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Degenerate Regularization of Forward–Backward Parabolic Equations: The Regularized Problem

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Abstract

We study a quasilinear parabolic equation of forward–backward type in one space dimension, under assumptions on the nonlinearity which hold for a number of important mathematical models (for example, the one-dimensional Perona–Malik equation), using a degenerate pseudoparabolic regularization proposed in Barenblatt et al. (SIAM J Math Anal 24:1414–1439, 1993), which takes time delay effects into account. We prove existence and uniqueness of positive solutions of the regularized problem in a space of Radon measures. We also study qualitative properties of such solutions, in particular concerning their decomposition into an absolutely continuous part and a singular part with respect to the Lebesgue measure. In this respect, the existence of a family of viscous entropy inequalities plays an important role.

1. Introduction

In this paper and its companion [17] we study the initial-boundary value problem

$$\begin{cases} U_t = [\varphi(U)]_{xx} & \text{in } \Omega \times (0,T] =: Q \\ \varphi(U) = 0 & \text{in } \partial\Omega \times (0,T] \\ U = U_0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{1.1}$$

Here $\Omega \subseteq \mathbb{R}$ is a bounded interval, $T > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a *nonmonotone* odd function. Its main feature is that there exists $\alpha > 0$ such that

$$(s-\alpha)\varphi'(s) \leqq 0 \quad \text{for any } s > 0 \tag{1.2}$$

(see assumption (H_1)). Therefore the first equation in (1.1) is a quasilinear parabolic equation of *forward–backward* type, and problem (1.1) is *ill-posed* whenever the solution U takes values where $\varphi' < 0$.

Our motivation comes from the Perona–Malik equation [11] in one space dimension

$$u_t = [\varphi(u_x)]_x, \tag{1.3}$$

which also appears in a mathematical model of oceanography [2]. Typical forms of φ in (1.3) are

$$\varphi(s) = \frac{s}{s^2+\alpha}, \quad \varphi(s) = s\exp\left(-\frac{s}{\alpha}\right) \qquad (\alpha > 0) \tag{1.4}$$

(usually the first expression in (1.4) with $\alpha = 1$ is used; observe that it does not belong to $L^1(\mathbb{R})$, thus it does not satisfy assumption (H_1)-(i) below).

Formally deriving equation (1.3) with respect to x and setting $U := u_x$ gives the first equation in (1.1). Since a natural function space in which to study (1.3) is that of real functions of bounded variation [2], the above formal argument suggests that we study problem (1.1) in the space of bounded *Radon measures on* $\bar{\Omega}$, as we do here.

Let us mention that problem (1.1) independently arises (with φ as in (1.4)) in a model of aggregating populations in population dynamics [10], and with a cubic-like φ in the theory of phase transitions (see [3,4,8] and references therein). In the latter case φ denotes the *response function* associated with the Landau energy density, and the first equation in (1.1) is derived from the continuity equation via the so-called *Cahn quasi-equilibrium principle*. In the following we shall borrow some terminology from this interpretation of (1.1), for instance when speaking of *stable* and *unstable branches* of the graph of φ.

Our purpose is to study a specific regularization of problem (1.1) (previously introduced in [2] for equation (1.3)), namely

$$\begin{cases} U_t = [\varphi(U)]_{xx} + \epsilon[\psi(U)]_{txx} & \text{in } Q \\ \varphi(U) + \epsilon[\psi(U)]_t = 0 & \text{in } \partial\Omega \times (0,T] \\ U = U_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{1.5}$$

and its limit as the regularization parameter $\epsilon > 0$ goes to zero. The *increasing* odd function $\psi : \mathbb{R} \to \mathbb{R}$ in (1.5) is related to φ by several assumptions (see (H_2) below); in fact, it arises as the coefficient of a formal first order approximation to a modified version of (1.1), which takes into account *time delay* effects ([2]; see also [1] and references therein). As in the theory of hyperbolic conservation laws, sometimes we shall call problem (1.5) the *viscous problem* associated with (1.1), and its limit as $\epsilon \to 0^+$ the *vanishing viscosity limit*. We address the viscous problem in this paper, while referring the reader to [17] for the vanishing viscosity limit.

A major feature of ψ is that $\psi'(s) \to 0$ as $s \to \infty$—therefore, the first equation in (1.5) is *degenerate quasiparabolic* [2]. This makes an important difference with respect to another regularization of (1.1), sometimes called *Sobolev regularization*, which formally corresponds to the choice $\psi(s) = s$ [5,9,12–15]. In fact, the behaviour of ψ at infinity does not play any role in the case of a cubic-like φ, since in this case any bounded, sufficiently large interval of values of U is an invariant domain for solutions of problem (1.5) [9]. Instead, for a function φ of Perona–Malik type (namely, satisfying assumption (H_1)) the only bounded invariant domain in $\mathbb{R}_+$ is $[0, \alpha]$, where problem (1.1) is well posed. Therefore, to study (1.1) in the general case, unbounded values of U must be considered.

Arguing as in [10], it is easily seen that using the Sobolev regularization $\psi(U) = U$ gives a unique solution $U \in C^1([0, T]; C(\overline{\Omega}))$ to problem (1.5) for any initial data function $U_0 \in C(\overline{\Omega})$. On the contrary, in view of the example given in [2, Section 8], we can expect that some solution of (1.5) *with smooth initial data* takes values in the space of Radon measures for positive times. Clearly, such a loss of regularity depends on the degenerate character of the regularization used in (1.5). This seems in satisfactory agreement with the physical interpretation of the model and reflects the more rigorous derivation (with respect to the Sobolev regularization) of the first equation of (1.5) in [2].

In the sequel we prove that for any $\epsilon > 0$ there exists a unique measure-valued function U (in the sense of the Radon measures on $\bar{\Omega}$), which solves problem (1.5) in a suitable sense (see Definition 2.1 and Theorem 2.1 below). The proof makes use of a family of approximating problems, defined by a regularization of ψ and U_0, and of uniform a priori estimates of their solutions.

Further, we prove that the *regular part* U_r (with respect to the Lebesgue measure of Ω) of the solution U satisfies a family of infinitely many inequalities, which we call *viscous entropy inequalities* by analogy with the case of hyperbolic conservation laws (see Theorem 2.4). Similar inequalities are known to hold for the Sobolev regularization, both for a cubic-like φ [9] and for a φ of Perona–Malik type [15], and play an important role when studying the vanishing viscosity limit. With respect to [9, 15] we prove here an improved version of these inequalities, which holds for almost every $t \in (0, T)$ (in this connection, see [16]). This plainly implies an interesting property of the solution U of problem (1.5), that is, that the support of its *singular part* U_s (with respect to the Lebesgue measure) is *nondecreasing in time* (see Theorem 2.5).

In view of the above monotonicity property, it is natural to ask whether the support of U_s becomes nonempty for some positive time, if the initial data function U_0 is smooth. By the degenerate character of the regularization used in (1.5), it is easy to conjecture that this will not happen, if $\psi'(s) \to 0$ "slow enough" as $s \to \infty$. This is indeed the case, if suitable assumptions on U_0 are satisfied (see Theorem 2.8).

Let us mention for completeness that the main result of the companion paper [17] is the proof of existence of a properly defined *weak solution* of the original problem (1.1) (see [17, Definition 2.2 and Theorem 2.8]). This is the natural generalization to the present case of the notion of *weak solution (in the sense of Young measures)* of problem (1.1), and of the corresponding existence result, given in [12] for the case of a cubic-like φ. However, at variance from the latter case, the presence of a singular term (that is, of a Radon measure $\mu \neq 0$) in the solution cannot be excluded (see [17].)

The paper is organized as follows. In Section 2 we describe the mathematical framework and state the main results. A number of preliminary a priori estimates are stated and proven in Section 3, whereas proofs of the main results are presented in Sections 4, 5 and 6.

2. Mathematical Framework and Results

Concerning the functions φ and ψ, we always make the following assumptions:

$$(H_1)\quad \begin{cases} \text{(i)}\quad \varphi \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R}),\ \varphi \text{ odd};\ \varphi(s) > 0 \text{ for } s > 0; \\ \text{(ii)}\ \varphi'(s) > 0 \text{ for } 0 < s < \alpha,\ \varphi'(s) < 0 \text{ for } s > \alpha \quad (\alpha > 0); \\ \text{(iii)}\ \varphi''(s) \geqq 0 \text{ for any } s \geqq s_0,\ \text{for some } s_0 > 0; \\ \text{(iv)}\quad \varphi^{(j)} \in L^\infty(\mathbb{R}) \text{ for any } j \in \mathbb{N}. \end{cases}$$

$$(H_2)\quad \begin{cases} \text{(i)}\quad \psi \in C^\infty(\mathbb{R}),\ \psi' > 0 \text{ in } \mathbb{R},\ \psi \text{ odd}, \\ \qquad \psi(s) \to \gamma \text{ as } s \to \infty \text{ for some } \gamma \in (0,\infty); \\ \text{(ii)}\quad \psi^{(j)} \in L^\infty(\mathbb{R}) \text{ for any } j \in \mathbb{N}; \\ \text{(iii)}\ \psi''(s) \leqq 0 \text{ for any } s \geqq s_0,\ \text{for some } s_0 > 0; \\ \text{(iv)}\ |\varphi'| \leqq k_1\psi' \text{ in } \mathbb{R} \text{ for some } k_1 > 0; \\ \text{(v)}\quad \left|\left(\dfrac{\varphi'}{\psi'}\right)'\right| \leqq k_2\psi' \text{ in } \mathbb{R} \text{ for some } k_2 > 0; \\ \text{(vi)}\quad \dfrac{|\varphi''|}{(\psi')^2} \leqq k_3\psi' \text{ in } \mathbb{R} \text{ for some } k_3 > 0. \end{cases}$$

Here $\varphi^{(j)}$, $\psi^{(j)}$ denote the j-th derivatives of the functions φ, ψ $(j \in \mathbb{N})$; however, the usual notation φ', φ'', ψ', ψ'' will be used for the first and second derivatives. Observe that (H_1)-(i) and (ii) imply $\varphi(s) \to 0$ as $s \to \infty$, $0 < \varphi(s) \leqq \varphi(\alpha)$ for $s > 0$, whereas (H_2)-(i) implies $\psi'(s) \to 0$ as $s \to \infty$.

By abuse of notation, we shall also denote by ψ the extension of ψ to $\overline{\mathbb{R}}$ defined by setting $\psi(\infty) := \gamma$.

We shall denote by $\mathcal{M}(\Omega)$ (respectively, $\mathcal{M}(\mathbb{R})$) the space of Radon measures on Ω (respectively, on $\mathbb{R}$), and by $\mathcal{M}^+(\Omega)$ (respectively, $\mathcal{M}^+(\mathbb{R})$) the cone of positive Radon measures on Ω (respectively, on $\mathbb{R}$). For any $\mu \in \mathcal{M}(\Omega)$ we shall denote by μ_r and μ_s the density of the absolutely continuous part, respectively the singular part of μ with respect to the Lebesgue measure on Ω. Moreover, we shall denote by $< \cdot, \cdot >_\Omega$ (respectively, $< \cdot, \cdot >_\mathbb{R}$) the duality map between the space $\mathcal{M}(\Omega)$ (respectively, $\mathcal{M}(\mathbb{R})$) and the space $C_c(\Omega)$ (respectively, $C_c(\mathbb{R})$) of continuous functions with compact support. The restriction to the interval $\bar{\Omega}$ of any $\zeta \in C_c(\mathbb{R})$ will be denoted by $\zeta_{\bar{\Omega}}$.

By $\mathcal{M}(\bar{\Omega})$ (respectively, $\mathcal{M}^+(\bar{\Omega})$) we shall denote the space of Radon measures $\mu \in \mathcal{M}(\mathbb{R})$ (respectively, the cone of positive Radon measures $\mu \in \mathcal{M}^+(\mathbb{R})$) such that $\operatorname{supp}\mu \subseteq \bar{\Omega}$. For any $\mu \in \mathcal{M}(\bar{\Omega})$ we set

$$\|\mu\|_{\mathcal{M}(\bar{\Omega})} := \|\mu\|_{\mathcal{M}(\mathbb{R})}$$

(observe that $|\mu|(\mathbb{R}) = |\mu|(\bar{\Omega}) < \infty$). For any $\mu \in \mathcal{M}(\bar{\Omega})$ and any $\zeta \in C(\bar{\Omega})$ we also define

$$\langle \mu, \zeta \rangle_{\bar{\Omega}} := \langle \mu, \tilde{\zeta} \rangle_\mathbb{R},$$

where $\tilde{\zeta} \in C_c(\mathbb{R})$ is any continuous function with compact support such that $\tilde{\zeta} = \zeta$ in $\bar{\Omega}$. Observe that the above definition is well posed, since the right-hand side does not depend on the choice of the extension $\tilde{\zeta} \in C_c(\mathbb{R})$. Also observe that the duality map $\langle \mu, \zeta \rangle_\Omega$ is well defined for any $\zeta \in C_0(\Omega) := \{\zeta \in C(\bar{\Omega}) \mid \zeta = 0 \text{ on } \partial\Omega\}$, and there holds

$$\langle \mu, \zeta \rangle_\Omega = \langle \mu, \zeta \rangle_{\bar{\Omega}}.$$

Similar notations will be used for the space of Radon measures on Q, $\bar{Q}$ and $\mathbb{R}^2$. To be specific, we shall denote by $\mathcal{M}(\bar{Q})$ the space of Radon measures $\mu \in \mathcal{M}(\mathbb{R}^2)$ such that $\operatorname{supp} \mu \subseteq \bar{Q}$ and for any $\zeta \in C(\bar{Q})$, $\mu \in \mathcal{M}(\bar{Q})$ we set

$$\langle \mu, \zeta \rangle_{\bar{Q}} := \langle \mu, \tilde{\zeta} \rangle_{\mathbb{R}^2},$$

where $\tilde{\zeta} \in C_c(\mathbb{R}^2)$ satisfies $\tilde{\zeta} = \zeta$ in $\bar{Q}$. For any $\zeta \in C_c(\mathbb{R}^2)$ its restriction to the rectangle $\bar{Q}$ will be denoted by $\zeta_{\bar{Q}}$.

Concerning the initial data U_0, we shall always assume the following:

$$(H_3) \quad \begin{cases} \text{(i) } U_0 \in \mathcal{M}^+(\bar{\Omega}); \\ \text{(ii) there exists a family } \{U_{0\kappa}\} \subseteq C_c^\infty(\Omega), U_{0\kappa} \geqq 0, \\ \|U_{0\kappa}\|_{L^1(\Omega)} \leqq \|U_0\|_{\mathcal{M}(\bar{\Omega})} \text{ for any } \kappa > 0, \text{ such that as } k \to 0: \\ \text{(a) } \displaystyle\int_\Omega U_{0\kappa}\, \zeta \, dx \to < U_0, \zeta >_{\bar{\Omega}} \text{ for any } \zeta \in C(\bar{\Omega}), \\ \text{(b) } \psi(U_{0\kappa}) \rightharpoonup \psi(U_{0r}) \text{ in } H_0^1(\Omega), \\ \text{(c) } \kappa\, U_{0\kappa} \rightharpoonup 0 \text{ in } H_0^1(\Omega). \end{cases}$$

Let us observe that assumption (H_3)-(i) corresponds to assumption $(A3)$ in [2] concerning the nondecreasing character of the initial data u_0 for equation (1.3), which is motivated on physical grounds.

A family of initial data satisfying (H_3) is exhibited below (see Proposition 2.6). It is worth observing that conditions (a) and (b) of assumption (H_3)-(ii) imply $\psi(U_{0r}) \in H_0^1(\Omega)$,

$$\operatorname{supp} U_{0s} \subseteq \mathcal{S}_0 := \{x \in \bar{\Omega} \mid \psi(U_{0r})(x) = \gamma\},$$

and $U_{0r} \in C(\bar{\Omega} \setminus \mathcal{S}_0)$ (see Proposition 6.1).

We shall denote by $L^\infty((0,T); \mathcal{M}^+(\bar{\Omega}))$ the set of positive Radon measures $U \in \mathcal{M}^+(\bar{Q})$ which satisfy the following property: for almost every $t \in \mathbb{R}$ there exists a measure $U(\cdot, t) \in \mathcal{M}^+(\bar{\Omega})$, $U(\cdot, t) = 0$ if $t \notin [0, T]$, such that

(i) for any $\zeta \in C(\bar{Q})$ the map $t \to \langle U(\cdot, t), \zeta(\cdot, t) \rangle_{\bar{\Omega}}$ is Lebesgue measurable, and

$$\langle U, \zeta \rangle_{\bar{Q}} = \int_0^T \langle U(\cdot, t), \zeta(\cdot, t) \rangle_{\bar{\Omega}} dt; \tag{2.1}$$

(ii) there exists a constant $C > 0$ such that

$$\operatorname*{ess\,sup}_{t \in (0,T)} \|U(\cdot, t)\|_{\mathcal{M}(\bar{\Omega})} \leqq C.$$

Denoting by $U_r \in L^1(Q), U_r \geqq 0$ and by $U_s \in \mathcal{M}^+(\bar{Q})$ the density of the absolutely continuous part, respectively the singular part of U with respect to the Lebesgue measure over $\mathbb{R}^2$, equality (2.1) implies for any $\zeta \in C(\bar{Q})$

$$\langle U_r, \zeta\rangle_{\bar{Q}} = \iint_Q U_r\, \zeta \,\mathrm{d}x\mathrm{d}t,$$

$$\langle U_s, \zeta\rangle_{\bar{Q}} = \int_0^T \langle U_s(\cdot,t), \zeta(\cdot,t)\rangle_{\bar{\Omega}}\mathrm{d}t. \tag{2.2}$$

Now we can state the following definition.

Definition 2.1. By a solution of problem (1.5) we mean any $U \in \mathcal{M}^+(\bar{Q})$ such that:

(i) $U \in L^\infty((0,T); \mathcal{M}^+(\bar{\Omega}))$;
(ii) $\varphi(U_r), \psi(U_r) \in L^\infty((0,T); H_0^1(\Omega))$, and $[\psi(U_r)]_t \in L^2((0,T); H_0^1(\Omega))$; moreover,

$$\psi(U_r)(x,0) = \psi(U_{0r})(x) \quad \text{for any } x \in \bar{\Omega}; \tag{2.3}$$

(iii) there holds

$$\operatorname{supp} U_s \subseteq \mathcal{S} := \left\{(x,t) \in \bar{Q} \;\middle|\; \psi(U_r)(x,t) = \gamma\right\}; \tag{2.4}$$

(iv) there holds

$$\iint_Q U_r \zeta_t \,\mathrm{d}x\mathrm{d}t + \int_0^T \langle U_s(\cdot,t), \zeta_t(\cdot,t)\rangle_\Omega \,\mathrm{d}t$$
$$= \iint_Q \left\{[\varphi(U_r)]_x\, \zeta_x + \epsilon[\psi(U_r)]_{tx}\, \zeta_x\right\}\mathrm{d}x\mathrm{d}t \;-\langle U_0, \zeta(\cdot,0)\rangle_\Omega \tag{2.5}$$

for any $\zeta \in C^1([0,T]; H_0^1(\Omega)),\ \zeta(\cdot,T) = 0$ in $\bar{\Omega}$.

Remark 2.1. In the above Definition 2.1, as always in the following, we identify $\psi(U_r) \in L^\infty(Q)$ with its *continuous representative*.

This continuous representative $w \in C(\bar{Q}), \psi(U_r) \equiv w$ exists by Definition 2.1-(ii).

Therefore, in view of these remarks the set $\mathcal{S}$ defined in (2.4) is closed.

Similarly, since $\varphi(U_r) \equiv \varphi(\psi^{-1}(\psi(U_r)))$ in Q, by assumptions (H_2)-(iv) and (v) there holds $\left[\varphi(U_r)\right]_t \in L^2((0,T); H_0^1(\Omega))$. Together with Definition 2.1-(ii), this implies $\varphi(U_r) \in C(\bar{Q})$. Since $\varphi(s) \to 0$ as $s \to \infty$, equality (2.7) below implies $\varphi(U_r) = 0$ on the set $\mathcal{S}$.

The following existence and uniqueness result will be proven.

Theorem 2.1. *Let assumptions* (H_1)–(H_3) *be satisfied. Then there exists a unique solution* U *of problem* (1.5). *Moreover,*

(i) *for almost every* $t \in (0,T)$ *there holds*

$$\|U_r(\cdot,t)\|_{L^1(\Omega)} + \|U_s(\cdot,t)\|_{\mathcal{M}(\bar{\Omega})} \leqq \|U_0\|_{\mathcal{M}(\bar{\Omega})}; \tag{2.6}$$

(ii) $U_r \in H^1(Q_0)$ *for any open subset* $Q_0 \subseteq Q$ *such that* $\operatorname{dist}\left(\bar{Q}_0, \mathcal{S}\right) > 0$. *Moreover,* $U_r \in C(\bar{Q} \setminus \mathcal{S})$ *and*

$$\lim_{dist((x,t),\mathcal{S})\to 0} U_r(x,t) = \infty. \tag{2.7}$$

Since $U_r \in L^1(Q)$, by (2.7) it is reasonable to expect that the set $\mathcal{S}$ defined in (2.4) has zero Lebesgue measure. On this subject we refer the reader to Theorem 2.2 below.

Remark 2.2. It will be proven below (see Lemma 4.6) that the function

$$V_r := \varphi(U_r) + \epsilon[\psi(U_r)]_t \qquad (\epsilon > 0) \tag{2.8}$$

belongs to $L^\infty(Q) \cap L^2((0,T); H_0^1(\Omega))$. Therefore equality (2.5) reads

$$\iint_Q U_r \zeta_t \,\mathrm{d}x\mathrm{d}t + \int_0^T \langle U_s(\cdot,t), \zeta_t(\cdot,t)\rangle_\Omega \,\mathrm{d}t = \iint_Q V_{rx}\, \zeta_x \,\mathrm{d}x\mathrm{d}t - \langle U_0, \zeta(\cdot,0)\rangle_\Omega \,.$$

Let describe precisely in which sense the solution U given by Theorem 2.1 satisfies the initial conditions of problem (1.5). Since $\psi(U_r) \in C(\bar{Q})$, $U_r \in C(\bar{Q} \setminus \mathcal{S})$ (see Remark 2.1 and Theorem 2.1-(ii)) and by (2.3) there holds

$$\psi(U_{0r})(x) = \psi(U_r)(x,0) = \lim_{t\to 0} \psi(U_r)(x,t) \qquad (x \in \bar{\Omega}),$$

we obtain

$$U_r(x,0) = \lim_{t\to 0} U_r(x,t) = U_{0r}(x) \quad \text{for any } x \in \bar{\Omega} \setminus \mathcal{S}_0,$$

where

$$\mathcal{S}_0 = \{x \in \bar{\Omega} \mid \psi(U_{0r})(x) = \gamma\} = \{x \in \bar{\Omega} \mid \psi(U_r)(x,0) = \gamma\}.$$

Let us mention that the set $\mathcal{S}_0$ is closed and has zero Lebesgue measure (see Proposition 6.1).

Moreover, since $\varphi(U_r)$, $\psi(U_r)_t \in L^2((0,T); H_0^1(\Omega))$ by Definition 2.1-(ii), in view of equality (2.5) the map

$$t \;\to\; \mathcal{J}_\zeta(t) := \int_\Omega U_r(x,t)\,\zeta(x)\,\mathrm{d}x + \langle U_s(\cdot,t), \zeta\rangle_\Omega \qquad (t \in (0,T))$$

belongs to the space $H^1(0,T)$, and

$$\mathcal{J}_\zeta(0) = \lim_{t\to 0} \mathcal{J}_\zeta(t) = \int_\Omega U_{0r}(x)\,\zeta(x)\,\mathrm{d}x + \langle U_{0s}, \zeta\rangle_\Omega \tag{2.9}$$

for any $\zeta \in H_0^1(\Omega)$.

Information about the set $\mathcal{S}$ in (2.4) is provided by the following

Theorem 2.2. *Let assumptions* (H_1)–(H_3) *be satisfied. Let* U *be the solution of problem* (1.5) *given by Theorem* 2.1. *Then the set* $\mathcal{S}$ *defined by* (2.4)*:*

(i) *has zero Lebesgue measure;*
(ii) *has a strictly positive distance from* $\partial\Omega \times [0, T] \subseteq \partial Q$.

Remark 2.3. In view of Definition 2.1-(iii), Theorem 2.2 gives information about the support of the singular part of the solution U. Observe that by (2.2) there holds

$$\operatorname{supp} U_s(\cdot, t) \subseteq (\operatorname{supp} U_s)_t$$

for almost every $t \in (0, T)$, $(\operatorname{supp} U_s)_t$ denoting the section at the time t of $\operatorname{supp} U_s$. Therefore,

$$\operatorname{supp} U_s(\cdot, t) \subseteq \mathcal{S}_t := \left\{x \in \bar{\Omega} \mid \psi(U_r)(x, t) = \gamma\right\} \tag{2.10}$$

and by Theorem 2.2-(ii) there holds

$$\operatorname{dist}\left(\operatorname{supp} U_s(\cdot, t), \partial\Omega\right) > 0 \tag{2.11}$$

for almost every $t \in (0, T)$. Clearly, by Remark 2.1 and Theorem 2.2-(i) the section $\mathcal{S}_t$ is closed and has zero Lebesgue measure for almost every $t \in (0, T)$.

