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Equilibrium Configurations of Epitaxially Strained Elastic Films: Second Order Minimality Conditions and Qualitative Properties of Solutions

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Abstract

We consider a variational model introduced in the physical literature to describe the epitaxial growth of an elastic film over a thick flat substrate when a lattice mismatch between the two materials is present. We study quantitative and qualitative properties of equilibrium configurations, that is, of local and global minimizers of the free-energy functional. More precisely, we determine analytically the critical threshold for the local minimality of the flat configuration and we also prove several results concerning its global minimality. The non-occurrence of singularities in non-flat global minimizers is also addressed. One of the main results of the paper is a new sufficient condition for local minimality, which provides the first extension of the classical criteria based on the positivity of the second variation to the context of functionals with bulk and surface energies.

Contents

1. Introduction

In recent years, the physical and computational communities have displayed a growing interest in the study of the morphological instabilities of interfaces between solids generated by elastic stress, the so-called Stress Driven Rearrangement Instabilities.

These occur, for instance, in hetero-epitaxial growth of thin films for systems with a lattice mismatch between film and substrate, such as InGaAs/GaAs or SiGe/Si, which are useful in the fabrication of nano-structures with specific optic and electronic properties. When the film is grown on a flat substrate, its profile remains flat until a critical value of the thickness is reached, after which the free surface develops corrugations or other kinds of irregularities. This is referred to as the Asaro–Grinfeld–Tiller (AGT) instability, after the scientists who started the theoretical investigations on this phenomenon [5,18].

This threshold effect is usually explained in terms of the minimization of two competing forms of energy: the surface energy and the bulk elastic energy. More precisely, due to the lattice mismatch between film and substrate, no stress-free states are admissible and the flat configuration (of the film) bears an amount of stored elastic energy that is proportional to the volume of the material deposited. Hence, nontrivial morphologies (such as wavy profiles or formations of strained islands separated by a thin wetting layer) become favorable when the higher energetic cost in terms of surface tension is compensated by the release of elastic energy.

Several numerical and theoretical studies have been carried out to study qualitative properties of equilibrium configurations of strained epitaxial films (see, for example, [16,23,26,27]). All these works are very insightful. Nevertheless, they rely upon formal methods that often lack rigorous mathematical content.

The paper [19] by Grinfeld casts the study of AGT instability in a more analytical perspective. Following the celebrated Gibbs variational approach, the author considers a suitable free-energy functional (given by the sum of the stored elastic energy of the film and the interfacial energy of its free surface) and studies when the second variation is positive definite, establishing various instability results for the flat morphology of the film. However, existence of minimizers and the problem of deriving minimality properties from the positive definiteness of the second variation are not addressed.

A first attempt to provide a sound variational formulation for the existence problem of minimizing configurations in the context of epitaxial growth has been carried out in [7], but for an unrealistic one-dimensional model. The determination of a proper functional setting for the more realistic energy introduced in [19] has been achieved in [6] and in [15] (for a slightly different model), using relaxation and geometric measure theory techniques. In both papers the framework is that of linear elasticity and only two-dimensional morphologies are considered, corresponding to three-dimensional configurations with planar symmetry (see also [10] for a partial extension of these relaxation results to higher dimensions). Besides the existence of minimizing configurations, in [15] a complete regularity theory is obtained and, in the wetting regime, a rigorous proof of the zero contact angle condition between film and substrate is achieved, providing an analytical confirmation of the formal analysis of [25] (see also [14] dealing with the case of anisotropic surface energy). Concerning the possible formation of singularities in the film, it is shown that they can only be finitely many and of cusp type. In fact, the appearance of cusps, possibly leading to vertical fractures in the material, is observed both numerically and experimentally (see [16,26]). The regularity theory developed in [15] also applies with minor changes to the model considered in [6] (see Section 2.1). Similar regularity results for related free boundary problems are also contained in [3,9,22,24].

However, the analysis carried out in [15] leaves out several important issues concerning the qualitative properties of equilibrium configurations. In this paper we mainly address the following two issues for the model considered in [6]:

– we seek to prove rigorous minimality results and to perform a detailed analysis of the energy landscape, focusing, in particular, on the analytical determination of the critical volume thresholds for the local and global minimality of the flat configuration;
– we investigate under which conditions cusp singularities or fractures do not form, once the flat configuration becomes unstable.

These theoretical investigations may be important for those applications where the formation of corrugations and singularities in the film is undesirable.

We now describe the model studied in [6]. We assume that the reference configuration of the film is

$$\Omega_h := \left\{ z = (x, y) \in \mathbb{R}^2 : 0 < x < b,\ 0 < y < h(x) \right\},$$

where $h : [0, b] \to [0, \infty)$ represents the free-profile of the film. We work within the theory of small deformations, so that

$$E(u) := \frac{1}{2}\left(\nabla u + \nabla^T u\right)$$

represents the strain, with $u : \Omega_h \to \mathbb{R}^2$ the planar displacement. We also prescribe a Dirichlet boundary condition of the form $u(x, 0) = (e_0 x, 0) + q(x)$ at the interface between film and substrate, which models the case of a film growing on an infinitely rigid substrate. This boundary condition forces the film to be strained, thus generating elastic energy. The positive constant e_0 measures the mismatch between the lattices of the two materials and q is a b-periodic function. As customary in the physical literature and following [6], we also impose the periodicity conditions $h(0) = h(b)$ and $u(b, y) = u(0, y) + (e_0 b, 0)$. The energy associated with a configuration (h, u) when h is smooth is given by

$$F(h, u) = \int_{\Omega_h} \left[\mu |E(u)|^2 + \frac{\lambda}{2}(\mathrm{div} u)^2\right] \mathrm{d}z + \sigma \mathcal{H}^1(\Gamma_h),$$

where μ and λ represent the Lamé coefficients of the material, σ is the surface tension on the profile of the film, Γ_h is the graph of h, and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. One seeks to minimize F among all admissible configurations (h, u) satisfying a volume constraint $|\Omega_h| = d$. However, smooth minimizing sequences may converge to irregular configurations, where the profile

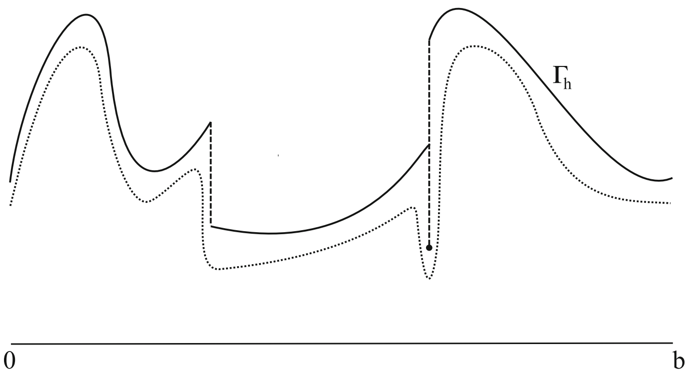

Fig. 1. An irregular profile h and a smooth approximation. The smooth profile surrounds the vertical cut so that in the limit its length is counted twice

h is just a lower semicontinuous function of bounded variation. In particular, the (extended) graph of h may contain vertical parts and cuts. The latter can be interpreted as vertical cracks in the film (see Fig. 1) and their union will be denoted by Σ_h. More precisely,

$$\Sigma_h := \{(x, y) : h(x) \leq y < \min\{h(x-), h(x+)\}\},$$

where $h(x\pm)$ denote the right and left limit at x. We denote this larger class of (possibly irregular) reachable configurations by X. Assume without loss of generality $\sigma = 1$. It is proved in [6] through a relaxation procedure that the energy associated to any pair $(h, u) \in X$ is given by

$$F(h, u) = \int_{\Omega_h} \Big[\mu|E(u)|^2 + \frac{\lambda}{2}(\mathrm{div} u)^2\Big]\, dz + \mathcal{H}^1(\Gamma_h) + 2\mathcal{H}^1(\Sigma_h). \tag{1.1}$$

Notice that in this formula the vertical cracks are counted twice since they arise as limits of regular profiles (see again Fig. 1). Equilibrium configurations are now identified with global or local minimizers of the energy (1.1) in the aforementioned larger class X under a prescribed volume constraint.

Coming to the main results of the paper, which are stated in precise form in Section 2, we first deal with the local minimality of the flat configuration with the Dirichlet datum $u(x, 0) = (e_0 x, 0)$. Roughly speaking, by a local minimizer we mean a configuration (h, u) minimizing the energy among all admissible competitors (g, v) such that $|\Omega_g| = |\Omega_h|$ and the two extended graphs $\Gamma_h \cup \Sigma_h$ and $\Gamma_g \cup \Sigma_g$ are close in the Hausdorff distance. When h is smooth this last condition is equivalent to requiring that h and g are close in the sup norm. In Theorem 2.9 we prove that if the size of the periodicity interval is sufficiently small, namely if

$$0 < b \leqq \frac{\pi}{4} \frac{2\mu + \lambda}{e_0^2 \mu(\mu + \lambda)} =: b_0, \tag{1.2}$$

then all flat configurations are local minimizers, no matter how thick the film is. Moreover, when b is larger than b_0, we show that the flat configuration remains a local minimizer if and only if the thickness of the film is smaller than a critical value depending on b and analytically determined (see (2.16)). Notice that formula (1.2) shows that the smaller the mismatch e_0, the larger the range of local minimality of the flat configuration. Also formula (2.16) displays a similar monotone dependence of the critical thickness on the mismatch.

Qualitatively similar results are shown to hold concerning the global minimality of the flat configuration. In Theorem 2.11 we prove that there exists $0 < b_{\text{crit}} \leqq b_0$ such that the flat configuration is the unique global minimizer for all values of the thickness if and only if $0 < b \leqq b_{\text{crit}}$. Moreover, when $b > b_{\text{crit}}$ we are also able to prove that the flat configuration is the unique global minimizer if and only if the thickness stays below a critical value, which depends on b. Differently from the case of local minimizers, here we do not determine the exact value of b_{crit} and of the critical thickness, since their existence follows from a more indirect argument. We also show that for large values of b, the critical thickness for the global minimality is strictly smaller than the one for the local minimality. When this happens a non-flat global minimizer beside the flat configuration is shown to exist at the critical level.

As mentioned before, another interesting issue concerns the occurrence of cusped or fractured configurations. It is important to establish under which circumstances (if any) non-flat minimal configurations present neither cusps nor vertical cracks. Numerical simulations and experiments suggest that this is the case when the sample is not too large in width and thickness. An analytical confirmation is provided in the third main result of the paper, Theorem 2.14.

One of the main tools needed to prove the previous results is a local minimality criterion based on the study of the second variation of F. More precisely, given a smooth critical configuration (h, u) and a b-periodic variation $\varphi \in C^\infty([0, b])$, with $\int_0^b \varphi \, dx = 0$, one may consider the one-parameter family of configurations (h_t, u_t), where $h_t := h + t\varphi$ and u_t is the elastic equilibrium in Ω_{h_t} under the proper Dirichlet and periodic boundary conditions. Then, the second variation at (h, u) along the direction φ is defined as

$$\frac{d^2}{dt^2} F(h_t, u_t)|_{t=0}.$$

The result is a non-local quadratic form $\partial^2 F(h, u)$, involving also the curvature of Γ_h and the trace of the gradient of $E(u)$ on Γ_h (see formula (3.5)). In Theorem 2.10 we prove that if $\partial^2 F(h, u)$ is positive definite, then (h, u) is a local minimizer. This theorem, together with the explicit calculation of the second variation at the flat configuration, contained in Section 5 and based on [19], allows us to deduce the local minimality properties of the flat configurations described above. Theorem 2.10 may be regarded as the counterpart in our framework of the classical sufficient conditions based on the positiveness of the second variation and of the Weierstrass excess function. However, here we do not resort to a suitably adapted field theory (it is not clear what the right notion of field would be in our context) but we follow a rather different approach, as explained below. To the best of our knowledge, this result provides the first extension of the classical sufficiency theorems for strong local minimizers to the context of functionals with bulk and surface energies.

The proof of Theorem 2.10 stretches for Sections 3, 4 and 6. In fact, finding sufficient minimality conditions is not at all an easy task for functionals with bulk and surface parts, due to the strong lack of convexity they display. Since this theorem is the central result of the paper and its proof is rather complex, we outline here the overall strategy, giving a flavor of the geometric arguments involved. A first crucial step consists in showing that the positivity of $\partial^2 F(h,u)$ implies that (h,u) is a local minimizer with respect to $W^{2,\infty}$-perturbations of the profile. This minimality property may be regarded as the analog of the classical notion of weak minimizer for the standard functionals of the Calculus of Variations. Its proof follows some ideas introduced in [8] to study a similar notion of second variation for the Mumford–Shah functional. However, the presence of the vectorial elastic energy in place of the scalar Dirichlet functional requires a much more involved argument. The key point is to study the continuity properties of the eigenvalues of the operator associated with the quadratic form $\partial^2 F(h,u)$ with respect to $W^{2,\infty}$-variations. Since the expression of $\partial^2 F(h,u)$ involves the trace of the gradient of $E(u)$ on Γ_h, this analysis requires delicate regularity estimates in the appropriate fractional Sobolev spaces. These estimates are carried out in Section 4.

The remaining part of the proof of Theorem 2.10 is devoted to showing that the $W^{2,\infty}$-local minimality is in fact equivalent to the local minimality with respect to any admissible (possibly irregular) profile sufficiently close in the sup norm. The argument goes as follows: Assume by contradiction that the $W^{2,\infty}$-local minimizer (h,u) is not a local minimizer. Then one can find a sequence of configurations (k_n, w_n) with $\sup_{[0,b]}|h-k_n| \leqq \frac{1}{n}$, $|\Omega_{k_n}| = |\Omega_h|$, and $F(k_n,w_n) < F(h,u)$. Consider the obstacle problems

$$\min\Big\{F(g,v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| : (g,v)\in X,\ g \geqq h - \tfrac{1}{n}\Big\}, \tag{1.3}$$

with $\Lambda > 0$, and let (g_n, v_n) be the corresponding minimizing configurations. Notice that we have replaced the volume constraint by a penalization term. Since (k_n, w_n) is an admissible competitor for (1.3), we have in particular

$$F(g_n,v_n) \leqq F(g_n,v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| \leqq F(k_n,w_n) < F(h,u). \tag{1.4}$$

We conclude by showing that if $\Lambda > \Lambda_0$, then g_n is regular and $g_n \to h$ in $W^{2,\infty}$. This fact, together with (1.4), gives a contradiction to the $W^{2,\infty}$-local minimality of (h,u). The proof of the regularity and convergence of g_n is the content of Section 6 and is obtained by refining in a quantitative fashion the regularity estimates for minimal configurations proved in [15]. The argument goes as follows. We first show that if Λ is sufficiently large, then that (h,u) is the unique minimizer to

$$\min\Big\{F(g,v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| : (g,v)\in X,\ g \geqq h\Big\}.$$

From this fact we deduce that (g_n, v_n) must converge (in a suitable sense) to (h,u). In particular, one can show that

$$g_n \to h \quad \text{in } L^\infty(0,b). \tag{1.5}$$

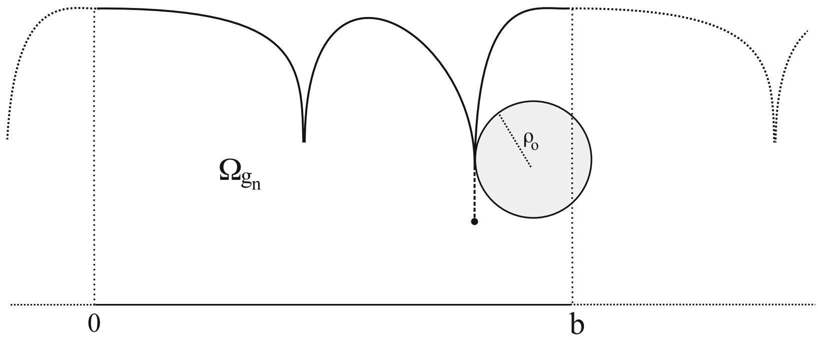

Fig. 2. The uniform inner ball condition: for each point of Γ_{g_n} there is a tangent ball of radius ρ_0 contained in $\Omega^{\#}_{g_n}$

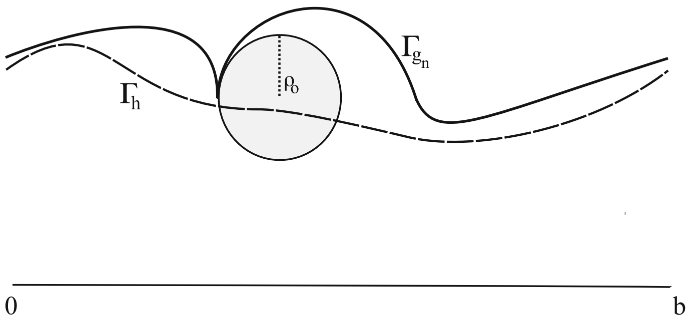

Fig. 3. If Γ_{g_n} contains a cusp or a vertical crack, then, due to the inner ball condition, the L^{∞}-distance between g_n and h is comparable to ρ_0, a contradiction to (1.5)

Next, we observe that from the definition (1.1) of F, the profile g_n minimizes the functional

$$g \to \mathcal{H}^1(\Gamma_g) + 2\mathcal{H}^1(\Sigma_g) + \Lambda\big||\Omega_g| - |\Omega_h|\big|$$

among all admissible g such that $h - \frac{1}{n} \leqq g \leqq g_n$.

