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On Nonlinear Stochastic Balance Laws

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Abstract

We are concerned with multidimensional stochastic balance laws. We identify a class of nonlinear balance laws for which uniform spatial BV bound for vanishing viscosity approximations can be achieved. Moreover, we establish temporal equicontinuity in L^1 of the approximations, uniformly in the viscosity coefficient. Using these estimates, we supply a multidimensional existence theory of stochastic entropy solutions. In addition, we establish an error estimate for the stochastic viscosity method, as well as an explicit estimate for the continuous dependence of stochastic entropy solutions on the flux and random source functions. Various further generalizations of the results are discussed.

1. Introduction

We are concerned with the well-posedness and continuous dependence estimates for the stochastic balance laws

$$\partial_t u(t, \mathbf{x}) + \nabla \cdot \mathbf{f}(u(t, \mathbf{x})) = \sigma(u(t, \mathbf{x}))\, \partial_t W(t), \quad \mathbf{x} \in \mathbb{R}^d,\ t > 0, \tag{1}$$

with initial data:

$$u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \tag{2}$$

We denote by ∇ and Δ the spatial gradient and Laplacian, respectively.

Equation (1) is a conservation law perturbed by a random force driven by a Brownian motion $W(t) = W(t, \omega), \omega \in \Omega$, over a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqq 0}, P)$, where P is a probability measure, $\mathcal{F}$ is a σ-algebra, and $\{\mathcal{F}_t\}_{t\geqq 0}$ is a right-continuous filtration on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_0$ contains all the P-negligible subsets.

The initial function $u_0(\mathbf{x})$ is assumed to be a random variable satisfying

$$E[\|u_0\|^p_{L^p(\mathbb{R}^d)} + |u_0|_{\mathrm{BV}(\mathbb{R}^d)}] < \infty, \quad p = 1, 2, \dots. \tag{3}$$

Regarding the flux $\mathbf{f} = (f_1, \dots, f_d) : \mathbb{R} \to \mathbb{R}^d$, we assume $f_i \in C^2(\mathbb{R})$, $i = 1, \dots, d$, and that each f_i has at most polynomial growth in u, that is,

$$|f_i(u)| \leqq C(1 + |u|^r) \quad \text{for some finite integer } r \geqq 0. \tag{4}$$

In this paper we focus mainly on the class of noise functions σ for which there exists a constant $C > 0$ such that

$$\sigma(0) = 0, \qquad |\sigma(u) - \sigma(v)| \leqq C|u - v| \quad \forall u, v \in \mathbb{R}. \tag{5}$$

This can be generalized to wider classes for different results in terms of existence, stability, and continuous dependence, respectively; see Section 6 for more details. One reason for requiring $\sigma(0) = 0$ is that it follows from the L^1-contraction principle that $E[\|u(t, \cdot)\|_{L^1(\mathbb{R}^d)}]$ is finite. Similarly, the Lipschitz continuity of $\sigma(u)$ is required for the existence and uniform L^p-estimates of solutions.

Stochastic partial differential equations arise in a number of problems concerning random phenomena occurring in biology, physics, engineering, and economics. In recent years, there has been an increased interest in studying the effect of stochastic forcing on solutions of nonlinear partial differential equations. Of specific interest is the effect of noise on discontinuous waves, since these are often the relevant solutions; an important issue concerns the well-posedness (existence, uniqueness, and stability) of discontinuous solutions.

The fundamental fluid dynamics models are based on the compressible Navier–Stokes equations and the Euler equations. However, abundant experimental observations suggest that the chaotic nature of many high-velocity fluid dynamics phenomena calls for their stochastic formulation. Indeed, in these flows with large Reynolds numbers, microscopic perturbations get amplified to macroscopic scales, giving rise to unsteady flow patterns which deviate significantly from those predicted by the classical Navier–Stokes/Euler models. The stochastic Euler or Navier–Stokes equations seem to be more viable models. In the present paper we are interested in nonlinear hyperbolic equations with stochastic forcing, so-called stochastic balance laws. These balance laws can be viewed as a simple caricature of the stochastic Euler equations.

When $\sigma \equiv 0$, Equation (1) becomes a nonlinear conservation law for which the maximum principle holds. A satisfactory well-posedness theory is now available (see [4]). Some efforts have been made toward the analysis of nonlinear stochastic balance laws. In [8], a one-dimensional stochastic balance law was analyzed for u_0 in L^∞ and compactly supported $\sigma = \sigma(u)$, which ensures an L^∞-bound. A splitting method was used to construct approximate solutions, and it was shown that a subsequence of these approximations converges to a (possible non-unique) weak solution.

For general σ, the maximum principle is not available. Indeed, even for initial data $u_0 \in L^\infty$, the solution is not in L^∞ generically. For $\sigma = \sigma(t, x)$ in $C_t(W_x^{1,\infty})$

and with compact support in x, Kim [10] established the existence and uniqueness of entropy solutions in the one-dimensional case; see also [21]. For general $\sigma = \sigma(x, u)$ depending on u and for multidimensional equations in the L^p-framework, the uniqueness of strong stochastic entropy solutions was first established in Feng–Nualart [7], but the existence result was restricted to one dimension. We refer to the recent paper Debussche–Vovelle [5] for multidimensional results via a kinetic formulation.[1] For the L^p–theory of deterministic conservations laws, see [20].

One of our main observations is that uniform spatial BV-bound is preserved for stochastic balance laws with noise functions $\sigma(u)$ satisfying (5). This yields the existence of strong stochastic entropy solutions in $L^p \cap \mathrm{BV}$, as well as in L^p, for multidimensional balance laws (1). Furthermore, we develop a "continuous dependence" theory for stochastic entropy solutions in BV, which can be used, for example, to derive an error estimate for the vanishing viscosity method. Whenever $\sigma = \sigma(\mathbf{x}, u)$ has a dependency on the spatial position $\mathbf{x}$, BV-estimates are no longer available, but we show that the continuous dependence framework can be used to derive local fractional BV-estimates, which, in turn, can be used as before via a temporal equicontinuity estimate to establish a multidimensional existence result.

Besides providing an existence result in a multidimensional context by standard methods, one reason for singling out the class of nonlinear balance laws defined by (5) is that it makes a natural test bed for numerical analysis, without having to account for all the added technical complications in a pure L^p-framework. Moreover, by assuming $\sigma(a) = \sigma(b) = 0$ for some constants $a < b$, one ensures that the solution remains bounded between a and b if the initial function u_0 does so. Consequently, it is possible to identify a class of stochastic balance laws for which $L^p \cap \mathrm{BV}$, or even $L^\infty \cap \mathrm{BV}$, supplies a relevant and technically simple functional setting, tailored for the construction and analysis of numerical methods.

For other related results, we refer to Sinai [19] and E–Khanin–Mazel–Sinai [6] for the existence, uniqueness, and weak convergence of invariant measures for the one-dimensional Burgers equation with stochastic forcing which is periodic in x, as well as the structure and regularity properties of the solutions that live on the support of this measure. We also refer to Lions–Souganidis [13–16] for Hamilton–Jacobi equations with stochastic forcing and the so-called "stochastic" viscosity solutions.

We employ the vanishing viscosity method to establish the existence of stochastic entropy solutions. To this end, consider the stochastic viscous conservation law

$$\partial_t u^\varepsilon(t, \mathbf{x}) + \nabla \cdot \mathbf{f}(u^\varepsilon(t, \mathbf{x})) = \sigma(u^\varepsilon(t, \mathbf{x}))\partial_t W(t) + \varepsilon \Delta u^\varepsilon(t, \mathbf{x}) \tag{6}$$

for any fixed $\varepsilon > 0$, with initial data

$$u^\varepsilon(0, \mathbf{x}) = u_0^\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{7}$$

[1] We became aware of this paper after our main results were obtained.

where $u_0^\varepsilon(\mathbf{x})$ is a smooth approximation to $u_0(\mathbf{x})$ with

$$E\left[\int_{\mathbb{R}^d} \left|u_0^\varepsilon(\mathbf{x})\right|^p \,\mathrm{d}\mathbf{x}\right] \leqq E\left[\int_{\mathbb{R}^d} |u_0(\mathbf{x})|^p \,\mathrm{d}\mathbf{x}\right]$$

and, if $u_0 \in \mathrm{BV}(\mathbb{R}^d)$,

$$E\left[\int_{\mathbb{R}^d} \left|\nabla u_0^\varepsilon(\mathbf{x})\right| \,\mathrm{d}\mathbf{x}\right] \leqq E\left[\int_{\mathbb{R}^d} |\nabla u_0(\mathbf{x})| \,\mathrm{d}\mathbf{x}\right].$$

In addition, $E[\int_{\mathbb{R}^d} |\nabla^2 u_0^\varepsilon(\mathbf{x})| \,\mathrm{d}\mathbf{x}] < \infty$ for each fixed ε.

With regard to (6), we should replace $(\mathbf{f}, \sigma)$ by appropriate smooth approximations $(\mathbf{f}^\varepsilon, \sigma^\varepsilon)$. However, mainly to ease the presentation throughout this paper, we will not do that but instead simply assume that $(\mathbf{f}, \sigma)$ are sufficiently smooth (see [7]) in order to ensure the validity of our calculations. At times, we will do the same with the initial data.

The existence of global smooth solutions to (6)–(7) is established in [7], along with the following uniform estimates for $p \geqq 1$ and $T > 0$:

$$\sup_{\varepsilon>0}\sup_{0\leqq t\leqq T} E\left[\|u^\varepsilon(t,\cdot)\|^p_{L^p(\mathbb{R}^d)}\right] + \sup_{\varepsilon>0} E\left[\varepsilon\int_0^T \|\nabla u^\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t\right] < \infty. \tag{8}$$

The solution satisfies

$$\begin{aligned} u^\varepsilon(t,\mathbf{x}) = &\int_{\mathbb{R}^d} G_\varepsilon(t, \mathbf{x}-\mathbf{y}) u_0(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &- \int_0^t \int_{\mathbb{R}^d} G_\varepsilon(t-s, \mathbf{x}-\mathbf{y}) \nabla\cdot \mathbf{f}(u^\varepsilon(t,\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^d} G_\varepsilon(t-s, \mathbf{x}-\mathbf{y}) \sigma(u^\varepsilon(s,\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}W(s), \end{aligned} \tag{9}$$

where $G_\varepsilon(t, \mathbf{x})$ is the heat kernel:

$$G_\varepsilon(t,\mathbf{x}) = \frac{1}{(4\pi\varepsilon t)^{d/2}} e^{-\frac{|\mathbf{x}|^2}{4\varepsilon t}}, \quad t > 0.$$

Using (3) and (8)–(9), it follows that, for each fixed $\varepsilon > 0$,

$$E\left[\|(\nabla, \Delta) u^\varepsilon\|_{L^1((0,T)\times\mathbb{R}^d)}\right] < \infty \quad \text{for any finite } T > 0, \tag{10}$$

that is, ∇u^ε and $\nabla^2 u^\varepsilon$ are integrable for each fixed $\varepsilon > 0$. With different methods, we will later prove an ε-uniform spatial BV-estimate.

The remaining part of this paper is organized as follows: In Section 2, we prove the uniform spatial BV-bound for stochastic viscous solutions $u^\varepsilon(t, \mathbf{x})$. In Section 3, based on the BV-bound, we establish the equicontinuity of $u^\varepsilon(t, \mathbf{x})$ in $t > 0$, uniformly in the viscosity coefficient $\varepsilon > 0$. With these uniform estimates, we establish in Section 4 the existence of stochastic entropy solutions in $L^p \cap \mathrm{BV}$, as the vanishing viscosity limits for problem (6)–(7) with initial data in $L^p \cap \mathrm{BV}$. Combining this existence result with the L^1-stability theory in Feng–Nualart [7] leads to the well-posedness in L^p for problem (1)–(2). We further establish estimates for the "continuous dependence on the nonlinearities" for BV stochastic entropy solutions in Section 5, which also lead to an error estimate for (6)–(7). Various further generalizations of the results are discussed in Section 6.

2. Uniform Spatial BV-Estimate

As indicated in Section 1, we have available regularity and uniform L^p-estimates (8) for the viscous solutions $u^\varepsilon(t, \mathbf{x})$ of (6)–(7). In this section, we establish the uniform L^1-estimate for ∇u^ε, that is, the uniform BV-estimate of $u^\varepsilon(t, \mathbf{x})$ in the spatial variables $\mathbf{x}$.

Before we do that, let us indicate why the BV-estimate does not seem to be available when the noise coefficient function $\sigma = \sigma(x, u)$ depends on the spatial position x, even if that dependence is C^∞ (see Section 6 for fractional BV-estimate). To this end, it suffices to consider the simple stochastic differential equation:

$$\mathrm{d}u = \sigma(x, u)\,\mathrm{d}W(t), \qquad u(0) = u_0(x), \quad x \in \mathbb{R},$$

where we have dropped nonlinear transport effects and restricted ourselves to one spatial dimension. The spatial derivative $v = \partial_x u$ satisfies

$$\mathrm{d}v = (\sigma_u(x, u)v + \sigma_x(x, u))\,\mathrm{d}W(t).$$

Let η be a C^2-function. By Ito's formula,

$$\begin{aligned}\mathrm{d}\eta(v) = \eta'(v)(\sigma_u(x, u)v + \sigma_x(x, u))\,\mathrm{d}W(t)\\ +\frac{1}{2}\eta''(v)(\sigma_u(x, u)v + \sigma_x(x, u))^2\,\mathrm{d}t.\end{aligned}$$

Integrating in x and taking expectations, it follows that

$$\begin{aligned}E\left[\int \eta(v(t))\,\mathrm{d}x\right] &= E\left[\int \eta(v(0))\,\mathrm{d}x\right]\\ &\quad +E\left[\int_0^t \int \frac{1}{2}\eta''(v)(\sigma_u(x, u)v + \sigma_x(x, u))^2\,\mathrm{d}x\,\mathrm{d}s\right].\end{aligned}$$

Modulo an approximation argument, we can take $\eta(\cdot)$ as $|\cdot|$. Unless $\sigma_x \equiv 0$, the second term on the right-hand side does not seem to be controllable (this term vanishes when $\sigma_x \equiv 0$).

Let us now continue with the derivation of the BV-estimate for (6). We will need a C^2-approximation of the Kruzkov entropy. Let $\bar{\eta} : \mathbb{R} \to \mathbb{R}$ be a C^2-function satisfying

$$\bar{\eta}(0) = 0, \quad \bar{\eta}(-r) = \bar{\eta}(r), \quad \bar{\eta}'(-r) = -\bar{\eta}'(r), \quad \bar{\eta}'' \geqq 0, \tag{11}$$

and

$$\bar{\eta}'(r) = \begin{cases} -1, & \text{when } r < -1,\\ \in [-1, 1], & \text{when } |r| \leqq 1,\\ +1, & \text{when } r > 1.\end{cases}$$

For any $\rho > 0$, define the function $\eta_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\eta_\rho(r) = \rho\bar{\eta}\left(\frac{r}{\rho}\right).$$

Then

$$|r| - M_1\rho \leqq \eta_\rho(r) \leqq |r|, \quad \left|\eta_\rho''(r)\right| \leqq \frac{M_2}{\rho}\mathbf{1}_{|r|<\rho}, \tag{12}$$

where

$$M_1 = \sup_{|r|\leqq 1} \big|\,|r| - \bar{\eta}(r)\big|, \quad M_2 = \sup_{|r|\leqq 1} \left|\bar{\eta}''(r)\right|. \tag{13}$$

We will frequently utilize the Burkholder–Davis–Gundy inequality, which we now recall. For $p > 0$, there exists a constant $C = C_p$ such that, if M_t is a continuous martingale and t a stopping time, then

$$E\left[\sup_{s\leqq t} |M_s|^p\right] \leqq C_p E[\langle M\rangle_t^{p/2}],$$

where $\langle M\rangle_t$ is the quadratic variation of M_t.

