

Convergence Rates in L^2 for Elliptic Homogenization Problems

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Abstract

We study rates of convergence of solutions in L^2 and $H^{1/2}$ for a family of elliptic systems $\{\mathcal{L}_\varepsilon\}$ with rapidly oscillating coefficients in Lipschitz domains with Dirichlet or Neumann boundary conditions. As a consequence, we obtain convergence rates for Dirichlet, Neumann, and Steklov eigenvalues of $\{\mathcal{L}_\varepsilon\}$. Most of our results, which rely on the recently established uniform estimates for the L^2 Dirichlet and Neumann problems in KENIG and SHEN (Math Ann 350:867–917, 2011; Commun Pure Appl Math 64:1–44, 2011) are new even for smooth domains.

1. Introduction

Let $u_\varepsilon \in H^1(\Omega)$ be the weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω subject to the Dirichlet condition $u_\varepsilon = f$ on $\partial\Omega$, where $F \in L^2(\Omega)$, $f \in H^{1/2}(\partial\Omega)$ and $\mathcal{L}_\varepsilon = -\operatorname{div}[A(x/\varepsilon)\nabla]$. Assuming that the coefficient matrix $A(y)$ is elliptic and periodic, it is well known that $u_\varepsilon \rightarrow u_0$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, where $u_0 \in H^1(\Omega)$ is the weak solution of the homogenized system $\mathcal{L}_0(u_0) = F$ in Ω and $u_0 = f$ on $\partial\Omega$ (see for example [4]). The same holds under the Neumann boundary conditions $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \frac{\partial u_0}{\partial \nu_0} = g \in H^{-1/2}(\partial\Omega)$ with $\langle g, 1 \rangle = -\int_\Omega F$, if one also requires $\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0$. The primary purpose of this paper is to study the rate of convergence of $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$, as $\varepsilon \rightarrow 0$, in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. As a consequence, we obtain convergence rates for Dirichlet, Neumann, and Steklov eigenvalues of \mathcal{L}_ε . Most of our results, which rely on the recently established uniform regularity estimates for the L^2 Dirichlet and Neumann problems in [14, 15], are new even for smooth domains.

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More precisely, we consider a family of elliptic systems in divergence form,

$$\mathcal{L}_\varepsilon = -\frac{\partial}{\partial x_i} \left[a_{ij}^{\alpha\beta} \left(\frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_j} \right], \quad \varepsilon > 0. \tag{1.1}$$

We will assume that $A(y) = (a_{ij}^{\alpha\beta}(y))$, $1 \leq i, j \leq d$, $1 \leq \alpha, \beta \leq m$ is real and satisfies the ellipticity condition,

$$\mu |\xi|^2 \leq a_{ij}^{\alpha\beta}(y) \xi_i^\alpha \xi_j^\beta \leq \frac{1}{\mu} |\xi|^2 \quad \text{for } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{dm}, \tag{1.2}$$

where $\mu > 0$, and the periodicity condition

$$A(y + z) = A(y) \quad \text{for } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1.3}$$

We shall also impose the smoothness condition,

$$|A(x) - A(y)| \leq \tau |x - y|^\lambda \quad \text{for some } \lambda \in (0, 1) \text{ and } \tau \geq 0, \tag{1.4}$$

and the symmetry condition $A = A^*$, that is, $a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y)$ for $1 \leq i, j \leq d$ and $1 \leq \alpha, \beta \leq m$. We say $A \in \Lambda(\mu, \lambda, \tau)$ if it satisfies conditions (1.2), (1.3) and (1.4).

The following are the main results of the paper.

Theorem 1.1. (Dirichlet condition) *Let Ω be a bounded Lipschitz domain, $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$. Given $F \in L^2(\Omega)$ and $f \in H^1(\partial\Omega)$, let $u_\varepsilon \in H^1(\Omega)$, $\varepsilon \geq 0$ be the unique weak solution of the Dirichlet problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $u_\varepsilon = f$ on $\partial\Omega$. Then for $0 < \varepsilon < (1/2)$,*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} + \|\mathcal{M}(u_\varepsilon - u_0)\|_{L^2(\partial\Omega)} \leq C \varepsilon \|u_0\|_{H^2(\Omega)} \tag{1.5}$$

if $u_0 \in H^2(\Omega)$, and

$$\begin{aligned} \|u_\varepsilon - u_0\|_{L^2(\Omega)} &\leq C_\sigma \varepsilon |\ln(\varepsilon)|^{\frac{1}{2}+\sigma} \{ \|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} \}, \\ \|\mathcal{M}(u_\varepsilon - u_0)\|_{L^2(\partial\Omega)} &\leq C_\sigma \varepsilon |\ln(\varepsilon)|^{\frac{3}{2}+\sigma} \{ \|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} \} \end{aligned} \tag{1.6}$$

for any $\sigma > 0$.

Theorem 1.2. (Neumann condition) *Let Ω be a bounded Lipschitz domain, $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$. Given $F \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ with $\int_\Omega F + \int_{\partial\Omega} g = 0$, let $u_\varepsilon \in H^1(\Omega)$, $\varepsilon \geq 0$ be the unique weak solution of the Neumann problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω , $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g$ on $\partial\Omega$ and $\int_\Omega u_\varepsilon = 0$. Then for $0 < \varepsilon < (1/2)$, estimate (1.5) holds if $u_0 \in H^2(\Omega)$, and*

$$\begin{aligned} \|u_\varepsilon - u_0\|_{L^2(\Omega)} + \|u_\varepsilon - u_0\|_{L^2(\partial\Omega)} &\leq C_\sigma \varepsilon |\ln(\varepsilon)|^{\frac{1}{2}+\sigma} \{ \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \}, \\ \|\mathcal{M}(u_\varepsilon - u_0)\|_{L^2(\partial\Omega)} &\leq C_\sigma \varepsilon |\ln(\varepsilon)|^{\frac{3}{2}+\sigma} \{ \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \} \end{aligned} \tag{1.7}$$

for any $\sigma > 0$.

Here and thereafter \mathcal{M} denotes the radial maximal operator (see Section 2 for its definition). Note that estimates of $\mathcal{M}(u_\varepsilon - u_0)$ in $L^2(\partial\Omega)$ in Theorems 1.1–1.2 imply, in particular, the convergence of u_ε to u_0 in $L^2(S)$ uniformly for any “parallel boundary” S of Ω . Also observe that in the case $F = 0$, estimates (1.6)–(1.7) give

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C_\sigma \varepsilon |\ln(\varepsilon)|^{\frac{1}{2} + \sigma} \|u_0\|_{H^1(\partial\Omega)}, \tag{1.8}$$

for any $\sigma > 0$. The estimate of $u_\varepsilon - u_0$ in $L^2(\Omega)$ by u_0 and its first-order derivatives is a natural question in the theory of homogenization (see [10] for a two-dimensional result). It is not known whether the logarithmic factor in (1.8) is necessary, even for smooth domains.

We now describe the existing results on L^2 convergence and our approach to Theorems 1.1–1.2. For the scalar equation ($m = 1$) with the Dirichlet condition, it is known that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \left\{ \|\nabla^2 u_0\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^\infty(\partial\Omega)} \right\} \tag{1.9}$$

holds without any smoothness or symmetry condition on $A(y)$ or smoothness of Ω . To see this, one considers

$$w_\varepsilon(x) = u_\varepsilon(x) - u_0(x) - \varepsilon \chi(x/\varepsilon) \nabla u_0(x) \quad \text{in } \Omega, \tag{1.10}$$

where $\chi(y)$ is the matrix of correctors for \mathcal{L}_ε . Let $w_\varepsilon(x) = \theta_\varepsilon(x) + z_\varepsilon(x)$, where θ_ε is the solution to the Dirichlet problem: $\mathcal{L}_\varepsilon(\theta_\varepsilon) = 0$ in Ω and $\theta_\varepsilon = -\varepsilon \chi(x/\varepsilon) \nabla u_0$ on $\partial\Omega$. It follows from the energy estimates that $\|z_\varepsilon\|_{H_0^1(\Omega)} \leq C\varepsilon \|\nabla^2 u_0\|_{L^2(\Omega)}$ (see [4, 12, 19] or Remark 3.7 below). This, together with the estimate $\|\theta_\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon \|\nabla u_0\|_{L^\infty(\partial\Omega)}$ obtained by the maximum principle, gives (1.9). For more recent work on error estimates for scalar equations with the Dirichlet condition $u_\varepsilon = u_0 = 0$ on $\partial\Omega$, we refer the reader to [5, 8, 9, 21]. In particular, it was proved by GRISO in [9] that $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|F\|_{L^2(\Omega)}$, where $u_\varepsilon \in H_0^1(\Omega)$ and $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in a $C^{1,1}$ domain Ω .

For elliptic systems in a $C^{1,\alpha}$ domain with $A \in \Lambda(\mu, \lambda, \tau)$, the uniform estimates in [1, 2] for the L^2 Dirichlet problem imply $\|\theta_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon \|\nabla u_0\|_{L^2(\partial\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}$. It follows that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq \|z_\varepsilon\|_{L^2(\Omega)} + \|\theta_\varepsilon\|_{L^2(\Omega)} + C\varepsilon \|\nabla u_0\|_{L^2(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)},$$

as noted in [19]. Using the recently established uniform L^2 estimates in [15], in the presence of symmetry ($A = A^*$), we extend this result to the case of Lipschitz domains in Section 3, where we in fact prove that

$$\|\mathcal{M}(w_\varepsilon)\|_{L^2(\partial\Omega)} + \|w_\varepsilon\|_{H^{1/2}(\Omega)} + \left\{ \int_\Omega |\nabla w_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{1.11}$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$ (see Theorem 3.4), and deduce (1.5) as a simple corollary of (1.11).

The proof of (1.6) is more involved than that of (1.5). Note that with boundary data $f \in H^1(\partial\Omega)$, one cannot expect $u_0 \in H^2(\Omega)$. Furthermore, if Ω is Lipschitz, u_0 may not be in $H^2(\Omega)$ even if F and f are smooth (it is known that $u_0 \in H^{3/2}(\Omega)$ [11]). To circumvent this difficulty, our basic idea is to replace u_0 in (1.10) by a solution v_ε to the Dirichlet problem for \mathcal{L}_0 in a slightly larger domain: $\mathcal{L}_0(v_\varepsilon) = \tilde{F}$ in Ω_ε and $v_\varepsilon = f_\varepsilon$ on $\partial\Omega_\varepsilon$, where Ω_ε is a Lipschitz domain such that $\Omega_\varepsilon \supset \Omega$ and $\text{dist}(\partial\Omega_\varepsilon, \partial\Omega) \approx \varepsilon$. Also, \tilde{F} an extension of F and $f_\varepsilon(Q) = f(\Lambda_\varepsilon^{-1}(Q))$, where $\Lambda_\varepsilon : \partial\Omega \rightarrow \partial\Omega_\varepsilon$ is bi-Lipschitz map. Let $\tilde{w}_\varepsilon = u_\varepsilon - v_\varepsilon - \varepsilon\chi(x/\varepsilon)\nabla v_\varepsilon = \tilde{z}_\varepsilon + \tilde{\theta}_\varepsilon$, where $\tilde{\theta}_\varepsilon$ solves

$$\begin{cases} \mathcal{L}_\varepsilon(\tilde{\theta}_\varepsilon) = 0 & \text{in } \Omega, \\ \tilde{\theta}_\varepsilon = f - v_\varepsilon - \varepsilon\chi(x/\varepsilon)\nabla v_\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{1.12}$$

The desired estimates of $\tilde{\theta}_\varepsilon$ follow from the estimates for the L^2 Dirichlet problem in [15]. To handle \tilde{z}_ε , one observes that $\mathcal{L}_\varepsilon(\tilde{z}_\varepsilon) = \varepsilon \text{div}(h_\varepsilon)$ in Ω and $\tilde{z}_\varepsilon = 0$ in $\partial\Omega$, where $|h_\varepsilon| \leq C|\nabla^2 v_\varepsilon|$ in Ω . Using weighted norm inequalities for singular integrals, we are able to bound $\|\tilde{z}_\varepsilon\|_{L^2(\Omega)}$ and $\|\mathcal{M}(\tilde{z}_\varepsilon)\|_{L^2(\partial\Omega)}$ as well as $\|\tilde{z}_\varepsilon\|_{H^{1/2}(\Omega)}$ by

$$C\varepsilon \left\{ \int_\Omega |\nabla^2 v_\varepsilon(x)|^2 \delta(x) \phi_a(\delta(x)) \, dx \right\}^{1/2} \leq C\varepsilon |\ln(\varepsilon)|^{\frac{a}{2}} \{ \|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} \},$$

for suitable choices of a 's, where $\phi_a(t) = \{\ln(\frac{1}{t} + e^a)\}^a$. See Section 4 for details.