This one-sided minimality property alone suffices to provide a lower bound for the curvature (in a generalized sense) of $\Gamma_{g_n} \cup \Sigma_{g_n}$. More precisely, using suitable isoperimetric estimates, we show that for all $z \in \Gamma_{g_n} \cup \Sigma_{g_n}$ there exists a ball $B_{\rho_0}(z_0) \subset \Omega^{\#}_{g_n}$ such that $z \in \partial B_{\rho_0}(z_0)$, with $\rho_0 \equiv \rho_0(\Lambda)$ independent of n. Here $\Omega^{\#}_{g_n}$ denotes the set obtained by repeating Ω_{g_n} periodically in the x-direction (see Fig. 2 above).

As a purely geometric consequence of this uniform inner ball condition and of (1.5), we deduce that g_n has no cusps nor vertical cuts for n large (see Fig. 3) and, in fact, $g_n \to h$ in $C^1([0, b])$.

Owing to this last convergence, we obtain by a blow-up argument the following decay estimate for the gradients of the displacements v_n:

$$\int_{B_r(z)\cap\Omega_{g_n}} |\nabla v_n|^2 \, dw \leqq C_0 r^{2-\delta}$$

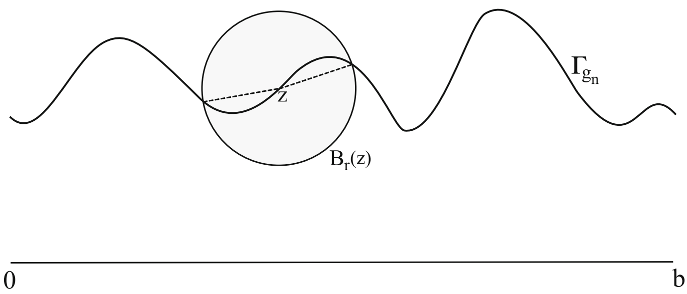

Fig. 4. The *dashed line* indicates the profile of the modified configuration used as a competitor to deduce the $C^{1,\alpha}$ uniform bounds

for all $z \in \Gamma_{g_n}$, $r \in (0, r_0)$, where C_0 and r_0 are independent of n and δ is any fixed small positive constant. With this estimate at hand, the deviation from flatness of Γ_{g_n} can be estimated by comparing the energy of (g_n, v_n) in $B_r(z) \cap \Omega_{g_n}$ with that of the modified configuration obtained by extending v_n to the whole ball and by replacing $\Gamma_{g_n} \cap B_r(z)$ with the segments connecting the center z to the points in $\Gamma_{g_n} \cap \partial B_r(z)$ (see Fig. 4).

The comparison argument yields an estimate of the oscillation of the unit normal vectors to Γ_{g_n} that implies a uniform bound of the $C^{1,\alpha}$-norms of $\{g_n\}$ for $\alpha \in (0, \frac{1}{2})$. In turn, by elliptic regularity, we deduce that $\{v_n\}$ is also uniformly bounded in the $C^{1,\alpha}$-norm. This allows us to use the Euler-Lagrange equations to finally deduce the desired $W^{2,\infty}$-convergence of g_n to h.

The proof of the global minimality properties stated before requires some additional arguments that are presented in Section 7. Finally, the last section of the paper collects some definitions and results on fractional Sobolev spaces and contains the simple proof of a regularity result for the Lamé system with homogeneous Neumann boundary condition, which is used in the proof of Theorem 2.10.

We would like to point out that the approach developed in this paper can be extended to a larger class of free boundary problems. Among these, we mention the one studied in [11], where a nonlocal perturbation of the isoperimetric problem, arising in the context of microphase separation of diblock copolymers, is considered. In that paper the authors compute the second variation of the functional and determine sufficient conditions for it to be positive definite. Even though our results do not apply directly, the overall strategy has been implemented in [1] to prove local minimality results also for that model.

We conclude this introduction by observing that the proof of the $W^{2,\infty}$-local minimality seems to be adaptable to more general surface and bulk energy densities, even in higher dimensions, while the arguments leading to local and global minimality rely upon two-dimensional geometric constructions. Hence, the extension of this last part to higher dimensions requires new ideas and will be the subject of future work.

2. Setting of the Problem and Statement of the Results

In this section we present the model studied by BONNETIER and CHAMBOLLE in [6] and the related functional setting. We also recall the regularity theorem proved in [15]. The main new results concerning local and global minimizers are stated in Sections 2.2 and 2.3, respectively.

2.1. The Model

We start by introducing the class of admissible profiles over the interval $(0, b)$. Roughly speaking, it consists of all functions with finite total variation in $(0, b)$ whose b-periodic extensions are lower semicontinuous in $\mathbb{R}$. For reasons that will become clear later on, it is convenient to identify such a function with its periodic extension. This motivates the following definition:

$$AP(0,b) := \{g : \mathbb{R} \to [0,+\infty) : \ g \text{ is lower semicontinuous and } b\text{-periodic}, \ \mathrm{Var}(g;0,b) < +\infty\}.$$

Here, $\mathrm{Var}(g; 0, b)$ denotes the *pointwise total variation* of g over the interval $(0, b)$, defined as

$$\mathrm{Var}(g;0,b) := \sup \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| < +\infty,$$

where the supremum is taken over all finite families $x_0, x_1, \ldots, x_k$, with $0 < x_0 < x_1 < \cdots < x_k < b$, $k \in \mathbb{N}$. Since $g \in AP(0, b)$ is b-periodic, its pointwise total variation is finite over any bounded interval of $\mathbb{R}$. Therefore, it admits right and left limits at every $x \in \mathbb{R}$ denoted by $g(x+)$ and $g(x-)$, respectively. In the following we use the notation

$$g^+(x) := \max\{g(x+), g(x-)\}, \qquad g^-(x) := \min\{g(x+), g(x-)\}. \tag{2.1}$$

We set

$$\begin{aligned} \Omega_g &:= \{(x,y) : x \in (0,b), \ 0 < y < g(x)\}, \\ \Omega_g^\# &:= \{(x,y) : x \in \mathbb{R}, \ 0 < y < g(x)\}. \end{aligned} \tag{2.2}$$

Thus, $\Omega_g^\#$ is the open set obtained by repeating Ω_g b-periodically in the x-direction.

We now define

$$\Gamma_g := \{(x,y) : x \in [0,b), \ g^-(x) \leqq y \leqq g^+(x)\}, \tag{2.3}$$

and

$$\Sigma_g := \{(x,y) : x \in [0,b), \ g(x) < g^-(x), \ g(x) \leqq y \leqq g^-(x)\}. \tag{2.4}$$

We refer to Σ_g as the *the set of vertical cracks*. We also set

$$\widetilde{\Gamma}_g := \Gamma_g \cup \Sigma_g,$$

and we will use the notation

$$\Gamma_g^\# := \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R},\ g^-(x) \leqq y \leqq g^+(x)\}.$$

In the same fashion we define $\Sigma_g^\#$ and $\widetilde{\Gamma}_g^\#$. Notice that we have decided not to highlight the dependence on b in the symbols Ω_g, Γ_g, and Σ_g, since throughout the most part of the paper b is fixed.

We now introduce a convergence in $AP(0, b)$. To this aim, for any pair (A, B) of subsets of $\mathbb{R}^2$ we set

$$d_H(A, B) := \inf\{\varepsilon > 0 : B \subset \mathcal{N}_\varepsilon(A) \text{ and } A \subset \mathcal{N}_\varepsilon(B)\},$$

where $\mathcal{N}_\varepsilon(A)$ denotes the ε-neighborhood of A. When restricted to the class of closed subsets, d_H reduces to the well-known Hausdorff distance.

We say that $h_n \to h$ in $AP(0, b)$ if

$$\sup_n \mathrm{Var}(h_n; 0, b) < +\infty \quad \text{and} \quad d_H(\mathbb{R}^2_+ \setminus \Omega^\#_{h_n}, \mathbb{R}^2_+ \setminus \Omega^\#_h) \to 0, \tag{2.5}$$

where $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y \geqq 0\}$. Given $g \in AP(0, b)$, we denote

$$LD_\#(\Omega_g; \mathbb{R}^2) := \Big\{ v \in L^2_{\mathrm{loc}}(\Omega^\#_g; \mathbb{R}^2) : v(x, y) = v(x+b, y) \\ \text{for } (x, y) \in \Omega^\#_g,\ E(v)|_{\Omega_g} \in L^2(\Omega_g; \mathbb{R}^2) \Big\},$$

where $E(v) := \frac{1}{2}(\nabla v + \nabla^T v)$, ∇v being the distributional gradient of v and $\nabla^T v$ its transpose. Given $e_0 \geqq 0$, we define

$$Y(e_0; 0, b) := \Big\{ (g, v) : g \in AP(0, b),\ v : \Omega^\#_g \to \mathbb{R}^2 \text{ s.t.} \\ v(\cdot, \cdot) - (e_0 \cdot, 0) \in LD_\#(\Omega_g; \mathbb{R}^2) \Big\}$$

and, given a b-periodic function q of class C^2,

$$X(e_0, q; 0, b) := \big\{ (g, v) \in Y(e_0; 0, b) : v(x, 0) = (e_0 x + q(x), 0) \text{ for } x \in \mathbb{R} \big\}.$$

If $q \equiv 0$ we simply write $X(e_0; 0, b)$ in place of $X(e_0, q; 0, b)$.

We introduce the following convergence in $Y(e_0; 0, b)$.

Definition 2.1. We say that $(h_n, u_n) \to (h, u)$ in $Y(e_0; 0, b)$ if and only if $h_n \to h$ in $AP(0, b)$ and $u_n \rightharpoonup u$ in $H^1_{\mathrm{loc}}(\Omega^\#_h; \mathbb{R}^2)$.

Notice that the definition is well posed, since by the second equation in (2.5) it follows that if $\Omega' \subset\subset \Omega^\#_h$ then $\Omega' \subset\subset \Omega^\#_{h_n}$ for n large enough. The notion of convergence just introduced is motivated by the following compactness theorem (see [6,15]).

Theorem 2.2. *Let* $(h_n, u_n) \in X(e_0, q; 0, b)$ *be such that*

$$\sup_n \left\{ \int_{\Omega_{h_n}} |E(u_n)|^2 \, dz + \mathrm{Var}(h_n; 0, b) + |\Omega_{h_n}| \right\} < +\infty.$$

Then there exist $(h, u) \in X(e_0, q; 0, b)$ *and a subsequence* $\{(h_{n_k}, u_{n_k})\}$ *such that* $(h_{n_k}, u_{n_k}) \to (h, u)$ *in* $Y(e_0; 0, b)$.

We work in the framework of linearized elasticity, and for simplicity we only consider isotropic and homogeneous materials. Hence, the elastic energy density $Q : \mathbb{M}^{2\times 2}_{\mathrm{sym}} \to [0, +\infty)$ takes the form

$$Q(\xi) := \frac{1}{2}\mathbb{C}\xi : \xi = \mu|\xi|^2 + \frac{\lambda}{2}\left[\mathrm{tr}(\xi)\right]^2,$$

where

$$\mathbb{C}\xi = \begin{pmatrix} (2\mu + \lambda)\xi_{11} + \lambda\xi_{22} & 2\mu\xi_{12} \\ 2\mu\xi_{12} & (2\mu + \lambda)\xi_{22} + \lambda\xi_{11} \end{pmatrix} \tag{2.6}$$

and the *Lamé coefficients* μ and λ satisfy the ellipticity condition

$$\mu > 0 \qquad \text{and} \qquad \lambda > -\mu. \tag{2.7}$$

Since

$$Q(\xi) \geqq \min\{\mu, \mu + \lambda\}|\xi|^2 \qquad \text{for all } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}},$$

condition (2.7) guarantees that Q is coercive.

We are ready to introduce the energy functional. If $(g, v) \in Y(e_0; 0, b)$ and g is Lipschitz it is defined as

$$G(g, v) := \int_{\Omega_g} Q(E(v)) \, dz + \mathcal{H}^1(\Gamma_g).$$

The following result, proved in [6] (see also [15]), gives a representation formula for the energy in the general case.

Theorem 2.3. *For any pair* $(g, v) \in Y(e_0; 0, b)$ *define*

$$\begin{aligned} F(g, v) := \inf\{\liminf_n G(g_n, v_n) :\ &(g_n, v_n) \to (g, v) \text{ in } Y(e_0; 0, b), \\ &g_n \text{ Lipschitz, } v_n(x, 0) = v(x, 0) \text{ for } x \in \mathbb{R},\ |\Omega_{g_n}| = |\Omega_g|\}. \end{aligned}$$

Then,

$$F(g, v) = \int_{\Omega_g} Q(E(v)) \, dz + \mathcal{H}^1(\Gamma_g) + 2\mathcal{H}^1(\Sigma_g). \tag{2.8}$$

When the extended graph of g contains a vertical segment in $\{0\} \times \mathbb{R}$, by periodicity the same occurs in $\{b\} \times \mathbb{R}$. Nevertheless, since in the definitions (2.3) and (2.4) only the part of the graph over the half-open interval $[0, b)$ is considered, the energy in (2.8) is invariant with respect to horizontal translations of (g, v).

Since by definition F is lower semicontinuous with respect to the convergence in $Y(e_0; 0, b)$, thanks to Theorem 2.2 we have that the minimum problem

$$\min\{F(g, v) : (g, v) \in X(e_0, q; 0, b), |\Omega_g| = d\} \tag{2.9}$$

always has a solution for any $d > 0$. Moreover, it is a well known result in relaxation theory (see [12, Theorem 3.8]) that the minimum value in (2.9) is equal to

$$\inf\{G(g, v) : (g, v) \in X(e_0, q; 0, b), |\Omega_g| = d, \; g \text{ Lipschitz}\}$$

and that limit points of minimizing sequences for the above problem are minimizers of (2.9).

Definition 2.4. We say that an admissible pair $(h, u) \in X(e_0, q; 0, b)$ is a *b-periodic global minimizer* for F if it solves the minimum problem (2.9) for $d := |\Omega_h|$. Moreover, we say that an admissible pair $(h, u) \in X(e_0, q; 0, b)$ is a *b-periodic local minimizer* for F if there exists $\delta > 0$ such that

$$F(h, u) \leqq F(g, v) \tag{2.10}$$

for all pairs $(g, v) \in X(e_0, q; 0, b)$, with $|\Omega_g| = |\Omega_h|$ and $d_H(\widetilde{\Gamma}_h, \widetilde{\Gamma}_g) \leqq \delta$. If, in addition, when $g \neq h$ (2.10) holds with strict inequality, then we say that (h, u) is an *isolated b-periodic local minimizer*.