Theorem 1 (Spatial BV-estimate). *Suppose that* (3)–(5) *hold. Let* $u^\varepsilon(t,\mathbf{x})$ *be the solution of* (6)–(7)*. Then, for* $t > 0$,

$$E\left[\int_{\mathbb{R}^d} |\nabla u^\varepsilon(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\right] \leqq E\left[\int_{\mathbb{R}^d} |\nabla u_0^\varepsilon(\mathbf{x})|\,\mathrm{d}\mathbf{x}\right] \leqq E\left[\int_{\mathbb{R}^d} |\nabla u_0(\mathbf{x})|\,\mathrm{d}\mathbf{x}\right].$$

Proof. Taking the derivative of (6) with respect to x_i, $1 \leqq i \leqq d$, we obtain

$$\partial_t(u_{x_i}^\varepsilon) + \nabla\cdot(\mathbf{f}'(u^\varepsilon(t,\mathbf{x}))u_{x_i}^\varepsilon) = \sigma'(u^\varepsilon(t,\mathbf{x}))u_{x_i}^\varepsilon\partial_t W(t) + \varepsilon\Delta(u_{x_i}^\varepsilon).$$

Applying Ito's formula to $\eta_\rho(u_{x_i}^\varepsilon)$ yields

$$\begin{aligned}\partial_t\eta_\rho(u_{x_i}^\varepsilon) &= \eta_\rho'(u_{x_i}^\varepsilon)\sigma'(u^\varepsilon)u_{x_i}^\varepsilon\partial_t W(t)\\ &\quad+\eta_\rho'(u_{x_i}^\varepsilon)(\varepsilon\Delta u_{x_i}^\varepsilon - \nabla\cdot(\mathbf{f}'(u^\varepsilon)u_{x_i}^\varepsilon))\\ &\quad+\frac{1}{2}\eta_\rho''(u_{x_i}^\varepsilon)(\sigma'(u^\varepsilon)u_{x_i}^\varepsilon)^2.\end{aligned} \tag{14}$$

We observe that

$$\begin{aligned}\varepsilon\eta_\rho'(u_{x_i}^\varepsilon)\Delta(u_{x_i}^\varepsilon) &= \varepsilon(\nabla\cdot(\eta_\rho'(u_{x_i}^\varepsilon)\nabla u_{x_i}^\varepsilon) - \eta_\rho''(u_{x_i}^\varepsilon)\left|\nabla u_{x_i}^\varepsilon\right|^2)\\ &= \varepsilon(\Delta\eta_\rho(u_{x_i}^\varepsilon) - \eta_\rho''(u_{x_i}^\varepsilon)|\nabla u_{x_i}^\varepsilon|^2)\\ &\leqq \varepsilon\Delta\eta_\rho(u_{x_i}^\varepsilon),\end{aligned} \tag{15}$$

by using the convexity of η_ρ and interpreting $\Delta\eta_\rho(u_{x_i}^\varepsilon)$ in the distributional sense. Here we have used that $\nabla u_{x_i}^\varepsilon$, $1 \leqq i \leqq d$, are integrable, see (10), so that they vanish at infinity.

Integrating (14) with respect to $\mathbf{x}$, using (10) and (15), and noting that

$$\int_{\mathbb{R}^d}\int_0^t \eta'(u^\varepsilon_{x_i})\sigma'(u^\varepsilon)u^\varepsilon_{x_i}\,\mathrm{d}W(s)\,\mathrm{d}\mathbf{x}$$

is a martingale, we arrive at

$$\begin{aligned} &E\left[\int_{\mathbb{R}^d}\eta_\rho(u^\varepsilon_{x_i}(t,\mathbf{x}))\,\mathrm{d}\mathbf{x}\right]-E\left[\int_{\mathbb{R}^d}\eta_\rho(u^\varepsilon_{x_i}(0,\mathbf{x}))\,\mathrm{d}\mathbf{x}\right]\\ &\quad\leqq E\left[-\int_0^t\int_{\mathbb{R}^d}\eta'_\rho(u^\varepsilon_{x_i})\nabla\cdot(\mathbf{f}'(u^\varepsilon)u^\varepsilon_{x_i})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right.\\ &\qquad\quad\left.+\frac{1}{2}\int_0^t\int\eta''_\rho(u^\varepsilon_{x_i})(\sigma'(u^\varepsilon)u^\varepsilon_{x_i})^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right]. \end{aligned}\tag{16}$$

Now we send $\rho\to 0$ in (16). By the dominated convergence theorem,

$$\begin{aligned} &E\left[\int_{\mathbb{R}^d}\left|u^\varepsilon_{x_i}(t,\mathbf{x})\right|\,\mathrm{d}\mathbf{x}\right]\\ &\quad\leqq E\left[\int_{\mathbb{R}^d}\left|u^\varepsilon_{x_i}(0,\mathbf{x})\right|\,\mathrm{d}\mathbf{x}\right]-\lim_{\rho\to0}E\left[\int_0^t\int_{\mathbb{R}^d}\eta'_\rho(u^\varepsilon_{x_i})\nabla\cdot(\mathbf{f}'(u^\varepsilon)u^\varepsilon_{x_i})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right]\\ &\qquad+\lim_{\rho\to0}\frac{1}{2}E\left[\int_0^t\int_{\mathbb{R}^d}\eta''_\rho(u^\varepsilon_{x_i})(\sigma'(u^\varepsilon)u^\varepsilon_{x_i})^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right]\\ &\quad=:E\left[\int_{\mathbb{R}^d}\left|u^\varepsilon_{x_i}(0,\mathbf{x})\right|\,\mathrm{d}\mathbf{x}\right]+I_1+I_2. \end{aligned}$$

For the term I_1,

$$\begin{aligned} |I_1|&=\lim_{\rho\to0}\left|E\left[\int_0^t\int_{\mathbb{R}^d}\nabla\cdot(\mathbf{f}'(u^\varepsilon)\eta'_\rho(u^\varepsilon_{x_i})u^\varepsilon_{x_i})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right]\right|\\ &\quad+\lim_{\rho\to0}\left|E\left[\int_0^t\int_{\mathbb{R}^d}\eta''_\rho(u^\varepsilon_{x_i})u^\varepsilon_{x_i}\nabla u^\varepsilon_{x_i}\cdot\mathbf{f}'(u^\varepsilon)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right]\right|\\ &\leqq C\lim_{\rho\to0}E\left[\int_0^t\int_{\mathbb{R}^d}\left|u^\varepsilon_{x_i}\right|\frac{1}{\rho}\chi_{[-\rho,\rho]}(u^\varepsilon_{x_i})\left|\nabla u^\varepsilon_{x_i}\right|\left|\mathbf{f}'(u^\varepsilon)\right|\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right]. \end{aligned}$$

Notice that

$$\left|u^\varepsilon_{x_i}\right|\frac{1}{\rho}\chi_{[-\rho,\rho]}(u^\varepsilon_{x_i})\to 0\quad\text{for almost everywhere }(t,\mathbf{x})\text{ almost surely as }\rho\to0,$$

and

$$\left|u^\varepsilon_{x_i}\right|\frac{1}{\rho}\chi_{[-\rho,\rho]}(u^\varepsilon_{x_i})\left|\nabla u^\varepsilon_{x_i}\right|\left|\mathbf{f}'(u^\varepsilon)\right|\leqq C\left(\left|\nabla u^\varepsilon_{x_i}\right|^2+\left|u^\varepsilon\right|^{2(r-1)}\right),$$

where the right-side term of the inequality is integrable and independent of $\rho>0$. Then the dominated convergence theorem implies that $|I_1|=0$.

Next we consider I_2. By condition (5) and estimate (12), we have

$$\left|\eta_\rho''(u^\varepsilon_{x_i})(\sigma'(u^\varepsilon)u^\varepsilon_{x_i})^2\right| = \left|\eta_\rho''(u^\varepsilon_{x_i})\right| \left|u^\varepsilon_{x_i}\right|^2 (\sigma'(u^\varepsilon))^2 \leqq C\left|u^\varepsilon_{x_i}\right| \mathbf{1}_{\{|u^\varepsilon_{x_i}|<\rho\}} \leqq C\left|u^\varepsilon_{x_i}\right| \in L^1((0,T)\times\mathbb{R}^d).$$

On the other hand, since $|u^\varepsilon_{x_i}|$ is integrable and independent of $\rho > 0$, and

$$\left|u^\varepsilon_{x_i}\right| \mathbf{1}_{\left|\left|u^\varepsilon_{x_i}\right|<\rho\right\}} \to 0 \quad \text{for almost everywhere } (t,\mathbf{x}) \text{ almost surely as } \rho \to 0,$$

the dominated convergence theorem again implies $|I_2| = 0$. □

3. Uniform Temporal L^1-Continuity

In this section, we establish the uniform temporal L^1-continuity of $u^\varepsilon(t,\mathbf{x})$, independent of the viscosity coefficient $\varepsilon > 0$.

Theorem 2 (Temporal L^1-continuity). *Suppose that* (3)–(5) *hold. Let* $u^\varepsilon(t,\mathbf{x})$ *be the solution of* (6)–(7). *Let* $D \subset \mathbb{R}^d$ *be a bounded domain in* $\mathbb{R}^d$ *and* $T > 0$ *finite. Then, for any small* $\Delta t > 0$, *there exists a constant* $C > 0$ *independent of* Δt *such that*

$$E\left[\int_0^{T-\Delta t}\int_D \left|u^\varepsilon(t+\Delta t,\mathbf{x}) - u^\varepsilon(t,\mathbf{x})\right| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right] \leqq C(\Delta t)^{1/3} \to 0 \quad \textit{as } \Delta t \to 0. \tag{17}$$

Proof. Fix $\Delta t > 0$. For $t \in [0, T-\Delta t]$, set $w^\varepsilon(t,\cdot) := u^\varepsilon(t+\Delta t,\cdot) - u^\varepsilon(t,\cdot)$. Then, for any $\varphi \in L^\infty(0,T;C_0^\infty(D))$, we have

$$\begin{aligned}
&\int_D w^\varepsilon(t,\mathbf{x})\varphi(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&= \int_D \left(\int_t^{t+\Delta t} \partial_s u^\varepsilon(s,\mathbf{x})\,\mathrm{d}s\right)\varphi(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&= \int_t^{t+\Delta t}\int_D \mathbf{f}(u^\varepsilon(s,\mathbf{x}))\cdot\nabla\varphi(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\\
&\quad -\varepsilon\int_t^{t+\Delta t}\int_D \nabla u^\varepsilon(s,\mathbf{x})\cdot\nabla\varphi(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\\
&\quad +\int_t^{t+\Delta t}\int_D \sigma(u^\varepsilon(s,\mathbf{x}))\varphi(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}W(s).
\end{aligned} \tag{18}$$

For each $t \in [0, T-\Delta t]$, take $\delta > 0$, set

$$D_{-\delta} := \{\mathbf{x} \in D : \mathrm{dist}(\mathbf{x},\partial D) \geqq \delta\},$$

and denote by $\chi_{D_{-\delta}}(\cdot)$ its characteristic function.

Let $J \in C_c^\infty(\mathbb{R}^d)$ be the standard mollifier defined by

$$J(\mathbf{x}) = \begin{cases} C \exp\left(\frac{1}{|\mathbf{x}|^2-1}\right) & \text{if } |\mathbf{x}| < 1, \\ 0 & \text{if } |\mathbf{x}| \geqq 1, \end{cases} \tag{19}$$

where the constant $C > 0$ is chosen so that $\int_{\mathbb{R}^d} J(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$. For each $\delta > 0$, we take

$$\varphi := \varphi_\delta(t,\mathbf{x}) = \delta^{-d} \int_{\mathbb{R}^d} J\left(\frac{\mathbf{x}-\mathbf{y}}{\delta}\right) \operatorname{sgn}(w(t,\mathbf{y}))\chi_{D_{-\delta}}(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

in (18). It is clear that $\|\varphi_\delta\|_{L^\infty(D)} + \delta\,\|\nabla\varphi_\delta\|_{L^\infty(D)} \leqq C$, uniformly in t, for some constant $C > 0$ independent of $\delta > 0$.

Integrating (18) in t from 0 to $T - \Delta t$ yields

$$\begin{aligned}
&\int_0^{T-\Delta t}\int_D \left|w^\varepsilon(t,\mathbf{x})\right|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \int_0^{T-\Delta t}\int_t^{t+\Delta t}\int_D \mathbf{f}(u^\varepsilon(s,\mathbf{x}))\cdot\nabla\varphi_\delta(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad - \int_0^{T-\Delta t}\int_t^{t+\Delta t}\int_D \varepsilon\nabla u^\varepsilon(s,\mathbf{x})\cdot\nabla\varphi_\delta(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^{T-\Delta t}\left(\int_t^{t+\Delta t}\left(\int_D \sigma(u^\varepsilon(s,\mathbf{x}))\varphi_\delta(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\,\mathrm{d}W(s)\right)\,\mathrm{d}t \\
&\quad + \int_0^{T-\Delta t}\int_D w^\varepsilon(t,\mathbf{x})(w^\varepsilon(t,\mathbf{x}) - \varphi_\delta(t,\mathbf{x}))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&:= \sum_{j=1}^4 I_j^\delta.
\end{aligned}$$

We examine these parts separately.

Thanks to the polynomial growth of $\mathbf{f}$ and (8),

$$\left|E[I_1^\delta]\right| \leqq C\frac{\Delta t}{\delta}\,\|\mathbf{f}\|_{L^1(D\times(0,T))} \leqq C(T,D)\frac{\Delta t}{\delta}.$$

For the term I_2^δ, we have

$$\begin{aligned}
\left|E[I_2^\delta]\right| &\leqq C\left(E\left[\int_0^{T-\Delta t}\int_D\left(\int_t^{t+\Delta t}\sqrt{\varepsilon}|\nabla u^\varepsilon(s,\mathbf{x})|\,\mathrm{d}s\right)^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right]\right)^{\frac{1}{2}} \\
&\qquad\times\left(E\left[\int_0^{T-\Delta t}\int_D \varepsilon|\nabla\varphi_\delta|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right]\right)^{\frac{1}{2}} \\
&\leqq C\Delta t\left(E\left[\int_0^{T-\Delta t}\int_D |\nabla\varphi_\delta|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right]\right)^{\frac{1}{2}} \\
&\leqq C(T,D)\frac{\Delta t}{\delta},
\end{aligned}$$

where the second inequality follows from the energy estimate (8):

$$\sup_{\varepsilon>0} E\left[\varepsilon \int_0^T \left\|\nabla u^\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R}^d)} \,\mathrm{d}t\right] < \infty.$$

For the term I_3^δ, by the Burkholder–Davis–Gundy inequality applied to the martingale $0 \leqq \Delta t \mapsto \int_t^{t+\Delta t} (\int_D \sigma(u^\varepsilon(s,\mathbf{x}))\varphi_\delta(t,\mathbf{x})\,\mathrm{d}\mathbf{x})\,\mathrm{d}W(s)$, we have

$$\begin{aligned}
\left|E[I_3^\delta]\right| &\leqq C\int_0^{T-\Delta t} E\left[\left(\int_t^{t+\Delta t}\left(\int_D \sigma(u^\varepsilon(s,\mathbf{x}))\varphi_\delta(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^2 \mathrm{d}s\right)^{\frac{1}{2}}\right]\mathrm{d}t\\
&\leqq C\left(E\left[\int_0^{T-\Delta t}\int_t^{t+\Delta t}\int_D (\sigma(u^\varepsilon(s,\mathbf{x}))\varphi_\delta(t,\mathbf{x}))^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\,\mathrm{d}t\right]\right)^{\frac{1}{2}}\\
&\leqq C\left(E\left[\int_0^{\Delta t}\int_0^{T-\Delta t}\int_D (\sigma(u^\varepsilon(s+t,\mathbf{x})))^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\,\mathrm{d}s\right]\right)^{\frac{1}{2}}\\
&\leqq C\sqrt{\Delta t}\left(E\left[\int_0^T\int_D (\sigma(u^\varepsilon(t,\mathbf{x})))^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right]\right)^{\frac{1}{2}}\\
&\leqq C\sqrt{\Delta t}\left(E\left[\int_0^T\int_D |u^\varepsilon(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right]\right)^{\frac{1}{2}}\\
&\leqq C\sqrt{\Delta t},
\end{aligned}$$

where we have used that $\sup_{\varepsilon>0} E[\|u^\varepsilon(t)\|_2^2] < \infty$, uniformly in $t > 0$.