Very few results are known for the convergence rates in the case of the Neumann boundary conditions. By multiplying by a cut-off function the third term in the right-hand side of (1.10), one may obtain an $O(\sqrt{\varepsilon})$ estimate of $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$, regardless of the boundary condition [4, 12]. As far as we know, the only other known results are contained in [9, 20], both of which deal with the scalar equation $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω with $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on $\partial\Omega$. In particular, the estimate $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|F\|_{H^2(\Omega)}$ was proved in [20] for curvilinear convex polygons Ω in \mathbb{R}^2 , and in [9] for $C^{1,1}$ domains with no smoothness condition on $A(y)$. In Section 5 we prove estimate (1.5) in bounded Lipschitz domains in \mathbb{R}^d , $d \geq 2$ for the Neumann boundary conditions. The proof uses an explicit computation of the conormal derivative $\frac{\partial w_\varepsilon}{\partial \nu_\varepsilon}$ on $\partial\Omega$ and relies on the uniform estimates for the L^2 Neumann problem in [14, 15]. The proof of estimate (1.7), which is given in Section 6 and also uses estimates for the L^2 Neumann problem in [14, 15], is similar to that of (1.6). It is interesting to point out that in this case the function v_ε , which replaces u_0 in (1.10), is a solution to the Dirichlet problem for \mathcal{L}_0 in Ω_ε , with boundary data given by a push-forward of $u_0|_{\partial\Omega}$.

By a spectral theorem on collectively compact operators [12, 22], the L^2 error estimates of $u_\varepsilon - u_0$ in Theorems 1.1–1.2 lead to error estimates for eigenvalues of $\{\mathcal{L}_\varepsilon\}$. For $\varepsilon \geq 0$, let $\{\mu_\varepsilon^k\}$ denote the sequence of Neumann eigenvalues in an increasing order of $\{\mathcal{L}_\varepsilon\}$ in Ω . We will show in Section 7 that $|\mu_\varepsilon^k - \mu_0^k| \leq C_k \varepsilon$ if Ω is $C^{1,1}$ (or convex in the case $m = 1$), and $|\mu_\varepsilon^k - \mu_0^k| \leq C_{k,\sigma} \varepsilon |\ln(\varepsilon)|^{\frac{1}{2} + \sigma}$ for any $\sigma > 0$ if Ω is Lipschitz. The same holds for Dirichlet and Steklov eigenvalues. To the best of the authors' knowledge, only results for Dirichlet eigenvalues in smooth

domains [12, 19] and Neumann eigenvalues in a two-dimensional curvilinear convex polygon [20] were previously known (see [16, 17, 23, 24] for related homogenized eigenvalue problems).

Finally, in Section 8, we prove several weighted L^2 potential estimates, which are used in earlier sections, for the operators $\{\mathcal{L}_\varepsilon\}$. Our proofs use asymptotic estimates of the fundamental solutions for \mathcal{L}_ε in [3] as well as some classical results from harmonic analysis.

The summation convention is used throughout this paper. Unless otherwise stated, we always assume that $A \in \Lambda(\mu, \lambda, \tau)$, $A^* = A$, and Ω is a bounded Lipschitz domain in \mathbb{R}^d , $d \geq 2$. Without loss of generality we will also assume that $\text{diam}(\Omega) = 1$. We will use C and c to denote positive constants that depend at most on d, m, μ, λ, τ and the Lipschitz character of Ω .

2. Uniform Regularity Estimates

In this section we recall several uniform regularity estimates for $\{\mathcal{L}_\varepsilon\}$, on which the proofs of our main results rely. We also give definitions of the non-tangential maximal function and radial maximal operator \mathcal{M} .

Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω . Then if $B(x, 2r) \subset \Omega$,

$$|\nabla u_\varepsilon(x)| \leq \frac{C}{r^{d+1}} \int_{B(x,r)} |u_\varepsilon(y)| \, dy. \tag{2.1}$$

This uniform gradient estimate was proved in [1] (the symmetry condition $A^* = A$ is not needed for this). Let $\Gamma_\varepsilon(x, y) = (\Gamma_\varepsilon^{\alpha\beta}(x, y))$ denote the fundamental solution matrix for \mathcal{L}_ε in \mathbb{R}^d , with pole at y . It follows from the gradient estimate (2.1) that $|\Gamma_\varepsilon(x, y)| \leq C|x - y|^{2-d}$, $|\nabla_x \Gamma_\varepsilon(x, y)| + |\nabla_y \Gamma_\varepsilon(x, y)| \leq C|x - y|^{1-d}$ and $|\nabla_x \nabla_y \Gamma_\varepsilon(x, y)| \leq C|x - y|^{-d}$ (see [3]).

For a function u in a bounded Lipschitz domain Ω , the non-tangential maximal function $(u)^*$ on $\partial\Omega$ is defined by

$$(u)^*(Q) = \sup \{ |u(x)| : x \in \Omega \text{ and } |x - Q| < C_0 \text{dist}(x, \partial\Omega) \}, \tag{2.2}$$

where C_0 , depending on d and the Lipschitz character of Ω , is sufficiently large.

Theorem 2.1. *Let $f \in L^2(\partial\Omega)$ and u_ε be the unique solution of the L^2 Dirichlet problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω , $u_\varepsilon = f$ non-tangentially on $\partial\Omega$ and $(u_\varepsilon)^* \in L^2(\partial\Omega)$. Then*

$$\|(u_\varepsilon)^*\|_{L^2(\partial\Omega)} + \|u_\varepsilon\|_{H^{1/2}(\Omega)} + \left\{ \int_{\Omega} |\nabla u_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C \|f\|_{L^2(\partial\Omega)}, \tag{2.3}$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. Furthermore, if $f \in H^1(\partial\Omega)$, the solution satisfies the estimate $\|(\nabla u_\varepsilon)^*\|_{L^2(\partial\Omega)} \leq C \|f\|_{H^1(\partial\Omega)}$.

Proof. The non-tangential maximal function estimate $\|(u_\varepsilon)^*\|_{L^2(\partial\Omega)} \leq \|f\|_{L^2(\partial\Omega)}$ in Lipschitz domains was proved in B. DAHLBERG (1990, personal communication) for $m = 1$ and in [15] for $m \geq 1$. In the case of smooth domains, the estimate was obtained earlier in [1,2]. The proof of $\|(\nabla u_\varepsilon)^*\|_{L^2(\partial\Omega)} \leq C\|f\|_{H^1(\partial\Omega)}$ may be found in [14] for $m = 1$ and in [15] for $m \geq 1$.

It was also proved in [15] that the solution of the Dirichlet problem with boundary data f in $L^2(\partial\Omega)$ is given by a double layer potential $\mathcal{D}_\varepsilon(g_\varepsilon)$, where the density g_ε satisfies $\|g_\varepsilon\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}$. This, together with Proposition 8.5, gives the square function estimate in (2.3),

$$\left\{ \int_\Omega |\nabla u_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C\|g_\varepsilon\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}.$$

Finally, the estimate $\|u_\varepsilon\|_{H^{1/2}(\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}$ follows from the square function estimate by real interpolation (see for example [11, pp. 181–182]). \square

The next theorem was proved in [15] (the case $m = 1$ was obtained in [14]). We refer the reader to [13–15] for references on L^p boundary value problems in Lipschitz domains in non-homogenized settings.

Theorem 2.2. *Let $g \in L^2(\partial\Omega)$ with $\int_{\partial\Omega} g = 0$. Let $u_\varepsilon \in H^1(\Omega)$ be the unique (up to an additive constant) weak solution of the L^2 Neumann problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω , $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g$ on $\partial\Omega$. Then $\|(\nabla u_\varepsilon)^*\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)}$.*

The radial maximal operator. Given a bounded Lipschitz domain Ω , one may construct a continuous family $\{\Omega_t, -c < t < c\}$ of Lipschitz domains with uniform Lipschitz characters such that $\Omega_0 = \Omega$ and $\overline{\Omega}_t \subset \Omega_s$ for $t < s$. We may further assume that there exist homeomorphisms $\Lambda_t : \partial\Omega \rightarrow \partial\Omega_t$ such that $\Lambda_0(Q) = Q$, $|\Lambda_t(Q) - \Lambda_s(P)| \sim |t - s| + |P - Q|$ and $|\Lambda_s(Q) - \Lambda_t(Q)| \leq C_0 \text{dist}(\Lambda_s(Q), \partial\Omega_t)$ for any $t < s$ (see for example [26]). For a function u in Ω , the radial maximal function $\mathcal{M}(u)$ on $\partial\Omega$ is defined by

$$\mathcal{M}(u)(Q) = \sup \{|u(\Lambda_t(Q))| : -c < t < 0\}. \tag{2.4}$$

Observe that $\mathcal{M}(u)(Q) \leq (u)^*(Q)$ and if $S \subset \Omega$ is a surface near $\partial\Omega$ and obtained from $\partial\Omega$ by a bi-Lipschitz map, then $\|u\|_{L^2(S)} \leq C\|\mathcal{M}(u)\|_{L^2(\partial\Omega)}$. Also, note that $\|u\|_{L^2(\Omega)} + \|\mathcal{M}(u)\|_{L^2(\partial\Omega)} \leq C\|(u)^*\|_{L^2(\partial\Omega)}$, and the converse holds if u satisfies the interior L^∞ estimate $\|u\|_{L^\infty(B)} \leq C|2B|^{-1}\|u\|_{L^1(2B)}$ for any $2B \subset \Omega$. In particular, if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω , then $\|(u_\varepsilon)^*\|_{L^2(\partial\Omega)} \approx \|u_\varepsilon\|_{L^2(\Omega)} + \|\mathcal{M}(u_\varepsilon)\|_{L^2(\partial\Omega)}$.

3. Homogenization of Elliptic Systems

Let $\mathcal{L}_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla)$ with $A(y)$ satisfying (1.2)–(1.3). The matrix of correctors $\chi(y) = (\chi_j^{\alpha\beta}(y))$ for $\{\mathcal{L}_\varepsilon\}$ is defined by the following cell problem:

$$\begin{cases} \frac{\partial}{\partial y_i} \left[a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} (\chi_j^{\gamma\beta}) \right] = 0 & \text{in } \mathbb{R}^d, \quad \alpha = 1, \dots, m, \\ \chi_j^{\alpha\beta}(y) \text{ is periodic with respect to } \mathbb{Z}^d, \\ \int_Y \chi_j^{\alpha\beta} dy = 0, \end{cases} \tag{3.1}$$

for each $1 \leq j \leq d$ and $1 \leq \beta \leq m$, where $Y = [0, 1)^d \simeq \mathbb{R}^d / \mathbb{Z}^d$. The homogenized operator is given by $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$, where $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ and

$$\hat{a}_{ij}^{\alpha\beta} = \int_Y \left[a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} (\chi_j^{\gamma\beta}) \right] dy \tag{3.2}$$

(see [4]).

Lemma 3.1. *Let $F = (F_1, \dots, F_d) \in L^2(Y)$. Suppose that $\int_Y F_j dy = 0$ and $\operatorname{div}(F) = 0$. Then there exist $w_{ij} \in H^1(Y)$ such that $w_{ij} = -w_{ji}$ and $F_j = \frac{\partial w_{ij}}{\partial y_i}$.*

Proof. Let $f_j \in H^2(Y)$ be the solution to the cell problem: $\Delta f_j = F_j$ in Y , f_j is periodic with respect to \mathbb{Z}^d and $\int_Y f_j dy = 0$. Since $\operatorname{div}(F) = 0$, we may deduce that $\frac{\partial f_i}{\partial y_i}$ is constant. From this it is easy to see that

$$w_{ij} = \frac{\partial f_j}{\partial y_i} - \frac{\partial f_i}{\partial y_j}$$

has the desired properties. \square

Let

$$\Phi_{ij}^{\alpha\beta}(y) = \hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(y) - a_{ik}^{\alpha\gamma}(y) \frac{\partial}{\partial y_k} (\chi_j^{\gamma\beta}). \tag{3.3}$$

It follows from (3.2) and (3.1) that

$$\int_Y \Phi_{ij}^{\alpha\beta} dy = 0 \quad \text{and} \quad \frac{\partial}{\partial y_i} (\Phi_{ij}^{\alpha\beta}) = 0 \quad \text{in } \mathbb{R}^d.$$

Hence we may apply Lemma 3.1 to $\Phi_{ij}^{\alpha\beta}(y)$ (with α, β, j fixed). This gives $\Psi_{kij}^{\alpha\beta} \in H^1(Y)$, where $1 \leq i, j, k \leq d$ and $1 \leq \alpha, \beta \leq m$, with the property that

$$\Phi_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k} \{ \Psi_{kij}^{\alpha\beta} \} \quad \text{and} \quad \Psi_{kij}^{\alpha\beta} = -\Psi_{ikj}^{\alpha\beta}. \tag{3.4}$$

Furthermore, it follows from the proof of Lemma 3.1 that if $\chi \in W^{1,p}(Y)$ for some $p > d$ or χ is Hölder continuous, then $\Psi \in L^\infty(Y)$.