Observe also that, since $\varphi(\psi^{-1}(\gamma)) = 0$, we have

$$\operatorname{supp} U_s \subseteq \mathcal{S} \subseteq \{(x, t) \in \Omega \times [0, T] \mid \varphi(U_r)(x, t) = 0\}; \tag{2.12}$$

(where we have made use of Theorem 2.2-(ii)). Similar inclusions hold for the sections at the time t, for almost every $t \in (0, T)$.

As a consequence of Theorems 2.1–2.2, we can prove that the density U_r satisfies the first equation of problem (1.5) in a suitable weak sense, out of a set of arbitrarily small Lebesgue measure. In fact, the following holds.

Theorem 2.3. *Let assumptions* (H_1)–(H_3) *be satisfied. Let* U *be the solution of problem* (1.5) *given by Theorem* 2.1*,* $\mathcal{S}$ *the set defined in* (2.4) *and* $A \subseteq Q$ *any open set such that* $\operatorname{dist}\left(\bar{A}, \mathcal{S}\right) > 0$*. Then:*

(i) U_{rt}, $V_{rxx} \in L^2(A)$*, and*

$$U_{rt} = V_{rxx} \quad \text{in } L^2(A); \tag{2.13}$$

(ii) *for almost every* $t \in (0, T)$ *there holds*

$$\operatorname{supp} U_s(\cdot, t) \subseteq \mathcal{S}_t \subseteq \{x \in \Omega \mid V_r(x, t) = 0\}\,. \tag{2.14}$$

Further, we prove that the couple (U_r, V_r) satisfies in a weak sense infinitely many *entropy inequalities*. Define for any $g \in C^1(\mathbb{R})$

$$G(z) := \int_0^z g(\varphi(s))\mathrm{d}s \qquad (z \in \mathbb{R}). \tag{2.15}$$

Then the following holds.

Theorem 2.4. *Let U be the solution of problem* (1.5) *given by Theorem* 2.1. *Let* $g \in C^1([0, \varphi(\alpha)])$, $g' \geqq 0$, $g(0) = 0$. *Then* $G(U_r) \in C(\bar{Q})$, *and for any* $t_1, t_2 \in [0, T]$, $t_1 < t_2$, *there holds*

$$\int_\Omega G(U_r)(x, t_2)\zeta(x, t_2)\mathrm{d}x - \int_\Omega G(U_r)(x, t_1)\zeta(x, t_1)\mathrm{d}x$$
$$\leqq \int_{t_1}^{t_2}\int_\Omega \left[G(U_r)\zeta_t - g(V_r)V_{rx}\zeta_x - g'(V_r)(V_{rx})^2\zeta\right]\mathrm{d}x\mathrm{d}t \qquad (2.16)$$

for any $\zeta \in C^1([0, T]; H_0^1(\Omega))$, $\zeta \geqq 0$.

As already remarked, the existence of a family of viscous entropy inequalities is an important property of problem (1.5). As a particular consequence of it, we prove that the singular measure $U_s(\cdot, t)$ is *nondecreasing in time* (see [2, Theorem 3.3] for a related result).

Theorem 2.5. *Let assumptions* (H_1)–(H_3) *be satisfied. Let U be the solution of problem* (1.5) *given by Theorem* 2.1. *Then for any* $\eta \in H_0^1(\Omega)$, $\eta \geqq 0$ *there holds*

$$\langle U_{0s}, \eta\rangle_\Omega \leqq \langle U_s(\cdot, t), \eta\rangle_\Omega \qquad (2.17)$$

for almost every $t \in (0, T)$, *and also*

$$\langle U_s(\cdot, t_1), \eta\rangle_\Omega \leqq \langle U_s(\cdot, t_2), \eta\rangle_\Omega \qquad (2.18)$$

for almost every $t_1 \leqq t_2, t_1, t_2 \in (0, T)$.

Therefore, if the singular measure $U_s(\cdot, t)$ exists at some time $\bar{t} \geqq 0$, then it also exists at any later time. However, it is natural to ask if the singular measure U_s exists at all.

As shown in [2], $U_s(\cdot, t)$ can arise at some time $t = \bar{t} > 0$ even if the initial data U_0 are regular. On the other hand, we prove below that for a class of smooth initial data and for a suitable choice of ψ, the singular measure is always absent (see Theorem 2.8). Expectedly, this depends on the order of degeneracy of ψ (namely, on the rate of growth of ψ') at infinity.

To address this point, it is informative first to exhibit a class of initial data $U_0 \in \mathcal{M}^+(\bar{\Omega})$ which satisfy assumption (H_3). To this purpose, assume that the function ψ (beside (H_2)) satisfies the following:

$$(H_2') \quad \begin{cases} \text{there exist } \sigma > 0 \text{ and } l_1 \in (0, \gamma\sigma) \text{ such that} \\ l_1 \leqq \psi'(s)(1+s)^{(\sigma+1)} \leqq \gamma\sigma \quad \text{for any } s > 0. \end{cases}$$

Concerning U_0, suppose that either assumption:

$$(A_1) \quad \begin{cases} \text{(i)} \;\; U_0 \in L^1(\Omega), \\ \text{(ii)} \;\; \psi(U_0) \in H_0^1(\Omega), \end{cases}$$

$$(A_2) \quad \begin{cases} \text{(i)} \quad \psi(U_{0r}) \in W_0^{1,\infty}(\Omega), \\ \text{(ii)} \quad \operatorname{supp} U_{0s} \subseteq \mathcal{S}_0, \\ \text{(iii)} \quad \text{there exist } x_1, \cdot, \cdot, x_p \in \Omega \text{ and } \alpha_1, \cdot, \cdot, \alpha_p \geqq 0 \text{ such that} \\ \qquad\qquad U_{0s} = \sum_{i=1}^p \alpha_i \delta(\cdot - x_i) \end{cases}$$

holds true. Observe that by assumption (A_2), if U_{0s} contains a delta function concentrated at $\bar{x}$, then $\lim_{x\to\bar{x}} U_{0r}(x) = \infty$.

Proposition 2.6. *Let ψ satisfy (H_2)-(i) and (H_2'), and $U_0 \in \mathcal{M}^+(\bar{\Omega})$ satisfy either (A_1) or (A_2). Then assumption (H_3) is satisfied.*

By the same token we can prove the following result.

Proposition 2.7. *Let ψ satisfy assumptions (H_2)-(i) and:*

$$(H_2'') \quad \begin{cases} \text{there exist } \sigma \in (0, 1/2] \text{ and } l_1 \in (0, \gamma\sigma) \text{ such that} \\ l_1 \leqq \psi'(s)(1+s)^{(\sigma+1)} \text{ for any } s > 0. \end{cases}$$

Then for any $U_0 \in \mathcal{M}^+(\bar{\Omega})$ the following conditions are equivalent:

(i) $U_0 \in H_0^1(\Omega)$;
(ii) $\psi(U_{0r}) \in H_0^1(\Omega)$ *and* $\operatorname{supp} U_{0s} \subseteq \mathcal{S}_0$.

The above assumption (H_2'') ensures that the function ψ "grows slowly at infinity", thus the effect of the regularization in problem (1.5) is strong. This explains the following regularity result, which in particular rules out the possibility that the singular part $U_s(\cdot, t)$ arise at some time $t > 0$.

Theorem 2.8. *Let assumptions (H_1), (H_2) and (H_2'') be satisfied. Suppose that $U_0 \in H_0^1(\Omega)$. Then the corresponding solution U of problem (1.5) has the following properties:*

(i) $U \in L^\infty(Q)$;
(ii) *there holds*

$$\max_{(x,t)\in\bar{Q}} \psi(U)(x,t) =: \gamma^* < \gamma; \tag{2.19}$$

(iii) $U \in L^\infty((0,T); H_0^1(\Omega))$, *and* $U_t \in L^2((0,T); H_0^1(\Omega))$.

3. A Priori Estimates

To prove the existence part of Theorem 2.1, consider the family of approximating problems

$$(P_\kappa) \quad \begin{cases} U_t = \left[\varphi(U)\right]_{xx} + \epsilon\left[\psi_\kappa(U)\right]_{txx} & \text{in } Q \\ \varphi(U) + \epsilon\left[\psi_\kappa(U)\right]_t = 0 & \text{in } \partial\Omega \times (0,T] \\ U = U_{0\kappa} & \text{in } \Omega \times \{0\}. \end{cases}$$

Here $U_{0\kappa}$ is any family with the properties stated in assumption (H_3). Concerning the family ψ_κ the following is assumed:

$$(A_k) \quad \begin{cases} \text{(i)} \quad \psi_\kappa \in C^\infty(\mathbb{R}), \quad \psi_\kappa \to \psi \text{ in } C^3_{loc}(\mathbb{R}) \text{ as } \kappa \to 0; \\ \text{(ii)} \ \psi_\kappa \text{ odd}, \ \psi' + \kappa \leqq \psi_\kappa' \leqq \psi' + 2\kappa \text{ on } \mathbb{R}; \\ \text{(iii)} \quad \psi_\kappa^{(j)} \in L^\infty(\mathbb{R}) \text{ for any } j \in \mathbb{N}; \\ \text{(iv)} \quad |\varphi'| \leqq k_1\, \psi_\kappa', \ \left|\left(\frac{\varphi'}{\psi_\kappa'}\right)'\right| \leqq k_2^*\, \psi_\kappa' \text{ on } \mathbb{R} \text{ for some } k_2^* > 0. \end{cases}$$

Observe that (H_3) and (A_k)-(ii) ensure that

$$\psi_\kappa(U_{0\kappa}) \rightharpoonup \psi(U_{0r}) \quad \text{in } H_0^1(\Omega) \tag{3.1}$$

as $\kappa \to 0$. Moreover, it is easily seen that the family

$$\psi_\kappa(s) := \psi(s) + \kappa s \qquad (\kappa > 0)$$

has the above properties; the only nontrivial point is to check the second inequality in (A_k)-(iv). In fact, by assumption (H_2)-(v) and (vi) we have

$$\left| \frac{1}{\psi_\kappa'(s)} \left(\frac{\varphi'}{\psi_\kappa'} \right)'(s) \right| = \frac{\left|\varphi''(s)(\psi'(s)+\kappa) - \varphi'(s)\psi''(s)\right|}{(\psi'(s)+\kappa)^3} \leqq k_2 + \frac{|\varphi''(s)|}{(\psi'(s))^2}$$
$$\leqq k_2 + k_3.$$

Hence, the second inequality in (A_k)-(iv) follows defining $k_2^* := k_2 + k_3$.

Let us state the following

Definition 3.1. For any $U_{0\kappa}$ satisfying assumption (H_3), by a solution to problem $(P_\kappa)(\kappa > 0)$ we mean any $U_\kappa \in C^1([0,T]; C(\bar{\Omega}))$ such that the function

$$V_\kappa := \varphi(U_\kappa) + \epsilon\, [\psi_\kappa(U_\kappa)]_t \tag{3.2}$$

belongs to $C([0,T]; C^2(\bar{\Omega}) \cap H_0^1(\Omega))$, and there holds

$$\iint_Q \left\{ U_\kappa \zeta_t - \left[\varphi\left(U_\kappa\right) + \epsilon \left[\psi_\kappa\left(U_\kappa\right)\right]_t \right]_x \zeta_x \right\} \mathrm{d}x\mathrm{d}t = -\int_\Omega U_{0\kappa} \zeta(x,0)\,\mathrm{d}x \tag{3.3}$$

for any $\zeta \in C^1([0,T]; H_0^1(\Omega)),\ \zeta(\cdot,T) = 0$ in Ω.

Let us prove the following well-posedness result.

Proposition 3.1. *Let assumption (A_k) hold. Then for any $\kappa > 0$ there exists a unique solution U_κ to problem (P_κ) in the sense of Definition 3.1. Moreover, for any $l \in \mathbb{N}$ there exists $T_l \in (0,T]$ such that $U_\kappa \in C^1([0,T_l]; C^l(\bar{\Omega}))$. Finally,*

(i) *$U_\kappa \geqq 0$ in Q;*

(ii) *there holds*

$$U_\kappa = [\psi_\kappa(U_\kappa)]_t = 0 \quad \textit{on } \partial\Omega \times [0,T]. \tag{3.4}$$

In addition, the function V_κ defined in (3.2) has the following properties:

(i′) *for any $t \in [0,T]$ $V_\kappa(\cdot,t)$ solves the problem:*

$$\begin{cases} V_\kappa(\cdot,t) - \epsilon \left[\psi_\kappa'(U_\kappa(\cdot,t))\right] V_{\kappa xx}(\cdot,t) = \varphi(U_\kappa(\cdot,t)) & \textit{in } \Omega \\ V_\kappa(\cdot,t) = 0 & \textit{on } \partial\Omega; \end{cases} \tag{3.5}$$

(ii′) *there holds*

$$U_{\kappa t} = V_{\kappa xx} \quad \textit{in } Q; \tag{3.6}$$

(iii′) *for any $t \in [0,T]$*

$$0 \leqq V_\kappa(\cdot,t) \leqq \varphi(\alpha) \quad \textit{in } \Omega; \tag{3.7}$$

(iv′) *for any* $t \in [0, T]$

$$\frac{\partial V_\kappa}{\partial \nu}(\cdot, t) < 0 \quad on\ \partial\Omega, \tag{3.8}$$

where $\frac{\partial}{\partial \nu}$ *denotes the outer derivative at* $\partial\Omega$.

Proof. (α) Following [9], let us formulate problem (P_κ) as an abstract evolution problem. In this direction, let $U_\kappa \in C^1([0, T]; C(\bar{\Omega}))$ be any solution to problem (P_κ) (in the sense of Definition 3.1), and observe that by assumption (A_k)-(ii) $\psi_\kappa' \geqq \kappa > 0$ on $\mathbb{R}$, thus there exists $\psi_\kappa^{-1} \in C^\infty(\mathbb{R})$. Therefore, setting

$$w_\kappa := \psi_\kappa(U_\kappa), \quad h_\kappa(w_\kappa) := \varphi(\psi_\kappa^{-1}(w_\kappa)), \tag{3.9}$$

we have

$$h_\kappa(w_\kappa) + \epsilon w_{\kappa t} = V_\kappa \in C([0, T]; C^2(\bar{\Omega}) \cap H_0^1(\Omega)) \tag{3.10}$$

(see (3.2)), and equation (3.3) reads

$$(P_\kappa') \quad \begin{cases} \dfrac{w_{\kappa t}}{\psi_\kappa'\big(\psi_\kappa^{-1}(w_\kappa)\big)} = [h_\kappa(w_\kappa) + \epsilon w_{\kappa t}]_{xx} & \text{in } Q \\ h_\kappa(w_\kappa) + \epsilon w_{\kappa t} = 0 & \text{in } \partial\Omega \times (0, T] \\ w_\kappa = w_{0\kappa} := \psi_\kappa(U_{0\kappa}) & \text{in } \Omega \times \{0\} \end{cases}$$

(observe that $w_{0\kappa} \in C^l(\bar{\Omega})$ for any $l \in \mathbb{N}$). For any $t \in [0, T]$ the first equation can be rewritten as follows:

$$w_{\kappa t}(\cdot, t) = \psi_\kappa'\big(\psi_\kappa^{-1}(w_\kappa(\cdot, t))\big)\big[h_\kappa(w_\kappa(\cdot, t)) + \epsilon w_{\kappa t}(\cdot, t)\big]_{xx}$$

in Ω, namely

$$-\epsilon\psi_\kappa'\big(\psi_\kappa^{-1}(w_\kappa(\cdot, t))\big)V_{\kappa xx}(\cdot, t) + V_\kappa(\cdot, t) = h_\kappa(w_\kappa(\cdot, t)) \tag{3.11}$$

in Ω (see (3.10)).

On the other hand, since $\psi_\kappa' \geqq \kappa > 0$ on $\mathbb{R}$, by standard results on elliptic equations the problem

$$\begin{cases} -\epsilon\psi_\kappa'\big(\psi_\kappa^{-1}(f)\big)v_{xx} + v = h_\kappa(f) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3.12}$$

has a unique classical solution $v_f \in C^2(\bar{\Omega}) \cap H_0^1(\Omega)$ for any $f \in C(\bar{\Omega})$. Moreover,

- if $f \in C^l(\bar{\Omega})$ then $v_f \in C^{2+l}(\bar{\Omega})$ ($l \in \mathbb{N}$), and

$$\|v_f\|_{C(\bar{\Omega})} \leqq \|h_\kappa(f)\|_{C(\bar{\Omega})} \leqq \varphi(\alpha) \tag{3.13}$$

  (where we have made use of (3.9) and assumption (H_1));
- the operator $\mathcal{L} : C^l(\bar{\Omega}) \to C^l(\bar{\Omega})$,

$$\mathcal{L}(f) := \frac{1}{\epsilon}\big[v_f - h_\kappa(f)\big], \tag{3.14}$$

  is well defined.

Since $V_\kappa(\cdot,t) \in C^2(\bar{\Omega}) \cap H_0^1(\Omega)$, it follows by (3.11) that for any $t \in [0,T]$

$$V_\kappa(\cdot,t) = v_{w_\kappa(\cdot,t)} \ \text{ in } \Omega, \tag{3.15}$$

where $v_{w_\kappa(\cdot,t)}$ denotes the unique solution of problem (3.12) with $f = w_\kappa(\cdot,t)$. Then from (3.10) we obtain that $w_\kappa \in C^1([0,T]; C(\bar{\Omega}))$ solves the abstract Cauchy problem in the Banach space $C(\bar{\Omega})$:

$$\begin{cases} w_{\kappa t} = \mathcal{L}(w_\kappa) = \dfrac{1}{\epsilon}\left[v_{w_\kappa} - h_\kappa(w_\kappa)\right] & \text{in } (0,T) \\ w_\kappa(0) = w_{0\kappa}. \end{cases} \tag{3.16}$$

Clearly, if $U_\kappa \in C^1([0,T_l]; C^l(\bar{\Omega}))$ (for some $T_l \in (0,T]$, $l \in \mathbb{N}$), arguing as above we can prove that w_κ defined in (3.9) solves problem (3.16) in the Banach space $C^l(\bar{\Omega})$ (with $T = T_l$).

(β) The above considerations show that for any solution $U_\kappa \in C^1([0,T]; C^l(\bar{\Omega}))$ of (P_κ) the function $w_\kappa \in C^1([0,T]; C^l(\bar{\Omega}))$ defined in (3.9) solves problem (3.16) ($l \in \mathbb{N}$). Conversely, if $w_\kappa \in C^1([0,T]; C^l(\bar{\Omega}))$ is a solution of problem (3.16), the function $U_\kappa := \psi_\kappa^{-1}(w_\kappa) \in C^1([0,T]; C^l(\bar{\Omega}))$ gives a solution of problem (P_κ) (in the sense of Definition 3.1), which satisfies (3.5) and (3.6).

In fact, by (3.16) there holds

$$v_{w_\kappa} = h_\kappa(w_\kappa) + \epsilon\mathcal{L}(w_\kappa) = h_\kappa(w_\kappa) + \epsilon w_{\kappa t} = V_\kappa \quad \text{in } (0,T); \tag{3.17}$$

where for any $t \in [0,T]$ the function $v_{w_\kappa}(\cdot,t) \in C^{l+2}(\bar{\Omega}) \cap H_0^1(\Omega)$ is the unique solution of problem (3.12) with $f = w_\kappa(\cdot,t)$. Therefore, by (3.17) we have $V_\kappa(\cdot,t) = v_{w_\kappa}(\cdot,t)$ in Ω and for any $t \in [0,T]$, hence (3.5) follows. Moreover, since $U_\kappa \in C^1([0,T]; C(\bar{\Omega}))$, thus both $\psi'_\kappa(U_\kappa)$ and $\varphi(U_\kappa)$ belong to the same space, by (3.5) and standard results on elliptic equations there holds $V_\kappa \in C([0,T]; C^2(\bar{\Omega}) \cap H_0^1(\Omega))$ (this follows from (3.20) and (3.22) below with $w_1 = w_\kappa(\cdot,t_1)$ and $w_2 = w_\kappa(\cdot,t_2)$, for any $t_1,\ t_2 \in [0,T]$; see also (3.13)). Further, by (3.16) and (3.17) there holds

$$U_{\kappa t} = \frac{w_{\kappa t}}{\psi'_\kappa(\psi_\kappa^{-1}(w_\kappa))} = \frac{1}{\epsilon\psi'_\kappa(\psi_\kappa^{-1}(w_\kappa))}\left[v_{w_\kappa} - h_\kappa(w_\kappa)\right] = \left[v_{w_\kappa}\right]_{xx} = V_{\kappa xx}, \tag{3.18}$$

namely equation (3.6) (where we have made use of (3.12) with $f = w_\kappa(\cdot,t)$). Finally, by (3.6) and the equality

$$U_\kappa(x,0) = \psi_\kappa^{-1}(w_{0\kappa}(x)) = U_{0\kappa}(x) \quad \text{for any } x \in \Omega,$$

equation (3.3) follows.

(γ) By $(\alpha) - (\beta)$ above, existence and uniqueness of a solution $U_\kappa \in C^1([0,T]; C(\bar{\Omega})) \cap C^1([0,T_l]; C^l(\bar{\Omega}))$ of problem (P_κ) (for some $T_l \in (0,T]$, $l \in \mathbb{N}$) satisfying (3.5)–(3.6) will follow, if we prove that problem (3.16) is *globally* well posed in $C^1([0,T]; C(\bar{\Omega}))$, and *locally* well posed in $C^1([0,T]; C^l(\bar{\Omega}))$ for any $l \in \mathbb{N}$.

To this purpose, let us show that the operator $\mathcal{L} : C^l(\bar\Omega) \to C^l(\bar\Omega)$ is globally Lipschitz continuous on $C(\bar\Omega)$, and locally Lipschitz continuous on $C^l(\bar\Omega)$ for any $l \in \mathbb{N}$, $l \geqq 1$. In this connection, observe that by assumption (A_k)-(ii) and (iii) for any $j \in \mathbb{N}$ the derivative $h_\kappa^{(j)}$ is bounded on $\mathbb{R}$. Then there exists $l_1 > 0$ such that for any $w_1,\ w_2 \in C(\bar\Omega)$ we have:

$$\|h_\kappa(w_1) - h_\kappa(w_2)\|_{C(\bar\Omega)} \leqq l_1 \|w_1 - w_2\|_{C(\bar\Omega)}. \tag{3.19}$$

Moreover, from (3.12) (with $f = w_i$, $i = 1, 2$) we obtain plainly

$$\begin{aligned} &-\left(v_{w_1}\right)_{xx} + \left(v_{w_2}\right)_{xx} + \frac{v_{w_1} - v_{w_2}}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_1))} \\ &= v_{w_2}\left(\frac{1}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_2))} - \frac{1}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_1))}\right) \\ &\quad + \frac{h_\kappa(w_1)}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_1))} - \frac{h_\kappa(w_2)}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_2))} \quad \text{in } \Omega. \end{aligned} \tag{3.20}$$

Multiplying the above equality by $v_{w_1} - v_{w_2}$ and integrating on Ω gives

$$\begin{aligned} &\int_\Omega \left[v_{w_1} - v_{w_2}\right]_x^2 \mathrm{d}x + \int_\Omega \frac{1}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_1))}\left[v_{w_1} - v_{w_2}\right]^2 \mathrm{d}x \\ &\leqq \int_\Omega \left|v_{w_2}\right| \left|\frac{1}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_1))} - \frac{1}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_2))}\right| \left|v_{w_1} - v_{w_2}\right| \mathrm{d}x \\ &\quad + \int_\Omega \left|\frac{h_\kappa(w_1)}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_1))} - \frac{h_\kappa(w_2)}{\epsilon \psi_\kappa'(\psi_\kappa^{-1}(w_2))}\right| \left|v_{w_1} - v_{w_2}\right| \mathrm{d}x. \end{aligned} \tag{3.21}$$

Since both h_κ and $1/\psi_\kappa'(\psi_\kappa^{-1})$ are Lipschitz continuous in $\mathbb{R}$ (recall that $\psi_\kappa' \geqq \kappa$ in $\mathbb{R}$ by assumption (A_k)-(ii)), by (3.13) (with $f = w_2$) and the above inequality we obtain

$$\begin{aligned} \|v_{w_1} - v_{w_2}\|_{H_0^1(\Omega)} &\leqq C_1 \left\|\frac{h_\kappa(w_1)}{\psi_\kappa'(\psi_\kappa^{-1}(w_1))} - \frac{h_\kappa(w_2)}{\psi_\kappa'(\psi_\kappa^{-1}(w_2))}\right\|_{C(\bar\Omega)} \\ &+ C_1 \left\|\frac{1}{\psi_\kappa'(\psi_\kappa^{-1}(w_1))} - \frac{1}{\psi_\kappa'(\psi_\kappa^{-1}(w_2))}\right\|_{C(\bar\Omega)} \leqq C_1 C_2 \|w_1 - w_2\|_{C(\bar\Omega)} \end{aligned} \tag{3.22}$$

for some $C_1, C_2 > 0$. Then defining $C := l_1 + C_1 C_2 |\Omega|^{1/2}$ gives

$$\begin{aligned} \|\mathcal{L}(w_1) - \mathcal{L}(w_2)\|_{C(\bar\Omega)} &\leqq \frac{1}{\epsilon}\left[\|v_{w_1} - v_{w_2}\|_{C(\bar\Omega)} + \|h_\kappa(w_1) - h_\kappa(w_2)\|_{C(\bar\Omega)}\right] \\ &\leqq \frac{C}{\epsilon}\|w_1 - w_2\|_{C(\bar\Omega)} \end{aligned}$$

for any $w_1,\ w_2 \in C(\bar\Omega)$. This shows that (3.16), hence problem (P_κ), is well posed in $C^1([0, T]; C(\bar\Omega))$.

