathcal{I}(u, A) := -\int_\Omega A \cdot ju \,\mathrm{d}x, \tag{1.20}$$

$$\begin{aligned}\mathcal{M}(A, h_{\mathrm{ex}}) &:= \int_\Omega \frac{|A|^2}{2}\mathrm{d}x + \int_{\mathbb{R}^3} \frac{|dA - h_{\mathrm{ex}}|^2}{2}\,\mathrm{d}x \\ &= \frac{1}{2}\|A\|^2_{L^2(\Lambda^1\Omega)} + \frac{1}{2}\|A - A_{\mathrm{ex}}\|^2_{\dot{H}^1_*(\Lambda^1\mathbb{R}^3)},\end{aligned} \tag{1.21}$$

and $\mathcal{R}(u, A) = \frac{1}{2}\int_\Omega(|u|^2 - 1)|A|^2\mathrm{d}x$ is a remainder term of lower order. Thus $\mathcal{F}_\epsilon(u, A)$ may be written as a continuous perturbation of $E_\epsilon(u) + \mathcal{M}(A, h_{\mathrm{ex}})$, and using the stability properties of Γ-convergence we deduce, as in [23] for the two-dimensional case, Γ-convergence for the functionals $\mathcal{F}_\epsilon$ in the critical energy regime $g_\epsilon = |\log\epsilon|^2$:

Theorem 4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, let $W(u)$ satisfy (H_q) with $q \geqq 3$, and assume $h_{\mathrm{ex}} = dA_{\mathrm{ex},\epsilon}$ and that there exists $A_{\mathrm{ex},o} \in H^1_{loc}(\Lambda^1\mathbb{R}^3)$ such that $\frac{A_{\mathrm{ex},\epsilon}}{|\log\epsilon|} - A_{\mathrm{ex},0} \to 0$ in $\dot{H}^1_*(\Lambda^1\mathbb{R}^3)$. Then the following hold.*

(i) Compactness. *For any sequence $(u_\epsilon, A_\epsilon) \in H^1(\Omega;\mathbb{C})\times[A_{\mathrm{ex},0}+\dot{H}^1_*(\Lambda^1\mathbb{R}^3)]$ such that $\mathcal{F}_\epsilon(u_\epsilon, A_\epsilon) \leqq K|\log\epsilon|^2$, we have, up to a subsequence,*

$$\frac{A_\epsilon}{|\log\epsilon|} - A \rightharpoonup 0 \quad \textit{weakly in } \dot{H}^1_*(\Lambda^1\mathbb{R}^3), \tag{1.22}$$

for some $A \in A_{\mathrm{ex},0} + \dot{H}^1_*(\Lambda^1\mathbb{R}^3)$ as well as the convergences (1.4), (1.5) and (1.6) of Theorem 2 in the case $g_\epsilon = |\log\epsilon|^2$.

(ii) Γ-convergence. *For $v \in L^2(\Lambda^1\Omega)$ and $A \in A_{\mathrm{ex},0} + \dot{H}^1_*(\Lambda^1\mathbb{R}^3)$, define*

$$\mathcal{F}(v, A) = \frac{1}{2}||dv|| + \frac{1}{2}||v - A||^2_{L^2(\Lambda^1\Omega)} + \frac{1}{2}||dA - dA_{\mathrm{ex},0}||^2_{L^2(\Lambda^2\mathbb{R}^3)}, \tag{1.23}$$

if $||dv|| = |dv|(\Omega)$ is finite, $\mathcal{F}(v, A) = +\infty$ otherwise.
Then under the convergences (1. 22), (1.4), (1.5) and (1.6), we have

$$\frac{\mathcal{F}_\epsilon(u_\epsilon, A_\epsilon)}{|\log\epsilon|^2} \xrightarrow{\Gamma} \mathcal{F}(v, A). \tag{1.24}$$

Remark 9. Assuming (H_∞), the Γ-limit (1. 24) is obtained with respect to the convergences (1. 22) and (1. 17).

Remark 10. The statement of Theorem 4 is not gauge-invariant, as the condition that $A_\epsilon \in A_{\mathrm{ex},\epsilon} + H^1_*(\Lambda^1\mathbb{R}^3)$ uniquely determines the function ϕ in (1. 18). Fixing this degree of freedom is clearly necessary for compactness. Note, however, that the limiting functional $\mathcal{F}$ has a gauge-invariance property: $\mathcal{F}(v, A) = \mathcal{F}(v + \gamma|_\Omega, A + \gamma)$ whenever $d\gamma = 0$.

The Euler–Lagrange equations of the functional $\mathcal{F}$ consist in the Ampère law $d^*H = j$ for the resulting magnetic field $H = dA - h$, generated by the (gauge-invariant) super-current $j = v - A$ in Ω (see (4.6)), and a curvature equation for the vortex filaments, that is, the streamlines of the limiting vortex distribution (see (4.7)), which reads, in the regular case,

$$\begin{cases} \boldsymbol{\kappa} = 2\boldsymbol{\tau} \times \boldsymbol{j} & \text{in } \Omega, \\ \boldsymbol{\tau}_\top = 0 & \text{on } \partial\Omega, \end{cases} \tag{1. 25}$$

with $\boldsymbol{\kappa}$ and $\boldsymbol{\tau}$ denoting, respectively, the curvature vector and the unit tangent to the vortex filament, $\boldsymbol{j}$ the vector field corresponding to the super-current $j = v - A$, and $\times$ the exterior product in $\mathbb{R}^3$. Formula (1. 25) generalizes the corresponding law in the case of a finite number of vortices (see [7], Theorem 3 (iv), and [13]).

Remark 11. In [2] we analyze in more detail the properties of minimizers of the limiting functional $\mathcal{F}$ through the introduction of a dual variational problem. We use this description to characterize to leading order the first critical field H_{c_1}.

These results extend to three dimensions facts about two-dimensional models of superconductivity first established by Sandier and Serfaty [29], see also [30] and other references cited therein. Following the initial work of Sandier and Serfaty, it was shown in [23] that their results can be recovered via the two-dimensional analog of the procedure we follow here and in [2].

As far as we know, the relevance of convex duality in these settings was first pointed out by Brezis and Serfaty [12].

Remark 12. In [2] we also apply Theorem 2 to study the Γ-limit of the Gross–Pitaevskii functional for superfluidity, and derive, in particular, a reduced vortex density model for rotating Bose–Einstien condensates, deducing the corresponding curvature equations and an expression for the critical angular velocity.

Remark 13. Theorem 4 is concerned with the description of the behavior of *global* minimizers. The convergence of *local* minimizers with bounded vorticity has been studied, under various assumptions, in [21,25,26], relying on techniques related to Theorem 1.

1.4. Plan of the Paper

This paper is organized as follows: in Section 2 we prove the lower bound and compactness statement (i) of Theorem 2, while Section 3 is devoted to the proof of the upper bound statement (ii). In Section 4 we prove Theorem 4 and derive the Euler–Lagrange equations of the Γ-limit, obtaining, in particular, formula (1. 25). Section 5 is an Appendix that collects some notation and the proofs of some auxiliary results.

2. Lower Bound and Compactness

In this section we prove statement (i) of Theorem 2, relying largely on our previous works [1,23]. We prove everything in $\Omega \subset \mathbb{R}^n$ for arbitrary $n \geqq 3$. We note, however, that the lower bound inequality (1.7) is not expected to be sharp when $n \geqq 4$, see Remark 8.

We first derive (1.4) and (1.5). Then, assuming (1.6), we derive the characterization of the limiting spaces $\mathcal{A}_k$ corresponding to the scaling regimes S_k identified in the statement of the Theorem. We next turn to the proof of the lower bound (1.7). The compactness statement (1.6) in the case $p = 1$ will be obtained during the proof of (1.7), and the case $1 < p < \frac{n}{n-1}$ (see Remark 6) will follow from the case $p = 1$ by a short interpolation argument.

Proof of (1.4), (1.5). Observe first that $|u_\epsilon| \to 1$ in $L^q(\Omega)$ by assumptions (H_q) on $W(u)$ and (H_g) on E_ϵ, since

$$\int_\Omega |1 - |u_\epsilon||^q \leqq C \int_\Omega W(u_\epsilon) \leqq C\epsilon^2 E_\epsilon(u_\epsilon) \leqq C\epsilon^2 g_\epsilon \to 0.$$

From the identity $|u|^2|\nabla u|^2 = |u|^2|\nabla|u||^2 + |ju|^2$ we deduce that

$$\int_\Omega \frac{|ju_\epsilon|^2}{|u_\epsilon|^2 g_\epsilon} \leqq 2 \cdot \frac{E_\epsilon(u_\epsilon)}{g_\epsilon} \leqq 2K, \tag{2.1}$$

which yields, up to a subsequence, $\frac{ju_\epsilon}{|u_\epsilon|\sqrt{g_\epsilon}} \rightharpoonup v$ weakly in $L^2(\Omega)$, completing the proof of (1.4). Now write

$$\frac{ju_\epsilon}{\sqrt{g_\epsilon}} = \frac{ju_\epsilon}{|u_\epsilon|\sqrt{g_\epsilon}} + (|u_\epsilon| - 1) \cdot \frac{ju_\epsilon}{|u_\epsilon|\sqrt{g_\epsilon}}.$$

Using (1.4) we deduce that $(|u_\epsilon| - 1) \cdot \frac{ju_\epsilon}{|u_\epsilon|\sqrt{g_\epsilon}} \rightharpoonup 0$ weakly in $L^{\frac{2q}{q+2}}(\Omega)$. This yields $\frac{ju_\epsilon}{\sqrt{g_\epsilon}} \rightharpoonup v$ weakly in $L^{\frac{2q}{q+2}}(\Omega)$, that is (1.5). $\square$

Next, the characterization of limiting spaces $\mathcal{A}_k$ follows from (1.4), (1.5) and (1.6), since by (1.5) we deduce that $d(\frac{ju_\epsilon}{\sqrt{g_\epsilon}}) \rightharpoonup dv$ weakly in $W^{-1,\frac{2q}{q+2}}(\Omega)$, hence, in the case $g_\epsilon \gg |\log\epsilon|^2$,

$$\frac{|\log\epsilon|}{g_\epsilon} Ju_\epsilon = \left(\frac{|\log\epsilon|}{\sqrt{g_\epsilon}}\right) d\left(\frac{ju_\epsilon}{\sqrt{g_\epsilon}}\right) \rightharpoonup 0 \cdot dv = 0 \quad \text{in } W^{-1,\frac{2q}{q+2}}(\Omega). \tag{2.2}$$

In view of (1.6), this implies $J = 0$ by uniqueness of the weak limit. On the other hand, in the case $g_\epsilon \ll |\log\epsilon|^2$,

$$d\left(\frac{ju_\epsilon}{\sqrt{g_\epsilon}}\right) = 2\left(\frac{\sqrt{g_\epsilon}}{|\log\epsilon|}\right) \cdot \left(\frac{|\log\epsilon|}{g_\epsilon} Ju_\epsilon\right) \to 0 \cdot J = 0 \quad \text{in } W^{-1,p}(\Omega),\ \ p < \frac{n}{n-1},$$

which implies $dv = 0$, again by uniqueness of the weak limit. The above formulas, in the case $g_\epsilon = |\log\epsilon|^2$, imply that $dv = 2J$.

We turn to the proof of (1.7) distinguishing two cases, namely $|\log\epsilon| \ll g_\epsilon \leqq |\log\epsilon|^2$, and $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$. We begin with the latter case.

Proof of (1.7) **in the case** $g_\epsilon \gg |\log \epsilon|^2$**.** In this energy regime, we have just shown that $J = 0$, and (1.4) and (2.1) immediately imply

$$\liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon)}{g_\epsilon} \geqq \frac{1}{2} \int_\Omega |v|^2, \tag{2.3}$$

yielding conclusion (1.7). □

If it is not true that $g_\epsilon \gg |\log \epsilon|^2$, then by passing to a subsequence we may suppose that $g_\epsilon \leqq C|\log \epsilon|^2$. By renaming the constant K in (H_g) we may also assume that $C = 1$. Thus the proof of (1.7) will be completed by the following.

Proof of (1.7) **in the case** $|\log \epsilon| \ll g_\epsilon \leqq |\log \epsilon|^2$**.** The main step in the proof is the following improvement of [1], Proposition 3.1. We establish it in greater generality than is needed for the proof of (1.7). □

We remark that (1.7) in the scaling $|\log \epsilon| \ll g_\epsilon \leqq |\log \epsilon|^2$ is already established in [31] and, moreover, that a key point in that proof is a result similar to the following proposition.

Proposition 1. *Let u_ϵ be a sequence of smooth maps on $\Omega \subset \mathbb{R}^n$, $n \geqq 2$, such that (H_g) holds, with $|\log \epsilon| \leqq g_\epsilon \leqq |\log \epsilon|^2$. Then we have, up to a subsequence,*

$$\frac{|\log \epsilon|}{g_\epsilon} J u_\epsilon \to J \qquad \text{in } W^{-1,1}(\Lambda^2 \Omega), \tag{2.4}$$

where J is an exact measure-valued 2-form[2] *with finite mass in Ω. Moreover, there exists a closet set $C_\epsilon \subset \Omega$ such that $|C_\epsilon| \to 0$, and such that for every simple 2-covector η such that $|\eta| = 1$ and for every open set $U \Subset \Omega$, it holds that*

$$\liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon; C_\epsilon)}{g_\epsilon} \geqq |(J, \eta)|(U), \tag{2.5}$$

where (J, η) is the signed measure defined according to (5.4).

Our proof of Proposition 1 differs from that of the corresponding point (Proposition IV.3) in [31]. One feature of our proof is that the set C_ϵ that we construct is manifestly a closed set, whereas in the construction of [31], a certain amount of work is required even to see that the corresponding set is measurable.

Taking for granted Proposition 1, we complete the proof of (1.7). First, a standard localization argument (see [1], p. 1436) gives, for any finite collection of pairwise disjoint open sets $U_j \Subset \Omega$ and simple unit 2-covectors η_j,

$$\sum_j |(J, \eta_j)|(U_j) \leqq \liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon; C_\epsilon)}{g_\epsilon}. \tag{2.6}$$

Taking the supremum over all choices of pairwise disjoint open sets U_j and unit simple 2-covectors, η_j on the left-hand side of (2.6) yields the total *comass norm* of

[2] In the case $g_\epsilon = |\log \epsilon|$, J has the structure of a rectifiable boundary with multiplicities in $\pi \cdot \mathbb{Z}$, according to Theorem 1.

J in the sense of [16], section 1.8.1. In the three-dimensional case[3] this coincides with the total variation (or L^1, accordingly) norm of J, since all 2-covectors in $\mathbb{R}^3$ are necessarily simple. Hence we may write, for $n = 3$,

$$|J|(\Omega) \leqq \liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon; C_\epsilon)}{g_\epsilon}. \tag{2.7}$$

Now let $\Omega_\epsilon \equiv \Omega \setminus C_\epsilon$, and let $\chi_\epsilon(x)$ be the characteristic function of Ω_ϵ. We may assume after passing to a subsequence that $\chi_\epsilon(x) \to 1$ as $\epsilon \to 0$ for almost everywhere $x \in \Omega$, since $|C_\epsilon| \to 0$. Then for any $h \in L^2$, $\chi_\epsilon \cdot h \to h$ in L^2 by the dominated convergence theorem, and so it follows from (1.4) that

$$\int_\Omega h \cdot \chi_\epsilon \cdot \frac{ju_\epsilon}{|u_\epsilon|\sqrt{g_\epsilon}} \to \int h \cdot v \qquad \text{as } \epsilon \to 0.$$

That is, $\chi_\epsilon \cdot \frac{ju_\epsilon}{|u_\epsilon|\sqrt{g_\epsilon}} \rightharpoonup v$ weakly in L^2. Since

$$\int_{\Omega_\epsilon} e_\epsilon(u) \geqq \frac{1}{2} \int_\Omega \chi_{\Omega_\epsilon} \frac{|ju_\epsilon|^2}{|u_\epsilon|^2},$$

we deduce that

$$\liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon; \Omega_\epsilon)}{g_\epsilon} \geqq \liminf_{\epsilon \to 0} \frac{1}{2} \int_\Omega \chi_{\Omega_\epsilon} \frac{|ju_\epsilon|^2}{|u_\epsilon|^2 g_\epsilon} \geqq \frac{1}{2} \int_\Omega v^2. \tag{2.8}$$

To conclude, observe that $E_\epsilon(u_\epsilon; \Omega) = E_\epsilon(u_\epsilon; C_\epsilon) + E_\epsilon(u_\epsilon; \Omega_\epsilon)$, so that

$$\liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon; \Omega)}{g_\epsilon} \geqq \liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon; C_\epsilon)}{g_\epsilon} + \liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon; \Omega_\epsilon)}{g_\epsilon}. \tag{2.9}$$

Combining (2.9) with (2.8) and (2.7) we obtain (1.7). □

We now supply the

Proof of Proposition 1. We will proceed in two steps: first, we apply the discretization procedure of [1], Section 3 at a suitable scale ℓ_ϵ to deduce (2.4) and to identify a small set $C'_\epsilon \subset \Omega$ where the Jacobian Ju_ϵ is essentially confined. Second, we apply the cited procedure again, this time imposing an additional condition that yields good control of the resulting 2-form ν'_ϵ (a discretization of the Jacobian) in a small neighborhood C_ϵ of C'_ϵ by the Ginzburg–Landau energy *in the same small neighborhood* C_ϵ. We then argue that the restriction of ν'_ϵ to a suitable subset of C_ϵ converges to the same limit as Ju_ϵ, so that from lower semicontinuity, bounds on $(\nu'_\epsilon, \eta) \llcorner C_\epsilon$ yield estimates on (J, η), thereby proving (2.5).

We carry out these arguments in detail in the case $n = 3$ and then we discuss the general case.

[3] And for any $n \geqq 3$ if $g_\epsilon = |\log \epsilon|$, then J is obtained as a limit of polygonal currents with uniformly bounded mass, and hence is rectifiable by the Federer–Fleming closure theorem.

Step 1 We follow [1], Section 3. Fix a unit simple 2-covector η, and an orthonormal basis $(\boldsymbol{e}_i)$ of $\mathbb{R}^3$ satisfying $\eta(\boldsymbol{e}_2 \wedge \boldsymbol{e}_3) = 1$. Consider a grid $\mathcal{G} = \mathcal{G}(a, \boldsymbol{e}_i, \ell)$, given by the collection of cubes with edges of size ℓ, and vertices having coordinates (with respect to a reference system with origin in $a \in \mathbb{R}^3$ and orthonormal directions $(\boldsymbol{e}_i)_{i=1,2,3}$) which are integer multiples of ℓ. For $h = 1, 2$ denote by R_h the h-skeleton of $\mathcal{G}$, that is, the union of all h-dimensional faces of the cubes of $\mathcal{G}$. Consider also the dual grid having vertices in the centers of the cubes of $\mathcal{G}$, and denote its h-skeleton by R'_h for $h = 1, 2$. From (H_g) and the assumption that $g_\epsilon \leqq |\log \epsilon|^2$ we have

$$E_\epsilon(u_\epsilon; \Omega) \leqq K|\log \epsilon|^2, \quad \text{and we set } \ell \equiv \ell_\epsilon := |\log \epsilon|^{-10}. \tag{2.10}$$

Observe that (2.10) replaces (3.22) and (3.23) in [1]. Choose $a \equiv a_\epsilon$ by a mean-value argument in such a way that Lemma 3.11 of [1] holds, so that, in particular, the restriction of the energy on the two-dimensional and one-dimensional skeleton of $\mathcal{G}$ is controlled by

$$\int_{R_h \cap \Omega} e_\epsilon(u_\epsilon) \mathrm{d}\mathcal{H}^h \leqq C_0 \ell^{h-3} E_\epsilon(u_\epsilon; \Omega), \quad h = 1, 2, \tag{2.11}$$

for a suitable constant $C_0 > 1$, and moreover

$$\ell \int_\Omega \frac{e_\epsilon(u_\epsilon)}{|\mathrm{dist}(x, R_1)|} \mathrm{d}x \leqq C_0 E_\epsilon(u_\epsilon; \Omega). \tag{2.12}$$

In view of (2.10), Lemma 3.4 in [1] is satisfied, hence $|u_\epsilon| \to 1$ uniformly on $R_1 \cap \Omega$. In particular, for any face $Q \in R_2$, the topological degree $d_Q := \deg\left(\frac{u_\epsilon}{|u_\epsilon|}, \partial Q, S^1\right) \in \mathbb{Z}$ is well-defined (modulo the choice of an orientation of Q in $\mathbb{R}^3$).

The discretization procedure of [1], Lemmas 3.7 to 3.10, may then take place on any fixed open set $U \Subset \Omega$, yielding an oriented polyhedral 1-cycle (actually, a relative boundary in $\bar{U}$) $M_\epsilon = \sum (-1)^{\sigma_i} d_{Q_i} \cdot Q'_i$, where $Q'_i \subset R'_1$ is the unique edge of the cubes of the dual grid intersecting the face $Q_i \subset R_2$, the sign $(-1)^{\sigma_i}$ depends on the orientations of both Q_i and Q'_i, and the sum is extended to any $Q_i \subset R_2$ such that $Q_i \cap U \neq \emptyset$. Notice that M_ϵ is supported in $R'_1 \cap U^{\sqrt{3}\ell}$, where $U^{\sqrt{3}\ell}$ denotes the tubular neighborhood of U of thickness $\sqrt{3}\ell$. The cycle M_ϵ gives rise to a (measure-valued) 2-form ν_ϵ, whose action on 2-forms in $C^\infty_c(\Lambda^2 \Omega)$ is defined by

$$\langle \nu_\epsilon, \varphi \rangle := \pi \cdot \sum_{\substack{Q_i \subset R_2 \\ Q_i \cap U \neq \emptyset}} (-1)^{\sigma_i} d_{Q_i} \int_{Q'_i} \star\varphi. \tag{2.13}$$

The 2-form ν_ϵ is exact in U, since M_ϵ is a relative boundary in $\bar{U}$, and enjoys the following properties: it is a measure-valued 2-form supported in $R'_1 \cap U^{\sqrt{3}\ell}$, such that its total variation $|\nu_\epsilon|$ is bounded on U by[4]

[4] See [1], (3.29).

$$|\nu_\epsilon|(U) = \sum_{\substack{Q_i \subset R_2 \\ Q_i \cap U \neq \emptyset}} \pi\ell \cdot |d_{Q_i}| \leqq C \frac{E_\epsilon(u_\epsilon;\Omega)}{|\log\epsilon|}, \tag{2.14}$$

with $C > 0$ independent of $U \Subset \Omega$, and such that ν_ϵ is close to Ju_ϵ in the $W^{-1,1}$ norm, namely[5]

$$||Ju_\epsilon - \nu_\epsilon||_{W^{-1,1}(\Lambda^2 U)} \leqq C\ell \cdot E_\epsilon(u_\epsilon;\Omega). \tag{2.15}$$

Moreover, the support of ν_ϵ is contained in the interior of a set $C'_\epsilon \subset U^{\sqrt{3}\ell}$ given by the union of those cubes of the grid $\mathcal{G}$ having at least one face $Q \subset R_2$, $Q \cap U \neq \emptyset$, such that $d_Q \neq 0$. Denote by I the set of indices i in (2.14) for which $d_{Q_i} \neq 0$, or equivalently, $|d_{Q_i}| \geqq 1$. By (2.14) we have

$$|C'_\epsilon| \leqq \ell^3 \cdot |I| \leqq \sum_{i\in I} \ell^3 \cdot |d_{Q_i}| \leqq C\ell^2 \frac{E_\epsilon(u_\epsilon;\Omega)}{|\log\epsilon|}, \tag{2.16}$$

so that by (2.10), $|C'_\epsilon| \to 0$ as $\epsilon \to 0$.