The next lemma is more or less known (see for example [12, Chapter 1] for the case $m = 1$ and $v_\varepsilon = u_0$). We provide the proof for the sake of completeness.

Lemma 3.2. *Let $u_\varepsilon^\alpha \in H^1(\Omega)$, $v_\varepsilon^\alpha \in H^2(\Omega)$ and*

$$w_\varepsilon^\alpha(x) = u_\varepsilon^\alpha(x) - v_\varepsilon^\alpha(x) - \varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial v_\varepsilon^\beta}{\partial x_k}, \tag{3.5}$$

where $1 \leq \alpha \leq m$. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(v_\varepsilon)$ in Ω . Then

$$(\mathcal{L}_\varepsilon(w_\varepsilon))^\alpha = \varepsilon \frac{\partial}{\partial x_i} \left\{ b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} \right\}, \tag{3.6}$$

where

$$b_{ijk}^{\alpha\gamma}(y) = \Psi_{jik}^{\alpha\gamma}(y) + a_{ij}^{\alpha\beta}(y) \chi_k^{\beta\gamma}(y) \tag{3.7}$$

and $\Psi_{jik}^{\alpha\gamma}(y)$ is given in (3.4).

Proof. It follows from the assumption $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(v_\varepsilon)$ that

$$\begin{aligned} (\mathcal{L}_\varepsilon(w_\varepsilon))^\alpha &= -\frac{\partial}{\partial x_i} \left\{ [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \frac{\partial v_\varepsilon^\beta}{\partial x_j} \right\} + \frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_j} [\chi_k^{\beta\gamma}(x/\varepsilon)] \right. \\ &\quad \times \left. \frac{\partial v_\varepsilon^\gamma}{\partial x_k} \right\} + \varepsilon \frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} \right\} \\ &= -\frac{\partial}{\partial x_i} \left\{ \Phi_{ik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial v_\varepsilon^\gamma}{\partial x_k} \right\} + \varepsilon \frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} \right\}, \end{aligned}$$

where the periodic function $\Phi_{ik}^{\alpha\gamma}(y)$ is given by (3.3). Using the first equation in (3.4), we obtain

$$\begin{aligned} (\mathcal{L}_\varepsilon(w_\varepsilon))^\alpha &= -\varepsilon \frac{\partial^2}{\partial x_i \partial x_j} \left\{ \Psi_{jik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial v_\varepsilon^\gamma}{\partial x_k} \right\} \\ &\quad + \varepsilon \frac{\partial}{\partial x_i} \left\{ \left(\Psi_{jik}^{\alpha\gamma}(x/\varepsilon) + a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) \right) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} \right\}. \tag{3.8} \end{aligned}$$

By the second equation in (3.4), the first term in the right-hand side of (3.8) is zero. This gives the equation (3.6). \square

Remark 3.3. Under the assumption $A \in \Lambda(\mu, \lambda, \tau)$, it is known that $\nabla\chi$ is Hölder continuous. This implies that $\nabla\Psi_{ijk}^{\alpha\beta}$ is Hölder continuous. In particular, $\Psi_{ijk}^{\alpha\beta}, b_{ijk}^{\alpha\gamma} \in L^\infty(Y)$. Furthermore, $\|\Psi_{ijk}^{\alpha\beta}\|_\infty + \|b_{ijk}^{\alpha\beta}\|_\infty$ is bounded by a constant depending only on m, d, μ, λ and τ . Note that in the scalar case ($m = 1$), the correctors χ_k are Hölder continuous by the De Giorgi-Nash estimate. This implies that Ψ_{ijk}, b_{ijk} are bounded.

Fix $F \in L^2(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$. Let $u_\varepsilon, u_0 \in H^1(\Omega)$ solve

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } \Omega, \\ u_\varepsilon = f & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}_0(u_0) = F & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

respectively.

Theorem 3.4. *Let Ω be a bounded Lipschitz domain. Suppose that $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$. Assume further that $u_0 \in H^2(\Omega)$. Then*

$$\|\mathcal{M}(w_\varepsilon)\|_{L^2(\partial\Omega)} + \|w_\varepsilon\|_{H^{1/2}(\Omega)} + \left\{ \int_{\Omega} |\nabla w_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{3.10}$$

where $w_\varepsilon^\alpha(x) = u_\varepsilon^\alpha(x) - u_0^\alpha(x) - \varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial u_0^\beta}{\partial x_k}$ and $\delta(x) = \text{dist}(x, \partial\Omega)$.

Observe that

$$\begin{aligned} & \|u_\varepsilon - u_0\|_{L^2(\Omega)} + \|\mathcal{M}(u_\varepsilon - u_0)\|_{L^2(\partial\Omega)} \\ & \leq \|w_\varepsilon\|_{L^2(\Omega)} + \|\mathcal{M}(w_\varepsilon)\|_{L^2(\partial\Omega)} + C\varepsilon \left\{ \|\nabla u_0\|_{L^2(\Omega)} + \|\mathcal{M}(\nabla u_0)\|_{L^2(\partial\Omega)} \right\} \\ & \leq \|w_\varepsilon\|_{L^2(\Omega)} + \|\mathcal{M}(w_\varepsilon)\|_{L^2(\partial\Omega)} + C\varepsilon \|u_0\|_{H^2(\Omega)}, \end{aligned} \tag{3.11}$$

where we have used the fact that $\|\mathcal{M}(\nabla u_0)\|_{L^2(\partial\Omega)} \leq C\|u_0\|_{H^2(\Omega)}$, which follows from the estimate (8.15).

As a corollary of Theorem 3.4, we obtain the following.

Corollary 3.5. *Under the same assumptions as in Theorem 3.4, we have*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} + \|\mathcal{M}(u_\varepsilon - u_0)\|_{L^2(\partial\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}. \tag{3.12}$$

Proof of Theorem 3.4. We first observe that by (3.6), w_ε satisfies

$$\begin{cases} \{\mathcal{L}_\varepsilon(w_\varepsilon)\}^\alpha = \varepsilon \frac{\partial}{\partial x_i} \left\{ b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 u_0^\gamma}{\partial x_j \partial x_k} \right\} & \text{in } \Omega, \\ w_\varepsilon^\alpha = -\varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial u_0^\beta}{\partial x_k} & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Let $w = \theta_\varepsilon + z_\varepsilon$, where

$$\begin{cases} \{\mathcal{L}_\varepsilon(\theta_\varepsilon)\}^\alpha = 0 & \text{in } \Omega, \\ \theta_\varepsilon^\alpha = -\varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial u_0^\beta}{\partial x_k} & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

and

$$\begin{cases} \{\mathcal{L}_\varepsilon(z_\varepsilon)\}^\alpha = \varepsilon \frac{\partial}{\partial x_i} \left\{ b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 u_0^\gamma}{\partial x_j \partial x_k} \right\} & \text{in } \Omega, \\ z_\varepsilon \in H_0^1(\Omega). \end{cases} \tag{3.15}$$

To estimate θ_ε , we apply Theorem 2.1 to obtain

$$\|\mathcal{M}(\theta_\varepsilon)\|_{L^2(\partial\Omega)} + \|\theta_\varepsilon\|_{H^{1/2}(\Omega)} + \left\{ \int_{\Omega} |\nabla \theta_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C\varepsilon \|\nabla u_0\|_{L^2(\partial\Omega)}. \tag{3.16}$$

Since $b_{ijk}^{\alpha\gamma} \in L^\infty(Y)$, by the usual energy estimates, we have $\|z_\varepsilon\|_{H_0^1(\Omega)} \leq C\varepsilon\|\nabla^2 u_0\|_{L^2(\Omega)}$. Thus,

$$\|\mathcal{M}(z_\varepsilon)\|_{L^2(\partial\Omega)} + \|z_\varepsilon\|_{H^{1/2}(\Omega)} + \left\{ \int_\Omega |\nabla z_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C\varepsilon\|\nabla^2 u_0\|_{L^2(\Omega)}. \tag{3.17}$$

Since $\|\nabla u_0\|_{L^2(\partial\Omega)} \leq C\|u_0\|_{H^2(\Omega)}$, the desired estimate (3.10) follows from (3.16) and (3.17). This completes the proof of Theorem 3.4. \square

Remark 3.6. Let Ω be a $C^{1,\alpha}$ domain in \mathbb{R}^d . As we mentioned in the Introduction, the estimate $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}$ was proved in [19], using the estimates for the L^2 Dirichlet problem in [1, 2]. Let θ_ε and z_ε be given by (3.14) and (3.15) respectively. It follows from [1, Theorem 3] that $\|\theta_\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon\|\nabla u_0\|_{L^\infty(\partial\Omega)}$. In view of (3.15) we have

$$|z_\varepsilon(x)| \leq C\varepsilon \int_\Omega |\nabla_y G_\varepsilon(x, y)| |\nabla^2 u_0(y)| \, dy, \tag{3.18}$$

where $G_\varepsilon(x, y)$ denotes the Green function for \mathcal{L}_ε in Ω . By [1] we have $|\nabla_y G_\varepsilon(x, y)| \leq C|x - y|^{1-d}$. It follows from (3.18) and Hölder inequality that $\|z_\varepsilon\|_{L^\infty(\Omega)} \leq C_p \varepsilon \|\nabla^2 u_0\|_{L^p(\Omega)}$ for any $p > d$. This gives

$$\|u_\varepsilon - u_0\|_{L^\infty(\Omega)} \leq C_p \varepsilon \|u_0\|_{W^{2,p}(\Omega)} \quad \text{for any } p > d, \tag{3.19}$$

where we also used the Sobolev imbedding $\|\nabla u_0\|_{C(\bar{\Omega})} \leq C_p \|u_0\|_{W^{2,p}(\Omega)}$ for $p > d$.

Remark 3.7. For scalar equations ($m = 1$) with bounded measurable coefficients, since $b_{ijk} \in L^\infty(Y)$, the estimate $\|z_\varepsilon\|_{H_0^1(\Omega)} \leq C\varepsilon\|\nabla^2 u_0\|_{L^2(\Omega)}$ continues to hold. This, together with $\|\theta_\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon\|\theta_\varepsilon\|_{L^\infty(\partial\Omega)}$ obtained by the maximum principle, gives the estimate (1.9).

4. Dirichlet Boundary Condition

Let Ω be a bounded Lipschitz domain. Let $\Omega_\varepsilon \supset \Omega$ and $\Lambda_\varepsilon : \partial\Omega \rightarrow \partial\Omega_\varepsilon$ be defined as in Section 2. Given $f \in H^1(\partial\Omega)$ and $F \in L^2(\Omega)$, let $v_\varepsilon \in H^1(\Omega_\varepsilon)$ be the weak solution of

$$\begin{cases} \mathcal{L}_0(v_\varepsilon) = \tilde{F} & \text{in } \Omega_\varepsilon, \\ v_\varepsilon = f_\varepsilon & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.1}$$

where $\tilde{F} = F$ in Ω and zero otherwise, and $f_\varepsilon(Q) = f(\Lambda_\varepsilon^{-1}(Q))$ for $Q \in \partial\Omega_\varepsilon$. The goal of this section is to prove the following.

Theorem 4.1. *Let Ω be a bounded Lipschitz domain. Suppose that $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$. Let*

$$w_\varepsilon^\alpha(x) = u_\varepsilon^\alpha(x) - u_0^\alpha(x) - \varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial v_\varepsilon^\beta}{\partial x_k}, \tag{4.2}$$

where v_ε is given by (4.1). Then, if $0 < \varepsilon < (1/2)$,

$$\begin{aligned} \|w_\varepsilon\|_{L^2(\Omega)} &\leq C\varepsilon |\ln(\varepsilon)|^a \{ \|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} \}, \quad \text{for any } a > 1/2, \tag{4.3} \\ \|\mathcal{M}(w_\varepsilon)\|_{L^2(\partial\Omega)} &\leq C\varepsilon |\ln(\varepsilon)|^a \{ \|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} \}, \quad \text{for any } a > 3/2, \tag{4.4} \end{aligned}$$

and

$$\|w_\varepsilon\|_{H^{1/2}(\Omega)} + \left\{ \int_\Omega |\nabla w_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C\varepsilon |\ln \varepsilon| \{ \|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} \}, \tag{4.5}$$

where C depends only on $\mu, \lambda, \tau, d, m, a$ and Ω .