This L^2-bound also implies

$$\begin{aligned}
&E\left[\int_0^T\int_{D\setminus D_{-2\delta}} |u^\varepsilon(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right]\\
&\quad\leqq C\,(E[\|u^\varepsilon\|_2^2])^{\frac{1}{2}}\left(E\left[\int_0^T\int_{D\setminus D_{-2\delta}}\mathrm{d}\mathbf{x}\,\mathrm{d}t\right]\right)^{\frac{1}{2}}\\
&\quad\leqq C\sqrt{\delta}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|E[I_4^\delta]\right| &\leqq 2E\left[\int_0^{T-\Delta t}\int_{D\setminus D_{-2\delta}} |w(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right]\\
&\quad + E\left[\int_0^{T-\Delta t}\int_{D_{-2\delta}}\Big||w(t,\mathbf{x})|\right.\\
&\qquad\left. - w(t,\mathbf{x})\int_{\mathbb{R}^d}\delta^{-d}J\left(\frac{\mathbf{x}-\mathbf{y}}{\delta}\right)\operatorname{sgn}(w(t,\mathbf{y}))\Big|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right]\\
&\leqq C\sqrt{\delta} + E\left[\int_0^{T-\Delta t}\int_{D_{-2\delta}}\int_{\mathbb{R}^d}\delta^{-d}J\left(\frac{\mathbf{x}-\mathbf{y}}{\delta}\right)\right.\\
&\qquad\left.\times\big|\,|w(t,\mathbf{x})| - w(t,\mathbf{x})\operatorname{sgn}(w(t,\mathbf{y}))\big|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right]
\end{aligned}$$

$$
\begin{aligned}
&\leqq C\sqrt{\delta} \\
&\quad + CE\left[\int_0^{T-\Delta t}\int_{D_{-2\delta}}\int_{\mathbb{R}^d}\delta^{-d}J\left(\frac{\mathbf{x}-\mathbf{y}}{\delta}\right)|w(t,\mathbf{x})-w(t,\mathbf{y})|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right] \\
&\leqq C\sqrt{\delta}+CE\left[\int J(z)\int_0^T\int_{D_{-2\delta}}|u^{\varepsilon}(t,\mathbf{x})-u^{\varepsilon}(t,\mathbf{x}-\delta z)|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\,\mathrm{d}z\right] \\
&\leqq C\sqrt{\delta}+4\delta\leqq C\sqrt{\delta},
\end{aligned}
$$

where the third inequality follows from $||a|-a\,\mathrm{sgn}(b)|\leqq 2|a-b|$ for any $a,b\in\mathbb{R}$. The fifth inequality follows, since u^{ε} belongs to BV in $\mathbf{x}$.

Setting $\rho(\Delta t)=\inf_{\delta>0}\{C_1\frac{\Delta t}{\delta}+C_2\sqrt{\Delta t}+C_3\sqrt{\delta}\}$, it follows that

$$
\int_0^{T-\Delta t}\int_D|w(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\leqq\rho(\Delta t).
$$

The function $\rho(\cdot)$ reaches the infimum at $\delta=C(\Delta t)^{\frac{2}{3}}$, and hence

$$
\int_0^{T-\Delta t}\int_D|w(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\leqq C(\Delta t)^{\frac{1}{3}}\to 0\quad\text{as }\Delta t\to 0.
$$

This concludes the proof of the theorem. □

Remark 1. Since the Brownian sample paths are α-Hölder continuous for every $\alpha<\frac{1}{2}$, a fractional order in the temporal L^1-continuity in (17) is expected. The proof of Theorem 2 uses an idea due to Kruzkov [11].

4. Well-Posedness Theory in L^p

Before we introduce the relevant notions of generalized solutions, let us define what is meant by an entropy–entropy flux pair $(\eta,\mathbf{q})$, or more simply an entropy pair, namely a C^2 function $\eta:\mathbb{R}\to\mathbb{R}$ such that η',η'' have at most polynomial growth, with corresponding entropy flux $\mathbf{q}$ defined by $\mathbf{q}'(u)=\eta'(u)\mathbf{f}'(u)$. An entropy pair is called convex if $\eta''(u)\geqq 0$.

Definition 1 (*Stochastic entropy solutions*). An $\{\mathcal{F}_t\}_{t\geqq 0}$-adapted and $L^2(\mathbb{R}^d)$-valued stochastic process $u=u(t)=u(t,\mathbf{x};\omega)$ is called a *stochastic entropy solution* of the balance law (1) with initial data (2) provided that the following conditions hold:

(i) for $p=1,2,\ldots$,

$$
\sup_{0\leqq t\leqq T}E\left[\|u(t)\|^p_{L^p(\mathbb{R}^d)}\right]<\infty,\quad\text{for any }T>0;
$$

(ii) for any convex entropy pair $(\eta, \mathbf{q})$ and any $0 < s < t$,

$$
\begin{aligned}
&-\left(\int_{\mathbb{R}^d} \eta(u(t,\mathbf{x}))\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^d} \eta(u(s,\mathbf{x}))\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\\
&\quad + \int_s^t \int_{\mathbb{R}^d} \mathbf{q}(u(\tau,\mathbf{x}))\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau\\
&\quad + \int_s^t \int_{\mathbb{R}^d} \frac{1}{2}\eta''(u(\tau,\mathbf{x}))(\sigma(u(\tau,\mathbf{x})))^2\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau\\
&\quad + \int_s^t \left(\int_{\mathbb{R}^d} \eta'(u(\tau,\mathbf{x}))\sigma(u(\tau,\mathbf{x}))\varphi\,\mathrm{d}\mathbf{x}\right)\mathrm{d}W(\tau) \geqq 0,
\end{aligned}
$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \geqq 0$, where $\int_s^t (\cdots)\,\mathrm{d}W(\tau)$ is an Ito integral.

To motivate the next definition, let us make a formal attempt to derive the L^1-contraction property for stochastic entropy solutions. To this end, consider smooth (in x) solutions to the one-dimensional problems:

$$
\begin{aligned}
\mathrm{d}u + \partial_x f(u)\,\mathrm{d}t &= \sigma(u)\,\mathrm{d}W, \quad u|_{t=0} = u_0,\\
\mathrm{d}v + \partial_x f(v)\,\mathrm{d}t &= \sigma(v)\,\mathrm{d}W, \quad v|_{t=0} = v_0.
\end{aligned}
$$

Subtracting the two stochastic conservation laws yields

$$
\mathrm{d}(u-v) = -[\partial_x(f(u)-f(v))]\,\mathrm{d}t + [\sigma(u)-\sigma(v)]\,\mathrm{d}W.
$$

Let $\eta(\cdot)$ be an entropy. An application of the chain rule (Ito's formula) now yields

$$
\begin{aligned}
\mathrm{d}\eta(u-v) = \Big[&-\partial_x(\eta'(u-v)(f(u)-f(v)))\\
&+\eta''(u-v)(f(u)-f(v))\partial_x(u-v)\\
&+\frac{1}{2}\eta''(u-v)(\sigma(u)-\sigma(v))^2\Big]\,\mathrm{d}t\\
&+\eta'(u-v)(\sigma(u)-\sigma(v))\,\mathrm{d}W,
\end{aligned}
$$

where the last term is a martingale. Choosing $\eta(\cdot) = |\cdot|$ yields $\eta''(\cdot) = \delta_0$ and the two "η'' terms" vanish. Consequently, after integrating and taking expectations, we arrive at the L^1-contraction (conservation) principle:

$$
E\left[\int |u(t)-v(t)|\,\mathrm{d}x\right] = E\left[\int |u_0 - v_0|\,\mathrm{d}x\right].
$$

Of course, for non-smooth solutions, the Ito formula is not available and we should instead derive the L^1-contraction principle from the (stochastic) entropy inequalities via Kruzkov's method.

Attempting precisely that, we write the entropy condition for $u(t) = u(t,x;\omega)$ with the entropy $\eta(u(t)-v(s,y;\omega))$, where $v(s,y;\omega)$ is being treated as a constant with respect to (t,x). Similarly, write the entropy condition for $v(s) = v(s,y;\omega)$ for the entropy $\eta(v(s)-u(t,x;\omega))$, with $u(t,x;\omega)$ being constant with respect to

(s, y). Take $\eta(\cdot) = |\cdot|$, and then $q(u, v) = \mathrm{sgn}(u - v)(f(u) - f(v))$. After adding together the two entropy inequalities, we formally obtain

$$\begin{aligned}(d_t + d_s)|u - v| \leqq \Big[&- (\partial_x + \partial_y)(\mathrm{sgn}(u - v)(f(u) - f(v))) \\ &+ \frac{1}{2}\delta(u - v)[(\sigma(u))^2 + (\sigma(v))^2]\Big] \,\mathrm{d}t\,\mathrm{d}s \\ &+ \mathrm{sgn}(u(t, x) - v(s, y))\sigma(u(t, x))\,\mathrm{d}W(t)\,\mathrm{d}s \\ &- \mathrm{sgn}(u(t, x) - v(s, y))\sigma(v(s, y))\,\mathrm{d}W(s)\,\mathrm{d}t.\end{aligned}$$

Depending on $t < s$ or $t > s$, one of the last two terms is not adapted, and this causes a problem for the Ito integral. In particular, by taking the expectation of the above inequality, only one of the last two terms vanishes. Moreover, to write $\frac{1}{2}\delta(u - v)[(\sigma(u))^2 + (\sigma(v))^2]$ in the favorable form:

$$\frac{1}{2}\delta(u - v)(\sigma(u) - \sigma(v))^2,$$

we are missing the cross term $2\sigma(u)\sigma(v)$. These difficulties can be effectively handled by the notion of "strong" stochastic entropy solutions.

Definition 2 (*Strong stochastic entropy solutions*). We refer to a stochastic entropy solution u of the balance law (1) with initial data (2) as a *strong stochastic entropy solution* if the following condition holds:

(iii) for each $\{\mathcal{F}_t\}_{t \geqq 0}$-adapted, $L^2(\mathbb{R})$-valued stochastic process $\tilde{u} = \tilde{u}(t) = \tilde{u}(t, \mathbf{x}; \omega)$ satisfying

$$\sup_{0 \leqq t \leqq T} E\left[\|\tilde{u}(t)\|^p_{L^p(\mathbb{R}^d)}\right] < \infty \quad \text{for any } T > 0,\ p = 1, 2, \ldots,$$

and for each entropy function $S : \mathbb{R} \to \mathbb{R}$, with

$$\overline{S}(r; v, \mathbf{y}) := \int_{\mathbb{R}^d} S'(\tilde{u}(r, \mathbf{x}) - v)\sigma(\tilde{u}(r, \mathbf{x}))\varphi(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{x},$$

where $r \geqq 0$, $v \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^d$, and $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, there exists a deterministic function $\Delta(s, t)$, $0 \leqq s \leqq t$, such that

$$\begin{aligned}&E\left[\int_{\mathbb{R}^d}\int_s^t \overline{S}(\tau; v = u(t, \mathbf{y}), \mathbf{y})\,\mathrm{d}W(\tau)\,\mathrm{d}\mathbf{y}\right] \\ &\leqq E\left[\int_s^t \int_{\mathbb{R}^d} \partial_v \overline{S}(\tau; v = \tilde{u}(\tau, \mathbf{y}), \mathbf{y})\sigma(u(\tau, \mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau\right] + \Delta(s, t),\end{aligned}$$

where $\Delta(\cdot, \cdot)$ is such that, for each $T > 0$, there exists a partition $\{t_i\}_{i=1}^m$ of $[0, T]$, $0 = t_0 < t_1 < \cdots < t_m = T$, so that

$$\lim_{\max_i |t_{i+1}-t_i|\to 0} \sum_{i=1}^{m} \Delta(t_i, t_{i+1}) = 0.$$

The notion of strong stochastic entropy solutions is due to Feng–Nualart [7], who proved the L^1-contraction property for these solutions:

$$E\left[\|u(t)-v(t)\|_{L^1(\mathbb{R}^d)}\right] \leqq E\left[\|u_0-v_0\|_{L^1(\mathbb{R}^d)}\right] \quad \text{for } t>0, \tag{20}$$

where $u(t)$ is any stochastic entropy solution with $u|_{t=0}=u_0$ and $v(t)$ is any *strong* stochastic entropy solution with $v|_{t=0}=v_0$. In (20), the entropy $|\cdot|$ can be replaced by $(\cdot)^+$, yielding the L^1-comparison principle.

Feng–Nualart [7] employed the compensated compactness method to prove an existence result in the one-dimensional context. The following theorem provides the existence of strong stochastic entropy solutions for a class of multidimensional equations.

Theorem 3 (Existence in $L^p \cap \mathrm{BV}$). *Suppose that* (3)–(5) *hold. Then there exists a strong stochastic entropy solution u of the balance law* (1) *with initial data* (2) *such that*

$$E\left[|u(t,\cdot)|_{\mathrm{BV}(\mathbb{R}^d)}\right] \leqq E\left[|u_0|_{\mathrm{BV}(\mathbb{R}^d)}\right] \quad \textit{for any } t \geqq 0. \tag{21}$$

Proof. For fixed $\varepsilon>0$, we mollify u_0 by $u_0^\varepsilon \in C^\infty$ so that $E[\|u_0^\varepsilon\|^2_{H^s(\mathbb{R}^d)}]$ is finite for any $s>0$, and

$$E\left[\|u_0^\varepsilon\|^p_{L^p(\mathbb{R}^d)} + |u_0^\varepsilon|_{\mathrm{BV}(\mathbb{R}^d)}\right] \leqq E\left[\|u_0\|^p_{L^p(\mathbb{R}^d)} + |u_0|_{\mathrm{BV}(\mathbb{R}^d)}\right] < \infty,$$

for any $p=1,2,\ldots$, and $u_0^\varepsilon(\mathbf{x}) \to u_0(\mathbf{x})$ for almost everywhere $\mathbf{x}$, almost surely as $\varepsilon \to 0$.

Now the same arguments as in Section 4 of Feng–Nualart [7] yield that there exists an $\mathcal{F}_t$-adapted stochastic process $u^\varepsilon = u^\varepsilon(t) \in C([0,\infty); L^2(\mathbb{R}^d))$ satisfying almost surely that

(i) $E[\|u^\varepsilon(t,\cdot)\|^2_{H^s(\mathbb{R}^d)}] < \infty$ for all $t>0$;
(ii) $\partial_{x_i x_j} u^\varepsilon(t,\cdot) \in C(\mathbb{R}^d)$ for all $i, j = 1, \ldots, d$;
(iii) For any $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \geqq 0$, and $0<s<t$,

$$\begin{aligned}
&\langle \eta(u^\varepsilon(t,\cdot)), \varphi\rangle - \langle \eta(u^\varepsilon(s,\cdot)), \varphi\rangle \\
&= \int_s^t \left\langle \mathbf{q}(u^\varepsilon(\tau,\cdot)), \nabla\varphi\right\rangle \mathrm{d}\tau + \frac{1}{2}\int_s^t \left\langle \eta''(u^\varepsilon(\tau,\cdot))(\sigma(u^\varepsilon(\tau,\cdot))^2, \varphi\right\rangle \mathrm{d}\tau \\
&\quad + \int_s^t \left\langle \eta'(u(\tau,\cdot))\sigma(u(\tau,\cdot)), \varphi\right\rangle \mathrm{d}W(\tau) \\
&\quad + \varepsilon \int_s^t \left(\left\langle \eta(u^\varepsilon(\tau,\cdot)), \Delta\varphi\right\rangle - \left\langle \eta''(u^\varepsilon(\tau,\cdot))|\nabla u^\varepsilon(\tau,\cdot)|^2, \varphi\right\rangle\right) \mathrm{d}\tau
\end{aligned}$$

$$\leqq \int_s^t \langle \mathbf{q}(u^\varepsilon(\tau,\cdot)), \nabla\varphi\rangle \, \mathrm{d}\tau + \frac{1}{2}\int_s^t \left\langle \eta''(u^\varepsilon(\tau,\cdot))(\sigma(u^\varepsilon(\tau,\cdot))^2, \varphi\right\rangle \mathrm{d}\tau$$

$$+ \int_s^t \langle \eta'(u^\varepsilon(\tau,\cdot))\sigma(u(\tau,\cdot)), \varphi\rangle \, \mathrm{d}W(\tau) + \mathcal{O}(\varepsilon),$$

where the first equality in (iii) follows from the Ito formula.