Notice, moreover, that (2.14) and (H_g) imply that $\frac{|\log\epsilon|}{g_\epsilon} \cdot \nu_\epsilon \rightharpoonup J$ weakly as measures, where J is a measure-valued 2-form in Ω, which is exact and has total variation $|J|(\Omega) \leqq C \liminf_{\epsilon\to 0} \dfrac{E_\epsilon(u_\epsilon;\Omega)}{g_\epsilon}$. By (2.15) we finally deduce that $\frac{|\log\epsilon|}{g_\epsilon} \cdot Ju_\epsilon \to J$ in $W^{-1,1}(\Lambda^2 U)$ for any $U \Subset \Omega$, which yields (2.4).

Step 2 For $N > 0$ to be chosen below, define $C_\epsilon \equiv C_{N,\epsilon} := \{x \in \Omega,\ \mathrm{dist}(x, C'_\epsilon) \leqq 2N\ell\}$ to be the tubular neighborhood of C'_ϵ of thickness $2N\ell$ intersected with Ω. By (2.16) we have

$$|C_\epsilon| \leqq 8N^3|C'_\epsilon| \leqq CN^3\ell^2 \frac{g_\epsilon}{|\log\epsilon|} \to 0 \qquad \text{as } \epsilon \to 0, \tag{2.17}$$

as long as $N^3 \leqq \ell^{-1}$. In view of (2.10), (2.17) is verified for instance by fixing

$$N \equiv N_\epsilon := |\log\epsilon|^3. \tag{2.18}$$

Observe, moreover, that

$$E_\epsilon(u_\epsilon; C_\epsilon) \leqq E_\epsilon(u_\epsilon;\Omega) \leqq K g_\epsilon \leqq |\log\epsilon|^2. \tag{2.19}$$

Consider the grid $\mathcal{G}^*_\epsilon = \mathcal{G}(b_\epsilon, \boldsymbol{e}_i, \ell)$, where $\ell = \ell_\epsilon = |\log\epsilon|^{-10}$ as above and b_ϵ is chosen such that for an arbitrarily fixed $\delta > 0$, (3.18), (3.19) and (3.20) in Lemma 3.11 of [1] hold true, and, moreover, (3.17) holds true with Ω replaced by C_ϵ. In

[5] Combine (2.10) and (2.12) with (3.7) and (3.14) of [1].

other words, denoting by R_h^* the h-skeleton of $\mathcal{G}_\epsilon^*$, $h = 1, 2$, and by $\tilde{R}_2^*$ the union of the faces of the 2-skeleton of $\mathcal{G}_\epsilon^*$ orthogonal to $\boldsymbol{e}_1$, we have,

$$\int_{\tilde{R}_2^* \cap (C_\epsilon)} e_\epsilon(u_\epsilon) \mathrm{d}\mathcal{H}^2 \leqq (1+\delta)\ell^{-1} E_\epsilon(u_\epsilon; C_\epsilon), \tag{2.20}$$

$$\int_{R_h^* \cap \Omega} e_\epsilon(u_\epsilon) \mathrm{d}\mathcal{H}^h \leqq C_0 \delta^{-1} \ell^{h-3} E_\epsilon(u_\epsilon; \Omega), \quad h = 1, 2, \tag{2.21}$$

$$\ell \int_\Omega \frac{e_\epsilon(u_\epsilon)}{|\mathrm{dist}(x, R_1^*)|} \mathrm{d}x \leqq C_0 \delta^{-1} E_\epsilon(u_\epsilon; \Omega). \tag{2.22}$$

Fix an open subset $U \Subset \Omega$. As in Step 1, the procedure of [1] yields a polyhedral cycle

$$M'_\epsilon = \sum_{\substack{Q_i \subset R_2^* \\ Q_i \cap U \neq \emptyset}} (-1)^{\sigma_i} d_{Q_i} \cdot Q'_i, \tag{2.23}$$

which is a relative boundary in $\bar{U}$ and is supported in $R_1^{*\prime} \cap U^{\sqrt{3}\ell}$, where $R_1^{*\prime}$ is the one-dimensional skeleton of the dual grid to $\mathcal{G}^*$. The corresponding measure-valued 2-form ν'_ϵ, defined as in (2.13) by

$$\langle \nu'_\epsilon, \varphi \rangle := \pi \cdot \sum_{\substack{Q_i \subset R_2^* \\ Q_i \cap U \neq \emptyset}} (-1)^{\sigma_i} d_{Q_i} \int_{Q'_i} \star\varphi, \qquad \forall \varphi \in C_c^\infty(\Lambda^2(\Omega)), \tag{2.24}$$

is exact on U and verifies $|\nu'_\epsilon|(U) \leqq C \frac{E_\epsilon(u_\epsilon; \Omega)}{|\log \epsilon|}$ with $C > 0$ independent of U.

For $x \in \Omega$ define $f(x) := \mathrm{dist}(x, M_\epsilon)$, so that f is 1-Lipschitz. Denoting by $C^t = \{x : \ f(x) \leqq t\} \cap \Omega$, we have that $C^{2N\ell} \subset C_\epsilon$.

Lemma 1. *There exists $t := t_\epsilon < N\ell$ such that*

$$||\nu'_\epsilon \llcorner C^t - \nu_\epsilon||_{W^{-1,1}(U)} \leqq C(\ell + N^{-1}) g_\epsilon, \tag{2.25}$$

with $C > 0$ independent of ϵ and U. In particular, the choices of ℓ and N (see (2.10) and (2.18)) imply that

$$\frac{|\log \epsilon|}{g_\epsilon} \cdot \nu'_\epsilon \llcorner C^t \to J \quad in\ W^{-1,1}(\Lambda^2 U) \tag{2.26}$$

and, for any 2-covector η,

$$\left(\frac{|\log \epsilon|}{g_\epsilon} \cdot \nu' \llcorner C^t, \eta \right) \to (J, \eta) \quad in\ W^{-1,1}(U). \tag{2.27}$$

We postpone the proof of Lemma 1 to Section 5.6 of the Appendix. By (2.27) and lower semicontinuity of total variation we deduce

$$
\begin{aligned}
|(J,\eta)|(U) &\leqq \liminf_{\epsilon\to 0}\left|\left(\frac{|\log\epsilon|}{g_\epsilon}\cdot \nu'_\epsilon \llcorner C^t,\eta\right)\right|(U)\\
&\leqq \liminf_{\epsilon\to 0}\left|\left(\frac{|\log\epsilon|}{g_\epsilon}\cdot \nu'_\epsilon \llcorner C^{N\ell},\eta\right)\right|(U).
\end{aligned} \tag{2.28}
$$

Observe that specializing (2.24) to the case $\varphi = \psi\,\eta$, with $\psi \in C_c^\infty(\Omega)$, and letting ψ approach the characteristic function of $C^{N\ell}\cap U$, we have

$$
|(\nu'_\epsilon \llcorner C^{N\ell},\eta)|(U) = |(\nu'_\epsilon,\eta)|(C^{N\ell}\cap U) = \pi\cdot \sum_{\substack{Q_i\subset R_2^*\\ Q_i\cap U\neq\emptyset}}\left| d_{Q_i}\int_{Q_i'\cap C^{N\ell}\cap U}\star\eta\right|. \tag{2.29}
$$

Notice that for any $Q'\subset R_1^{*\prime}$ such that $Q'\cap C^{N\ell}\neq\emptyset$, its dual element Q is contained in the tubular nighborhood of thickness $\sqrt{3}\ell$ of $C^{N\ell}$, which is a subset of $C^{2N\ell}$, so that, in particular, $Q\subset C_\epsilon$. Recalling from the definitions that $\star\eta = dx^1$, which is the oriented arclength element along Q_i' for $Q_i\in\tilde{R}_2^*$, we obtain from (2.29) that

$$
|(\nu'_\epsilon \llcorner C^{N\ell},\eta)|(U) \leqq \sum_{Q\subset \tilde{R}_2^*\cap C_\epsilon}\pi\ell\cdot|d_Q|. \tag{2.30}
$$

One readily verifies, following [1], p. 1435, that (2.10) and (2.19) allowed us to apply Lemma 3.10 there (which relied in turn on a fundamental estimate in [20,28]), to efficiently estimate the sum of the degrees $|d_Q|$ in terms of $E_\epsilon(u_\epsilon;C_\epsilon)$. Namely, for any $r>0$, and any $Q\subset R_2^*\cap\Omega$ we have

$$
(1-c_r(\epsilon))\pi\cdot|d_Q| \leqq \frac{1}{|\log\epsilon|}\int_Q e_\epsilon(u_\epsilon)\mathrm{d}\mathcal{H}^2 + \frac{Kr\ell}{|\log\epsilon|}\int_{\partial Q} e_\epsilon(u_\epsilon)\mathrm{d}\mathcal{H}^1, \tag{2.31}
$$

where $c_r(\epsilon)$ is independent of Q, and $c_r(\epsilon)\to 0$ as $\epsilon\to 0$ (see [1], p. 1435). We may thus write

$$
(1-c_r(\epsilon))\sum_{Q\subset\tilde{R}_2^*\cap C_\epsilon}\pi\cdot|d_Q| \leqq \frac{1}{|\log\epsilon|}\int_{\tilde{R}_2^*\cap C_\epsilon} e_\epsilon(u_\epsilon)\mathrm{d}\mathcal{H}^2 + \frac{Kr\ell}{|\log\epsilon|}\int_{R_1^*\cap C_\epsilon} e_\epsilon(u_\epsilon)\mathrm{d}\mathcal{H}^1. \tag{2.32}
$$

Combining (2.30) with (2.32), and taking into account (2.20), (2.21), we are led to

$$
(1-c_r(\epsilon))|\left(\frac{|\log\epsilon|}{g_\epsilon}\cdot\nu'_\epsilon \llcorner C^{N\ell},\eta\right)|(U) \leqq \left(1+\delta+\frac{Kr}{\delta}\right)\frac{E_\epsilon(u_\epsilon;C_\epsilon)}{g_\epsilon}. \tag{2.33}
$$

Passing to the limit as $\epsilon\to 0$, we have, in view of (2.28),

$$
|(J,\eta)|(U) \leqq \left(1+\delta+\frac{Kr}{\delta}\right)\liminf_{\epsilon\to 0}\frac{E_\epsilon(u_\epsilon;C_\epsilon)}{g_\epsilon}. \tag{2.34}
$$

Taking $r<\delta^2$ and δ arbitrarily small yields (2.5). $\square$

Proof in the general case $n \geqq 3$. The main tool used above is the algorithm from [1] for constructing a polyhedral approximation of the Jacobian Ju, and hence a measure-valued 2-form ν_ϵ, with good estimates of $\|Ju - \nu_\epsilon\|_{W^{-1,1}}$ and of $|(\nu_\epsilon, \eta)|(W)$ for suitable subsets $W \subset \Omega$. The procedure in [1], in fact, is presented in $\mathbb{R}^n$, $n \geqq 3$, and so can be employed in the general case as for $n = 3$, with purely cosmetic differences. For example, in $\mathbb{R}^n$, the analog of Q_i' in (2.13) and elsewhere is now the unique $n-2$ face of the dual grid that intersects Q_i. Also, different scalings make it convenient to choose $\ell = |\log \epsilon|^{-(3n+1)}$, say, while we still take $N = |\log \epsilon|^3$. Then it remains true that $g_\epsilon \ll N$, which is needed for the proof of Lemma 1, and that $|C_\epsilon'| \to 0$, which follows from the fact that $N^n \ell^2 \frac{g_\epsilon}{|\log \epsilon|} \to 0$ as $\epsilon \to 0$, compare (2.17). Modulo changes of this sort, the argument is identical in the general case. □

Proof of (1.6). Recall that we have assumed that $g_\epsilon \leqq |\log \epsilon|^2$. Since

$$||Ju_\epsilon - \nu_\epsilon||_{L^1(\Lambda^2 U)} \leqq ||Ju_\epsilon||_{L^1(\Lambda^2 U)} + ||\nu_\epsilon||_{L^1(\Lambda^2 U)} \leqq C E_\epsilon(u_\epsilon; \Omega) \leqq C g_\epsilon \tag{2.35}$$

for any $U \Subset \Omega$, we deduce, by interpolation with (2.15),

$$||Ju_\epsilon - \nu_\epsilon||_{W^{-1,p}(\Lambda^2 U)} \leqq C(\ell_\epsilon \cdot g_\epsilon)^{1-\frac{n(p-1)}{p}} g_\epsilon^{\frac{n(p-1)}{p}} \leqq C \ell_\epsilon^{1-\frac{n(p-1}{p}} \cdot |\log \epsilon|^2. \tag{2.36}$$

The conclusion (1.6) follows by choosing $\ell_\epsilon = \ell_{\epsilon,p} = |\log \epsilon|^{-\frac{3p}{n-p(n-1)}}$, so that the right-hand side of (2.36) vanishes. □

3. Upper Bound

In this section we prove statement (ii) of Theorem 2.

3.1. Strategy of Proof

The proof is subdivided into steps. First of all, we reduce our focus in Section 3.2 to considering an appropriate dense class of the domain of the Γ-limit, using a suitable finite elements approximation. The construction of the recovery sequence will be based on a Hodge decomposition of the limiting momentum p, described in Section 3.3, and a discretization of the limiting vorticity dp in terms of a system of lines where the vorticity is concentrated and quantized; this, and associated estimates of the discretized vorticity and related quantities, are the main points in the proof. An argument *à la* Biot-Savart then allows us to construct S^1-valued maps whose Jacobians are concentrated precisely on the discretized vorticity lines, and we obtain our maps u_ϵ by adjusting the modulus around the vortex cores. The proof is completed by the verification of the upper bound inequality, which relies crucially on good properties of the discretized vortex lines and estimates satisfied by associated auxiliary functions.

3.2. Nice Dense Class

We say that a 1-form p on a domain $\Omega \subset \mathbb{R}^3$ is *rational piecewise linear* if p is continuous, and there exists a family of closed simplices $\{P_i\}$ with pairwise disjoint interiors such that

1. $\Omega \subset \cup P_i$;
2. for every i, the restriction to P_i of p has the form $\sum(a_i^{jk}x_k + b_i^j)dx^j$ for $a_i^{jk}, b_i^j \in \mathbb{Q}$; and
3. all the vertices of each P_i are rational (that is, belong to $\mathbb{Q}^3$.)

Rational piecewise linear 1-forms have the following useful property:

Lemma 2. *Let p be a rational piecewise linear 1-form on $\Omega \subset \mathbb{R}^3$, and let $\{P_i\}$ denote the associated family of simplices, as described above.*

Then for any (two-dimensional) face T of any of the simplices P_i,

$$\int_T dp \in \mathbb{Q}.$$

Proof. We fix P_i and write $V_1, \ldots, V_4$ to denote its vertices. Consider some face F of P_i, say $F = \mathrm{co}\{V_j, V_k, V_l\} \equiv$ the convex hull of $\{V_i, V_j, V_k\}$, for some distinct $\{j, k, l\} \in \{1, \ldots, 4\}$, and let W denote the constant 2-covector such that $dp = W$ in P. Then

$$\int_F dp = \pm\frac{1}{2}((V_l - V_j) \wedge (V_l - V_k), W), \tag{3.1}$$

where the sign depends on the orientation that F inherits from P_i, and this is clearly rational. □

We say that a closed set is *rational polygonal* if it is a finite union of closed simplices with pairwise disjoint interiors and rational vertices. A rational polygonal open set is the interior of a rational polygonal closed set. We write "$\simeq$" to mean "is homeomorphic to".

Lemma 3. *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded open subset and that $\partial\Omega$ is of class C^1. Given $p \in L^2(\Lambda^1(\Omega))$ such that dp is a measure, and given $\delta > 0$ small, there exists a rational polygonal set Ω_δ^P with $\Omega \Subset \Omega_\delta^P \Subset \Omega^\delta = \{\mathrm{dist}(x, \Omega) < \delta\}$, and such that $\Omega \simeq \Omega_\delta^P \simeq \Omega^\delta$, and a rational, piecewise linear 1-form $p_\delta \in L^2(\Lambda^1\Omega_\delta^P)$, such that $dp_\delta \in L^1(\Lambda^2\Omega_\delta^P)$ and*

$$\|p - p_\delta\|_{L^2(\Omega)} \leqq \delta \tag{3.2}$$

$$\|p_\delta\|^2_{L^2(\Omega_\delta^P)} \leqq \|p\|^2_{L^2(\Omega)} + \delta \tag{3.3}$$

$$\int_{\Omega_\delta^P} |dp_\delta| \leqq |dp|(\Omega) + \delta. \tag{3.4}$$

Proof. Since $\partial\Omega$ is of class C^1, it is clear that we may find $\delta > 0$ such that $\Omega^\delta \simeq \Omega$. By adapting standard approximation techniques for BV functions as in [18], we can find a set $\Omega' \subset \Omega^\delta$ such that $\Omega \Subset \Omega'$, and a 1-form $p' \in C^\infty(\Lambda^1(\Omega'))$, such that $\|p - p'\|_{L^2(\Omega)} \leqq \delta/2$, $\|p'\|^2_{L^2(\Omega')} \leqq \|p\|^2_{L^2(\Omega)} + \delta/2$ and $|dp'|(\Omega') \leqq |dp|(\Omega) + \delta/2$.

Now choose a rational polygonal domain Ω^P_δ such that $\Omega \Subset \Omega^P_\delta \Subset \Omega'$ with $\Omega \simeq \Omega^P_\delta$. This can be achieved by setting $\mathbf{Q}_h$ to be the collection of cubes of side-length h with vertices at points in $h\mathbb{Z}^3$, and defining

$$\Omega^P_\delta := \mathrm{int}(\overline{\cup_{\{Q\in\mathbf{Q}_h : Q\cap\Omega\neq\emptyset\}} Q})$$

for some sufficiently small (rational) h. (Clearly each such cubes can be subdivided into simplices with rational vertices.) If h is small enough, then $Q \cap \Omega$ is contractible for every $Q \in \mathbf{Q}_h$, and then it is not hard to check that $\Omega^P_\delta \simeq \Omega$.

By taking h smaller as necessary, we may also obtain rational triangulations with arbitrarily small mesh size.

By standard interpolation theory from the finite elements method (see for instance [14, Chapter 3]), we can find piecewise linear 1-forms which are arbitrarily close to p' in $W^{1,2}(\Omega_\delta)$: it suffices to choose a sufficiently fine triangulation constructed as above, and to take the (unique) piecewise linear form p_δ which interpolates p' in the vertices of the triangulation. Moreover, an arbitrarily small change of p_δ in the vertices makes it rational. □

We will also need the following variant of the above.

Lemma 3′. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open subset and that $\partial\Omega$ is of class C^1. Given an exact measure-valued 2-form J, and given $\delta > 0$ small, there exists a rational polygonal set Ω^P_δ such that $\Omega \Subset \Omega^P_\delta \Subset \Omega^\delta = \{dist(x, \Omega) < \delta\}$, and such that $\Omega \simeq \Omega^P_\delta \simeq \Omega^\delta$, and a rational, piecewise linear 1-form $p'_\delta \in L^2(\Lambda^1\Omega^P_\delta)$, such that $dp'_\delta \in L^1(\Lambda^2\Omega^P_\delta)$ and such that*

$$\|p - p_\delta\|_{W^{-1,1}(\Omega)} \leqq \delta, \qquad \int_{\Omega^P_\delta} |dp'_\delta| \leqq |J|(\Omega) + \delta. \tag{3.5}$$

The proof is a straightforward modification of the proof of Lemma 3, once we note from Corollary 1 in the Appendix that any exact measure-valued 2-form J in Ω can be written in the form $J = dp'$ for some $p' \in \cap_{1\leqq q<\frac{n}{n-1}} L^q(\Lambda^1\Omega)$.

3.3. Hodge Decomposition of p_δ

Here we refer for notation and basic theory to Section 5.2 of the Appendix. We henceforth write p instead of p_δ.

Since basic results on Hodge theory to which we appeal require some smoothness of the domain, we fix an open set Ω_δ with smooth boundary, such that $\Omega \Subset \Omega_\delta \Subset \Omega^P_\delta$, and such that $\Omega \simeq \Omega_\delta \simeq \Omega^P_\delta$. In particular, we assume that if $\partial\Omega^P_\delta$ has connected components $(\partial\Omega^P_\delta)_i$, $i = 1, \dots, b$, then there exist disjoint connected open sets $W_1, \dots, W_b$ such that

$$\Omega^P_\delta \setminus \bar{\Omega}_\delta = \cup_{i=1}^b W_i, \qquad \partial W_i = (\partial\Omega^P_\delta)_i \cup (\partial\Omega_\delta)_i \qquad \forall i. \tag{3.6}$$

Consider the Hodge decomposition $p = \gamma + d\alpha + d^*\beta$ on Ω_δ satisfying the boundary conditions (5.11). Thanks to Corollary 1 in the Appendix, we know that $\beta = -\Delta_N^{-1}(dp)$, so that in particular $||\beta||_q \leqq C_q||dp||_1 \,\forall\, q < 3/2$. Recall that by L^2-orthogonality of the Hodge decomposition we have

$$\int_{\Omega_\delta} |p|^2 = \int_{\Omega_\delta} |\gamma|^2 + |d^*\beta|^2 + |d\alpha|^2. \tag{3.7}$$

We emphasize that, in what follows, we will carry out most geometric arguments on the polygonal set Ω_δ^P, but the Hodge decomposition always refers to the smooth set $\Omega_\delta \subset \Omega_\delta^P$.