To prove that the operator $\mathcal{L}$ is locally Lipschitz continuous on $C^l(\bar{\Omega})$ for $l = 1$ (when $l > 1$ the claim follows by analogous arguments), let us multiply (3.20) by $\left(v_{w_2} - v_{w_1}\right)_{xx}$ $(w_1, \ w_2 \in C^1(\bar{\Omega}))$. This gives

$$\int_\Omega \left[v_{w_1} - v_{w_2}\right]_{xx}^2 \mathrm{d}x \leqq \int_\Omega \frac{1}{\epsilon\psi_\kappa'(\psi_\kappa^{-1}(w_1))} \left|v_{w_1} - v_{w_2}\right| \left|\left[v_{w_1} - v_{w_2}\right]_{xx}\right| \mathrm{d}x$$
$$+ \int_\Omega \left|v_{w_2}\right| \left|\frac{1}{\epsilon\psi_\kappa'(\psi_\kappa^{-1}(w_1))} - \frac{1}{\epsilon\psi_\kappa'(\psi_\kappa^{-1}(w_2))}\right| \left|\left[v_{w_1} - v_{w_2}\right]_{xx}\right| \mathrm{d}x$$
$$+ \int_\Omega \left|\frac{h_\kappa(w_1)}{\epsilon\psi_\kappa'(\psi_\kappa^{-1}(w_1))} - \frac{h_\kappa(w_2)}{\epsilon\psi_\kappa'(\psi_\kappa^{-1}(w_2))}\right| \left|\left[v_{w_1} - v_{w_2}\right]_{xx}\right| \mathrm{d}x.$$

By inequalities (3.13), (3.22), and Young's inequality, from the above inequality we obtain

$$\begin{aligned}
\|v_{w_1} - v_{w_2}\|_{C^1(\bar{\Omega})} &\leqq L_1 \|v_{w_1} - v_{w_2}\|_{H^2(\Omega)} \\
&\leqq L_2 \left\| \frac{h_\kappa(w_1)}{\psi_\kappa'(\psi_\kappa^{-1}(w_1))} - \frac{h_\kappa(w_2)}{\psi_\kappa'(\psi_\kappa^{-1}(w_2))} \right\|_{C(\bar{\Omega})} \\
&\quad + L_2 \left\| \frac{1}{\psi_\kappa'(\psi_\kappa^{-1}(w_1))} - \frac{1}{\psi_\kappa'(\psi_\kappa^{-1}(w_2))} \right\|_{C(\bar{\Omega})} \\
&\leqq L_3 \|w_1 - w_2\|_{C(\bar{\Omega})}
\end{aligned}$$

for some $L_3 > 0$. On the other hand, using assumption (A_k)-(iv) we have:

$$\begin{aligned}
&\left\|\left[h_\kappa(w_1)\right]_x - \left[h_\kappa(w_2)\right]_x\right\|_{C(\overline{\Omega})} \leqq \left\|h_\kappa'(w_1)w_{1x} - h_\kappa'(w_1)w_{2x}\right\|_{C(\overline{\Omega})} \\
&+ \left\|h_\kappa'(w_1)w_{2x} - h_\kappa'(w_2)w_{2x}\right\|_{C(\overline{\Omega})} \leqq k_1\|w_{1x} - w_{2x}\|_{C(\overline{\Omega})} \\
&+ k_2 \|w_{2x}\|_{C(\overline{\Omega})} \|w_1 - w_2\|_{C(\overline{\Omega})}.
\end{aligned}$$

By the above inequalities the operator $\mathcal{L}$ is locally Lipschitz continuous in $C^1(\bar{\Omega})$. The rest of the proof follows by standard calculation; we omit the details. Therefore, (3.16), hence problem (P_κ), has a unique local solution in $C^1([0, T_1]; C^1(\bar{\Omega}))$ for some $T_1 \in (0, T]$.

(δ) Since $U_{0\kappa} \geqq 0$ in Ω and $\varphi(s) \geqq 0$ for $s \geqq 0$, $\varphi(s) < 0$ for $s < 0$ (see assumptions (H_1) and (H_3)-(ii)), it is easily seen that the interval $[0, \infty)$ is positively invariant for problem (P_κ). The proof is almost the same as that of [10, Proposition 3], thus we omit it (see also [2,9], and [15]).

Equalities (3.5) and (3.6) have already been proven (see (3.17) and (3.18)). Further, let us address inequality (3.7). To this aim, since $\psi_\kappa'(U_\kappa) \geqq \kappa > 0$, observe that for every $t \in [0, T]$ we can apply the weak maximum principle to the operator

$$\mathcal{A}(f) := -\psi_\kappa'(U_\kappa(\cdot, t))\, f_{xx} + f \qquad (f \in C^2(\Omega)).$$

Thus, by the condition $0 \leqq \varphi(U_\kappa) \leqq \varphi(\alpha)$ (see assumption (H_1) and claim (i)), for every $t \in [0, T]$ the function $V_\kappa(\cdot, t) \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies

$$0 \leqq \mathcal{A}(V_\kappa(\cdot, t)) \leqq \mathcal{A}(\varphi(\alpha)), \quad V_\kappa(\cdot, t) = 0 \text{ on } \partial\Omega.$$

Hence, inequality (3.7) follows by the weak maximum principle. Since $V_\kappa(\cdot,t) \geqq 0$ in Ω and $V_\kappa(\cdot,t) = 0$ on $\partial\Omega$, we easily obtain that $\frac{\partial V_\kappa(\cdot,t)}{\partial\nu} \leqq 0$ on $\partial\Omega$ for every $t \in [0,T]$. Finally, the strong inequality $V_\kappa(\cdot,t) > 0$ in Ω and (3.8) are a consequence of the strong maximum principle for the operator $\mathcal{A}$ (for example, see [7]).

(ϵ) Let us finally prove the equalities (3.4). Set $\Omega \equiv (\omega_1,\omega_2)$. If $t_0 = 0$ the claim follows since $U_\kappa(\omega_i,0) = U_{0\kappa}(\omega_i) = 0$ and $V_\kappa(\omega_i,0) = 0 (i = 1,2)$. Hence, by contradiction, let there exists $t_0 \in (0,T]$ such that

$$U_\kappa(\omega_1,t_0) > 0 \tag{3.23}$$

(the proof for the case $U_\kappa(\omega_2,t_0) > 0$ is the same). Since by assumption (H_1)-(i) $\varphi > 0$ on $\mathbb{R}^+$, inequality (3.23) implies

$$\big[\varphi\big(U_\kappa\big)\big](\omega_1,t_0) > 0. \tag{3.24}$$

On the other hand, we have

$$\big[\varphi(U_\kappa)\big](\omega_1,t_0) + \epsilon\big[\psi_\kappa(U_\kappa)\big]_t(\omega_1,t_0) = V_\kappa(\omega_1,t_0) = 0 \tag{3.25}$$

(see (3.2)). Combining (3.23)–(3.25) gives:

$$\big[\psi_\kappa(U_\kappa)\big]_t(\omega_1,t_0) < 0.$$

Also, observe that since $U_{0\kappa} \in C_c^\infty(\Omega)$ (see (H_3)-(ii)) we have $\varphi(U_{0\kappa})(\omega_1) = 0$, hence

$$\epsilon\big[\psi_\kappa(U_\kappa)\big]_t(\omega_1,0) = V_\kappa(\omega_1,0) - \varphi(U_{0\kappa})(\omega_1) = 0$$

(recall that $V_\kappa(\cdot,0)$ solves problem (3.5) for $t = 0$, hence belongs to $H_0^1(\Omega)$). Therefore, by the continuity of $\big[\psi_\kappa\big(U_\kappa\big)\big]_t(\omega_1,\cdot)$ in $[0,T]$ there exists $t_1 \in [0,t_0)$ such that

$$\big[\psi_\kappa(U_\kappa)\big]_t(\omega_1,t) \begin{cases} < 0 & \text{if } t \in (t_1,t_0) \\ = 0 & \text{if } t = t_1. \end{cases} \tag{3.26}$$

Since $V_\kappa(\omega_1,t_1) = 0$, (3.26) implies $\big[\varphi\big(U_\kappa\big)\big](\omega_1,t_1) = 0$, namely:

$$U_\kappa(\omega_1,t_1) = \psi_\kappa(U_\kappa)(\omega_1,t_1) = 0. \tag{3.27}$$

Using (3.26) and (3.27) gives:

$$\big[\psi_\kappa\big(U_\kappa\big)\big](\omega_1,t_0) = \int_{t_1}^{t_0} \big[\psi_\kappa\big(U_\kappa\big)\big]_t(\omega_1,s)\,\mathrm{d}s < 0.$$

Since ψ_κ is odd, the above inequality implies $U_\kappa(\omega_1,t_0) < 0$, a contradiction. This completes the proof. □

It will be proven below (see Lemma 3.4) that the solution U_κ of problem (P_κ) given by the above Proposition 3.1 exists *globally* in the space $C^1([0,T];H_0^1(\Omega))$.

The next step is to obtain a priori estimates of the families $\{U_\kappa\}$ and $\{\psi_\kappa(U_\kappa)\}$, which are uniform with respect to $\kappa > 0$. This is the content of the following four lemmata (in this connection, see [2, Lemma 5.2]).

Lemma 3.2. *Let* (A_k) *be satisfied. Then there exists a constant* $C > 0$ (*independent of* ϵ) *such that for any* $\kappa > 0$

$$\|U_\kappa\|_{L^\infty((0,T);L^1(\Omega))} \leqq C. \tag{3.28}$$

Proof. Integrating equality (3.6) on Ω and using inequality (3.8) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U_\kappa(x,t)\mathrm{d}x = \int_\Omega V_{\kappa xx}(x,t)\mathrm{d}x \leqq 0 \qquad (t \in (0,T)),$$

whence

$$\int_\Omega U_\kappa(x,t)\mathrm{d}x \leqq \int_\Omega U_{0\kappa}(x)\mathrm{d}x. \tag{3.29}$$

Since $U_\kappa \geqq 0$ (see Proposition 3.1), the result follows. □

Lemma 3.3. *Let* (A_k) *be satisfied. Then there exists a constant* $C > 0$ (*independent of* ϵ) *such that for any* $\kappa > 0$

$$\|V_{\kappa x}\|_{L^2(Q)} \leqq C, \tag{3.30}$$

$$\iint_Q \frac{\left[\psi_\kappa(U_\kappa)\right]_t^2}{\psi_\kappa'(U_\kappa)}\,\mathrm{d}x\mathrm{d}t \leqq \frac{C}{\epsilon}, \tag{3.31}$$

where the function V_k *is defined by* (3.2).

Proof. By assumption (A_k)-(ii) the function

$$\frac{\left[\psi_\kappa(U_\kappa)\right]_t^2}{\psi_\kappa'(U_\kappa)} = \psi_\kappa'(U_\kappa)(U_{\kappa t})^2$$

is well defined in Q.

Multiplying by $\varphi(U_\kappa)$ equality (3.6), integrating on Ω for any fixed $t \in (0,T)$ and using (3.2) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{d}x\int_0^{U_\kappa}\varphi(s)\mathrm{d}s = \int_\Omega \varphi(U_\kappa)U_{\kappa t}\,\mathrm{d}x = -\int_\Omega \epsilon\psi_\kappa'(U_\kappa)(U_{\kappa t})^2\mathrm{d}x \\ -\int_\Omega (V_{\kappa x})^2\mathrm{d}x.$$

Integrating the above equality with respect to t and using the nonnegativity of U_κ, we have

$$\begin{aligned}&\iint_Q (V_{\kappa x})^2 + \epsilon\psi_\kappa'(U_\kappa)(U_{\kappa t})^2\mathrm{d}x\mathrm{d}t \\ &= \int_\Omega \mathrm{d}x\int_0^{U_{0\kappa}(x)}\varphi(s)\mathrm{d}s - \int_\Omega \mathrm{d}x\int_0^{U_\kappa(x,T)}\varphi(s)\mathrm{d}s \\ &\leqq \int_\Omega \mathrm{d}x\int_0^{U_{0\kappa}(x)}\varphi(s)\mathrm{d}s \leqq \varphi(\alpha)\|U_0\|_{\mathcal{M}(\bar\Omega)}.\end{aligned} \tag{3.32}$$

(see assumptions (H_1) and (H_3)). Then the result follows. □

Remark 3.1. Observe that by assumptions (H_2)-(ii) and (A_k)-(ii) inequality (3.31) implies

$$\| [\psi_\kappa(U_\kappa))]_t \|_{L^2(Q)} \leqq \frac{C}{\sqrt{\epsilon}}. \tag{3.33}$$

Lemma 3.4. *Let (A_k) be satisfied. Then $U_\kappa \in C^1([0,T]; H_0^1(\Omega))$ and there exists a constant $C > 0$(depending on $\epsilon > 0$, $\|U_0\|_{\mathcal{M}(\bar\Omega)}$, and $\|\psi(U_{0r})\|_{H_0^1(\Omega)}$) such that for any $\kappa > 0$*

$$\|\psi_\kappa(U_\kappa)\|_{L^\infty((0,T);H_0^1(\Omega))} \leqq C, \tag{3.34}$$

$$\| [\psi_\kappa(U_\kappa)]_{tx} \|_{L^2(Q)} \leqq C, \tag{3.35}$$

$$\|\varphi(U_\kappa)\|_{L^\infty((0,T);H_0^1(\Omega))} \leqq C. \tag{3.36}$$

Proof. Existence and uniqueness of a local solution $U_\kappa \in C^1([0,\tau]; C^1(\bar\Omega))$ to problem (P_κ) $(\kappa > 0)$ for some $\tau > 0$ follow by Proposition 3.1. By standard argument $\psi_\kappa(U_\kappa) \in C^1([0,T]; H_0^1(\Omega))$, thus $U_\kappa = \psi_\kappa^{-1}(\psi_\kappa(U_\kappa)) \in C^1([0,T]; H_0^1(\Omega))$, if we prove the uniform estimate (3.34).

Multiplying equality (3.6) by $\psi_\kappa(U_\kappa)$ and integrating over Ω for any fixed $t \in (0,\tau)$ gives

$$\begin{aligned}\int_\Omega [\psi_\kappa(U_\kappa)\, U_{\kappa t}]\,(x,t)\,\mathrm{d}x &= -\int_\Omega \left[\varphi(U_\kappa)\right]_x(x,t)\left[\psi_\kappa(U_\kappa)\right]_x(x,t)\,\mathrm{d}x \\ &\quad -\epsilon\int_\Omega \left[\psi_\kappa(U_\kappa)\right]_x(x,t)\left[\psi_\kappa(U_\kappa)\right]_{tx}(x,t)\,\mathrm{d}x \\ &\leqq k_1\int_\Omega \left[\psi_\kappa(U_\kappa)\right]_x^2(x,t)\mathrm{d}x \\ &\quad -\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\left[\psi_\kappa(U_\kappa)\right]_x^2}{2}(x,t)\,\mathrm{d}x;\end{aligned}$$

where we have made use of assumption (A_k)-(iv).

Integrating the above inequality with respect to time, we obtain

$$\begin{aligned}\frac{\epsilon}{2}\int_\Omega \left[\psi_\kappa(U_\kappa)\right]_x^2(x,t)\,\mathrm{d}x &\leqq \frac{\epsilon}{2}\int_\Omega \left[\psi_\kappa(U_{0\kappa})\right]_x^2(x)\,\mathrm{d}x + k_1\iint_{Q_t}\left[\psi_\kappa(U_\kappa)\right]_x^2\mathrm{d}x \\ &\quad -\iint_{Q_t}\psi_\kappa(U_\kappa)\,U_{\kappa t}\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.37}$$

(where $t \in (0,\tau)$, $Q_t := \Omega \times (0,t)$). Since $\kappa U_{0\kappa} \to 0$ in $C(\bar\Omega)$ by assumption (H_3)-(ii), there exists a constant $l_0 > 0$ such that $\kappa U_{0\kappa}(x) \leqq l_0$ for any $x \in \bar\Omega$; moreover, by assumption (A_k)-(ii) and since $\psi_\kappa(0) = \psi(0) = 0$ there holds:

$$\psi(s) + \kappa s \leqq \psi_\kappa(s) \leqq \psi(s) + 2\kappa s \qquad (s \in \mathbb{R}).$$

By the above considerations, (H_3)-(ii), and (3.1), there exists a constant $l_1 > 0$ such that for any $\kappa > 0$ and $t \in (0, \tau)$

$$\int_\Omega \left[\psi_\kappa(U_{0\kappa})\right]_x^2(x)\,\mathrm{d}x \leqq l_1,$$

$$-\iint_{Q_t} \psi_\kappa(U_\kappa)\,U_{\kappa t}\,\mathrm{d}x\mathrm{d}t = \int_\Omega\left(\int_0^{U_{0\kappa}(x)} \psi_\kappa(s)\,\mathrm{d}s\right)\mathrm{d}x$$

$$-\int_\Omega\left(\int_0^{U_\kappa(x,t)} \psi_\kappa(s)\,\mathrm{d}s\right) \leqq \int_\Omega\left(\int_0^{U_{0\kappa}(x)} \psi_\kappa(s)\,\mathrm{d}s\right)\mathrm{d}x$$

$$\leqq \int_\Omega\left(\int_0^{U_{0\kappa}(x)} \psi(s) + 2\kappa s\,\mathrm{d}s\right)\mathrm{d}x \leqq \gamma\|U_{0\kappa}\|_{L^1(\Omega)} + l_0\|U_{0\kappa}\|_{L^1(\Omega)} \leqq l_1$$

(where we have also used the conditions $\psi(s) > 0$ for $s > 0$ and $U_\kappa \geqq 0$ in Q). Combining the above estimates and inequality (3.37) gives

$$\frac{\epsilon}{2}\int_\Omega \left[\psi_\kappa(U_\kappa)\right]_x^2(x,t)\,\mathrm{d}x \leqq k_1 \iint_{Q_t} \left[\psi_\kappa(U_\kappa)\right]_x^2\,\mathrm{d}x + l_1\left(1+\frac{\epsilon}{2}\right)$$

for any $t \in (0, \tau)$, whence by Gronwall's inequality

$$\int_\Omega \left[\psi_\kappa(U_\kappa)\right]_x^2(x,t)\,\mathrm{d}x \leqq \frac{2\,l_1}{\epsilon}\left(1+\frac{\epsilon}{2}\right)\mathrm{e}^{2k_1T/\epsilon} \quad \text{for any } t \in (0,\tau).$$

Therefore $\tau = T$, and inequality (3.34) follows.

Next, by assumption (A_k)-(iv) and inequality (3.34) there holds

$$\begin{aligned}\int_\Omega \left[\varphi(U_\kappa)\right]_x^2(x,t)\,\mathrm{d}x &= \int_\Omega \left[\varphi'(U_\kappa)\,U_{\kappa x}\right]^2(x,t)\,\mathrm{d}x \\ &\leqq k_1^2\int_\Omega \left[\psi_\kappa'(U_\kappa)\,U_{\kappa x}\right]^2(x,t)\,\mathrm{d}x \\ &= k_1^2\int_\Omega \left[\psi_\kappa(U_\kappa)\right]_x^2(x,t) \leqq k_1^2\,C \quad (t \in (0,T))\end{aligned} \tag{3.38}$$

for some constant $C > 0$. Since $\varphi(U_\kappa) = 0$ on $\partial\Omega \times [0, T]$, inequality (3.38) implies that the family $\{\varphi(U_\kappa)\}$ is uniformly bounded in $L^\infty((0,T);H_0^1(\Omega))$, thus inequality (3.36) follows.

It remains to prove (3.35). Multiplying by $\left[\psi_\kappa(U_\kappa)\right]_t$ equality (3.6) and integrating over the rectangle Q gives

$$\iint_Q \left[\psi_\kappa(U_\kappa)\right]_t U_{\kappa\,t}\,\mathrm{d}x\mathrm{d}t = -\iint_Q \left[\varphi(U_\kappa)\right]_x \left[\psi_\kappa(U_\kappa)\right]_{tx}\mathrm{d}x\mathrm{d}t$$

$$-\epsilon\iint_Q \left[\psi_\kappa(U_\kappa)\right]_{tx}^2(x,t)\,\mathrm{d}x\mathrm{d}t,$$

since $\left[\psi_\kappa(U_\kappa)\right]_t = 0$ on $\partial\Omega \times [0, T]$ (see (3.4)). Then

$$\begin{aligned}
&\epsilon \int\int_Q \left[\psi_\kappa(U_\kappa)\right]_{tx}^2(x,t)\,\mathrm{d}x\mathrm{d}t \\
&\leqq -\int\int_Q \frac{\left[\psi_\kappa(U_\kappa)\right]_t^2}{\psi_\kappa'(U_\kappa)}\mathrm{d}x\mathrm{d}t + \left(\int\int_Q \left[\varphi(U_\kappa)\right]_x^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad \times \left(\int\int_Q \left[\psi_\kappa(U_\kappa)\right]_{tx}^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}.
\end{aligned}$$

The estimate (3.35) follows from the above inequality and (3.36). This completes the proof. □

Denote by $C^{1/2}(Q)$ the space of Hölder continuous functions with exponent $1/2$ in Q endowed with the usual norm. Then the following holds.

Lemma 3.5. *Let* (A_k) *be satisfied. Then there exists a constant* $C > 0$ (*depending on* $\epsilon > 0$) *such that for any* $\kappa > 0$

$$\|\psi_\kappa(U_\kappa)\|_{H^1(Q)} \leqq C, \tag{3.39}$$

$$\|\psi_\kappa(U_\kappa)\|_{C^{1/2}(Q)} \leqq C. \tag{3.40}$$

Proof. Inequality (3.39) follows from (3.33) and (3.34). To prove (3.40), observe that by (3.35) there exists $C > 0$ such that for any $\kappa > 0$

$$\left\|\left[\psi_\kappa\left(U_\kappa\right)\right]_t\right\|_{L^2((0,T);L^\infty(\Omega))} \leqq C. \tag{3.41}$$

Then for any $(x_1, t_1),\ (x_2, t_2) \in \bar{Q}$ we obtain

$$\begin{aligned}
&|\psi_\kappa(U_\kappa)(x_2,t_2) - \psi_\kappa(U_\kappa)(x_1,t_1)| \\
&\quad \leqq |\psi_\kappa(U_\kappa)(x_2,t_2) - \psi_\kappa(U_\kappa)(x_1,t_2)| + |\psi_\kappa(U_\kappa)(x_1,t_2) - \psi_\kappa(U_\kappa)(x_1,t_1)| \\
&\quad \leqq \left\|\left[\psi_\kappa(U_\kappa)\right]_x\right\|_{L^\infty((0,T);L^2(\Omega))} |x_2 - x_1|^{1/2} + \left|\int_{t_1}^{t_2} \left|\left[\psi_\kappa(U_\kappa)\right]_t(x_1,s)\right| \mathrm{d}s\right| \\
&\quad \leqq \left\|\left[\psi_\kappa(U_\kappa)\right]_x\right\|_{L^\infty((0,T);L^2(\Omega))} |x_2 - x_1|^{1/2} \\
&\qquad + \left\|\left[\psi_\kappa(U_\kappa))\right]_t\right\|_{L^2((0,T);L^\infty(\Omega))} |t_2 - t_1|^{1/2}.
\end{aligned}$$

By (3.34), (3.41) and the the above inequality, we obtain (3.40). Then the result follows. □

Let us now draw some conclusions from the above estimates.

Proposition 3.6. *Let assumption* (A_k) *be satisfied. Then there exists a sequence* $\kappa_j \to 0$ *with the following properties:*

(i) *there exists* $U \in \mathcal{M}^+(\bar{Q})$ *such that*

$$\iint_Q U_{\kappa_j}\,\zeta\,\mathrm{d}x\mathrm{d}t \;\to\; \langle U, \zeta_{\bar{Q}}\rangle_{\bar{Q}} \tag{3.42}$$

for any $\zeta \in C_c(\mathbb{R}^2)$*;*

(ii) *there exists* $w \in L^\infty((0,T); H_0^1(\Omega)) \cap H^1(Q) \cap C(\bar{Q})$, $0 \leqq w \leqq \gamma$ *in* $\overline{Q}$, *with* $w_t \in L^2((0,T); H_0^1(\Omega))$, *such that*

$$\psi_{\kappa_j}\left(U_{\kappa_j}\right) \rightharpoonup w \quad in \;\; H^1(Q), \tag{3.43}$$

$$\psi_{\kappa_j}\left(U_{\kappa_j}\right) \to w \quad in \;\; C(\bar{Q}), \tag{3.44}$$

$$\left[\psi_{\kappa_j}\left(U_{\kappa_j}\right)\right]_t \rightharpoonup w_t \quad in \;\; L^2((0,T); H_0^1(\Omega)); \tag{3.45}$$

(iii) *there holds*

$$\psi(U_{\kappa_j}) \to w \quad in \;\; L^\infty((0,T); L^1(\Omega)), \tag{3.46}$$

$$\psi(U_{\kappa_j}) \to w \quad almost\ everywhere\ in \;\; Q, \tag{3.47}$$

$$U_{\kappa_j} \to \psi^{-1}(w) \quad almost\ everywhere\ in \;\; Q. \tag{3.48}$$

Proof. (i) For simplicity, set

$$\tilde{U}_\kappa(x,t) := \begin{cases} U_\kappa(x,t) & \text{if } (x,t) \in \bar{Q} \\ 0 & \text{if } (x,t) \in \mathbb{R}^2 \setminus \bar{Q}. \end{cases}$$

It follows that $\|\tilde{U}_\kappa\|_{L^1(\mathbb{R}^2)} = \|U_\kappa\|_{L^1(Q)} \leqq C$ (see (3.28)), hence there exist a subsequence $\left\{\tilde{U}_{\kappa_j}\right\} \subseteq \{\tilde{U}_\kappa\}$, and a Radon measure $U \in \mathcal{M}^+(\mathbb{R}^2)$ such that

$$\tilde{U}_{\kappa_j} \overset{*}{\rightharpoonup} U \quad \text{in } \mathcal{M}(\mathbb{R}^2).$$

Clearly, $\operatorname{supp} U \subseteq \bar{Q}$, and the above convergence reads:

$$\lim_{j\to\infty} \iint_Q U_{\kappa_j}\, \zeta \, \mathrm{d}x\mathrm{d}t = \lim_{j\to\infty} \iint_{\mathbb{R}^2} \tilde{U}_{\kappa_j}\, \zeta \, \mathrm{d}x\mathrm{d}t$$
$$= \langle U, \zeta \rangle_{\mathbb{R}^2} = \langle U, \zeta_{\bar{Q}} \rangle_{\bar{Q}}$$

for any $\zeta \in C_c(\mathbb{R}^2)$. Then (3.42) holds.