As a corollary, we obtain the following convergence rates of u_ε to u_0 in L^2 .

Corollary 4.2. *Under the same conditions as in Theorem 4.1, we have*

$$\begin{aligned} \|u_\varepsilon - u_0\|_{L^2(\Omega)} &\leq C\varepsilon |\ln(\varepsilon)|^{\frac{1}{2}+\sigma} \{ \|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} \}, \tag{4.6} \\ \|\mathcal{M}(u_\varepsilon - u_0)\|_{L^2(\partial\Omega)} &\leq C\varepsilon |\ln \varepsilon|^{\frac{3}{2}+\sigma} \{ \|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} \}, \tag{4.7} \end{aligned}$$

for any $\sigma > 0$.

Without loss of generality we shall assume that $\|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} = 1$ in the rest of this section. We begin with an estimate on ∇v_ε .

Lemma 4.3. *Let v_ε be defined by (4.1). Then*

$$\|\mathcal{M}_\varepsilon(\nabla v_\varepsilon)\|_{L^2(\partial\Omega_\varepsilon)} + \|\nabla v_\varepsilon\|_{H^{1/2}(\Omega_\varepsilon)} + \left\{ \int_{\Omega_\varepsilon} |\nabla^2 v_\varepsilon(x)|^2 \delta_\varepsilon(x) \, dx \right\}^{1/2} \leq C,$$

where $\delta_\varepsilon(x) = \text{dist}(x, \partial\Omega_\varepsilon)$ and $\mathcal{M}_\varepsilon(v_\varepsilon)(\Lambda_\varepsilon(Q)) = \sup \{|v_\varepsilon(\Lambda_s(Q))| : -c < t < \varepsilon\}$.

Proof. Let $G = \Gamma_0 * \tilde{F}$ in \mathbb{R}^d , where $\Gamma_0(x)$ is the matrix of fundamental solutions for the operator \mathcal{L}_0 , with pole at the origin. Clearly, $\|G\|_{H^2(\mathbb{R}^d)} \leq C\|F\|_{L^2(\Omega)}$. This implies that $\|\mathcal{M}_\varepsilon(\nabla G)\|_{L^2(\partial\Omega_\varepsilon)} + \|G\|_{H^1(\partial\Omega_\varepsilon)} \leq C\|F\|_{L^2(\Omega)}$.

Next, we note that $\mathcal{L}_0(v_\varepsilon - G) = 0$ in Ω_ε and $v_\varepsilon - G = f_\varepsilon - G$ on $\partial\Omega_\varepsilon$. Hence, by Theorem 2.1 (see [7] for operators with constant coefficients),

$$\begin{aligned} \|(\nabla(v_\varepsilon - G))^*\|_{L^2(\partial\Omega_\varepsilon)} + \|\nabla(v_\varepsilon - G)\|_{H^{1/2}(\Omega_\varepsilon)} + \left\{ \int_{\Omega_\varepsilon} |\nabla^2(v_\varepsilon - G)|^2 \delta_\varepsilon(x) \, dx \right\}^{1/2} \\ \leq C\|f_\varepsilon - G\|_{H^1(\partial\Omega_\varepsilon)} \leq C. \end{aligned}$$

It follows that

$$\begin{aligned} & \|\mathcal{M}_\varepsilon(\nabla v_\varepsilon)\|_{L^2(\partial\Omega_\varepsilon)} + \|\nabla v_\varepsilon\|_{H^{1/2}(\Omega_\varepsilon)} + \left\{ \int_{\Omega_\varepsilon} |\nabla^2 v_\varepsilon(x)|^2 \delta_\varepsilon(x) \, dx \right\}^{1/2} \\ & \leq C + \|\mathcal{M}_\varepsilon(\nabla G)\|_{L^2(\partial\Omega_\varepsilon)} + \|\nabla G\|_{H^{1/2}(\Omega_\varepsilon)} + \left\{ \int_{\Omega_\varepsilon} |\nabla^2 G(x)|^2 \delta_\varepsilon(x) \, dx \right\}^{1/2} \\ & \leq C. \end{aligned}$$

Remark 4.4. By Lemma 4.3 we have $\|\nabla v_\varepsilon\|_{L^2(\Omega)} + \|\mathcal{M}(\nabla v_\varepsilon)\|_{L^2(\partial\Omega)} \leq C$. It follows that

$$\begin{aligned} \|u_\varepsilon - u_0\|_{L^2(\Omega)} & \leq \|w_\varepsilon\|_{L^2(\Omega)} + C\varepsilon, \\ \|\mathcal{M}(u_\varepsilon - u_0)\|_{L^2(\partial\Omega)} & \leq \|\mathcal{M}(w_\varepsilon)\|_{L^2(\partial\Omega)} + C\varepsilon. \end{aligned} \tag{4.8}$$

This, together with Theorem 4.1, gives the estimates in Corollary 4.2.

Lemma 4.5. Let $f_\varepsilon(Q) = f(\Lambda_\varepsilon^{-1}(Q))$ and v_ε be defined by (4.1). Then $\|f - v_\varepsilon\|_{L^2(\partial\Omega)} \leq C\varepsilon$ and

$$\|(v_\varepsilon - u_0)^*\|_{L^2(\partial\Omega)} + \|v_\varepsilon - u_0\|_{H^{1/2}(\Omega)} + \left\{ \int_{\Omega} |\nabla(v_\varepsilon - u_0)|^2 \delta(x) \, dx \right\}^{1/2} \leq C\varepsilon.$$

Proof. Note that for $Q \in \partial\Omega$,

$$|f(Q) - v_\varepsilon(Q)| = |v_\varepsilon(\Lambda_\varepsilon(Q)) - v_\varepsilon(Q)| \leq C\varepsilon \mathcal{M}_\varepsilon(\nabla v_\varepsilon)(\Lambda_\varepsilon(Q)).$$

This gives $\|f - v_\varepsilon\|_{L^2(\partial\Omega)} \leq C\varepsilon \|\mathcal{M}_\varepsilon(\nabla v_\varepsilon)\|_{L^2(\partial\Omega_\varepsilon)} \leq C\varepsilon$, where the last inequality follows from Lemma 4.3. Since $\mathcal{L}_0(v_\varepsilon - u_0) = 0$ in Ω and $v_\varepsilon - u_0 = v_\varepsilon - f$ on $\partial\Omega$, we may apply Theorem 2.1 (for the case of constant coefficients) to obtain

$$\begin{aligned} \|(v_\varepsilon - u_0)^*\|_{L^2(\partial\Omega)} + \|v_\varepsilon - u_0\|_{H^{1/2}(\Omega)} + \left\{ \int_{\Omega} |\nabla(v_\varepsilon - u_0)|^2 \delta(x) \, dx \right\}^{1/2} \\ \leq C \|v_\varepsilon - f\|_{L^2(\partial\Omega)} \leq C\varepsilon. \end{aligned}$$

This completes the proof. \square

$$\text{Let } \phi_a(t) = \left\{ \ln\left(\frac{1}{t} + e^a\right) \right\}^a.$$

Lemma 4.6. Let $W_\varepsilon \in H^1(\Omega)$ be a solution of $\mathcal{L}_\varepsilon(W_\varepsilon) = \text{div}(h)$ in Ω and $W_\varepsilon = g$ on $\partial\Omega$ for some $h \in L^2(\Omega)$ and $g \in H^1(\partial\Omega)$. Then

$$\|W_\varepsilon\|_{L^2(\Omega)} \leq C \|g\|_{L^2(\partial\Omega)} + C_a \left\{ \int_{\Omega} |h(x)|^2 \delta(x) \phi_a(\delta(x)) \, dx \right\}^{1/2} \text{ for any } a > 1, \tag{4.9}$$

$$\left\{ \int_{\Omega} |\nabla W_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C \|g\|_{L^2(\partial\Omega)} + C \left\{ \int_{\Omega} |h(x)|^2 \delta(x) \phi_2(\delta(x)) \, dx \right\}^{1/2}, \tag{4.10}$$

and

$$\|\mathcal{M}(W_\varepsilon)\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)} + C_a \left\{ \int_\Omega |h(x)|^2 \delta(x) \phi_a(\delta(x)) \, dx \right\}^{1/2} \quad (4.11)$$

for any $a > 3$.

Proof. Let $h = (h_i^\alpha)$ and

$$H_\varepsilon^\alpha(x) = - \int_\Omega \frac{\partial}{\partial y_i} \{ \Gamma_\varepsilon^{\alpha\beta}(x, y) \} h_i^\beta(y) \, dy, \quad (4.12)$$

where $\Gamma_\varepsilon(x, y) = (\Gamma_\varepsilon^{\alpha\beta}(x, y))$ is the matrix of fundamental solutions for \mathcal{L}_ε in \mathbb{R}^d , with pole at y . Note that $\mathcal{L}_\varepsilon(W_\varepsilon - H_\varepsilon) = 0$ in Ω and $W_\varepsilon - H_\varepsilon = g - H_\varepsilon$ on $\partial\Omega$. It follows by Theorem 2.1 that

$$\begin{aligned} & \|W_\varepsilon - H_\varepsilon\|_{L^2(\Omega)} + \|(W_\varepsilon - H_\varepsilon)^*\|_{L^2(\partial\Omega)} + \left\{ \int_\Omega |\nabla(W_\varepsilon - H_\varepsilon)|^2 \delta(x) \, dx \right\}^{1/2} \\ & \leq C\|g\|_{L^2(\partial\Omega)} + C\|H_\varepsilon\|_{L^2(\partial\Omega)}. \end{aligned}$$

Hence,

$$\begin{aligned} & \|W_\varepsilon\|_{L^2(\Omega)} \leq C \{ \|g\|_{L^2(\partial\Omega)} + \|H_\varepsilon\|_{L^2(\partial\Omega)} \} + \|H_\varepsilon\|_{L^2(\Omega)}, \\ & \|\mathcal{M}(W_\varepsilon)\|_{L^2(\partial\Omega)} \leq C \{ \|g\|_{L^2(\partial\Omega)} + \|\mathcal{M}(H_\varepsilon)\|_{L^2(\partial\Omega)} \}, \\ & \left\{ \int_\Omega |\nabla W_\varepsilon|^2 \delta(x) \, dx \right\}^{1/2} \leq C \{ \|g\|_{L^2(\partial\Omega)} + \|H_\varepsilon\|_{L^2(\partial\Omega)} \} + \left\{ \int_\Omega |\nabla H_\varepsilon|^2 \delta(x) \, dx \right\}^{1/2}. \end{aligned}$$

The desired estimates now follow from Propositions 8.1, 8.2, 8.3 and 8.4. \square

We are now in a position to give the proof of Theorem 4.1.

Proof of Theorem 4.1. Let

$$\tilde{w}_\varepsilon^\alpha = u_\varepsilon^\alpha(x) - v_\varepsilon^\alpha(x) - \varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial v_\varepsilon^\beta}{\partial x_k} = w_\varepsilon^\alpha - (v_\varepsilon - u_0) \quad (4.13)$$

in Ω . In view of Lemma 4.5, it suffices to show that \tilde{w}_ε satisfies the estimates in Theorem 4.1.