3.4. Discretization of $dp = dd^*\beta$

We will use different arguments to approximate the different terms in the Hodge decomposition of p. Most of our efforts will be devoted to $d^*\beta$. As noted above, the first step in our construction is to discretize $dp = dd^*\beta$, which one can think of as the vorticity.

Proposition 2. *Let p be a rational piecewise linear 1-form supported on $\Omega_\delta^P \subset \mathbb{R}^3$, and fix $\eta \in (0, 1)$. For any $h \leqq \eta^2$ there exists an exact measure-valued 2-form q_h in Ω_δ^P such that:*

(i)

$$q_h = dd^*\beta_h, \ \textit{where}\ \beta_h = -\Delta_N^{-1} q_h \ in \Omega_\delta.$$

(ii)

$$||q_h - dp||_{W^{-1,1}(\Omega_\delta^P)} \leqq C\eta,$$

(iii)

$$|q_h|(\Omega_\delta^P) \leqq ||dp||_{L^1(\Omega_\delta^P)} + C\eta,$$

(iv)

$$||d^*\beta_h||_{L^p(\Omega_\delta)} \leqq C_p|q_h|(\Omega_\delta), \quad d^*\beta_h \rightharpoonup d^*\beta^\eta \ in\ L^p(\Omega_\delta)\ \forall\, p < 3/2,$$
$$||d^*\beta^\eta - d^*\beta||^2_{L^2(\Omega_\delta)} \leqq C\eta,$$

where $C > 0$ is independent of h, η, U. For any $\varphi \in C^0(\Lambda^2\Omega_\delta^P)$ we have the integral representation

(v)

$$\langle q_h, \varphi\rangle = h \int_{\Gamma_h} \star\varphi = h \sum_{\ell=1}^{m(h)} \int_{\Gamma_h^\ell} \star\varphi,$$

where $\Gamma_h = \cup_{s=1}^{n(h)} L_h^s \subset \Omega_\delta^P$, L_h^s is an oriented line segment $\forall\, s, h$, $m(h) < n(h) \leqq Kh^{-1}$, and for any ℓ, h, Γ_h^ℓ is an oriented simple piecewise linear curve in Ω_δ^P such that $\partial\Gamma_h^\ell \cap U = \emptyset\, \forall\, U \subset \Omega_\delta^P$. In particular, we have $|q_h|(U) = h|\Gamma_h \cap U|$ for any $U \subset \Omega_\delta^P$. Moreover,

(vi)

$$\mathrm{dist}(L_1, L_2) > c_0 \eta h^{1/2} \qquad \textit{if } L_1, L_2 \textit{ are disjoint closed line segments of } \Gamma_h,$$

with $c_0 > 0$ *independent of* h, η.

Finally, if L_1, L_2 *are two line segments of* Γ_h^ℓ *with exactly one endpoint in common, and* τ_1, τ_2 *are unit tangents consistent with the orientations (fixed in (v)) of* L_1, L_2 *respectively, then*

(vii)

$$\tau_1 \cdot \tau_2 > -1 + C\eta^2,$$

for some $C > 0$ *independent of* h, η.

Remark 14. The discretized vorticity q_h has a 1-dimensional character, in that it is supported on a union of line segments, so that in realizing it as a (measure-valued) 2-form, rather than a 1-form or vector field, we are departing both from the convention discussed in (5.6) and from standard practice in geometric measure theory. However, this departure is natural in that q_h is an approximation of the 2-form dp, and it is very useful when we want to appeal to Hodge Theory to solve elliptic equations with q_h on the right-hand side, as in conclusion (i) above.

Remark 15. The role of the parameter η is to guarantee that q_h enjoys certain properties such as a good lower bound on distances between distinct piecewise linear curves in the support of q_h, see conclusion (vi) above. These are essential for the verification of the upper bound inequality.

Remark 16. Our arguments (in particular the proof of (iv)) show that there exists 2-form q^η such that $q_h \rightharpoonup q^\eta$ weakly as measures as $h \to 0$. In fact, our construction is designed to yield an explicit description of q^η, see (3.19). This complicates the construction of q_h but immediately yields uniform estimates of q^η, needed for (iv), that would otherwise require some work to obtain.

Proof. The proof of Proposition 2 will be divided into several steps. □

Proof of (v). We start by constructing q_h, which amounts to constructing a collection Γ_h of oriented line segments, see (v). Let $\eta \in (0, 1)$ be fixed, and let p be a piecewise linear rational 1-form with respect to the triangulation $\{S_i\}$ of Ω_δ^P as fixed in the proof of Lemma 3. In particular, for each i there exists a vector $v_i = (v_i^1, v_i^2, v_i^3)$ such that $dp \llcorner S_i = \sum_j v_i^j \star dx_j$. For any simplex S_i, let b_i its barycentre, and let

$$\tilde{S}_i = (1 - \eta) \cdot S_i + \eta \cdot b_i \subset S_i \tag{3.8}$$

be a homothetic copy of S_i, and let $T_{ij}, \tilde{T}_{ij}, \; j = 1, \ldots, 4$ be the 2-faces of $S_i, \tilde{S}_i$ respectively, with the induced orientations.

We will arrange that within each $\tilde{S}_i$, our discretization of dp is supported on a finite union of line segments exactly parallel to v_i. In order to to this and to match fluxes across the faces of each S_i, we discretize the flux through the faces of each S_i and each $\tilde{S}_i$ in related, though different, ways.

For every i and for $j \neq k \in \{1, \ldots, 4\}$, define $T_{ijk} \equiv \pi^{-1}(\pi(T_{ij}) \cap \pi(T_{ik})) \cap T_{ij}$ (with the orientation of T_{ij}), where $\pi \equiv \pi_i$ is the projection on the 2-plane $(v_i)^{\perp}$. One may think of T_{ijk} as the portion of T_{ij} connected to T_{ik} by flux lines of dp. Further, define

$$\phi_{ij} = \int_{T_{ij}} dp, \qquad \phi_{ijk} = \int_{T_{ijk}} dp \;=\; \frac{|T_{ij}|}{|T_{ijk}|}\phi_{ij}.$$

It follows from Lemma 2 that $\phi_{ij} \in \mathbb{Q}$ for every i, j, and we will prove shortly that $\phi_{ijk} \in \mathbb{Q}$ for every i, j, k. For now, we accept this fact and continue with the construction of q_h. Thus, let ϕ^{-1} be the least common denominator of $\{|\phi_{ijk}|\} \in \mathbb{N}$, so that $\phi_{ijk}\phi^{-1} \in \mathbb{Z}$.

For $N \in \mathbb{N}$, we define $h_N := \frac{\phi}{N}$, so that $\frac{\phi_{ijk}}{h_N} \in \mathbb{Z}$ for all i, j, k, and similarly $\frac{\phi_{ij}}{h_N} \in \mathbb{Z}$ for every i, j. We will prove the proposition for every h_N such that $h_N < \eta^2$; for arbitrary $h < \eta^2$, the conclusions of the proposition then hold if we define $q_h := q_{h_N}$, $\beta_h := \beta_{h_N}$, for N such that $h_N \leqq h < h_{N-1}$.

We henceforth fix an arbitrary N such that $h_N < \eta^2$, and we drop the subscript and write simply h.

We first discretize dp on every T_{ij}. In order to avoid discretizing any 2-face twice in inconsistent ways, we define

$$\mathcal{T} := \{T_{ij} \;:\; \phi_{ij} > 0 \text{ or } T_{ij} \subset \partial\Omega_\delta^P\}.$$

For $T_{ij} \in \mathcal{T}$, let $m = m_{ij} := \frac{\phi_{ij}}{h} \in \mathbb{Z}$, and let $\ell = \ell_{ij}$ verify $(\ell_{ij} - 1)^2 < m \leqq \ell_{ij}^2$. Now partition T_{ij} into ℓ_{ij}^2 closed triangular pieces $\{T_{ij}^a\}_{a=1}^{\ell^2}$ with pairwise disjoint interiors, each one isometric to $\ell_{ij}^{-1} T_{ij}$. Select m of these triangles, and let $\{s_{ij}^a\}_{a=1}^m$ be the barycentres of the chosen triangles.

If $T_{ij} \notin \mathcal{T}$, then $T_{ij} = -T_{i'j'}$ for some $T_{i'j'} \in \mathcal{T}$, we set $m = m_{ij} := m_{i'j'}$, and $s_{ij}^a = s_{i'j'}^a$ for $a = 1 \ldots m_{ij}$.

Next we consider $\{\tilde{T}_{ij}\}$. For i, j, k, let $\tilde{T}_{ijk} \equiv (1 - \eta) \cdot T_{ijk} + \eta \cdot b$ (with the orientation of T_{ijk}) and define

$$\tilde{\mathcal{T}} := \{\tilde{T}_{ijk} \;:\; \phi_{ijk} > 0\}.$$

Now proceed as above: for each $\tilde{T}_{ijk} \in \tilde{\mathcal{T}}$, let $m = m_{ijk} := \frac{\phi_{ijk}}{h} \in \mathbb{Z}$ and $\ell_{ijk} := \lceil \sqrt{m}\, \rceil$, and partition $\tilde{T}_{ijk}$ into ℓ_{ijk}^2 closed triangular pieces $\{\tilde{T}_{ijk}^a\}_{a=1}^{\ell_{ijk}^2}$ with pairwise disjoint interiors, each one isometric to $\ell_{ijk}^{-1}\tilde{T}_{ijk}$. Select m of these triangles, and let $\{\tilde{s}_{ijk}^a\}_{a=1}^m$ be the barycentres of the chosen triangles.

If $T_{ijk} \notin \tilde{\mathcal{T}}$, then $\phi_{ijk} \leqq 0$. If $\phi_{ijk} = 0$ (which in particular happens if $T_{ijk} = \emptyset$) we do nothing. If $\phi_{ijk} < 0$, then noting that our orientation conventions imply that $\phi_{ijk} = -\phi_{ikj}$, we see that $\tilde{T}_{ikj} \in \tilde{\mathcal{T}}$, and we define $\tilde{s}_{ijk}^a = \pi_i^{-1}\pi_i(\tilde{s}_{ikj}^a) \cap T_{ijk}$.

We now define piecewise linear curves as follows. First, for every $T_{ijk} \in \tilde{\mathcal{T}}$, we define

$$\tilde{\Gamma}_{ijk}^a := [\pi_i^{-1}(\pi(\tilde{s}_{ijk}^a))] \cap \tilde{S}_i, \quad \text{oriented so that } \partial\tilde{\Gamma}_{ijk}^a = \tilde{s}_{ijk}^a - \tilde{s}_{ikj}^a.$$

Here and below, if c is an oriented piecewise smooth curve, we write $\partial c = p - q$ to mean that $\int_c df = f(p) - f(q)$ whenever f is a smooth function. We define $\Gamma_i = \sum_{j,k,a} \tilde{\Gamma}^a_{ijk}$, so that $\Gamma_i \subset \tilde{S}_i$, and

$$\partial \Gamma_i = \sum_{j,k,a} \operatorname{sign}(\phi_{ijk}) \tilde{s}^a_{ijk}. \tag{3.9}$$

Moreover, let Γ_i be the collection of segments with the smallest total arclength satisfying this condition (as the segments of Γ_i are all parallel to each other).

Now for each i, j, let $P_{ij} := \{(1-\lambda)x + \lambda b_i : x \in T_{ij}, 0 < \lambda < \eta\}$ be the pyramidal frustum having bases T_{ij} and $\tilde{T}_{ij}$, and let Γ_{ij} be a collection of (oriented) line segments such that

$$\partial \Gamma_{ij} = \sum_a \operatorname{sign}(\phi_{ij}) s^a_{ij} - \sum_{k,a} \operatorname{sign}(\phi_{ijk}) \tilde{s}^a_{ijk}, \tag{3.10}$$

and that minimizes the total arclength among the set of all collections of line segments satisfying the constraint (3.10). Such collections exist, since $\operatorname{sign}(\phi_{ij}) = \operatorname{sign}(\phi_{ijk})$ and $m_{ij} = \sum_{k \neq j} m_{ijk}$, so that

$$\sum_a \operatorname{sign}(\phi_{ij}) - \sum_{k,a} \operatorname{sign}(\phi_{ijk}) = \operatorname{sign}(\phi_{ij}) m_{ij} - \sum_k \operatorname{sign}(\phi_{ijk}) m_{ijk} = 0.$$

Hence Γ_{ij} is well-defined, and clearly $\Gamma_{ij} \subset P_{ij}$.

We define Γ_h to be the union $\cup \Gamma_i \cup \Gamma_{ij}$ of the families of segments constructed above, and $n(h)$ to be the total number of segments comprising Γ_h. We also define Γ^ℓ_h, for $\ell = 1, \ldots, m(h)$, where $m(h) \leqq n(h)$, to be the polyhedral curves realizing the connected components of Γ_h. It follows from (3.11), proved below, that $\partial \Gamma^\ell_h = 0$ in Ω^P_δ.

Finally, we define the measure-valued 2-form q_h to satisfy statement (v).

In the following we will write "a region" to refer to either one of the $\tilde{S}_i$ or one of the P_{ij}. We remark that the definition of Γ_h states that, in the language of Brezis et al. [11], its restriction to any region is a *minimal connection*, subject to the condition (3.9) in $\tilde{S}_i$ and (3.10) in P_{ij}.

Proof that $\phi_{ijk} \in \mathbb{Q}$. Fix i, j, k, let $V_1, \ldots, V_4 \in \mathbb{Q}^3$ denote the vertices of S_i, anality) that $T_{ij} = \operatorname{co}\{V_1, V_2, V_3\}$, $T_{ik} = \operatorname{co}\{V_1, V_2, V_4\}$, where $\operatorname{co} A$ denotes the convex hull of A. Let $\tilde{V}_l := \pi(V_l)$ for $l = 1, \ldots, 4$, so that

$$\pi(T_{ij}) \cap \pi(T_{ik}) = \operatorname{co}\{\tilde{V}_1, \tilde{V}_2, \tilde{V}_3\} \cap \operatorname{co}\{\tilde{V}_1, \tilde{V}_2, \tilde{V}_4\}.$$

Clearly this set is a (possibly degenerate) triangle containing the segment co $\{\tilde{V}_1, \tilde{V}_2\}$. From elementary geometry we see that one of the following three cases must hold:

Case 1 $\pi(T_{ij}) \cap \pi(T_{ik}) = \operatorname{co}\{\tilde{V}_1, \tilde{V}_2\}$. Then $\phi_{ijk} = 0 \in \mathbb{Q}$.

Case 2 $\pi(T_{ij}) \subset \pi(T_{ik})$ or $\pi(T_{ik}) \subset \pi(T_{ij})$. Then $\phi_{ijk} = \phi_{ij}$ or ϕ_{ik}, so $\phi_{ijk} \in \mathbb{Q}$.

Case 3 $\pi(T_{ij}) \cap \pi(T_{ik}) = \mathrm{co}(\tilde{V}_1, \tilde{V}_2, Z)$, where, after possibly switching the labels on $\tilde{V}_1$ and $\tilde{V}_2$, $\{Z\} = \mathrm{co}\{\tilde{V}_1, \tilde{V}_3\} \cap \mathrm{co}\{\tilde{V}_2, \tilde{V}_4\}$. If this holds, then let $Z_{ij} := \pi^{-1}(Z) \cap T_{ij}$, and $Z_{ik} := \pi^{-1}(Z) \cap T_{ik}$. Then since $\pi(Z_{ij}) = Z = \pi(Z_{ik})$, there exist some numbers a_1, a_2, a_3 such that

$$V_1 + a_1(V_3 - V_1) = Z_{ij} = Z_{ik} + a_3 v_i = V_2 + a_2(V_4 - V_2) + a_3 v_i.$$

This is a system of three equations for (a_1, a_2, a_3) with rational coefficients, and moreover it is nondegenerate in the case we are considering, so $a_l \in \mathbb{Q}$ for $l = 1, \ldots, 3$. It follows that $Z_{ij} \in \mathbb{Q}^3$, and hence that $T_{ijk} = \mathrm{co}\{V_1, V_2, Z_{ij}\}$ has rational vertices. Then it follows as in the proof of Lemma 2 (see in particular (3.1)) that $\phi_{ijk} \in \mathbb{Q}$.

Proof of (i). By Lemma 12 and Corollary 1 in the Appendix, it suffices to check that $dq_h = 0$ in Ω_δ and that $\int_{(\partial\Omega)_i} (q_h)_\top = 0$ for every connected component $(\partial\Omega_\delta)_i$ of $\partial\Omega_\delta$.

To do this, fix any $f \in C_c^\infty(\mathbb{R}^3)$, and note that (v), (3.9), (3.10) imply that

$$\begin{aligned}\langle dq_h, \star f\rangle = \langle q_h, d^* \star f\rangle = \langle q_h, \star df\rangle &= h\sum_i \int_{\Gamma_i} df + \sum_{i,j}\int_{\Gamma_{ij}} df \\ &= h\sum_{i,j,a} (\mathrm{sign}\,\phi_{ij}) f(s_{ij}^a).\end{aligned}$$

Here all terms of the form $f(\tilde{s}_{ijk}^a)$ have cancelled, since they occur twice, with opposite signs, in (3.9) and (3.10). If $s_{ij}^a \in \Omega_\delta^P$, then our construction implies that there exists exactly one $(i', j', a') \neq (i, j, a)$ such that $s_{ij}^a = s_{i'j'}^{a'}$, and moreover that $\mathrm{sign}\,\phi_{ij} = -\,\mathrm{sign}\,\phi_{i'j'}$. Thus all contributions from Ω_δ^P vanish, and the above reduces to

$$\langle dq_h, \star f\rangle = h \sum_{\{i,j,a : s_{ij}^a \in \partial\Omega_\delta^P\}} (\mathrm{sign}\,\phi_{ij})\ f(s_{ij}^a). \tag{3.11}$$

In particular, by considering $f \in C_c^\infty(\Omega_\delta)$ we see that $dq_h = 0$ in Ω_δ.

Now fix some component $(\partial\Omega_\delta)_k$ of $\partial\Omega_\delta$. Then (3.6) implies that

$$0 = \int_{W_k} d1 = \int_{\partial W_k} 1 = \int_{(\partial\Omega_\delta^P)_k} (q_h)_\top - \int_{(\partial\Omega_\delta)_k} (q_h)_\top.$$

Moreover, it follows from (3.10), (3.11), and the definition of $(q_h)_\top$ (see (5.8) in the Appendix) that

$$\int_{(\partial\Omega_\delta^P)_k} (q_h)_\top = \sum_{(i,j) : T_{ij} \subset (\partial\Omega_\delta)_k} h(\mathrm{sign}\,\phi_{ij}) m_{ij}.$$

However, the definitions of m_{ij} and ϕ_{ij} imply that the above quantity equals

$$\sum_{(i,j):T_{ij}\subset(\partial\Omega_\delta^P)_k} \phi_{ij} = \sum_{(i,j):T_{ij}\subset(\partial\Omega_\delta^P)_k} \int_{T_{ij}} dp = \int_{(\partial\Omega_\delta^P)_k} dp = 0.$$

Then, as remarked above, (i) follows from Lemma 12 and Corollary 1.

Proof of (iii). We next estimate the mass of q_h. We will bound the mass on each region R, and then sum up the estimates. We begin by comparing the fluxes of q_h and dp across ∂R.

Lemma 4. *Let R be a region, and let $(dp)_\top$ and $(q_h)_\top$ be the tangential parts of dp and q_h, respectively, on ∂R, ie, the measures in $\mathbb{R}^3$, supported in ∂R, defined as discussed in the Appendix, see (5.8). Then there exists a constant $C = C(dp, \Omega_\delta^P)$, independent of η and h, such that*

$$||(q_h - dp)_\top||_{W^{-1,1}(\mathbb{R}^3)} \leqq C(\eta + h^{1/2}) \leqq C\eta. \tag{3.12}$$

Proof. First consider the case of a pyramidal frustrum P_{ij}. Then, arguing as in the proof of (i), we find from (3.10) that $(q_h)_\top = h\sum_a \operatorname{sign}(\phi_{ij}) \delta_{s^a_{ij}} - h\sum_{k,a} \operatorname{sign}(\phi_{ijk})\delta_{\tilde{s}^a_{ijk}}$. Similarly, the definition of ϕ_{ij} and the fact that T_{ij} and $\tilde{T}_{ij}$ are parallel implies that

$$\int_{\partial P_{ij}} f(dp)_\top = \frac{\phi_{ij}}{|T_{ij}|}\int_{T_{ij}} f\,\mathrm{d}\mathcal{H}^2 - \frac{\phi_{ij}}{|T_{ij}|}\int_{\tilde{T}_{ij}} f\,\mathrm{d}\mathcal{H}^2 + O(\|f\|_\infty \eta)$$

where the error term comes from neglecting $\partial P_{ij} \setminus (T_{ij} \cup \tilde{T}_{ij})$, which has an area bounded by $C\eta$.