Combining the results established in Sections 2 and 3, we conclude that there exist a subsequence (still denoted) $\{u^\varepsilon(t,\mathbf{x})\}_{\varepsilon>0}$ and a limit $u(t,\mathbf{x})$ such that, as $\varepsilon \to 0$,

$$u^\varepsilon(t,\mathbf{x}) \to u(t,\mathbf{x}) \quad \text{for almost everywhere } (t,\mathbf{x}), \text{ almost surely,}$$

and the limit $u(t,\mathbf{x})$ satisfies (21). Arguing as in Feng–Nualart [7], we can pass to the limit in the entropy inequality (iii) to conclude that the limit $u(t,\mathbf{x})$ is a stochastic entropy solution (see Definition 1). Moreover, we can prove that u is a *strong* stochastic entropy solution (see Definition 2). □

Combining Theorem 3 with the L^1-stability result established in Feng–Nualart [7], we conclude

Theorem 4 (Well-posedness in L^p). *Suppose that* (4) *and* (5) *hold, and that* u_0 *satisfies*

$$E\left[\|u_0\|^p_{L^p(\mathbb{R}^d)}\right] < \infty, \quad p = 1, 2, \ldots.$$

(i) *Existence: There exists a strong stochastic entropy solution of the balance law* (1) *with initial data* (2) *such that, for any* $t \geqq 0$,

$$E\left[\|u(t,\cdot)\|^p_{L^p(\mathbb{R}^d)}\right] < \infty, \quad p = 1, 2, \ldots. \tag{22}$$

(ii) *Stability: Let* $u(t,\mathbf{x})$ *be a strong stochastic entropy solution of* (1) *with initial data* $u_0(\mathbf{x})$*, and let* $v(t,\mathbf{x})$ *be a stochastic entropy solution with initial data* $v_0(\mathbf{x})$*. Then, for any* $t > 0$,

$$E\left[\int_{\mathbb{R}^d} |u(t,\mathbf{x}) - v(t,\mathbf{x})| \, \mathrm{d}\mathbf{x}\right] \leqq E\left[\int_{\mathbb{R}^d} |u_0(\mathbf{x}) - v_0(\mathbf{x})| \, \mathrm{d}\mathbf{x}\right]. \tag{23}$$

Proof. For the $\cap_{p=1}^\infty L^p(\mathbb{R}^d)$-valued random variable u_0, we can approximate u_0 by $u_0^\delta(\mathbf{x})$ in L^1 as $\delta \to 0$, with $E[\|u_0^\delta\|_p^p + |u_0^\delta|_{BV}] < \infty$ for fixed $\delta > 0$. Then, according to Theorem 3, there exists a corresponding family of global strong entropy solutions $u^\delta(t,\mathbf{x})$ for $\delta > 0$.

The L^1-stability (contraction) result established in Feng–Nualart [7] implies that $u^\delta(t,\mathbf{x})$ is a Cauchy sequence in L^1, which yields the strong convergence of $u^\delta(t,\mathbf{x})$ to $u(t,\mathbf{x})$ almost everywhere, almost surely. Since

$$E\left[\|u^\delta(t,\cdot)\|^p_{L^p(\mathbb{R}^d)}\right] \leqq E\left[\|u_0^\delta(\cdot)\|^p_{L^p(\mathbb{R}^d)}\right] \leqq C, \quad p = 1, 2, \ldots,$$

where C is independent of δ, one can check that $u(t,\mathbf{x})$ is a strong stochastic entropy solution, and (22) holds. For the stability result (23), see [7]. □

5. Continuous Dependence Estimates

The aim of this section is to establish an explicit "continuous dependence on the nonlinearities" estimate in the BV class. Let $u(t) = u(t, \mathbf{x}; \omega)$ be a strong stochastic entropy solution of

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = \sigma(u)\, \partial_t W, \quad u|_{t=0} = u_0. \tag{24}$$

Let $v(t) = v(t, \mathbf{x}; \omega)$ be a strong stochastic entropy solution of

$$\partial_t v + \nabla \cdot \hat{\mathbf{f}}(v) = \hat{\sigma}(v)\, \partial_t W, \quad v|_{t=0} = v_0. \tag{25}$$

We are interested in estimating $E[\|u(t) - v(t)\|_{L^1}]$ in terms of $u_0 - v_0$, $\mathbf{f} - \hat{\mathbf{f}}$ and $\sigma - \hat{\sigma}$. Relevant continuous dependence results for deterministic conservation laws have been obtained in [17,1], and in [3] for strongly degenerate parabolic equations; see also [2,9].

We start with the following important lemma.

Lemma 1. *Suppose that (3)–(5) hold for the two data sets* $(u_0, \mathbf{f}, \sigma)$ *and* $(v_0, \hat{\mathbf{f}}, \hat{\sigma})$. *For any fixed* $\varepsilon > 0$, *let* $u(t) = u(t, \mathbf{x}; \omega)$ *be the solution to the stochastic parabolic problem*

$$\mathrm{d}u + [\nabla_{\mathbf{x}} \cdot \mathbf{f}(u) - \varepsilon \Delta_{\mathbf{x}} u]\, \mathrm{d}t = \sigma(u)\, \mathrm{d}W(t), \quad u|_{t=0} = u_0. \tag{26}$$

For any fixed $\hat{\varepsilon} > 0$, *let* $v(t) = v(t, \mathbf{y}; \omega)$ *be the solution to the stochastic parabolic problem*

$$\mathrm{d}v + [\nabla_{\mathbf{y}} \cdot \hat{\mathbf{f}}(v) - \hat{\varepsilon} \Delta_{\mathbf{y}} v]\, \mathrm{d}t = \hat{\sigma}(v)\, \mathrm{d}W(t), \quad v|_{t=0} = v_0. \tag{27}$$

Take $0 \leqq \phi_\delta = \phi_\delta(\mathbf{x}, \mathbf{y}) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ *to be of the form:*

$$\phi_\delta(\mathbf{x}, \mathbf{y}) = \frac{1}{\delta^d} J\left(\frac{\mathbf{x} - \mathbf{y}}{2\delta}\right) \psi\left(\frac{\mathbf{x} + \mathbf{y}}{2}\right) =: J_\delta\left(\frac{\mathbf{x} - \mathbf{y}}{2}\right) \psi\left(\frac{\mathbf{x} + \mathbf{y}}{2}\right), \tag{28}$$

where $J(\cdot)$ *is a regularization kernel as in (19) and* $0 \leqq \psi \in C_c^\infty(\mathbb{R}^d)$. *Moreover, given any entropy function* $\eta(\cdot)$ *with* $\eta(0) = 0$ *and* $\eta'(\cdot)$ *odd, introduce the associated entropy fluxes for* $u, v \in \mathbb{R}$:

$$\mathbf{q}^{\mathbf{f}}(u, v) = \int_v^u \eta'(\xi - v)\mathbf{f}'(\xi)\, \mathrm{d}\xi, \quad \mathbf{q}^{\hat{\mathbf{f}}}(u, v) = \int_v^u \eta'(\xi - v)\hat{\mathbf{f}}'(\xi)\, \mathrm{d}\xi.$$

Then, for any $t > 0$,

$$\begin{aligned}
&\iint \eta(u(t, \mathbf{x}) - v(t, \mathbf{y}))\phi_\delta(\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} - \iint \eta(u_0(\mathbf{x}) - v_0(\mathbf{y}))\phi_\delta(\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \\
&\quad \leqq I^{\mathbf{f}}(\phi_\delta) + I^{\mathbf{f}, \hat{\mathbf{f}}}(\phi_\delta) + I^{\sigma, \hat{\sigma}}(\phi_\delta) + I^{\varepsilon, \hat{\varepsilon}}(\phi_\delta) \\
&\qquad + \iint \int_s^t \eta'(u(s, \mathbf{x}) - v(s, \mathbf{y}))(\sigma(u(s, \mathbf{x})) - \hat{\sigma}(v(s, \mathbf{y}))) \\
&\qquad\qquad \times \phi_\delta(\mathbf{x}, \mathbf{y})\, \mathrm{d}W(s)\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y},
\end{aligned}$$

where

$$
\begin{aligned}
I^{\mathbf{f}}(\phi_\delta) &= \iint \int_0^t \mathbf{q}^{\mathbf{f}}(u(s,\mathbf{x}),v(s,\mathbf{y}))\cdot\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y},\\
I^{\mathbf{f},\hat{\mathbf{f}}}(\phi_\delta) &= \iint \int_0^t (\mathbf{q}^{\hat{\mathbf{f}}}(v(s,\mathbf{y}),u(s,\mathbf{x})) - \mathbf{q}^{\mathbf{f}}(u(s,\mathbf{x}),v(s,\mathbf{y})))\\
&\qquad\qquad \times\nabla_{\mathbf{y}}\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y},\\
I^{\varepsilon,\hat{\varepsilon}}(\phi_\delta) &= (\sqrt{\varepsilon}-\sqrt{\hat{\varepsilon}})^2 \iint \int_0^t \eta(u(s,\mathbf{x})-v(s,\mathbf{y}))\\
&\qquad\qquad\qquad \times\Delta_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\quad +\frac{1}{4}(\sqrt{\varepsilon}+\sqrt{\hat{\varepsilon}})^2 \iint \int_0^t \eta(u(s,\mathbf{x})-v(s,\mathbf{y}))\\
&\qquad\qquad\qquad \times J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\Delta\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\quad +(\hat{\varepsilon}-\varepsilon)\iint \int_0^t \eta(u(s,\mathbf{x})-v(s,\mathbf{y}))\\
&\qquad\qquad\qquad \times\nabla_{\mathbf{y}} J_\delta(\mathbf{x}-\mathbf{y})\cdot\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y},\\
I^{\sigma,\hat{\sigma}}(\phi_\delta) &= \iint \int_0^t \frac{1}{2}\eta''(u(s,\mathbf{x})-v(s,\mathbf{y}))\\
&\qquad\qquad \times(\sigma(u(s,\mathbf{x}))-\hat{\sigma}(v(s,\mathbf{y})))^2\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.
\end{aligned}
$$

Proof. Subtracting (27) from (26) and subsequently applying Ito's formula to $\eta(u(t)-v(t))$, we obtain

$$
\begin{aligned}
\mathrm{d}\eta(u-v) = \Big[&-\eta'(u-v)(\nabla_{\mathbf{x}}\cdot\mathbf{f}(u)-\nabla_{\mathbf{y}}\cdot\hat{\mathbf{f}}(v)) + \eta'(u-v)(\varepsilon\Delta_{\mathbf{x}}u-\hat{\varepsilon}\Delta_{\mathbf{y}}v)\\
&+\frac{1}{2}\eta''(u-v)(\sigma(u)-\sigma(v))^2\Big]\,\mathrm{d}t\\
&+\eta'(u-v)(\sigma(u)-\sigma(v))\,\mathrm{d}W(t).
\end{aligned}
\tag{29}
$$

Observe that

$$
\eta'(u-v)\nabla_{\mathbf{x}}\cdot\mathbf{f}(u) = \nabla_{\mathbf{x}}\cdot\mathbf{q}^{\mathbf{f}}(u,v), \quad \eta'(u-v)\nabla_{\mathbf{y}}\cdot\hat{\mathbf{f}}(v) = \nabla_{\mathbf{y}}\cdot\mathbf{q}^{\hat{\mathbf{f}}}(v,u),
$$

and thus

$$
\begin{aligned}
&-\eta'(u-v)(\nabla_{\mathbf{x}}\cdot\mathbf{f}(u)-\nabla_{\mathbf{y}}\cdot\hat{\mathbf{f}}(v))\\
&\quad = -(\nabla_{\mathbf{x}}+\nabla_{\mathbf{y}})\cdot\mathbf{q}^{\mathbf{f}}(u,v) + \nabla_{\mathbf{y}}\cdot(\mathbf{q}^{\mathbf{f}}(u,v)-\mathbf{q}^{\hat{\mathbf{f}}}(v,u)).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\eta'(u-v)(\varepsilon\Delta_{\mathbf{x}}u-\hat{\varepsilon}\Delta_{\mathbf{y}}v)\\
&\quad = (\varepsilon\Delta_{\mathbf{x}}+\hat{\varepsilon}\Delta_y)\eta(u-v) - \eta''(u-v)(\varepsilon|\nabla_{\mathbf{x}}u|^2+\hat{\varepsilon}|\nabla_{\mathbf{y}}v|^2)\\
&\quad = (\varepsilon\Delta_{\mathbf{x}}+2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}+\hat{\varepsilon}\Delta_y)\eta(u-v) - \eta''(u-v)|\sqrt{\varepsilon}\nabla_{\mathbf{x}}u-\sqrt{\hat{\varepsilon}}\nabla_{\mathbf{y}}v|^2.
\end{aligned}
$$

Inserting the last two relations into (29), we arrive at

$$
\begin{aligned}
\mathrm{d}\eta(u-v) = \Big[& -(\nabla_{\mathbf{x}}+\nabla_{\mathbf{y}})\cdot\mathbf{q}^{\mathbf{f}}(u,v)+\nabla_{\mathbf{y}}\cdot(\mathbf{q}(u,v)-\mathbf{q}^{\hat{\mathbf{f}}}(v,u)) \\
& +(\varepsilon\Delta_{\mathbf{x}}+2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}+\hat{\varepsilon}\Delta_{\mathbf{y}})\eta(u-v) \\
& -\eta''(u-v)\left|\sqrt{\varepsilon}\nabla_{\mathbf{x}}u-\sqrt{\hat{\varepsilon}}\nabla_{\mathbf{y}}v\right|^2 \\
& +\frac{1}{2}\eta''(u-v)(\sigma(u)-\sigma(v))^2\Big]\,\mathrm{d}t \\
& +\eta'(u-v)(\sigma(u)-\sigma(v))\,\mathrm{d}W(t). \qquad (30)
\end{aligned}
$$

We integrate (30) against the test function ϕ_δ defined in (28) to yield

$$
\begin{aligned}
&\iint \eta(u(t,\mathbf{x})-v(t,\mathbf{y}))\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} - \iint \eta(u_0(\mathbf{x})-v_0(\mathbf{y}))\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\quad \leqq I_c^1 + I_c^2 + I_d + I^{\sigma,\hat{\sigma}}(\phi_\delta) \\
&\qquad + \iint\int_0^t \eta'(u(s,\mathbf{x})-v(s,\mathbf{y}))(\sigma(u(s,\mathbf{x}))-\sigma(v(s,\mathbf{y}))) \\
&\qquad\qquad \times\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}W(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y},
\end{aligned}
$$

where

$$
\begin{aligned}
I_c^1 &:= -\iint\int_0^t (\nabla_{\mathbf{x}}+\nabla_{\mathbf{y}})\cdot\mathbf{q}^{\mathbf{f}}(u,v)\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}, \\
I_c^2 &:= \iint\int_0^t \nabla_{\mathbf{y}}\cdot(\mathbf{q}^{\mathbf{f}}(u(s,\mathbf{x}),v(s,\mathbf{y}))-\mathbf{q}^{\hat{\mathbf{f}}}(v(s,\mathbf{y}),u(s,\mathbf{x}))) \\
&\qquad \times\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}, \\
I_d &:= \iint\int_0^t (\varepsilon\Delta_{\mathbf{x}}+2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}+\hat{\varepsilon}\Delta_{\mathbf{y}})\eta(u(s,\mathbf{x})-v(s,\mathbf{y})) \\
&\qquad \times\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.
\end{aligned}
$$