To this end, we first observe that by Lemma 3.2,

$$\begin{cases} (\mathcal{L}_\varepsilon(\tilde{w}_\varepsilon))^\alpha = \varepsilon \frac{\partial}{\partial x_i} \left\{ b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} \right\}, & \text{in } \Omega \\ \tilde{w}_\varepsilon^\alpha = f^\alpha - v_\varepsilon^\alpha - \varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial v_\varepsilon^\beta}{\partial x_k} & \text{on } \partial\Omega. \end{cases}$$

Let $\tilde{w}_\varepsilon = \tilde{\theta}_\varepsilon + \tilde{z}_\varepsilon$, where $\mathcal{L}_\varepsilon(\tilde{\theta}_\varepsilon) = 0$ in Ω , $\tilde{\theta}_\varepsilon = \tilde{w}_\varepsilon$ on $\partial\Omega$, and \tilde{z}_ε satisfies

$$\begin{cases} (\mathcal{L}_\varepsilon(\tilde{z}_\varepsilon))^\alpha = \varepsilon \frac{\partial}{\partial x_i} \left\{ b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} \right\} & \text{in } \Omega \\ \tilde{z}_\varepsilon \in H_0^1(\Omega). \end{cases}$$

To estimate $\tilde{\theta}_\varepsilon$, we apply Theorem 2.1 to obtain

$$\begin{aligned} \|\tilde{\theta}_\varepsilon\|_{L^2(\Omega)} + \|\mathcal{M}(\tilde{\theta}_\varepsilon)\|_{L^2(\partial\Omega)} + \left\{ \int_\Omega |\nabla\tilde{\theta}_\varepsilon|^2 \delta(x) \, dx \right\}^{1/2} &\leq C \|\tilde{\theta}_\varepsilon\|_{L^2(\partial\Omega)} \\ &= C \|\tilde{w}_\varepsilon\|_{L^2(\partial\Omega)} \leq C \left\{ \|f - v_\varepsilon\|_{L^2(\partial\Omega)} + \varepsilon \|\nabla v_\varepsilon\|_{L^2(\partial\Omega)} \right\} \leq C\varepsilon, \end{aligned}$$

where the last inequality follows from Lemmas 4.3 and 4.5.

Finally, we use Lemma 4.6 to handle \tilde{z}_ε . In particular, this gives

$$\|\tilde{z}_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon \left\{ \int_\Omega |\nabla^2 v_\varepsilon|^2 \delta(x) \phi_a(\delta(x)) \, dx \right\}^{1/2}. \tag{4.14}$$

for any $a > 1$. Note that $\phi_a(t)$ is decreasing and $t\phi_a(t)$ is increasing on $(0, \infty)$ for any $a \geq 0$. Hence, for any $x \in \Omega$ and $0 < \varepsilon < c_0$,

$$\delta(x)\phi_a(x) \leq \delta_\varepsilon(x)\phi_a(\delta_\varepsilon(x)) \leq \delta_\varepsilon(x)\phi_a(\varepsilon/C) \leq C\delta_\varepsilon(x)|\ln(\varepsilon)|^a,$$

where $\delta_\varepsilon(x) = \text{dist}(x, \partial\Omega_\varepsilon)$. In view of (4.14) we obtain

$$\begin{aligned} \int_\Omega |\nabla^2 v_\varepsilon|^2 \delta(x)\phi_a(\delta(x)) \, dx &\leq C\varepsilon |\ln(\varepsilon)|^a \int_\Omega |\nabla^2 v_\varepsilon|^2 \delta_\varepsilon(x) \, dx \\ &\leq C\varepsilon |\ln(\varepsilon)|^a \int_{\Omega_\varepsilon} |\nabla^2 v_\varepsilon|^2 \delta_\varepsilon(x) \, dx \\ &\leq C\varepsilon |\ln(\varepsilon)|^a, \end{aligned} \tag{4.15}$$

for any $a > 1$, where the last inequality follows from Lemma 4.3. Thus $\|\tilde{z}_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon |\ln(\varepsilon)|^{a/2}$ for any $a > 1$. This, together with the estimates of $\tilde{\theta}_\varepsilon$ and $v_\varepsilon - u_0$ in $L^2(\Omega)$, gives (4.3). Estimates (4.4) and (4.5) follow from Lemma 4.6 in the same manner. We omit the details. \square

Proof of Theorem 1.1. Estimate (1.5) is given in Corollary 3.5 and estimate (1.6) in Corollary 4.2. \square

5. Neumann Boundary Condition, Part I

Fix $F \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. Suppose that $\int_\Omega F + \int_{\partial\Omega} g = 0$. Let $u_\varepsilon, u_0 \in H^1(\Omega)$ solve

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}_0(u_0) = F & \text{in } \Omega, \\ \frac{\partial u_0}{\partial \nu_0} = g & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

respectively. Recall that

$$\left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right)^\alpha = n_i(x) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial u_\varepsilon^\beta}{\partial x_j} \quad \text{and} \quad \left(\frac{\partial u_0}{\partial \nu_0} \right)^\alpha = n_i(x) \hat{a}_{ij}^{\alpha\beta} \frac{\partial u_0^\beta}{\partial x_j}, \tag{5.2}$$

where $n = (n_1, \dots, n_d)$ denotes the outward unit normal to $\partial\Omega$.

Lemma 5.1. *Let $w_\varepsilon^\alpha = u_\varepsilon^\alpha - v_\varepsilon^\alpha - \varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial v_\varepsilon^\beta}{\partial x_k}$, where $u_\varepsilon \in H^1(\Omega)$ and $v_\varepsilon \in H^2(\Omega)$. Then*

$$\begin{aligned} \left(\frac{\partial w_\varepsilon}{\partial v_\varepsilon} \right)^\alpha &= n_i(x) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial u_\varepsilon^\beta}{\partial x_j} - n_i(x) \hat{a}_{ij}^{\alpha\beta} \frac{\partial v_\varepsilon^\beta}{\partial x_j} \\ &\quad + \frac{\varepsilon}{2} \left\{ n_i(x) \frac{\partial}{\partial x_j} - n_j(x) \frac{\partial}{\partial x_i} \right\} \left\{ \Psi_{jik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial v_\varepsilon^\gamma}{\partial x_k} \right\} \\ &\quad - \varepsilon n_i(x) b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k}, \end{aligned} \tag{5.3}$$

where $\Psi_{jik}^{\alpha\gamma}(y)$ and $b_{ijk}^{\alpha\gamma}(y)$ are the same as in Lemma 3.2.

Proof. A direct computation shows that

$$\begin{aligned} n_i(x) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial w_\varepsilon^\beta}{\partial x_j} &= n_i(x) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial u_\varepsilon^\beta}{\partial x_j} - n_i(x) \hat{a}_{ij}^{\alpha\beta} \frac{\partial v_\varepsilon^\beta}{\partial x_j} \\ &\quad + n_i(x) \Phi_{ik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial v_\varepsilon^\gamma}{\partial x_k} \\ &\quad - \varepsilon n_i(x) a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k}, \end{aligned} \tag{5.4}$$

where $\Phi_{ik}^{\alpha\gamma}(y)$ is defined by (3.3). By (3.4), we obtain

$$\begin{aligned} n_i(x) \Phi_{ik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial v_\varepsilon^\gamma}{\partial x_k} &= \varepsilon n_i(x) \frac{\partial}{\partial x_j} \left\{ \Psi_{jik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial v_\varepsilon^\gamma}{\partial x_k} \right\} \\ &\quad - \varepsilon n_i(x) \Psi_{jik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} \\ &= \frac{\varepsilon}{2} \left\{ n_i(x) \frac{\partial}{\partial x_j} - n_j(x) \frac{\partial}{\partial x_i} \right\} \left\{ \Psi_{jik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial v_\varepsilon^\gamma}{\partial x_k} \right\} \\ &\quad - \varepsilon n_i(x) \Psi_{jik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k}. \end{aligned} \tag{5.5}$$

Equation (5.3) now follows from (5.4) and (5.5). \square

Theorem 5.2. *Let Ω be a bounded Lipschitz domain. Suppose that $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$. Let (u_ε, u_0) be a solution of (5.1) with $\int_{\partial\Omega} u_\varepsilon = \int_{\partial\Omega} u_0 = 0$. Assume further that $u_0 \in H^2(\Omega)$. Then*

$$\|\mathcal{M}(w_\varepsilon)\|_{L^2(\partial\Omega)} + \|w_\varepsilon\|_{H^{1/2}(\Omega)} + \left\{ \int_\Omega |\nabla w_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{5.6}$$

where $w_\varepsilon^\alpha = u_\varepsilon^\alpha(x) - u_0^\alpha(x) - \varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial u_0^\beta}{\partial x_k}$.

As in the case of Dirichlet boundary conditions, Theorem 5.2 gives the following convergence rate of u_ε to u_0 in L^2 . As we mentioned in the Introduction, the estimate $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}$ was proved in [20] when Ω is a curvilinear convex domain in \mathbb{R}^2 .

Corollary 5.3. *Under the same assumptions as in Theorem 5.2, we have*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} + \|\mathcal{M}(u_\varepsilon - u_0)\|_{L^2(\partial\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}. \tag{5.7}$$

Proof of Theorem 5.2. In view of Lemmas 3.2 and 5.1, we may write $w_\varepsilon = \theta_\varepsilon + z_\varepsilon + \rho$, where

$$\begin{cases} \mathcal{L}_\varepsilon(\theta_\varepsilon) = 0 & \text{in } \Omega, \\ \left(\frac{\partial\theta_\varepsilon}{\partial\nu_\varepsilon}\right)^\alpha = \frac{\varepsilon}{2} \left\{ n_i(x) \frac{\partial}{\partial x_j} - n_j(x) \frac{\partial}{\partial x_i} \right\} \left\{ \Psi_{jik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial u_0^\gamma}{\partial x_k} \right\} & \text{on } \partial\Omega, \\ \theta_\varepsilon \in H^1(\Omega) \text{ and } \int_{\partial\Omega} \theta_\varepsilon = 0, \end{cases} \tag{5.8}$$

$$\begin{cases} (\mathcal{L}_\varepsilon(z_\varepsilon))^\alpha = \varepsilon \frac{\partial}{\partial x_i} \left\{ b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 u_0^\gamma}{\partial x_j \partial x_k} \right\} & \text{in } \Omega, \\ \left(\frac{\partial z_\varepsilon}{\partial\nu_\varepsilon}\right)^\alpha = -\varepsilon n_i(x) b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 u_0^\gamma}{\partial x_j \partial x_k} & \text{on } \partial\Omega \\ z_\varepsilon \in H^1(\Omega) \text{ and } \int_\Omega z_\varepsilon = 0, \end{cases} \tag{5.9}$$

and

$$\rho = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} (w_\varepsilon - z_\varepsilon)$$

is a constant. It follows from the energy estimates that $\|z_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}$. Also note that

$$|\rho| \leq C \int_{\partial\Omega} |z_\varepsilon| + C\varepsilon \int_{\partial\Omega} |\nabla u_0| \leq C\varepsilon\|u_0\|_{H^2(\Omega)},$$

where we have used the condition $\int_{\partial\Omega} u_\varepsilon = \int_{\partial\Omega} u_0 = 0$. Thus it remains only to estimate θ_ε .

To this end we use a duality argument and consider the L^2 Neumann problem

$$\begin{cases} \mathcal{L}_\varepsilon(\Theta_\varepsilon) = 0 & \text{in } \Omega, \\ \frac{\partial\Theta_\varepsilon}{\partial\nu_\varepsilon} = h & \text{on } \partial\Omega, \\ \Theta_\varepsilon \in H^1(\Omega) \text{ and } \int_{\partial\Omega} \Theta_\varepsilon = 0, \end{cases} \tag{5.10}$$

where $h \in L^2(\partial\Omega)$ and $\int_{\partial\Omega} h = 0$. It follows from integration by parts that

$$\begin{aligned} \left| \int_{\partial\Omega} \theta_\varepsilon \cdot h \right| &= \left| \int_{\partial\Omega} \theta_\varepsilon \cdot \frac{\partial \Theta_\varepsilon}{\partial \nu_\varepsilon} \right| = \left| \int_{\partial\Omega} \Theta_\varepsilon \cdot \frac{\partial \theta_\varepsilon}{\partial \nu_\varepsilon} \right| \\ &= \frac{\varepsilon}{2} \left| \int_{\partial\Omega} \left\{ n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i} \right\} \Theta_\varepsilon^\alpha \cdot \Psi_{jik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial u_0^\gamma}{\partial x_k} \right| \\ &\leq C\varepsilon \|\nabla \Theta_\varepsilon\|_{L^2(\partial\Omega)} \|\nabla u_0\|_{L^2(\partial\Omega)}, \end{aligned} \tag{5.11}$$

where we have used the fact that $n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i}$ is a tangential derivative for $1 \leq i, j \leq d$. In view of Theorem 2.2 we have $\|\nabla \Theta_\varepsilon\|_{L^2(\partial\Omega)} \leq C \|h\|_{L^2(\partial\Omega)}$. Hence, by (5.11) and duality, we obtain $\|\theta_\varepsilon\|_{L^2(\partial\Omega)} \leq C\varepsilon \|\nabla u_0\|_{L^2(\partial\Omega)}$. Here we also use the fact $\int_{\partial\Omega} \theta_\varepsilon = 0$.