Thus for any continuous f,

$$\begin{aligned}\int_{\partial P_{ij}} f(dp - q_h)_\top = &\left[\frac{\phi_{ij}}{|T_{ij}|}\int_{T_{ij}} f\,\mathrm{d}\mathcal{H}^2 - h\sum_a \operatorname{sign}(\phi_{ij}) f(s^a_{ij})\right] \\ &- \left[\frac{\phi_{ij}}{|T_{ij}|}\int_{\tilde{T}_{ij}} f\,\mathrm{d}\mathcal{H}^2 - h\sum_{a,k}\operatorname{sign}(\phi_{ij}) f(\tilde{s}^a_{ijk})\right] + O(\|f\|_\infty\eta).\end{aligned}$$

We will consider only the second term on the right-hand side (which is slightly harder). We assume for simplicity that $\phi_{ij} > 0$; the case $\phi_{ij} < 0$ is essentially identical. Noting that $\frac{\phi_{ij}}{|\tilde{T}_{ij}|} = \frac{\phi_{ijk}}{|\tilde{T}_{ijk}|}$ and that $|\tilde{T}^a_{ijk}| = \ell_{ijk}^{-2}|\tilde{T}_{ijk}|$, and using notation from the first step above, we have

$$\begin{aligned}\int_{\tilde{T}_{ij}} f(dp - q_h)_\top \;&=\; \frac{\phi_{ij}}{|T_{ij}|}\int_{\tilde{T}_{ij}} f\,\mathrm{d}\mathcal{H}^2 - h\sum_{a,k} f(\tilde{s}^a_{ijk}) \\ &= \left(\frac{\phi_{ij}}{|T_{ij}|} - \frac{\phi_{ij}}{|\tilde{T}_{ij}|}\right)\int_{\tilde{T}_{ij}} f\,\mathrm{d}\mathcal{H}^2 \;+\; \sum_{k,a}\frac{\phi_{ijk}}{|\tilde{T}_{ijk}|}\int_{\tilde{T}^a_{ijk}} f - f(\tilde{s}^a_{ijk})\mathrm{d}\mathcal{H}^2 \\ &\quad + \sum_{a,k}\left[\frac{|\phi_{ijk}|}{\ell^2_{ijk}} - h\right] f(\tilde{s}^a_{ijk}) + \sum_k \frac{\phi_{ij}}{|T_{ij}|}\sum_k \int_{\tilde{T}_{ijk}\setminus\cup_a \tilde{T}^a_{ijk}} f\,\mathcal{H}^2.\end{aligned} \tag{3.13}$$

It is clear from the definition of ϕ_{ij} that $|\phi_{ij}| \leqq \|dp\|_\infty |T_{ij}| \leqq C$, and since by definition $(\ell_{ijk} - 1)^2 < m_{ijk} = h^{-1}\phi_{ijk} \leqq \ell^2_{ijk}$,

$$\left|\frac{\phi_{ijk}}{\ell^2_{ijk}} - h\right| \leqq \frac{2}{m_{ijk}}\frac{\phi_{ijk}}{\ell_{ijk}} \leqq \frac{C}{m_{ijk}}(h\phi_{ijk})^{1/2} \leqq C\frac{\sqrt{h}}{m_{ijk}}.$$

Similarly, one checks that $|T_{ijk} \setminus \cup_a T^a_{ijk}| = |T_{ijk}||1 - \frac{m_{ijk}}{\ell^2_{ijk}}| \leqq C|T_{ijk}|\sqrt{h}$. Note also that $|f(x) - f(\tilde{s}^a_{ijk})| \leqq \|df\|_\infty \mathrm{diam}(\tilde{T}^a_{ijk}) \leqq C\|df\|_\infty\sqrt{h}$ for $x \in \tilde{T}^a_{ijk}$. Taking these into account, elementary calculations yield

$$\left| \int_{\tilde{T}_{ij}} f(dp - q_h)_\top \right| \leqq C(\eta + \sqrt{h})\|f\|_{W^{1,\infty}}.$$

Since similar computations apply to T_{ij}, we deduce that $|\int_{\partial P_{ij}} f(dp - q)_\top| \leqq C\eta\|f\|_{W^{1,\infty}}$ for every P_{ij}. If the region R is a simplex $\tilde{S}_i$, then $\int_{\partial S_i} f(dp - _h)_\top$ is a sum of terms of exactly the form $\int_{\tilde{T}_{ij}} f(dp - q_h)_\top$ already estimated (now with the opposite orientation) and so the conclusion follows in this case, as well. □

For future reference, we remark that the above proof shows that that

$$\int_{T_{ij}} f(dp - q_h)_\top \leqq C\sqrt{h}\|f\|_{W^{1,\infty}}, \quad \int_{\tilde{T}_{ij}} f\Big(\frac{dp}{(1-\eta)^2} - q_h\Big)_\top \\ \leqq C\sqrt{h}\|f\|_{W^{1,\infty}}. \tag{3.14}$$

Indeed, every term on the right-hand side of (3.13) can be bounded by $Ch^{1/2}$ except for the term $(\frac{\phi_{ij}}{|T_{ij}|} - \frac{\phi_{ij}}{|\tilde{T}_{ij}|}) \int_{\tilde{T}_{ij}} f \, \mathrm{d}\mathcal{H}^2$. This term is not present when one considers T_{ij} rather than $\tilde{T}_{ij}$, and it is also not present if one considers $\tilde{T}_{ij}$, but with dp replaced by $\frac{dp}{(1-\eta)^2}$, since $(1-\eta)^2 = |\tilde{T}_{ij}|/|T_{ij}|$. Thus (3.14) follows from our earlier arguments.

We will need the following result about continuous dependence of the minimal connection upon its boundary datum.

Lemma 5. *Let K be a compact convex domain in $\mathbb{R}^3$, ζ a measure supported on ∂K such that $\int_{\partial K} \zeta = 0$. Then we have*

$$\min\{||\alpha|| \equiv |\alpha|(K),\ d\alpha = 0 \text{ in } K,\ \alpha_\top = \zeta \text{ on } \partial K\} \leqq C\,||\zeta||_{W^{-1,1}(\mathbb{R}^3)}.$$

The proof of this lemma is postponed to Section 5.5 in the Appendix. Let us apply Lemma 5 first with $K = P_{ij}$, $\zeta = (q_h - dp)_\top$ and let α_h be the measure 2-form that realizes the minimum. By (3.12) and Lemma 5 we deduce $|\alpha_h|(P_{ij}) \leqq C\eta$.

As remarked above, the restriction of Γ_h to any region R is a minimal connection, and as a consequence, it follows from Theorems 5.3 an 5.4 in BREZIS ET AL. [11] that $q_h \llcorner R$ has minimal mass among all 2-form-valued measures q' in R such that $(q')_\top = (q_h)_\top$ on ∂R (not merely those corresponding to a union of oriented line segments). We thus have

$$|q_h|(P_{ij}) \leqq ||\alpha_h + dp|| \leqq |\alpha_h|(P_{ij}) + \int_{P_{ij}} |dp| \leqq \int_{P_{ij}} |dp| + C\eta. \tag{3.15}$$

Next, applying Lemma 5 with $K = \tilde{S}_i$, $\zeta = (q_h - dp)_\top$ and arguing exactly as above, we obtain

$$|q_h|(\tilde{S}_i) \leqq \int_{\tilde{S}_i} |dp| + C\eta. \tag{3.16}$$

Statement (iii) follows by summing over all regions.

Proof of (ii). It suffices to show that for every region R,

$$\langle \varphi, (dp - q_h) \llcorner R \rangle = \int_R (\varphi, dp) - \langle \varphi, q_h \llcorner R \rangle \leqq C\eta \|\varphi\|_{W^{1,\infty}} \tag{3.17}$$

for every $\varphi \in C_c^\infty(\Lambda^2 \mathbb{R}^3)$. This is clear if $R = P_{ij}$, since $|P_{ij}| \leqq C\eta$ for all i, j, so that $\|dp\|_{L^1(P_{ij})} \leqq C\eta$, and hence $|q_h|(P_{ij}) \leqq C\eta$ by (3.15).

If $R = \tilde{S}_i$, then we assume, after changing coordinates, that $dp = \lambda dx^2 \wedge dx^3$ on $\tilde{S}_i$ for some $\lambda \in \mathbb{R}$. Now fix $\varphi \in C_c^\infty(\Lambda^2 \mathbb{R}^3)$ and let $\Phi \in C_c^\infty(\mathbb{R}^3)$ be a function such that $(\star d\Phi, dx^2 \wedge dx^3) = (\varphi, dx^2 \wedge dx^3)$ in S_i, and such that $\|\Phi\|_{W^{1,\infty}} \leqq C\|\varphi\|_{W^{1,\infty}}$. Indeed, $(\star d\Phi, dx^2 \wedge dx^3) = \Phi_{x_1}$, so we can take

$$\Phi(x) := \chi(x) \int_{-\infty}^{x_1} \Big(\varphi(s, x_2, x_3), dx^2 \wedge dx^3 \Big)\, \mathrm{d}s$$

where $\chi \in C_c^\infty(\mathbb{R}^3)$ satisfies $\chi \equiv 1$ on S_i and $\|\nabla \chi\|_{L^\infty} \leqq 1$. Then clearly $\langle dp \llcorner \tilde{S}_i, \varphi \rangle = \langle dp \llcorner \tilde{S}_i, \star d\Phi \rangle$, and it follows from the form of dp and the definition (that is, statement (v)) of q_h that $\langle q_h \llcorner \tilde{S}_i, \varphi \rangle = \langle q_h \llcorner \tilde{S}_i, \star d\Phi \rangle$. Thus Lemma 4 implies that

$$\langle \varphi, (dp - q_h) \llcorner \tilde{S}_i \rangle = \langle \star d\Phi, (dp - q_h) \llcorner \tilde{S}_i \rangle = \int_{\partial \tilde{S}_i} \Phi (dp - q_h)_\top \leqq C\eta \|\varphi\|_{W^{1,\infty}}.$$

Thus $\|(dp - q_h) \llcorner S_i\|_{W^{-1,1}(\mathbb{R}^3)} \leqq C\eta$. $\square$

Proof of (iv). The estimate $\|d^*\beta_h\|_{L^p(\Omega_\delta)} \leqq C_p |q_h|(\Omega_\delta) \leqq C$, $1 \leqq p < 3/2$, follows immediately from Corollary 1 in the Appendix. Thus $d^*\beta_h$ is weakly precompact in these L^p spaces, and we need only to identify the limit, prove that it is unique, and estimate its L^2 distance from $d^*\beta$.

To do this we will show that $q_h \to q^\eta$ in $W^{-1,1}(\Omega_\delta)$, where $q^\eta = (1-\eta)^{-2} dp$ on $\tilde{S}_i$, while on P_{ij}, q^η is defined to be the unique minimizer of the problem

$$\min\{|\alpha|(P_{ij}),\ d\alpha = 0 \text{ in } P_{ij},\ \alpha_\top = \zeta \text{ on } \partial P_{ij}\}, \tag{3.18}$$

where $\zeta = (dp)_\top$ on T_{ij}, $\zeta = (1-\eta)^{-2}(dp)_\top$ on $\tilde{T}_{ij}$ and $\zeta = 0$ on the remaining faces of ∂P_{ij}. Since then $\beta^\eta = -\Delta^{-1} q^\eta$, the uniqueness of β^η will follow, and we will deduce the estimates of β^η from the explicit form of q^η, which we find below.

We consider first a truncated pyramidal region P_{ij}, which is the harder case. The uniform mass bounds (3.16) imply that $q_h \llcorner P_{ij}$ is precompact in $W^{-1,1}(\mathbb{R}^3)$. Let q denote a limit of a convergent subsequence. It follows from (3.14) that $(q_h)_\top$ on ∂P_{ij} converges to ζ as defined above, and hence that $q_\top = \zeta$ on ∂P_{ij}. Next, if q did not solve the minimization problem (3.18), we could use the estimate $\|(q_h)_\top - \zeta\|_{W^{-1,1}} \leqq C\sqrt{h}$ (which is (3.14)) together with Lemma 5 to create a sequence q_h' such that $(q_h')_\top = (q_h)_\top$, and with $|q_h'|(P_{ij}) < |q_h|(P_{ij})$ for all small enough h, contradicting the minimality of q_h. Thus $q = q^\eta$, a minimizer of (3.18).

We now argue that the unique minimizer (3.18) is given by

$$q^*(x) = a \frac{(x - b_i)_\ell}{((x - b_i) \cdot \nu_{ij})^3} \star dx^\ell, \tag{3.19}$$

where b_i denotes the barycentre of S_i, ν_{ij} is the unit normal to T_{ij}, and $a \in \mathbb{R}$ is adjusted so that $q^*_\top = \zeta$. (A calculation shows that such a number a exists and also that $dq^* = 0$.) The (unique) minimality of q^* now follows from a calibration argument. We briefly recall the idea: Let $f(x) = |x - b_i|$, so that $df = \sum \frac{(x-b_i)_\ell}{|x-b_i|} dx^\ell$, and $(\star df, q^*) = |q^*|$ in P_{ij}. For any other 2-form valued measure q' supported in P_{ij} such that $dq' = 0$ in P_{ij} and $q'_\top = \zeta$ on ∂P_{ij}, we have

$$|q^*|(P_{ij}) \;=\; \langle q^* \llcorner P_{ij}, \star df\rangle \;=\; \int_{\partial P_{ij}} f\zeta \;=\; \langle q', \star df\rangle \;\leqq\; |q'|(P_{ij}),$$

since $|\star df| \leqq 1$ everywhere. Hence q^* is a minimizer. Furthermore, if equality holds, then, heuristically, q' is parallel to $\star df$, or more precisely, q' has the form $\langle q', \psi\rangle = \int_{P_{ij}} (\frac{(x-b_i)_\ell \star dx^\ell}{|x-b_i|}, \psi) d\mu'$ for some measure μ'. Then one can check that q^* is the only measure-valued 2-form of this form such that $dq' = 0$ in P_{ij}, $q'_\top = \zeta$ on ∂P_{ij}. Hence $q^\eta = q^*$ as asserted.

The proof that $q_h \llcorner \tilde{S}_i$ converges in $W^{-1,1}$ to $(1-\eta)^{-2} dp \llcorner \tilde{S}_i$ can be carried out on exactly the same lines, except that the limit has a simpler form. It can also be proved by arguing as in the proof of (ii), but using (3.14) instead of (iii). Thus we have proved that $q_h \to q^\eta$ in $W^{-1,1}(\Omega^P_\delta)$.

From the explicit form of q^η, noting that $\sum_{i,j} |P_{ij}| \leqq C\eta$, we see that

$$||q^\eta - dp||^2_{L^2(\Omega^P_\delta)} \leqq C\eta. \tag{3.20}$$

Thus $\|d^*\beta^\eta - d^*\beta\|_2^2 = \|d^*\Delta_N^{-1}(q^\eta - dp)\|_2^2 \leqq C\eta$, by (3.20) and standard elliptic estimates. This concludes the proof of statement (iv).

Proof of (vi). We now prove the separation properties of the polyhedral curves Γ^ℓ_h. Let L_1 and L_2 be closed line segments of Γ_h, with endpoints $s_1^\pm$ and $s_2^\pm$, and assume that L_1 and L_2 are disjoint, so that in particular $\{s_1^\pm\} \cap \{s_2^\pm\} = \emptyset$.

If L_1, L_2 belongs to non-adjacent regions of the family $\{\tilde{S}_i, P_{ij}\}$ then the conclusion is obvious, so we assume that this is not the case, and we claim that

$$\operatorname{dist}(s_m^\pm, L_n) \geqq c_2 \eta h^{1/2} \qquad \text{for } m \neq n, m, n \in \{1, 2\}. \tag{3.21}$$

To see this, let F denote the face (some T_{ij} or $\tilde{T}_{ij}$) containing s_1^+ say. If F also contains an endpoint of L_2 (for example s_2^+), then by construction

$$L_2 \text{ forms an angle of at least } c\eta \text{ with } F, \tag{3.22}$$

by which we mean that $|t \cdot n| \geqq c\eta$, where t and n here denote a unit tangent to L_2 and a unit normal to F. Indeed, the set of angles between segments $\tilde{\Gamma}^a_{ijk}$ interior to a simplex $\tilde{S}_i$ and a face of that simplex are independent of η and h and are nonzero, and since there are only finitely many such angles, (3.22) holds for such segments if c is sufficiently small. All other segments connect two parallel faces of a pyramidal frustrum P_{ij}, and for these segments, (3.22) follows from the fact that these faces are, by construction, separated by a distance $c\eta$, for c independent of η, h.

Combining (3.22) with the fact that $|s_1^+ - s_2^+| \geqq ch^{1/2}$, we deduce (3.21) from elementary geometry in the case when F contains an endpoint of L_2. The claim (3.21) is still clearer if neither endpoint of L_2 is contained in F.

It is evident that (3.21) implies (vi) if L_1 and L_2 belong to distinct but adjacent regions. If L_1 and L_2 belong to the same region, then in view of the minimality property of q_h, we obtain statement (vi) from (3.21) and the following Lemma:

Lemma 6. *Let $\{s_m^\pm\}_{m=1,2}$ satisfy $|s_1^+ - s_1^-| + |s_2^+ - s_2^-| \leqq |s_1^+ - s_2^-| + |s_2^+ - s_1^-|$. Also, let L_m be the segment joining s_m^+ and s_m^-, for $m = 1, 2$. Then*

$$\operatorname{dist}(L_1, L_2) \geqq \frac{1}{\sqrt{2}} \min_{m \neq n} \operatorname{dist}(s_m^\pm, L_n). \tag{3.23}$$

Proof. Let $Q_m \in L_m, m = 1, 2$ be such that $\operatorname{dist}(L_1, L_2) = |Q_1 - Q_2| = d$. If either Q_m is an endpoint, then the conclusion is clear, so we assume that both are interior points, in which case the segment from Q_1 to Q_2 is orthogonal to both L_1, L_2. We may then assume without loss of generality that the midpoint $\frac{Q_1+Q_2}{2}$ is the origin, and that $Q_1 = (0, 0, \frac{d}{2}),\ Q_2 = (0, 0, -\frac{d}{2})$, and, moreover, that L_1 and L_2 are parallel to the directions $(\cos\theta, \sin\theta, 0)$, $(\cos\theta, -\sin\theta, 0)$, respectively, for some θ. Define $\tilde{s}_1^\pm = (\pm\lambda\cos\theta, \pm\lambda\sin\theta, \frac{d}{2})$, $\tilde{s}_2^\pm = (\pm\lambda\cos\theta, \mp\lambda\sin\theta, -\frac{d}{2})$, for $\lambda > 0$, chosen so that one of the $\tilde{s}_m^\pm$ coincides with the closest point to 0 among the original endpoints.

Our hypothesis and the triangle inequality imply that $|\tilde{s}_1^+ - \tilde{s}_2^-| + |\tilde{s}_2^+ - \tilde{s}_1^-| \geqq |\tilde{s}_1^+ - \tilde{s}_1^-| + |\tilde{s}_2^+ - \tilde{s}_2^-|$, which reduces to

$$2\sqrt{4\lambda^2\cos^2\theta + d^2} \geqq 4\lambda = 2\sqrt{4\lambda^2(\cos^2\theta + \sin^2\theta)}, \qquad \text{so that } d^2 \geqq 4\lambda\sin^2\theta.$$

On the other hand, assuming for concreteness that $\tilde{s}_1^+$ agrees with the original endpoint s_1^+, then since $\tilde{s}_2^+ \in L_2$, we use the above inequality to find that we

$$\operatorname{dist}(s_1^+, L_2) \leqq |\tilde{s}_1^+ - \tilde{s}_2^+| \ = \ \sqrt{4\lambda^2\sin^2\theta + d^2} \leqq \sqrt{2}d.$$

□

Proof of (vii). Finally, suppose that L_1 and L_2 are adjacent, and that L_1 precedes L_2 in the ordering induced by their respective orienting unit tangents $\tau_1,\ \tau_2$. Decompose τ_i as $\tau_i^\perp + \tau_i^\parallel$, where for $i = 1, 2,\ \tau_i^\perp$ is orthogonal to the face T_{ij} that contains the common endpoint of L_1 and L_2. The orientation conventions imply that $\tau_1^\perp \cdot \tau_2^\perp > 0$, and, as noted above, each segment forms an angle of at least $c\eta$ with T_{ij}, which implies that $|\tau_i^\perp| \geqq c\eta$ for $i = 1, 2$. Statement (vii) follows directly.

The proof of Proposition 2 is now complete. □

3.5. Pointwise Estimates for $d^*\beta_h$

Let $G(x) = (4\pi)^{-1}|x|^{-1}$ be the Poisson kernel in $\mathbb{R}^3$. We may write

$$d^*\beta_h = d^*(G * q_h) + \Psi_h \qquad \Psi_h = d^*(-\Delta_N^{-1} q_h - G * q_h). \tag{3.24}$$

In view of statement (i), we deduce that $d\Psi_h = d^*\Psi_h = 0$ in Ω_δ, that is, $-\Delta\Psi = 0$ in Ω_δ and $\Psi_N = -d^*(G * q_h)_N$ on $\partial\Omega_\delta$. From the decomposition (3.24) we will deduce pointwise and integral estimates for $d^*\beta_h$.

We begin with the term $d^*(G * q_h) = G * d^* q_h$. The integral representation of $d^*(G * q_h)$ through the Biot-Savart law takes the form

$$d^*(G * q_h)(x) = h \sum_{\ell=1}^{m(h)} \sum_{i,j,k=1}^{3} \frac{1}{4\pi} dx^i \epsilon_{ijk} \int_{\Gamma_h^\ell} \frac{(x_j - y_j)dy^k}{|x-y|^3}, \tag{3.25}$$

where ϵ_{ijk} is the usual totally antisymmetric tensor. This can be justified, for example, by noting that $\langle d^*(G * q_h), \varphi\rangle = \langle q_h, G * d\varphi\rangle$, since G is even, and then using statement (v) of Proposition 2 to explicitly write out the right-hand side. From (3.25) we readily deduce

Lemma 7. *Let $l_1, l_2 > 0$, $L = \{(0,0,z),\ -l_1 \leqq z \leqq l_2\} \subset \mathbb{R}^3$, q the associated measure 2-form, that is $\langle q, \varphi\rangle = \int_L \star\varphi$ for $\varphi \in C^0(\Lambda^2\mathbb{R}^3)$. Then*

$$d^*(G * q) = \frac{x dy - y dx}{4\pi(x^2+y^2)} \left(\frac{l_2 - z}{\sqrt{x^2+y^2+(l_2-z)^2}} + \frac{l_1 + z}{\sqrt{x^2+y^2+(l_1+z)^2}} \right). \tag{3.26}$$

As a result,

$$|d^*(G * q)(p_0)| \leqq \frac{1}{2\pi \cdot \mathrm{dist}\,(p_0, L)} \qquad \text{for every } p_0 \in \mathbb{R}^3. \tag{3.27}$$

Proof. We obtain (3.26) by particularizing (3.25) to the case $\Gamma_h = L$. We easily deduce (3.27) from (3.26) if $p_0 = (x_0, y_0, z_0)$ with $-l_1 \leqq z_0 \leqq l_2$, in which case $\mathrm{dist}(p_0, L) = \sqrt{x_0^2 + y_0^2}$. If $z_0 > l_2$ then, writing $r_0 = (x_0^2 + y_0^2)^{1/2}$, since $\lambda \mapsto \frac{\lambda}{\sqrt{r_0^2+\lambda^2}} < 1$ is an increasing function and $0 < z_0 - l_2 < z_0 + l_1$, we find from (3.26) that

$$\begin{aligned} |d^*(G * q)(p_0)| &\leqq \frac{1}{4\pi r_0}\left(1 - \frac{z_0 - l_2}{\sqrt{r_0^2 + (l_2 - z_0)^2}}\right) \\ &= \left(\frac{\sqrt{r_0^2 + (l_2 - z_0)^2} - (z_0 - l_2)}{r_0}\right)\left(\frac{1}{4\pi\,\mathrm{dist}(p_0, L)}\right), \end{aligned}$$

and (3.27) follows, since $\sqrt{a^2+b^2} \leqq a + b$ for $a, b \geqq 0$. The same reasoning of course holds if $z_0 < -l_1$. $\square$

Lemma 8. *Let $x \in \Omega_\delta$ be such that* $\mathrm{dist}\,(x, \Gamma_h) \leqq \frac{c_0}{2}\eta h^{1/2}$, *where $c_0 > 0$ is defined in statement* (vi) *of Proposition* 2. *Then there exists a constant $K > 0$ independent of η, h such that if $\eta \leqq 1$, then*

$$|d^*\beta_h(x)| \leqq \frac{h}{2\pi \cdot \mathrm{dist}\,(x, \Gamma_h)} + \frac{K}{\eta^2} \quad \text{if} \quad \mathrm{dist}(x, \cup_{i,j}\partial\tilde{S}_i \cup \partial P_{ij}) \geqq \frac{c_0}{2}\eta h^{1/2}, \tag{3.28}$$

$$|d^*\beta_h(x)| \leqq \frac{h}{\pi \cdot \mathrm{dist}\,(x, \Gamma_h)} + \frac{K}{\eta^2} \quad \text{if} \quad \mathrm{dist}(x, \cup_{i,j}\partial\tilde{S}_i \cup \partial P_{ij}) < \frac{c_0}{2}\eta h^{1/2}. \tag{3.29}$$

Proof. The definition (3.24) of Ψ_h implies that for any measure-valued 2-form q,

$$|d^*\beta_h| \leqq |d^*(G * q)| + |d^*(G * q_h - G * q)| + |\Psi_h|. \tag{3.30}$$

Fix $x \in \Omega_\delta \setminus \Gamma_h$ and let $r = \frac{c_0}{2}\eta h^{1/2}$. Define a measure-valued 2-form by $\langle q, \varphi\rangle = h\sum_{\{s: B_r(x)\cap L_h^s \neq \emptyset\}} \int_{L_h^s} \star\varphi$, where $\{L_h^s\}$ is the collection of line segments whose union gives Γ_h, see Proposition 2 (v). By Proposition 2 (vi), there is at most one term in the sum that defines q if $\mathrm{dist}(x, \cup_{i,j}\partial P_{ji} \cup \partial\tilde{S}_i) \geqq r$, and otherwise at most two terms. Then $|d^*(G * q)|$ is estimated via Lemma 7 to give the first term on the right-hand sides of (3.28) and (3.29), respectively, and we must show that the other two terms in (3.30) can be bounded by K/η^2.