Remark 2.5. Note that if h is continuous, then the above definition of a b-periodic local minimizer is equivalent to assuming that there exists $\delta > 0$ such that (2.10) holds for all pairs $(g, v) \in X(e_0, q; 0, b)$, with $|\Omega_g| = |\Omega_h|$ and

$$\sup_{x \in [0,b]} |g(x) - h(x)| \leqq \delta.$$

We notice here that a (sufficiently regular) b-periodic local or global minimizer $(h, u) \in X(e_0, q; 0, b)$ satisfies the following set of Euler-Lagrange conditions:

$$\begin{cases} \operatorname{div} \mathbb{C}E(u) = 0 & \text{in } \Omega_h; \\ \mathbb{C}E(u)[\nu] = 0 & \text{on } \Gamma_h \cap \{y > 0\}; \\ \mathbb{C}E(u)(0, y)[\nu] = -\mathbb{C}E(u)(b, y)[\nu] & \text{for } 0 < y < h(0) = h(b); \\ k + Q(E(u)) = \text{const} & \text{on } \Gamma_h \cap \{y > 0\}, \end{cases} \tag{2.11}$$

where ν denotes the outer unit normal to Ω_h, k is the curvature of Γ_h, and the constant appearing on the right-hand side of the last equation may be interpreted as the Lagrange multiplier associated with the volume constraint. Due to (2.7), equation $(2.11)_1$ is a linear elliptic system satisfying the Legendre–Hadamard condition.

Definition 2.6. Let $(h, u) \in X(e_0, q; 0, b)$ be such that $h \in C^2([0, b])$. We say that the pair (h, u) is *a critical point* for F if it satisfies (2.11).

The regularity theory developed in [15] also applies with minor changes to the model under consideration. To recall the main result, for $g \in AP(0,b)$ we denote the set of *cusp points* by

$$\Sigma_{g,c} := \{(x, g(x)) : x \in [0,b), g^-(x) = g(x), \text{ and } \\ g'_+(x) = -g'_-(x) = +\infty\}, \tag{2.12}$$

where g^- is defined in (2.1), while g'_+ and g'_- denote the right and left derivatives, respectively.

As usual, the set $\Sigma^\#_{g,c}$ is obtained by replacing $[0,b)$ by $\mathbb{R}$ in the previous formula and coincides with the b-periodic extension of $\Sigma_{g,c}$.

Theorem 2.7. (Regularity of local minimizers, see [15]) *Let* $(h,u) \in X(e_0, q; 0, b)$ *be a b-periodic local minimizer for F. Then the following regularity results hold:*

(i) *cusp points and vertical cracks are at most finite in* $[0,b)$*; that is,*

$$\text{card}\left(\{x \in [0,b) : (x,y) \in \Sigma_h \cup \Sigma_{h,c} \textit{ for some } y \geqq 0\}\right) < +\infty\,;$$

(ii) *the curve* $\Gamma^\#_h$ *is of class* C^1 *away from* $\Sigma^\#_h \cup \Sigma^\#_{h,c}$ *and*

$$\lim_{x\to x_0^\pm} h'(x) = \pm\infty \qquad \textit{for every } x_0 \in \Sigma^\#_h \cup \Sigma^\#_{h,c};$$

(iii) $\Gamma^\#_h \cap \{y > 0\}$ *is of class* $C^{1,\alpha}$ *away from* $\Sigma^\#_h \cup \Sigma^\#_{h,c}$ *for all* $\alpha \in (0, 1/2)$*;*
(iv) *let* $A := \{x \in \mathbb{R} : h(x) > 0$ *and h is continuous at* $x\}$*. Then A is an open set of full measure in* $\{h > 0\}$ *and h is analytic in A.*

Statement (ii) of Theorem 2.7 implies in particular that the *zero contact angle condition* between film and substrate holds. Though the regularity results proved in [15] refer to a slightly different model and to a slightly stronger notion of local minimality, the theorem above can be deduced from that paper. In particular, as in [15, Proposition 3.5], (i) follows from the inner ball condition that for our model has an even simpler proof (see Corollary 6.4 and Lemma 6.7 in this paper). Concerning (ii), the C^1 regularity of $\Gamma^\#_h \cup \{y > 0\}$ can be proved exactly as in [15, Theorem 3.14]. Only the contact angle condition at $\{y = 0\}$ requires extra care due to the presence of the Dirichlet condition, which is proven in [13] in the case $q \equiv 0$. However, the same proof given in [13] applies with minor modifications to the general case $q \in C^2$. Finally (iii) and (iv) are proved exactly as [15, Theorems 3.17 and 3.19], respectively.

Remark 2.8. If $h > 0$, Γ_h is of class $C^{1,\alpha}$ for all $\alpha \in (0, 1/2)$, and $(h,u) \in X(e_0, q; 0, b)$ satisfies the first three equations in (2.11), then the elliptic regularity (see Proposition 8.9) implies that $u \in C^{1,\alpha}(\overline{\Omega}_h)$ for all $\alpha \in (0, 1/2)$. Moreover, if also $(2.11)_4$ holds in the distributional sense, then the results contained in [21, Subsection 4.2] imply that (h,u) is analytic.

2.2. Local Minimizers

In this subsection we state the main results of the paper concerning local minimizers.

Given $d > 0$, the pair $(h, u_{e_0}) \in X(e_0; 0, b)$ defined as

$$h \equiv \frac{d}{b}, \qquad u_{e_0}(x, y) := \Big(e_0 x, \frac{-\lambda e_0}{2\mu + \lambda} y\Big), \tag{2.13}$$

will be referred to as the *flat configuration with volume d*. With a slight abuse of notation we simply write $(d/b, u_{e_0})$ to denote such a configuration. Note that $(d/b, u_{e_0})$ is a critical point for the functional F, that is, it satisfies (2.11). We warn the reader that whenever the flat configuration comes into play it is understood that the Dirichlet datum is $u_0(x, 0) = (e_0 x, 0)$.

For the applications it is important to know when the film starts developing irregularities and corrugations. When the growth of the film is quasistatic, this corresponds to determining the critical thickness at which the flat configuration ceases to be a (local) minimizer.

One of the main results of the paper is the exact description of the the local minimality threshold. This is done in terms of the *Grinfeld function K* defined for $y \geqq 0$ by

$$K(y) := \max_{n \in \mathbb{N}} \frac{1}{n} J(ny), \tag{2.14}$$

where

$$J(y) := \frac{y + (3 - 4\nu_p) \sinh y \cosh y}{4(1 - \nu_p)^2 + y^2 + (3 - 4\nu_p) \sinh^2 y},$$

ν_p being the *Poisson modulus* of the elastic material, that is,

$$\nu_p := \frac{\lambda}{2(\lambda + \mu)}. \tag{2.15}$$

It turns out (see Corollary 5.3) that K is strictly increasing and continuous, $K(y) \leqq Cy$ and $\lim_{y \to +\infty} K(y) = 1$, for some positive constant C.

Theorem 2.9. (Local minimality of the flat configuration) *Let* $d_{\mathrm{loc}} : (0, +\infty) \to (0, +\infty]$ *be defined as* $d_{\mathrm{loc}}(b) := +\infty$*, if* $0 < b \leqq \frac{\pi}{4} \frac{2\mu+\lambda}{e_0^2 \mu(\mu+\lambda)}$*, and as the solution to*

$$K\Big(\frac{2\pi d_{\mathrm{loc}}(b)}{b^2}\Big) = \frac{\pi}{4} \frac{2\mu + \lambda}{e_0^2 \mu(\mu + \lambda)} \frac{1}{b}, \tag{2.16}$$

otherwise. Then the flat configuration $(d/b, u_{e_0})$ *is an isolated b-periodic local minimizer for F in the sense of Definition 2.4 if* $0 < d < d_{\mathrm{loc}}(b)$.

The threshold d_{loc} *is critical; indeed, for* $d > d_{\mathrm{loc}}(b)$ *there exists* $(g, v) \in X(e_0; 0, b)$*, with* $|\Omega_g| = d$*, and* $\sup_{[0,b]} |g - d/b|$ *arbitrarily small such that* $F(g, v) < F(d/b, u_{e_0})$.

In particular, if $0 < b \leqq \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$, then the flat configuration is always a local minimizer. It is an open problem to establish whether $(d_{\text{loc}}(b)/b, u_{e_0})$ remains a local minimizer or not.

We remark that the last part of the statement was already established in [19] where the function (2.14) has been computed in connection with a certain notion of the second variation of the energy functional F. Instead, the first part of the statement, that is, the local minimality result below the critical threshold, is new. It is achieved by establishing a new criterion for local minimality expressed in terms of the positive definiteness of a suitable quadratic form. More precisely, let $(h, u) \in X(e_0, q; 0, b)$ be a critical point, with $h \in C^\infty_\#([0, b])$ and $h > 0$.[1] Given $\psi \in C^\infty_\#([0, b])$ with $\int_0^b \psi \, \mathrm{d}x = 0$, for $t \in \mathbb{R}$ we set $h_t := h + t\psi$, we let u_t be the elastic equilibrium corresponding to Ω_{h_t} under the usual periodicity and boundary conditions, and we define the *second variation of* F *at* (h, u) *along the direction* ψ to be the value of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} F(h_t, u_t)|_{t=0}. \tag{2.17}$$

If (2.17) is positive for all $\psi \in C^\infty_\#([0, b])$, with $\int_0^b \psi \, \mathrm{d}x = 0$ and $\psi \neq 0$, then we say that *the second variation of* F *at the critical point* (h, u) *is positive definite*. The main result of Section 6 is the following.

Theorem 2.10. (Local minimality criterion) *Let* $(h, u) \in X(e_0, q; 0, b)$ *be a critical point for* F*, with* $h \in C^\infty_\#([0, b])$ *and* $h > 0$*, and assume that the second variation of* F *at* (h, u) *is positive definite. Then* (h, u) *is an isolated b-periodic local minimizer in the sense of Definition* 2.4.

Note that the regularity assumption on h is not so restrictive, thanks to Remark 2.8. To the best of our knowledge, Theorem 2.10 provides the first extension of the classical local minimality criteria based on the second variation to the framework of functionals with bulk and surface energies.

2.3. Global Minimizers

We describe here further qualitative properties that we are able to prove for global minimizers. The first question we deal with is whether the flat configuration is an absolute minimizer. We shall show two types of results: (1) given $b > 0$, the flat configuration is the unique b-periodic global minimizer, provided that the thickness d/b is small enough; (2) if the period b is sufficiently small then the flat configuration is the unique b-periodic global minimizer no matter how thick the film is. More precisely, we have the following.

---

[1] Throughout the paper the notation $C^k_\#([0, b])$, $k \in \mathbb{N} \cup \{\infty\}$, stands for the space of functions in $C^k([0, b])$ that admit a b-periodic extension in $C^k(\mathbb{R})$. The space $C^k_\#(\Gamma_h)$ is defined similarly.

Theorem 2.11. (Global minimality of the flat configuration) *The following two statements hold.*

(i) *For every $b > 0$, there exists $0 < d_{\rm glob}(b) \leqq d_{\rm loc}(b)$ (see Theorem 2.9) such that the flat configuration $(d/b, u_{e_0})$ is a b-periodic global minimizer if and only if $0 < d \leqq d_{\rm glob}(b)$. Moreover, if $0 < d < d_{\rm glob}(b)$, then $(d/b, u_{e_0})$ is the unique b-periodic global minimizer.*[2]

(ii) *There exists $0 < b_{\rm crit} \leqq \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$ such that $d_{\rm glob}(b) = +\infty$ if and only if $0 < b \leqq b_{\rm crit}$, that is, the flat configuration $(d/b, u_{e_0})$ is the unique b-periodic global minimizer for all $d > 0$ if and only if $0 < b \leqq b_{\rm crit}$.*

The results of Theorem 2.11 are more qualitative in nature than those of Theorem 2.9. In particular, the function $d_{\rm glob}$ and the constant $b_{\rm crit}$ are not analytically determined and it is an open problem to establish whether or not $b_{\rm crit} < \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$ and $d_{\rm glob}(b) < d_{\rm loc}(b)$. However, the next result shows that the latter inequality holds, at least for b large.

Proposition 2.12. ($d_{\rm glob}(b) < d_{\rm loc}(b)$ for b large) *There exists a constant $c_0 \equiv c_0(\lambda, \mu) > 0$ such that*

$$\frac{d_{\rm loc}(b)}{b} \geqq \frac{c_0}{e_0^2} \qquad \text{for all } b > 0. \tag{2.18}$$

Moreover,

$$\lim_{b\to\infty}\frac{d_{\rm glob}(b)}{b} = 0.$$

As a consequence of the previous proposition, we have a non-uniqueness result.

Theorem 2.13. (Non-uniqueness) *Let $b > 0$ such that $d_{\rm glob}(b) < d_{\rm loc}(b)$. Then the minimum problem (2.9) with $d = d_{\rm glob}(b)$ has at least another solution besides the flat configuration $(d_{\rm glob}(b)/b, u_{e_0})$.*

Next, we address the occurrence of regular non-flat minimal configurations. The following theorem gives an analytical confirmation of the numerical and experimental observations that singularities do not form when the sample is not too large in width and thickness.

Theorem 2.14. (Regular non-flat minimal configurations) *Let $b_{\rm crit}$ be the constant introduced in Theorem 2.11. Then the following two statements hold.*

(i) *If $b_{\rm crit} < b < \frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$, then for every b-periodic non-flat global minimizer $(h, u) \in X(e_0; 0, b)$ we have $h \in C^1([0, b])$.*

[2] The function $d_{\rm glob}$, as well as $d_{\rm loc}$ and the constant $b_{\rm crit}$ introduced in the second part of the statement, depend also on the data μ, λ, and e_0 but this dependence is not highlighted since such quantities are considered here to be fixed.

(ii) *Assume* $\lambda \geqq -\frac{17}{18}\mu$. *There exist* $b_{\mathrm{reg}} > \frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$ *and* $d_0 > 0$ *with the following property: If* $\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)} \leqq b < b_{\mathrm{reg}}$ *and* $d_{\mathrm{glob}}(b) \leqq d < d_{\mathrm{glob}}(b)+d_0$, *then for every b-periodic global minimizer* $(h, u) \in X(e_0; 0, b)$ *with* $|\Omega_h| = d$ *we have* $h \in C^1([0, b])$.

In both cases (h, u) *satisfies all the conclusions of Theorem* 2.7, *with* $\Sigma_h^{\#} = \Sigma_{h,c}^{\#} = \emptyset$.

Remark 2.15. Note that in statement (ii) of the previous theorem we impose a condition on the Lamé coefficients that is slightly more restrictive than (2.7).

The last result that we want to highlight deals with the existence of nontrivial analytic minimal configurations. It states that if b is small enough, then b-periodic non-flat global minimizers are analytic.

Theorem 2.16. (Analytic non-flat minimal configurations) *Let* b_{crit} *be the constant introduced in Theorem* 2.11. *There exists* $\eta_0 > 0$ *such that if* $b = b_{\mathrm{crit}} + \eta$, *with* $\eta \in (0, \eta_0)$, *and* $(h, u) \in X(e_0; 0, b)$ *is any non-flat b-periodic global minimizer, then* (h, u) *is analytic; more precisely, h is strictly positive and analytic over* $\mathbb{R}$ *and, in turn, u is analytic in* $\overline{\Omega}_h^{\#}$.