Finally, we use the estimates for the L^2 Dirichlet problem in Theorem 2.1 to see that

$$\begin{aligned} \|\theta_\varepsilon\|_{L^2(\Omega)} + \|\mathcal{M}(\theta_\varepsilon)\|_{L^2(\Omega)} + \left\{ \int_{\Omega} |\nabla \theta_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \\ \leq C \|\theta_\varepsilon\|_{L^2(\partial\Omega)} \leq C\varepsilon \|\nabla u_0\|_{L^2(\partial\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}. \end{aligned}$$

This, together with the estimates of z_ε and ρ , completes the proof of Theorem 5.2. □

Remark 5.4. The estimates in Theorem 5.2 and Corollary 5.3 also hold under the condition $\int_{\Omega} u_\varepsilon = \int_{\Omega} u_0 = 0$. In this case the constant ρ is given by $\rho = \frac{1}{|\Omega|} \int_{\Omega} (w_\varepsilon - \theta_\varepsilon)$, and we have

$$|\rho| \leq C\varepsilon \int_{\Omega} |\nabla u_0| + C \int_{\Omega} |\theta_\varepsilon| \leq C\varepsilon \|u_0\|_{H^2(\Omega)}.$$

This will be used in the proof of the error estimate for the Neumann eigenvalues for \mathcal{L}_ε .

6. Neumann Boundary Condition, Part II

In this section we extend the results on convergence rates in Section 4 to the case of Neumann boundary conditions.

Construction of the first-order term. Fix $F \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ such that $\int_{\Omega} F + \int_{\partial\Omega} g = 0$. Let (u_ε, u_0) be the solution of (5.1) with $\int_{\partial\Omega} u_\varepsilon = \int_{\partial\Omega} u_0 = 0$. Consider

$$w_\varepsilon^\alpha = u_\varepsilon^\alpha - u_0^\alpha - \varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial v_\varepsilon^\beta}{\partial x_k}, \tag{6.1}$$

where v_ε is the solution of (4.1) in Ω_ε with f given by $u_0|_{\partial\Omega}$ and $f_\varepsilon(Q) = f(\Lambda_\varepsilon^{-1}(Q))$. Note that by Theorem 2.2 (for operators with constant coefficients),

$$\|f\|_{H^1(\partial\Omega)} \leq C \left\{ \|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\partial\Omega)} \right\}. \tag{6.2}$$

Theorem 6.1. *Let Ω be a bounded Lipschitz domain. Suppose that $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$. Let w_ε be defined by (6.1). Then, if $0 < \varepsilon < (1/2)$,*

$$\|w_\varepsilon\|_{L^2(\partial\Omega)} + \|w_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon |\ln(\varepsilon)|^a \{ \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \}, \quad \text{for any } a > 1/2, \tag{6.3}$$

$$\|\mathcal{M}(w_\varepsilon)\|_{L^2(\partial\Omega)} \leq C\varepsilon |\ln(\varepsilon)|^a \{ \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \}, \quad \text{for any } a > 3/2, \tag{6.4}$$

and

$$\|w_\varepsilon\|_{H^{1/2}(\Omega)} + \left\{ \int_\Omega |\nabla w_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C\varepsilon |\ln \varepsilon| \{ \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \}, \tag{6.5}$$

where C depends only on $\mu, \lambda, \tau, d, m, a$ and Ω .

Observe that

$$\begin{aligned} \|\nabla v_\varepsilon\|_{L^2(\Omega)} + \|\mathcal{M}(\nabla v_\varepsilon)\|_{L^2(\partial\Omega)} &\leq C \{ \|F\|_{L^2(\Omega)} + \|f\|_{H^1(\partial\Omega)} \} \\ &\leq C \{ \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \}. \end{aligned}$$

Thus, as a corollary of Theorem 6.1, we obtain the following convergence rates of u_ε to u_0 in L^2 .

Corollary 6.2. *Under the same conditions as in Theorem 6.1, we have*

$$\|u_\varepsilon - u_0\|_{L^2(\partial\Omega)} + \|\tilde{u}_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon |\ln(\varepsilon)|^{\frac{1}{2}+\sigma} \{ \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \}, \tag{6.6}$$

$$\|\mathcal{M}(u_\varepsilon - u_0)\|_{L^2(\partial\Omega)} \leq C\varepsilon |\ln \varepsilon|^{\frac{3}{2}+\sigma} \{ \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \}, \tag{6.7}$$

for any $\sigma > 0$.

Without loss of generality we will assume that $\|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \leq 1$ in the rest of this section. We remark that because of (6.2), the estimates of ∇v_ε in Lemmas 4.3 and 4.5 continue to hold.

Proof of Theorem 6.1. We proceed as in the case of the Dirichlet condition and write

$$w_\varepsilon^\alpha = \tilde{w}_\varepsilon^\alpha + \{v_\varepsilon^\alpha - u_0^\alpha\} \quad \text{and} \quad \tilde{w}_\varepsilon^\alpha = u_\varepsilon^\alpha(x) - v_\varepsilon^\alpha(x) - \varepsilon \chi_k^{\alpha\beta}(x/\varepsilon) \frac{\partial v_\varepsilon^\beta}{\partial x_k}.$$

The desired estimates for $v_\varepsilon - u_0$ follow directly from Lemmas 4.3–4.5 and (6.2).

Next we let $\tilde{w}_\varepsilon = \tilde{\theta}_\varepsilon + \tilde{z}_\varepsilon + \rho$, where

$$\begin{cases} \mathcal{L}_\varepsilon(\tilde{\theta}_\varepsilon) = 0 & \text{in } \Omega, \\ \left(\frac{\partial \tilde{\theta}_\varepsilon}{\partial v_\varepsilon} \right)^\alpha = \frac{\varepsilon}{2} \left\{ n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i} \right\} \left\{ \Psi_{jik}^{\alpha\gamma}(x/\varepsilon) \frac{\partial v_\varepsilon^\gamma}{\partial x_k} \right\} & \text{on } \partial\Omega, \\ \tilde{\theta}_\varepsilon \in H^1(\Omega) \quad \text{and} \quad \int_{\partial\Omega} \tilde{\theta}_\varepsilon = 0, \end{cases} \tag{6.8}$$

$$\begin{cases} (\mathcal{L}_\varepsilon(\tilde{z}_\varepsilon))^\alpha = \varepsilon \frac{\partial}{\partial x_i} \left\{ b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} \right\} & \text{in } \Omega, \\ \left(\frac{\partial \tilde{z}_\varepsilon}{\partial \nu_\varepsilon} \right)^\alpha = -\varepsilon n_i(x) b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} & \text{on } \partial\Omega \\ \tilde{z}_\varepsilon \in H^1(\Omega) \quad \text{and} \quad \int_{\partial\Omega} \tilde{z}_\varepsilon = 0, \end{cases} \quad (6.9)$$

and

$$\rho = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \tilde{w}_\varepsilon = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} w_\varepsilon - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} (v_\varepsilon - u_0)$$

is a constant. Note that

$$|\rho| \leq C\varepsilon \|\nabla v_\varepsilon\|_{L^2(\partial\Omega)} + C \|v_\varepsilon - u_0\|_{L^2(\partial\Omega)} \leq C\varepsilon,$$

where we have used the fact $\int_{\partial\Omega} u_\varepsilon = \int_{\partial\Omega} u_0 = 0$ as well as Lemmas 4.3 and 4.5.

By a duality argument similar to that in the proof of Theorem 5.2, we have

$$\|\tilde{\theta}_\varepsilon\|_{L^2(\partial\Omega)} \leq C\varepsilon \|\nabla v_\varepsilon\|_{L^2(\partial\Omega)} \leq C\varepsilon.$$

It then follows from the estimates for the L^2 Dirichlet problem in Theorem 2.1 that

$$\begin{aligned} \|\mathcal{M}(\tilde{\theta}_\varepsilon)\|_{L^2(\partial\Omega)} + \|\tilde{\theta}_\varepsilon\|_{H^{1/2}(\Omega)} + \left\{ \int_\Omega |\nabla \tilde{\theta}_\varepsilon|^2 \delta(x) \, dx \right\}^{1/2} \\ \leq C \|\tilde{\theta}_\varepsilon\|_{L^2(\partial\Omega)} \leq C\varepsilon. \end{aligned}$$

The estimates of \tilde{z}_ε also relies on a duality estimate. Indeed, let $\Theta_\varepsilon \in H^1(\Omega)$ be the solution of (5.10) with $h \in L^2(\partial\Omega)$ and $\int_{\partial\Omega} h = 0$. It follows from integration by parts that

$$\begin{aligned} \left| \int_{\partial\Omega} \tilde{z}_\varepsilon \cdot h \right| &= \left| \int_{\partial\Omega} \tilde{z}_\varepsilon \cdot \frac{\partial \Theta_\varepsilon}{\partial \nu_\varepsilon} \right| \\ &= \left| \int_\Omega a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial \tilde{z}_\varepsilon^\alpha}{\partial x_i} \cdot \frac{\partial \Theta_\varepsilon^\beta}{\partial x_j} \right| \\ &= \varepsilon \left| \int_\Omega b_{ijk}^{\alpha\gamma}(x/\varepsilon) \frac{\partial^2 v_\varepsilon^\gamma}{\partial x_j \partial x_k} \cdot \frac{\partial \Theta_\varepsilon^\alpha}{\partial x_i} \right|. \end{aligned} \quad (6.10)$$

By the Cauchy inequality this gives

$$\begin{aligned} \left| \int_{\partial\Omega} \tilde{z}_\varepsilon \cdot h \right| \\ \leq \varepsilon \left\{ \int_\Omega |\nabla^2 v_\varepsilon|^2 \delta(x) \phi_a(\delta(x)) \, dx \right\}^{1/2} \\ \left\{ \int_\Omega |\nabla \Theta_\varepsilon|^2 \frac{dx}{\delta(x) \phi_a(\delta(x))} \right\}^{1/2}. \end{aligned} \quad (6.11)$$

Observe that if $a > 1$,

$$\left\{ \int_{\Omega} |\nabla \Theta_{\varepsilon}|^2 \frac{dx}{\delta(x)\phi_a(\delta(x))} \right\}^{1/2} \leq C \|(\nabla \Theta_{\varepsilon})^*\|_{L^2(\partial\Omega)} \leq C \|h\|_{L^2(\partial\Omega)}.$$

Hence, by (6.11) and duality, we obtain

$$\|\tilde{z}_{\varepsilon}\|_{L^2(\partial\Omega)} \leq C\varepsilon \left\{ \int_{\Omega} |\nabla^2 v_{\varepsilon}|^2 \delta(x)\phi_a(\delta(x)) dx \right\}^{1/2} \tag{6.12}$$

for any $a > 1$. With (6.12) at our disposal we may apply Lemma 4.6 to obtain

$$\|\tilde{z}_{\varepsilon}\|_{L^2(\Omega)} \leq C\varepsilon \left\{ \int_{\Omega} |\nabla^2 v_{\varepsilon}|^2 \delta(x)\phi_a(\delta(x)) dx \right\}^{1/2} \leq C\varepsilon |\ln(\varepsilon)|^{a/2},$$

where we have used (4.15) for the last inequality. This, together with estimates of $\tilde{\theta}_{\varepsilon}$, ρ and $v_{\varepsilon} - u_0$, gives (6.3). Estimates (6.4) and (6.5) follow from Lemma 4.6 and (6.12) in the same manner. This completes the proof of Theorem 6.1. \square

Remark 6.3. The estimates in Theorem 6.1 and Corollary 6.2 continue to hold under the condition $\int_{\Omega} u_{\varepsilon} = \int_{\Omega} u_0 = 0$. In this case one has

$$\rho = \frac{1}{|\Omega|} \int_{\Omega} \{\tilde{w}_{\varepsilon} - \tilde{\theta}_{\varepsilon} - \tilde{z}_{\varepsilon}\} = \frac{1}{|\Omega|} \int_{\Omega} w_{\varepsilon} - \frac{1}{|\Omega|} \int_{\Omega} \{v_{\varepsilon} - u_0\} - \frac{1}{|\Omega|} \int_{\Omega} \{\tilde{\theta}_{\varepsilon} + \tilde{z}_{\varepsilon}\}.$$

Hence,

$$\begin{aligned} |\rho| &\leq C\varepsilon \|\nabla v_{\varepsilon}\|_{L^2(\Omega)} + C\|v_{\varepsilon} - u_0\|_{L^2(\Omega)} + C\|\tilde{\theta}_{\varepsilon}\|_{L^2(\Omega)} + C\|\tilde{z}_{\varepsilon}\|_{L^2(\Omega)} \\ &\leq C\varepsilon |\ln(\varepsilon)|^{a/2}, \end{aligned}$$

for any $a > 1$. The rest of the proof is the same.