Interior regularity for harmonic functions, together with Proposition 2, statements (iii) and (iv) allow us to fix some $q \in (1, 3/2)$ and argue as follows:

$$\begin{aligned} ||\Psi_h||_{L^\infty(\Omega)} &\leqq C||\Psi_h||_{W^{2,2}(\Omega)} \\ &\leqq C||\Psi_h||_{L^q(\Omega_\delta)} \\ &= C||d^*\beta_h - d^*(G * q_h)||_{L^q(\Omega_\delta)} \\ &\leqq C(1 + C\eta)||dp||_{L^1(\Omega_\delta)} \leqq C. \end{aligned} \tag{3.31}$$

To estimate the remaining term in (3.30), observe that

$$\begin{aligned} |d^*(G * q_h - G * q)(x)| &\leqq \frac{6}{4\pi}h\sum_{k=1}^{3}\sum_{\ell=1}^{m(h)}\int_{\Gamma_h^\ell \cap B_r(x)^c}\frac{dy^k}{|x-y|^2} \\ &\leqq C\sum_{k=1}^{3}\int_{-M}^{M}\left(\sum_{\ell'=1}^{m'(h)}\frac{h}{|x-y_{\ell'}^t|^2}\right)\mathrm{d}t \end{aligned} \tag{3.32}$$

where $M > 0$ is such that $\Omega_\delta \subset B_M(0)$ and $\{y_{\ell'}^t\}_{\ell'} = \cup_\ell \Gamma_h^\ell \cap \{y_k = t, |y-x| > r\}$, for $|t| \leqq M$. For every k and t ,

$$\sum_{\ell'=1}^{m'(h)}\frac{h}{|x-y_{\ell'}^t|^2} \leqq \textstyle\sum_{j=1}^{M/r}\frac{h}{r^2j^2}\,\#\{\ell' : jr \leqq |x - y_{\ell'}^t| < (j+1)r\}.$$

Consider the collection of (two dimensional) balls

$$\{z : z^k = t, |z - y_{\ell'}^t| < r\}, \quad \text{for} \quad y_{\ell'}^t \quad \text{such that} \quad jr \leqq |x - y_{\ell'}^t| < (j+1)r.$$

These balls are pairwise disjoint by Proposition 2 (vi), and are contained in the annulus $\{z : z^k = t, (j-1)r \leqq |x - z| < (j+2)r\}$, which has area $(6j+3)\pi r^2$. Thus

$\#\{\ell' : jr \leqq |x-y^t_{\ell'}| < (j+1)r\} \leqq 6j+3$ for all j. In addition, if we write x^t for the projection of x onto the plane $\{z^k = t\}$, then $\#\{\ell' : jr \leqq |x-y^t_{\ell'}| < (j+1)r\} = 0$ if $(j+1)r < |x-x^t|$. Then elementary estimates lead to the conclusion

$$\sum_{\ell'=1}^{m'(h)} \frac{h}{|x-y^t_{\ell'}|^2} \leqq C\frac{h}{r^2}\log\Big(\frac{M}{|x-x^t|}\Big).$$

Substituting this into (3.32), we see that $|d^*(G*q_h - G*q)(x)| \leqq C\frac{h}{r^2} = C(c_0\eta)^{-2}$, completing the proof of the lemma. □

The next lemma shows that we get uniform estimates of certain quantities if we mollify on a scale comparable to the minimum distance between the discretized vortex lines.

Lemma 9. *Let $0 < \mu < 1$ and $r = \mu c_0 \eta h^{1/2}$, for c_0 as in statement* (vi) *of Proposition 2. Then there exists a nonnegative radial function ϕ supported in the unit ball, with $\int\phi = 1$, and such that, in addition, $\phi_r(x) := r^{-3}\phi(x/r)$ satisfies*

$$||\phi_r * d^*\beta_h||_{W^{1,p}(\Lambda^1\Omega)} \leqq K \tag{3.33}$$

for any $p < \infty$, where $K = K(\mu, \eta, \|\phi\|_\infty, p)$ is independent of h.

Proof. First, let ψ be any radial mollifier with support in the unit ball, such that $\psi \geqq 0$ and $\int\psi = 1$, and let $\psi_r(x) := r^{-3}\psi(x/r)$. Then for $x \in \Omega_\delta$, in view of statement (vi) of Proposition 2, either $B_r(x)\cap\Gamma_h = \emptyset$ or $B_r(x)\cap\Gamma_h = B_r(x)\cap\{L_1\}$, or $B_r(x)\cap\Gamma_h = B_r(x)\cap\{L_1, L_2\}$, where L_i are segments of Γ_h. Hence we have

$$|\psi_r * q_h(x)| \leqq r^{-3}||\psi||_\infty \sum_i h|L_i \cap B_r(x)| \leqq 4hr^{-2}||\psi||_\infty \leqq \frac{4}{(c_0\mu\eta)^2}||\psi||_\infty. \tag{3.34}$$

Now fix open sets $\Omega = \Omega_3 \Subset \Omega_2 \Subset \Omega_1 \Subset \Omega_0 = \Omega_\delta$ and functions χ_m for $m = 1, 2, 3$ such that $\chi_m \in C^\infty_c(\Omega_{m-1})$ and $\chi_m \equiv 1$ on an open neighborhood of $\bar\Omega_m$. Fix a mollifier ψ^1 as above, but such that $\mathrm{spt}(\psi^1) \subset B_{1/3}$, and define $\psi^2 = \psi^1 * \psi^1$ and $\psi^3 = \psi^1 * \psi^2$. Thus ψ^m is radial with support in B_1 for $m = 1, 2, 3$, so that (3.34) applies to ψ^m_r. Now write $\zeta_0 = d^*\beta$, and for $m = 1, 2, 3$ define $\zeta_m = \psi^1_r * (\chi_m\zeta_{m-1})$.

If h, and thus r, is small enough (which we will henceforth take to be the case), then

$$\zeta_m = \psi^1_r * \zeta_{m-1} = \psi^m_r * d^*\beta \mathrm{on}\Omega_m, \text{ and } \zeta_m \text{ has support in } \Omega_{m-1}. \tag{3.35}$$

We claim that

$$\begin{aligned} \|d\zeta_m\|_{L^p(\Omega_{m-1})} &\leqq C_m\|\zeta_{m-1}\|_{L^p(\Omega_{m-1})} + C(p, \mu, \psi^1, \Omega_\delta)\,, \\ \|d^*\zeta_m\|_{L^p(\Omega_{m-1})} &\leqq C_m\|\zeta_{m-1}\|_{L^p(\Omega_{m-1})}. \end{aligned} \tag{3.36}$$

To see these, note first that $d\zeta_m = \psi_r^1 * (d\chi_m \wedge \zeta_{m-1}) + \psi_r^1 * (\chi_m d\zeta_{m-1})$. Then Jensen's inequality implies that

$$\|\psi_1 * (d\chi_m \wedge \zeta_{m-1})\|_{L^p(\Omega_{m-1})} \leqq \|d\chi_m \wedge \zeta_{m-1}\|_{L^p(\Omega_{m-1})} \leqq C_m\|\zeta_{m-1}\|_{L^p(\Omega_{m-1})}.$$

We estimate $\psi_r^1 * (\chi_m d\zeta_{m-1})$ first in the case $m = 1$, when it follows from statement (i) of Proposition 2 that $\psi_r^1 * (\chi_1 d\zeta_0) = \psi_r^1 * (\chi_1 q_h)$. Then arguing as in (3.34) we find that for any $p < \infty$,

$$\|\psi_r^1 * (\chi_1 q_h)\|_{L^p(\Omega)} \leqq C(p, \Omega_\delta)\|\psi_r^1 * (\chi_1 q_h)\|_{L^\infty(\Omega)} \leqq C(p, \psi^1, \Omega_\delta)(c_0\mu\eta)^{-2},$$

proving the first part of (3.36) for $m = 1$. For $m = 2, 3$,

$$\|\psi_r^1 * (\chi_m d\zeta_{m-1})\|_{L^p(\Omega_{m-1})} \leqq \|d\zeta_{m-1}\|_{L^p(\Omega_{m-1})} \overset{(3.35)}{=} \|\psi_{m-1} * q_h\|_{L^p(\Omega_{m-1})},$$

and we conclude (3.36) much as in the case $m = 1$. The second claim of (3.36) is similar but easier, since (3.35) implies that $d^*\zeta_m = \psi_r^1 * [\star d\chi_m \wedge \star\zeta_{m-1}]$, so that $\|d^*\zeta_m\|_p \leqq \||d\chi_m|\,|\zeta_{m-1}|\|_{L^p(\Omega_{m-1})} \leqq C_m\|\zeta_{m-1}\|_{L^p(\Omega_{m-1})}$.

Now recall the Gaffney–Gårding inequality

$$\|\zeta\|_{W^{1,p}(U)} \leqq C_p(U)\left(\|\zeta\|_{L^p(U)} + \|d\zeta\|_{L^p(U)} + \|d^*\zeta\|_{L^p(U)}\right), \quad 1 < p < +\infty, \tag{3.37}$$

valid for a differential form ζ with compact support in $U \subset \mathbb{R}^n$. Applying this to ζ_m, taking into account (3.36) and noting that $\|\zeta_m\|_{L^p} \leqq \|\zeta_{m-1}\|_{L^p}$, we find that

$$\|\zeta_m\|_{W^{1,p}(\Omega_{m-1})} \leqq C\|\zeta_{m-1}\|_{L^p(\Omega_{m-1})} + C. \tag{3.38}$$

Recall that Proposition 2, statement (iv), provides uniform estimates of $\zeta_0 = d^*\beta$ in $L^p(\Omega_0)$ for every $p < 3/2$, so (3.38) implies uniform estimates of $\|\zeta_1\|_{W^{1,p}(\Omega_0)}$ for every $p < 3/2$, and hence of $\|\zeta_1\|_{L^p(\Omega_0)}$ for ever $p < 3$. Iterating this argument twice more and recalling (3.35), we find that (3.33) holds with $\phi = \psi^3$. □

3.6. Construction of the Sequence u_ϵ in Case $g_\epsilon \geqq |\log\epsilon|^2$

Assume that the sequence g_ϵ satisfies either $g_\epsilon = |\log\epsilon|^2$ or $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$. Suppose that we are given $(J, v) \in \mathcal{A}_0$ as defined in (1.3), and moreover that $J = \frac{1}{2}dv$ if $g_\epsilon = |\log\epsilon|^2$, and that $J = 0$ if $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$.

Set $p = \frac{1}{2\pi}v$. Fix $\delta > 0$ and let p_δ be the piecewise linear approximation provided by Lemma 3, and recall the Hodge decomposition $p_\delta = \gamma + d\alpha + d^*\beta$ in Ω_δ introduced in Section 3.3. Fix $\eta > 0$, and $h = h_\epsilon = (g_\epsilon)^{-1/2}$, and let q_h be the discretized vorticity, with support Γ_h, and $\beta_h = -\Delta_N^{-1}q_h$ the approximation to β constructed in Proposition 2.

As we discuss in Remark 22, if c is any cycle in $\Omega_\delta \setminus \Gamma_h$, then $h^{-1}\int_c d^*\beta_h$ is an integer for every h. Thus, if we fix $\bar{x} \in \Omega$ and let $c_{\bar{x},x}$ denote a path in $\Omega_\delta \setminus \Gamma_h$ from $\bar{x}$ to x, it follows that

$$\phi_h(x) := \frac{1}{h}\int_{c_{\bar{x},x}} d^*\beta_h \quad \text{is well-defined function } \Omega_\delta \setminus \Gamma_h \to \mathbb{R}/\mathbb{Z}, \tag{3.39}$$

independent of the choice of $c_{\bar{x},x}$, and is hence well-defined almost everywhere in Ω.

Moreover, according to Lemma 11, we may write $\gamma = \sum_{j=1}^{\kappa} a_j \cdot d\phi_j$, where ϕ_j is well-defined in $\mathbb{R}/\mathbb{Z}$ for $j = 1, \dots, \kappa$. For any j let $n_j = [h^{-1}a_j] \in \mathbb{Z}$ be the integer part of $h^{-1}a_j$, and consider $h^{-1}\gamma_h \equiv d\psi_h = \sum_{j=1}^{\kappa} n_j d\phi_j$, so that ψ_h is well-defined in $\mathbb{R}/\mathbb{Z}$. Let finally $\alpha_h = h^{-1}\alpha$. The map

$$v_h = \exp(i2\pi(\phi_h + \psi_h + \alpha_h)) \tag{3.40}$$

is thus a well-defined map $\Omega_\delta \to S^1$, with

$$jv_h = 2\pi(d\phi_h + d\psi_h + d\alpha_h) = \frac{2\pi}{h}(d^*\beta_h + \gamma_h + d\alpha), \tag{3.41}$$

and $Jv_h = \frac{\pi}{h} dd^*\beta_h = \frac{\pi}{h} \cdot q_h$. Now let

$$\rho_\epsilon(x) \equiv \rho_{\epsilon,h}(x) = \min\left\{\frac{\operatorname{dist}(x,\Gamma_h)}{\epsilon},\ 1\right\}, \tag{3.42}$$

for Γ_h as in Proposition 2, statement (v) and set, finally,

$$u_\epsilon \equiv u_{\epsilon,h} = \rho_\epsilon \cdot v_h. \tag{3.43}$$

3.7. *Completion of Proof of* (1.8) *in Case* $g_\epsilon \geqq |\log \epsilon|^2$

We first claim that

$$\frac{ju_\epsilon}{\sqrt{g_\epsilon}} \rightharpoonup 2\pi(d\alpha + d^*\beta^\eta + \gamma) \quad \text{weakly in } L^q \text{ for every } q \in (1, 3/2). \tag{3.44}$$

for β^η as in statement (iv) of Proposition 2. To see this, we write

$$\frac{ju_\epsilon}{\sqrt{g_\epsilon}} = 2\pi(d^*\beta_h + \gamma_h + \alpha) + 2\pi(\rho_\epsilon^2 - 1)(d^*\beta_h + \gamma_h + d\alpha). \tag{3.45}$$

It is clear from the definition of γ_h that $\gamma_h \to \gamma$ uniformly as ϵ (and thus h) tend to 0, and we know from Proposition 2 that $d^*\beta_h \rightharpoonup d^*\beta^\eta$ in the relevant L^q spaces. So we need to show only that the last term in (3.45) vanishes. For this, we use statements (vi), (v), and (iii) of Proposition 2 to see that

$$|\{\operatorname{dist}(x,\Gamma_h) \leqq \epsilon\}| \leqq C\epsilon^2|\Gamma_h| = C\frac{\epsilon^2}{h}|q_h|(\Omega_\delta) \leqq C\frac{\epsilon^2}{h}. \tag{3.46}$$

It easily follows from this and from the definition of ρ_ϵ that $(\rho_\epsilon^2 - 1) \to 0$ in L^r for every $r < \infty$. Thus, fixing $q \in (1, 3/2)$ and r such that $\frac{1}{q} + \frac{1}{r} = 1$, in view of uniform estimates of $\|d^*\beta_h\|_q$ in Proposition 2 (iv), we find from Hölder's inequality that $(\rho_\epsilon^2 - 1)(d^*\beta_h + \gamma_h + d\alpha) \to 0$ in L^1 as $\epsilon \to 0$, proving (3.44).

We now turn to the proof of the upper bound. Since $h = g_\epsilon^{-1/2}$, we have

$$\frac{E_\epsilon(u_\epsilon;\Omega)}{g_\epsilon} = \frac{h^2}{2}\int_\Omega |\nabla\rho_\epsilon|^2 + \rho_\epsilon^2|jv_h|^2 + \frac{W(\rho_\epsilon)}{\epsilon^2}. \tag{3.47}$$

Let us estimate the various terms contributing to $g_\epsilon{}^{-1}E_\epsilon(u_\epsilon;\Omega)$. First note that

$$\frac{h^2}{2}\int_\Omega |\nabla\rho_\epsilon|^2 + \frac{W(\rho_\epsilon)}{\epsilon^2} \leqq \frac{Ch^2}{\epsilon^2}|\{\mathrm{dist}\,(x,\Gamma_h)\leqq\epsilon\}|$$

for $C=\frac{1}{2}(1+\|W\|_{L^\infty(B_1)})$. It follows from this and (3.46) that

$$\frac{h^2}{2}\int_\Omega |\nabla\rho_\epsilon|^2 + \frac{W(\rho_\epsilon)}{\epsilon^2} \ \leqq\ Ch. \tag{3.48}$$

Moreover,

$$\frac{h^2}{2}\int_\Omega \rho_\epsilon^2|jv_h|^2 = 2\pi^2\int_\Omega \rho_\epsilon^2(|d^*\beta_h|^2+|d\alpha+\gamma_h|^2+2\,d^*\beta_h\cdot(d\alpha+\gamma_h)). \tag{3.49}$$

We have just shown in the proof of (3.44) that $\rho_\epsilon^2(d\alpha+\gamma_h)\to d\alpha+\gamma$ in $L^p\ \forall\, p<+\infty$ and that $d^*\beta_h \rightharpoonup d^*\beta^\eta$ weakly in $L^q\ \forall\, q<3/2$. Thus, recalling the estimate $\|d^*\beta^\eta-d^*\beta\|_2^2\leqq C\eta$ from statement (iv) in Proposition 2, we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega \rho_\epsilon^2(d\alpha+\gamma_h)\cdot d^*\beta_h &= \int_\Omega d^*\beta^\eta\cdot(d\alpha+\gamma)\\ &= C\sqrt{\eta}+\int_\Omega d^*\beta\cdot(d\alpha+\gamma),\end{aligned} \tag{3.50}$$

$$\lim_{\epsilon\to 0}\int_\Omega \rho_\epsilon^2|d\alpha+\gamma_h|^2 \leqq \int_{\Omega_\delta}|d\alpha+\gamma|^2. \tag{3.51}$$

For the remaining term, fix $0<\mu<1$ and set $r=c_0\mu\eta h^{1/2}$. Denote $G_h^\lambda=\{\mathrm{dist}\,(x,\Gamma_h)\leqq\lambda\}\cap\Omega$. We have

$$2\pi^2\int_{\mathbb{R}^3}\rho_\epsilon^2|d^*\beta_h|^2 = A_\epsilon+B_\epsilon+C_\epsilon, \tag{3.52}$$

where

$$\begin{aligned}A_\epsilon &= 2\pi^2\int_{G_h^\epsilon}\rho_\epsilon^2|d^*\beta_h|^2,\quad B_\epsilon = 2\pi^2\int_{G_h^r\setminus G_h^\epsilon}|d^*\beta_h|^2,\\ C_\epsilon &= 2\pi^2\int_{\Omega\setminus G_h^r}|d^*\beta_h|^2.\end{aligned} \tag{3.53}$$

Let us estimate A_ϵ. By (3.28), (3.29), and (3.42), $\rho_\epsilon^2|d^*\beta_h|^2\leqq\frac{h^2}{\epsilon^2}+\frac{2K^2}{\eta^4}$ in G_h^ϵ, so (3.46) implies that

$$A_\epsilon \leqq |G_h^\epsilon|\Big(\frac{h^2}{\epsilon^2}+\frac{2K^2}{\eta^4}\Big)\leqq C(h+K\frac{\epsilon^2}{\eta^4h}) \tag{3.54}$$

so that, since $h=g_\epsilon^{-1/2}$ and $|\log\epsilon|^2\leqq g_\epsilon\ll\epsilon^{-2}$, we have

$$\limsup_{\epsilon\to 0} A_\epsilon = 0. \tag{3.55}$$

Let us turn to C_ϵ. Let ϕ_r be the radial mollifier found in Lemma 9. Observe that $d^*\beta_h$ is harmonic on $\Omega \setminus G_h^r$, and hence coincides there with $\phi_r * d^*\beta_h$, by the mean-value property of harmonic functions. By (3.33) and Rellich's Theorem we deduce that $\phi_r * d^*\beta_h$ is strongly compact in $L^2(\Omega)$, and hence by Proposition 2, statement (iv) that $\phi_r * d^*\beta_h \to d^*\beta^\eta$ in $L^2(\Omega)$ as $\epsilon \to 0$. We deduce that

$$\begin{aligned}\limsup_{\epsilon\to 0} C_\epsilon &= \limsup_{\epsilon\to 0} 2\pi^2 \int_{\Omega\setminus G_h^r} |\phi_r * d^*\beta_h|^2 \leqq \lim_{\epsilon\to 0} 2\pi^2 \int_\Omega |\phi_r * d^*\beta_h|^2 \\ &= 2\pi^2 \int_\Omega |d^*\beta^\eta|^2 \qquad (3.56) \\ &\leqq 2\pi^2 \int_\Omega |d^*\beta|^2 + C\eta.\end{aligned}$$