Integrating by parts gives $I_c^2 = I^{\mathbf{f},\hat{\mathbf{f}}}(\phi_\delta)$, and also $I_c^1 = I^{\mathbf{f}}(\phi_\delta)$, since

$$
\begin{aligned}
(\nabla_{\mathbf{x}}+\nabla_{\mathbf{y}})\phi_\delta(\mathbf{x},\mathbf{y}) &= J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)(\nabla_{\mathbf{x}}+\nabla_{\mathbf{y}})\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \\
&= J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right).
\end{aligned}
$$

We now investigate the term I_d. A calculation shows that

$$
\begin{aligned}
&(\varepsilon\Delta_{\mathbf{x}}+2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}+\hat{\varepsilon}\Delta_{\mathbf{y}})\phi_\delta(\mathbf{x},\mathbf{y}) \\
&\quad = (\varepsilon\Delta_{\mathbf{x}}+2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}+\hat{\varepsilon}\Delta_{\mathbf{y}})J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \\
&\qquad + J_\delta(\mathbf{x}-\mathbf{y})(\varepsilon\Delta_{\mathbf{x}}+2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}+\hat{\varepsilon}\Delta_{\mathbf{y}})\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) + R,
\end{aligned}
$$

and

$$
\begin{aligned}
R &= 2\varepsilon \nabla_{\mathbf{x}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) \cdot \nabla_{\mathbf{x}} \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) + 2\hat{\varepsilon} \nabla_{\mathbf{y}} J_\delta(\mathbf{x}-\mathbf{y}) \cdot \nabla_{\mathbf{y}} \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \\
&\quad + 2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}} \nabla_{\mathbf{x}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) \cdot \nabla_{\mathbf{y}} \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \\
&\quad + 2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}} \nabla_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) \cdot \nabla_{\mathbf{x}} \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \\
&= \Bigg(2\varepsilon \nabla_{\mathbf{x}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) + 2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}} \nabla_{\mathbf{x}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) + 2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}} \nabla_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) \\
&\qquad + 2\hat{\varepsilon} \nabla_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\Bigg) \cdot \nabla_{\mathbf{y}} \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \\
&= 2\nabla_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) \cdot \nabla_{\mathbf{y}} \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)(\hat{\varepsilon}-\varepsilon) \\
&= \nabla_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) \cdot \nabla \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)(\hat{\varepsilon}-\varepsilon).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&(\varepsilon \Delta_{\mathbf{x}} + 2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}} \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{y}} + \hat{\varepsilon}\Delta_{\mathbf{y}}) J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) = (\sqrt{\varepsilon}-\sqrt{\hat{\varepsilon}})^2 \Delta_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right), \\
&(\varepsilon \Delta_{\mathbf{x}} + 2\sqrt{\varepsilon}\sqrt{\hat{\varepsilon}} \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{y}} + \hat{\varepsilon}\Delta_{\mathbf{y}}) \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) = \frac{1}{4}(\sqrt{\varepsilon}+\sqrt{\hat{\varepsilon}})^2 \Delta \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right).
\end{aligned}
$$

Consequently, after integrating by parts, I_d becomes $I^{\varepsilon,\hat{\varepsilon}}(\phi_\delta)$. □

Theorem 5 (Continuous dependence estimates). *Suppose that* (3)–(5) *hold for the two data sets* $(u_0, \mathbf{f}, \sigma)$ *and* $(v_0, \hat{\mathbf{f}}, \hat{\sigma})$. *Let* $u(t)$ *and* $v(t)$ *be the strong stochastic entropy solutions of* (24)–(25)*, respectively, for which*

$$
E\left[|v(t)|_{\mathrm{BV}(\mathbb{R}^d)}\right] \leqq E\left[|v_0|_{\mathrm{BV}(\mathbb{R}^d)}\right] \quad \text{for } t > 0.
$$

In addition, we assume that either

$$
u, v \in L^\infty((0,T) \times \mathbb{R}^d \times \Omega) \quad \text{for any } T > 0,
$$

or

$$
\mathbf{f}'', \mathbf{f}' - \hat{\mathbf{f}}', \sigma - \hat{\sigma} \in L^\infty.
$$

Then

(i) *There is a constant* $C_T > 0$ *such that, for any* $0 < t < T$ *with* T *finite,*

$$
\begin{aligned}
&E\left[\int_{\mathbb{R}^d} |u(t,\mathbf{x}) - v(t,\mathbf{x})| \psi(\mathbf{x})\, \mathrm{d}\mathbf{x}\right] \\
&\quad \leqq C_T \Bigg(E\left[\int_{\mathbb{R}^d} |u_0(\mathbf{x}) - v_0(\mathbf{x})| \psi(\mathbf{x})\, \mathrm{d}\mathbf{x}\right] + \sqrt{t}\,\|\psi\|_{L^1(\mathbb{R}^d)} \|\sigma - \hat{\sigma}\|_{L^\infty} \\
&\qquad\qquad + t\, E\left[|v_0|_{\mathrm{BV}(\mathbb{R}^d)}\right] (\|\mathbf{f}' - \hat{\mathbf{f}}'\|_{L^\infty} + \|\sigma - \hat{\sigma}\|_{L^\infty})\Bigg),
\end{aligned}
$$

where the constant $C_T > 0$ is independent of $|u_0|_{BV(\mathbb{R}^d)}$ *and* $|v_0|_{BV(\mathbb{R}^d)}$*, and may grow exponentially in* T*. Moreover,* $\psi = \psi(\mathbf{x}) \geqq 0$ *is any function satisfying* $|\psi| \leqq C_0$ *and* $|\nabla\psi| \leqq C_0\psi$*, which includes* $\psi(\mathbf{x}) = e^{-C_0|\mathbf{x}|}$ *and, more generally,* $\psi(\mathbf{x}) = 1$ *when* $|\mathbf{x}| \leqq R$ *and* $\psi(\mathbf{x}) = e^{-C_0(|\mathbf{x}|-R)}$ *when* $|\mathbf{x}| \geqq R$*. In particular, for any* $R > 0$*, this choice implies*

$$
\begin{aligned}
&E\left[\int_{|\mathbf{x}|<R} |u(t,\mathbf{x}) - v(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\right] \\
&\leqq C_{T,R}\left(E\left[\int_{\mathbb{R}^d} |u_0(\mathbf{x}) - v_0(\mathbf{x})|\,\mathrm{d}\mathbf{x}\right] + \sqrt{t}\|\sigma - \hat{\sigma}\|_{L^\infty}\right. \\
&\qquad\left. + t\,E\left[|v_0|_{BV(\mathbb{R}^d)}\right](\|\mathbf{f}' - \hat{\mathbf{f}}'\|_{L^\infty} + \|\sigma - \hat{\sigma}\|_{L^\infty})\right).
\end{aligned}
$$

(ii) *There is a constant* C_T *such that, for any* $0 < t < T < \infty$*,*

$$
\begin{aligned}
&E\left[\int_{\mathbb{R}^d} |u(t,\mathbf{x}) - v(t,\mathbf{x})|\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right] \\
&\leqq C_T\left(E\left[\int_{\mathbb{R}^d} |u_0(\mathbf{x}) - v_0(\mathbf{x})|\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right] + \sqrt{t}\|\psi\|_{L^1(\mathbb{R}^d)}\Delta(\sigma,\hat{\sigma})\right. \\
&\qquad\left. + t\,E\left[|v_0|_{BV(\mathbb{R}^d)}\right](\|\mathbf{f}' - \hat{\mathbf{f}}'\|_{L^\infty} + \Delta(\sigma,\hat{\sigma}))\right),
\end{aligned}
$$

where $\psi(\mathbf{x})$ *is as before and*

$$
\Delta(\sigma,\hat{\sigma}) := \sup_{\xi\neq 0} \frac{\left|\sigma(\xi) - \hat{\sigma}(\xi)\right|}{|\xi|}.
$$

Remark 2. If, in addition to the assumptions listed in Theorem 5, $u_0(\mathbf{x})$ and $v_0(\mathbf{x})$ are periodic in $\mathbf{x}$ with the same period, we can "remove" ψ from the above estimates, since integrations are then over a bounded domain.

Proof. As the vanishing viscosity method converges (see Theorem 3), it suffices to prove the result for (26)–(27) with $\hat{\varepsilon} = \varepsilon$.

For $\rho > 0$, let $\eta_\rho : \mathbb{R} \to \mathbb{R}$ be the function defined by (11)–(13). Then the function

$$
\mathbf{q}^{\mathbf{f}}_\rho(u,v) = \int_v^u \eta'_\rho(\xi - v)\mathbf{f}'(\xi)\,\mathrm{d}\xi, \quad u, v \in \mathbb{R},
$$

satisfies

$$
\left|\partial_u(\mathbf{q}^{\mathbf{f}}_\rho(u,v) - \mathbf{q}^{\mathbf{f}}_\rho(v,u))\right| \leqq \frac{M_2}{2}\left\|\mathbf{f}''\right\|_{L^\infty}\rho, \tag{31}
$$

where $M_2 = \sup_{|u|\leqq 1}\left|\bar{\eta}''(u)\right|$.

In view of Lemma 1 with $\hat{\varepsilon} = \varepsilon$,

$$
\begin{aligned}
& E\left[\iint \eta_\rho(u(t,\mathbf{x}) - v(t,\mathbf{y}))\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad - E\left[\iint \eta_\rho(u_0(\mathbf{x}) - v_0(\mathbf{y}))\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\leqq E\left[\iint\int_0^t \mathbf{q}_\rho^{\mathbf{f}}(u(s,\mathbf{x}), v(s,\mathbf{y}))\cdot\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad + E\left[\iiint_0^t (\mathbf{q}_\rho^{\hat{\mathbf{f}}}(v(s,\mathbf{y}), u(s,\mathbf{x})) \right. \\
&\qquad\qquad \left. - \mathbf{q}_\rho^{\mathbf{f}}(u(s,\mathbf{x}), v(s,\mathbf{y})))\cdot\nabla_{\mathbf{y}}\phi_\delta\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad + E\left[\iiint_0^t \frac{1}{2}\eta_\rho''(u(s,\mathbf{x}) - v(s,\mathbf{y})) \right. \\
&\qquad\qquad \left. \times(\sigma(u(s,\mathbf{x})) - \hat{\sigma}(v(s,\mathbf{y})))^2\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad + \varepsilon\, E\left[\iiint_0^t \eta_\rho(u(s,\mathbf{x}) - v(s,\mathbf{y})) J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\Delta_{\mathbf{x}}\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right].
\end{aligned}
\tag{32}
$$

Observe that

$$
\begin{aligned}
& -\nabla_{\mathbf{y}}\cdot(\mathbf{q}_\rho^{\hat{\mathbf{f}}}(v(s,\mathbf{y}), u(s,\mathbf{x})) - \mathbf{q}_\rho^{\mathbf{f}}(u(s,\mathbf{x}), v(s,\mathbf{y}))) \\
&\quad = \nabla_{\mathbf{y}} v\cdot\partial_v(\mathbf{q}_\rho^{\mathbf{f}}(u,v) - \mathbf{q}_\rho^{\hat{\mathbf{f}}}(v,u))|_{(u,v)=(u(s,\mathbf{x}),v(s,\mathbf{y}))},
\end{aligned}
$$

and, thanks to (31),

$$
\begin{aligned}
& \left|\partial_v(\mathbf{q}_\rho^{\mathbf{f}}(u,v) - \mathbf{q}_\rho^{\hat{\mathbf{f}}}(v,u))\right| \\
&\quad = \left|\partial_v(\mathbf{q}_\rho^{\mathbf{f}}(v,u) - \mathbf{q}_\rho^{\hat{\mathbf{f}}}(v,u)) + \partial_v(\mathbf{q}_\rho^{\mathbf{f}}(u,v) - \mathbf{q}_\rho^{\mathbf{f}}(v,u))\right| \\
&\quad \leqq |\mathbf{f}'(v) - \hat{\mathbf{f}}'(v)| + \frac{M_2}{2}\|\mathbf{f}''\|_{L^\infty}\rho.
\end{aligned}
$$

Hence, after an integration by parts,

$$
\begin{aligned}
& \left| E\left[\iiint_0^t (\mathbf{q}_\rho^{\hat{\mathbf{f}}}(v(s,\mathbf{y}), u(s,\mathbf{x})) - \mathbf{q}_\rho^{\mathbf{f}}(u(s,\mathbf{x}), v(s,\mathbf{y})))\cdot\nabla_{\mathbf{y}}\phi_\delta\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\right| \\
&\quad \leqq t\, E\left[|v_0|_{BV(\mathbb{R}^d)}\right]\|\psi\|_{L^\infty(\mathbb{R}^d)}\left(\|\mathbf{f}' - \hat{\mathbf{f}}'\|_{L^\infty} + \frac{M_2}{2}\|\mathbf{f}''\|_{L^\infty}\rho\right).
\end{aligned}
$$

Consequently, again thanks to (31) and also (12), we can write (32) as

$$
\begin{aligned}
&E\left[\iint |u(t,\mathbf{x})-v(t,\mathbf{y})|\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad -E\left[\iint |u_0(\mathbf{x})-v_0(\mathbf{y})|\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\leqq E\left[\iiint_0^t \mathbf{q}_\rho^{\mathbf{f}}(u(s,\mathbf{x}),v(s,\mathbf{y}))\cdot\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad +E\left[\iiint_0^t \frac{1}{2}\eta_\rho''(u(s,\mathbf{x})-v(s,\mathbf{y}))\right. \\
&\qquad\qquad \left.\times(\sigma(u(s,\mathbf{x}))-\hat{\sigma}(v(s,\mathbf{y})))^2\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad +t\;|v_0|_{\mathrm{BV}(\mathbb{R}^d)}\,\|\psi\|_{L^\infty(\mathbb{R}^d)}\,(\|\mathbf{f}'-\hat{\mathbf{f}}'\|_{L^\infty}+\mathcal{O}(\rho)) \\
&\quad +\mathcal{O}(\|\psi\|_{L^1(\mathbb{R}^d)}\,\rho)+\mathcal{O}(\varepsilon).
\end{aligned}
\tag{33}
$$

Sending $\delta\to 0$ and using $|\nabla\psi(\mathbf{x})|\leqq C_0\psi(\mathbf{x})$, we obtain

$$
\begin{aligned}
&\lim_{\delta\to 0}\left|E\left[\iiint_0^t \mathbf{q}_\rho^{\mathbf{f}}(u(s,\mathbf{x}),v(s,\mathbf{y}))\cdot\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)J_\delta(\mathbf{x}-\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\right| \\
&\leqq C_2\|\mathbf{f}'\|_{L^\infty}\int_0^t E\left[\int |u(s,\mathbf{x})-v(s,\mathbf{x})|\,\psi(\mathbf{x})\mathrm{d}\mathbf{x}\right]\mathrm{d}s;
\end{aligned}
$$

hence, sending $\delta\to 0$ in (33) returns

$$
\begin{aligned}
&E\left[\int |u(t,\mathbf{x})-v(t,\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right]-E\left[\int |u_0(\mathbf{x})-v_0(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right] \\
&\leqq C_2\|\mathbf{f}'\|_\infty\int_0^t E\left[\int |u(s,\mathbf{x})-v(s,\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right]\mathrm{d}s \\
&\quad +E\left[\iint_0^t \frac{1}{2}\eta_\rho''(u(s,\mathbf{x})-v(s,\mathbf{x}))(\sigma(u(s,\mathbf{x}))-\hat{\sigma}(v(s,\mathbf{x})))^2\psi(\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\right] \\
&\quad +t\;E\left[|v_0|_{\mathrm{BV}(\mathbb{R}^d)}\right]\|\psi\|_{L^\infty(\mathbb{R}^d)}\,(\|\mathbf{f}'-\hat{\mathbf{f}}'\|_{L^\infty}+\mathcal{O}(\rho)) \\
&\quad +\mathcal{O}(\|\psi\|_{L^1(\mathbb{R}^d)}\,\rho)+\mathcal{O}(\varepsilon).
\end{aligned}
$$