Proof of Theorem 1.2. Estimate (1.5) for the Neumann boundary conditions is given in Corollary 5.3 and estimate (1.7) in Corollary 6.2. \square

7. Convergence Rates for Eigenvalues

In this section we study the convergence rates for Dirichlet, Neumann, and Steklov eigenvalues associated with $\{\mathcal{L}_{\varepsilon}\}$. Our approach relies on the following theorem, whose proof may be found in [12, pp. 338–345].

Theorem 7.1. *Let $\{T_{\varepsilon}, \varepsilon \geq 0\}$ be a family of bounded, positive, self-adjoint, compact operators on a Hilbert space \mathcal{H} . Suppose that (1) $\|T_{\varepsilon}\| \leq C$ and $\|T_{\varepsilon}f - T_0f\| \rightarrow 0$ as $\varepsilon \rightarrow 0$ for all $f \in H$; (2) $\{T_{\varepsilon}f_{\varepsilon}, \varepsilon > 0\}$ is pre-compact in \mathcal{H} , whenever $\{f_{\varepsilon}\}$ is bounded in \mathcal{H} . Let $\{u_{\varepsilon}^k\}$ be an orthonormal basis of \mathcal{H} consisting of eigenvectors of T_{ε} with the corresponding eigenvalues $\{\mu_{\varepsilon}^k\}$ in a decreasing order,*

$$\mu_{\varepsilon}^1 \geq \mu_{\varepsilon}^2 \geq \dots \geq \mu_{\varepsilon}^k \geq \dots > 0.$$

Then $\mu_\varepsilon^k \geq c_k > 0$, and if $\varepsilon > 0$ is sufficiently small,

$$|\mu_\varepsilon^k - \mu_0^k| \leq 2 \sup \{ \|T_\varepsilon u - T_0 u\| : u \in N(\mu_0^k, T_0) \text{ and } \|u\| = 1 \},$$

where $N(\mu_0^k, T_0)$ is the eigenspace of T_0 associated with eigenvalue μ_0^k .

Dirichlet eigenvalues. Given $f \in \mathcal{H} = L^2(\Omega)$, let $T_\varepsilon^D(f) = u_\varepsilon \in H_0^1(\Omega)$ be the weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = f$ in Ω . It is easy to see that $\{T_\varepsilon^D, \varepsilon \geq 0\}$ satisfies the assumptions in Theorem 7.1. Recall that λ_ε is a Dirichlet eigenvalue for \mathcal{L}_ε in Ω if there exists a nonzero $u_\varepsilon \in H_0^1(\Omega)$ such that $\mathcal{L}_\varepsilon(u_\varepsilon) = \lambda_\varepsilon u_\varepsilon$ in Ω . Let $\{\lambda_\varepsilon^k\}$ denote the sequence of Dirichlet eigenvalues in an increasing order for \mathcal{L}_ε . Then $\{(\lambda_\varepsilon^k)^{-1}\}$ is the sequence of eigenvalues in a decreasing order for T_ε^D on $L^2(\Omega)$. It follows from Theorem 7.1 that if ε is sufficiently small,

$$\left| \frac{1}{\lambda_\varepsilon^k} - \frac{1}{\lambda_0^k} \right| \leq 2 \sup \{ \|u_\varepsilon - u_0\|_{L^2(\Omega)} \}, \tag{7.1}$$

where the supremum is taken over all $u_\varepsilon, u_0 \in H_0^1(\Omega)$ with the property that $\lambda_0^k \|u_0\|_{L^2(\Omega)} = 1$ and $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0) = \lambda_0^k u_0$ in Ω . Note that if Ω is $C^{1,1}$ (or convex in the case $m = 1$), then $\|u_0\|_{H^2(\Omega)} \leq C_k$ by the standard regularity theory for second-order elliptic systems with constant coefficients. Also, by Theorem 7.1, we see that $\lambda_\varepsilon^k \leq C_k$. Hence, we may deduce from (7.1) and Corollary 3.5 that $|\lambda_\varepsilon^k - \lambda_0^k| \leq c_k \varepsilon$ (see for example [12, p. 347]). However, if Ω is a general Lipschitz domain, $u_0 \in H^2(\Omega)$ no longer holds. Nevertheless, Corollary 4.2 gives us the following.

Theorem 7.2. *Let $0 \leq \varepsilon \leq (1/2)$ and $\{\lambda_\varepsilon^k\}$ be the sequence of Dirichlet eigenvalues in an increasing order of \mathcal{L}_ε in a bounded Lipschitz domain. Then for any $\sigma > 0$,*

$$|\lambda_\varepsilon^k - \lambda_0^k| \leq c \varepsilon |\ln(\varepsilon)|^{\frac{1}{2} + \sigma},$$

where c depends on k and σ , but not ε .

Neumann eigenvalues. Given $f \in L^2(\Omega)$ with $\int_\Omega f = 0$, let $T_\varepsilon^N(f) = u_\varepsilon \in H^1(\Omega)$ be the weak solution of the Neumann problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = f$ in Ω , $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on $\partial\Omega$ and $\int_\Omega u_\varepsilon = 0$. Again, it is easy to verify that the family of operators $\{T_\varepsilon^N\}$ on the Hilbert space $\{f \in L^2(\Omega) : \int_\Omega f = 0\}$ satisfies the assumptions of Theorem 7.1. Recall that ρ_ε is a Neumann eigenvalue for \mathcal{L}_ε in Ω if there exists a nonzero $u_\varepsilon \in H^1(\Omega)$ such that $\mathcal{L}_\varepsilon(u_\varepsilon) = \rho_\varepsilon u_\varepsilon$ in Ω and $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on $\partial\Omega$. Let $\{\rho_\varepsilon^k\}$ be the sequence of nonzero Neumann eigenvalues in an increasing order for \mathcal{L}_ε in Ω . Then $\{(\rho_\varepsilon^k)^{-1}\}$ is the sequence of eigenvalues in a decreasing order for T_ε^N . Thus, in view of Theorems 7.1 and Remarks 5.4 and 6.3, we obtain the following.

Theorem 7.3. *Let $0 \leq \varepsilon \leq (1/2)$ and $\{\rho_\varepsilon^k\}$ denote the sequence of nonzero Neumann eigenvalues in an increasing order for \mathcal{L}_ε in a bounded Lipschitz domain Ω . Then for any $\sigma > 0$,*

$$|\rho_\varepsilon^k - \rho_0^k| \leq c \varepsilon |\ln(\varepsilon)|^{\frac{1}{2} + \sigma},$$

where c depends on k and σ , but not ε . Furthermore, the estimate $|\rho_\varepsilon^k - \rho_0^k| \leq c_k \varepsilon$ holds if Ω is $C^{1,1}$ (or convex in the case $m = 1$).

Steklov eigenvalues. We say s_ε is a Steklov eigenvalue for \mathcal{L}_ε in Ω if there exists a nonzero $u_\varepsilon \in H^1(\Omega)$ such that $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω and $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = s_\varepsilon |\partial\Omega|^{-1} u_\varepsilon$ on $\partial\Omega$. Note that $s_\varepsilon |\partial\Omega|^{-1}$ is also an eigenvalue of the Dirichlet-to-Neumann map associated with \mathcal{L}_ε . Given $g \in L^2(\partial\Omega)$ with $\int_{\partial\Omega} g = 0$, let $S_\varepsilon(g) = u_\varepsilon|_{\partial\Omega}$, where u_ε is the weak solution to the L^2 Neumann problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω , $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g$ on $\partial\Omega$ and $\int_{\partial\Omega} u_\varepsilon = 0$. It is not hard to verify that the family of operators $\{S_\varepsilon\}$ on the Hilbert space $\{g \in L^2(\partial\Omega) : \int_{\partial\Omega} g = 0\}$ satisfies the assumptions in Theorem 7.1. Consequently, the $L^2(\partial\Omega)$ convergence estimates in Corollaries 5.3 and 6.2 give the following.

Theorem 7.4. *Let $0 \leq \varepsilon \leq (1/2)$ and $\{s_\varepsilon^k\}$ denote the sequence of nonzero Steklov eigenvalues in an increasing order for \mathcal{L}_ε in a bounded Lipschitz domain Ω . Then for any $\sigma > 0$,*

$$|s_\varepsilon^k - s_0^k| \leq c \varepsilon |\ln(\varepsilon)|^{\frac{1}{2} + \sigma},$$

where c depends on k and σ , but not ε . Furthermore, the estimate $|s_\varepsilon^k - s_0^k| \leq c_k \varepsilon$ holds if Ω is $C^{1,1}$ (or convex in the case $m = 1$).

Remark 7.5. The operator S_ε introduced above is, in fact, the inverse of the Dirichlet–Neumann map associated with \mathcal{L}_ε . Note that by Corollaries 5.3 and 6.2,

$$\|S_\varepsilon - S_0\|_{L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)} \leq \begin{cases} C \varepsilon & \text{if } \Omega \text{ is } C^{1,1}, \\ C_\sigma \varepsilon |\ln(\varepsilon)|^{\frac{1}{2} + \sigma} & \text{if } \Omega \text{ is Lipschitz,} \end{cases} \tag{7.2}$$

for any $\sigma > 0$.

8. Weighted Potential Estimates

Let $H_\varepsilon(x) = (H_\varepsilon^1(x), \dots, H_\varepsilon^m(x))$ be defined by

$$H_\varepsilon^\alpha(x) = \int_\Omega \frac{\partial}{\partial y_k} \{ \Gamma_\varepsilon^{\alpha\beta}(x, y) \} h^\beta(y) \, dy, \tag{8.1}$$

where $h = (h^1, \dots, h^m) \in L^2(\Omega)$. It follows from [3] that $\|\nabla H_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C \|h\|_{L^2(\Omega)}$.

Proposition 8.1. *The estimate*

$$\|H_\varepsilon\|_{L^2(\partial\Omega)} \leq C_a \left\{ \int_\Omega |h(x)|^2 \delta(x) \phi_a(\delta(x)) \, dx \right\}^{1/2} \tag{8.2}$$

holds for any $a > 1$.

Proof. Recall that $\delta(x) = \text{dist}(x, \partial\Omega)$ and $\phi_a(t) = \{\ln(\frac{1}{t} + e^a)\}^a$. Let $g = (g^1, \dots, g^m) \in L^2(\partial\Omega)$ and $u_\varepsilon = (u_\varepsilon^1, \dots, u_\varepsilon^m)$, where

$$u_\varepsilon^\beta(y) = \int_{\partial\Omega} \frac{\partial}{\partial y_k} \{ \Gamma_\varepsilon^{\alpha\beta}(x, y) \} g^\alpha(x) \, d\sigma(x).$$

It follows from [15, Theorem 4.3] that $\|(u_\varepsilon)^*\|_{L^2(\partial\Omega)} \leq C \|g\|_{L^2(\partial\Omega)}$. Observe that

$$\begin{aligned} \left| \int_{\partial\Omega} H_\varepsilon^\alpha(x) g^\alpha(x) \, d\sigma(x) \right| &= \left| \int_{\Omega} h^\beta(y) u_\varepsilon^\beta(y) \, dy \right| \\ &\leq \left\{ \int_{\Omega} |h(y)|^2 \delta(y) \phi_a(\delta(y)) \, dy \right\}^{1/2} \left\{ \int_{\Omega} |u_\varepsilon(y)|^2 \{ \delta(y) \phi_a(\delta(y)) \}^{-1} \, dy \right\}^{1/2} \end{aligned} \tag{8.3}$$

and that if $a > 1$,

$$\int_{\Omega} |u_\varepsilon(y)|^2 \{ \delta(y) \phi_a(\delta(y)) \}^{-1} \, dy \leq C \int_{\partial\Omega} |(u_\varepsilon)^*|^2 \, d\sigma \leq C \|g\|_{L^2(\partial\Omega)}^2. \tag{8.4}$$

Estimate (8.2) follows from (8.3)–(8.4) by duality. \square

Proposition 8.2. *The estimate*

$$\|H_\varepsilon\|_{L^2(\Omega)} \leq C_a \left\{ \int_{\Omega} |h(x)|^2 \delta(x) \phi_a(\delta(x)) \, dx \right\}^{1/2} \tag{8.5}$$

holds for any $a > 1$.