To estimate B_ϵ we proceed as follows: let $V_1 = (G_h^r \setminus G_h^\epsilon) \setminus U_{r_0}$, where $U_{r_0} = \{\mathrm{dist}\,(x, \cup_{i,j}\partial\tilde{S}_i \cup \partial P_{ij}) < r_0\} \cap \Omega$ and $r_0 = \frac{c_0}{2}\eta h^{1/2}$, and set $V_2 = (G_h^r \setminus G_h^\epsilon) \cap U_{r_0}$. For any $\sigma > 0$ we have, using for $d^*\beta_h$ the bound (3.28) on V_1 and (3.29) on V_2,

$$\begin{aligned}2\pi^2 \int_{V_1} |d^*\beta_h|^2 &\leqq (1+\sigma)\frac{h^2}{2}\int_{V_1} \frac{\mathrm{d}x}{|\mathrm{dist}\,(x,\Gamma_h)|^2} + (1+\frac{1}{\sigma})\frac{2\pi^2K^2}{\eta^4}|V_1| \\ &\leqq (1+\sigma)h^2\pi \log\left(\frac{r}{\epsilon}\right)|\Gamma_h \setminus U_{r_0}| + (1+\frac{1}{\sigma})\frac{C\mu^2}{\eta^2}h|\Gamma_h \setminus U_{r_0}|, \qquad (3.57)\\ 2\pi^2 \int_{V_2} |d^*\beta_h|^2 &\leqq 4(1+\sigma)\frac{h^2}{2}\int_{V_2} \frac{\mathrm{d}x}{|\mathrm{dist}\,(x,\Gamma_h)|^2} + (1+\frac{1}{\sigma})\frac{2\pi^2K^2}{\eta^4}|V_2|, \\ &\leqq 4(1+\sigma)h^2\pi \log\left(\frac{r}{\epsilon}\right)|\Gamma_h \cap U_{r_0}| + (1+\frac{1}{\sigma})\frac{C\mu^2}{\eta^2}h|\Gamma_h \cap U_{r_0}|, \qquad (3.58)\end{aligned}$$

so that

$$B_\epsilon \leqq (1+\sigma)h^2\pi \log\left(\frac{r}{\epsilon}\right)\left(|\Gamma_h| + 3|\Gamma_h \cap U_{r_0}|\right) + (1+\frac{1}{\sigma})\frac{C\mu^2}{\eta^2}h|\Gamma_h|. \qquad (3.59)$$

If $g_\epsilon = h^{-2} = |\log\epsilon|^2$, then statements (iii) and (v) of Proposition 2 and (3.59) give

$$\limsup_{\epsilon\to 0} B_\epsilon \leqq \left[(1+\sigma)\pi + (1+\frac{1}{\sigma})\frac{C\mu^2}{\eta^2}\right]\cdot(C\eta + ||dp_\delta||_{L^1(\Omega_\delta)}), \qquad (3.60)$$

while if $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$ (that is $\epsilon \ll h \ll |\log\epsilon|^{-1}$), we have

$$\limsup_{\epsilon\to 0} B_\epsilon \leqq (1+\frac{1}{\sigma})\frac{C\mu^2}{\eta^2}\cdot(C\eta + ||dp_\delta||_{L^1(\Omega_\delta)}). \qquad (3.61)$$

We sum up all the contributions (3.48), (3.50), (3.51), (3.55), (3.56), (3.60) and (3.61), noting that the terms estimated in (3.50), (3.51), and (3.56) add up to $2\pi^2\int_\Omega |d\alpha+\gamma+d^*\beta|^2 + C\sqrt{\eta} = 2\pi^2\int_\Omega |p_\delta|^2 + C\sqrt{\eta}$. Thus, letting first $\mu \to 0$, then $\sigma \to 0$, in (3.60) and (3.61), we obtain

$$\limsup_{\epsilon\to 0} \frac{E_\epsilon(u_\epsilon,\Omega)}{g_\epsilon} \leqq \pi\int_{\Omega_\delta} |dp_\delta| + 2\pi^2\int_\Omega |p_\delta|^2 + C\sqrt{\eta}, \qquad (3.62)$$

if $g_\epsilon = |\log\epsilon|^2$, and

$$\limsup_{\epsilon\to 0} \frac{E_\epsilon(u_\epsilon, \Omega)}{g_\epsilon} \leqq 2\pi^2 \int_{\Omega_\delta} |p_\delta|^2 + C\sqrt{\eta}, \tag{3.63}$$

if $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$. In these estimates C is independent of η. Thus, since $p = 2\pi v$, and recalling (3.7), (3.3), (3.4), and statement (iv) of Proposition 2, we see that as first η and then δ tend to 0, the right-hand sides above converge to $\frac{1}{2}|dv|(\Omega) + \frac{1}{2}\|v\|^2_{L^2(\Omega)}$ in the case $g_\epsilon = |\log\epsilon|^2$, and $\frac{1}{2}\|v\|^2_{L^2(\Omega)}$ in the case $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$. Thus, we can find sequences $\eta = \eta_\epsilon$ and $\delta = \delta_\epsilon$ tending to zero slowly enough that, if we define $U_\epsilon := u_\epsilon$ with parameters δ_ϵ in the piecewise linear approximation (Lemma 3) and η_ϵ in the discretization of the vorticity (Proposition 2) , then

$$\limsup_{\epsilon\to 0} \frac{E_\epsilon(U_\epsilon, \Omega)}{g_\epsilon} \leqq \frac{1}{2}|dv|(\Omega) + \frac{1}{2}\|v\|^2_{L^2(\Omega)} \quad \text{if } g_\epsilon = |\log\epsilon|^2 \tag{3.64}$$

$$\limsup_{\epsilon\to 0} \frac{E_\epsilon(U_\epsilon, \Omega)}{g_\epsilon} \leqq \frac{1}{2}\|v\|^2_{L^2(\Omega)} \qquad \text{if } |\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}. \tag{3.65}$$

This finally proves the upper bound (1.8), recalling that $J = \frac{1}{2}dv$ for $g_\epsilon = |\log\epsilon|^2$ and $J = 0$ when $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$.

Finally, having established the energy upper bound for U_ϵ, the compactness assertions (1.4), (1.5) and (1.6) imply that $\frac{1}{\sqrt{g_\epsilon}} jU_\epsilon$, $\frac{1}{\sqrt{g_\epsilon |U_\epsilon|}} jU_\epsilon$ and JU_ϵ converge to limits in the required spaces, so it suffices only to identify the limits. In fact, it suffices to show, for example, that $\frac{1}{\sqrt{g_\epsilon}} jU_\epsilon \to v$ in the sense of distributions, and this follows (after taking η_ϵ in the definition of U_ϵ to converge to zero more slowly, if necessary) from (3.44). □

3.8. Construction of the Sequence u_ϵ in Case $g_\epsilon \ll |\log\epsilon|^2$

Let J be an exact measure-valued 2-form in Ω and $v \in L^2(\Lambda^1\Omega)$ such that $dv = 0$. Fix $\delta > 0$, and let p_δ be the rational piecewise linear approximation of $p := \frac{v}{2\pi}$ from Lemma 3. Furthermore, let p'_δ be the rational piecewise linear function from Lemma 3', so that dp' approximates J. Our Hodge decomposition gives, respectively, $p_\delta = \gamma + d\alpha + d^*\beta'$, and $p'_\delta = \gamma' + d\alpha' + d^*\beta$. Let $h = \frac{1}{\sqrt{g_\epsilon}}$ and $h' = \frac{|\log\epsilon|}{g_\epsilon}$, so that $h = h' \frac{\sqrt{g_\epsilon}}{|\log\epsilon|} \ll h'$. Fix $\eta > 0$, and for $h' < \eta^2$ let $d^*\beta_{h'}$ be the discretization of $d^*\beta$ via Proposition 2. Let $\phi_{h'}$ be defined as in (3.39), so that $d\phi_{h'} = \frac{1}{h'} d^*\beta_{h'}$, let $h^{-1}\gamma_h = d\psi_h$ be as in Section 3.6, and set $\alpha_h = h^{-1}\alpha$. Finally, let ρ_ϵ be as in (3.42) and define

$$u_\epsilon = \rho_\epsilon \exp(i2\pi \cdot (\phi_{h'} + \psi_h + \alpha_h)). \tag{3.66}$$

3.9. Completion of Proof of (1.8) in Case $g_\epsilon \ll |\log\epsilon|^2$

We have to estimate

$$\frac{E_\epsilon(u_\epsilon; \Omega)}{g_\epsilon} = \frac{h^2}{2} \int_\Omega |\nabla\rho_\epsilon|^2 + \frac{W(\rho_\epsilon)}{\epsilon^2} + 4\pi^2\rho_\epsilon^2 \left| \frac{1}{h'} d^*\beta_{h'} + \frac{1}{h}(\gamma_h + d\alpha) \right|^2. \tag{3.67}$$

Then $|\operatorname{dist}(x,\Gamma_h)\leqq\epsilon\}|\leqq\frac{\epsilon^2}{h'}$ as in (3.46), so we find as in (3.48) that

$$\frac{h^2}{2}\int_\Omega|\nabla\rho_\epsilon|^2+\frac{W(\rho_\epsilon)}{\epsilon^2}\leqq C\frac{h^2}{h'}\longrightarrow 0. \tag{3.68}$$

For the remaining terms we have

$$2\pi^2\int_\Omega\rho_\epsilon|d\alpha+\gamma_h|^2\to 2\pi^2\int_\Omega|d\alpha+\gamma|^2\leqq 2\pi^2\int_{\Omega_\delta}|p_\delta|^2, \tag{3.69}$$

$$2\pi^2\frac{h}{h'}\int_\Omega\rho_\epsilon^2 d^*\beta_{h'}\cdot(d\alpha+\gamma_h)\to 0, \tag{3.70}$$

$$2\pi^2\frac{h^2}{h'^2}\int_\Omega\rho_\epsilon^2|d^*\beta_{h'}|^2=A'_\epsilon+B'_\epsilon+C'_\epsilon, \tag{3.71}$$

where, in the notation corresponding to (3.71),

$$\begin{aligned}
A'_\epsilon&=2\pi^2\frac{h^2}{h'^2}\int_{G^\epsilon_{h'}}\rho_\epsilon^2|d^*\beta_{h'}|^2,\\
B'_\epsilon&=2\pi^2\frac{h^2}{h'^2}\int_{G^r_{h'}\setminus G^\epsilon_{h'}}|d^*\beta_{h'}|^2,\\
C'_\epsilon&=2\pi^2\frac{h^2}{h'^2}\int_{\Omega\setminus G^r_{h'}}|d^*\beta_{h'}|^2
\end{aligned} \tag{3.72}$$

for $r=c_0\eta(h')^{1/2}$. Reasoning as in (3.54) and (3.56), we deduce *a fortiori* that $\limsup A_\epsilon=\limsup_{\epsilon\to 0}C_\epsilon=0$, while following (3.57) and (3.58) we deduce

$$B'_\epsilon\leqq(1+\sigma)h^2\pi\log(\frac{r}{\epsilon})\,(|\Gamma_{h'}|+C|\Gamma_{h'}\cap U_r|)+(1+\frac{1}{\sigma})\frac{h^2}{h'}|\Gamma_{h'}|, \tag{3.73}$$

so that $\limsup B'_\epsilon\leqq(1+\sigma)\pi\int_{\Omega_\delta}|dp'_\delta|+C\eta$ by Proposition 2 (iii). Summing up the various contributions and then letting $\sigma\to 0$, we obtain

$$\limsup_{\epsilon\to 0}\frac{E_\epsilon(u_\epsilon)}{g_\epsilon}\leqq\pi\int_{\Omega_\delta}|dp'_\delta|\;+2\pi^2\int_{\Omega_\delta}|p_\delta|^2\;+C\eta. \tag{3.74}$$

We conclude the proof as in the previous cases, by defining $U_\epsilon:=u_{(\epsilon,\eta_\epsilon,\delta_\epsilon)}$ (that is, defining u_ϵ as above, but with parameters δ_ϵ in the piecewise linear approximation of Lemma 3, and η_ϵ in the discretization of the vorticity of Proposition 2) for η_ϵ and δ_ϵ converging to zero sufficiently slowly, so that U_ϵ satisfies the Gamma-limsup inequality (1.8), and then verifying the convergence as before. □

4. Applications to Superconductivity

In this section we prove Theorem 4 and begin the analysis of the limiting functional $\mathcal{F}$, deriving the curvature equation for the vortex filaments. We use a good deal of notation that was introduced in Section 1.3.

In the companion paper [2] we analyze the properties of $\mathcal{F}$ in more detail and derive further applications such as a general expression for the first critical field, H_{c_1}.

4.1. Proof of Theorem 4

First, recalling that $h_{\text{ex}} = dA_{\text{ex},\epsilon}$, we see immediately from the definition of $\mathcal{F}_\epsilon$ and of the $\dot{H}^1_*(\Lambda^1\mathbb{R}^3)$ norm that

$$\|A_\epsilon - A_{\text{ex},\epsilon}\|^2_{\dot{H}^1_*} \leqq 2\mathcal{F}_\epsilon(u_\epsilon, A_\epsilon) \leqq K|\log\epsilon|^2.$$

It immediately follows that $\frac{1}{|\log\epsilon|}(A_\epsilon - A_{\text{ex},\epsilon})$ is weakly precompact in $\dot{H}^1_*(\Lambda^1\mathbb{R}^3)$, and since $|\log\epsilon|^{-1}A_{\text{ex},\epsilon} \to A_{\text{ex},0}$ in $\dot{H}^1_*(\Lambda^1\mathbb{R}^3)$, we deduce (1. 22).

The above bounds on A_ϵ and the Sobolev embedding $\dot{H}^1_* \hookrightarrow L^6$ imply that

$$\||\log\epsilon|^{-1}A_\epsilon\|_{L^6(\Lambda^1\Omega)} \leqq K. \tag{4.1}$$

In order to establish the remaining compactness assertions, we use the decomposition (1. 19), which implies that

$$E_\epsilon(u_\epsilon) \leqq \mathcal{F}_\epsilon(u_\epsilon, A_\epsilon) + \left|\int_\Omega A_\epsilon \cdot ju_\epsilon\right| \leqq K|\log\epsilon|^2 + \left|\int_\Omega A_\epsilon \cdot ju_\epsilon\right|,$$

using the fact that $\mathcal{M}(A; dA_{\text{ex},\epsilon}) + \mathcal{R}(u_\epsilon, A_\epsilon) \geqq 0$. To estimate the right-hand side, note that, in general,

$$\begin{aligned}|ju\cdot A| &\leqq |u|\,|Du|\,|A| \leqq \frac{1}{4}|Du|^2 + |u|^2|A|^2 \leqq \frac{1}{4}|Du|^2 + 2|A|^2\\ &\quad +2(|u|-1)^2|A|^2 \leqq \frac{1}{4}|Du|^2 + 2|A|^2 + \frac{c}{\epsilon^2}\,|\,|u|-1\,|^3 + C\epsilon^2|A|^6.\end{aligned}$$

And hypothesis (H_q) with $q \geqq 3$ implies that $c\,|\,|u|-1\,|^3 \leqq \frac{1}{2}W(u)$ if c is small enough, so that

$$\left|\int_\Omega A_\epsilon \cdot ju_\epsilon\right| \leqq \frac{1}{2}E_\epsilon(u_\epsilon) + C\int_\Omega |A_\epsilon|^2 + \epsilon^2|A_\epsilon|^6\,\mathrm{d}x.$$

By combining the above inequalities and using (4.1), we find that $E_\epsilon(u_\epsilon) \leqq K'|\log\epsilon|^2$, which in view of Theorem 2 implies that (1.4), (1.5), (1.6) hold with $g_\epsilon = |\log\epsilon|$.

To prove statement (ii), consider the decomposition of $\mathcal{F}_\epsilon$ given by (1. 19), (1. 20), which may be rewritten

$$\begin{aligned}\frac{\mathcal{F}_\epsilon(u_\epsilon, A_\epsilon)}{|\log\epsilon|^2} &= \frac{E_\epsilon(u_\epsilon)}{|\log\epsilon|^2} + \mathcal{M}\Big(\frac{A_\epsilon}{|\log\epsilon|}, \frac{h_{\text{ex}}}{|\log\epsilon|}\Big)\\ &\quad + \mathcal{I}\Big(\frac{ju_\epsilon}{|\log\epsilon|}, \frac{A_\epsilon}{|\log\epsilon|}\Big) + \frac{\mathcal{R}(u_\epsilon, A_\epsilon)}{|\log\epsilon|^2}.\end{aligned} \tag{4.2}$$

Recall that (1. 15) asserts

$$\frac{1}{|\log\epsilon|^2}E_\epsilon(u_\epsilon) \xrightarrow{\Gamma} E(v),$$

with $E(v)$ defined in (1. 16). Note further that $\mathcal{M}$ is lower semicontinuous with respect to the weak $\dot{H}^1_*$ convergence of $\frac{A_\epsilon}{|\log \epsilon|}$, and hence, taking into account (1. 22), we readily deduce

$$\mathcal{M}\Big(\frac{A_\epsilon}{|\log \epsilon|}, \frac{h_{\text{ex}}}{|\log \epsilon|}\Big) \xrightarrow{\Gamma} \mathcal{M}(A, h). \tag{4.3}$$

Moreover, by Sobolev embedding, (1. 22) implies $\frac{A_\epsilon}{|\log \epsilon|} \to A$ strongly in $L^p(\Omega)$, for any $1 \leqq p < 6$, whereas (1.5) gives $\frac{ju_\epsilon}{|\log \epsilon|} \rightharpoonup v$ weakly in $L^{2q/(q+2)}(\Omega)$. For $q \geqq 3$ we have $2q/(q+2) \geqq 6/5$, so that for any admissible sequence (u_ϵ, A_ϵ) we have

$$\mathcal{I}\Big(\frac{ju_\epsilon}{|\log \epsilon|}, \frac{A_\epsilon}{|\log \epsilon|}\Big) \to \mathcal{I}(v, A). \tag{4.4}$$

Note, finally, that for the remainder term $\mathcal{R}(u_\epsilon, A_\epsilon)$, since $|1-|u|^2|^{3/2} \leqq CW(u)$,

$$\begin{aligned}
|\mathcal{R}(u_\epsilon, A_\epsilon)| &\leqq \int_\Omega \Big|1-|u|^2\Big|\, |A_\epsilon|^2 \mathrm{d}x \\
&\leqq C\epsilon^{4/3}\left(\int_\Omega \frac{W(u_\epsilon)}{\epsilon^2}\mathrm{d}x\right)^{2/3}\left(\int_\Omega |A_\epsilon|^6 \mathrm{d}x\right)^{1/3} \\
&\leqq C\epsilon^{4/3} E_\epsilon(u_\epsilon)^{2/3} \|A_\epsilon\|^2_{L^6(\Omega)} \\
&\leqq C\epsilon^{4/3}|\log \epsilon|^{10/3},
\end{aligned}$$

so that $\frac{1}{|\log \epsilon|^2}\mathcal{R}(u_\epsilon, A_\epsilon) \leqq C(\epsilon|\log \epsilon|)^{4/3}$ converges uniformly to 0.

From the above considerations it follows immediately that

$$\frac{\mathcal{F}_\epsilon(u_\epsilon, A_\epsilon)}{|\log \epsilon|^2} \xrightarrow{\Gamma} E(v) + \mathcal{I}(v, A) + \mathcal{M}(A, h), \tag{4.5}$$

which is formula (1. 23). $\square$

4.2. Some Properties of the Γ-Limit $\mathcal{F}$

In this section we derive the Euler–Lagrange equations for the functional $\mathcal{F}$ and deduce a curvature equation for the limiting vortex filaments. First of all, notice that $\mathcal{F}$ is strictly convex and hence admits a unique minimizer (v, A). We first make variations of $\mathcal{F}$ with respect to A. Standard computations yield

$$\begin{cases} d^*(dA - h) = \mathbf{1}_\Omega \cdot (v - A) & \text{in } \mathbb{R}^3 \\ [(\star(dA-h))_\top] = [(dA-h)_N] = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

where $\mathbf{1}_\Omega$ denotes the characteristic function of Ω and $[(dA-h)_N]$ denotes the jump across $\partial\Omega$ of the normal component of $(dA-h)$. Denoting $j = \mathbf{1}_\Omega \cdot (v - A)$ the gauge-invariant supercurrent in Ω and $H = dA - h$, we recover from (4.6) Ampère's law $d^*H = j$ in $\mathbb{R}^3$ for the magnetic field H, which has to be coupled with Gauss's law for electromagnetism $dH = d(dA-h) = 0$ in $\mathbb{R}^3$, and with the continuity condition $[H] = 0$ on $\partial\Omega$, which is a consequence of $[H_N] = 0$ (by (4.6)) and $[H_\top] = 0$ on $\partial\Omega$ (by Gauss's law $dH = 0$).