(ii) The existence of $w \in L^\infty((0,T); H_0^1(\Omega)) \cap H^1(Q) \cap C(\bar{Q})$, with the asserted properties (see (3.43)–(3.45)) is a consequence of estimates (3.34), (3.35), (3.39), and (3.40) (see also (3.4)).

(iii) Assumption (A_k)-(ii) and the equality $\psi_\kappa(0) = \psi(0) = 0$ (recall that both ψ_κ and ψ are odd) imply

$$\psi(U_{\kappa_j}) + \kappa_j U_{\kappa_j} \leqq \psi_{\kappa_j}(U_{\kappa_j}) \leqq \psi(U_{\kappa_j}) + 2\kappa_j U_{\kappa_j}. \tag{3.49}$$

Since $U_{\kappa_j} \geqq 0$, there holds

$$\|\psi(U_{\kappa_j}) - \psi_{\kappa_j}(U_{\kappa_j})\|_{L^\infty((0,T);L^1(\Omega))}$$
$$= \sup_{t\in(0,T)} \int_\Omega \left[\psi_{\kappa_j}(U_{\kappa_j}) - \psi(U_{\kappa_j})\right](x,t)\mathrm{d}x \leqq 2\kappa_j \|U_{\kappa_j}\|_{L^\infty((0,T);L^1(\Omega))}.$$

Using (3.28), (3.44), from the above inequality we obtain (3.46).

From (3.46) (possibly extracting a subsequence, still denoted by $\{U_{\kappa_j}\}$) we also obtain (3.47), whence (3.48) follows (recall that we have set $\psi^{-1}(\gamma) = \infty$).

Finally, observe that the left inequality in (3.49) and (3.44) imply $w \geqq 0$ in $\bar{Q}$ (since $U_{\kappa_j} \geqq 0$), whereas (3.47) and the fact that $w \in C(\bar{Q})$ give $w \leqq \gamma$ in $\bar{Q}$. This proves the result. □

Remark 3.2. It will be shown below that the solution to problem (1.5) is unique (see Proposition 4.5). Therefore the convergences in (3.43)–(3.48) hold *as* $\kappa \to 0$, since the limiting values are the same along any converging sequence.

4. Proof of Well-Posedness

4.1. Proof of Theorems 2.1–2.2

The following result plays an important role in the sequel.

Proposition 4.1. *Let assumptions* (H_1)–(H_3) *be satisfied. Let* $U \in \mathcal{M}^+(\bar{Q})$ *and* $w \in L^\infty((0,T); H_0^1(\Omega)) \cap H^1(Q) \cap C(\bar{Q})$ *as in Proposition 3.6. Then the set*

$$\tilde{\mathcal{S}} := \left\{(x,t) \in \bar{Q} \mid w(x,t) = \gamma\right\} \tag{4.1}$$

has zero Lebesgue measure. Moreover,

$$\operatorname{supp} U_s \subseteq \tilde{\mathcal{S}}, \tag{4.2}$$

$$U_r(x,t) = \psi^{-1}(w(x,t)) \quad \textit{for almost every } (x,t) \in \bar{Q} \setminus \tilde{\mathcal{S}}. \tag{4.3}$$

Proof. (i) Set:

$$B_n := \left\{(x,t) \in \overline{Q} \mid w(x,t) \geqq \gamma - \frac{1}{n}\right\} \qquad (n \in \mathbb{N}). \tag{4.4}$$

Then

$$B_{n+1} \subseteq B_n, \quad \tilde{\mathcal{S}} = \bigcap_{n=1}^{\infty} B_n, \quad |\tilde{\mathcal{S}}| = \lim_{n\to\infty} |B_n|, \tag{4.5}$$

where $|\cdot|$ denotes the Lebesgue measure. Let us prove that:

$$\lim_{n\to\infty} |B_n| = 0. \tag{4.6}$$

Since by Proposition 3.6 there exists a sequence $\left\{\psi_{\kappa_j}\left(U_{\kappa_j}\right)\right\}$ such that $\psi_{\kappa_j}\left(U_{\kappa_j}\right) \to w$ uniformly in $\overline{Q}$, thus in B_n (see (3.44)), there holds

$$\sup_{(x,t)\in B_n} \left|\psi_{\kappa_j}\left(U_{\kappa_j}\right)(x,t) - w(x,t)\right| < \frac{1}{n} \tag{4.7}$$

for any $\kappa_j > 0$ sufficiently small. On the other hand, by (3.28) there exists a subsequence, denoted again $\{\kappa_j\}$, such that $\kappa_j U_{\kappa_j} \to 0$ almost everywhere in Q; thus, by the Severini–Egorov Lemma for any $\sigma > 0$ there exists $Q_\sigma \subseteq Q$ such that $|Q \setminus Q_\sigma| \leqq \sigma$ and $\kappa_j U_{\kappa_j} \to 0$ uniformly in Q_σ. From inequalities (4.7) and (3.49), for any $\sigma > 0$ we obtain:

$$U_{\kappa_j} > \psi^{-1}\left(\gamma - \frac{2}{n} - 2\kappa_j U_{\kappa_j}\right) \quad \text{in } B_n \cap Q_\sigma \tag{4.8}$$

for any j sufficiently large (such that $\gamma - 2/n - 2\kappa_j U_{\kappa_j} \geqq -\gamma$ in Q_σ). Moreover, the above considerations ensure that

$$\psi^{-1}\left(\gamma - \frac{2}{n} - 2\kappa_j U_{\kappa_j}\right) \to \psi^{-1}\left(\gamma - \frac{2}{n}\right)$$

almost everywhere in $B_n \cap Q_\sigma$. Then by the Lebesgue Theorem we have:

$$\iint_{B_n \cap Q_\sigma} \psi^{-1}\left(\gamma - \frac{2}{n} - 2\kappa_j U_{\kappa_j}\right) \mathrm{d}x\mathrm{d}t \to \psi^{-1}\left(\gamma - \frac{2}{n}\right) |B_n \cap Q_\sigma| \tag{4.9}$$

for any $n \in \mathbb{N}$. By (4.8)–(4.9), we obtain

$$\begin{aligned} \psi^{-1}\left(\gamma - \frac{2}{n}\right)|B_n| &= \psi^{-1}\left(\gamma - \frac{2}{n}\right)\left(|B_n \setminus Q_\sigma| + |B_n \cap Q_\sigma|\right) \\ &= \psi^{-1}\left(\gamma - \frac{2}{n}\right)|B_n \setminus Q_\sigma| + \lim_{\kappa_j \to 0} \iint_{B_n \cap Q_\sigma} \psi^{-1}\left(\gamma - \frac{2}{n} - 2\kappa_j U_{\kappa_j}\right) \mathrm{d}x\mathrm{d}t \\ &\leqq \psi^{-1}\left(\gamma - \frac{2}{n}\right)\sigma + \liminf_{\kappa_j \to 0} \iint_{B_n} U_{\kappa_j} \mathrm{d}x\mathrm{d}t \leqq \psi^{-1}\left(\gamma - \frac{2}{n}\right)\sigma + C \end{aligned}$$

for some constant $C > 0$. Thus, by the arbitrariness of σ we have

$$|B_n| < \frac{C}{\psi^{-1}\left(\gamma - \frac{2}{n}\right)} \tag{4.10}$$

for any $n \in \mathbb{N}$. Letting $n \to \infty$ in the previous inequality gives (4.6), whence $|\tilde{S}| = 0$ by (4.5).

(ii) Let $\mathcal{R} \subseteq Q$ be the open set defined as follows:

$$\mathcal{R} := \left\{(x, t) \in Q \mid w(x, t) < \gamma\right\}. \tag{4.11}$$

We now claim that

$$\langle U_s, \zeta \rangle_Q = \iint_Q \left\{\psi^{-1}(w) - U_r\right\} \zeta \,\mathrm{d}x\mathrm{d}t \quad \text{for any } \zeta \in C_c(\mathcal{R}). \tag{4.12}$$

Fix any $\zeta \in C_c(\mathcal{R})$, denote by $K \subset \mathcal{R}$ the support of ζ and set:

$$M_K := \max_{(x,t) \in K} w(x, t) < \gamma, \quad \delta_K := \gamma - M_K.$$

Since $\psi_{\kappa_j}\left(U_{\kappa_j}\right) \to w$ uniformly in $C(\bar{Q})$ (see (3.44)), there holds:

$$\max_K \psi_{\kappa_j}\left(U_{\kappa_j}\right) \leqq M_K + \frac{\delta_K}{2} = \gamma - \frac{\delta_K}{2}$$

for any κ_j sufficiently small. By the left inequality in (3.49), this plainly implies

$$U_{\kappa_j} \leqq \psi^{-1}\left(\gamma - \frac{\delta_K}{2}\right) \quad \text{in } K,$$

if κ_j is sufficiently small. From the latter inequality and the limit (3.48), by the Lebesgue Theorem we obtain:

$$\iint_Q U_{\kappa_j} \zeta \, \mathrm{d}x\mathrm{d}t \to \iint_Q \psi^{-1}(w) \zeta \, \mathrm{d}x\mathrm{d}t \quad \text{for any } \zeta \in C_c(\mathcal{R}) \tag{4.13}$$

(recall that by definition $w(x,t) < \gamma$ for any $(x,t) \in \mathcal{R}$, and $w(x,t) \leqq M_K = \gamma - \delta_K$ for any $(x,t) \in K$). On the other hand, by (3.28) and (3.42), there holds

$$\iint_Q U_{\kappa_j} \zeta \, \mathrm{d}x\mathrm{d}t \to \langle U, \zeta \rangle_Q = \iint_Q U_r \zeta \, \mathrm{d}x\mathrm{d}t + \langle U_s, \zeta \rangle_Q \tag{4.14}$$

for any $\zeta \in C_c(Q)$. From (4.13)–(4.14) we obtain (4.12).

(iii) Let us now prove (4.2). By (4.12) we have $U_s(K) = 0$ for any compact subset $K \subseteq \mathcal{R}$, hence $U_s(\mathcal{R}) = 0$. This implies that $\operatorname{supp} U_s \subseteq \bar{Q} \setminus \mathcal{R}$ (recall that $\operatorname{supp} U_s \subseteq \bar{Q}$). Since

$$\bar{Q} \setminus \mathcal{R} = \tilde{\mathcal{S}} \cup \left\{ \partial Q \setminus \left\{ \tilde{\mathcal{S}} \cap \partial Q \right\} \right\},$$

the claim will follow if we show that any point $(x_0, t_0) \in \partial Q \setminus \left\{ \tilde{\mathcal{S}} \cap \partial Q \right\}$ does not belong to $\operatorname{supp} U_s$.

In fact, by the very definition of $\tilde{\mathcal{S}}$ (see (4.1)) for any such (x_0, t_0) there holds $w(x_0, t_0) < \gamma$. Since $w \in C(\bar{Q})$ (see Proposition 3.6-(ii)), for any $\delta > 0$ sufficiently small there exists an open neighbourhood $U_{0,\delta} \subseteq \mathbb{R}^2$ such that $(x_0, t_0) \in U_{0,\delta}$, $\overline{U}_{0,\delta} \cap \tilde{\mathcal{S}} = \emptyset$, and

$$w(x,t) \leqq w(x_0, t_0) + \delta \leqq \gamma - \delta$$

for any $(x,t) \in U_{0,\delta} \cap \bar{Q}$. Arguing as in (ii) above, by the uniform convergence (3.44) and the left inequality in (3.49), we obtain

$$U_{\kappa_j} \leqq \psi^{-1}\left(\gamma - \frac{\delta}{2} \right) \quad \text{in } U_{0,\delta} \cap \bar{Q}$$

for any κ_j small enough. As above, by such an estimate we obtain the convergence in (4.13) for any $\zeta \in C_c(U_{0,\delta})$, hence in view of (3.42) we have

$$\langle U_s, \zeta_{\bar{Q}} \rangle_{\bar{Q}} = \iint_Q \left\{ \psi^{-1}(w) - U_r \right\} \zeta \, \mathrm{d}x\mathrm{d}t \tag{4.15}$$

for any $\zeta \in C_c(U_{0,\delta})$, whence $U_s(K \cap \bar{Q}) = 0$ for any compact subset $K \subseteq U_{0,\delta}$. Thus the claim follows.

(iv) Finally we prove (4.3). Observe that

$$\bar{Q} \setminus \tilde{\mathcal{S}} = \mathcal{R} \cup \left\{ \partial Q \setminus \left\{ \tilde{\mathcal{S}} \cap \partial Q \right\} \right\}.$$

Since $\operatorname{supp} U_s \subseteq \tilde{\mathcal{S}}$, by (4.12)–(4.15) and the above equality we have

$$\iint_Q \left\{ \psi^{-1}(w) - U_r \right\} \zeta \, \mathrm{d}x\mathrm{d}t = 0$$

for any $\zeta \in C_c(\mathbb{R}^2)$ such that $\operatorname{supp} \zeta \cap \tilde{\mathcal{S}} = \emptyset$. Then the conclusion follows. $\square$

A useful disintegration of the singular term $U_s \in \mathcal{M}^+(\bar{Q})$ with respect to the Lebesgue measure on $(0, T)$ is the content of the following

Proposition 4.2. *Let $U_s \in \mathcal{M}^+(\bar{Q})$ be the singular term of the measure $U \in \mathcal{M}^+(\bar{Q})$ given by Proposition 3.6. Then:*

(i) *for almost every $t \in \mathbb{R}$ there exists a measure $U_s(\cdot, t) \in \mathcal{M}^+(\bar{\Omega})$(hence in particular* $\operatorname{supp} U_s(\cdot, t) \subseteq \bar{\Omega}$), $U(\cdot, t) = 0$ *if $t \notin [0, T]$, such that*

$$U_s(E) = \int_0^T U_s(\cdot, t)(E_t)\, \mathrm{d}t, \tag{4.16}$$

where $E \subseteq \mathbb{R}^2$ is any Borel set and $E_t := \{x \in \mathbb{R} \mid (x, t) \in E\}$ its section at the time t. Moreover, for any $\zeta \in C_c(\mathbb{R}^2)$ the map $t \to \langle U_s(\cdot, t), \zeta(\cdot, t)_{\bar{\Omega}}\rangle_{\bar{\Omega}}$ is Lebesgue measurable and there holds

$$\langle U_s, \zeta_{\bar{Q}}\rangle_{\bar{Q}} = \int_0^T \langle U_s(\cdot, t), \zeta(\cdot, t)_{\bar{\Omega}}\rangle_{\bar{\Omega}} \mathrm{d}t; \tag{4.17}$$

(ii) *for almost every $t \in (0, T)$ inequality (2.6) holds.*

Proof. (i) Since $U_s \in \mathcal{M}^+(\bar{Q})$—namely, $U_s \in \mathcal{M}(\mathbb{R}^2)$ and $\operatorname{supp} U_s \subseteq \bar{Q}$—there exists a measure $\lambda \in \mathcal{M}^+(\mathbb{R})$, $\operatorname{supp} \lambda \subseteq [0, T]$, and for λ-a.e. $t \in \mathbb{R}$ a measure $\gamma_t \in \mathcal{M}^+(\mathbb{R})$, $\operatorname{supp} \gamma_t \subseteq \bar{\Omega}$, with the following properties (for example, see [6, Vol. I, Proposition 8 on p. 35]):

(a) for any Borel set $E \subseteq \mathbb{R}^2$ we have

$$U_s(E) = \int_{\mathbb{R}} \gamma_t(E_t)\, \mathrm{d}\lambda(t) = \int_{[0,T]} \gamma_t(E_t)\, \mathrm{d}\lambda(t); \tag{4.18}$$

(b) for any $\zeta \in C_c(\mathbb{R}^2)$ there holds

$$\langle U_s, \zeta\rangle_{\mathbb{R}^2} = \int_{\mathbb{R}} \mathrm{d}\lambda(t) \int_{\mathbb{R}} \zeta(x, t)\, \mathrm{d}\gamma_t = \int_{[0,T]} \mathrm{d}\lambda(t) \int_{\bar{\Omega}} \zeta(x, t)\, \mathrm{d}\gamma_t. \tag{4.19}$$

Moreover, since $U_s(\bar{Q}) < \infty$ (because U_s is a Radon measure on $\mathbb{R}^2$), and $\operatorname{supp} U_s \subseteq \bar{Q}$, we have $U_s(\mathbb{R}^2) < \infty$. Therefore we can choose λ and γ_t such that $\lambda(I) = U_s(\mathbb{R} \times I)$ for any Borel set $I \subseteq \mathbb{R}$, and $\gamma_t(\mathbb{R}) = \gamma_t(\bar{\Omega}) = 1$ for λ-a.e. $t \in \mathbb{R}$.

Let us prove that the measure λ is absolutely continuous with respect to the Lebesgue measure. To this purpose, fix any $0 < t_0 < T$ (the cases $t_0 = 0$ and $t_0 = T$ can be treated in an analogous way), then choose $r > 0$ and $\sigma > 0$ such that $I_{r+\sigma} := [t_0 - r - \sigma, t_0 + r + \sigma] \subseteq [0, T]$. Moreover, fix any $\eta_{r,\sigma} \in C_c^1(\mathbb{R})$ such that $\eta_{r+\sigma} \equiv 1$ in $[t_0 - r, t_0 + r]$, $0 \leqq \eta_{r+\sigma} \leqq 1$ and $\operatorname{supp} \eta_{r+\sigma} \subseteq I_{r+\sigma}$. Finally set:

$$\tilde{\eta}_{r+\sigma}(t) := -\int_t^{t_0+r+\sigma} \eta_{r+\sigma}(s)\, \mathrm{d}s \qquad (t \in \mathbb{R}).$$

As before, let U_{κ_j} be the solution of problem (P_{κ_j}) and V_{κ_j} the function defined by (3.2) with $\kappa = \kappa_j > 0$. By equality (3.6), inequality (3.8) and assumption (H_3) there holds

$$\iint_Q U_{\kappa_j}(x,t)\eta_{r+\sigma}(t)\,\mathrm{d}x\mathrm{d}t = -\int_0^T \tilde{\eta}_{r+\sigma}(t)\,\mathrm{d}t \int_\Omega V_{\kappa_j xx}\,\mathrm{d}x$$
$$-\tilde{\eta}_{r+\sigma}(0)\int_\Omega U_{0\kappa_j}\mathrm{d}x \leqq (2r+2\sigma)\int_\Omega U_{0\kappa_j}\mathrm{d}x \leqq (2r+2\sigma)\|U_0\|_{\mathcal{M}(\bar{\Omega})} \tag{4.20}$$

(observe that $\tilde{\eta}_{r+\sigma}(t) = 0$ for any $t \geqq t_0 + r + \sigma$). In view of (3.42), passing to the limit in (4.20) as $\kappa_j \to 0$, and using (4.18) we obtain:

$$\begin{aligned}
&\int_{t_0-r}^{t_0+r}\int_\Omega U_r(x,t)\,\mathrm{d}x\mathrm{d}t + \int_{[t_0-r,t_0+r]}\mathrm{d}\lambda(t)\\
&= \int_{t_0-r}^{t_0+r}\int_\Omega U_r(x,t)\,\mathrm{d}x\mathrm{d}t + \int_{[t_0-r,t_0+r]}\gamma_t(\bar{\Omega})\,\mathrm{d}\lambda(t)\\
&= \int_{t_0-r}^{t_0+r}\int_\Omega U_r(x,t)\,\mathrm{d}x\mathrm{d}t + U_s\big(\bar{\Omega}\times[t_0-r,t_0+r]\big)\\
&\leqq \iint_Q U_r(x,t)\eta_{r+\sigma}(t)\,\mathrm{d}x\mathrm{d}t + \langle U_s,\eta_{r+\sigma}\rangle_{\mathbb{R}^2} \leqq (2r+2\sigma)\|U_0\|_{\mathcal{M}(\bar{\Omega})}
\end{aligned}$$

(recall that $\gamma_t(\bar{\Omega}) = 1$ for λ-a.e. $t \in \mathbb{R}$). By the arbitrariness of σ the above inequality implies

$$\int_{t_0-r}^{t_0+r}\mathrm{d}t\int_\Omega U_r(x,t)\,\mathrm{d}t + \int_{[t_0-r,t_0+r]}\mathrm{d}\lambda(t) \leqq 2r\|U_0\|_{\mathcal{M}(\bar{\Omega})}, \tag{4.21}$$

hence

$$\int_{[t_0-r,t_0+r]}\mathrm{d}\lambda(t) \leqq 2r\|U_0\|_{\mathcal{M}(\bar{\Omega})} \tag{4.22}$$

since $U_r \geqq 0$ almost everywhere in Q. This proves the claim.

Therefore, there exists $h \in L^1(\mathbb{R})$, $\operatorname{supp} h \subseteq [0,T]$, such that $\mathrm{d}\lambda(t) = h(t)\mathrm{d}t$. Moreover, since $U_s \in \mathcal{M}^+(\bar{Q})$ is a positive Radon measure, from (4.22) we have

$$0 \leqq h(t) \leqq \|U_0\|_{\mathcal{M}(\bar{\Omega})} \quad \text{for almost every } t \in (0,T),$$

hence $h \in L^\infty(\mathbb{R})$. Finally, claim (i) follows defining

$$U_s(\cdot,t) := h(t)\gamma_t. \tag{4.23}$$

(ii) By (4.23) and the positivity of U_s, for almost every $t_0 \in (0,T)$ there holds

$$h(t_0) = U_s(\cdot,t_0)(\bar{\Omega}) = \|U_s(\cdot,t_0)\|_{\mathcal{M}(\bar{\Omega})}$$

(recall that γ_{t_0} is a probability measure). Then inequality (4.21) reads

$$\frac{1}{2r}\int_{t_0-r}^{t_0+r}\int_\Omega U_r(x,t)\,\mathrm{d}x\mathrm{d}t + \frac{1}{2r}\int_{t_0-r}^{t_0+r}\|U_s(\cdot,t)\|_{\mathcal{M}(\bar{\Omega})}\,\mathrm{d}t \leqq \|U_0\|_{\mathcal{M}(\bar{\Omega})},$$

whence inequality (2.6) follows, letting $r \to 0$. This completes the proof. □

Now we can prove the existence part of Theorem 2.1. This is the content of the following

Proposition 4.3. *Let assumptions* (H_1)–(H_3) *be satisfied. Then there exists a solution* U *of problem* (1.5)*, which has the properties asserted in Theorem* 2.1*.*

Proof. Let us show that the Radon measure $U \in \mathcal{M}^+(\bar{Q})$ mentioned in Proposition 3.6-(i) is a solution U of problem (1.5) with the asserted properties.

(i) Claim (i) of Theorem 2.1 follows by Proposition 4.2, thus in particular $U \in L^\infty((0,T);\mathcal{M}^+(\bar{\Omega}))$.

(ii) Consider the function $w \in L^\infty((0,T); H_0^1(\Omega)) \cap H^1(Q) \cap C(\bar{Q})$ mentioned in Proposition 3.6-(ii). Since the set $\tilde{\mathcal{S}}$ defined in (4.1) has zero Lebesgue measure and $\psi(U_r) = w$ almost everywhere in $Q \setminus \tilde{\mathcal{S}}$ by Proposition 4.1, w is the unique continuous representative of the function $\psi(U_r) \in L^\infty(Q)$. Identifying $\psi(U_r)$ with w in $\bar{Q}$ (see Remark 2.1), we obtain that $\psi(U_r) \in C(\bar{Q}) \cap L^\infty((0,T); H_0^1(\Omega))$ and $[\psi(U_r)]_t \in L^2((0,T); H_0^1(\Omega))$ by Proposition 3.6-(ii).

Moreover, observe that for any $\kappa > 0$

$$\psi_\kappa(U_\kappa(x,0)) = \psi_\kappa(U_{0\kappa})(x) \quad \text{for any } x \in \bar{\Omega},$$

U_κ being the unique solution of problem (P_κ)). Therefore, since $\psi_\kappa(U_{0\kappa}) \to \psi(U_{0r})$ in $C(\bar{\Omega})$ by assumption (H_3)-(ii) (see also (3.1)), and $\psi_\kappa(U_\kappa) \to w \equiv \psi(U_r)$ in $C(\bar{Q})$ by Proposition 3.6-(ii)—hence $\psi_\kappa(U_\kappa)(\cdot,0) \to \psi(U_r)(\cdot,0)$ in $C(\bar{\Omega})$—we obtain $\big[\psi(U_r)\big](x,0) = \psi(U_{0r})(x)$ for any $x \in \bar{\Omega}$.