Next, with our choice of η_ρ, it follows that

$$
\begin{aligned}
&\left|E\left[\iint_0^t \frac{1}{2}\eta_\rho''(u(s,\mathbf{x})-v(\tau,\mathbf{x}))(\sigma(u(s,\mathbf{x}))-\hat{\sigma}(v(s,\mathbf{x})))^2\psi(\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\right]\right| \\
&\leqq E\left[\iint_0^t \frac{M_2}{\rho}\mathbf{1}_{|u(s,\mathbf{x})-v(s,\mathbf{x})|<\rho}(\sigma(u(s,\mathbf{x}))-\hat{\sigma}(u(s,\mathbf{x})))^2\psi(\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\right] \\
&\quad +E\left[\iint_0^t \frac{M_2}{\rho}\mathbf{1}_{|u(s,\mathbf{x})-v(s,\mathbf{x})|<\rho}(\hat{\sigma}(u(s,\mathbf{x}))-\hat{\sigma}(v(s,\mathbf{x})))^2\psi(\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\right] \\
&=: A+B.
\end{aligned}
\tag{34}
$$

Clearly,

$$
\begin{aligned}
|A| &\leqq C_3 E\left[\iint_0^t \frac{\left|\sigma(u(s,\mathbf{x})) - \hat{\sigma}(u(s,\mathbf{x}))\right|^2}{\rho}\psi(\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\right] \\
&\leqq C_3 \|\psi\|_{L^1(\mathbb{R}^d)} \frac{t\|\sigma - \hat{\sigma}\|_{L^\infty}^2}{\rho}
\end{aligned}
$$

and, in view of (5),

$$
|B| \leqq C_4 \int_0^t E\left[\int |u(s,\mathbf{x}) - v(s,\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right]\mathrm{d}s.
$$

In summary, we have arrived at

$$
\begin{aligned}
&E\left[\int |u(t,\mathbf{x}) - v(t,\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right] - E\left[\int |u_0(\mathbf{x}) - v_0(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right] \\
&\leqq C\Bigg(\|\mathbf{f}'\|_{L^\infty}\int_0^t E\left[\int |u(s,\mathbf{x}) - v(s,\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right]\mathrm{d}s \\
&\qquad + \|\psi\|_{L^\infty(\mathbb{R}^d)} E[|v_0|_{BV(\mathbb{R}^d)}]t(\|\mathbf{f}' - \hat{\mathbf{f}}'\|_{L^\infty} + \rho) \\
&\qquad + \|\psi\|_{L^1(\mathbb{R}^d)} \frac{t\|\sigma - \hat{\sigma}\|_{L^\infty}^2}{\rho} + \|\psi\|_{L^1(\mathbb{R}^d)}\,\rho + \varepsilon\Bigg),
\end{aligned}
$$

which implies via the Gronwall inequality that, for any $t > 0$,

$$
\begin{aligned}
&E\left[\int |u(t,\mathbf{x}) - v(t,\mathbf{x})|\;\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right] \\
&\leqq e^{C\|\mathbf{f}'\|_{L^\infty} t} E\left[\int |u_0(\mathbf{x}) - v_0(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right] \\
&\quad + C e^{C\|\mathbf{f}'\|_{L^\infty} t}\Bigg(\|\psi\|_{L^\infty(\mathbb{R}^d)} E\left[|v_0|_{BV(\mathbb{R}^d)}\right] t(\|\mathbf{f}' - \hat{\mathbf{f}}'\|_{L^\infty} + \rho) \\
&\qquad\qquad + \|\psi\|_{L^1(\mathbb{R}^d)} \frac{t\|\sigma - \hat{\sigma}\|_{L^\infty}^2}{\rho} + \|\psi\|_{L^1(\mathbb{R}^d)}\,\rho + \varepsilon\Bigg). \qquad (35)
\end{aligned}
$$

Choosing $\rho = \sqrt{t}\|\sigma - \hat{\sigma}\|_{L^\infty}$ and sending $\varepsilon \to 0$ supplies part (i).

About part (ii), the only difference in the proof comes from the estimate of the A-term in (34), which is replaced by

$$
\begin{aligned}
|A| &\leqq C_3 E\left[\iint_0^t \frac{|\sigma(u(s,\mathbf{x})) - \hat{\sigma}(u(s,\mathbf{x}))|^2}{\rho |u(s,\mathbf{x})|^2} |u(s,\mathbf{x})|^2 \psi(\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\right] \\
&= C_3 E\left[\iint_0^t \frac{(\Delta(\sigma,\hat{\sigma}))^2}{\rho} |u(s,\mathbf{x})|^2 \psi(\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\right] \\
&\leqq C_3 \|\psi\|_{L^\infty(\mathbb{R}^d)} E\left[\|u\|_{L^\infty(0,T;L^2(\mathbb{R}^d))}\right] \frac{t\,(\Delta(\sigma,\hat{\sigma}))^2}{\rho}.
\end{aligned}
$$

With this estimate at our disposal, (35) is replaced by

$$
\begin{aligned}
&E\left[\int |u(t,\mathbf{x})-v(t,\mathbf{x})|\ \psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right]\\
&\leqq e^{C\|\mathbf{f}\|_{L^\infty} t} E\left[\int |u_0(\mathbf{x})-v_0(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right]\\
&\quad + Ce^{C\|\mathbf{f}'\|_{L^\infty} t}\bigg(\|\psi\|_{L^\infty(\mathbb{R}^d)} E\left[|v_0|_{BV(\mathbb{R}^d)}\right] t\,(\|\mathbf{f}'-\hat{\mathbf{f}}'\|_{L^\infty}+\rho)\\
&\qquad + \|\psi\|_{L^\infty(\mathbb{R}^d)}\frac{t\,(\Delta(\sigma,\hat\sigma))^2}{\rho} + \|\psi\|_{L^1(\mathbb{R}^d)}\,\rho+\varepsilon\bigg).
\end{aligned}
$$

Part (ii) follows by choosing $\rho=\sqrt{t}\Delta(\sigma,\hat\sigma)$ and sending $\varepsilon\to 0$. □

Theorem 6 (Error estimate). *Suppose that* (3)–(5) *hold. Let $u(t)$ be the strong stochastic entropy solutions of* (24), *for which*

$$
E\left[|u(t)|_{BV(\mathbb{R}^d)}\right]\leqq |u_0|_{BV(\mathbb{R}^d)}\quad \textit{for } t>0, \tag{36}
$$

and let u^ε be the solution to the parabolic problem

$$
\mathrm{d}u^\varepsilon+[\nabla_{\mathbf{x}}\cdot\mathbf{f}(u^\varepsilon)-\varepsilon\Delta_{\mathbf{x}}u^\varepsilon]\,\mathrm{d}t=\sigma(u^\varepsilon)\,\mathrm{d}W(t),\quad u^\varepsilon|_{t=0}=u_0.
$$

In addition, we assume that

$$
\textit{either } u,v\in L^\infty((0,T)\times\mathbb{R}^d\times\Omega)\ \textit{ for any } T>0,\ \textit{or } \mathbf{f}''\in L^\infty.
$$

Then there exists a constant $C_T>0$ such that, for any $0<t<T$ with T finite,

$$
E\left[\int_{\mathbb{R}^d}|u(t,\mathbf{x})-u^\varepsilon(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\right]\leqq C_T\,E\left[|u_0|_{BV(\mathbb{R}^d)}\right]t\sqrt{\varepsilon}.
$$

Proof. We proceed as in the proof of Theorem 5, starting from Lemma 1 with $\hat\sigma=\sigma,\hat{\mathbf{f}}=\mathbf{f},\hat\varepsilon\neq\varepsilon,u^\varepsilon=u,u^{\hat\varepsilon}=v$, leading to

$$
\begin{aligned}
&E\left[\iint \left|u^\varepsilon(t,\mathbf{x})-u^{\hat\varepsilon}(t,\mathbf{y})\right|\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&\leqq E\left[\iiint_0^t \mathbf{q}_\rho^{\mathbf{f}}(u^\varepsilon(s,\mathbf{x}),u^{\hat\varepsilon}(s,\mathbf{y}))\cdot\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&\quad+E\bigg[\iiint_0^t \eta_\rho''(u^\varepsilon(s,\mathbf{x})-u^{\hat\varepsilon}(s,\mathbf{y}))\\
&\qquad\qquad\times(\sigma(u^\varepsilon(s,\mathbf{x}))-\sigma(u^{\hat\varepsilon}(s,\mathbf{y})))^2\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\bigg]\\
&\quad+t\ |u_0|_{BV(\mathbb{R}^d)}\,\|\psi\|_{L^\infty(\mathbb{R}^d)}\,\mathcal{O}(\rho)+\mathcal{O}(\|\psi\|_{L^1(\mathbb{R}^d)}\,\rho)\\
&\quad+(\sqrt{\varepsilon}-\sqrt{\hat\varepsilon})^2E\bigg[\iiint_0^t \eta_\rho(u^\varepsilon(s,\mathbf{x})-u^{\hat\varepsilon}(s,\mathbf{y}))\\
&\qquad\qquad\times\Delta_{\mathbf{y}}J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\bigg]
\end{aligned}
$$

$$+\frac{1}{4}(\sqrt{\varepsilon}+\sqrt{\hat{\varepsilon}})^2 E\left[\iiint_0^t \eta_\rho(u^\varepsilon(s,\mathbf{x})-u^{\hat{\varepsilon}}(s,\mathbf{y})) \times J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\Delta\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]$$

$$+(\hat{\varepsilon}-\varepsilon)E\left[\iiint_0^t \eta_\rho(u^\varepsilon(s,\mathbf{x})-u^{\hat{\varepsilon}}(s,\mathbf{y})) \times \nabla_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\cdot\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]$$

$$=: I_1+I_2+I_3+I_4+I_5+I_6. \tag{37}$$

As before,

$$|I_2| \leqq C_1\int_0^t E\left[\int \left|u^\varepsilon(s,\mathbf{x})-u^\varepsilon(s,\mathbf{y})\right| J_\delta(x-y)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\mathrm{d}s.$$

Noting that the right-hand side is independent of ρ, we can first send $\rho\to 0$ in (37), and then let ψ tend to $\mathbf{1}_{\mathbb{R}^d}$, keeping in mind the L^p-estimates (21), with the outcome that $I_1, I_3, I_5, I_6 \to 0$. The resulting estimate reads

$$E\left[\iint \left|u^\varepsilon(t,\mathbf{x})-u^{\hat{\varepsilon}}(t,\mathbf{y})\right| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \leqq C_1\int_0^t E\left[\iint |u(s,\mathbf{x})-v(s,\mathbf{y})|\, J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\mathrm{d}\mathbf{x}\right]\mathrm{d}s+I, \tag{38}$$

where

$$I=(\sqrt{\varepsilon}-\sqrt{\hat{\varepsilon}})^2 E\left[\iiint_0^t |u^\varepsilon(s,\mathbf{x})-u^{\hat{\varepsilon}}(s,\mathbf{y})|\Delta_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right].$$

An integration by parts, followed by application of the spatial BV-estimate (36), yields

$$|I| \leqq C_2\, t\, E\left[|u_0|_{BV(\mathbb{R}^d)}\right]\frac{(\sqrt{\varepsilon}-\sqrt{\hat{\varepsilon}})^2}{\delta}.$$

In view of this, it follows from (38) in a completely standard way that

$$E\left[\int |u^\varepsilon(t,\mathbf{x})-u^{\hat{\varepsilon}}(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\right] \leqq C_1\int_0^t E\left[\int \left|u^\varepsilon(s,\mathbf{x})-v^\varepsilon(s,\mathbf{x})\right|\mathrm{d}\mathbf{x}\right]\mathrm{d}s +C_3 E\left[|u_0|_{BV(\mathbb{R}^d)}\right]\left(\delta+t\frac{(\sqrt{\varepsilon}-\sqrt{\hat{\varepsilon}})^2}{\delta}\right).$$

Choosing $\delta=\sqrt{\varepsilon}-\sqrt{\hat{\varepsilon}}$ gives

$$E\left[\int_{\mathbb{R}^d}|u^\varepsilon(t,\mathbf{x})-u^{\hat{\varepsilon}}(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\right] \leqq C_T\, E\left[|u_0|_{BV(\mathbb{R}^d)}\right] t(\sqrt{\varepsilon}-\sqrt{\hat{\varepsilon}}).$$

Sending $\hat{\varepsilon}\to 0$ concludes the proof of the theorem. □

Remark 3. Theorem 6 indicates that the sequence $\{u^\varepsilon(t,\mathbf{x})\}$ is a Cauchy sequence in $C(0,T;L^1)$, which directly implies its strong convergence.

6. More General Equations

We now briefly discuss diverse generalizations.

First of all, as in [7], the stochastic term in (1) can be replaced by the more general term

$$\int_{z\in Z} \sigma(u(t,x);z)\partial_t W(t,\mathrm{d}z),$$

where Z is a metric space, $\sigma:\mathbb{R}\times Z\to\mathbb{R}$, $W(t,\mathrm{d}z)$ is a space-time Gaussian white noise martingale random measure with respect to a filtration $\{\mathcal{F}_t\}$ (see for example, Walsh [22], Kurtz–Protter [12]) with

$$E[W(t,A)\cap W(t,B)] = \mu(A\cap B)t$$

for measurable $A, B\subset Z$, where μ is a (deterministic) σ-finite Borel measure on the metric space Z. In particular, when $Z=\{1,2,\ldots,m\}$ and μ is a counting measure on Z, then the stochastic term reduces to

$$\sum_{k=1}^{m} \sigma_k(u(t,\mathbf{x}))\partial_t W_k(t).$$

For the spatial BV and temporal L^1-continuity estimates and stability results, we can allow for more general flux functions $\mathbf{f}(t,\mathbf{x},u)$ with spatial dependence, by combining the present methods with those in [2,9].

Next, let us discuss the case where the noise coefficient $\sigma(\mathbf{x},u)$ has a spatial dependence, focusing on the stochastic balance law:

$$\partial_t u + \nabla\cdot\mathbf{f}(u) = \sigma(\mathbf{x},u)\,\partial_t W(t), \tag{39}$$

where the noise coefficient is assumed to satisfy $\sigma(\mathbf{x},0)=0$ and

$$\begin{aligned} &|\sigma(\mathbf{x},u)-\sigma(\mathbf{x},v)| \leqq C\,|u-v|\,, \quad \forall\, u,v\in\mathbb{R},\ \forall\,\mathbf{x}\in\mathbb{R}^d,\\ &|\sigma(\mathbf{x},u)-\sigma(\mathbf{y},u)| \leqq C\,|\mathbf{x}-\mathbf{y}|\,|u|\,, \quad \forall\, u\in\mathbb{R},\ \forall\,\mathbf{x},\mathbf{y}\in\mathbb{R}^d, \end{aligned} \tag{40}$$

where C is a deterministic constant.