Now let $J(v)$ denote the convex and positively 1-homogeneous function $J(v) := ||dv||$, and let ∂J be its subdifferential. Making variations of $\mathcal{F}$ with respect to v yields the differential inclusion

$$0 \in \frac{1}{2}\,\partial J(v) + v - A. \tag{4.7}$$

Assume the minimizer v is regular and $\operatorname{spt}|dv| = \bar{U}$, with U an open subset of Ω. In particular, if U is a proper subset of Ω, then one may view $\Omega \cap \partial U$ as a kind of free boundary. This situation has a counterpart in the two-dimensional case (see [23,30]). Then (4.7) corresponds to

$$\frac{1}{2}\int_U \frac{dv}{|dv|} \wedge \star d\phi + \int_\Omega (v - A) \wedge \star\phi = 0 \tag{4.8}$$

for any $\phi \in C^\infty(\Lambda^1\Omega)$ such that $\operatorname{spt}\phi \subset \Omega \setminus \partial U$. Testing (4.8) with $\phi \in C_c^\infty(\Lambda^1(\Omega \setminus \bar{U}))$ we deduce $v = A$ in $\Omega \setminus \bar{U}$. Testing now with those $\phi \in C^\infty(\Lambda^1(\Omega))$ such that $\operatorname{spt}\phi \subset \bar{U} \setminus (\Omega \cap \partial U)$ and integrating by parts (4.8) we further deduce

$$\int_U \left[\frac{1}{2} d^*\left(\frac{dv}{|dv|}\right) + v - A\right] \wedge \star\phi + \int_{\partial\Omega\cap\bar{U}} \left(\phi \wedge \star\frac{dv}{|dv|}\right)_\top = 0, \tag{4.9}$$

whence

$$\begin{cases} d^*\left(\frac{dv}{|dv|}\right) = 2(A - v) & \text{in } U, \\ (\star\frac{dv}{|dv|})_\top = 0 & \text{on } \bar{U} \cap \partial\Omega. \end{cases} \tag{4.10}$$

Notice that $\tau = \star\frac{dv}{|dv|}$ is the unit tangent covector field to the streamlines of the covector distribution $\star dv$, which correspond to the limiting vorticity. From (4.10) we obtain, in particular,

$$\begin{cases} \tau \wedge \star d\tau = 2\tau \wedge (v - A) = 2\tau \wedge j & \text{in } U, \\ \tau_\top = 0 & \text{on } \bar{U} \cap \partial\Omega. \end{cases} \tag{4.11}$$

Denoting, respectively, by $\boldsymbol{\tau}$ and $\boldsymbol{j}$ the vector fields corresponding to τ and j, we notice that $\star(\tau \wedge j)$ corresponds to $\boldsymbol{\tau} \times \boldsymbol{j}$, and $\star d\tau$ corresponds to the vector field $\nabla \times \boldsymbol{\tau}$, so that $\star(\tau \wedge \star d\tau)$ corresponds to the curvature vector $\boldsymbol{\kappa} = \boldsymbol{\tau} \times (\nabla \times \boldsymbol{\tau})$. We thus deduce the curvature equation (1. 25).

Remark 17. Notice that $d^*\tau = \star d(\frac{dv}{|dv|}) = 0$ (or equivalently $\nabla \cdot \boldsymbol{\tau} = 0$) in Ω. From (4.10) we deduce that τ satisfies the Hodge system

$$\begin{cases} d\tau = \star 2j & \text{in } \Omega \\ d^*\tau = 0 & \text{in } \Omega \\ \tau_\top = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

or respectively

$$\begin{cases} \nabla \times \boldsymbol{\tau} = 2\boldsymbol{j} & \text{in } \Omega \\ \nabla \cdot \boldsymbol{\tau} = 0 & \text{in } \Omega \\ \boldsymbol{\tau}_{\top} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

under the pointwise constraint $|\tau| = 1$ (resp. $|\boldsymbol{\tau}| = 1$) in spt j.

Remark 18. From (4.6), (4.10) we recover, in particular, the continuity equation $d^* j = d^*(v - A) = 0$ (or equivalently, $\nabla \cdot \boldsymbol{j} = 0$). If A is in the Coulomb gauge $d^* A = 0$ (which happens in particular if $A_{\mathrm{ex}} = cx^1 dx^2 - x^2 dx_1$) and $A \in H^1_*$, so that $d^*(A - A_{\mathrm{ex}}) = 0$), then it follows that v satisfies

$$\begin{cases} d^* v = 0 & \text{in } \Omega \\ v_N = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

5. Appendix

In this Appendix we recollect basic facts and notation that we use throughout the paper, as well as background on differential forms, Hodge decompositions and minimal connections. We also provide the proofs of Lemma 1 and Lemma 5.

5.1. Differential Forms

For $0 \leqq k \leqq n$, let $\Lambda^k \mathbb{R}^n$ be the space of k-covectors in $\mathbb{R}^n$, that is, $\theta \in \Lambda^k \mathbb{R}^n$ if $\theta = \sum \theta_I dx^I$, where $dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $1 \leqq i_1 < \cdots < i_k \leqq n$. For $\theta, \beta \in \Lambda^k \mathbb{R}^n$, their inner product is given by $(\theta, \beta) := \sum \theta_I \cdot \beta_I$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. We will denote by $C^\infty(\Lambda^k \Omega) := C^\infty(\Omega; \Lambda^k \mathbb{R}^n)$ the space of smooth k-forms on Ω. Similarly, we denote by $L^p(\Lambda^k \Omega)$, $W^{1,p}(\Lambda^k \Omega)$ the spaces of k-forms of class L^p and $W^{1,p}$ respectively. For $\omega \in C^\infty(\Lambda^k \Omega)$, denote by $\omega_\top \in C^\infty(\Lambda^k \partial\Omega)$ its tangential component[6] on $\partial\Omega$, and by $\omega_N := \omega_{|\partial\Omega} - \omega_\top$ its normal component on $\partial\Omega$. The operators $\omega \mapsto \omega_\top$ and $\omega \mapsto \omega_N$ extend to bounded linear operators $W^{1,p}(\Lambda^k \Omega) \to L^p(\partial\Omega; \Lambda^k \mathbb{R}^n)$. The Hodge star operator $\star : \Lambda^k \mathbb{R}^n \to \Lambda^{n-k} \mathbb{R}^n$ is defined in such a way that $\theta \wedge \star\varphi = (\theta, \varphi) dx^1 \wedge \cdots \wedge dx^n$. The L^2 inner product of $\omega, \eta \in C^\infty(\Lambda^k \Omega)$ is defined by

$$\langle \omega, \eta \rangle := \int_\Omega (\omega, \eta) \mathrm{d}\mathcal{L}^n = \int_\Omega \omega \wedge \star\eta.$$

Let $T \subset \Omega$ be a piecewise smooth m-dimensional submanifold with boundary. Integration of (the tangential component of) a smooth m-form ω on T will be denoted by $\int_T \omega \equiv \int_T \omega_\top = \int_T i^* \omega$, with $i : T \to \Omega$ the inclusion map.

The adjoint with respect to $\langle \cdot, \cdot \rangle$ of the Hodge $\star$ operator on k-forms is $(-1)^{k(n-k)} \star$.

[6] That is $\omega_\top := i^* \omega$, where $i : \partial\Omega \to \Omega$ is the inclusion map.

5.1.1. Measure-Valued Forms A distribution-valued k-form μ is an element of the dual space[7] of $C^\infty(\Lambda^k\Omega)$, and we express the duality pairing through the notation $\langle\cdot,\cdot\rangle$. In particular, we will say that μ is a measure-valued k-form (see [3], Definition 2.1) if

$$\langle\mu,\varphi\rangle \leqq C||\varphi||_\infty \qquad \forall\,\varphi\in C_c^\infty(\Lambda^k\Omega). \tag{5.1}$$

A measure-valued k-form μ can be represented by integration (see [3], Proposition 2.2) as follows:

$$\langle\mu,\varphi\rangle = \int_\Omega (\nu,\varphi)\,\mathrm{d}|\mu|, \tag{5.2}$$

where $|\mu|$ is the total variation measure of (the vector measure) μ and ν is a $|\mu|$-measurable k-form such that $(\nu,\nu)^{1/2} =: |\nu| = 1$ $|\mu|$-almost everywhere in Ω. We denote by $||\mu|| := |\mu|(\Omega)$ the total variation norm of $|\mu|$. It coincides with the L^1 norm $||\mu||_1 = \int_\Omega|\mu|$ if $\mu\in L^1(\Lambda^k\Omega)$. We denote by $\mu\llcorner U$ the restriction of μ to $U\subset\Omega$, defined by

$$\langle\mu\llcorner U,\varphi\rangle = \int_U (\nu,\varphi)\,\mathrm{d}|\mu|. \tag{5.3}$$

Moreover, for η a unit k-covector and μ a measure k-form in Ω, the component along η of μ is a signed measure denoted (μ,η) defined by

$$(\mu,\eta)(U) := (\mu(U),\eta) = \int_U (\nu,\eta)\mathrm{d}|\mu| \qquad \forall U\Subset\Omega, \tag{5.4}$$

with variation measure $|(\mu,\eta)|$ given by

$$|(\mu,\eta)|(U) = \int_U |(\nu,\eta)|\mathrm{d}|\mu| \qquad \forall U\Subset\Omega. \tag{5.5}$$

Notice that an oriented piecewise smooth k-dimensional submanifold $T\subset\Omega$ can be identified with a measure k-form $\widehat{T}$, whose action on smooth k-forms φ is given by

$$\langle\widehat{T},\varphi\rangle = \int_T \varphi. \tag{5.6}$$

Let d be the exterior differentiation operator, and $d^* = (-1)^{n(k+1)+1}\star d\star$ its adjoint with respect to $\langle\cdot,\cdot\rangle$, that is, $\langle d\omega,\eta\rangle = \langle\omega,d^*\eta\rangle$ for ω a k-form, and η an $(n-k-1)$-form. We define the action of d and d^* on a measure-valued distribution μ by duality, so that $\langle d\mu,\eta\rangle := \langle\mu,d^*\eta\rangle$ and $\langle d^*\mu,\eta\rangle := \langle\mu,d\eta\rangle$ for η with compact support.

Stokes' Theorem reads $\int_T d\varphi = \int_{\partial T}\varphi_\top$, for φ a smooth $(k-1)$-form and T as above. Notice that by (5.6) we have

$$\langle\widehat{T},d\varphi\rangle = \langle d^*\widehat{T},\varphi\rangle = \langle\widehat{\partial T},\varphi\rangle, \quad \text{so that } \widehat{\partial T} = d^*\widehat{T}. \tag{5.7}$$

A measure-valued k-form μ is said to be *closed* if $d\mu = 0$, and it is *exact* if there exists a measure-valued $k-1$-form ψ such that $\mu = d\psi$.

[7] One can thus identify a distribution-valued k-form with a k-current, see [16], although we generally choose not to do so.

5.1.2. The Tangential Part of Measure-Valued Forms Suppose that ω is a closed measure-valued $n-1$-form defined on an open subset $\Omega \subset \mathbb{R}^n$. If we fix an open $U \subset \Omega$ with piecewise smooth boundary ∂U, we will use the notation $\omega_\top$ to denote the distribution defined by

$$\int f\omega_\top := \int_U df \wedge \omega \quad \text{for all } f \in C^\infty(U) \cap C(\bar{U}). \tag{5.8}$$

Thus our definition states that $\omega_\top := \star d(\chi_U \omega)$ in the sense of distributions, where χ_U is the characteristic function of U. Although the notation $\omega_\top$ does not explicitly indicate the set U, it will normally be clear from the context, and when it is not, we will write, for example, "$\omega_\top$ on ∂U".

In general, $\omega_\top$ is a distribution supported on ∂U. We claim that

$$\int f\omega_T \text{ depends only on } f|_{\partial U}, \text{ for smooth } f. \tag{5.9}$$

To verify this, it suffices to check that $\int_U df \wedge \omega = 0$ for ω as above, whenever $f = 0$ on ∂U. Toward this end, let χ_ϵ denote a smooth function with compact support in U, such that $0 \leqq \chi_\epsilon \leqq 1$, $|\nabla \chi_\epsilon| \leqq C/\epsilon$, $\chi_\epsilon(x) = 1$ if $\operatorname{dist}(x, \partial U) \geqq \epsilon$, and $\chi_\epsilon = 0$ if $\operatorname{dist}(x, \partial U) \leqq \epsilon/2$. Then

$$\int_U df \wedge \omega = \lim_{\epsilon \to 0} \int_U \chi_\epsilon df \wedge \omega = \lim_{\epsilon \to 0} \int_U f d\chi_\epsilon \wedge \omega,$$

since ω is closed. Since f is smooth and $f = 0$ on ∂U, $|f d\chi_\epsilon| \leqq (C\epsilon)(C/\epsilon) \leqq C$ when $\operatorname{dist}(x, \partial U) < \epsilon$, so the right-hand side is bounded by $|\omega|(\operatorname{supp} d\chi_\epsilon)$. Since $|\omega|$ has finite total mass by assumption, we easily conclude that there exists a sequence $\epsilon_k \searrow 0$ such that $\lim_{k\to\infty} \int_U \chi_{\epsilon_k} df \wedge \omega = 0$, proving (5.9).

It follows from (5.9) that expressions such as $\int_{\partial U} \omega_\top$ are well-defined.

In this paper it will often be the case that $\omega_\top$ is a measure supported on ∂U, and when this holds, we may also think of $\omega_\top$ as a measure-valued $(n-1)$-form on ∂U. In particular, if ω is smooth enough, then $\int f\omega_\top$ agrees with the classical expression discussed above, $\int_{\partial U} f(x) i^*\omega(x)$, where $i : \partial U \to \Omega$ is the inclusion map.

5.1.3. Harmonic Forms If $d\omega = d^*\omega = 0$, then ω is said to be harmonic. Denote by

$$\mathcal{H}^k \equiv \mathcal{H}^k(\Omega) := \{\omega \in L^2 \cap C^\infty(\Lambda^k\Omega), \ d\omega = 0, \ d^*\omega = 0\}$$

the space of harmonic k-forms on Ω, and by

$$\mathcal{H}^k_\top = \{\omega \in \mathcal{H}^k, \ \omega_\top = 0\}, \qquad \mathcal{H}^k_N = \{\omega \in \mathcal{H}^k, \ \omega_N = 0\}$$

the spaces of harmonic forms with vanishing tangential and normal components on $\partial\Omega$. Since $\star\omega_N = (\star\omega)_\top$ and $\star\star = (-1)^{k(n-k)}$, we have the bijections

$$\star : \mathcal{H}^k_\top \to \mathcal{H}^{n-k}_N, \qquad \star : \mathcal{H}^k_N \to \mathcal{H}^{n-k}_\top.$$

Harmonic forms in $\mathcal{H}^k_\top \cup H^k_N$ are smooth up to $\partial\Omega$. Denote by $H(\omega)$ (resp. $H_\top(\omega)$, $H_N(\omega)$) the orthogonal projection of a k-form ω on $\mathcal{H}^k$ (resp. $\mathcal{H}^k_\top$, $\mathcal{H}^k_N$). With respect to an orthonormal basis $\{\gamma_i\}_{i=1,\dots,\ell}$ of $\mathcal{H}^k$ (resp. $\mathcal{H}^k_\top$, $\mathcal{H}^k_N$), the orthogonal projection is of course given by $\sum_{i=1}^{\ell} \langle \omega, \gamma_i \rangle \gamma_i$.

The Laplace operator $-\Delta = dd^* + d^*d$ on smooth k-forms is positive semi-definite, commutes with $\star$, d, d^*, and $h \in \mathcal{H}^k \Rightarrow -\Delta h = 0$.

5.2. Hodge Decompositions

For $\omega \in L^p(\Lambda^k\Omega)$, $1 < p < +\infty$, we have the following Hodge decomposition, orthogonal with respect to $\langle \cdot, \cdot \rangle$ (see for example [19], Theorem 5.7, or [27] for $p \geqq 2$):

$$\omega = \gamma + d\alpha + d^*\beta, \tag{5.10}$$

where

$$\gamma \in \mathcal{H}^k_N,\ \alpha \in W^{1,p}(\Lambda^{k-1}\Omega),\ \beta \in W^{1,p}(\Lambda^{k+1}\Omega),\ \beta_N = 0. \tag{5.11}$$

Then $\gamma = H_N(\omega)$. Moreover there exists a unique $\Psi \in W^{2,p}(\Lambda^k\Omega)$ such that

$$-\Delta\Psi = \omega - H_N(\omega), \quad \Psi_N = 0, \quad (d\Psi)_N = 0, \tag{5.12}$$

and

$$||d\Psi||_{1,p} + ||d^*\Psi||_{1,p} \leqq C_p||\omega||_p. \tag{5.13}$$

We will write $\Psi = -\Delta_N^{-1}(\omega - H_N(\omega))$.

We may also decompose $\omega = \gamma + d\alpha + d^*\beta$ with

$$\gamma \in \mathcal{H}^k_\top,\ \alpha \in W^{1,p}(\Lambda^{k-1}\Omega),\ \beta \in W^{1,p}(\Lambda^{k+1}\Omega),\ \alpha_\top = 0, \tag{5.14}$$

so that $\gamma = H_\top(\omega)$. In this case there exists a unique $\Psi \in W^{2,p}(\Lambda^k\Omega)$ such that

$$-\Delta\Psi = \omega - H_\top(\omega), \quad \Psi_\top = 0, \quad (d^*\Psi)_\top = 0. \tag{5.15}$$

Moreover, (5.13) holds. We write in this case $\Psi = -\Delta_\top^{-1}(\omega - H_\top(\omega))$.

The operator $-\Delta_\top^{-1}$ is self-adjoint on $\mathcal{H}^\perp_\top$, and similarly $-\Delta_N^{-1}$ is self-adjoint on $\mathcal{H}^\perp_N$.

Remark 19. In case $\Omega = \mathbb{R}^n$, basic properties of harmonic functions imply that $\mathcal{H}^k = \{0\}$. For ω compactly supported, the potential Ψ is given in particular by $\Psi = G*\omega$, where $G(x) = c_n|x|^{n-2}$ is the Poisson kernel on $\mathbb{R}^n$, $n \geqq 3$. The Hodge decomposition of ω reads $\omega = d\alpha + d^*\beta$, with $\beta = G * d\omega$ and $\alpha = G * d^*\omega$. In this case $\alpha, \beta \in \dot{W}^{1,p}$ rather than $W^{1,p}$.

For $\omega \in L^1(\Lambda^k\Omega)$ or, more generally, a measure-valued k-form, the decomposition (5.10) fails in general, but decompositions of the form (5.12), (5.15) still hold, in view of this variant of [3], Theorem 2.10:

Proposition 3. *Let μ be a measure-valued k-form in Ω. If $H_N(\mu) = 0$, there exists a unique $\Psi \in W^{1,q}(\Lambda^k\Omega)\,\forall\, q < n/(n-1)$, denoted by $\Psi = -\Delta_N^{-1}(\mu)$, such that*

$$-\Delta\Psi = \mu, \quad \Psi_N = 0, \quad (d\Psi)_N = 0,$$

so that, in particular, $H_N(\Psi) = 0$.

If $H_\top(\mu) = 0$, then there exists a unique $\Psi \in W^{1,q}(\Lambda^k\Omega)\,\forall\, q < n/(n-1)$, denoted by $\Psi = -\Delta_\top^{-1}(\mu)$, such that

$$-\Delta\Psi = \mu, \quad \Psi_\top = 0, \quad (d^*\Psi)_\top = 0,$$

and in particular $H_\top(\Psi) = 0$.

In both cases, we have

$$||d\Psi||_q + ||d^*\Psi||_q \leqq C_q||\mu|| \qquad \forall q < \frac{n}{n-1}. \tag{5.16}$$

Proof. The proof of Proposition 3 follows exactly the duality argument *à la* Stampacchia carried out in [3], taking into account the elliptic estimates (5.13) for the operators $-\Delta_N$ and $-\Delta_\top$, and observing that they are self-adjoint. □

Corollary 1. *A measure-valued k-form μ is exact if and only if $d\mu = 0$ and $H_N(\mu) = 0$. In addition, if μ is exact then $\mu = d\zeta$, for $\zeta := d^*(-\Delta_N)^{-1}\mu \in \bigcap_{1\leqq q<n/n-1} L^q(\Lambda^{k-1}(\Omega))$, and $||\zeta||_q \leqq C_q||\mu||$.*

Similarly, a measure-valued k form μ is co-exact (that is, can be written $\mu = d^*\psi$ for some measure-valued $k+1$-form ψ) if and only if $d^*\mu = 0$ and $H_\top(\mu) = 0$, and if these conditions hold, then $\mu = d^*\zeta$ for $\zeta = d(-\Delta_\top)^{-1}\mu \in \bigcap_{1\leqq q<n/n-1} L^q(\Lambda^{k+1}\Omega)$, and $||\zeta||_q \leqq C_q||\mu||$.

Proof. If $d\mu = 0$ and $H_N(\mu) = 0$, then we appeal to Proposition 3 and define $\zeta = d^*(-\Delta_N^{-1}\mu)$, and it follows that $\mu = d\zeta$. Conversely, $\mu = d\psi$ in Ω for some measure-valued $k-1$-form ψ, then it is clear that $d\mu = 0$ in Ω, and if $\varphi \in \mathcal{H}_N^k$, then for χ_ϵ as in the proof of (5.9),

$$\int \phi \cdot \mu = \lim_{\epsilon\to 0} \int \chi_\epsilon\varphi \cdot d\psi = \lim_{\epsilon\to 0} \int d^*(\chi_\epsilon\varphi) \cdot \psi.$$

Next, the fact that $\varphi \in \mathcal{H}_N^k$ and properties of χ_ϵ imply that $|d^*(\chi_\epsilon\varphi)| = |d\chi_\epsilon \wedge \star\varphi| \leqq C$, independent of ϵ. We then conclude as in the proof of (5.9) that $\int \phi \cdot \mu = 0$, and hence that $H_N(\mu) = 0$.

The assertions about co-exact forms are proved in exactly the same way. □

Remark 20. In case $\Omega = \mathbb{R}^n$, μ compactly supported, we have in particular $\zeta = d^*(G * \mu)$ (resp. $\zeta = d(G * \mu)$).

Remark 21. If φ is a smooth k-form and $\varphi_N = 0$ (resp. $\varphi_\top = 0$), then $(d^*\varphi)_N = 0$ (resp. $(d\varphi)_\top = 0$). The form ζ of Corollary 1 is only in L^q, and so does not have a normal (resp. tangential) trace, but can be shown to satisfy $\zeta_N = 0$ (resp. $\zeta_\top = 0$) in a sort of distributional sense, as a consequence of the fact that $\zeta = d^*\Psi$ (resp. $\beta = d\Psi$) for $\Psi = -\Delta_N^{-1}\mu \in W^{1,q}$, with $\Psi_N = 0$ (resp. $\Psi = -\Delta_\top^{-1}\mu,\ \Psi_\top = 0$).

This distributional trace (of which our definition (5.8) of $q_\top$ for a closed measure-valued $n-1$-form q is a special case) is strong enough to provide uniqueness assertions in the setting of Corollary 1. For example, if $d\mu = 0$, then there is a *unique* $\zeta \in L^q(\Lambda^{k-1}\Omega)$ satisfying $d\zeta = \mu$, $d^*\zeta = 0$, and $\zeta_N = 0$ in the distributional sense.