By Proposition 3.6 (see (3.48)) and equality (4.3), we have $U_{\kappa_j} \to U_r$ almost everywhere in Q. Therefore,

$$\varphi(U_{\kappa_j}) \to \varphi(U_r) \quad \text{almost everywhere in } Q. \tag{4.24}$$

Hence, by estimate (3.36) we see that any subsequence of $\big\{\varphi(U_{\kappa_j})\big\}$ admits a sub-subsequence which converges weakly to $\varphi(U_r)$ in $L^p((0,T); H_0^1(\Omega))$ for any $p \in [1,\infty)$ (where we have made use of (4.24)). Moreover, since the sequence $\{\varphi(U_{\kappa_j})\}$ is uniformly bounded in $L^\infty((0,T); H_0^1(\Omega))$, there also holds $\varphi(U_r) \in L^\infty((0,T); H_0^1(\Omega))$.

(iii) Clearly, the identification $\psi(U_r) \equiv w$ in $\bar{Q}$ implies $\mathcal{S} = \tilde{\mathcal{S}}$ (see (4.1) and the very definition of $\mathcal{S}$ in (2.4)). Then the property (2.4) follows from (4.2).

(iv) It remains to prove equality (2.5). To this purpose, consider for any $\kappa_j > 0$ the solution U_{κ_j} of problem (P_{κ_j}). For any $\rho \in H_0^1(\Omega)$ set

$$\mathcal{I}_{\kappa_j,\rho}(t) := \int_\Omega U_{\kappa_j}(x,t)\rho(x)\,\mathrm{d}x \qquad (t \in [0,T]).$$

By equality (3.6) we have

$$\mathcal{I}'_{\kappa_j,\rho}(t) := -\int_\Omega V_{\kappa_j x}(x,t)\rho'(x)\,\mathrm{d}x,$$

thus $\mathcal{I}_{\kappa_j,\rho} \in H^1(0,T)$ (where V_{κ_j} is the function defined by (3.2); recall that $V_{\kappa_j} \in C([0,T]; C^2(\bar{\Omega}))$). Moreover, by (3.28) and (3.30) there exists a constant $C_\rho > 0$ such that for any $\kappa_j > 0$

$$\left\|\mathcal{I}_{\kappa_j,\rho}\right\|_{C([0,T])} \leqq C_\rho, \tag{4.25}$$

$$\left|\mathcal{I}_{\kappa_j,\rho}(t_2) - \mathcal{I}_{\kappa_j,\rho}(t_1)\right| = \left|\int_{t_1}^{t_2} \int_\Omega V_{\kappa_j x}(x,t)\rho'(x)\,\mathrm{d}x\mathrm{d}t\right| \leqq C_\rho |t_2 - t_1|^{1/2} \tag{4.26}$$

for any $t_1,\ t_2 \in [0,T]$. Estimates (4.25)–(4.26) imply that the sequence $\left\{\mathcal{I}_{\kappa_j,\rho}\right\}$ is uniformly bounded in $C([0,T])$ and equicontinuous. Therefore, by the Ascoli–Arzelá Theorem there exist a subsequence $\left\{\kappa_{j_l}\right\} \subseteq \left\{\kappa_j\right\}$ and a function $\mathcal{I}_\rho \in C([0,T])$ such that:

$$\mathcal{I}_{\kappa_{j_l},\rho} \to \mathcal{I}_\rho \quad \text{in } C([0,T]) \quad \text{as } l \to \infty. \tag{4.27}$$

On the other hand, for any $k \in C_c(0,T)$ we have:

$$\begin{aligned} &\lim_{l\to\infty} \int_0^T k(t)\,\mathrm{d}t \int_\Omega U_{\kappa_{j_l}}(x,t)\rho(x)\,\mathrm{d}x \\ &\quad = \int_0^T k(t)\,\mathrm{d}t \left\{\int_\Omega U_r(x,t)\rho(x)\,\mathrm{d}x + \langle U_s(\cdot,t),\rho\rangle_\Omega\right\} \end{aligned} \tag{4.28}$$

(where we have made use of (3.42) and Proposition 4.2-(i)). By the arbitrariness of the function k, combining (4.27) and (4.28) gives

$$\mathcal{I}_\rho(t) = \int_\Omega U_r(x,t)\rho(x)\,\mathrm{d}x + \langle U_s(\cdot,t),\rho\rangle_\Omega \tag{4.29}$$

for almost every $t \in (0,T)$. It is easily seen that the same limit is reached along any converging subsequence $\left\{\mathcal{I}_{\kappa_{j_m},\rho}\right\} \subseteq \left\{\mathcal{I}_{\kappa_j,\rho}\right\}$. Therefore, we have:

$$\lim_{j\to\infty} \int_\Omega U_{\kappa_j}(x,t)\rho(x)\,\mathrm{d}x = \int_\Omega U_r(x,t)\rho(x)\,\mathrm{d}x + \langle U_s(\cdot,t),\rho\rangle_\Omega \tag{4.30}$$

for any $\rho \in H_0^1(\Omega)$ and for almost every $t \in (0,T)$.

We can now prove equality (2.5). Recall that for any $\kappa_j > 0$ and for any $\zeta \in C^1([0,T]; H_0^1(\Omega)),\ \zeta(\cdot,T) = 0$ in Ω there holds

$$\iint_Q \left\{U_{\kappa_j}\zeta_t - \left[\varphi\left(U_{\kappa_j}\right)\right]_x \zeta_x - \epsilon\left[\psi_{\kappa_j}\left(U_{\kappa_j}\right)\right]_{tx}\zeta_x\right\}\mathrm{d}x\mathrm{d}t = -\int_\Omega U_{0\kappa_j}\zeta(x,0)\,\mathrm{d}x \tag{4.31}$$

(see (3.3)). By (3.28) and (4.30) we obtain respectively

$$\left|\int_\Omega U_{\kappa_j}(x,t)\zeta_t(x,t)\,\mathrm{d}x\right| \leqq C\|\zeta_t\|_{C(\bar{Q})}, \tag{4.32}$$

and

$$\lim_{j\to\infty}\int_\Omega U_{\kappa_j}(x,t)\zeta_t(x,t)\,\mathrm{d}x = \int_\Omega U_r(x,t)\zeta_t(x,t)\,\mathrm{d}x + \langle U_s(\cdot,t),\zeta_t(\cdot,t)\rangle_\Omega \tag{4.33}$$

for almost every $t \in (0,T)$. Therefore, by the Lebesgue Theorem there holds

$$\lim_{j\to\infty}\iint_Q U_{\kappa_j}(x,t)\zeta_t(x,t)\,\mathrm{d}x\mathrm{d}t = \iint_Q U_r(x,t)\zeta_t(x,t)\,\mathrm{d}x\mathrm{d}t + \int_0^T \langle U_s(\cdot,t),\zeta_t(\cdot,t)\rangle_\Omega\,\mathrm{d}t. \tag{4.34}$$

Finally, by assumption (H_3)-(ii), Proposition 3.6-(ii) and the weak convergence of the sequence $\{\varphi\left(U_{\kappa_j}\right)\}$ to $\varphi(U_r)$ in $L^p((0,T);H_0^1(\Omega))$, ($p \in [1,\infty)$; see (ii) above), letting $j \to \infty$ in equality (4.31) and using (4.34) we obtain equality (2.5).

(v) Let us finally prove claim (ii) of Theorem 2.1. For any open subset $Q_0 \subseteq Q$ such that dist $\left(\bar{Q}_0,\mathcal{S}\right) > 0$, by the continuity of the function $\psi(U_r)$ in $\bar{Q}$ and by the very definition of the set $\mathcal{S}$ there exists $0 < \gamma_0 < \gamma$ such that

$$\max_{(x,t)\in\overline{Q}_0} \psi(U_r)(x,t) \leqq \gamma_0,$$

hence

$$\begin{aligned}&\operatorname{ess\,sup}_{(x,t)\in Q_0} U_r(x,t) \leqq \psi^{-1}(\gamma_0) < \infty,\\ &\operatorname{ess\,inf}_{(x,t)\in Q_0}\left[\psi'(U_r)\right](x,t) \geqq a\end{aligned}$$

for some $a > 0$ (see assumption (H_2)-(i)). Since $\psi(U_r) \in H^1(Q)$ by the identification $w \equiv \psi(U_r)$ (see Proposition 3.6-(ii)), from the above estimates and the identification $U_r \equiv \psi^{-1}(\psi(U_r))$ we obtain that $U_r \in H^1(Q_0)$.

To prove that $U_r \in C(\bar{Q}\setminus\mathcal{S})$, fix any $(\tilde{x},\tilde{t}) \in \bar{Q}\setminus\mathcal{S}$. By the very definition of $\mathcal{S}$, there holds $\psi(U_r)(\tilde{x},\tilde{t}) = \tilde{\gamma} < \gamma$. Since $\psi(U_r) \in C(\bar{Q})$, for any $\epsilon \in (0,\gamma-\tilde{\gamma})$ there exists a neighbourhood of $(\tilde{x},\tilde{t})$ where $\psi(U_r) < \tilde{\gamma}+\epsilon$. On the other hand, ψ^{-1} is well defined and continuous in $(0,\tilde{\gamma}+\epsilon)$, for $\tilde{\gamma}+\epsilon < \gamma$. Hence $U_r \equiv \psi^{-1}\left(\psi(U_r)\right)$ is continuous in the same neighbourhood, and the claim follows.

Since $\psi(U_r) \in C(\bar{Q})$, for any $\epsilon \in (0,\gamma)$ there exists $\delta > 0$ such that

$$0 < dist\,((x,t),\mathcal{S}) < \delta \quad\Rightarrow\quad \psi(U_r) > \gamma-\epsilon \quad\Rightarrow\quad U_r > \psi^{-1}(\gamma-\epsilon).$$

Then by assumption (H_2)-(i) we obtain (2.7). This completes the proof. □

Remark 4.1. Since the limiting value in (4.24) does not depend on the (the choice of any subsequence of the) sequence $\left\{U_{\kappa_j}\right\}$, the arguments in part (ii) of the above proof show that as $\kappa_j \to 0$

$$\varphi(U_{\kappa_j}) \rightharpoonup \varphi(U_r) \quad \text{in } L^p((0,T);H_0^1(\Omega)) \quad (1 \leqq p < \infty). \tag{4.35}$$

It can be further observed that

$$\varphi\left(U_{\kappa_j}\right) \to \varphi(U_r) \quad \text{in } C(\bar{Q}) \tag{4.36}$$

as $\kappa_j \to 0$. In fact,

$$\left|\left[\varphi(U_{\kappa_j})\right]_t\right| = \left|\varphi'(U_{\kappa_j})\, U_{\kappa_j t}\right| \leqq k_1 \left|\psi'_{\kappa_j}(U_{\kappa_j})\, U_{\kappa_j t}\right| = k_1 \left|\left[\psi_{\kappa_j}(U_{\kappa_j})\right]_t\right|$$

by assumption (A_k)-(iv). From the above estimate and inequality (3.41) we obtain

$$\left\|\left[\varphi\left(U_{\kappa_j}\right)\right]_t\right\|_{L^2((0,T);L^\infty(\Omega))} \leqq C \tag{4.37}$$

for some constant $C > 0$. Arguing as in the proof of (3.40), with (3.34) and (3.41) replaced by (3.36) and (4.37), respectively, we obtain

$$\|\varphi\left(U_{\kappa_j}\right)\|_{C^{1/2}(Q)} \leqq C.$$

Hence there exists a subsequence (denoted again $\left\{\varphi(U_{\kappa_j})\right\}$ for simplicity) uniformly converging in $\bar{Q}$. Then by (4.24) we obtain (4.36).

Let us proceed to prove the uniqueness part of Theorem 2.1. To this purpose we need the following lemma, whose proof is postponed.

Lemma 4.4. *Let $U \in L^\infty((0,T);\mathcal{M}^+(\bar{\Omega}))$ be a solution of problem (1.5). Then for any $\zeta \in C([0,T];L^2(\Omega))$, $\zeta_t \in L^2(Q)$,*
(i) *the function*

$$\mathcal{J}_\zeta(t) := \int_\Omega \left[\psi(U_r)\right]_x(x,t)\,\zeta(x,t)\,\mathrm{d}x \qquad (t \in (0,T)) \tag{4.38}$$

belongs to $H^1(0,T)$, with weak derivative

$$\mathcal{J}'_\zeta(t) = \int_\Omega \left\{\left[\psi(U_r)\right]_{tx}(x,t)\,\zeta(x,t) + \left[\psi(U_r)\right]_x(x,t)\,\zeta_t(x,t)\right\}\,\mathrm{d}x; \tag{4.39}$$

(ii) *there holds:*

$$\mathcal{J}_\zeta(0) = \int_\Omega \left[\psi(U_{0r})\right]_x(x)\,\zeta(x,0)\,\mathrm{d}x. \tag{4.40}$$

Proposition 4.5. *Let assumptions (H_1)–(H_3) be satisfied. Then there exists at most one solution U of problem (1.5).*

Proof. (i) Let U_1, U_2 be two solutions to problem (1.5). Then the difference $U_1 - U_2$ satisfies the equality

$$\begin{aligned}
&\iint_{Q_\tau} \left[U_{1r} - U_{2r}\right]\zeta_t\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle U_{1s}(\cdot,t) - U_{2s}(\cdot,t),\ \zeta_t(\cdot,t)\right\rangle_\Omega \mathrm{d}t \\
&= \iint_{Q_\tau} \left\{\left[\varphi(U_{1r}) - \varphi(U_{2r})\right]_x + \epsilon\left[\psi(U_{1r}) - \psi(U_{2r})\right]_{tx}\right\}\zeta_x\,\mathrm{d}x\mathrm{d}t \\
&= \iint_{Q_\tau} \left[V_{1r} - V_{2r}\right]_x \zeta_x\,\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{4.41}$$

for any $\zeta \in C^1([0,\tau]; H_0^1(\Omega))$, $\zeta(\cdot,\tau) = 0$ (see (2.5)). Here V_{jr} is defined by (2.8) ($j = 1, 2$), and $Q_\tau := \Omega \times (0,\tau)$ with $\tau > 0$ to be chosen.

By standard regularization arguments, we can use in (4.41) the test function

$$\tilde{\zeta}(x,t) := -\int_t^\tau \big[\psi(U_{1r}) - \psi(U_{2r})\big](x,s)\,\mathrm{d}s \qquad \big((x,t) \in Q_\tau\big).$$

Then we obtain

$$\begin{aligned}
&\iint_{Q_\tau} \big[U_{1r} - U_{2r}\big]\big[\psi(U_{1r}) - \psi(U_{2r})\big]\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau \big\langle U_{1s}(\cdot,t) - U_{2s}(\cdot,t),\ \psi(U_{1r})(\cdot,t) - \psi(U_{2r})(\cdot,t)\big\rangle_\Omega \mathrm{d}t \\
&= -\iint_{Q_\tau} \big[V_1 - V_2\big]_x \left(\int_t^\tau \big[\psi(U_{1r}) - \psi(U_{2r})\big]_x(x,s)\,\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t. \qquad (4.42)
\end{aligned}$$

Let us study the three terms separately:

$$\begin{aligned}
I_1 &:= \iint_{Q_\tau} \big[U_{1r} - U_{2r}\big]\big[\psi(U_{1r}) - \psi(U_{2r})\big]\,\mathrm{d}x\mathrm{d}t, \\
I_2 &:= \int_0^\tau \big\langle U_{1s}(\cdot,t) - U_{2s}(\cdot,t),\ \psi(U_{1r})(\cdot,t) - \psi(U_{2r})(\cdot,t)\big\rangle_\Omega \mathrm{d}t, \\
I_3 &:= -\iint_{Q_\tau} \big[V_1 - V_2\big]_x \left(\int_t^\tau \big[\psi(U_{1r}) - \psi(U_{2r})\big]_x(x,s)\,\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

We have:

$$\begin{aligned}
I_1 &= \iint_{Q_\tau} \big[\psi^{-1}(\psi(U_{1r})) - \psi^{-1}(\psi(U_{2r}))\big]\big[\psi(U_{1r}) - \psi(U_{2r})\big]\,\mathrm{d}x\mathrm{d}t \\
&\geqq \frac{1}{\|\psi'\|_{L^\infty(\mathbb{R})}} \iint_{Q_\tau} \big[\psi(U_{1r}) - \psi(U_{2r})\big]^2 \mathrm{d}x\mathrm{d}t; \qquad (4.43) \\
I_2 &= \int_0^\tau \big\langle U_{1s}(\cdot,t), \psi(U_{1r})(\cdot,t) - \psi(U_{2r})(\cdot,t)\big\rangle_\Omega \mathrm{d}t \\
&\quad + \int_0^\tau \big\langle U_{2s}(\cdot,t), \psi(U_{2r})(\cdot,t) - \psi(U_{1r})(\cdot,t)\big\rangle_\Omega \mathrm{d}t \\
&= \int_0^\tau \big\langle U_{1s}(\cdot,t),\ \big[\gamma - \psi(U_{2r})(\cdot,t)\big]\big\rangle_\Omega \mathrm{d}t \\
&\quad + \int_0^\tau \big\langle U_{2s}(\cdot,t),\ \big[\gamma - \psi(U_{1r})(\cdot,t)\big]\big\rangle_\Omega \mathrm{d}t \geqq 0 \qquad (4.44)
\end{aligned}$$

by the characterization (2.4), since $U_{1s},\ U_{2s} \in \mathcal{M}^+(\bar{Q})$;

$$
\begin{aligned}
I_3 = &- \iint_{Q_\tau} \left[\varphi(U_{1r}) - \varphi(U_{2r})\right]_x \left(\int_t^\tau \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x (x,s)\,\mathrm{d}s \right) \mathrm{d}x\mathrm{d}t \\
&+ \epsilon \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x \tilde{\zeta}_x \,\mathrm{d}x\mathrm{d}t \\
&- \epsilon \int_0^\tau \int_\Omega \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x \tilde{\zeta}_{xt} \,\mathrm{d}x\mathrm{d}t \\
= &- \iint_{Q_\tau} \left[\varphi(U_{1r}) - \varphi(U_{2r})\right]_x \left(\int_t^\tau \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x (x,s)\,\mathrm{d}s \right) \mathrm{d}x\mathrm{d}t \\
&- \epsilon \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{4.45}
$$

by Lemma **4.4** (observe that $\psi(U_{1r}),\ \psi(U_{2r}) \in L^\infty((0,T);H_0^1(\Omega))$ imply that $\tilde{\zeta}_x \in C([0,\tau];L^2(\Omega))$ and $\tilde{\zeta}_{xt} \in L^2(Q_\tau)$).

(ii) From (4.42)–(4.45) we obtain

$$
\begin{aligned}
&\frac{1}{\|\psi'\|_\infty} \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]^2 \mathrm{d}x\mathrm{d}t + \epsilon \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \\
&\leqq - \iint_{Q_\tau} \left[\varphi(U_{1r}) - \varphi(U_{2r})\right]_x \left(\int_t^\tau \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x (x,s)\,\mathrm{d}s \right) \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.46}
$$

Concerning the right-hand side of the above inequality, there holds

$$
\begin{aligned}
&\iint_{Q_\tau} \left| \left[\varphi(U_{1r}) - \varphi(U_{2r})\right]_x \left(\int_t^\tau \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x (x,s)\,\mathrm{d}s \right) \right| \mathrm{d}x\mathrm{d}t \\
&\leqq \frac{1}{2\delta} \iint_{Q_\tau} \left(\int_t^\tau \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x (x,s)\,\mathrm{d}s \right)^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \frac{\delta}{2} \iint_{Q_\tau} \left[\varphi(U_{1r}) - \varphi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \leqq \frac{\tau^2}{2\delta} \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \frac{\delta}{2} \iint_{Q_\tau} \left[\varphi(U_{1r}) - \varphi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{4.47}
$$

for any $\delta > 0$. On the other hand, writing $\varphi(U_{ir}) = \left(\varphi \circ \psi^{-1}\right)(\psi(U_{ir}))(i = 1,2)$ and using assumptions (H_2)-(iv), (v) we easily find

$$
\begin{aligned}
&\frac{1}{2} \iint_{Q_\tau} \left[\varphi(U_{1r}) - \varphi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \leqq k_1^2 \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \\
&\quad + k_2^2 \iint_{Q_\tau} \left[\psi(U_{2r})\right]_x^2 \left[\psi(U_{1r}) - \psi(U_{2r})\right]^2 \leqq k_1^2 \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \\
&\quad + k_2^2 \sup_{t\in(0,\tau)} \left(\int_\Omega \left[\psi(U_{2r})\right]_x^2 (x,t)\mathrm{d}x \right) \int_0^\tau \left\| \left[\psi(U_{1r}) - \psi(U_{2r})\right]^2 (\cdot,t) \right\|_{C(\bar{\Omega})} \mathrm{d}t.
\end{aligned} \tag{4.48}
$$

Since $\left[\psi(U_{1r}) - \psi(U_{2r})\right](\cdot, t) \in H_0^1(\Omega)$ for almost every $t \in (0, \tau)$, there also holds

$$\begin{aligned}
&\left\| \left[\psi(U_{1r}) - \psi(U_{2r})\right]^2(\cdot, t) \right\|_{C(\bar{\Omega})} \\
&\leqq 2 \int_\Omega \left| \left[\psi(U_{1r}) - \psi(U_{2r})\right](\xi, t) \right| \left| \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x(\xi, t) \right| \mathrm{d}\xi \\
&\leqq \int_\Omega \left[\psi(U_{1r}) - \psi(U_{2r})\right]^2(\xi, t)\, \mathrm{d}\xi + \int_\Omega \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2(\xi, t)\, \mathrm{d}\xi.
\end{aligned}$$

Inserting the above inequality into the last term of (4.48) gives

$$\begin{aligned}
&\frac{1}{2} \iint_{Q_\tau} \left[\varphi(U_{1r}) - \varphi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \leqq k_1^2 \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \\
&\quad + k_2^2 \sup_{t \in (0,\tau)} \left(\int_\Omega \left[\psi(U_{2r})\right]_x^2(x, t)\mathrm{d}x \right) \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]^2 \mathrm{d}x\mathrm{d}t \\
&\quad + k_2^2 \sup_{t \in (0,\tau)} \left(\int_\Omega \left[\psi(U_{2r})\right]_x^2(x, t)\mathrm{d}x \right) \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t. \qquad (4.49)
\end{aligned}$$

(iii) We can now conclude the proof. By (4.47) and (4.49), inequality (4.46) reads:

$$\begin{aligned}
&\frac{1}{\|\psi'\|_\infty} \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]^2 + \epsilon \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \\
&\leqq \frac{\tau^2}{2\delta} \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t + \delta k_1^2 \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \delta k_2^2 \sup_{t \in (0,\tau)} \left(\int_\Omega \left[\psi(U_{2r})\right]_x^2(x, t)\mathrm{d}x \right) \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \delta k_2^2 \sup_{t \in (0,\tau)} \left(\int_\Omega \left[\psi(U_{2r})\right]_x^2(x, t)\mathrm{d}x \right) \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

whence

$$C_1 \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]^2 \mathrm{d}x\mathrm{d}t + C_2 \iint_{Q_\tau} \left[\psi(U_{1r}) - \psi(U_{2r})\right]_x^2 \mathrm{d}x\mathrm{d}t \leqq 0,$$

where

$$C_1 := \frac{1}{\|\psi'\|_\infty} - \delta k_2^2 \sup_{t \in (0,\tau)} \left(\int_\Omega \left[\psi(U_{2r})\right]_x^2(x, t)\mathrm{d}x \right), \qquad (4.50)$$

$$C_2 := \epsilon - \frac{\tau^2}{2\delta} - \delta k_1^2 - \delta k_2^2 \sup_{t \in (0,\tau)} \left(\int_\Omega \left[\psi(U_{2r})\right]_x^2(x, t)\mathrm{d}x \right). \qquad (4.51)$$

Choosing δ and τ so that

$$\begin{cases}
\delta k_2^2 \, \sup_{t \in (0,\tau)} \left(\int_\Omega \left[\psi(U_{2r})\right]_x^2(x, t)\mathrm{d}x \right) < \|\psi'\|_\infty^{-1}, \\
\tau^2 < \delta\epsilon, \\
\delta k_1^2 + \delta k_2^2 \, \sup_{t \in (0,\tau)} \left(\int_\Omega \left[\psi(U_{2r})\right]_x^2(x, t)\mathrm{d}x \right) < \epsilon/2,
\end{cases}$$

we obtain $C_1 > 0$, $C_2 > 0$. Then by (4.50)

$$\|\psi(U_{1r}) - \psi(U_{2r})\|_{L^2((0,\tau);H_0^1(\Omega))} \leqq 0,$$

whence the result follows. □

Proof of Lemma 4.4. Let $U \in L^\infty((0,T);\mathcal{M}^+(\bar{\Omega}))$ be a solution of problem (1.5). Then $\psi(U_r) \in L^\infty((0,T);H_0^1(\Omega))$, thus $\mathcal{J}_\zeta \in L^2(0,T)$ for any $\zeta \in C([0,T];L^2(\Omega))$. Moreover, since $\left[\psi(U_r)\right]_t \in L^2((0,T);H_0^1(\Omega))$, it is easily seen that for any $\zeta \in C^\infty(\bar{Q})$ with $\zeta(\cdot,t) \in C_c^\infty(\Omega)$ and for any $h \in C_c^1(0,T)$ there holds

$$\begin{aligned}\int_0^T h(t)\,\mathrm{d}t \int_\Omega \left\{\left[\psi(U_r)\right]_{tx}(x,t)\,\zeta(x,t) + \left[\psi(U_r)\right]_x(x,t)\,\zeta_t(x,t)\right\}\mathrm{d}x\\ = -\int_0^T h_t(t)\mathcal{J}_\zeta(t)\mathrm{d}t. \qquad (4.52)\end{aligned}$$

Since $\left[\psi(U_r)\right]_{tx} \in L^2(Q)$ and $\left[\psi(U_r)\right]_x \in L^\infty((0,T);L^2(\Omega))$, by standard regularization results the above equality holds for any $\zeta \in C([0,T];L^2(\Omega))$, $\zeta_t \in L^2(Q)$, and we have

$$\begin{aligned}&\int_0^T \left(\int_\Omega \left\{\left[\psi(U_r)\right]_{tx}(x,t)\,\zeta(x,t) + \left[\psi(U_r)\right]_x(x,t)\,\zeta_t(x,t)\right\}\mathrm{d}x\right)^2 \mathrm{d}t\\ &\leqq 2\int_0^T\left(\int_\Omega \left[\psi(U_r)\right]_{tx}(x,t)\,\zeta(x,t)\,\mathrm{d}x\right)^2\mathrm{d}t + 2\int_0^T\left(\int_\Omega \left[\psi(U_r)\right]_x(x,t)\,\zeta_t(x,t)\,\mathrm{d}x\right)^2\mathrm{d}t\\ &\leqq 2\int_0^T \left(\int_\Omega \left[\psi(U_r)\right]_{tx}^2(x,t)\,\mathrm{d}x\right)\left(\int_\Omega \zeta^2(x,t)\,\mathrm{d}x\right)\mathrm{d}t\\ &\quad + 2\int_0^T \left(\int_\Omega \left[\psi(U_r)\right]_x^2(x,t)\,\mathrm{d}x\right)\left(\int_\Omega \zeta_t^2(x,t)\,\mathrm{d}x\right)\mathrm{d}t\\ &\leqq 2\|\zeta\|^2_{L^\infty((0,T);L^2(\Omega))}\left\|\left[\psi(U_r)\right]_{tx}\right\|^2_{L^2(Q)}\\ &\quad + 2\left\|\left[\psi(U_r)\right]_x\right\|^2_{L^\infty((0,T);L^2(\Omega))}\|\zeta_t\|^2_{L^2(Q)} \leqq C.\end{aligned}$$

Therefore, the map

$$t \longmapsto \int_\Omega \left\{\left[\psi(U_r)\right]_{tx}(x,t)\,\zeta(x,t) + \left[\psi(U_r)\right]_x(x,t)\,\zeta_t(x,t)\right\}\mathrm{d}x$$

belongs to $L^2(0,T)$, whence by equality (4.52)

$$\mathcal{J}_\zeta \in H^1(0,T) \qquad (4.53)$$

for any ζ as above. This proves claim (i).