In the previous sections, we have established the existence of a strong stochastic entropy solution in the multidimensional context. The proof was based on deriving the BV-estimates. However, as mentioned before, the BV-estimates are no longer available when the noise term σ depends on the spatial location $\mathbf{x}$. However, it is possible to derive fractional BV estimates. For fixed $\varepsilon>0$, let $u^\varepsilon(t,\mathbf{x})$ be the solution to the stochastic parabolic problem:

$$\mathrm{d}u^\varepsilon + [\nabla_{\mathbf{x}}\cdot\mathbf{f}(u^\varepsilon) - \varepsilon\Delta_{\mathbf{x}}u^\varepsilon]\,\mathrm{d}t = \sigma(\mathbf{x},u^\varepsilon)\,\mathrm{d}W(t), \quad u^\varepsilon|_{t=0} = u_0, \tag{41}$$

where we tactically assume that $\mathbf{f}$, σ, u_0 are sufficiently smooth to ensure the existence of a regular solution [7]. Utilizing the continuous dependence framework (Lemma 1) which also holds when the noise term σ depends on $\mathbf{x}$, we will prove that, for any $\delta > 0$,

$$
\begin{aligned}
&E\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|u^\varepsilon(t,\mathbf{x}+\mathbf{z}) - u^\varepsilon(t,\mathbf{x}-\mathbf{z})\right| J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\right] \\
&\quad \leqq C_T\, E\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u_0(\mathbf{x}+\mathbf{z}) - u_0(\mathbf{x}-\mathbf{z})|\, J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\right] \\
&\quad\quad + C_T\sqrt{\delta}\,(1 + \|\psi\|_{L^1(\mathbb{R}^d)}), \quad 0 < t < T,
\end{aligned}
\tag{42}
$$

for some finite constant C_T independent of ε, where J_δ is a symmetric mollifier and $\psi \geqq 0$ is a compactly supported smooth function. In what follows, we assume that the cut-off function $\psi \geqq 0$ satisfies

$$
|\nabla\psi(\mathbf{x})| \leqq C_0\psi(\mathbf{x}), \quad |\Delta\psi(\mathbf{x})| \leqq C_0\psi(\mathbf{x}), \quad \psi \equiv 1 \text{ on } K_R := \{|\mathbf{x}| < R\},
$$

for some constants $C_0 > 0$ and $R > 0$. One example of such a function, at least after an easy approximation argument, is the compactly supported function $\psi \in W^{2,\infty}(\mathbb{R}^d)$ defined by

$$
\psi(\mathbf{x}) = \begin{cases} 1, & |\mathbf{x}| \leqq R, \\ \frac{1}{e^\pi+1}\left(\sqrt{2}e^{\pi-(|\mathbf{x}|-R)}\sin(|\mathbf{x}| - R + \frac{\pi}{4}) + 1\right), & R \leqq |\mathbf{x}| \leqq R+\pi, \\ 0, & |\mathbf{x}| \geqq R+\pi. \end{cases}
$$

Estimate (42) can be turned into a fractional BV estimate thanks to the following deterministic lemma, which is related to known links between Sobolev, Besov, and Nikolskii fractional spaces (see, for example, [18]); a proof can be found in the appendix.

Lemma 2. *Let $h : \mathbb{R}^d \to \mathbb{R}$ be a given integrable function, $r, s \in (0,1)$ with $r < s$, $\psi \in C_c^\infty(\mathbb{R}^d)$, and $\{J_\delta\}_{\delta>0}$ a sequence of symmetric mollifiers, that is, $J_\delta(x) = \frac{1}{\delta^d}J(\frac{|x|}{\delta})$, $0 \leqq J \in C_c^\infty(\mathbb{R})$, supp $(J) \subset [-1,1]$, $J(-\cdot) = J(\cdot)$, and $\int J = 1$. Then*

(i) *There exists a positive constant $C_1 = C_1(J,d,r,s) < \infty$ such that, for any $\delta > 0$,*

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z}) - h(\mathbf{x}-\mathbf{z})|\, J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z} \\
&\quad \leqq C_1\,\delta^r \sup_{|\mathbf{z}|\leqq\delta}\left(|\mathbf{z}|^{-s}\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z}) - h(\mathbf{x}-\mathbf{z})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right).
\end{aligned}
\tag{43}
$$

(ii) *There exists a positive constant $C_2 = C_2(J, d, r, s) < \infty$ such that, for any $\delta > 0$,*

$$\sup_{|z|\leqq\delta}\left(\int_{\mathbb{R}^d}|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)$$
$$\leqq C_2\,\delta^r \sup_{0<\delta\leqq 1}\left(\delta^{-s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x}-\mathbf{z})|\,J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\right)$$
$$+C_2\,\delta^r\,\|h\|_{L^1(\mathbb{R}^d)}\,. \tag{44}$$

Suppose that u_0 is, say, a deterministic function belonging to $\mathrm{BV}(\mathbb{R}^d)$, or more generally to the Besov space $B^\ell_{1,\nu}(\mathbb{R}^d)$ for $\nu \in (\frac{1}{2}, 1)$.

Starting from (42) with $\delta > 0$,

$$\frac{1}{\sqrt{\delta}}E\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|u^\varepsilon(t,\mathbf{x}+\mathbf{z})-u^\varepsilon(t,\mathbf{x}-\mathbf{z})\right|J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\right]$$
$$\leqq C_T\,\frac{1}{\sqrt{\delta}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|u_0(\mathbf{x}+\mathbf{z})-u_0(\mathbf{x}-\mathbf{z})|\,J_\delta(\mathbf{z})\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}$$
$$+C_T\,(1+\|\psi\|_{L^1(\mathbb{R}^d)})$$
$$\leqq 2\,C_T\,C_1\,\|\psi\|_{L^\infty(\mathbb{R}^d)}\sup_{|\mathbf{z}|\leqq\delta}\left(|z|^{-s}\int_{\mathbb{R}^d}|u_0(\mathbf{x}+\mathbf{z})-u_0(\mathbf{x})|\,\mathrm{d}\mathbf{x}\right)$$
$$+C_T\,(1+\|\psi\|_{L^1(\mathbb{R}^d)})$$
$$\leqq C(T,R), \tag{45}$$

where (43) with $r = \frac{1}{2}$ and $s > \frac{1}{2}$ was used to arrive at the second inequality.

In view of (44) with $s = \frac{1}{2}$ and $r < \frac{1}{2}$,

$$\sup_{|\mathbf{z}|\leqq\frac{\delta}{2}} E\left[\int_{\mathbb{R}^d}\left|u^\varepsilon(t,\mathbf{x}+\mathbf{z})-u^\varepsilon(t,\mathbf{x})\right|\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right]$$
$$\leqq C_2\,\delta^r\sup_{0<\delta\leqq 1}\left(\frac{1}{\sqrt{\delta}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|u^\varepsilon(t,\mathbf{x}+\mathbf{z})-u^\varepsilon(t,\mathbf{x}-\mathbf{z})\right|J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\right)$$
$$+C_2\,\delta^r\,\left\|u^\varepsilon(t,\cdot)\right\|_{L^1(\mathbb{R}^d)}\,. \tag{46}$$

Combining (45) with (46) yields

Theorem 7 (Fractional BV-estimate). *For fixed $\varepsilon > 0$, let u^ε solve the stochastic parabolic problem* (41) *with initial data u_0 belonging to the Besov space $B^\nu_{1,\infty}(\mathbb{R}^d)$ for some $\nu \in (\frac{1}{2}, 1)$. In addition, we assume that*

$$\textit{either } u^\varepsilon \in L^\infty((0,T)\times\mathbb{R}^d\times\Omega) \ \textit{ for any } T>0, \quad \textit{or} \ \ \mathbf{f}'' \in L^\infty.$$

Fix $T > 0$ and $R > 0$. There exists a constant $C_{T,R}$ independent of ε such that, for any $0 < t < T$,

$$\sup_{|\mathbf{z}|\leqq\delta} E\left[\int_{K_R} \left|u^\varepsilon(t,\mathbf{x}+\mathbf{z}) - u^\varepsilon(t,\mathbf{x})\right| \,\mathrm{d}\mathbf{x}\right] \leqq C_{T,R}\,\delta^r, \quad r \in \left(0, \frac{1}{2}\right).$$

Proof of (42). We start off from Lemma 1 with $\hat{\mathbf{f}} = \mathbf{f}$, $\hat{\varepsilon} = \varepsilon$, $\hat{\sigma} = \sigma$, $v_0 = u_0$, and $v = u$ (this lemma also holds when σ depends on $\mathbf{x}$):

$$\begin{aligned}
&E\left[\iint \eta_\rho(u^\varepsilon(t,\mathbf{x}) - u^\varepsilon(t,\mathbf{y}))J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&\quad -E\left[\iint \eta_\rho(u_0(\mathbf{x}) - u_0(\mathbf{y}))J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&\leqq E\left[\iiint_0^t \mathbf{q}^{\mathbf{f}}_\rho(u^\varepsilon(s,\mathbf{x}), u^\varepsilon(s,\mathbf{y}))\cdot\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&\quad +E\left[\iiint_0^t (\mathbf{q}^{\mathbf{f}}_\rho(u^\varepsilon(s,\mathbf{y}), u^\varepsilon(s,\mathbf{x}))\right.\\
&\qquad\qquad \left.-\mathbf{q}^{\mathbf{f}}_\rho(u^\varepsilon(s,\mathbf{x}), u^\varepsilon(s,\mathbf{y})))\cdot\nabla_{\mathbf{y}}\phi_\delta\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&\quad +E\left[\iiint_0^t \frac{1}{2}\eta''_\rho(u^\varepsilon(s,\mathbf{x}) - u^\varepsilon(s,\mathbf{y}))\right.\\
&\qquad\qquad \left.\times(\sigma(\mathbf{x},u^\varepsilon(s,\mathbf{x})) - \sigma(\mathbf{y},u^\varepsilon(s,\mathbf{y})))^2\phi_\delta(\mathbf{x},\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&\quad +\varepsilon\,E\left[\iiint_0^t \eta_\rho(u^\varepsilon(s,\mathbf{x})-u^\varepsilon(s,\mathbf{y}))J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\Delta_{\mathbf{x}}\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{47}$$

Finally, denoting the left-hand side of (47) by LHS and utilizing (12), we have

$$\begin{aligned}
\mathrm{LHS} &= E\left[\iint \left|u^\varepsilon(t,\mathbf{x}) - u^\varepsilon(t,\mathbf{y})\right| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&\quad -E\left[\iint |u_0(\mathbf{x}) - u_0(\mathbf{y})|\, J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\\
&\quad +\mathcal{O}(\rho)\,\|\psi\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Since $|\nabla\psi(\mathbf{x})| \leqq C_0\psi(\mathbf{x})$,

$$|I_1| \leqq C\int_0^t E\left[\iint \left|u^\varepsilon(s,\mathbf{x}) - u^\varepsilon(s,\mathbf{y})\right| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\mathrm{d}s.$$

Note that, thanks to (31) and the boundedness of $\mathbf{f}''$,

$$\begin{aligned}
\mathbf{q}^{\mathbf{f}}_\rho(v,u) &= \mathbf{q}^{\mathbf{f}}_\rho(u,v) + \int_v^u \partial_\xi\left(\mathbf{q}^{\mathbf{f}}_\rho(\xi,v) - \mathbf{q}^{\mathbf{f}}_\rho(v,\xi)\right)\mathrm{d}\xi\\
&= \mathbf{q}^{\mathbf{f}}_\rho(u,v) + |u-v|\,O(\rho),
\end{aligned}$$

so that

$$
\begin{aligned}
|I_2| &\leqq C\,\rho\, E\left[\iiint_0^t \left|u^\varepsilon(s,\mathbf{x})-u^\varepsilon(s,\mathbf{y})\right| \left|\nabla_{\mathbf{y}} J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\right| \psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad + C\,\rho\, E\left[\iiint_0^t \left|u^\varepsilon(s,\mathbf{x})-u^\varepsilon(s,\mathbf{y})\right| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right) \left|\nabla\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\right| \mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\leqq C\,t\, \|\psi\|_{L^\infty(\mathbb{R}^d)} \left(\frac{\rho}{\delta}+\rho\right),
\end{aligned}
$$

where we have used the estimate

$$
\sup_{0\leqq t\leqq T} E\left[\left\|u^\varepsilon(t)\right\|_{L^1(\mathbb{R}^d)}\right] < \infty \quad \text{for any } T>0,
$$

and exploited $|\nabla\psi(\mathbf{x})| \leqq C_0\psi(\mathbf{x})$.

Regarding I_3,

$$
\begin{aligned}
|I_3| &\leqq E\Bigg[\iiint_0^t \frac{M_2}{\rho}\mathbf{1}_{|u^\varepsilon(s,\mathbf{x})-u^\varepsilon(s,\mathbf{x})|<\rho}\left(\sigma(\mathbf{x},u^\varepsilon(s,\mathbf{x}))-\sigma(\mathbf{y},u^\varepsilon(s,\mathbf{x}))\right)^2 \\
&\qquad\qquad \times J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\Bigg] \\
&\quad + E\Bigg[\iiint_0^t \frac{M_2}{\rho}\mathbf{1}_{|u^\varepsilon(s,\mathbf{x})-u^\varepsilon(s,\mathbf{y})|<\rho}\left(\sigma(\mathbf{y},u^\varepsilon(s,\mathbf{x}))-\sigma(\mathbf{y},u^\varepsilon(s,\mathbf{y}))\right)^2 \\
&\qquad\qquad \times J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\Bigg] =: A+B,
\end{aligned}
$$

where, with reference to the second part of (40),

$$
\begin{aligned}
|A| &\leqq M_2 E\Bigg[\iiint_0^t \frac{|\sigma(\mathbf{x},u^\varepsilon(s,\mathbf{x}))-\sigma(\mathbf{y},u^\varepsilon(s,\mathbf{x}))|^2}{\rho} \\
&\qquad\qquad \times J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\Bigg] \\
&\leqq C E\left[\iiint_0^t \frac{|\mathbf{y}-\mathbf{x}|^2}{\rho}|u^\varepsilon(s,\mathbf{x})|^2 J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\leqq C\,\|\psi\|_{L^\infty(\mathbb{R}^d)}\, t\,\frac{\delta^2}{\rho},
\end{aligned}
$$

where we have put to use the estimate

$$
\sup_{0\leqq t\leqq T} E\left[\left\|u^\varepsilon(t)\right\|^2_{L^2(\mathbb{R}^d)}\right] \leqq C_T \quad \text{for any } T>0.
$$

Moreover, with reference to the first part of (40),

$$
|B| \leqq C\int_0^t E\left[\iint \left|u^\varepsilon(s,\mathbf{x})-u^\varepsilon(s,\mathbf{y})\right| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right]\mathrm{d}s.
$$

Regarding I_4, using $|\Delta\psi(\mathbf{x})| \leqq C_0\psi(\mathbf{x})$, we have

$$|I_4| \leqq C \int_0^t E\left[\iint \left|u^\varepsilon(s,\mathbf{x}) - u^\varepsilon(s,\mathbf{y})\right| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \mathrm{d}s.$$

Summarizing, we have arrived at

$$\begin{aligned}
&E\left[\iint \left|u^\varepsilon(t,\mathbf{x}) - u^\varepsilon(t,\mathbf{y})\right| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad\leqq E\left[\iint |u_0(\mathbf{x}) - u_0(\mathbf{y})| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\qquad+C\int_0^t E\left[\iint \left|u^\varepsilon(s,\mathbf{x}) - u^\varepsilon(s,\mathbf{y})\right| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \mathrm{d}s \\
&\qquad+C\,t\,\|\psi\|_{L^\infty(\mathbb{R}^d)}\left(\frac{\rho}{\delta}+\rho\right) + C\,\|\psi\|_{L^\infty(\mathbb{R}^d)}\,t\,\frac{\delta^2}{\rho} + C\rho\,\|\psi\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Optimizing with respect to ρ (take $\rho = \mathcal{O}(\delta^{3/2})$) and applying Gronwall's inequality give

$$\begin{aligned}
&E\left[\iint \left|u^\varepsilon(t,\mathbf{x}) - u^\varepsilon(t,\mathbf{y})\right| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\quad\leqq C_T\,E\left[\iint |u_0(\mathbf{x}) - u_0(\mathbf{y})| J_\delta\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\psi\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right] \\
&\qquad+C_T\left(1+\|\psi\|_{L^1(\mathbb{R}^d)}\right)\sqrt{\delta}, \quad 0 < t < T,
\end{aligned} \tag{48}$$

for some constant C_T independent of ε.