3. Computation of the Second Variation

In this section we study a suitable notion of the second variation for the functional F. We look at regular configurations, where the displacement minimizes the elastic energy. More precisely, throughout this section we assume that $(h, u) \in X(e_0, q; 0, b)$, $h \in C^\infty_\#([0, b])$, $h > 0$, and u satisfies

$$\int_{\Omega_h} \mathbb{C}E(u) : E(w)\,\mathrm{d}z = 0 \qquad \text{for all } w \in A(\Omega_h), \tag{3.1}$$

where

$$A(\Omega_h) := \{w \in LD_\#(\Omega_h; \mathbb{R}^2) : \ w(\cdot, 0) \equiv 0\}.$$

Given $\psi \in C^\infty_\#([0, b])$ with $\int_0^b \psi\,\mathrm{d}x = 0$, for $t \in \mathbb{R}$ we set $h_t := h + t\psi$ and we let u_t be the elastic equilibrium corresponding to Ω_{h_t} under the usual periodicity and boundary conditions, that is, $(h_t, u_t) \in X(e_0, q; 0, b)$ and

$$\int_{\Omega_{h_t}} \mathbb{C}E(u_t) : E(w)\,\mathrm{d}z = 0 \qquad \text{for all } w \in A(\Omega_{h_t}). \tag{3.2}$$

We define the *second variation of* F *at* (h, u) *along the direction* ψ to be the value of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} F(h_t, u_t)|_{t=0}.$$

It is convenient to introduce the following subspace of $H^1(\Gamma_h)$:

$$\widetilde{H}^1_\#(\Gamma_h) := \Big\{\varphi \in H^1(\Gamma_h) : \varphi(0, h(0)) = \varphi(b, h(b)), \int_{\Gamma_h} \varphi \, \mathrm{d}\mathcal{H}^1 = 0\Big\}.$$

In the following we will make use of the following notational convention: For any one-parameter family of function $\{g_t\}_t$ the symbol $\dot{g}_t(x)$ denotes the partial derivative with respect to t of the function $(t, x) \mapsto g_t(x)$. We omit the subscript when $t = 0$. In particular we let

$$\dot{u}_t := \frac{\partial u_t}{\partial t} \qquad \dot{u} := \frac{\partial u_t}{\partial t}\bigg|_{t=0}$$

We denote by ν the exterior normal to Ω_h and we let $\tau := \nu^\perp$ be the unit tangent vector to Γ_h, where $^\perp$ stands for the clockwise rotation by $\frac{\pi}{2}$. As usual, ∂_τ, ∂_ν denote the tangential and normal derivatives, while D_τ, D_ν stand for the tangential and normal gradient, respectively. If α is a vector field from Γ_h to $\mathbb{R}^2$, we denote its (distributional) tangential divergence by $\mathrm{div}_\tau\alpha$. Recall that if α is sufficiently smooth and $\alpha(0, h(0)) = \alpha(b, h(b))$, then

$$\int_{\Gamma_h} \mathrm{div}_\tau\alpha \, \mathrm{d}\mathcal{H}^1 = \int_{\Gamma_h} k(\alpha \cdot \nu) \, \mathrm{d}\mathcal{H}^1,$$

where $k = \mathrm{div}_\tau \nu$ is the scalar curvature of Γ_h. In particular, if α is a tangential field and φ is a sufficiently regular scalar function such that $\varphi(0, h(0)) = \varphi(b, h(b))$, then

$$\int_{\Gamma_h} \varphi \mathrm{div}_\tau\alpha \, \mathrm{d}\mathcal{H}^1 = -\int_{\Gamma_h} D_\tau\varphi \cdot \alpha \, \mathrm{d}\mathcal{H}^1. \tag{3.3}$$

Remark 3.1. Formula (3.3) still holds when $\varphi(0, h(0)) \neq \varphi(b, h(b))$, provided the tangential field vanishes at the points of Γ_h, that is, $\alpha(0, h(0)) = \alpha(b, h(b)) = 0$.

We are now ready to state one of the main results of this section.

Theorem 3.2. *Let (h, u), ψ, and (h_t, u_t) be as above, let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection on the x-axis, and let $\varphi \in \widetilde{H}^1_\#(\Gamma_h)$ be defined as $\varphi := \frac{\psi}{\sqrt{1+h'^2}} \circ \pi_1$. Then the function $\dot{u}$ belongs to $A(\Omega_h)$ and satisfies the equation*

$$\int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(w) \, \mathrm{d}z = \int_{\Gamma_h} \mathrm{div}_\tau(\varphi \, \mathbb{C}E(u)) \cdot w \, \mathrm{d}\mathcal{H}^1 \tag{3.4}$$

for all $w \in A(\Omega_h)$. Moreover, the second variation of F at (h, u) along the direction ψ is given by

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} F(h_t, u_t)|_{t=0} = &-2\int_{\Omega_h} Q(E(\dot{u})) \, \mathrm{d}z \\ &+ \int_{\Gamma_h} (\partial_\tau\varphi)^2 \, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma_h} (\partial_\nu[Q(E(u))] - k^2)\varphi^2 \, \mathrm{d}\mathcal{H}^1 \\ &- \int_{\Gamma_h} (Q(E(u)) + k)\partial_\tau\big((h' \circ \pi_1)\varphi^2\big) \, \mathrm{d}\mathcal{H}^1. \end{aligned} \tag{3.5}$$

Proof. It is convenient to introduce the one-parameter family of C^∞-diffeomorphisms $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi_t(x,y) := (x, y + t\psi(x))$. We will make repeated use of the following fact: There exists $\varepsilon_0 > 0$ such that the map $(t,x) \mapsto u_t \circ \Phi_t(x)$ is of class C^∞ in $(-\varepsilon_0, \varepsilon_0) \times \overline{\Omega}_h$. This can be proved by quite standard elliptic estimates arguing, for instance, as in [8, Proposition 8.1].

We divide the proof into several steps.

Step 1 We prove (3.4). Fix $w \in A(\Omega_h) \cap C^\infty(\overline{\Omega}_h)$. Then w may be extended outside Ω_h in such a way that $w \in A(\Omega_{h_t}) \cap C^\infty(\overline{\Omega}_{h_t})$ for t small. Hence we can differentiate (3.2) with respect to t and evaluate the result at $t = 0$ to obtain

$$\begin{aligned} 0 &= \int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(w)\,\mathrm{d}z + \int_0^b \psi(x)\,[\mathbb{C}E(u) : E(w)](x, h(x))\,\mathrm{d}x \\ &= \int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(w)\,\mathrm{d}z + \int_{\Gamma_h} \varphi\,\mathbb{C}E(u) : E(w)\,\mathrm{d}\mathcal{H}^1. \end{aligned} \tag{3.6}$$

Since $\mathbb{C}E(u)[\nu] = 0$, by (3.3) the second integral in the above formula can be rewritten as

$$\begin{aligned} \int_{\Gamma_h} \varphi\,\mathbb{C}E(u) : E(w)\,\mathrm{d}\mathcal{H}^1 &= \int_{\Gamma_h} \varphi\,\mathbb{C}E(u) : \nabla w\,\mathrm{d}\mathcal{H}^1 \\ &= \int_{\Gamma_h} \varphi\,\mathbb{C}E(u) : D_\tau w\,\mathrm{d}\mathcal{H}^1 = -\int_{\Gamma_h} \mathrm{div}_\tau(\varphi\,\mathbb{C}E(u)) \cdot w\,\mathrm{d}\mathcal{H}^1. \end{aligned}$$

If w is any function in $A(\Omega_h)$ we conclude by approximation.

Step 2 In this step we introduce suitable functions carrying useful geometric information and we prove some identities for later use. Let d_t denote the signed distance function from $\Gamma^\#_{h_t}$; more precisely,

$$d_t(z) := \begin{cases} -\mathrm{dist}(z, \Gamma^\#_{h_t}) & \text{if } z \in \Omega^\#_{h_t}, \\ \mathrm{dist}(z, \Gamma^\#_{h_t}) & \text{if } z \notin \Omega^\#_{h_t}. \end{cases}$$

By our assumptions on h and ψ there exist $\varepsilon_0 > 0$ and a tubular neighborhood $\mathcal{N}$ of $\Gamma^\#_h$ such that the map $(t,z) \mapsto d_t(z)$ is smooth in $(-\varepsilon_0, \varepsilon_0) \times \mathcal{N}$. For $(t,z) \in (-\varepsilon_0, \varepsilon_0) \times \mathcal{N}$ we set $\nu_t(z) := \nabla d_t(z)$ and $k_t(z) := (\mathrm{div}\nu_t)(z)$. Note that by construction $\nu_t|_{\Gamma_{h_t}}$ coincides with the outer unit normal to Ω_{h_t}, while $k_t|_{\Gamma_{h_t}}$ represents the curvature of Γ_{h_t}. More precisely, since ν_t points outward we have

$$k_t|_{\Gamma_{h_t}} = \Big(-\frac{h' + t\psi'}{\sqrt{1 + (h' + t\psi')^2}}\Big)'. \tag{3.7}$$

We recall that we omit the subscript t when $t = 0$; in particular, we write ν and k instead of ν_0 and k_0, respectively. Differentiating the identity $|\nu|^2 = 1$ with respect to ν we get $D\nu[\nu] = 0$. This immediately yields

$$D\nu = D_\tau \nu = k\tau \otimes \tau \quad \text{and} \quad \mathrm{div}\nu = \mathrm{div}_\tau \nu \qquad \text{on } \Gamma_h. \tag{3.8}$$

The same relations clearly hold for ν_t on Γ_{h_t}. Moreover, differentiating the identity $D\nu[\nu] = 0$ we obtain that $0 = \sum_{j=1}^2 (\partial^2_{jk}\nu_i\nu_j + \partial_j\nu_i\partial_k\nu_j)$ for $k,\ i = 1, 2,$

where ∂_1, ∂_2 stand for the partial derivative with respect to x and with respect to y, respectively. Hence, $\big(\partial_\nu(D\nu)\big)_{ik} = \sum_{j=1}^2 \partial^2_{jk}\nu_i\nu_j = -\sum_{j=1}^2 \partial_j\nu_i\partial_k\nu_j = -\big((D\nu)^2\big)_{ik}$ for $k,\ i = 1, 2$. From the last identity, recalling that $k = (\mathrm{div}\nu)(z) = \mathrm{tr}(D\nu)$, we deduce

$$\partial_\nu k = \partial_\nu\big(\mathrm{tr}(D\nu)\big) = \mathrm{tr}\big(\partial_\nu(D\nu)\big) = \mathrm{tr}\big(-(D\nu)^2\big) = -k^2, \tag{3.9}$$

where the last equality follows from (3.8). Note now that

$$\nu_t \circ \Phi_t = \bigg(-\frac{h' + t\psi'}{\sqrt{1 + (h' + t\psi')^2}}, \frac{1}{\sqrt{1 + (h' + t\psi')^2}}\bigg).$$

Differentiating this equality with respect to t and evaluating at $t = 0$, we get

$$\dot\nu + \partial_2\nu(\psi \circ \pi_1) = -\Big(\frac{\psi'}{1 + (h')^2} \circ \pi_1\Big)\tau \quad \text{on } \Gamma_h.$$

Multiplying both sides by τ and using (3.8), we obtain

$$\dot\nu \cdot \tau + k\tau_2\psi \circ \pi_1 = -\frac{\psi'}{1 + (h')^2} \circ \pi_1,$$

that is,

$$\dot\nu \cdot \tau = -\partial_\tau\varphi \quad \text{on } \Gamma_h. \tag{3.10}$$

Step 3 We start by computing the first variation. Straightforward computations lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}F(h_t, u_t) = \int_{\Omega_{h_t}} \mathbb{C}E(u_t) : E(\dot u_t)\,\mathrm{d}z + \int_0^b (Q \circ E(u_t))(x, h_t(x))\psi(x)\,\mathrm{d}x + \int_0^b \frac{(h' + t\psi')\psi'}{\sqrt{1 + (h' + t\psi')^2}}\,\mathrm{d}x.$$

Since $\dot u_t \in A(\Omega_{h_t})$, by (3.2) the first integral in the previous formula vanishes. Hence, integrating by parts and recalling (3.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}F(h_t, u_t) = \int_0^b \big[(Q \circ E(u_t))(x, h_t(x)) + k_t(x, h_t(x))\big]\psi(x)\,\mathrm{d}x. \tag{3.11}$$

Step 4 We finally compute the second variation. Differentiating (3.11) with respect to t and evaluating the result at $t = 0$, we get

$$\begin{aligned}
&\frac{d^2}{dt^2} F(h_t, u_t)|_{t=0} \\
&= \frac{d}{dt}\left(\int_0^b \big[(Q \circ E(u_t))(x, h_t(x)) + k_t(x, h_t(x))\big]\psi(x)\, dx\right)\Big|_{t=0} \\
&= \int_0^b \big(\mathbb{C}E(u) : E(\dot{u})\big)(x, h(x))\psi(x)\, dx \\
&\quad + \int_0^b \nabla(Q \circ E(u))(x, h(x)) \cdot (0, \psi(x))\psi(x)\, dx \\
&\quad + \int_0^b \dot{k}(x, h(x))\psi(x)\, dx + \int_0^b \nabla k(x, h(x)) \cdot (0, \psi(x))\psi(x)\, dx \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.12}$$

We now treat each integral I_i separately. From (3.6) we obtain

$$\begin{aligned}
I_1 = \int_{\Gamma_h} \mathbb{C}E(u) : E(\dot{u})\varphi\, d\mathcal{H}^1 &= -\int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(\dot{u})\varphi\, dz \\
&= -2\int_{\Omega_h} Q(E(\dot{u}))\, dz.
\end{aligned} \tag{3.13}$$

Writing $\nabla(Q \circ E(u)) = \partial_\nu[Q(E(u))]\nu + \partial_\tau[Q(E(u))]\tau$ we have

$$\begin{aligned}
I_2 &= \int_{\Gamma_h} \partial_\nu[Q(E(u))]\varphi^2\, d\mathcal{H}^1 + \int_{\Gamma_h} \partial_\tau[Q(E(u))](h' \circ \pi_1)\varphi^2\, d\mathcal{H}^1 \\
&= \int_{\Gamma_h} \partial_\nu[Q(E(u))]\varphi^2\, d\mathcal{H}^1 - \int_{\Gamma_h} Q(E(u))\partial_\tau\big((h' \circ \pi_1)\varphi^2\big)\, d\mathcal{H}^1,
\end{aligned} \tag{3.14}$$

where the last integration by parts is justified thanks to the periodicity of the functions involved. Analogously, using also (3.9), we may rewrite

$$\begin{aligned}
I_4 &= \int_{\Gamma_h} \partial_\nu k\varphi^2\, d\mathcal{H}^1 + \int_{\Gamma_h} \partial_\tau k(h' \circ \pi_1)\varphi^2\, d\mathcal{H}^1 \\
&= -\int_{\Gamma_h} k^2\varphi^2\, d\mathcal{H}^1 - \int_{\Gamma_h} k\partial_\tau\big((h' \circ \pi_1)\varphi^2\big)\, d\mathcal{H}^1.
\end{aligned} \tag{3.15}$$

Differentiating the identity $\dot{\nu} \cdot \nu = 0$ and recalling (3.8) we obtain $\partial_\nu \dot{\nu} \cdot \nu = -\dot{\nu} \cdot \partial_\nu \nu = 0$ that, in turn, implies $\dot{k} = \mathrm{div}\dot{\nu} = \mathrm{div}_\tau \dot{\nu}$. Hence,

$$I_3 = \int_{\Gamma_h} \mathrm{div}_\tau \dot{\nu}\varphi\, d\mathcal{H}^1 = -\int_{\Gamma_h} \dot{\nu} \cdot \tau\partial_\tau\varphi\, d\mathcal{H}^1 = \int_{\Gamma_h} (\partial_\tau\varphi)^2\, d\mathcal{H}^1, \tag{3.16}$$

where in the last equality we have used (3.10). Collecting (3.12)–(3.16) we finally obtain (3.5).

Remark 3.3. If (h, u) is a critical pair for F, then $Q(E(u))+k$ is constant along Γ_h. This implies that the last integral in (3.5) vanishes since the function $(h' \circ \pi_1)\varphi^2$ takes the same values at the endpoints of Γ_h.