Concerning (ii), observe that by its very definition

$$\mathcal{J}_\zeta(t) = -\int_\Omega [\psi(U_r)](x,t)\,\zeta_x(x,t)\,\mathrm{d}x \qquad (4.54)$$

for any $\zeta \in C([0,T]; H_0^1(\Omega))$, $\zeta_t \in L^2(Q)$ and $t > 0$. Then by the continuity of $\psi(U_r)$ in the rectangle $\bar{Q}$ and the initial condition there holds

$$\mathcal{J}_\zeta(0) = \lim_{t\to 0} \mathcal{J}_\zeta(t) = -\int_\Omega \psi(U_{0r})(x)\, \zeta_x(x,t)\, \mathrm{d}x = \int_\Omega [\psi(U_{0r})]_x(x)\, \zeta(x,0)\, \mathrm{d}x,$$

(here we have made use of assumption (H_3)). Finally, by regularization results the above equality holds for any $\zeta \in C([0,T]; L^2(\Omega))$, $\zeta_t \in L^2(Q)$. This completes the proof. □

Proof of Theorem 2.1. The existence of a solution to problem (1.5) with the asserted properties in (i)–(ii) follows by Proposition 4.3, whereas the uniqueness of this solution has been proven in Proposition 4.5 above. □

Let us now prove Theorem 2.2.

Proof of Theorem 2.2. Claim (i) follows from Proposition 4.1 and the identification $\psi(U_r) \equiv w$, whereas (ii) is a consequence of the very definition of $\mathcal{S}$ and the boundary condition $\psi(U_r) = 0$ on $\partial\Omega \times [0,T]$ (see Definition 2.1-(ii)), for the set $\mathcal{S}$ is closed (see Remark 2.1). □

4.2. Proof of Theorem 2.3

To prove Theorem 2.3 we need some preliminary results.

Lemma 4.6. *The function V_r defined by (2.8) belongs to the space $L^\infty(Q) \cap L^2((0,T); H_0^1(\Omega))$. Moreover,*

(i) *there exists $C > 0$ (independent of ϵ) such that*

$$\|V_{rx}\|_{L^2(Q)} \leqq C; \tag{4.55}$$

(ii) *there holds*

$$0 \leqq V_r \leqq \varphi(\alpha). \tag{4.56}$$

Proof. Let $\{\kappa_j\}$ be the sequence in Proposition 3.6. Since $\varphi(U_r) \in L^\infty((0,T); H_0^1(\Omega))$ and $\left[\psi(U_r)\right]_t \in L^2((0,T); H_0^1(\Omega))$, we have that $V_r \in L^2((0,T); H_0^1(\Omega))$ and

$$V_{\kappa_j} \rightharpoonup V_r \quad \text{in } L^2((0,T); H_0^1(\Omega)) \tag{4.57}$$

(see Proposition 3.6-(ii) and (4.35)). Then inequality (4.55) follows from (3.30) by the lower semicontinuity of the norm. On the other hand, inequality (4.56) immediately follows from (3.7), observing that

$$0 \leqq \lim_{\kappa_j\to 0} \iint_Q \left\{\varphi(\alpha) - V_{\kappa_j}\right\}\zeta\, \mathrm{d}x\mathrm{d}t = \iint_Q \left\{\varphi(\alpha) - V_r\right\}\zeta\, \mathrm{d}x\mathrm{d}t$$

for any $\zeta \in L^2(Q)$, $\zeta \geqq 0$. Hence the result follows. □

Lemma 4.7. *There exists a constant $C > 0$ such that for any $\epsilon > 0$ there holds:*

$$\iint_Q \frac{\left[\psi(U_r)\right]_t^2}{\psi'(U_r)}\,\mathrm{d}x\mathrm{d}t \leqq \frac{C}{\epsilon}. \tag{4.58}$$

Proof. First observe that by (3.31) there exist a sequence $\kappa_j \to 0$ and $g \in L^2(Q)$ such that

$$\frac{\left[\psi_{\kappa_j}(U_{\kappa_j})\right]_t}{\sqrt{\psi'_{\kappa_j}(U_{\kappa_j})}} \rightharpoonup g \quad \text{in } L^2(Q). \tag{4.59}$$

Let $\tilde{\mathcal{S}}$ and B_n be the sets defined by (4.1), (4.4). Denote by A_n the complement of B_n in Q, namely

$$A_n := \left\{(x,t) \in Q \mid \psi(U_r) < \gamma - \frac{1}{n}\right\} \tag{4.60}$$

(recall that by choice $w \equiv \psi(U_r)$ in $\overline{Q}$, thus $\tilde{\mathcal{S}} = \mathcal{S}$; see (2.4)). Then $A_n \subseteq A_{n+1}$, $B_n \cap Q = Q \setminus A_n$ for any $n \in \mathbb{N}$. For any $j \in \mathbb{N}$ sufficiently large there holds:

$$U_{k_j} \leqq \psi^{-1}\left(\gamma - \frac{1}{2n}\right) \quad \text{in } A_n. \tag{4.61}$$

In fact, since $\psi_{k_j}(U_{k_j}) \to \psi(U_r)$ in $C(\bar{Q})$ as $k_j \to 0$ (see (3.44) and (4.3)), we have

$$\psi_{k_j}(U_{k_j}) \leqq \gamma - \frac{1}{2n} \quad \text{in } A_n$$

for any κ_j sufficiently small. Then inequality (3.49) and the nonnegativity of U_{k_j} give

$$\psi(U_{k_j}) \leqq k_j U_{k_j} + \psi(U_{k_j}) \leqq \psi_{k_j}(U_{k_j}) \leqq \gamma - \frac{1}{2n} \quad \text{in } A_n.$$

By inequality (4.61), (H_2)-(i) and (A_k)-(ii), there exists $C_n > 0$ such that

$$\psi'_{\kappa_j}(U_{\kappa_j}) \geqq \kappa_j + \psi'(U_{\kappa_j}) \geqq \frac{1}{C_n} \quad \text{in } A_n,$$

thus

$$0 \leqq \frac{1}{\psi'_{\kappa_j}(U_{\kappa_j})} \leqq C_n \quad \text{in } A_n, \tag{4.62}$$

for any $j \in \mathbb{N}$ large enough.

Moreover, since A_n is open, and $\cup_n A_n = Q \setminus \mathcal{S}$, for any $\zeta \in C_c^1(Q \setminus \mathcal{S})$ there exists $n \in \mathbb{N}$ such that supp $\zeta \subseteq A_n$. Since $\psi'_{\kappa_j} \to \psi'$ in $C_{loc}(\mathbb{R})$ (see assumption (A_k)-(i)), by (3.48), (4.3), and inequality (4.62) there holds

$$\frac{1}{\sqrt{\psi'_{\kappa_j}(U_{\kappa_j})}} \to \frac{1}{\sqrt{\psi'(U_r)}} \quad \text{in } L^2(\text{supp } \zeta).$$

On the other hand, by (3.45)

$$\left[\psi_{\kappa_j}(U_{\kappa_j})\right]_t \rightharpoonup \left[\psi(U_r)\right]_t \quad \text{in } L^2(Q).$$

Therefore

$$\iint_Q \frac{\left[\psi_{\kappa_j}(U_{\kappa_j})\right]_t}{\sqrt{\psi'_{\kappa_j}(U_{\kappa_j})}}\,\zeta\,\mathrm{d}x\mathrm{d}t \to \iint_Q \frac{\left[\psi(U_r)\right]_t}{\sqrt{\psi'(U_r)}}\,\zeta\,\mathrm{d}x\mathrm{d}t$$

for any $\zeta \in C^1_c(Q \setminus \mathcal{S})$. Since $|\mathcal{S}| = 0$, in view of (4.59) we obtain

$$g = \frac{\left[\psi(U_r)\right]_t}{\sqrt{\psi'(U_r)}} \quad a.e. \text{ in } Q. \tag{4.63}$$

Then inequality (4.58) follows from (3.31), (4.59) and (4.63) by the lower semi-continuity of the norm. This proves the result. □

Proposition 4.8. *For any $n \in \mathbb{N}$ U_{rt}, $V_{rxx} \in L^2(A_n)$, where A_n is the open set defined by* (4.60)*, and there holds*

$$U_{rt} = V_{rxx} \quad in\ L^2(A_n). \tag{4.64}$$

Moreover,

$$U_{rt} = \frac{\left[\psi(U_r)\right]_t}{\psi'\left(U_r\right)} \quad a.e.\ in\ A_n. \tag{4.65}$$

Proof. Let $\kappa_j \to 0$ be any sequence such that $U_{\kappa_j} \to U_r$ almost everywhere in Q (this sequence exists by Proposition 3.6-(iii); see (3.48) and (4.3)). By the equality

$$V_{\kappa_j xx} = U_{\kappa_j t} = \frac{[\psi_{\kappa_j}(U_{\kappa_j})]_t}{\psi'_{\kappa_j}(U_{\kappa_j})} \tag{4.66}$$

and inequalities (3.33), (4.62) (which holds for any $j \in \mathbb{N}$ sufficiently large), we have

$$\begin{aligned}\iint_{A_n} (U_{\kappa_j t})^2 \mathrm{d}x\mathrm{d}t &= \iint_{A_n} (V_{\kappa_j xx})^2 \mathrm{d}x\mathrm{d}t \\ &= \iint_{A_n} \left(\frac{[\psi_{\kappa_j}(U_{\kappa_j})]_t}{\psi'_{\kappa_j}(U_{\kappa_j})}\right)^2 \mathrm{d}x\mathrm{d}t \leqq C_n^2 \left\|[\psi_{\kappa_j}(U_{\kappa_j})]_t\right\|^2_{L^2(Q)} \leqq C_n^2\,\frac{C}{\epsilon}.\end{aligned}$$

Therefore the families $\left\{U_{\kappa_j t}\right\}$, $\left\{V_{\kappa_j xx}\right\}$ are uniformly bounded in $L^2(A_n)$, thus U_{rt}, $V_{rxx} \in L^2(A_n)$ and

$$U_{k_j t} \rightharpoonup U_{rt}, \quad V_{k_j xx} \rightharpoonup V_{rxx} \quad \text{in } L^2(A_n) \qquad (n \in \mathbb{N}). \tag{4.67}$$

Hence (4.64) follows from the first equality in (4.66).

Finally, arguing as in the proof of Lemma 4.7 we obtain

$$\frac{[\psi_{\kappa_j}(U_{\kappa_j})]_t}{\psi'_{\kappa_j}(U_{\kappa_j})} \rightharpoonup \frac{\left[\psi(U_r)\right]_t}{\psi'\left(U_r\right)} \quad \text{in } L^2(A_n).$$

By the second equality in (4.66), the above convergence gives equality (4.65). Then the conclusion follows. □

Remark 4.2. By Proposition 4.8 the distributions $U_{rt}, V_{rxx} \in \mathcal{D}'(Q)$ "restricted to the open set $Q \setminus \mathcal{S}$" can be identified with the function $\frac{[\psi(U_r)]_t}{\psi'(U_r)} \in L^1_{loc}(Q \setminus \mathcal{S})$, namely

$$U_{rt} \equiv V_{rxx} \equiv \frac{[\psi(U_r)]_t}{\psi'(U_r)} \quad \text{in } \mathcal{D}'(Q \setminus \mathcal{S}). \tag{4.68}$$

In fact, using (4.64)–(4.65), the very definition of the sets $\mathcal{S}$ and A_n (see (2.4) and (4.60), respectively) and the continuity of the function $\psi(U_r)$ in $\bar{Q}$, it is easily seen that

$$U_{rt}(\zeta) = V_{rxx}(\zeta) = \int_{Q\setminus\mathcal{S}} \frac{[\psi(U_r)]_t}{\psi'(U_r)}\, \zeta \, \mathrm{d}x\mathrm{d}t \tag{4.69}$$

for any $\zeta \in C_c^\infty(Q \setminus \mathcal{S})$.

Since the set $\mathcal{S}$ has zero Lebesgue measure, in view of (4.68) we can associate to the distributions $U_{rt}, V_{rxx} \in \mathcal{D}'(Q)$ two *measurable functions*, again denoted U_{rt}, V_{rxx} for simplicity, which "represent" the distributions U_{rt}, V_{rxx} in $Q \setminus \mathcal{S}$ in the sense of (4.69), such that

$$V_{rxx}(x,t) = U_{rt}(x,t) = \frac{[\psi(U_r)]_t}{\psi'(U_r)}(x,t) \tag{4.70}$$

for almost every $(x,t) \in Q$. Keeping in mind the above considerations, in the following we shall always use the notations U_{rt}, V_{rxx} to indicate the measurable functions in (4.70) (in particular, see Propositions 4.10 and 5.1).

To prove Theorem 2.3 we also need the following technical lemma, whose standard proof is omitted.

Lemma 4.9. *Let $\zeta \in L^2((0,T); H^1(\Omega))$, where $\Omega \subseteq \mathbb{R}$ is a bounded interval. Then:*

(i) *for any $x_0 \in \Omega$ the function $\zeta(x_0, \cdot) : (0,T) \to \mathbb{R}$ belongs to the space $L^1(0,T)$;*
(ii) *there exists a set $H \subseteq (0,T)$ of Lebesgue measure $|H| = 0$ such that for any $t_0 \in (0,T) \setminus H$ and for any $x_0 \in \Omega$*

$$\lim_{h\to 0} \frac{1}{h} \int_{t_0}^{t_0+h} \zeta(x_0,t)\, \mathrm{d}t = \zeta(x_0,t_0). \tag{4.71}$$

Let us remark that the null set $H \subseteq (0,T)$ which we exclude in the limit (4.71) is *independent* of the choice of $x_0 \in \Omega$.

Proof of Theorem 2.3. (i) Let $A \subseteq Q$ be any open set such that $\mathrm{dist}\big(\bar{A}, \mathcal{S}\big) > 0$. By the continuity of $\psi(U_r)$ in $\bar{Q}$ and the very definition of the set $\mathcal{S}$, there exists A_n such that $A \subseteq A_n$ (where A_n is the set defined in (4.60)). Then by Proposition 4.8 the claim follows.

(ii) Since V_r and $\left[\psi(U_r)\right]_t$ belong to $L^2((0,T);H_0^1(\Omega))$, there exists a set $H \subseteq (0,T)$ of zero Lebesgue measure such that for any $t \in (0,T)\setminus H$ there holds $V_r(\cdot,t) \in H_0^1(\Omega)$, $\left[\psi(U_r)\right]_t(\cdot,t) \in H_0^1(\Omega) \subseteq C(\bar{\Omega})$ and

$$\lim_{h\to 0}\frac{1}{h}\int_{t_0}^{t_0+h}\left[\psi(U_r)\right]_t(x_0,t)\mathrm{d}t = \left[\psi(U_r)\right]_t(x_0,t_0) \tag{4.72}$$

for any $x_0 \in \Omega$.

Set

$$B_\delta(t) := \left\{x \in \Omega \mid V_r(x,t) \geqq \delta\right\} \qquad (\delta > 0;\, t \in (0,T)). \tag{4.73}$$

Arguing by contradiction, let there exist $t_0 \in (0,T)\setminus H$ and $x_0 \in \Omega$ such that $x_0 \in B_\delta(t_0) \cap \mathcal{S}_{t_0}$ for some $\delta > 0$. For any $r > 0$ and any $h > 0$ sufficiently small we have

$$\begin{aligned}&\frac{1}{r}\int_{I_r(x_0)}\psi(U_r)(x,t_0+h)\mathrm{d}x - \frac{1}{r}\int_{I_r(x_0)}\psi(U_r)(x,t_0)\mathrm{d}x\\&\quad = \frac{1}{r}\int_{t_0}^{t_0+h}\int_{I_r(x_0)}[\psi(U_r)]_t(x,t)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $I_r(x_0) \equiv (x_0 - r, x_0 + r)$. Since $\psi(U_r) \in C(\bar{Q}), \left[\psi(U_r)\right]_t \in L^2((0,T);H_0^1(\Omega))$, thus $\left[\psi(U_r)\right]_{tx} \in L^2(Q)$, and $\left[\psi(U_r)\right]_t(x_0,\cdot) \in L^1(0,T)$ by Lemma 4.9-(i), letting $r \to 0$ in the above equality plainly gives:

$$\psi(U_r)(x_0,t_0+h) - \psi(U_r)(x_0,t_0) = \int_{t_0}^{t_0+h}[\psi(U_r)]_t(x_0,t)\,\mathrm{d}t. \tag{4.74}$$

Since $\psi(U_r) \leqq \gamma$ in $\bar{Q}$ and $\psi(U_r)(x_0,t_0) = \gamma$ by assumption, the left-hand side of the above equality is nonpositive. Then by (4.72) we have

$$\lim_{h\to 0}\frac{1}{h}\int_{t_0}^{t_0+h}[\psi(U_r)]_t(x_0,t)\mathrm{d}t = \left[\psi(U_r)\right]_t(x_0,t_0) \leqq 0.$$

On the other hand, there holds $\varphi(U_r)(x_0,t_0) = 0$ (see Remark 2.1). Therefore, we obtain

$$0 < \delta \leqq V_r(x_0,t_0) = \varphi(U_r)(x_0,t_0) + \epsilon[\psi(U_r)]_t(x_0,t_0) \leqq 0.$$

The contradiction proves that

$$B_\delta(t) \cap \mathcal{S}_t = \emptyset \tag{4.75}$$

for any $\delta > 0$ and for any $t \in (0,T)\setminus H$; here $B_\delta(t)$ is the set defined in (4.73). Then, since $\operatorname{supp} U_s(\cdot,t) \subseteq \mathcal{S}_t$ and $\mathcal{S}_t \subseteq \Omega$ by the boundary condition $\psi(U_r) = 0$ in $\partial\Omega \times [0,T]$, by the arbitrariness of $\delta > 0$ the claim follows. This completes the proof. □

Let us prove, for further purposes, an additional result concerning the regularity of the function $V_r(\cdot,t)$ for almost every fixed $t \in (0,T)$. The following proposition can be regarded as a pointwise version (with respect to t) of Proposition 4.8.

Proposition 4.10. *Let U_{rt}, V_{rxx} be the functions defined in Remark 4.2. Then there exists a null set $F \subseteq (0,T)$, $|F| = 0$, such that for any $t \in (0,T) \setminus F$ and for any $n \in \mathbb{N}$ there holds $U_{rt}(\cdot,t)$, $V_{rxx}(\cdot,t) \in L^2(A_n^t)$, $V_r(\cdot,t) \in H^2(A_n^t)$ and*

$$U_{rt}(\cdot,t) = V_{rxx}(\cdot,t) = [V_r(\cdot,t)]_{xx} \quad \text{a.e. in } A_n^t, \tag{4.76}$$

where

$$A_n^t := \left\{ x \in \Omega \mid \psi(U_r)(x,t) < \gamma - \frac{1}{n} \right\} \qquad (n \in \mathbb{N}, t \in (0,T)). \tag{4.77}$$

Proof. (i) Since the functions U_{rt}, V_{rxx} are measurable in Q and V_r, $\left[\psi(U_r)\right]_t \in L^2((0,T); H_0^1(\Omega)) \subseteq L^1(Q)$, it is easily seen that there exists a set $F \subseteq (0,T)$, $|F| = 0$, such that for any $t \in (0,T) \setminus F$

- $U_{rt}(\cdot,t)$ and $V_{rxx}(\cdot,t)$ are measurable in Ω and the equalities in (4.70) hold for almost every $x \in \Omega$;
- $V_r(\cdot,t)$, $\left[\psi(U_r)\right]_t(\cdot,t) \in H_0^1(\Omega)$;
- for any $\zeta \in C(\bar{\Omega})$

$$\lim_{R \to 0} \frac{1}{2R} \int_{t-R}^{t+R} \int_\Omega V_r(x,s)\zeta(x)\,\mathrm{d}x\mathrm{d}s = \int_\Omega V_r(x,t)\zeta(x)\,\mathrm{d}x, \tag{4.78}$$

$$\lim_{R \to 0} \frac{1}{2R} \int_{t-R}^{t+R} \int_\Omega \left[\psi(U_r)\right]_t(x,s)\zeta(x)\,\mathrm{d}x\mathrm{d}s = \int_\Omega \left[\psi(U_r)\right]_t(x,t)\zeta(x)\,\mathrm{d}x, \tag{4.79}$$

$$\lim_{R \to 0} \frac{1}{2R} \int_{t-R}^{t+R} \int_\Omega \left|\left[\psi(U_r)\right]_t(x,s)\right|\zeta(x)\,\mathrm{d}x\mathrm{d}s = \int_\Omega \left|\left[\psi(U_r)\right]_t(x,t)\right|\zeta(x)\,\mathrm{d}x. \tag{4.80}$$

Also observe that for any $n \in \mathbb{N}$ there exists $\delta > 0$ (depending on n) such that for any $t \in (0,T)$

$$Q_n^t := A_n^t \times (t - \delta, t + \delta) \subseteq A_{2n} \tag{4.81}$$

(where $A_{2n} \subseteq Q$ is the set in (4.60)). In fact, by Lemma 3.5, (3.43)–(3.45) and (4.3), there holds $\psi(U_r) \in C^{1/2}(\bar{Q})$. Hence there exists $C > 0$ such that for any (x_1,t_1), $(x_2,t_2) \in \bar{Q}$ there holds:

$$|\psi(U_r)(x_2,t_2) - \psi(U_r)(x_1,t_1)| \leqq C \left(|x_2 - x_1|^{1/2} + |t_2 - t_1|^{1/2} \right). \tag{4.82}$$

Then for any $x \in A_n^t$ ($n \in \mathbb{N}$ fixed) and for any $s \in (0,T)$ we have

$$\psi(U_r)(x,s) \leqq C\ |s-t|^{1/2} + \psi(U_r)(x,t) < C\ |s-t|^{1/2} + \gamma - \frac{1}{n}. \tag{4.83}$$

Set $\delta := \left(\frac{1}{2nC}\right)^2$. Then the above inequality gives

$$\psi(U_r)(x,s) < \gamma - \frac{1}{2n}$$

for any $x \in A_n^t$ and any $s \in (t-\delta, t+\delta)$, thus the claim follows.

(ii) Fix any $n \in \mathbb{N}, t \in (0,T) \setminus F$. Let $A_n^t \subseteq \Omega$ be the corresponding set defined by (4.77). Clearly, $V_r(\cdot,t) \in H^1(A_n^t)$ and $[V_r(\cdot,t)]_x = V_{rx}(\cdot,t)$ almost everywhere in Ω, for $V_r(\cdot,t) \in H_0^1(\Omega)$. Therefore the conclusion will follow, if we show that $U_{rt}(\cdot,t) \in L^2(A_n^t)$ and

$$\int_{A_n^t} U_{rt}(x,t)\,\zeta(x)\,\mathrm{d}x = \int_{A_n^t} V_r(x,t)\,\zeta''(x)\,\mathrm{d}x \tag{4.84}$$

for any $\zeta \in C_c^\infty(A_n^t)$ (let us also recall that $V_{rxx}(\cdot,t) = U_{rt}(\cdot,t)$ almost everywhere in Ω by (4.70)).