Introducing new variables, $\tilde{\mathbf{x}} = \frac{\mathbf{x}+\mathbf{y}}{2}$ and $z = \frac{\mathbf{x}-\mathbf{y}}{2}$ in (48), so $\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{z}$ and $\mathbf{y} = \tilde{\mathbf{x}} + \mathbf{z}$, we finally obtain (42) (dropping the tildes). □

Combining Theorem 7 with the argument in Section 3, we conclude

Theorem 8 (Existence and regularity). *Suppose that* (40) *holds and also that* $\|\mathbf{f}''\|_{L^\infty} < \infty$.

(i) *Let the initial data* u_0 *belong to the Besov space* $B^\nu_{1,\infty}(\mathbb{R}^d)$ *for some* $\nu \in (\frac{1}{2}, 1)$ *and*

$$E\left[\|u_0\|^p_{L^p(\mathbb{R}^d)}\right] < \infty, \quad p = 1, 2, \ldots. \tag{49}$$

Then there exists a strong stochastic entropy solution of the balance law (39) *with initial data* u_0 *such that, for fixed* $T > 0$ *and* $R > 0$*, there exists a constant* $C_{T,R}$ *such that, for any* $0 < t < T$,

$$\sup_{|\mathbf{z}|\leqq\delta} E\left[\int_{K_R} |u(t,\mathbf{x}+\mathbf{z}) - u(t,\mathbf{x})|\,\mathrm{d}\mathbf{x}\right] \leqq C_{T,R}\,\delta^r$$

for some $r \in (0, \frac{1}{2})$ *and*

$$E\left[\|u(t,\cdot)\|^p_{L^p(\mathbb{R}^d)}\right] < \infty, \quad p = 1, 2, \ldots. \tag{50}$$

(ii) *Let* u_0 *satisfy only* (49). *Then there exists a strong stochastic entropy solution of the balance law* (39) *with initial data* u_0 *satisfying* (50).

Finally, we remark in passing that the results and techniques extend easily to stochastic balance laws with additional nonhomogeneous terms, by combining with the Gronwall inequality, such as

$$\partial_t u(t,\mathbf{x}) + \nabla\cdot\mathbf{f}(\mathbf{x},u(t,\mathbf{x})) = \sigma(\mathbf{x},u(t,\mathbf{x}))\,\partial_t W(t) + g(\mathbf{x},u(t,\mathbf{x})),$$

for a large class of non-homogeneous terms $\mathbf{f}(\mathbf{x},u)$, $g(\mathbf{x},u)$.

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Appendix A. Proof of Lemma 2

Since $r < s$, we can prove (43) as follows:

$$\begin{aligned}
&\delta^{-r}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x}-\mathbf{z})|\,J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{x}\\
&\quad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x}-\mathbf{z})|}{\delta^{d+r}}J\left(\frac{|\mathbf{z}|}{\delta}\right)\psi(\mathbf{x})\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{x}\\
&\quad\leqq \|J\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}^d}\int_{|z|\leqq\delta}\frac{|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x}-\mathbf{z})|}{|\mathbf{z}|^{d+r}}\psi(x)\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{x}.\\
&\quad\leqq \|J\|_{L^\infty(\mathbb{R})}\sup_{|\mathbf{z}|\leqq\delta}\left(\mathbf{z}^{-s}\,\|(h(\cdot+\mathbf{z})-h(\cdot-\mathbf{z}))\psi\|_{L^1(\mathbb{R}^d)}\int_{|\mathbf{z}|\leqq\delta}\frac{1}{|\mathbf{z}|^{d+r-s}}\,\mathrm{d}\mathbf{z}\right)\\
&\quad\leqq C_{J,d,r,s}\sup_{|\mathbf{z}|\leqq\delta}\left(\mathbf{z}^{-s}\,\|(h(\cdot+\mathbf{z})-h(\cdot-\mathbf{z}))\psi\|_{L^1(\mathbb{R}^d)}\right),
\end{aligned}$$

where we have used the integrability of $1/|z|^{d+r-s}$ (since $d+r-s<d$).

We continue with the proof of (44). To this end, let us introduce the modulus of continuity

$$\omega(\delta) := \sup_{|\mathbf{z}|\leqq\delta}\left(\int_{\mathbb{R}^d}|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right),\quad \delta>0.$$

Clearly, $\omega(\cdot)$ is a non-decreasing function and thus

$$\int_0^\infty\kappa^{-r-1}\omega(\kappa)\,\mathrm{d}\kappa \geqq \int_\delta^\infty\kappa^{-r-1}\omega(\kappa)\,\mathrm{d}\kappa \geqq \omega(\delta)\int_\delta^\infty\kappa^{-r-1}\,\mathrm{d}\kappa = \frac{1}{r}\delta^{-r}\omega(\delta);$$

therefore

$$\omega(\delta) \leqq r\,\delta^r \int_0^\infty \kappa^{-r-1}\omega(\kappa)\,\mathrm{d}\kappa. \tag{A.1}$$

Set

$$h_\delta(\mathbf{x}) := \int_{\mathbb{R}^d} J_{\frac{\delta}{2}}(\mathbf{y})h(\mathbf{x}+\mathbf{y})\,\mathrm{d}\mathbf{y},$$

and note that

$$\begin{aligned}
&\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z}) - h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&\leqq \int_{\mathbb{R}^d} |h_\delta(\mathbf{x}+\mathbf{z}) - h_\delta(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} |h_\delta(\mathbf{x}+\mathbf{z}) - h(\mathbf{x}+\mathbf{z})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&\quad + \int_{\mathbb{R}^d} |h_\delta(\mathbf{x}) - h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x},
\end{aligned} \tag{A.2}$$

We estimate the first two terms on the right-hand side as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d} |h_\delta(\mathbf{x}) - h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&= \int_{\mathbb{R}^d} \left| 2^d\delta^{-d} \int_{\mathbb{R}^d} J\left(\frac{2\,|\mathbf{y}|}{\delta}\right)(h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x}))\,\mathrm{d}\mathbf{y}\right| \psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&\leqq \|J\|_{L^\infty(\mathbb{R})}\,\delta^{-d} \int_{|\mathbf{y}|\leqq\frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |h_\delta(\mathbf{x}+\mathbf{z}) - h(\mathbf{x}+\mathbf{z})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&\leqq \|J\|_{L^\infty(\mathbb{R})}\,\delta^{-d} \int_{|\mathbf{y}|\leqq\frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z}+\mathbf{y}) - h(\mathbf{x}+\mathbf{z})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&= \|J\|_{L^\infty(\mathbb{R})}\,\delta^{-d} \int_{|\mathbf{y}|\leqq\frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x})|\,\psi(\mathbf{x}-\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\leqq C\,\delta^{-d} \int_{|\mathbf{y}|\leqq\frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} + I_1(\delta),
\end{aligned}$$

where, for $\delta \geqq 0$,

$$\begin{aligned}
I_1(\delta) &:= \delta^{-d} \sup_{|\mathbf{z}|\leqq\frac{\delta}{2}} \left(\int_{|\mathbf{y}|\leqq\delta} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x})|\,|\psi(\mathbf{x}) - \psi(\mathbf{x}-\mathbf{z})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right)\\
&\leqq \delta\,C\,\|\nabla\psi\|_{L^\infty(\mathbb{R}^d)}\,\|h\|_{L^1(\mathbb{R}^d)}\,\mathbf{1}_{0\leqq\delta\leqq1}(\delta)\\
&\quad + C\,\|\psi\|_{L^\infty(\mathbb{R}^d)}\,\|h\|_{L^1(\mathbb{R}^d)}\,\mathbf{1}_{\delta>1}(\delta).
\end{aligned}$$

For each $\mathbf{z} \in \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^d$,

$$h_\delta(\mathbf{x}+\mathbf{z}) - h_\delta(\mathbf{x}) = \int_0^1 \nabla h_\delta(\mathbf{x}+\theta\mathbf{z}) \cdot \mathbf{z} \, \mathrm{d}\theta.$$

Observe that, for each $x \in \mathbb{R}^d$,

$$\nabla h_\delta(\mathbf{x}) = \int_{\mathbb{R}^d} \nabla J_{\frac{\delta}{2}}(\mathbf{y})(h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x})) \, \mathrm{d}\mathbf{y}.$$

by the symmetry of the mollifier. Thus, with $|\mathbf{z}| \leqq \delta$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |h_\delta(\mathbf{x}+\mathbf{z}) - h_\delta(\mathbf{x})| \, \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\
&\quad = \int_{\mathbb{R}^d} \left| \int_0^1 \nabla h_\delta(\mathbf{x}+\theta\mathbf{z}) \cdot \mathbf{z} \, \mathrm{d}\theta \right| \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\
&\quad \leqq C\,\delta^{-d} \sup_{|\mathbf{z}| \leqq \delta,\, \theta \in [0,1]} \left(\int_{|y| \leqq \frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\theta\mathbf{z}+\mathbf{y}) - h(\mathbf{x}+\theta\mathbf{z})| \, \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \right) \\
&\quad = C\,\delta^{-d} \sup_{|\mathbf{z}| \leqq \delta,\, \theta \in [0,1]} \left(\int_{|y| \leqq \frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x})| \, \psi(\mathbf{x}-\theta\mathbf{z}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \right) \\
&\quad \leqq C\,\delta^{-d} \int_{|y| \leqq \frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x})| \, \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} + I_2(\delta),
\end{aligned}$$

where $I_2(\delta)$ denotes the expression

$$C\,\delta^{-d} \sup_{|\mathbf{z}| \leqq \delta,\, \theta \in [0,1]} \left(\int_{|y| \leqq \frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x})| \, |\psi(\mathbf{x}) - \psi(\mathbf{x}-\theta\mathbf{z})| \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \right),$$

and

$$\begin{aligned}
I_2(\delta) \leqq \delta\, C \, \|\nabla\psi\|_{L^\infty(\mathbb{R}^d)} \, \|h\|_{L^1(\mathbb{R}^d)} \, \mathbf{1}_{0 \leqq \delta \leqq 1}(\delta) \\
+ C \, \|\psi\|_{L^\infty(\mathbb{R}^d)} \, \|h\|_{L^1(\mathbb{R}^d)} \, \mathbf{1}_{\delta > 1}(\delta),
\end{aligned}$$

with reference to the term $I_1(\delta)$.

In view of the estimates derived above, taking the supremum in (A.2) over $|\mathbf{z}| \leqq \delta$, we have established

$$\begin{aligned}
\omega(\delta) \leqq C\,\delta^{-d} \int_{|y| \leqq \frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x})| \, \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \\
+ C \, \|h\|_{L^1(\mathbb{R}^d)} \, (\delta \, \mathbf{1}_{0 \leqq \delta \leqq 1}(\delta) + \mathbf{1}_{\delta > 1}(\delta)).
\end{aligned}$$

Multiplying this by δ^{-r-1} and integrating in δ from 0 to ∞ yield (replacing $\mathbf{y}$ by $\mathbf{z}$)

$$\begin{aligned}
&\int_0^\infty \delta^{-r-1}\omega(\delta)\,\mathrm{d}\delta \\
&\quad \leqq C \int_0^\infty \delta^{-r-1-d} \int_{|\mathbf{z}|\leqq\frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\delta \\
&\qquad + C\,\|h\|_{L^1(\mathbb{R}^d)} \left(\int_0^1 \delta^{-r}\,\mathrm{d}\delta + \int_1^\infty \delta^{-r-1}\,\mathrm{d}\delta\right) =: A+B, \qquad \text{(A.3)}
\end{aligned}$$

where the integrals on the last line are bounded since $r \in (0,1)$:

$$B \leqq C_r\ \|h\|_{L^1(\mathbb{R}^d)}.$$

Noticing that $\frac{|\mathbf{z}|}{\delta} \leqq \frac{1}{2} \Rightarrow J(\frac{|\mathbf{z}|}{\delta}) > 0$ and using $r < s$, we have

$$\begin{aligned}
A &\leqq C_J \int_0^1 \delta^{-r-1-d} \int_{|\mathbf{z}|\leqq\frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\, J\left(\frac{|\mathbf{z}|}{\delta}\right) \psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\delta \\
&\leqq C_J \int_0^1 \delta^{-s}\delta^{s-r-1} \int_{|\mathbf{z}|\leqq\frac{\delta}{2}} \int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\, J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\delta \\
&\leqq C_J \left(\int_0^1 \frac{1}{\delta^{1+r-s}}\,\mathrm{d}\delta\right) \sup_{0<\delta\leqq 1} (\delta^{-s} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\, J_\delta(\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}) \\
&\leqq C_{J,r,s} \sup_{0<\delta\leqq 1} (\delta^{-s} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\, J_\delta(\mathbf{z})\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}),
\end{aligned}$$

where $C_{J,r,s} = C_J\,\frac{1}{s-r}$.

Consequently, from (A.1) and (A.3), it follows that, for any $\delta > 0$,

$$\begin{aligned}
&\sup_{|z|\leqq\delta} \left(\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\right) \\
&\quad \leqq C\,\delta^r \sup_{0<\delta\leqq 1} \left(\delta^{-s} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\, J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\right) \\
&\qquad + C\,\delta^r\,\|h\|_{L^1(\mathbb{R}^d)}, \qquad \text{(A.4)}
\end{aligned}$$

for some finite constant C.

Finally, observe that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\, J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z} \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x}-\mathbf{z})|\, J_\delta(2\mathbf{z})\psi(\mathbf{x}-\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z} \\
&\quad = \frac{1}{2^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x}-\mathbf{z})|\, J_{\frac{\delta}{2}}(\mathbf{z})\psi(\mathbf{x}-\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z} \\
&\quad \leqq \frac{1}{2^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |h(\mathbf{x}+\mathbf{z})-h(\mathbf{x}-\mathbf{z})|\, J_{\frac{\delta}{2}}(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z} + I_3(\delta),
\end{aligned}$$

where $I_3(\delta)$ denotes the expression

$$\frac{1}{2^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x}-\mathbf{z})|\,J_{\frac{\delta}{2}}(\mathbf{z})\,|\psi(\mathbf{x})-\psi(\mathbf{x}-\mathbf{z})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}.$$

As with $I_1(\delta)$,

$$I_3(\delta)\leqq C\,\|h\|_{L^1(\mathbb{R}^d)}\,(\delta\,\mathbf{1}_{0\leqq\delta\leqq 1}(\delta)+\mathbf{1}_{\delta>1}(\delta)),$$

which implies

$$\begin{aligned}&\sup_{0<\delta\leqq 1}\left(\delta^{-s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\,J_\delta(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\right)\\ &\quad\leqq C\sup_{0<\delta\leqq 1}\left(\delta^{-s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x}-\mathbf{z})|\,J_{\frac{\delta}{2}}(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\right)\\ &\qquad+C\,\|h\|_{L^1(\mathbb{R}^d)}\,.\end{aligned}$$

We can therefore replace (A.4) by

$$\begin{aligned}&\sup_{|z|\leqq\delta}\int_{\mathbb{R}^d}|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &\quad\leqq C\,\delta^r\sup_{0<\delta\leqq 1}\left(\delta^{-s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h(\mathbf{x}+\mathbf{z})-h(\mathbf{x})|\,J_{\frac{\delta}{2}}(\mathbf{z})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\right)\\ &\qquad+C\,\delta^r\,\|h\|_{L^1(\mathbb{R}^d)}\,,\end{aligned}$$

for some finite constant C, which implies (44).

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