The theorem together with the previous remark suggests that we may associate with every critical pair $(h, u) \in X(e_0, q; 0, b)$ the quadratic form $\partial^2 F(h, u)$ defined for all $\varphi \in \widetilde{H}^1_\#(\Gamma_h)$ as

$$\partial^2 F(h,u)[\varphi] := -2\int_{\Omega_h} Q(E(v_\varphi))\,\mathrm{d}z + \int_{\Gamma_h} (\partial_\tau \varphi)^2\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_h} (\partial_\nu[Q(E(u))] - k^2)\varphi^2\,\mathrm{d}\mathcal{H}^1, \tag{3.17}$$

where v_φ is the unique solution in $A(\Omega_h)$ to the equation

$$\int_{\Omega_h} \mathbb{C}E(v_\varphi) : E(w)\,\mathrm{d}x = \int_{\Gamma_h} \mathrm{div}_\tau(\varphi\,\mathbb{C}E(u)) \cdot w\,\mathrm{d}\mathcal{H}^1 \tag{3.18}$$

for all $w \in A(\Omega_h)$. Our purpose is to provide necessary and sufficient conditions for local minimality in terms of such a quadratic form. We immediately have the following corollary.

Corollary 3.4. *Let $(h, u) \in X(e_0, q; 0, b)$ be a local minimizer. Then*

$$\partial^2 F(h,u)[\varphi] \geqq 0 \qquad \textit{for all } \varphi \in \widetilde{H}^1_\#(\Gamma_h). \tag{3.19}$$

Proof. If $\varphi \in \widetilde{H}^1_\#(\Gamma_h) \cap C^\infty_\#(\Gamma_h)$ then (3.19) is a consequence of Theorem 3.2 and Remark 3.3. The conclusion follows by approximating any $\varphi \in \widetilde{H}^1_\#(\Gamma_h)$ with functions in $\widetilde{H}^1_\#(\Gamma_h) \cap C^\infty_\#(\Gamma_h)$ and observing that the map $\varphi \mapsto \partial^2 F(h,u)[\varphi]$ is continuous with respect to the strong convergence in $\widetilde{H}^1_\#(\Gamma_h)$.

In one of the main theorems of the paper we will show that if the quadratic form is in fact *positive definite*; that is,

$$\partial^2 F(h,u)[\varphi] > 0 \qquad \text{for all } \varphi \in \widetilde{H}^1_\#(\Gamma_h) \setminus \{0\}, \tag{3.20}$$

then (h, u) is an isolated local minimizer. The next proposition starts paving the way for this result. It provides two different equivalent formulations of condition (3.20): The first one is related to the first eigenvalue of a suitable compact operator, while the second one is expressed in terms of a dual minimum problem. To this aim, we fix a critical pair $(h, u) \in X(e_0, q; 0, b)$ and we introduce the following bilinear form, defined for every $\varphi, \psi \in \widetilde{H}^1_\#(\Gamma_h)$:

$$(\varphi, \psi)_\sim := \int_{\Gamma_h} a\,\varphi\psi\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_h} \partial_\tau\varphi\,\partial_\tau\psi\,\mathrm{d}\mathcal{H}^1, \tag{3.21}$$

where $a := \partial_\nu[Q(E(u))] - k^2$. It is easy to check that whenever

$$(\varphi, \varphi)_\sim > 0 \quad \text{for all } \varphi \in \widetilde{H}^1_\#(\Gamma_h) \setminus \{0\}, \tag{3.22}$$

the bilinear form (3.21) defines an equivalent scalar product on $\widetilde{H}^1_\#(\Gamma_h)$.

Remark 3.5. Note that for the flat configuration $(d/b, u_{e_0})$ the coefficient a vanishes, and thus (3.22) is always satisfied.

Proposition 3.6. *The following statements are equivalent.*

(i) $\partial^2 F(h,u)$ *satisfies* (3.20).

(ii) *Condition* (3.22) *holds and the compact monotone self-adjoint operator* T *acting from* $(\widetilde{H}^1_\#(\Gamma_h), \sim)$ *to itself, defined by duality as*

$$(T\varphi, \psi)_\sim := \int_{\Omega_h} \mathbb{C}E(v_\varphi) : E(v_\psi)\,\mathrm{d}z = \int_{\Omega_h} \mathbb{C}E(v_\psi) : E(v_\varphi)\,\mathrm{d}z \qquad (3.23)$$

for all $\varphi, \psi \in \widetilde{H}^1_\#(\Gamma_h)$*, satisfies*

$$\lambda_1 := \max_{\|\varphi\|_\sim = 1} (T\varphi, \varphi)_\sim = \|T\|_\sim < 1. \qquad (3.24)$$

Here v_φ *and* v_ψ *are defined as in* (3.18).

(iii) *Condition* (3.22) *holds and, setting*

$$\mu_1 := \min\left\{2\int_{\Omega_h} Q(E(v))\,\mathrm{d}z : \ v \in A(\Omega_h),\ \|\Phi_v\|_\sim = 1\right\}, \qquad (3.25)$$

where Φ_v *is the unique solution in* $\widetilde{H}^1_\#(\Gamma_h)$ *to*

$$\int_{\Gamma_h} \partial_\tau \Phi_v\, \partial_\tau \psi\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_h} a\,\Phi_v\,\psi\,\mathrm{d}\mathcal{H}^1 = \int_{\Gamma_h} \mathrm{div}_\tau(\psi\,\mathbb{C}E(u)) \cdot v\,\mathrm{d}\mathcal{H}^1 \qquad (3.26)$$

for all $\psi \in \widetilde{H}^1_\#(\Gamma_h)$*, we have*

$$\mu_1 = \frac{1}{\lambda_1} > 1.$$

Proof. Assume now (3.20). As

$$\partial^2 F(h,u)[\varphi] = (\varphi,\varphi)_\sim - (T\varphi,\varphi)_\sim, \qquad (3.27)$$

we have $(\varphi,\varphi)_\sim > (T\varphi,\varphi)_\sim \geqq 0$ whenever $\varphi \neq 0$. Hence, $(\cdot,\cdot)_\sim$ defines a scalar product on $\widetilde{H}^1_\#(\Gamma_h)$ that is equivalent to the standard one, as observed before. The monotonicity and the self-adjointness of T are evident from the definition.

We first observe that if $\psi_n \rightharpoonup \psi$ weakly in $\widetilde{H}^1_\#(\Gamma_h)$, then $\mathrm{div}_\tau(\psi_n\,\mathbb{C}E(u)) \rightharpoonup \mathrm{div}_\tau(\psi, \mathbb{C}E(u))$ weakly in $L^2(\Gamma_h)$. Hence, recalling (3.18) and using Korn's inequality we get that $v_{\psi_n} \rightharpoonup v_\psi$ weakly in A_{Ω_h}. Then, by the compactness of the trace operator, we conclude that

$$\psi_n \rightharpoonup \psi \ \text{ weakly in } \widetilde{H}^1_\#(\Gamma_h) \implies v_{\psi_n} \to v_\psi \ \text{ strongly in } L^2(\Gamma_h). \qquad (3.28)$$

To check the compactness of T let $\varphi_n \rightharpoonup \varphi$ weakly in $\widetilde{H}^1_\#(\Gamma_h)$. Recalling (3.18) again, we have

$$\int_{\Gamma_h} \mathrm{div}_\tau(\varphi_n\,\mathbb{C}E(u)) \cdot v_\psi\,\mathrm{d}\mathcal{H}^1 \to \int_{\Gamma_h} \mathrm{div}_\tau(\varphi\,\mathbb{C}E(u)) \cdot v_\psi\,\mathrm{d}\mathcal{H}^1,$$

that is,

$$(T\varphi_n, \psi)_\sim \to (T\varphi, \psi)_\sim. \tag{3.29}$$

Thus, $T\varphi_n \rightharpoonup T\varphi$ weakly in $\widetilde{H}^1_\#(\Gamma_h)$. Choosing $\psi = T\varphi_n$ in (3.29) and recalling that by (3.28) $v_{T\varphi_n} \to v_{T\varphi}$ in $L^2(\Gamma_h)$, we discover that $\|T\varphi_n\|_\sim \to \|T\varphi\|_\sim$ and, in turn, $T\varphi_n \to T\varphi$ strongly in $\widetilde{H}^1_\#(\Gamma_h)$. We conclude that T is compact so that λ_1 in (3.24) is well-defined and coincides with the first eigenvalue of T. By (3.27) the condition $\lambda_1 < 1$ is equivalent to (3.20). This concludes the proof of the equivalence of (i) and (ii).

To finish up the proof of the proposition it is enough to show that under (3.22) we have $\mu_1 = \frac{1}{\lambda_1}$. First of all, arguing as before, one can check that the map $v \mapsto \Phi_v$ is a linear compact operator from $A(\Omega_h)$ to $\widetilde{H}^1_\#(\Gamma_h)$. Exploiting this observation, the existence of a solution to (3.25) can be established by the direct method of the Calculus of Variations.

Let now $\varphi \in \widetilde{H}^1_\#(\Gamma_h)$ be such that $\|\varphi\|_\sim = 1$ and $T\varphi = \lambda_1\varphi$. Then by (3.18), (3.21), and (3.23) we have

$$\lambda_1 \int_{\Gamma_h} \partial_\tau\varphi\, \partial_\tau\psi \, d\mathcal{H}^1 + \lambda_1 \int_{\Gamma_h} a\, \varphi\, \psi \, d\mathcal{H}^1 = \int_{\Gamma_h} \mathrm{div}_\tau(\psi\, \mathbb{C}E(u)) \cdot v_\varphi \, d\mathcal{H}^1 \tag{3.30}$$

for all $\psi \in \widetilde{H}^1_\#(\Gamma_h)$. Hence, recalling (3.26), $\Phi_{v_\varphi} = \lambda_1\varphi$. Moreover, choosing $\psi = \varphi$ in (3.30) and using (3.18), we have

$$\lambda_1 = \lambda_1 \|\varphi\|^2_\sim = 2\int_{\Omega_h} Q(E(v_\varphi))\, dz.$$

We conclude that v_φ/λ_1 is admissible for problem (3.25) and

$$\mu_1 \leqq 2\int_{\Omega_h} Q\Big(E\Big(\frac{v_\varphi}{\lambda_1}\Big)\Big)\, dz = \frac{1}{\lambda_1}.$$

To show the converse inequality, let $\bar{v}$ be a solution of (3.25). Then there exists a Lagrange multiplier μ_0 such that

$$\int_{\Omega_h} \mathbb{C}E(\bar{v}) : E(w)\, dz = \mu_0(\Phi_{\bar{v}}, \Phi_w)_\sim \quad \text{for all } w \in A(\Omega_h). \tag{3.31}$$

Choosing $w = \bar{v}$ we immediately deduce $\mu_0 = \mu_1$. Moreover, recalling (3.26), we get from (3.31)

$$\int_{\Omega_h} \mathbb{C}E(\bar{v}) : E(w)\, dz = \mu_1 \int_{\Gamma_h} \mathrm{div}_\tau(\Phi_{\bar{v}}\mathbb{C}E(u)) \cdot w \, d\mathcal{H}^1 \quad \text{for all } w \in A(\Omega_h),$$

which means, by (3.18), $\bar{v}/\mu_1 = v_{\Phi_{\bar{v}}}$. Hence, by (3.23), (3.18), and (3.26) we have for all $\psi \in \widetilde{H}^1_\#(\Gamma_h)$,

$$\begin{aligned}(T\Phi_{\bar{v}}, \psi)_\sim &= \int_{\Omega_h} \mathbb{C}E\Big(\frac{\bar{v}}{\mu_1}\Big) : E(v_\psi)\, dz = \int_{\Omega_h} \mathbb{C}E(v_\psi) : E\Big(\frac{\bar{v}}{\mu_1}\Big)\, dz \\ &= \int_{\Gamma_h} \mathrm{div}_\tau(\psi\mathbb{C}E(u)) \cdot \frac{\bar{v}}{\mu_1}\, d\mathcal{H}^1 = \frac{1}{\mu_1}(\Phi_{\bar{v}}, \psi)_\sim.\end{aligned}$$

We conclude that $1/\mu_1$ is an eigenvalue of T and thus $1/\mu_1 \leqq \lambda_1$.

Corollary 3.7. *Assume* (3.20)*. Then there exists a constant* $C > 0$ *such that*

$$\partial^2 F(h,u)[\varphi] \geqq C\|\varphi\|^2_{H^1(\Gamma_h)}$$

for every $\varphi \in \widetilde{H}^1_\#(\Gamma_h)$.

Proof. By Proposition 3.6, (3.24) holds. Hence, recalling (3.27), we have

$$\begin{aligned}\partial^2 F(h,u)[\varphi] &= \|\varphi\|^2_\sim - (T\varphi,\varphi)_\sim \geqq \|\varphi\|^2_\sim - \|T\|_\sim \|\varphi\|^2_\sim \\ &= (1-\lambda_1)\|\varphi\|^2_\sim \geqq C\|\varphi\|^2_{H^1(\Gamma_h)},\end{aligned}$$

which proves the corollary.

Corollary 3.8. *For* $d > 0$ *let* $\lambda_1(d)$ *be the first eigenvalue of the operator* T *associated with the quadratic form* $\partial^2 F(d/b, u_{e_0})$*, according to Proposition* 3.6 (*see Remark* 3.5)*. Then* λ_1 *is a strictly increasing function of* d.

Proof. Let $0 < d_1 < d_2$ and let $v_1 \in \mathcal{A}\big((0,b)\times(0,d_1/b)\big)$ be a solution to problem (3.25), with $h \equiv d_1/b$. Observe that the function

$$v_2(x,y) := \begin{cases} 0 & \text{if } 0 < y < \dfrac{d_2-d_1}{b}, \\ v_1\Big(x, y - \dfrac{d_2-d_1}{b}\Big) & \text{if } \dfrac{d_2-d_1}{b} \leqq y < \dfrac{d_2}{b} \end{cases}$$

belongs to $\mathcal{A}\big((0,b)\times(0,d_2/b)\big)$ and is an admissible competitor for problem (3.25), with $h \equiv d_2/b$. Moreover, v_2 cannot be a minimizer, since otherwise it would be 0 everywhere by analyticity. Hence, by Proposition 3.6(iii),

$$\frac{1}{\lambda_1(d_1)} = \int_0^b \int_0^{d_1/b} Q(E(v_1))\,\mathrm{d}y\mathrm{d}x = \int_0^b \int_0^{d_2/b} Q(E(v_2))\,\mathrm{d}y\mathrm{d}x > \frac{1}{\lambda_1(d_2)},$$

which concludes the proof.

4. Second Variation and $W^{2,\infty}$-Local Minimality

We now come to a crucial point of the paper, namely the proof that condition (3.20) implies the $W^{2,\infty}$-local minimality. This is essentially achieved in Proposition 4.5 from which Theorem 4.6 quickly follows. The section also contains a variant of this theorem which will be used in Section 7 to prove the second statement in Theorem 2.11. We start with some technical lemmas. In the representation formula (3.17) of the second variation, the last term involves the normal derivative of $Q(E(u))$ on Γ_h, where u is the elastic equilibrium in Ω_h. Therefore, an important step in the proof is provided by Lemma 4.1, which shows that the $H^{-\frac{1}{2}}(\Gamma_h)$-norm of the trace of $\nabla \mathbb{C} E(u)$ can be controlled uniformly with respect to C^2-perturbations of the boundary. Lemma 4.3 deals with the construction of a suitable harmonic lifting from Γ_h to Ω_h, which is then used in Lemma 4.4 to prove a higher integrability result for $E(u)$.

For the definitions and the properties of fractional Sobolev spaces needed in this section we refer to the Appendix.