To this purpose, observe that

$$U_r(x,t) \leqq \psi^{-1}(\gamma - 1/n)$$

for almost every x in the closure $\bar{A}_n^t$ of the set A_n^t. Since $\psi(U_r)(\cdot,t) \in H_0^1(\Omega) \subseteq C(\bar{\Omega})$ and $\psi^{-1} \in C^1([0,\gamma - 1/n])$, it follows that

$$U_r(\cdot,t) \equiv \psi^{-1}(\psi(U_r)(\cdot,t)) \in C(\bar{A}_n^t).$$

Moreover, the function $\psi'(U_r(\cdot,t))$ is bounded away from zero in $\bar{A}_n^t$, hence

$$\frac{1}{\psi'(U_r)}(\cdot,t) \in C(\bar{A}_n^t) \subseteq L^\infty(A_n^t).$$

Since $\left[\psi(U_r)\right]_t(\cdot,t) \in H_0^1(\Omega)$, by the very definition of the functions U_{rt}, V_{rxx} in Remark 4.2 (in particular, see (4.70)) we obtain

$$V_{rxx}(\cdot,t) = U_{rt}(\cdot,t) = \frac{\left[\psi(U_r)\right]_t}{\psi'\big(U_r\big)}(\cdot,t) \in L^2(A_n^t). \tag{4.85}$$

It remains to prove (4.84). To this purpose, fix any $\zeta \in C_c^\infty(A_n^t)$ and choose $\delta = \delta(n) > 0$ as in (i) above, so that (4.81) holds. Then by Proposition 4.8 $U_{rt}, V_{rxx} \in L^2(Q_n^t)$ and

$$U_{rt} = V_{rxx} \quad \text{in} \quad L^2(Q_n^t). \tag{4.86}$$

In particular, using (4.65) there holds

$$\frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{A_n^t} \frac{\left[\psi(U_r)\right]_t}{\psi'(U_r)}(x,s)\,\zeta(x)\,\mathrm{d}x = \frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{A_n^t} V_r(x,s)\,\zeta''(x)\,\mathrm{d}x \tag{4.87}$$

for any $R \in (0,\delta)$ and $\zeta \in C_c^\infty(A_n^t)$.

The left-hand side of the above equality can be rewritten as follows:

$$\begin{aligned} &\frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{A_n^t} \frac{\left[\psi(U_r)\right]_t}{\psi'(U_r)}(x,s)\,\zeta(x)\,\mathrm{d}x \\ &= \frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{A_n^t} \left[\psi(U_r)\right]_t(x,s)\frac{\zeta(x)}{\psi'(U_r)(x,t)}\,\mathrm{d}x \\ &\quad + \frac{1}{2R}\int_{t-R}^{t+R}\mathrm{d}s\int_{A_n^t}\left\{\frac{\zeta(x)}{\left[\psi'(U_r)\right](x,s)} - \frac{\zeta(x)}{\left[\psi'(U_r)\right](x,t)}\right\}\left[\psi(U_r)\right]_t(x,s)\mathrm{d}x. \end{aligned} \tag{4.88}$$

Since $Q_n^t \subseteq A_{2n}$, arguing as in the proof of (4.85), we have

$$\frac{1}{\psi'(U_r)} \in C(\bar{Q}_n^t) \subseteq L^\infty(Q_n^t).$$

In particular, the function $1/\psi'(U_r)$ is uniformly continuous in the compact set $\bar{Q}_n^t$. Hence for any $\epsilon > 0$ there exists $0 < \sigma_\epsilon \leqq \delta$ such that for any $x \in A_n^t$ and $s \in (t-\delta, t+\delta)$ there holds

$$\left| \frac{1}{\big[\psi'(U_r)\big](x,t)} - \frac{1}{\big[\psi'(U_r)\big](x,s)} \right| \leqq \epsilon$$

whenever $|s-t| \leqq \sigma_\epsilon$. Hence for any R sufficiently small

$$\begin{aligned}
&\frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{A_n^t} \left| \frac{\zeta(x)}{\big[\psi'(U_r)\big](x,s)} - \frac{\zeta(x)}{\big[\psi'(U_r)\big](x,t)} \right| \left|\big[\psi(U_r)\big]_t(x,s)\right| \mathrm{d}x \\
&\quad \leqq \epsilon \, \frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{A_n^t} \left|\big[\psi(U_r)\big]_t(x,s)\zeta(x)\right| \, \mathrm{d}x \\
&\quad \leqq \epsilon \, \frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{\Omega} \left|\big[\psi(U_r)\big]_t(x,s)\zeta(x)\right| \, \mathrm{d}x.
\end{aligned}$$

By (4.80) and the arbitrariness of ϵ, letting $R \to 0$ in the above inequality gives

$$\lim_{R\to 0} \frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{A_n^t} \left| \frac{\zeta(x)}{\big[\psi'(U_r)\big](x,s)} - \frac{\zeta(x)}{\big[\psi'(U_r)\big](x,t)} \right| \left|\big[\psi(U_r)\big]_t(x,s)\right| \mathrm{d}x = 0. \tag{4.89}$$

Moreover, the function $\phi := \frac{\zeta(\cdot)}{\big[\psi'(U_r)\big](\cdot,t)}$ belongs to $C_c(A_n^t) \subseteq C(\bar{\Omega})$. Therefore by (4.79), (4.88) and (4.89) we have

$$\lim_{R\to 0} \frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{A_n^t} \frac{\big[\psi(U_r)\big]_t}{\psi'(U_r)}(x,s)\,\zeta(x)\,\mathrm{d}x = \int_{A_n^t} U_{rt}(x,t)\,\zeta(x)\,\mathrm{d}x$$

(see (4.70)). On the other hand, by (4.78) we also have

$$\lim_{R\to 0} \frac{1}{2R}\int_{t-R}^{t+R} \mathrm{d}s \int_{A_n^t} V_r(x,s)\,\zeta''(x)\,\mathrm{d}x = \int_{A_n^t} V_r(x,t)\,\zeta''(x)\,\mathrm{d}x.$$

Then letting $R \to 0$ in equality (4.87) the conclusion follows (see also (4.85)). □

5. Proof of Entropy Inequalities

To prove Theorem 2.4 we need the following result.

Proposition 5.1. *Let G be the function defined by (2.15), with $g \in C^1([0, \varphi(\alpha)])$ $g(0) = 0$. Then $G(U_r) \in C(\bar{Q})$. Moreover, if $g = 0$ in $[0, S_g]$ for some $S_g \in (0, \varphi(\alpha))$, then*

(i) *$G(U_{\kappa_j}) \to G(U_r)$ almost everywhere in Q, $\{\kappa_j\}$ being the sequence mentioned in Proposition 3.6;*

(ii) *$\left[G(U_r)\right]_t \in L^2(Q)$ and*

$$\left[G(U_r)\right]_t = g(\varphi(U_r))\, U_{rt} \quad \text{almost everywhere in } Q, \tag{5.1}$$

where U_{rt} is the function defined in Remark 4.2. Moreover,

$$[G(U_{\kappa_j})]_t \rightharpoonup \left[G(U_r)\right]_t \quad \text{in } L^2(Q). \tag{5.2}$$

Proof. For almost every $(x, t) \in Q$ we have

$$|G(U_r(x, t))| \leqq \int_0^\infty |g(\varphi(s))|\, \mathrm{d}x \leqq \|g'\|_\infty \int_0^\infty \varphi(s)\, \mathrm{d}s < \infty,$$

since $\varphi \in L^1(\mathbb{R})$ by assumption (H_1)-(i). Therefore, $G(U_r) \in L^\infty(Q)$. By Theorem 2.1-(ii) the function

$$\bar{G}(x, t) := \begin{cases} G(U_r) & \text{in } \bar{Q} \setminus \mathcal{S}, \\ \displaystyle\int_0^\infty g(\varphi(s))\, \mathrm{d}s & \text{in } \mathcal{S} \end{cases} \tag{5.3}$$

is a continuous representative in $\bar{Q}$ of $G(U_r)$, again denoted $G(U_r)$ for simplicity (recall that $|\mathcal{S}| = 0$ and G is continuous by definition).

Claim (i) follows from (3.48), (4.3) and the continuity of G. To prove (ii), let $0 < s_1(S_g) < s_2(S_g)$ denote the roots of the equation $\varphi(z) = S_g$, with $S_g \in (0, \varphi(\alpha))$. Set

$$E_{\kappa_j} := \left\{(x, t) \in Q \,\middle|\, U_{\kappa_j}(x, t) < s_2(S_g)\right\}.$$

Since $g = 0$ in $[0, S_g]$, we have

$$\left[G(U_{\kappa_j})\right]_t = \begin{cases} g(\varphi(U_{\kappa_j}))U_{\kappa_j t} & \text{in } E_{\kappa_j}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\left[G(U_{\kappa_j})\right]_t \in L^2(Q)$ (recall that $U_{\kappa_j t} \in L^2(Q)$ by definition) and

$$\iint_Q \left[G(U_{\kappa_j})\right]_t \zeta \,\mathrm{d}x\mathrm{d}t = \iint_{E_{\kappa_j}} \left[G(U_{\kappa_j})\right]_t \zeta \,\mathrm{d}x\mathrm{d}t \tag{5.4}$$

for any $\zeta \in L^2(Q)$.

We shall prove the following

Claim There exists $n \in \mathbb{N}$ (possibly depending on g) such that $E_{\kappa_j} \subseteq A_n$ for any $j \in \mathbb{N}$ sufficiently large, A_n being the open set defined in (4.60).

Since

- $U_{k_j t} \rightharpoonup U_{rt}$ in $L^2(A_n)$ by (4.67) (see also Remark 4.2),
- $\varphi(U_{k_j}) \to \varphi(U_r)$ uniformly in $\bar{A}_n$ (see Remark 4.1),
- g is continuous,

the above Claim and equality (5.4) imply claim (ii). Hence the conclusion follows.

It remains to prove the Claim. To this purpose, observe that by assumptions (A_k)-(i), (ii)

$$E_{\kappa_j} = \left\{(x,t) \in Q \,\middle|\, \psi_{\kappa_j}(U_{\kappa_j})(x,t) < \psi_{\kappa_j}(s_2(S_g))\right\},$$
$$\psi_{\kappa_j}(s_2(S_g)) \to \psi(s_2(S_g)).$$

Therefore there exists $\bar{\kappa} > 0$ such that for any $\kappa_j < \bar{\kappa}$

$$\psi_{\kappa_j}(s_2(S_g)) \leqq \psi(s_2(S_g)) + \sigma,$$

where $\sigma := \frac{\gamma - \psi(s_2(S_g))}{4}$. Hence

$$E_{\kappa_j} \subseteq \left\{(x,t) \in Q \,\middle|\, \psi_{\kappa_j}(U_{\kappa_j})(x,t) < \psi(s_2(S_g)) + \sigma\right\} \tag{5.5}$$

for any $\kappa_j < \bar{\kappa}$. On the other hand, since by (3.44) and (4.3)

$$\psi_{\kappa_j}\left(U_{\kappa_j}\right) \to \psi(U_r) \quad \text{in } C(\bar{Q}),$$

there exists $\tilde{\kappa} > 0$ such that for any $\kappa_j < \tilde{\kappa}$ and any $(x,t) \in Q$

$$\psi_{\kappa_j}\left(U_{\kappa_j}(x,t)\right) \geqq \psi(U_r)(x,t) - \sigma.$$

Therefore by (5.5)

$$E_{\kappa_j} \subseteq \left\{(x,t) \in Q \,\middle|\, \psi(U_r)(x,t) - \sigma < \psi(s_2(S_g)) + \sigma\right\} \tag{5.6}$$

for any $\kappa_j < \min\{\bar{\kappa}, \tilde{\kappa}\}$. It is easily checked that for any $n > \left[\frac{1}{2\sigma}\right]$

$$\left\{(x,t) \in Q \,\middle|\, \psi(U_r)(x,t) - \sigma < \psi(s_2(S_g)) + \sigma\right\} \subseteq A_n. \tag{5.7}$$

Then by (5.6)–(5.7) the Claim follows. This completes the proof. □

Now we can prove Theorem 2.4.

Proof of Theorem 2.4. It follows by Proposition 5.1 that $G(U_r) \in C(\bar{Q})$. Let us prove inequalities (2.16), assuming first $g = 0$ in $[0, S_g]$ for some $S_g \in (0, \varphi(\alpha))$ (steps (i)–(iii)). This assumption will be removed in step (iv).

(i) Let $F \subseteq (0, T)$, $|F| = 0$, be the set considered in Proposition 4.10. Fix any $t \in (0, T) \setminus F$ and any $g \in C^1([0, \varphi(\alpha)])$ such that $g' \geqq 0$, $g = 0$ in $[0, S_g]$ for some $S_g \in (0, \varphi(\alpha))$. Then $g(\varphi(U_r))(\cdot, t) \in C(\bar{\Omega})$ (see Remark 2.1) and there holds:

$$\begin{aligned} &\operatorname{supp} g(\varphi(U_r)(\cdot, t)) \subseteq \left\{x \in \bar{\Omega} \,|\, \varphi(U_r)(x, t) \geqq S_g\right\} \\ &\quad \subseteq \left\{x \in \Omega \,|\varphi(U_r)(x, t) > S_g/2\right\} \subseteq \left\{x \in \Omega \,|\, \psi(U_r)(x, t) < \psi(s_2(S_g/2))\right\} \end{aligned} \tag{5.8}$$

(where $s_2(S_g/2)$ is the second root of the equation $S_g/2 = \varphi(z)$, $S_g \in (0, \varphi(\alpha))$); here we have made use of the equality $\varphi(U_r) = 0$ on $\partial\Omega \times (0, T)$ and of the properties of the functions φ, ψ. Since $\psi(s_2(S_g/2)) < \gamma$, by the continuity of $\psi(U_r)(\cdot, t)$ in Ω there exists $n \in \mathbb{N}$ such that

$$\left\{x \in \Omega \,|\, \psi(U_r)(x, t) < \psi(s_2(S_g/2))\right\} \subseteq A_n^t. \tag{5.9}$$

From (5.8)–(5.9) we obtain $\operatorname{supp} g(\varphi(U_r)(\cdot, t)) \subseteq A_n^t$, thus $g(\varphi(U_r))U_{rt} \in L^2(\Omega)$ by Proposition 4.10.

(ii) Let V_{rxx} be the function defined in Remark 4.2. Arguing as in (i) above, using (4.75) and the boundary condition $V_r(\cdot, t) = 0$ on $\partial\Omega$, it is similarly seen that

$$\operatorname{supp} g(V_r(\cdot, t)) \subseteq A_{n_g}^t \tag{5.10}$$

for some $n_g \in \mathbb{N}$ sufficiently large. Then, since $V_r \in H^2(A_n^t)$ and $[V_r(\cdot, t)]_{xx} = V_{rxx}(\cdot, t)$ almost everywhere in A_n^t for any n by Proposition 4.10, the function

$$F_g(x) := \begin{cases} g(V_r(x, t))V_{rxx}(\cdot, t) & \text{for } x \in A_{n_g}^t \\ 0 & \text{for } x \in \overline{\Omega} \setminus A_{n_g}^t \end{cases}$$

belongs to the space $L^2(\Omega)$; moreover, a standard calculation shows that for any $\eta \in H_0^1(\Omega)$

$$\begin{aligned} &\int_\Omega \left\{g(V_r(x, t))\, \eta'(x)\, V_{rx}(x, t) + \eta(x)\, g'(V_r(x, t))\, V_{rx}^2(x, t)\right\} \mathrm{d}x \\ &\quad = -\int_\Omega \eta(x)\, g(V_r(x, t))\, V_{rxx}(x, t)\, \mathrm{d}x. \end{aligned} \tag{5.11}$$

(iii) By equalities (4.76) and (5.11) we have

$$\begin{aligned} &\int_\Omega [g(\varphi(U_r))U_{rt}\, \zeta]\, (x, t)\, \mathrm{d}x \\ &\quad = \int_\Omega \left\{\left[g(\varphi(U_r)) - g(V_r)\right] \frac{[\psi(U_r)]_t}{\psi'(U_r)} \zeta\right\} (x, t)\, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_\Omega \big[g(V_r)V_{rxx}\zeta\big](x,t)\,\mathrm{d}x \\
&= \int_\Omega \left\{\big[g(\varphi(U_r)) - g(V_r)\big]\frac{V_r - \varphi(U_r)}{\epsilon\psi'(U_r)}\zeta\right\}(x,t)\,\mathrm{d}x \\
&\quad - \int_\Omega \big[g'(V_r)(V_{rx})^2\zeta + g(V_r)V_{rx}\zeta_x\big](x,t)\,\mathrm{d}x \\
&\leqq - \int_\Omega \big[g'(V_r)(V_{rx})^2\zeta + g(V_r)V_{rx}\zeta_x\big](x,t)\mathrm{d}x
\end{aligned} \tag{5.12}
$$

for any t as above and any $\zeta \in C^1([0,T]; H^1_0(\Omega))$, $\zeta \geqq 0$ (where we have made use of (4.70) and of the assumption $g' \geqq 0$).

Observe that the right-hand side of inequality (5.12) belongs to the space $L^1(0,T)$, since $V_r \in L^2((0,T); H^1_0(\Omega)) \cap L^\infty(Q)$. Moreover, by Proposition 5.1-(ii) for any $\zeta \in C^1([0,T]; H^1_0(\Omega))$ the map

$$
t \longmapsto \int_\Omega \big[G(U_r)\,\zeta\big](x,t)\,\mathrm{d}x =: \mathcal{G}(t)
$$

belongs to the space $H^1(0,T)$, with weak derivative

$$
\mathcal{G}'(t) = \int_\Omega [g(\varphi(U_r))U_{rt}\zeta]\,(x,t)\,\mathrm{d}x + \int_\Omega \big[G(U_r)\,\zeta_t\big](x,t)\,\mathrm{d}x.
$$

Integrating the above equality between t_1 and t_2, with $0 \leqq t_1 < t_2 \leqq T$, we obtain

$$
\begin{aligned}
&\int_\Omega G(U_r)(x,t_2)\zeta(x,t_2)\,\mathrm{d}x - \int_\Omega G(U_r)(x,t_1)\zeta(x,t_1)\,\mathrm{d}x \\
&\quad = \int_{t_1}^{t_2}\int_\Omega \{g(\varphi(U_r))U_{rt}\zeta + G(U_r)\zeta_t\}\,\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{5.13}
$$

for any g and ζ as above. Combining (5.12) and (5.13) we obtain inequalities (2.16) for any smooth, nondecreasing g such that $g = 0$ in $[0, S_g]$ for some $S_g \in (0, \varphi(\alpha))$.

(iv) Finally, let us remove the auxiliary assumption $g = 0$ in $[0, S_g] (S_g \in (0, \varphi(\alpha)))$. To this purpose, fix any $g \in C^1([0,\varphi(\alpha)])$, $g' \geqq 0$, $g(0) = 0$. Choose any sequence $\{g_n\} \subseteq C^1([0,\varphi(\alpha)])$ such that $g_n' \geqq 0$, $g_n = 0$ in $[0, S_n]$ for some $S_n \in (0, \varphi(\alpha))$, and

$$
g_n \to g \quad \text{in } C^1([0,\varphi(\alpha)]). \tag{5.14}
$$

By the Dominated Convergence Theorem it is easily seen that

$$
g_n(\varphi) \to g(\varphi) \quad \text{in } L^1(\mathbb{R}), \tag{5.15}
$$

$$
\big[G_n(U_r)\big](\cdot,t) \to \big[G(U_r)\big](\cdot,t) \quad \text{in } L^1(\Omega) \qquad (t \in [0,T]), \tag{5.16}
$$

$$
G_n(U_r) \to G(U_r) \quad \text{in } L^1(Q). \tag{5.17}
$$

Moreover, we also have:

$$
g_n(V_r)V_{rx} \to g(V_r)V_{rx} \quad \text{in } L^2(Q), \tag{5.18}
$$

$$
g_n'(V_r) \to g'(V_r) \quad \text{in } L^2(Q). \tag{5.19}
$$

On the other hand, the above steps of the proof ensure that

$$\int_\Omega G_n(U_r)(x,t_2)\zeta(x,t_2)\mathrm{d}x - \int_\Omega G_n(U_r)(x,t_1)\zeta(x,t_1)\mathrm{d}x$$
$$\leqq \int_{t_1}^{t_2}\int_\Omega \left[G_n(U_r)\zeta_t - g_n(V_r)V_{rx}\zeta_x - g_n'(V_r)(V_{rx})^2\zeta\ \right]\mathrm{d}x\mathrm{d}t$$

for any $\zeta \in C^1([0,T]; H_0^1(\Omega))$, $\zeta \geqq 0$. In view of (5.15)–(5.19), letting $n \to \infty$ in the above inequality the conclusion follows. □

Proof of Theorem 2.5. Consider the sequence $\{g_n\} \subseteq C^1([0,\varphi(\alpha)])$, defined as follows:

$$g_n(s) = \begin{cases} 0 & \text{if } s \in [0, 1/2n] \\ 2ns - 1 & \text{if } s \in (1/2n, 1/n) \\ 1 & \text{if } s \in [1/n, \varphi(\alpha)]. \end{cases}$$

Denote by Γ_n the function (2.15) with $g = g_n$, namely

$$\Gamma_n(z) := \int_0^z g_n(\varphi(s))\mathrm{d}s \qquad (z \in \mathbb{R}).$$

Since $\Gamma_n(U_r)$ and $\psi(U_r)$ are continuous in $\bar{Q}$, and since $\psi(U_r)(\cdot,0) = \psi(U_{0r})$ in $\bar{\Omega}$ (see Definition 2.1-(ii)), we obtain:

$$\left[\Gamma_n(U_r)\right](x,0) = \Gamma_n(U_{0r}(x))$$

for almost every $x \in \Omega$. Thus, although $g_n \notin C^1([0,\varphi(\alpha)])$, by standard convolution arguments, writing the entropy inequalities (2.16) for $g = g_n$ and $t_1 = 0$ and $t_2 = T$ gives:

$$\iint_Q [\Gamma_n(U_r)\zeta_t - g_n(V_r)V_{rx}\zeta_x]\mathrm{d}x\mathrm{d}t \tag{5.20}$$
$$\geqq -\int_\Omega [\Gamma_n(U_{0r})](x)\zeta(x,0)\,\mathrm{d}x$$

for any $\zeta \in C^1([0,T]; H_0^1(\Omega))$, $\zeta \geqq 0$, $\zeta(\cdot,T) = 0$ in Ω.

Since $0 \leqq \Gamma_n(U_r) \leqq U_r$, U_r belongs to $L^1(Q)$ and $\Gamma_n(U_r) \to U_r$ almost everywhere in Q, by the Lebesgue Theorem we obtain

$$\iint_Q \Gamma_n(U_r)\zeta_t\mathrm{d}x\mathrm{d}t \to \iint_Q U_r\zeta_t\mathrm{d}x\mathrm{d}t \tag{5.21}$$

for any ζ as above. Moreover, there holds

$$g_n(V_r)V_{rx} = \left[\int_0^{V_r} g_n(s)\mathrm{d}s\right]_x, \tag{5.22}$$

and

$$\|g_n(V_r)V_{rx}\|_{L^2(Q)} \leqq \|V_{rx}\|_{L^2(Q)}.$$

Therefore the sequence $\{g_n(V_r)V_{rx}\}$ is weakly relatively compact in $L^2(Q)$. By (5.22), since for almost every $(x,t) \in Q$ there holds

$$\int_0^{V_r(x,t)} g_n(s)\mathrm{d}s \to V_r(x,t)$$

as $n \to \infty$, we obtain

$$g_n(V_r)V_{rx} \rightharpoonup V_{rx} \quad \text{in } L^2(Q). \tag{5.23}$$

Using (5.21) and (5.23), passing to the limit as $n \to \infty$ in (5.20) gives:

$$\iint_Q \{U_r\zeta_t - V_{rx}\zeta_x\}\,\mathrm{d}x\mathrm{d}t \geqq -\int_\Omega U_{0r}(x)\,\zeta(x,0)\,\mathrm{d}x \tag{5.24}$$

for any $\zeta \in C^1([0,T]; H_0^1(\Omega))$, $\zeta \geqq 0$, $\zeta(\cdot,T) = 0$ in Ω (in fact, it can be easily seen that $\Gamma_n(U_{0r}) \to U_{0r}$ in $L^1(\Omega)$ as $n \to \infty$). Combining (2.5) and (5.24), this gives

$$\int_0^T \langle U_s(\cdot,t), \zeta_t(\cdot,t)\rangle_\Omega\,\mathrm{d}t \leqq -\langle U_{0s}, \zeta(\cdot,0)\rangle_\Omega\,, \tag{5.25}$$

for any ζ as above.