Remark 22. Through the Green operators $-\Delta_N^{-1}$ (resp. $-\Delta_\top^{-1}$), one obtains an integral expression for the linking number of a k-cycle and a (relative) $(n-k-1)$-boundary (resp. a relative k-cycle with a $(n-k-1)$-boundary) in Ω (see for example [15]). For instance, let Γ be a relative $(n-k-1)$-boundary in Ω, that is, $\Gamma = \partial R + \Gamma'$ with $R \subset \Omega$ and $\Gamma' \subset \partial\Omega$. One immediately verifies that $H_\top(\widehat{\Gamma}) = 0$, and hence $H_N(\star\widehat{\Gamma}) = 0$. Let $\beta = -\Delta_N^{-1}(\star\widehat{\Gamma})$. Hence we have $d^*\beta \in L^p(\Lambda^1\Omega)$ for $p < \frac{n}{n-1}$ and β is smooth outside Γ. Hence, for a k-cycle $\gamma \subset \Omega \setminus \Gamma$ we have $0 = \widehat{\partial\gamma} = d^*\widehat{\gamma}$, and moreover

$$\begin{aligned}\int_\gamma d^*\beta &= \langle d^*\Delta_N^{-1}(\star\hat{\Gamma}), \widehat{\gamma}\rangle = \langle\widehat{\Gamma}, \star d(-\Delta_N^{-1}\widehat{\gamma})\rangle = \langle\widehat{\partial R}, \star d(-\Delta_N^{-1}\widehat{\gamma})\rangle \\ &= \langle\widehat{R}, \star d^* d(-\Delta_N^{-1}\widehat{\gamma})\rangle = \langle\widehat{R}, \star\widehat{\gamma} + \star\Delta_N^{-1}(dd^*\widehat{\gamma})\rangle \\ &= \langle\widehat{R}, \star\widehat{\gamma}\rangle = \langle\widehat{\gamma} \llcorner R, \star 1\rangle = \sum_{a_i \in \gamma\cap R} \star(\tau_\gamma \wedge \star\tau_R(a_i)) \in \mathbb{Z}.\end{aligned} \tag{5.17}$$

Observe that in case $\Gamma = \partial R \subset \Omega$ is a $(n-k-1)$-boundary in Ω, we have $H(\widehat{\Gamma}) = 0$, hence we may consider $\beta = -\Delta^{-1}(\star\widehat{\Gamma}) = G * (\star\widehat{\Gamma})$ with G the Poisson kernel in $\mathbb{R}^n$, and deduce for $d^*\beta$ the integral representation

$$d^*\beta = G * (\star d\widehat{\Gamma}) = (\star dG) * \widehat{\Gamma} = \int_\Gamma \star dG(x - \cdot), \tag{5.18}$$

which in the case $n = 3,\ k = 1$ reads more familiarly

$$d^*\beta = \sum_{i,j,k=1}^{3} 4\pi\, dx^i\ \epsilon_{ijk} \int_{\Gamma_h^\ell} \frac{(x_j - y_j)dy^k}{|x-y|^3}. \tag{5.19}$$

Following (5.17), we thus deduce the Biot-Savart formula for the linking number $\mathrm{link}(\Gamma, \gamma)$ of $\Gamma = \partial R$ with a k-cycle γ in Ω, namely

$$\int_\gamma d^*\beta = \int_{\gamma_x}\int_{\Gamma_y} \star dG(x-y) = \langle\widehat{R}, \star\widehat{\gamma}\rangle = \sum_{a_i\in\gamma\cap R} \star(\tau_\gamma \wedge \star\tau_R(a_i)) \in \mathbb{Z}. \tag{5.20}$$

Notice that the integral formula (5.20) also gives $\mathrm{link}(\Gamma, \gamma)$ when Γ is just a cycle, that is, $\partial\Gamma = 0$, not necessarily a boundary. In fact, considering $\gamma \times \Gamma \subset \mathbb{R}^n_x \times \mathbb{R}^n_y$, we have $\partial(\gamma\times\Gamma) = 0$ in $\mathbb{R}^n \times \mathbb{R}^n$, and $\star dG(x-y) = |S^{n-1}|^{-1} \cdot \psi^*(d\sigma)$, where $\psi : \gamma \times \Gamma \to S^{n-1} \subset \mathbb{R}^n$ is given by $\psi(x, y) = \frac{x-y}{|x-y|}$ and $d\sigma$ is the volume form of S^{n-1}. Hence

$$\int_{\gamma_x}\int_{\Gamma_y} \star dG(x-y) = \frac{1}{|S^{n-1}|}\int_{\gamma\times\Gamma} \psi^*(d\sigma) = \deg(\psi) \in \mathbb{Z}. \tag{5.21}$$

5.3. Representation of Harmonic 1-Forms

Next, we describe the spaces $\mathcal{H}^1_N$, (resp. $\mathcal{H}^1_\top$), of harmonic 1-forms on $\Omega \subset \mathbb{R}^n$ with zero normal (resp. tangential) component on $\partial\Omega$. Since $\mathcal{H}^{n-1}_N = \star\mathcal{H}^1_\top$ (resp. $\mathcal{H}^{n-1}_\top = \star\mathcal{H}^1_N$), this also yields a representation for harmonic $(n-1)$-forms.

Lemma 10. (Description of $\mathcal{H}^1_\top$) *Let $(\partial\Omega)_i$, $i = 0, \dots, b$ denote the connected components of $\partial\Omega$. Then $\gamma \in \mathcal{H}^1_\top$ if and only there exist constants $c_1, \dots, c_b$ such that $\gamma = d\phi$, where ϕ is the unique harmonic function in Ω such that $\phi \equiv c_i$ on $(\partial\Omega)_i$ for $i \geqq 1$, and $\phi = 0$ on $(\partial\Omega)_0$.*

Proof. In fact, $\mathcal{H}^1_\top$ is isomorphic to the first relative de Rham cohomology group of Ω, that is $H^1_{dR}(\Omega; \partial\Omega)$, (see for example [17] vol. 1, Corollary 1, section 5.2.6) and $H^1_{dR}(\Omega, \partial\Omega) \simeq \mathbb{R}^b$, as it is shown in Lemma 13 below. Finally, the family of 1-forms described in the above statement span a b-dimensional subspace of $\mathcal{H}^1_\top$. $\square$

Lemma 11. (Description of $\mathcal{H}^1_N$) *Let κ denote the dimension of $\mathcal{H}^1_N$. Then there exists an an orthogonal basis $\{H_j\}_{j=1}^\kappa$ for $\mathcal{H}^1_N$ normalized so that for each j, there exists a $\mathbb{R}/\mathbb{Z}$-valued function ϕ_j such that $H_j = d\phi_j$, so that $e^{i2\pi\phi_j}$ is well-defined.*

Proof. In fact, $\mathcal{H}^1_N$ is isomorphic to the first de Rham cohomology group $H^1_{dR}(\Omega)$, which in turn is isomorphic to $\mathrm{Hom}(H_1(\Omega, \mathbb{Z}), \mathbb{R})$, and these are all finitely generated. (See for example [17] vol.1, Corollary 1 in section 5.2.6 and Theorem 3 in Section 5.3.2). It follows that if $\{\gamma_i\}_{i=1}^\kappa$ are cycles that form a basis for $H_1(\Omega; \mathbb{Z})$, then there exists a (unique) basis $\{H_i\}_{i=1}^\kappa$ for $\mathcal{H}^1_N$ such that $\int_{\gamma_j} H_j = \delta_{ij}$ for $i, j = 1, \dots, \kappa$. We now fix $x_0 \in \Omega$ and define $\phi_j(x) := \int_{\gamma(x_0,x)} H_j$, $j = 1 \dots, \kappa$, where $\gamma(x_0, x)$ is any path in Ω that starts at x_0 and ends at x. If $\gamma'(x_0, x)$ is another such path, then $\gamma(x_0, x) - \gamma'(x_0, x)$ is homologous to an integer linear combination of the γ_i's, so that $\int_{\gamma(x_0,x)} H_j - \int_{\gamma'(x_0,x)} H_j \in \mathbb{Z}$. Thus ϕ_j is well-defined as a function $\Omega \to \mathbb{R}/\mathbb{Z}$. It is immediate that $H_j = d\phi_j$. $\square$

Next, we describe an exactness criterion for closed $(n-1)$-forms in $\Omega \subset \mathbb{R}^n$.

Lemma 12. *A measure-valued $(n-1)$ form q on a smooth bounded open set $\Omega \subset \mathbb{R}^n$ is exact if and only if $dq = 0$ and $\int_{(\partial\Omega)_i} q_\top = 0$ for every connected component $(\partial\Omega)_i$ of $\partial\Omega$.*

Proof. Let $\gamma \in \mathcal{H}^{n-1}_N$, so that $\star\gamma \in \mathcal{H}^1_\top$ and hence, by Lemma 10, $\star\gamma = d\varphi$, where $\Delta\varphi = 0$ in Ω and $\varphi \equiv c_i$ on the ith connected component $(\partial\Omega)_i$. Then

$$\langle q, \gamma \rangle = \int_\Omega q \wedge \star\gamma = \int_\Omega q \wedge d\varphi \overset{(5.8),(5.9)}{=} \sum_{i=1}^b c_i \int_{(\partial\Omega)_i} q_\top \quad . \tag{5.22}$$

We deduce that $H_N(q) = 0$ if and only if $\int_{(\partial\Omega)_i} q_\top = 0$ for every i. The conclusion now follows from Corollary 1. $\square$

5.4. Proof of Lemma 10 Completed

We need the following easy result, whose proof uses the language of algebraic topology (see for example [32]).

Lemma 13. *Let U be a connected Lipschitz domain in $\mathbb{R}^n$, such that ∂U has $b+1$ connected components. Then $H^1_{dR}(U, \partial U) \simeq \mathbb{R}^b$.*

Proof. From the exact sequence in singular homology for the pair $(\bar{U}, \partial U)$, we have

$$H_1(\partial U) \xrightarrow{i_*} H_1(\bar{U}) \xrightarrow{\Phi_*} H_1(\bar{U}, \partial U) \xrightarrow{\partial_*} H_0(\partial U) \xrightarrow{i^0_*} H_0(\bar{U}) \to 0, \quad (5.23)$$

which gives rise to the short exact sequence

$$0 \to \operatorname{Im} \Phi_* \to H_1(\bar{U}, \partial U) \to \operatorname{Ker} i^0_* \to 0. \quad (5.24)$$

By hypothesis we have $H_0(U) = \mathbb{Z},\ H_0(\partial U) = \mathbb{Z}^{b+1}$, and (5.23) implies $\operatorname{Ker} i^0_* = \mathbb{Z}^b$. By the Mayer–Vietoris exact sequence for $V = \bar{U},\ W = \mathbb{R}^n \setminus U$ we have

$$H_2(V \cup W) \to H_1(V \cap W) \xrightarrow{(i_*, i_*)} H_1(V) \oplus H_1(W) \to H_1(V \cup W), \quad (5.25)$$

which yields, since $V \cup W = \mathbb{R}^n$ is contractible,

$$0 \to H_1(\partial U) \xrightarrow{(i_*, i_*)} H_1(\bar{U}) \oplus H_1(\mathbb{R}^n \setminus \bar{U}) \to 0, \quad (5.26)$$

so that (i_*, i_*) is an isomorphism. In particular, $i_* = \pi_1 \circ (i_*, i_*)$ is onto, hence $H_1(\bar{U}) = \operatorname{Im} i_* = \operatorname{Ker} \Phi_*$, which yields $\operatorname{Im} \Phi_* = 0$, so that (5.24) implies that $H_1(\bar{U}, \partial U)$ is isomorphic to $\operatorname{Ker} i^0_* = \mathbb{Z}^b$. From the regularity assumption[8] on U we have, in particular, $H_1(\bar{U}, \partial U) \simeq H_1(U, \partial U)$. Finally, from the relation

$$H^1(U, \partial U; \mathbb{R}) = \operatorname{Hom}(H_1(U, \partial U); \mathbb{R}) = \operatorname{Hom}(\mathbb{Z}^b; \mathbb{R}) \simeq \mathbb{R}^b, \quad (5.27)$$

the conclusion follows, since the first singular relative cohomology group with real coefficients $H^1(U, \partial U; \mathbb{R})$ is isomorphic to the first de Rham relative cohomology group $H^1_{dR}(U, \partial U)$. $\square$

5.5. Proof of Lemma 5

Step 1 We have: $\inf\{||\alpha||_{L^1(\Lambda^2 K)},\ d\alpha = 0 \text{ in } K,\ \alpha_\top = \zeta \text{ on } \partial K\} = ||\zeta||_{\dot{W}^{-1,1}(K)}$, where

$$||\zeta||_{\dot{W}^{-1,1}(K)} = \sup\left\{\int \varphi\, \zeta \ :\ \varphi \in W^{1,\infty}_c(\mathbb{R}^3),\ ||d\varphi||_{L^\infty(K)} \leqq 1\right\}.$$

[8] Actually it is sufficient for U to be a Lipschitz neighborhood retract in $\mathbb{R}^n$.

This follows by a straightforward modification of an argument in Federer [16]. We provide a sketch: define a linear functional acting on $C^\infty_c(\mathbb{R}^3)$ by

$$A(\varphi) := \int_{\partial K} \varphi\, \zeta, \qquad \varphi \in C^\infty_c(\mathbb{R}^3).$$

Given any measure-valued 2-form α, we similarly define a linear functional B_α acting on $C^\infty_c(\Lambda^1\mathbb{R}^3)$ by

$$B_\alpha(\psi) = \int_K \psi \wedge \alpha, \qquad \psi \in C^\infty_c(\Lambda^1\mathbb{R}^3).$$

And generally, for a linear functional C on $C^\infty_c(\Lambda^1\mathbb{R}^3)$, we define $\partial C(\varphi) := C(d\varphi)$ for $\varphi \in C^\infty_c(\mathbb{R}^3)$. Then the definitions (see (5.8) in particular) imply that $A = \partial C$ and $\|C\| < \infty$ if and only if $C = B_\alpha$ for some measure-valued 2-form α such that $d\alpha = 0$ in K and $\alpha_\top = \zeta$ on ∂K. Next, we note that $||\zeta||_{\dot W^{-1,1}(K)} = \mathbf{F}_{\mathrm{hom},K}(A)$, where $\mathbf{F}_{\mathrm{hom},S}(A)$ denotes the *homogeneous* flat norm of A in K, see [16]. Then, as observed in section 4.1.12 of [16] in a slightly different setting, the Hahn-Banach Theorem implies that

$$\mathbf{F}_{\mathrm{hom},K}(A) = \min\{\, ||C||,\ \operatorname{spt} C \subset K,\ \partial C = A\,\},$$

and this translates to our claim, in view of our earlier remarks.

Step 2 We claim that $||\zeta||_{\dot W^{-1,1}(K)} \leqq C||\zeta||_{W^{-1,1}(\mathbb{R}^3)}$, where

$$||\zeta||_{W^{-1,1}(\mathbb{R}^3)} = \sup\left\{ \int_{\mathbb{R}^3} \varphi\zeta,\ \varphi \in W^{1,\infty}_c(\mathbb{R}^3),\ ||\varphi||_{W^{1,\infty}(\mathbb{R}^3)} \leqq 1\right\}.$$

It suffices to show that there exists $C > 0$ such that, for any $\varphi \in W^{1,\infty}_c(\mathbb{R}^3)$ with $\|d\varphi\|_{L^\infty(K)} \leqq 1$, there exists $\psi \in W^{1,\infty}_c(\mathbb{R}^3)$ such that

$$\int \varphi\zeta = \int \psi\zeta \quad \text{and} \quad \|\psi\|_{W^{1,\infty}(\mathbb{R}^3)} \leqq C. \tag{5.28}$$

Indeed, given φ such that $\|d\varphi\|_{L^\infty(K)} < \infty$, we fix $x_0 \in K$ and we define $\psi(x) = \varphi(x) - \varphi(x_0)$ for $x \in K$.

Since K is convex, φ and hence ψ are 1-Lipschitz on K, so that $|\psi(x)| \leqq |x - x_0| \leqq \operatorname{diam}(K)$ in K. Next, we extend ψ to $\mathbb{R}^3 \setminus K$, such that the extended function is still 1-Lipschitz and moreover satisfies $\|\psi\|_{L^\infty(\mathbb{R}^3)} \leqq$ diam(K), and has compact support.

Since ζ is a measure supported on ∂K, clearly $\int \psi\zeta$ depends only on the behavior of ψ in ∂K, and hence $\int \psi\,\zeta = \int(\varphi - \varphi(x_0))\,\zeta = \int \varphi\,\zeta$, since $\int_{\partial K} \zeta = 0$, proving (5.28). $\square$

5.6. Proof of Lemma 1

Step 1 We will show below that there exists a piecewise smooth oriented 2-manifold with boundary $S = S_\epsilon$ such that

$$\partial S = M_\epsilon - M'_\epsilon \qquad \text{in } U \quad \text{and } \mathcal{H}^2(S \cap U) \leqq C\ell \cdot E_\epsilon(u_\epsilon; \Omega) \leqq C\ell g_\epsilon, \tag{5.29}$$

with $C > 0$ independent of ϵ and U. (See the proof of Proposition 1 for notation used here and below.) We first complete the proof of the lemma, assuming (5.29).

We may assume that S transversally intersects the level set $f^{-1}(t)$ for almost everywhere t, since if not, we can arrange that this condition is satisfied after an arbitrarily small perturbation of S that leaves ∂S fixed. Noting that $f^{-1}(t)$ coincides with ∂C^t for almost everywhere t, we deduce that $S \cap \partial C^t$ is piecewise smooth for almost everywhere $t > 0$.

Since f is 1-Lipschitz, the same is true for $f \llcorner S$, so that $|\nabla(f \llcorner S)| \leqq 1$ almost everywhere, and

$$\begin{aligned} \mathcal{H}^2((S \cap C^{N\ell}) \cap U) &\geqq \int_{(S\cap C^{N\ell})\cap U} |\nabla(f \llcorner S)| \mathrm{d}\mathcal{H}^2 \\ &= \int_0^{N\ell} \mathcal{H}^1((S \cap \partial C^t) \cap U) \mathrm{d}t, \end{aligned}$$

by the coarea formula. We deduce that there exists t_ϵ such that

$$\mathcal{H}^1((S \cap \partial C^{t_\epsilon}) \cap U) \leqq (N\ell)^{-1}\mathcal{H}^2(S \cap U) \leqq CN^{-1}g_\epsilon. \tag{5.30}$$

In U it holds

$$\begin{aligned} \partial(S \cap C^{t_\epsilon}) &= (\partial S) \cap C^{t_\epsilon} + S \cap (\partial C^{t_\epsilon}) \\ &= (M_\epsilon - M'_\epsilon) \cap C^{t_\epsilon} + S \cap (\partial C^{t_\epsilon}) \\ &= M_\epsilon - M'_\epsilon \cap C^{t_\epsilon} + S \cap (\partial C^{t_\epsilon}). \end{aligned} \tag{5.31}$$

In particular, for $\phi \in C_c^\infty(\Lambda^1 U)$, we have

$$\langle \nu_\epsilon - \nu'_\epsilon \llcorner C^{t_\epsilon}, \phi \rangle = \int_{S\cap C^{t_\epsilon}} d \star \phi - \int_{S\cap\partial C^{t_\epsilon}} \star\phi,$$

(using the definitions (2.13) and (2.24)), whence

$$\begin{aligned} ||\nu_\epsilon - \nu'_\epsilon \llcorner C^{t_\epsilon}||_{W^{-1,1}(U)} &\leqq \mathcal{H}^2(S \cap C^{t_\epsilon} \cap U) + \mathcal{H}^1(S \cap \partial C^{t_\epsilon} \cap U) \\ &\leqq (1 + (N\ell)^{-1})\mathcal{H}^2(S \cap U) \leqq C(\ell + N^{-1})g_\epsilon, \end{aligned} \tag{5.32}$$

by (5.30) and (5.29). This gives precisely (2.25).

Step 2 To conclude, we supply the proof of our earlier claim (5.29).

Let $g(x) = |\mathrm{dist}(x, R_1)|^{-1} + |\mathrm{dist}(x, R_1^*)|^{-1}$. By the coarea formula, we have

$$\int_{B_1} ds \int_{u_\epsilon^{-1}(s)} g(x)\mathrm{d}\mathcal{H}^1(x) = \int_\Omega g(x)|Ju_\epsilon|\mathrm{d}x \leqq \int_\Omega g(x)e_\epsilon(u_\epsilon)\mathrm{d}x, \quad (5.33)$$

so that by a mean-value argument, (2.12), and (2.22), we deduce from (5.33) that there exists a regular value s of u_ϵ such that $|s| < 1/2$ and, denoting $M_s := u_\epsilon^{-1}(s)$, we have

$$\int_{M_s} g(x)\mathrm{d}\mathcal{H}^1(x) = \int_{M_s} \frac{\mathrm{d}\mathcal{H}^1(x)}{|\mathrm{dist}(x, R_1)|} + \int_{M_s} \frac{\mathrm{d}\mathcal{H}^1(x)}{|\mathrm{dist}(x, R_1^*)|} \leqq \frac{KE_\epsilon(u_\epsilon;\Omega)}{\pi\delta\ell}. \quad (5.34)$$

Define as in [1], Lemma 3.8 (i), the map $\Phi : \mathbb{R}^3 \setminus R_1 \to R_1'$ and, accordingly, the map $\Phi^* : \mathbb{R}^3 \setminus R_1^* \to R_1^{*\prime}$. Set $\Psi(t, x) = (1-t)x + t\Phi(x)$, $\Psi^*(t, x) = (1-t)x + t\Phi^*(x)$, and define $S_1 = \Psi([0, 1] \times M_s)$ and $S_2 = \Psi^*([0, 1] \times M_s)$. Note, following [1], Lemma 3.8 (ii), that since M_s has no boundary in U, we have $\partial S_1 = \Phi_\# M_s - M_s$ and $\partial S_2 = \Phi^*_\# M_s - M_s$ in U. However, from [1], Lemma 3.8 (i), we know that $\Phi_\# M_s = M_\epsilon$ the point being that the intersection number of M_s with any 2-face Q_i agrees with $(-1)^{\sigma_i} d_{Q_i}$, due to orientation conventions and elementary properties of topological degree. Similarly $\Phi^*_\# M_s = M_\epsilon'$, so if we define $S := S_1 - S_2$, then $\partial S = M_\epsilon - M_\epsilon'$ in U, which is the first part of (5.29). Following the proof of [1], Lemma 3.8 (ii), we readily deduce that

$$\mathcal{H}^2(S \cap U) = \mathcal{H}^2(S_1 \cap U) + \mathcal{H}^1(S_2 \cap U) \leqq C\ell^2 \int_{M_s} g(x)\mathrm{d}\mathcal{H}^1(x). \quad (5.35)$$

Combining (5.35) and (5.34), claim (5.29) follows. □

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