Lemma 4.1. *Let* $(g, v) \in Y(e_0; 0, b)$ *be such that* $g > 2c_0 > 0$ *in* $[0, b]$, $g \in C^2_\#([0, b])$ *and*

$$\int_{\Omega_g} \mathbb{C}E(v) : E(w)\,\mathrm{d}z = \int_{\Omega_g} d : \nabla w\,\mathrm{d}z \quad \textit{for all } w \in A(\Omega_g),$$

where $d \in C^1(\overline{\Omega}^\#_g; \mathbb{M}^{2\times 2})$ *is b-periodic in x. Then, setting* $D := \Omega_g \setminus \overline{\Omega}_{g-c_0}$, $D' := \Omega_g \setminus \overline{\Omega}_{g-2c_0}$, *for all* $p > 1$

$$\|E(v)\|_{W^{1,p}(D;\mathbb{M}^{2\times 2})} + \|\nabla \mathbb{C}E(v)\|_{H^{-\frac{1}{2}}_\#(\Gamma_g;\mathbb{T})} \leqq C(\|E(v)\|_{L^2(D';\mathbb{M}^{2\times 2})} + \|d\|_{C^1(\overline{D}';\mathbb{M}^{2\times 2})}),$$

where $\mathbb{T}$ *denotes the space of third order tensors and* C *is a positive constant depending only on the* C^2*-norm of* g*, on* c_0 *and* p*.*

Proof. Without loss of generality we may assume that d is C^2 in $\overline{\Omega}_g$. For $i, j \in \{1, 2\}$ we define

$$\sigma_{ijx} = \left(\mathbb{C}E\left(\frac{\partial v}{\partial x}\right)\right)_{ij}, \qquad d_{ijx} = \left(\frac{\partial d}{\partial x}\right)_{ij}$$

and similarly for σ_{ijy}, d_{ijy}. Let $\varphi \in H^{\frac{1}{2}}_\#(\Gamma_g)$. With the same letter we denote a lifting of φ such that $(\varphi, 0) \in A(\Omega_g)$, φ vanishes in Ω_{g-c_0} and $\|\varphi\|_{H^1(\Omega_g)} \leqq C\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_g)}$, with C depending only on the C^1-norm of g, p and on c_0. Such a lifting exists thanks to Theorem 8.5. Differentiating the equation $\mathrm{div}(\mathbb{C}E(v)) = \mathrm{div} d$ with respect to x, multiplying by φ, and integrating by parts, we get

$$\begin{aligned}
\int_{\Gamma_g} (\sigma_{11x}, \sigma_{12x}) \cdot \nu\varphi\,\mathrm{d}\mathcal{H}^1 &= \int_D \mathrm{div}(d_{11x}, d_{12x})\varphi\,\mathrm{d}z + \int_D (\sigma_{11x}, \sigma_{12x}) \cdot \nabla\varphi\,\mathrm{d}z \\
&= \int_D (\sigma_{11x} - d_{11x}, \sigma_{12x} - d_{12x}) \cdot \nabla\varphi\,\mathrm{d}z + \int_{\Gamma_g} (d_{11x}, d_{12x}) \cdot \nu\varphi\,\mathrm{d}\mathcal{H}^1 \\
&\leqq C(\|\nabla E(v)\|_{L^2(D;\mathbb{T})} + \|d\|_{C^1(\overline{D};\mathbb{M}^{2\times 2})})(\|\nabla\varphi\|_{L^2(D;\mathbb{R}^2)} + \|\varphi\|_{L^2(\Gamma_g)}) \\
&\leqq C(\|\nabla E(v)\|_{L^2(D;\mathbb{T})} + \|d\|_{C^1(\overline{D};\mathbb{M}^{2\times 2})})\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_g)}.
\end{aligned}$$

Therefore, setting $\alpha := (\sigma_{11x}, \sigma_{12x}) \cdot \nu$, from the inequality above we deduce that

$$\|\alpha\|_{H^{-\frac{1}{2}}_\#(\Gamma_g)} \leqq C(\|\nabla E(v)\|_{L^2(D;\mathbb{T})} + \|d\|_{C^1(\overline{D};\mathbb{M}^{2\times 2})}), \tag{4.1}$$

where C depends only on the C^1-norm of g, on p, and on c_0. Similarly,

$$\|\beta\|_{H^{-\frac{1}{2}}_\#(\Gamma_g)} + \|\gamma\|_{H^{-\frac{1}{2}}_\#(\Gamma_g)} + \|\delta\|_{H^{-\frac{1}{2}}_\#(\Gamma_g)} \leqq C(\|\nabla E(v)\|_{L^2(D;\mathbb{T})} + \|d\|_{C^1(\overline{D};\mathbb{M}^{2\times 2})}), \tag{4.2}$$

where

$$\beta := (\sigma_{12x}, \sigma_{22x}) \cdot \nu, \quad \gamma := (\sigma_{11y}, \sigma_{12y}) \cdot \nu, \quad \delta := (\sigma_{12y}, \sigma_{22y}) \cdot \nu.$$

Recall that by Theorem 8.6, if $\sigma \in H^1(D)$ is b-periodic in x, then

$$\|\partial_\tau \sigma\|_{H_\#^{-\frac{1}{2}}(\Gamma_g)} \leqq C\|\nabla\sigma\|_{L^2(D;\mathbb{R}^2)}$$

for some constant C depending only on the C^1-norm of g, on p, and on c_0. Hence, setting $\eta := \partial_\tau(\mathbb{C}E(v))_{11}$ and $\vartheta := \partial_\tau\left(\frac{\partial v_1}{\partial y}\right)$, we have

$$\begin{aligned}\|\eta\|_{H_\#^{-\frac{1}{2}}(\Gamma_g)} + \|\vartheta\|_{H_\#^{-\frac{1}{2}}(\Gamma_g)} &\leqq C\|\nabla\mathbb{C}E(v)\|_{L^2(D;\mathbb{T})} + \|\nabla\left(\frac{\partial v_1}{\partial y}\right)\|_{L^2(D;\mathbb{R}^2)}\\ &\leqq C\|\nabla\mathbb{C}E(v)\|_{L^2(D;\mathbb{T})}\\ &\leqq C\|\nabla E(v)\|_{L^2(D;\mathbb{T})},\end{aligned} \tag{4.3}$$

where we also used the fact that

$$\begin{aligned}\frac{\partial^2 v_1}{\partial y^2} &= \frac{4(\mu+\lambda)\sigma_{12y} + \lambda\sigma_{11x} - (2\mu+\lambda)\sigma_{22x}}{4\mu(\mu+\lambda)},\\ \frac{\partial^2 v_1}{\partial x\partial y} &= \frac{(2\mu+\lambda)\sigma_{11y} - \lambda\sigma_{22y}}{4\mu(\mu+\lambda)}.\end{aligned} \tag{4.4}$$

Observe that $\vartheta = \frac{\partial^2 v_1}{\partial x\partial y}\nu_2 - \frac{\partial^2 v_1}{\partial y^2}\nu_1$. Therefore, recalling also (4.4), $\sigma_{ijx}, \sigma_{ijy}$, i, $j \in \{1, 2\}$ satisfy the following linear system

$$A\begin{pmatrix}\sigma_{11x}\\ \sigma_{12x}\\ \sigma_{22x}\\ \sigma_{11y}\\ \sigma_{12y}\\ \sigma_{22y}\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\\ \eta\\ 4\mu\vartheta(\mu+\lambda)\end{pmatrix},$$

with

$$A := \begin{pmatrix}\nu_1 & \nu_2 & 0 & 0 & 0 & 0\\ 0 & \nu_1 & \nu_2 & 0 & 0 & 0\\ 0 & 0 & 0 & \nu_1 & \nu_2 & 0\\ 0 & 0 & 0 & 0 & \nu_1 & \nu_2\\ \nu_2 & 0 & 0 & -\nu_1 & 0 & 0\\ -\lambda\nu_1 & 0 & (2\mu+\lambda)\nu_1 & (2\mu+\lambda)\nu_2 & -4(\mu+\lambda)\nu_1 & -\lambda\nu_2\end{pmatrix}$$

A lengthy but elementary computation yields that the determinant of the 6×6 matrix of the system is $(2\mu+\lambda)\nu_2^2 > c > 0$, where c is a constant depending only on the C^1-norm of g. Thus, $\sigma_{ijx}, \sigma_{ijy}$ can be written as linear combinations of $\alpha, \beta, \gamma, \delta, \eta, \vartheta$ with coefficients given by suitable polynomials in ν_1, ν_2 divided by ν_2^2. Then from (4.1)–(4.3), using Lemma 8.8, it follows that

$$\|\nabla\mathbb{C}E(v)\|_{H_\#^{-\frac{1}{2}}(\Gamma_g;\mathbb{T})} \leqq C(\|\nabla E(v)\|_{L^2(D;\mathbb{T})} + \|d\|_{C^1(\overline{D};\mathbb{M}^{2\times 2})}), \tag{4.5}$$

with a constant C depending on c_0 and the C^1-norm of ν_1, ν_2, hence on the C^2-norm of g. It remains to estimate $\|E(v)\|_{W^{1,p}(D;\mathbb{M}^{2\times 2})}$. To this aim fix $p > 1$, set

$D'' = \Omega_g \setminus \overline{\Omega}_{g-3c_0/2}$, and fix a ball $B \subset\subset D'' \setminus \overline{D}$. By adding an infinitesimal rigid motion, if necessary, we may assume without loss of generality that

$$\int_B (\nabla v - \nabla^T v)\,\mathrm{d}x = 0 \quad \text{and} \quad \int_B v\,\mathrm{d}x = 0. \tag{4.6}$$

Then standard elliptic regularity results (see [2, Section 10.2]) imply

$$\begin{aligned}\|E(v)\|_{W^{1,p}(D;\mathbb{M}^{2\times 2})} &\leqq C\big(\|v\|_{W^{1,p}(D'';\mathbb{R}^2)} + \|d\|_{C^1(\overline{D}'';\mathbb{M}^{2\times 2})}\big)\\ &\leqq C\big(\|E(v)\|_{L^p(D'';\mathbb{M}^{2\times 2})} + \|d\|_{C^1(\overline{D}'';\mathbb{M}^{2\times 2})}\big)\\ &\leqq C\big(\|E(v)\|_{H^1(D'';\mathbb{M}^{2\times 2})} + \|d\|_{C^1(\overline{D}'';\mathbb{M}^{2\times 2})}\big), \end{aligned} \tag{4.7}$$

where the second inequality follows from Korn's inequality and (4.6), while in the last one we used the Sobolev imbedding and the fact that we are in dimension two. Arguing similarly with D and D'' replaced by D'' and D', respectively, by elliptic regularity and Korn's inequality, we get

$$\|E(v)\|_{H^1(D'';\mathbb{M}^{2\times 2})} \leqq C\big(\|E(v)\|_{L^2(D';\mathbb{M}^{2\times 2})} + \|d\|_{C^1(\overline{D}';\mathbb{M}^{2\times 2})}\big). \tag{4.8}$$

The conclusion of the lemma then follows by combining (4.5), (4.7), and (4.8).

Remark 4.2. An obvious modification of the final part of the previous proof also shows that if $v \in A(\Omega_g)$, then

$$\begin{aligned}&\|E(v)\|_{W^{1,p}(\Omega_g;\mathbb{M}^{2\times 2})} + \|\nabla \mathbb{C} E(v)\|_{H_\#^{-\frac{1}{2}}(\Gamma_g;\mathbb{T})}\\ &\quad\leqq C(\|v\|_{H^1(\Omega_g;\mathbb{M}^{2\times 2})} + \|d\|_{C^1(\overline{\Omega}_g;\mathbb{M}^{2\times 2})})\end{aligned} \tag{4.9}$$

holds with a constant depending only on the C^2-norm of g, on c_0, and p.

Lemma 4.3. *Let g, D be as in Lemma 4.1 and let $p > 1$. Given $\psi \in C^1_\#(\Gamma_g;\mathbb{R}^2)$, $a \in C^1_\#(\Gamma_g)$, there exists a harmonic function $f \in W^{1,p}(D)$ such that ∇f is b-periodic in x and $\partial_\nu f = a\,\mathrm{div}_\tau \psi$ on Γ_g. Moreover,*

$$\|\nabla f\|_{L^p(D;\mathbb{R}^2)} \leqq C\|\psi\|_{W^{1-\frac{1}{p},p}(\Gamma_g;\mathbb{R}^2)}, \tag{4.10}$$

where C is a constant depending only on the C^2-norm of g and on the C^1-norm of a.

Proof. Write $a\,\mathrm{div}_\tau \psi = \partial_\tau(a\psi_1\tau_1 + a\psi_2\tau_2) - \psi_1\partial_\tau(a\tau_1) - \psi_2\partial_\tau(a\tau_2)$. Thanks to Theorem 8.5 there exists a function $\Psi \in W^{1,p}(D)$, b-periodic in x, whose trace on Γ_g is $a\psi_1\tau_1 + a\psi_2\tau_2$ and such that

$$\|\Psi\|_{W^{1,p}(D)} \leqq C\|\psi\|_{W^{1-\frac{1}{p},p}(\Gamma_g;\mathbb{R}^2)},$$

with C depending only on the C^2-norm of g and on the C^1-norm of a. Let $\widetilde{G} \in W^{1,p}(\Gamma_g)$ be such that $\widetilde{G}(0, g(0)) = 0$ and $\partial_\tau \widetilde{G} = \psi_1\partial_\tau(a\tau_1) + \psi_2\partial_\tau(a\tau_2)$ and denote by G the function in $W^{1,p}(D)$ obtained by extending $\widetilde{G}$ in Ω by $G(x, y) := \widetilde{G}(x, g(x))$. Clearly ∇G is b-periodic in x and we have the estimate

$$\|G\|_{W^{1,p}(D)} \leqq C\|\psi\|_{W^{1-\frac{1}{p},p}(\Gamma_g;\mathbb{R}^2)},$$

again with a constant depending only on the C^2-norm of g and on the C^1-norm of a. Let $v \in H^1_{0,\#}(D)$ denote the solution of the following problem

$$\int_D \nabla v \cdot \nabla w \, dz = \int_D (\nabla \Psi - \nabla G) \cdot \nabla w \, dz \qquad \text{for all } w \in H^1_{0,\#}(D),$$

where

$$H^1_{0,\#}(D) := \{w \in H^1(D) : \\ w(x, g(x)) = w(x, g(x) - c_0) = 0, \ w(0, y) = w(b, y)\}.$$

By classical regularity results we have that $v \in W^{1,p}(D)$ and $\|v\|_{W^{1,p}(D)} \leqq C\|\nabla\Psi - \nabla G\|_{L^p(D;\mathbb{R}^2)}$, with a constant C depending only on the C^2-norm of g. Setting $w := \Psi - G - v$, then w is harmonic in D, ∇w is b-periodic, $\partial_\tau w = a\,\mathrm{div}_\tau \psi$ and $\|\nabla w\|_{L^p(D;\mathbb{R}^2)} \leqq C\|\psi\|_{W^{1-\frac{1}{p},p}(\Gamma_g;\mathbb{R}^2)}$. The conclusion then follows by letting f be a harmonic conjugate of w.