Let us prove (2.18), the proof of (2.17) being formally analogous. Fix any $0 < t_1 < t_2 < T$ and consider $\chi_r \in \mathrm{Lip}([0,T])$ defined as follows:

$$\chi_r(t) := \begin{cases} \frac{1}{r}(t - t_1 + \frac{r}{2}) & \text{if } t \in (t_1 - \frac{r}{2}, t_1 + \frac{r}{2}) \\ 1 & \text{if } t \in [t_1 + \frac{r}{2}, t_2 - \frac{r}{2}] \\ -\frac{1}{r}(t - t_2 - \frac{r}{2}) & \text{if } t \in (t_2 - \frac{r}{2}, t_2 + \frac{r}{2}) \\ 0 & \text{otherwise,} \end{cases}$$

with $r \in (0, t_2 - t_1)$ such that $[t_1 - \frac{r}{2}, t_2 + \frac{r}{2}] \subset (0,T)$. For any $\eta \in H_0^1(\Omega)$, $\eta \geqq 0$, choose $\eta(x)\chi_r(t)$ as test function in inequality (5.25). We obtain:

$$\frac{1}{r}\int_{t_1-\frac{r}{2}}^{t_1+\frac{r}{2}} \langle U_s(\cdot,t), \eta\rangle_\Omega\,\mathrm{d}t \leqq \frac{1}{r}\int_{t_2-\frac{r}{2}}^{t_2+\frac{r}{2}} \langle U_s(\cdot,t), \eta\rangle_\Omega\,\mathrm{d}t.$$

For almost every $0 < t_1 < t_2 < T$, letting $r \to 0$ in the above inequality gives:

$$\langle U_s(\cdot,t_1), \eta\rangle_\Omega \leqq \langle U_s(\cdot,t_2), \eta\rangle_\Omega$$

for any η as above. Then the conclusion follows. □

6. Proof of Regularity Results

Let us first prove Proposition 2.6.

Proof of Proposition 2.6. We shall distinguish two cases: $(\alpha) U_0$ satisfies (A_1), or $(\beta) U_0$ satisfies (A_2).

(α) In this case $U_{0s} \equiv 0$, $U_0 \equiv U_{0r} \in L^1(\Omega)$ and $\psi(U_0) \in H_0^1(\Omega)$. Let us first prove the following

Claim Let $\sigma \in (0, 1/2]$. Then $U_0 \in H_0^1(\Omega)$.

Clearly, by standard regularization results the above Claim implies the conclusion (if $\sigma \in (0, 1/2]$). In order to prove this, define for any $\kappa > 0$

$$\tilde{U}_{0\kappa}(x) := T_{1/\kappa}(U_0(x)), \qquad (x \in \Omega) \tag{6.1}$$

where

$$T_{1/\kappa}(s) := \begin{cases} s & \text{if } s \in [0, 1/\kappa), \\ 1/\kappa & \text{if } s \in [1/\kappa, \infty). \end{cases} \tag{6.2}$$

Defining

$$L_\kappa(z) := T_{1/\kappa}(\psi^{-1}(z)) = \begin{cases} \psi^{-1}(z) & \text{if } z \in [0, \psi(1/\kappa)), \\ 1/\kappa & \text{if } z \in [\psi(1/\kappa), \gamma], \end{cases}$$

we also have

$$\tilde{U}_{0\kappa}(x) = L_\kappa\big(\psi(U_0)(x)\big) \tag{6.3}$$

for almost every $x \in \Omega$. Moreover, L_κ is Lipschitz continuous in $[0, \gamma]$ and $L_\kappa(0) = 0$ (recall that the function $\psi^{-1} \in C^1([0, \psi(1/\kappa)])$ since $\psi' > 0$ in $\mathbb{R}$ by assumption (H_2)-(i)). By standard results on Sobolev functions, since $\psi(U_0) \in H_0^1(\Omega)$, equality (6.3) and the above considerations guarantee that $\tilde{U}_{0\kappa} \in H_0^1(\Omega)$ for any $\kappa > 0$, with weak derivative

$$\tilde{U}_{0\kappa x} = \frac{\big[\psi(U_0)\big]_x}{\psi'(U_0)} \chi_{E_\kappa}, \qquad E_\kappa := \{x \in \Omega \mid \psi(U_0)(x) < \psi(1/\kappa)\}. \tag{6.4}$$

Set

$$z_\kappa(x) := \ln\Big[1 + \tilde{U}_{0\kappa}(x)\Big] \qquad (x \in \Omega).$$

Then $z_\kappa \in H_0^1(\Omega)$ for any $\kappa > 0$ and

$$\begin{aligned} \int_\Omega |z_{\kappa x}| \, \mathrm{d}x &= \int_\Omega \frac{|\tilde{U}_{0\kappa x}|}{1 + \tilde{U}_{0\kappa}} \, \mathrm{d}x = \int_{E_\kappa} \frac{\big|\big[\psi(U_0)\big]_x\big| \, (1 + U_0)^\sigma}{\psi'(U_0)\, (1 + U_0)^{\sigma+1}} \, \mathrm{d}x \\ &\leqq \frac{1}{l_1} \int_\Omega (1 + U_0)^\sigma \, \big|\big[\psi(U_0)\big]_x\big| \, \mathrm{d}x \\ &\leqq \frac{1}{l_1} \left(\int_\Omega (1 + U_0)^{2\sigma} \, \mathrm{d}x\right)^{1/2} \left(\int_\Omega \big[\psi(U_0)\big]_x^2 \mathrm{d}x\right)^{1/2} \end{aligned} \tag{6.5}$$

(where we have made use of assumption (H_2')). Since by assumption $\sigma \leqq 1/2$, there holds

$$\int_\Omega (1 + U_0)^{2\sigma} \, \mathrm{d}x \leqq \int_\Omega (1 + U_0) \, \mathrm{d}x \leqq C. \tag{6.6}$$

Then from (6.5) we obtain $\|z_{\kappa x}\|_{L^1(\Omega)} \leqq C$ for some $C > 0$. Since $z_\kappa = 0$ on $\partial\Omega$, it follows that the family $\{z_\kappa\}$ is uniformly bounded in $W_0^{1,1}(\Omega)$, thus in

$L^\infty(\Omega)$. By the very definition of z_κ, we also have $\|\tilde{U}_{0\kappa}\|_{L^\infty(\Omega)} \leqq \bar{C}$ for some constant $\bar{C} > 0$. Therefore, there exists $\bar{\kappa} > 0$ such that for any $\kappa < \bar{\kappa}$ we have $U_0 = \tilde{U}_{0\kappa}$ almost everywhere in Ω (see (6.1)). Since $\tilde{U}_{0\kappa} \in H_0^1(\Omega)$ for any κ, the above equality implies that $U_0 \in H_0^1(\Omega)$. This proves the Claim.

Let us now suppose $\sigma > 1/2$. In this case we construct a family $\{\tilde{U}_{0\kappa}\} \subseteq H_0^1(\Omega)$ along which the convergences in (H_3)-(ii) hold. By standard regularization arguments, this guarantees the existence of a family of smooth functions with the same properties. Hence the conclusion follows in this case, too.

For any $\kappa > 0$ set

$$\tilde{U}_{0\kappa}(x) := T_{1/\kappa^\theta}(U_0(x)) \qquad (x \in \Omega),$$

where the function T_{1/κ^θ} is defined by (6.2) with $1/\kappa$ replaced by $1/\kappa^\theta$, $\theta > 0$ to be chosen. Arguing as in the case $\sigma \in (0, 1/2]$ shows that $\tilde{U}_{0\kappa} \in H_0^1(\Omega)$ for any $\kappa > 0$, with weak derivative given by (6.4) where $1/\kappa$ is replaced by $1/\kappa^\theta$. Since $\tilde{U}_{0\kappa} \to U_0$ as $\kappa \to 0$ almost everywhere in Ω, and $\tilde{U}_{0\kappa} \leqq U_0 \in L^1(\Omega)$ for any $\kappa > 0$, there holds

$$\tilde{U}_{0\kappa} \to U_0 \ \text{ in } \ L^1(\Omega). \tag{6.7}$$

Hence

$$\int_\Omega U_{0\kappa}\, \zeta \, \mathrm{d}x \ \to \ \langle U_0, \zeta\rangle_{\bar{\Omega}} \quad \text{for any} \ \zeta \in C(\bar{\Omega})$$

namely, condition (a) in (H_3)-(ii) follows.

Let us address condition (b) therein. To this purpose, observe that by (6.1), (6.2) and (6.4) with κ replaced by κ^θ, there holds

$$\int_\Omega \left[\psi(\tilde{U}_{0\kappa})\right]_x^2 \mathrm{d}x = \int_{E_{\kappa^\theta}} \left[\psi(U_0)\right]_x^2 \mathrm{d}x \leqq \int_\Omega \left[\psi(U_0)\right]_x^2 \mathrm{d}x. \tag{6.8}$$

Therefore, the family $\{\psi(\tilde{U}_{0\kappa})\}$ is weakly relatively compact in $H_0^1(\Omega)$. Moreover, by (6.7) for any converging subsequence $\{\psi(\tilde{U}_{0\kappa_j})\} \subseteq \{\psi(\tilde{U}_{0\kappa})\}$ there holds

$$\psi(\tilde{U}_{0\kappa_j}) \rightharpoonup \psi(U_0) \ \text{ in } \ H_0^1(\Omega).$$

Hence condition (b) follows.

It remains to prove condition (c). To this end, observe that by (6.4) and assumption (H_2')

$$\begin{aligned}
\left\|\kappa \tilde{U}_{0\kappa}\right\|^2_{H_0^1(\Omega)} &= \kappa^2 \int_{E_{\kappa^\theta}} \frac{\left[\psi(U_0)\right]_x^2}{\left[\psi'(U_0)\right]^2} \mathrm{d}x \\
&\leqq \frac{\kappa^2}{l_1^2} \int_{E_{\kappa^\theta}} \left[\psi(U_0)\right]_x^2 \, (1 + U_0)^{2\sigma+2} \, \mathrm{d}x \\
&\leqq \frac{\kappa^2}{l_1^2} \left(1 + \frac{1}{\kappa^\theta}\right)^{2\sigma+2} \int_\Omega \left[\psi(U_0)\right]_x^2 \mathrm{d}x.
\end{aligned} \tag{6.9}$$

Choosing $\theta < 1/(\sigma+1)$, from the above inequality we obtain the convergence in part (c) of (H_3)-(ii). This concludes the proof in case (α).

(β) Assume for simplicity that

$$U_{0s} = \delta(\cdot - x_0) \qquad (x_0 \in \Omega).$$

For any $\kappa > 0$ set

$$I_\kappa := [x_0 - \kappa^\theta, x_0 + \kappa^\theta],$$

with $\theta > 0$ to be chosen. Then, since $\psi(U_{0r}) \in W_0^{1,\infty}(\Omega)$ and $\psi(U_{0r})(x_0) = \gamma$ by the condition supp $U_{0s} \subseteq \mathcal{S}_0$, there exists $L > 0$ such that for any $x \in I_\kappa$

$$\psi(U_{0r})(x) \geqq \gamma - L\kappa^\theta. \tag{6.10}$$

On the other hand, by assumption (H_2') we also have

$$l_1\sigma^{-1} - \frac{l_1\sigma^{-1}}{(1+s)^\sigma} \leqq \psi(s) \leqq \gamma - \frac{\gamma}{(1+s)^\sigma}$$

for any $s \geqq 0$. Hence by inequality (6.10) we obtain for almost every $x \in I_\kappa$

$$1 + U_{0r}(x) \geqq \frac{\left(\gamma L^{-1}\right)^{1/\sigma}}{\kappa^{\theta/\sigma}} = C_\sigma\, \kappa^{-\theta/\sigma}. \tag{6.11}$$

Next, for any $\kappa > 0$ and almost every $x \in \Omega$ set

$$\begin{aligned}
&\tilde U_{0r\kappa}(x) := T_{C_\sigma \kappa^{-\theta/\sigma}-1}(U_{0r}(x));\\
&\tilde U_{0s\kappa}(x) := \begin{cases} \kappa^{-2\theta}\left(x - x_0 + \kappa^\theta\right) & \text{if } x \in [x_0 - \kappa^\theta, x_0],\\ \kappa^{-2\theta}\left(-x + x_0 + \kappa^\theta\right) & \text{if } x \in (x_0, x_0 + \kappa^\theta],\\ 0 & \text{otherwise;} \end{cases}\\
&\tilde U_{0\kappa}(x) := \tilde U_{0r\kappa}(x) + \tilde U_{0s\kappa}(x)
\end{aligned}$$

($\theta > 0$ to be chosen).

As in the above case (α) we have $\tilde U_{0r\kappa} \in H_0^1(\Omega)$ and $\tilde U_{0s\kappa} = 0$ on $\partial\Omega$, thus $\tilde U_{0s\kappa} \in H_0^1(\Omega)$, for any κ sufficiently small. Let us show that the family $\{\tilde U_{0\kappa}\}$ satisfies the convergences in (H_3)-(ii). Arguing as in (6.7), by the definition of the family $\{\tilde U_{0s\kappa}\}$ there holds

$$\tilde U_{0r\kappa} \to U_{0r} \;\text{ in }\; L^1(\Omega), \tag{6.12}$$

$$\tilde U_{0s\kappa} \overset{*}{\rightharpoonup} \delta(\cdot - x_0) \;\text{ in }\; \mathcal{M}(\mathbb{R}) \tag{6.13}$$

as $\kappa \to 0$. From (6.12) and (6.13), the convergence in (a) follows.

Concerning (b), observe that since supp $\tilde U_{0s\kappa} \subseteq I_\kappa$ and $\tilde U_{0r\kappa} = C_\sigma\kappa^{-\theta/\sigma} - 1$ in I_κ (see (6.11)), there holds

$$[\psi(\tilde U_{0\kappa})]_x(x) = \begin{cases} \psi'(\tilde U_{0r\kappa}(x))\tilde U_{0r\kappa x}(x) & \text{if } x \notin I_\kappa,\\ \psi'(\tilde U_{0r\kappa}(x) + \tilde U_{0s\kappa}(x))\tilde U_{0s\kappa x}(x) & \text{if } x \in I_\kappa. \end{cases} \tag{6.14}$$

Using (6.14) and arguing as in (6.8) we get:

$$\int_{\Omega\setminus I_\kappa} \left[\psi(\tilde{U}_{0\kappa})\right]_x^2 \mathrm{d}x = \int_\Omega \left[\psi(\tilde{U}_{0r\kappa})\right]_x^2 \mathrm{d}x \leqq \|\psi(U_{0r})\|^2_{H_0^1(\Omega)}. \tag{6.15}$$

On the other hand, by assumption (H_2') and (6.14) we have

$$\begin{aligned}
&\int_{I_\kappa} \left[\psi(\tilde{U}_{0\kappa})\right]_x^2 \mathrm{d}x = \int_{I_\kappa} \left[\psi'(\tilde{U}_{0\kappa})\right]^2 (\tilde{U}_{0s\kappa x})^2 \,\mathrm{d}x \\
&\leqq \frac{1}{\kappa^{4\theta}} \int_{I_\kappa} \gamma^2\sigma^2 \left(1 + \tilde{U}_{0r\kappa} + \tilde{U}_{0s\kappa}\right)^{-2(\sigma+1)} \mathrm{d}x \\
&\leqq \frac{2\gamma^2\sigma^2}{\kappa^{4\theta}} \int_{x_0}^{x_0+\kappa^\theta} \left(C_\sigma \kappa^{-\theta/\sigma} + \kappa^{-\theta} + \kappa^{-2\theta} x_0 - \kappa^{-2\theta} x\right)^{-2(\sigma+1)} \mathrm{d}x \\
&= \frac{2\gamma^2\sigma^2}{(2\sigma+1)\kappa^{2\theta}} \left[\frac{1}{\left(C_\sigma \kappa^{-\theta/\sigma}\right)^{2\sigma+1}} - \frac{1}{\left(C_\sigma \kappa^{-\theta/\sigma} + \kappa^{-\theta}\right)^{2\sigma+1}}\right] \leqq \tilde{C} \kappa^{\theta/\sigma} \qquad (6.16)
\end{aligned}$$

for some $\tilde{C} > 0$. By (6.15) and (6.16) the family $\{\psi(\tilde{U}_{0\kappa})\}$ is uniformly bounded in $H_0^1(\Omega)$, hence the convergence (b) follows. To prove (c), observe that by (H_2'), arguing as in (6.9) we obtain

$$\kappa^2 \|\tilde{U}_{0r\kappa}\|^2_{H_0^1(\Omega)} \leqq \frac{\kappa^2}{l_1^2} \left(\frac{C_\sigma}{\kappa^{\theta/\sigma}}\right)^{2\sigma+2} \int_\Omega \left[\psi(U_{0r})\right]_x^2 \mathrm{d}x$$

and

$$\kappa^2 \|\tilde{U}_{0s\kappa}\|^2_{H_0^1(\Omega)} = \kappa^2 \int_{I_\kappa} (\tilde{U}_{0s\kappa x})^2 \,\mathrm{d}x = 2\kappa^{2-3\theta}.$$

Choosing $\theta < \min\left\{\sigma/(\sigma+1), 2/3\right\}$ the convergence (c) follows. This completes the proof. $\square$

Now we can prove Proposition 2.7.

Proof of Proposition 2.7. (i) $\Rightarrow$ (ii) Obvious.
(ii) $\Rightarrow$ (i) The same proof of the Claim in the proof of Proposition 2.6 shows that $U_{0r} = \tilde{U}_{0\bar{\kappa}}$ for some $\bar{\kappa} > 0$, where $\tilde{U}_{0\kappa} := T_{1/\kappa}(U_{0r})$ ($\kappa > 0$; see (6.1)). Hence $U_{0r} \in H_0^1(\Omega)$, and

$$\max_{x\in\bar{\Omega}} \left[\psi(U_{0r})\right](x) = \psi\left(\frac{1}{\bar{\kappa}}\right) < \gamma.$$

Hence

$$\mathcal{S}_0 = \emptyset \quad \Rightarrow \quad \operatorname{supp} U_{0s} = \emptyset.$$

Therefore $U_{0s} = 0$, thus the conclusion follows. $\square$

Let us prove Theorem 2.8.

Proof of Theorem 2.8. Arguing by contradiction, and using the continuity of $\psi(U_r)$ in the rectangle $\bar{Q}$ (see Remark 2.1), let there exist $(x_0, t_0) \in \bar{Q}, t_0 > 0$ such that

$$\psi(U_r)(x_0, t_0) = \gamma$$

(observe that $\psi(U_r)(x, 0) = \psi(U_0(x)) < \gamma$ for any $x \in \bar{\Omega}$, since $U_0 \in H_0^1(\Omega)$). Observe that by the continuity of $\psi(U_r)$ in $\bar{Q}$ and by Definition 2.1, for any $\epsilon > 0$ there exists $t_1 \in (0, T)$ such that

$$\psi(U_r)(x_0, t_1) \geqq \gamma - \epsilon, \tag{6.17}$$

and

$$\psi(U_r)(\cdot, t_1) \in H_0^1(\Omega).$$

Then the argument used to prove the Claim in the proof of Proposition 2.6 shows that

$$U_r(\cdot, t_1) = T_{1/\bar{\kappa}}(U_r(\cdot, t_1)),$$

where

$$\begin{aligned} \frac{1}{\bar{\kappa}} &= \exp\left\{ l_1^{-1} \left(\int_\Omega (1 + U_r(x, t_1)) \mathrm{d}x \right)^{1/2} \left(\int_\Omega [\psi(U_r)]_x^2(x, t_1) \mathrm{d}x \right)^{1/2} \right\} - 1 \\ &\leqq \exp\left\{ l_1^{-1} \|1 + U_0\|_{\mathcal{M}(\bar{\Omega})} + l_1^{-1} \|\psi(U_r)\|_{L^\infty(0,T;H_0^1(\Omega))} \right\} - 1 =: \bar{L} \end{aligned}$$

(see (6.5)). Hence $U_r(\cdot, t_1) \equiv \psi^{-1}(\psi(U_r)(\cdot, t_1)) \in H_0^1(\Omega)$ and

$$\max_{x \in \bar{\Omega}} \left[\psi(U_r)(x, t_1)\right] \leqq \psi\left(\bar{L}\right) < \gamma - \epsilon$$

for any $\epsilon > 0$ sufficiently small. From the contradiction (see (6.17)) we have $\mathcal{S} = \emptyset$—hence $U_s = 0$ in $\mathcal{M}(\bar{Q})$ by (2.4)—and claim (ii) follows. Claim (i) is a consequence of (2.19), since

$$\|U\|_{L^\infty(Q)} \leqq \psi^{-1}(\gamma^*) < \infty.$$

Concerning (iii) observe that, since $\psi(U) = \psi(U_r) \in L^\infty((0, T); H_0^1(\Omega))$ (see Definition 2.1) and ψ' is strictly positive on every bounded subset of $\mathbb{R}$,

$$\begin{aligned} \|U(\cdot, t)\|_{L^\infty((0,T);H_0^1(\Omega))} &= \left\| \psi^{-1}\left\{\left[\psi(U)\right](\cdot, t)\right\} \right\|_{L^\infty((0,T);H_0^1(\Omega))} \\ &\leqq \frac{1}{\min_{s \in [0, \psi^{-1}(\gamma^*)]} \psi'(s)} \left\| \left[\psi(U)\right](\cdot, t) \right\|_{L^\infty((0,T);H_0^1(\Omega))} \leqq C. \end{aligned}$$

Since $U \in L^\infty(Q)$ and $\left[\psi(U)\right]_t \in L^2((0, T); H_0^1(\Omega))$, it is similarly seen that $U_t \in L^2((0, T); H_0^1(\Omega))$. This completes the proof. □

Let us finally mention the following result. The proof is analogous to that of Theorem 2.1 and Proposition 4.1, thus we omit it.

Proposition 6.1. *Let* $\mu \in \mathcal{M}^+(\bar{\Omega})$ *and* $\psi \in C^\infty(\mathbb{R})$ *satisfy assumption* (H_2). *Let there exists a family* $\{\mu_\kappa\} \subseteq C_c^\infty(\Omega)$, $\mu_\kappa \geqq 0 (\kappa > 0)$, *such that as* $\kappa \to 0$

$$\psi(\mu_\kappa) \rightharpoonup \psi(\mu_r) \ \text{in} \ H_0^1(\Omega), \tag{6.18}$$

$$\int_\Omega \mu_\kappa(x)\zeta(x)\,\mathrm{d}x \to \langle \mu, \zeta \rangle_{\bar{\Omega}} \tag{6.19}$$

for any $\zeta \in C(\bar{\Omega})$. *Then:*

(i) $\psi(\mu_r) \in H_0^1(\Omega)$;
(ii) *the set* $\mathcal{S}^\mu := \{x \in \bar{\Omega} \mid [\psi(\mu_r)](x) = \gamma\}$ *is closed and has zero Lebesgue measure;*
(iii) $\operatorname{supp} \mu_s \subseteq \mathcal{S}^\mu$;
(iv) $\mu_r \in C(\bar{\Omega} \setminus \mathcal{S}^\mu)$.

References

1. AMANN, H.: Time-delayed Perona–Malik type problems. *Acta. Math. Univ. Comenianae* **76**, 15–38 (2007)
2. BARENBLATT, G.I., BERTSCH, M., DAL PASSO, R., UGHI, M.: A degenerate pseudoparabolic regularization of a nonlinear forward–backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow. *SIAM J. Math. Anal.* **24**, 1414–1439 (1993)
3. BROKATE, M., SPREKELS, J.: *Hysteresis and Phase Transitions*. Appl. Math. Sci., vol. 121. Springer, 1996
4. EVANS, L.C.: A survey of entropy methods for partial differential equations. *Bull. Am. Math. Soc.* **41**, 409–438 (2004)
5. EVANS, L.C., PORTILHEIRO, M.: Irreversibility and hysteresis for a forward–backward diffusion equation. *Math. Mod. Methods Appl. Sci.* **14**, 1599–1620 (2004)
6. GIAQUINTA, M., MODICA, G., SOUCEK, J.: *Cartesian Currents in the Calculus of Variations*. Springer, 1998
7. GILBARG, D., TRUDINGER, N.: *Elliptic Partial Differential Equations of Second Order*. Springer, 1983
8. MASCIA, C., TERRACINA, A., TESEI, A.: Two-phase entropy solutions of a forward–backward parabolic equation. *Arch. Rational Mech. Anal.* **194**, 887–925 (2009)
9. NOVICK-COHEN, A., PEGO, R.L.: Stable patterns in a viscous diffusion equation. *Trans. Am. Math. Soc.* **324**, 331–351 (1991)
10. PADRÓN, V.: Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations. *Commun. Partial Differ. Equ.* **23**, 457–486 (1998)
11. PERONA, P., MALIK, J.: Scale space and edge detection using anisotropic diffusion. *IEEE Trans. Pattern Anal. Mach. Intell.* **12**, 629–639 (1990)
12. PLOTNIKOV, P.I.: Passing to the limit with respect to viscosity in an equation with variable parabolicity direction. *Differ. Equ.* **30**, 614–622 (1994)
13. PLOTNIKOV, P.I.: Equations with alternating direction of parabolicity and the hysteresis effect. *Russian Acad. Sci. Dokl. Math.* **47**, 604–608 (1993)
14. PLOTNIKOV, P.I.: Forward–backward parabolic equations and hysteresis. *J. Math. Sci.* **93**, 747–766 (1999)

15. Smarrazzo, F.: On a class of equations with variable parabolicity direction. *Discrete Contin. Dyn. Syst.* **22**, 729–758 (2008)
16. Smarrazzo, F., Tesei, A.: Long-time behaviour of solutions to a class of forward–backward parabolic equations. *SIAM J. Math. Anal.* **42**, 1046–1093 (2010)
17. Smarrazzo, F., Tesei, A.: *Degenerate Regularization of Forward–Backward Parabolic Equations: The Vanishing Viscosity Limit*

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