Proof. Let $K \subset K_1$ be two compact subsets of Ω such that $\text{dist}(K, \Omega \setminus K_1) \geq c_0 > 0$. Since $|\nabla_y \Gamma_\varepsilon(x, y)| \leq C|x - y|^{1-d}$, we have

$$|H_\varepsilon(x)| \leq C \int_{K_1} \frac{|h(y)| \, dy}{|x - y|^{d-1}} + C \int_{\Omega \setminus K_1} |h(y)| \, dy \quad \text{for any } x \in K.$$

This implies that $\|H_\varepsilon\|_{L^2(K)}$ is bounded by the right-hand side of (8.5) if $a > 1$.

To estimate $\|H_\varepsilon\|_{L^2(\Omega \setminus K)}$, it suffices to show that $\|H_\varepsilon\|_{L^2(\partial\Omega_t)}$ is bounded uniformly in t by the right-hand side of (8.5) for $-c < t < 0$, where Ω_t is defined in Section 2. This may be done by a duality argument, as in the proof of Proposition 8.1. Indeed, the argument reduces the problem to the following estimate

$$\int_{\Omega} |u_{\varepsilon,t}(y)|^2 \{ \delta(y) \phi_a(\delta(y)) \}^{-1} \, dy \leq C \int_{\partial\Omega_t} |g|^2 \, d\sigma, \tag{8.6}$$

where

$$u_{\varepsilon,t}^\beta(y) = \int_{\partial\Omega_t} \frac{\partial}{\partial y_k} \{ \Gamma_\varepsilon^{\alpha\beta}(x, y) \} g^\alpha(x) \, d\sigma(x).$$

Finally, the estimate (8.6) follows from the observation that $\|u_{\varepsilon,t}\|_{L^2(K)} \leq C_K \|g\|_{L^2(\partial\Omega_t)}$ for compact $K \subset \Omega_t$, and that $\|u_{\varepsilon,t}\|_{L^2(\partial\Omega_s)} \leq C \| (u_{\varepsilon,t})^* \|_{L^2(\partial\Omega_t)} \leq C \|g\|_{L^2(\partial\Omega_t)}$ for $-c < t, s < 0$. This completes the proof. \square

Proposition 8.3. *The estimate*

$$\left\{ \int_{\Omega} |\nabla H_{\varepsilon}(x)|^2 \delta(x) \phi_a(\delta(x)) \, dx \right\}^{1/2} \leq C_a \left\{ \int_{\Omega} |h(x)|^2 \delta(x) \phi_{a+2}(\delta(x)) \, dx \right\}^{1/2} \tag{8.7}$$

holds for any $a \geq 0$.

Proof. Using $|\nabla_x \nabla_y \Gamma_{\varepsilon}(x, y)| \leq C|x-y|^{-d}$ and a partition of unity, we may reduce the estimate (8.7) to the case where $\Omega = \{(x', x_d) : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi(x')\}$ is the region above a Lipschitz graph and $\delta(x)$ is replaced by $\tilde{\delta}(x) = |x_d - \psi(x')|$. Since $\Gamma_{\varepsilon}(x, y) = \varepsilon^{2-d} \Gamma_1(x/\varepsilon, y/\varepsilon)$, by a rescaling argument, we may further reduce the problem to the following weighted L^2 inequality for a singular integral operator,

$$\int_{\mathbb{R}^d} |T(f)|^2 \omega_1 \, dx \leq C \int_{\mathbb{R}^d} |f|^2 \omega_2 \, dx, \tag{8.8}$$

where $\omega_1(x) = \tilde{\delta}(x) \phi_a(\varepsilon \tilde{\delta}(x))$, $\omega_2(x) = \tilde{\delta}(x) \phi_{a+2}(\varepsilon \tilde{\delta}(x))$ and

$$T(f)(x) = \int_{\mathbb{R}^d} \nabla_x \nabla_y \Gamma_1(x, y) f(y) \, dy. \tag{8.9}$$

We point out that the constant C in (8.8) should depend only on $d, m, \mu, \lambda, \tau, a$ and $\|\nabla \psi\|_{\infty}$.

To establish (8.8), we first use the asymptotic estimates on $\nabla_x \nabla_y \Gamma_1(x, y)$ for $|x - y| \leq 1$ and $|x - y| \geq 1$ in [3] to obtain

$$|T(f)(x)| \leq C \{ |T_1^*(g_1)(x)| + |T_2^*(g_2)(x)| + M(f)(x) \}, \tag{8.10}$$

where T_1^*, T_2^* are L^2 bounded maximal singular integral operators with standard Calderón–Zygmund kernels, $M(f)$ is the Hardy–Littlewood maximal function of f in \mathbb{R}^d , and $|g_1|, |g_2|$ are bounded pointwise by $C|f|$. Next we observe that ω_1 is an A_{∞} weight in \mathbb{R}^d . This allows us to use a classical result of COIFMAN and FEFFERMAN [6] and (8.10) to deduce that

$$\int_{\mathbb{R}^d} |T(f)|^2 \omega_1 \, dx \leq C \int_{\mathbb{R}^d} |M(f)|^2 \omega_1 \, dx. \tag{8.11}$$

As a result, it remains only to show that

$$\int_{\mathbb{R}^d} |M(f)|^2 \omega_1 \, dx \leq C \int_{\mathbb{R}^d} |f|^2 \omega_2 \, dx. \tag{8.12}$$

This is a two-weight norm inequality for the Hardy–Littlewood maximal operator, which has been studied extensively. In particular, SAWYER [25] was able to characterize all pairs of (ω_1, ω_2) for which (8.12) holds.

Finally, to prove (8.12), by a bi-Lipschitz transformation, we may assume that $\psi = 0$. Consequently, it suffices to consider the case $d = 1$. This is because $M(f) \leq M_1 \circ M_2 \circ \dots \circ M_d(f)$, where M_i denotes the Hardy–Littlewood maximal function in the x_i variable. Furthermore, by rescaling, we may assume $\varepsilon = 1$.

With $\omega_1(x) = |x|\phi_a(|x|)$ and $\omega_2(x) = |x|\phi_{2+a}(|x|)$ in \mathbb{R} , it is not very hard to verify that (ω_1, ω_2) satisfies the necessary and sufficient condition in [25] for any $a \geq 0$. We omit the details. \square

Proposition 8.4. *The estimate*

$$\|\mathcal{M}(H_\varepsilon)\|_{L^2(\partial\Omega)} \leq C_a \left\{ \int_\Omega |h(x)|^2 \delta(x) \phi_a(\delta(x)) \, dx \right\}^{1/2} \tag{8.13}$$

holds for any $a > 3$.

Proof. By the fundamental theorem of calculus and definition of the radial maximal operator, it is easy to see that for any $Q \in \partial\Omega$,

$$\begin{aligned} \mathcal{M}(u)(Q) &\leq C \int_{-c}^0 \{ |\nabla u(\Lambda_t(Q))| + |u(\Lambda_t(Q))| \} \, dt \\ &\leq C_a \left\{ \int_{-c}^0 \{ |\nabla u(\Lambda_t(Q))|^2 + |u(\Lambda_t(Q))|^2 \} |t| \phi_a(|t|) \, dt \right\}^{1/2}, \end{aligned} \tag{8.14}$$

for any $a > 1$. This yields that

$$\int_{\partial\Omega} |\mathcal{M}(u)|^2 \, d\sigma \leq C_a \int_\Omega \{ |\nabla u(x)|^2 + |u(x)|^2 \} \delta(x) \phi_a(\delta(x)) \, dx. \tag{8.15}$$

Letting $u(x) = H_\varepsilon(x)$ in (8.15), we obtain estimate (8.13) by Propositions 8.2–8.3. \square

Proposition 8.5. *Let $f = (f^1, \dots, f^m) \in L^2(\partial\Omega)$ and $u_\varepsilon = (u_\varepsilon^1, \dots, u_\varepsilon^m)$ be given by*

$$u_\varepsilon^\alpha(x) = \int_{\partial\Omega} \frac{\partial}{\partial y_k} \{ \Gamma_\varepsilon^{\alpha\beta}(x, y) \} f^\beta(y) \, d\sigma(y).$$

Then

$$\left\{ \int_\Omega |\nabla u_\varepsilon(x)|^2 \delta(x) \, dx \right\}^{1/2} \leq C \|f\|_{L^2(\partial\Omega)}. \tag{8.16}$$

Proof. By a partition of unity we may assume that $\Omega = \{(x', x_d) : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi(x')\}$ is the region above a Lipschitz graph. By a rescaling argument we may further assume that $\varepsilon = 1$.

We first estimate the integral of $|\nabla u_1(x)|^2 \delta(x)$ on

$$D = \Omega + (0, \dots, 1) = \{(x', x_d) : x_d > \psi(x') + 1\}.$$

By the asymptotic estimates of $\nabla_x \nabla_y \Gamma_1(x, y)$ for $|x - y| \geq 1$ in [3, p. 906], we may deduce that if $x \in D$,

$$|\nabla u_1(x) - W(x)| \leq C \int_{\partial\Omega} \frac{|f(y)| \, d\sigma(y)}{|x - y|^{d+\eta}} \tag{8.17}$$

for some $\eta > 0$, where $W(x)$ is a finite sum of functions of form

$$e_{ij}(x) \int_{\partial\Omega} \frac{\partial^2}{\partial x_i \partial y_j} \{ \Gamma_0^{\alpha\beta}(x, y) \} g^\beta(y) \, d\sigma(y),$$

with $|e_{ij}(x)| \leq C$ and $|g^\beta| \leq C|f|$. Recall that $\Gamma_0(x, y)$ is the matrix of fundamental solutions for the operator \mathcal{L}_0 (with constant coefficients), for which the estimate (8.16) is well known [7]. It follows that

$$\int_{\Omega} |W(x)|^2 \delta(x) \, dx \leq C \int_{\partial\Omega} |f|^2 \, d\sigma. \tag{8.18}$$

Let $I(x)$ denote the integral in the right-hand side of (8.17). By the Cauchy inequality,

$$|I(x)|^2 \leq C \{ \delta(x) \}^{-1-\eta} \int_{\partial\Omega} \frac{|f(y)|^2 \, d\sigma(y)}{|x - y|^{d+\eta}}.$$

This gives $\int_D |I(x)|^2 \delta(x) \, dx \leq C \|f\|_{L^2(\partial\Omega)}^2$ and thus $\int_D |\nabla u_1(x)|^2 \delta(x) \, dx \leq C \|f\|_{L^2(\partial\Omega)}^2$.

To handle ∇u_1 in $\Omega \setminus D$, we let

$$\begin{aligned} \Delta(r) &= \{ (x', \psi(x')) : |x'| < r \}, \\ T(r) &= \{ (x', x_d) : |x'| < r \text{ and } \psi(x') < x_d < \psi(x') + C_0 r \}. \end{aligned} \tag{8.19}$$

We will show that if $\mathcal{L}_1(u) = 0$ in the Lipschitz domain $T(2)$ and $(u)^* \in L^2$, then

$$\int_{T(1)} |\nabla u(x)|^2 |x_d - \psi(x')| \, dx \leq C \int_{\Delta(2)} |u|^2 \, d\sigma + C \int_{T(2)} |u|^2 \, dx, \tag{8.20}$$

which is bounded by $C \int_{\Delta(2)} |(u)^*|^2 \, d\sigma$. By a simple covering argument one may deduce from (8.20) that

$$\int_{\Omega \setminus D} |\nabla u_1|^2 \delta(x) \, dx \leq C \int_{\partial\Omega} |(u_1)^*|^2 \, d\sigma \leq C \int_{\partial\Omega} |f|^2 \, d\sigma, \tag{8.21}$$

where the last inequality was proved in [15].

Finally, to see (8.20), we use the square function estimate for \mathcal{L}_1 on $T(r)$ for $3/2 < r < 2$,

$$\int_{T(r)} |\nabla u(x)|^2 \text{dist}(x, \partial T(r)) \, dx \leq C \int_{\partial T(r)} |u|^2 \, d\sigma, \tag{8.22}$$

to obtain

$$\int_{T(1)} |\nabla u(x)|^2 |x_d - \psi(x')| \, dx \leq C \int_{\Delta(2)} |u|^2 \, d\sigma + C \int_{\partial T(r) \setminus \Delta(2)} |u|^2 \, d\sigma. \tag{8.23}$$

Estimate (8.20) follows by integrating both sides of (8.23) in $r \in (3/2, 2)$. We remark that under the condition $A \in \Lambda(\mu, \lambda, \tau)$, the square function estimate (8.22) follows from the double layer potential representation obtained in [15] for solutions of the L^2 Dirichlet problem by a $T(b)$ -theorem argument (see for example [18, pp. 9–11]). This completes the proof. \square

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