Lemma 4.4. *Let g, c_0, D, and D' be as in Lemma 4.1, let $p > 2$, and let $M \in C^1_\#(\Gamma_g; \mathbb{M}^{2\times 2})$. There exist δ and $C > 0$, depending only on p, c_0, and on the C^2-norm of g, such that if $v \in A(\Omega_g)$ is the solution to the problem*

$$\int_{\Omega_g} \mathbb{C}E(v) : E(w) \, dz = \int_{\Gamma_g} \mathrm{div}_\tau M \cdot w \, d\mathcal{H}^1$$

for all $w \in A(\Omega_g)$, then the following estimate holds:

$$\|E(v)\|_{L^{2+\delta}(D;\mathbb{M}^{2\times 2})} \leqq C\big(\|E(v)\|_{L^2(D';\mathbb{M}^{2\times 2})} + \|M\|_{W^{1-\frac{1}{p},p}(\Gamma_g;\mathbb{M}^{2\times 2})}\big).$$

Proof. Adding an infinitesimal rigid motion, if necessary, we may assume without loss of generality that (4.6) holds for some ball $B \subset\subset D' \setminus \overline{D}$. Let $f \in W^{1,p}(D'; \mathbb{R}^2)$ be the harmonic map whose components f_1 and f_2 are constructed in Lemma 4.3, with D replaced by D' and taking $a = 1$ and ψ coinciding with the rows M_1 and M_2 of M, respectively. From the assumption, integrating by parts, we get that

$$\int_{\Omega_g} \mathbb{C}E(v) : E(w) \, dz = \int_{\Omega_g} \nabla f : \nabla w \, dz$$

for all $w \in A(\Omega_g)$ vanishing in $\Omega_g \setminus D'$. Then by a standard argument (see [17, Chapter V]) one obtains that there exists $r_0 > 0$, depending on the C^1-norm of g and on c_0 such that if $z_0 \in D^\#$ and $r < r_0$

$$\int_{B_r(z_0)\cap\Omega_g^\#} |\nabla v|^2 \, dz \leqq \frac{C}{r^2}\Big(\int_{B_{2r}(z_0)\cap\Omega_g^\#} |\nabla v| \, dz\Big)^2 + C\int_{B_{2r}(z_0)\cap\Omega_g^\#} |\nabla f|^2 \, dz,$$

where the constant C depends on c_0 and the C^1-norm of g. Setting $\alpha := |\nabla v|^2 \chi_{\Omega_g^\#}$, $\beta := |\nabla f|^2 \chi_{\Omega_g^\#}$, the above inequality reads

$$\fint_{B_r(z_0)} \alpha \, dz \leqq C\left(\fint_{B_{2r}(z_0)} \sqrt{\alpha} \, dz\right)^2 + C\fint_{B_{2r}(z_0)} \beta \, dz$$

for all $z_0 \in D^\#$ and $r < r_0$. Since $\beta \in L^{p/2}(D')$, a standard application of Gehring's Lemma (see [17, Proposition 1.1]) yields that

$$\|\nabla v\|_{L^{2+\delta}(D;\mathbb{M}^{2\times 2})} \leqq C(\|\nabla v\|_{L^2(D';\mathbb{M}^{2\times 2})} + \|\nabla f\|_{L^p(D';\mathbb{M}^{2\times 2})}).$$

Hence the conclusion follows from Korn's Inequality and (4.10).

The next proposition will be crucial in proving Theorem 2.10.

Proposition 4.5. *Let* $(h, u) \in X(e_0, q; 0, b)$ *be a critical point for* F *such that* $h > 2c_0 > 0$ *in* $[0, b]$, $h \in C^\infty_\#([0, b])$ *and*

$$\partial^2 F(h, u)[\varphi] \geqq C_1 \|\varphi\|^2_{H^1(\Gamma_h)} \tag{4.11}$$

for all $\varphi \in \widetilde{H}^1_\#(\Gamma_h)$ *with* $C_1 > 0$. *Let* (g_n, v_n) *be any sequence in* $X(e_0, q; 0, b)$ *such that* $g_n \in C^\infty_\#([0, b])$, $\int_0^b g_n \,\mathrm{d}x = \int_0^b h \,\mathrm{d}x$, *and* $g_n \to h$ *in* $C^2([0, b])$ *and let* $\psi_n \in \widetilde{H}^1_\#(\Gamma_{g_n})$ *be defined as* $\psi_n := \frac{g_n - h}{\sqrt{1+g_n'^2}} \circ \pi_1$. *Then there exists a constant* $C_2 > 0$ *depending only on* h, *such that*

$$F(h, u) + C_2 \|\psi_n\|^2_{H^1(\Gamma_{g_n})} \leqq F(g_n, v_n) \tag{4.12}$$

for n *large enough.*

Proof. In order to prove (4.12) we may assume without loss of generality that v_n is the elastic equilibrium in Ω_{g_n}, that is, the solution to (3.1) with h replaced by g_n. We will use the bilinear form $(\cdot,\cdot)_\sim : \widetilde{H}^1_\#(\Gamma_h) \times \widetilde{H}^1_\#(\Gamma_h) \to \mathbb{R}$, introduced in (3.21). We also introduce the bilinear forms $(\cdot,\cdot)_{\sim, g_n} : \widetilde{H}^1_\#(\Gamma_{g_n}) \times \widetilde{H}^1_\#(\Gamma_{g_n}) \to \mathbb{R}$ defined as in (3.21) with Γ_h and a replaced by Γ_{g_n} and $a_{g_n} := \partial_{\nu_{g_n}}[Q(E(v_n))] - k^2_{g_n}$. Here ν_{g_n} denotes the outer unit normal vector to Ω_{g_n} while k_{g_n} is the curvature of Γ_{g_n}. Finally, let $\Phi_n : \Omega_h \to \Omega_{g_n}$ be a diffeomorphism of class $C^2(\overline{\Omega}_h; \mathbb{R}^2)$, such that $\Phi_n - Id$ is b-periodic in x together with its first and second derivatives, $\Phi_n(x, 0) = (x, 0)$ for all $x \in [0, b]$, $\|\Phi_n - Id\|_{C^2(\overline{\Omega}_h;\mathbb{R}^2)} \leqq 2\|g_n - h\|_{C^2([0,b])}$ and $\Phi_n(x, y) := (x, y + g_n - h)$ in a neighborhood of $\overline{\Gamma}_h$. Finally, to avoid confusion we shall denote by ν_h the outer unit normal to Ω_h on Γ_h and we will write a_h to denote the quantity a defined in (3.21).

We now split the proof of the proposition into several steps.

Step 1 From the equations satisfied by u and v_n and an easy change of variable, we obtain that for all $w \in A(\Omega_{g_n})$

$$\int_{\Omega_{g_n}} \mathbb{C}(E(u \circ \Phi_n^{-1}) - E(v_n)) : E(w) \,\mathrm{d}z = \int_{\Omega_{g_n}} d_n : \nabla w \,\mathrm{d}z, \tag{4.13}$$

where

$$\|d_n\|_{C^1(\overline{\Omega}_{g_n};\mathbb{M}^{2\times 2})} \leqq C\|\Phi_n - Id\|_{C^2(\overline{\Omega}_h;\mathbb{R}^2)} \leqq C\|g_n - h\|_{C^2([0,b])},$$

and C is a constant depending only on the C^2-norm of u. From (4.9) and Korn's inequality (notice that $u \circ \Phi_n^{-1} - v_n = 0$ on $\{y = 0\}$), we get that for all $p > 1$

$$\begin{aligned}
&\|E(u \circ \Phi_n^{-1}) - E(v_n)\|_{W^{1,p}(\Omega_{g_n};\mathbb{M}^{2\times 2})} \\
&\quad + \|\nabla \mathbb{C}(E(u \circ \Phi_n^{-1}) - E(v_n))\|_{H_\#^{-\frac{1}{2}}(\Gamma_{g_n};\mathbb{T})} \\
&\leqq C\big(\|E(u \circ \Phi_n^{-1}) - E(v_n)\|_{L^2(\Omega_{g_n};\mathbb{M}^{2\times 2})} + \|g_n - h\|_{C^2([0,b])}\big) \\
&\leqq C\|g_n - h\|_{C^2([0,b])}, \qquad (4.14)
\end{aligned}$$

for a constant C depending only on $p > 1$ and the C^2-norms of h and u. Note that the last of the previous chain of inequalities follows by choosing $w = u \circ \Phi_n^{-1} - v_n$ in (4.13) and by Korn's inequality. Using Lemma 8.8, from (4.14), we conclude that

$$\|E(u) - E(v_n) \circ \Phi_n\|_{W^{1,p}(\Omega_h;\mathbb{M}^{2\times 2})} \leqq C\|g_n - h\|_{C^2([0,b])} \qquad (4.15)$$

and

$$\|\nabla \mathbb{C} E(u) - (\nabla \mathbb{C} E(v_n)) \circ \Phi_n\|_{H_\#^{-\frac{1}{2}}(\Gamma_h;\mathbb{T})} \leqq C\|g_n - h\|_{C^2([0,b])}. \qquad (4.16)$$

Let ν_{h_n} and ν_{g_n} denote the outer normal unit vectors along Γ_h and Γ_{g_n}, respectively, and $J_1\Phi_n := \frac{\sqrt{1+g_n'^2}}{\sqrt{1+h'^2}}$ be the 1-dimensional Jacobian of Φ_n on Γ_h. We claim that

$$\|\partial_{\nu_{g_n}}\big[Q(E(v_n))\big] \circ \Phi_n J_1\Phi_n - \partial_{\nu_h}\big[Q(E(u))\big]\|_{H_\#^{-\frac{1}{2}}(\Gamma_h)} \to 0. \qquad (4.17)$$

Indeed, we have for all $\varphi \in H_\#^{\frac{1}{2}}(\Gamma_h)$

$$\begin{aligned}
&\int_{\Gamma_h} \Big[\frac{\partial}{\partial x} Q(E(u)) - \Big(\frac{\partial}{\partial x} Q(E(v_n))\Big) \circ \Phi_n\Big]\varphi \,\mathrm{d}\mathcal{H}^1 \\
&= \int_{\Gamma_h} \Big[\mathbb{C}E\Big(\frac{\partial}{\partial x} u\Big) - \Big(\mathbb{C}E\Big(\frac{\partial}{\partial x} v_n\Big)\Big) \circ \Phi_n\Big] : (E(v_n) \circ \Phi_n)\varphi \,\mathrm{d}\mathcal{H}^1 \\
&\quad + \int_{\Gamma_h} \mathbb{C}E\Big(\frac{\partial}{\partial x} u\Big) : \big(E(u) - E(v_n) \circ \Phi_n\big)\varphi \,\mathrm{d}\mathcal{H}^1 \\
&\leqq C\|\nabla \mathbb{C} E(u) - (\nabla \mathbb{C} E(v_n)) \circ \Phi_n\|_{H_\#^{-\frac{1}{2}}(\Gamma_h;\mathbb{T})} \|(E(v_n) \circ \Phi_n)\varphi\|_{H^{\frac{1}{2}}(\Gamma_h;\mathbb{M}^{2\times 2})} \\
&\quad + \|E(u) - E(v_n) \circ \Phi_n\|_{L^\infty(\Gamma_h;\mathbb{M}^{2\times 2})} \|\varphi\|_{L^2(\Gamma_h)}, \qquad (4.18)
\end{aligned}$$

where the constant C depends only on the C^2-norms of u and h and on the length of Γ_h. Fix $p > 2$. Recalling the definition of Gagliardo seminorm (8.1) and using Hölder's inequality, the Sobolev Imbedding Theorem 8.3 (notice that $H^{\frac{1}{2}}(\Gamma_h)$ is continuously imbedded in $L^p(\Gamma)$ for all $p > 1$), and the trace Theorem 8.4, one obtains that

$$\begin{aligned}
&\|(E(v_n) \circ \Phi_n)\varphi\|_{H^{\frac{1}{2}}(\Gamma_h;\mathbb{M}^{2\times 2})} \leqq C\|E(v_n) \circ \Phi_n\|_{L^\infty(\Gamma_h;\mathbb{M}^{2\times 2})} \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_h)} \\
&\qquad + \|\varphi\|_{L^p(\Gamma_h)} \|E(v_n) \circ \Phi_n\|_{W^{\frac{p+2}{2p},\frac{2p}{p-2}}(\Gamma_h;\mathbb{M}^{2\times 2})} \\
&\quad \leqq C(\|E(v_n) \circ \Phi_n\|_{L^\infty(\Gamma_h;\mathbb{M}^{2\times 2})} + \|E(v_n) \circ \Phi_n\|_{W^{1,\frac{2p}{p-2}}(\Omega_h;\mathbb{M}^{2\times 2})}) \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_h)},
\end{aligned}$$

where the constant C depends only on p and on the C^1-norm of h. From the last inequality and (4.18), using (4.15) and (4.16), and performing a similar estimate for the derivative with respect to y, we conclude

$$\|\nabla(Q(E(v_n)))\circ\Phi_n - \nabla Q(E(u))\|_{H_\#^{-\frac{1}{2}}(\Gamma_h)} \leqq C\|g_n - h\|_{C^2([0,b])} \to 0.$$

Hence, (4.17) follows, taking into account that $\|\nu_{g_n}\circ\Phi_n - \nu_h\|_{C^1(\Gamma_h)} \to 0$, $\|\Phi_n - Id\|_{C^2(\overline{\Omega}_h;\mathbb{R}^2)} \to 0$, and using Lemma 8.8.

Step 2 We show that for n large enough

$$\|\varphi\|^2_{H^1(\Gamma_{g_n})} \leqq 3C_1^{-1}\|\varphi\|^2_{\sim,g_n} \qquad \text{for all } \varphi\in\widetilde{H}^1_\#(\Gamma_{g_n}) \tag{4.19}$$

(C_1 is the constant appearing in (4.11)).

Note that by (3.17) and (4.11) we have

$$\|\varphi\|^2_\sim \geqq \partial^2 F(h,u)[\varphi] \geqq C_1\|\varphi\|^2_{\widetilde{H}^1_\#(\Gamma_h)} \tag{4.20}$$

for all $\varphi\in\widetilde{H}^1_\#(\Gamma_h)$. Given $\varphi\in\widetilde{H}^1_\#(\Gamma_{g_n})$, we set

$$\tilde{\varphi} := (\varphi\circ\Phi_n)J_1\Phi_n.$$

Then $\tilde{\varphi}\in\widetilde{H}^1_\#(\Gamma_h)$ and

$$\begin{aligned}\|\varphi\|^2_{H^1(\Gamma_{g_n})} &= \int_{\Gamma_h}\left(|\varphi\circ\Phi_n|^2 + |(\partial_{\tau_{g_n}}\varphi)\circ\Phi_n|^2\right)J_1\Phi_n\,\mathrm{d}\mathcal{H}^1\\ &\leqq (1+\delta_n)\int_{\Gamma_h}\left(\tilde{\varphi}^2 + (\partial_{\tau_h}\tilde{\varphi})^2\right)\mathrm{d}\mathcal{H}^1\\ &\leqq (1+\delta_n)C_1^{-1}\|\tilde{\varphi}\|^2_\sim,\end{aligned} \tag{4.21}$$

where in the last inequality we used (4.20). The constant δ_n appearing in the above formulas depends only on $\|g_n - h\|_{C^2([0,b])}$ and tends to zero as $n\to\infty$. To continue, note that from (4.17) and Lemma 8.8 we have

$$\|(a_{g_n}\circ\Phi_n)J_1\Phi_n - a_h(J_1\Phi_n)^2\|_{H_\#^{-\frac{1}{2}}(\Gamma_h)} \to 0. \tag{4.22}$$

Moreover, we have

$$\begin{aligned}\|\tilde{\varphi}\|^2_\sim &= \int_{\Gamma_h}\left(a_h\tilde{\varphi}^2 + (\partial_{\tau_h}\tilde{\varphi})^2\right)\mathrm{d}\mathcal{H}^1\\ &\leqq \int_{\Gamma_h}(a_{g_n}\circ\Phi_n)(\varphi\circ\Phi_n)^2 J_1\Phi_n\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_{g_n}}(\partial_{\tau_{g_n}}\varphi)^2\,\mathrm{d}\mathcal{H}^1 + \delta_n\|\varphi\|^2_{H^1(\Gamma_{g_n})}\\ &\quad + \|(a_{g_n}\circ\Phi_n)J_1\Phi_n - a_h(J_1\Phi_n)^2\|_{H_\#^{-\frac{1}{2}}(\Gamma_h)}\|(\varphi\circ\Phi_n)^2\|_{H^{\frac{1}{2}}(\Gamma_h)}\\ &= \|\varphi\|^2_{\sim,g_n} + \|(a_{g_n}\circ\Phi_n)J_1\Phi_n - a_h(J_1\Phi_n)^2\|_{H_\#^{-\frac{1}{2}}(\Gamma_h)}\|(\varphi\circ\Phi_n)^2\|_{H^{\frac{1}{2}}(\Gamma_h)}\\ &\quad + \delta_n\|\varphi\|^2_{H^1(\Gamma_{g_n})},\end{aligned} \tag{4.23}$$

where, as before, δ_n depends only on $\|g_n - h\|_{C^2([0,b])}$ and tends to zero as $n \to \infty$. Finally, note that

$$\|(\varphi \circ \Phi_n)^2\|_{H^{\frac{1}{2}}(\Gamma_h)} \leqq C'\|(\varphi \circ \Phi_n)^2\|_{H^1(\Gamma_h)} \leqq C''\|\varphi \circ \Phi_n\|^2_{H^1(\Gamma_h)} \leqq C''(1+\delta_n)\|\varphi\|^2_{H^1(\Gamma_{g_n})}, \quad (4.24)$$

with δ_n as before and C'' independent of n. Note that the second inequality in the above formula can be proved taking into account the imbedding of $H^1(\Gamma_h)$ into $L^\infty(\Gamma_h)$. Combining (4.23) with (4.21) and taking into account (4.22) and (4.24), we conclude that for n large enough $\|\varphi\|^2_{H^1(\Gamma_{g_n})} \leqq 3C_1^{-1}\|\varphi\|^2_{\sim,g_n}$ for all $\varphi \in \widetilde{H}^1_\#(\Gamma_{g_n})$.

Step 3 Let T_{g_n} be the operators defined by (3.23) with $(\cdot,\cdot)_\sim$ and h replaced by $(\cdot,\cdot)_{\sim,g_n}$ and g_n, respectively. Let T_h be the operator corresponding to h. Note that the definition of T_{g_n} is well-posed, thanks to (4.19). Denote the first eigenvalues of T_h and T_{g_n} (see (3.24)) by $\lambda_{1,h}$ and λ_{1,g_n}, respectively. We claim that

$$\limsup_{n\to\infty} \lambda_{1,g_n} \leqq \lambda_{1,h}. \quad (4.25)$$

Assume without loss of generality that $\limsup_{n\to\infty}\lambda_{1,g_n} = \lim_{n\to\infty}\lambda_{1,g_n} =: \lambda_\infty$. Then there exist $\varphi_n \in C^\infty_\#(\Gamma_{g_n}) \cap \widetilde{H}^1_\#(\Gamma_{g_n})$, with $\|\varphi_n\|_{\sim,g_n} = 1$, and $v_{\varphi_n,g_n} \in A(\Omega_{g_n})$, solution to

$$\int_{\Omega_{g_n}} \mathbb{C}E(v_{\varphi_n,g_n}) : E(w)\,dz = \int_{\Gamma_{g_n}} \mathrm{div}_{\tau_{g_n}}(\varphi_n \mathbb{C}E(v_n)) \cdot w\, d\mathcal{H}^1 \quad (4.26)$$

for all $w \in A(\Omega_{g_n})$, such that

$$(T_{g_n}\varphi_n, \varphi_n)_{\sim,g_n} = 2\int_{\Omega_{g_n}} Q(E(v_{\varphi_n,g_n}))\,dz \to \lambda_\infty. \quad (4.27)$$

Note that by (4.19)

$$\sup_n \|\varphi_n\|_{H^1(\Gamma_{g_n})} < +\infty. \quad (4.28)$$

By the Imbedding Theorem 8.3 and the C^2-equiboundedness of the functions g_n it follows that $\sup_n \|\varphi_n\|_{W^{\frac{3}{4},4}(\Gamma_{g_n})} < +\infty$. From (4.15) it follows that $\sup_n \|\mathbb{C}E(v_n)\|_{C^{0,\alpha}(\Gamma_{g_n};\mathbb{M}^{2\times2})} < \infty$ for all $\alpha \in (0,1)$. Then, using the definition of the Gagliardo seminorm of $W^{\frac{3}{4},4}(\Gamma_{g_n})$ and the α-Hölder continuity with α sufficiently close to 1 (compare with Lemma 8.8 where α is taken equal to 1), one can check that

$$\sup_n \|\varphi_n \mathbb{C}E(v_n)\|_{W^{\frac{3}{4},4}(\Gamma_{g_n};\mathbb{M}^{2\times2})} < +\infty. \quad (4.29)$$

Using Lemma 4.4 we then have that

$$\|E(v_{\varphi_n,g_n})\|_{L^{2+\delta}(\Omega_{g_n}\setminus\Omega_{g_n-c_0};\mathbb{M}^{2\times2})} \leqq C(\|E(v_{\varphi_n,g_n})\|_{L^2(\Omega_{g_n}\setminus\Omega_{g_n-2c_0};\mathbb{M}^{2\times2})} + \|\varphi_n\mathbb{C}E(v_n)\|_{W^{\frac{3}{4},4}(\Gamma_{g_n})}) \quad (4.30)$$

for some δ and C independent of n. Choosing $w = v_{\varphi_n,g_n}$ in equation (4.26) we have, thanks to Corollary 8.7 and trace Theorem 8.4,

$$\begin{aligned}\int_{\Omega_{g_n}} &|E(v_{\varphi_n,g_n})|^2\,\mathrm{d}z \\ &\leqq C\|\mathrm{div}_{\tau_{g_n}}(\varphi_n\mathbb{C}E(v_n))\|_{W_{\#}^{-\frac14,4}(\Gamma_{g_n};\mathbb{R}^2)}\|v_{\varphi_n,g_n}\|_{W^{\frac14,\frac43}(\Gamma_{g_n};\mathbb{R}^2)} \\ &\leqq C\|\varphi_n\mathbb{C}E(v_n)\|_{W^{\frac34,4}(\Gamma_{g_n};\mathbb{R}^2)}\|v_{\varphi_n,g_n}\|_{W^{1,\frac43}(\Omega_{g_n};\mathbb{R}^2)} \\ &\leqq C\|E(v_{\varphi_n,g_n})\|_{L^{\frac43}(\Omega_{g_n};\mathbb{R}^2)} \leqq C\|E(v_{\varphi_n,g_n})\|_{L^2(\Omega_{g_n};\mathbb{R}^2)},\end{aligned} \tag{4.31}$$

where the last inequality follows from (4.29) and Korn's inequality with a constant independent on n. Hence, from (4.30), we deduce that

$$\sup_n \|E(v_{\varphi_n,g_n})\|_{L^{2+\delta}(\Omega_{g_n}\setminus\Omega_{g_n-c_0};\mathbb{M}^{2\times2})} < \infty.$$

We may extend v_{φ_n,g_n} to a function in $A(\Omega_{g_n}\cup\Omega_{h_n})$ in such a way that

$$\sup_n \|E(v_{\varphi_n,g_n})\|_{L^{2+\delta}((\Omega_{g_n}\cup\Omega_h)\setminus\Omega_{g_n-c_0};\mathbb{M}^{2\times2})} < +\infty. \tag{4.32}$$

We finally set $\tilde\varphi_n := \ell_n(\varphi_n\circ\Phi_n)J_1\Phi_n$, where $\ell_n := \|(\varphi_n\circ\Phi_n)J_1\Phi_n\|_{\sim}^{-1}$. From (4.23) applied with φ_n in place of φ and $\tilde\varphi$ replaced by $\frac{\tilde\varphi_n}{\ell_n}$ and recalling (4.28), we deduce that

$$\ell_n \to 1. \tag{4.33}$$

Let $v_{\tilde\varphi_n,h}$ defined as in (3.18) with φ replaced by $\tilde\varphi_n$. Arguing as before, we may extend $v_{\tilde\varphi_n,h}$ to a function in $A(\Omega_{g_n}\cup\Omega_h)$ in such a way that

$$\sup_n \|E(v_{\tilde\varphi_n,h})\|_{L^{2+\delta}((\Omega_{g_n}\cup\Omega_h)\setminus\Omega_{g_n-c_0};\mathbb{M}^{2\times2})} < +\infty. \tag{4.34}$$

To conclude the proof of (4.25) it will be enough to show that

$$\lim_{n\to\infty}\int_{\Omega_h} Q(E(v_{\tilde\varphi_n,h}-v_{\varphi_n,g_n}))\,\mathrm{d}z = 0. \tag{4.35}$$

Indeed, by (4.27) this would imply

$$\begin{aligned}\lambda_{1,h} \geqq \lim_{n\to\infty}(T_h\tilde\varphi_n,\tilde\varphi_n)_\sim &= \lim_{n\to\infty} 2\int_{\Omega_h} Q(E(v_{\tilde\varphi_n,h}))\,\mathrm{d}z \\ &= \lim_{n\to\infty} 2\int_{\Omega_h} Q(E(v_{\varphi_n,g_n}))\,\mathrm{d}z \\ &= \lim_{n\to\infty} 2\int_{\Omega_{g_n}} Q(E(v_{\varphi_n,g_n}))\,\mathrm{d}z = \lambda_\infty,\end{aligned}$$

where in the third equality we have used the equi-integrability of the functions $Q(E(v_{\varphi_n,g_n}))$ (implied by (4.32)) together with the fact that $|\Omega_{g_n}\triangle\Omega_h| \to 0$. In order to prove (4.35) we observe that $v_{\tilde\varphi_n,h} - v_{\varphi_n,g_n} \in A(\Omega_h)\cap A(\Omega_{g_n})$ is an admissible test function for (4.26) and for the equation satisfied by $v_{\tilde{\varphi}_n,h}$. Using such a test function and subtracting the two equations, we obtain

$$
\begin{aligned}
2\int_{\Omega_h} & Q(E(v_{\tilde{\varphi}_n,h} - v_{\varphi_n,g_n}))\,\mathrm{d}z \\
&= -\int_{\Omega_h\setminus\Omega_{g_n}} \mathbb{C}E(v_{\varphi_n,g_n}) : E(v_{\tilde{\varphi}_n,h} - v_{\varphi_n,g_n})\,\mathrm{d}z \\
&\quad + \int_{\Omega_{g_n}\setminus\Omega_h} \mathbb{C}E(v_{\varphi_n,g_n}) : E(v_{\tilde{\varphi}_n,h} - v_{\varphi_n,g_n})\,\mathrm{d}z \\
&\quad + \int_{\Gamma_h} \mathrm{div}_{\tau_h}(\tilde{\varphi}_n\mathbb{C}E(u))\cdot(v_{\tilde{\varphi}_n,h} - v_{\varphi_n,g_n})\,\mathrm{d}\mathcal{H}^1 \\
&\quad - \int_{\Gamma_h} \left[\mathrm{div}_{\tau_{g_n}}(\varphi_n\mathbb{C}E(v_n))\cdot(v_{\tilde{\varphi}_n,h} - v_{\varphi_n,g_n})\right]\circ\Phi_n J_1\Phi_n\,\mathrm{d}\mathcal{H}^1 \\
&=: I_n^1 + I_n^2 + I_n^3 + I_n^4.
\end{aligned}
$$

Thanks to (4.32), (4.34), and the fact that $|\Omega_{g_n}\triangle\Omega_h| \to 0$, we get $I_n^1 + I_n^2 \to 0$. Note now that I_n^4 can be rewritten as

$$
-\int_{\Gamma_h} |\nabla\Phi_n[\tau_h]|^{-1}\partial_{\tau_h}(\varphi_n\mathbb{C}E(v_n)\circ\Phi_n)[\tau_{g_n}\circ\Phi_n]\cdot(v_{\tilde{\varphi}_n,h} - v_{\varphi_n,g_n})\circ\Phi_n J_1\Phi_n\,\mathrm{d}\mathcal{H}^1.
$$

Notice that by (4.31) v_{φ_n,g_n} is bounded in $H^1(\Omega_h)$ (and similarly for $v_{\tilde{\varphi}_n,h}$). Therefore, by the trace Theorem 8.4, $v_{\tilde{\varphi}_n,h} - v_{\varphi_n,g_n}$ is bounded in $H^{\frac{1}{2}}(\Gamma_h)$. We may now apply Lemma 8.8 to deduce $\|\,|\nabla\Phi_n[\tau_h]|^{-1}(v_{\tilde{\varphi}_n,h} - v_{\varphi_n,g_n})\circ\Phi_n J_1\Phi_n - (v_{\tilde{\varphi}_n,h} - v_{\varphi_n,g_n})\|_{H^{\frac{1}{2}}(\Gamma_h)} \to 0$. Hence, (4.35) is proved once we show that $\|\mathrm{div}_{\tau_h}(\tilde{\varphi}_n\mathbb{C}E(u)) - \partial_{\tau_h}(\varphi_n\mathbb{C}E(v_n)\circ\Phi_n)[\tau_{g_n}\circ\Phi_n]\|_{H_\#^{-\frac{1}{2}}(\Gamma_h)} \to 0$. This, in turn, follows from Theorem 8.6 and Lemma 8.8 if we check that

$$
\|\tilde{\varphi}_n\mathbb{C}E(u) - \varphi_n\mathbb{C}E(v_n)\circ\Phi_n\|_{H^{\frac{1}{2}}(\Gamma_h)} \to 0. \tag{4.36}
$$

Recall that, also thanks to (4.33),

$$
\|\tilde{\varphi}_n - \varphi_n\circ\Phi_n\|_{H^1(\Gamma_h)} \to 0. \tag{4.37}
$$

Moreover, by (4.15),

$$
\|\mathbb{C}E(u) - \mathbb{C}E(v_n)\circ\Phi_n\|_{C^{0,\alpha}(\Gamma_h;\mathbb{M}^{2\times 2})} \to 0 \qquad \text{for all} \quad \alpha\in(0,1). \tag{4.38}
$$

Combining (4.37) and (4.38), claim (4.36) easily follows from the Gagliardo definition of the $H^{\frac{1}{2}}$- seminorm.

Step 4 For $t \in [0,1]$ consider $(h_{n,t}, u_{n,t}) \in X(e_0, q; 0, b)$, where $h_{n,t} := h + t(g_n - h)$ and $u_{n,t}$ is the corresponding elastic equilibrium. Note that $(h_{n,1}, u_{n,1}) = (g_n, v_n)$. Let $(\cdot,\cdot)_{\sim,h_{n,t}}$, $T_{h_{n,t}}$, and $\lambda_{1,h_{n,t}}$ be the corresponding bilinear forms, operators, and first eigenvalues. Let $\Phi_{n,t} : \Omega_h \to \Omega_{h_{n,t}}$ defined as at the beginning of the proof with g_n replaced by $h_{n,t}$. We claim that for n large enough and for all $t \in [0,1]$

$$\|\varphi\|^2_{H^1(\Gamma_{h_{n,t}})} \leqq 3C_1^{-1}\|\varphi\|^2_{\sim,h_{n,t}} \qquad \text{for all } \varphi \in \widetilde{H}^1_{\#}(\Gamma_{h_{n,t}}), \tag{4.39}$$

that

$$\limsup_{n\to\infty} \sup_{t\in(0,1]} \lambda_{1,h_{n,t}} \leqq \lambda_{1,h}, \tag{4.40}$$

and that for all $p > 1$

$$\sup_{t\in[0,1]} \|E(u) - E(u_{n,t}) \circ \Phi_{n,t}\|_{W^{1,p}(\Omega_h;\mathbb{M}^{2\times 2})} \to 0. \tag{4.41}$$

Indeed if not, then we may find a subsequence (not relabeled), $t_n \in (0,1]$, and φ_n such that

$$\|\varphi_n\|^2_{H^1(\Gamma_{h_{n,t_n}})} > 3C_1^{-1}\|\varphi_n\|^2_{\sim,h_{n,t_n}}. \tag{4.42}$$

Since the sequence (h_{n,t_n}, u_{n,t_n}) satisfies the same assumptions as (g_n, v_n), we may apply all the previous steps. In particular, by Step 3 we contradict (4.42) and conclude that (4.39) holds. In a similar fashion we may prove (4.40) and (4.41), using (4.25) and (4.15), respectively.

Step 5 We are now in a position to conclude the proof of the proposition. Let $f_n(t) := F(h_{n,t}, u_{n,t})$. We claim that

$$f_n''(t) > \frac{1}{24}C_1(1-\lambda_{1,h})\|\psi_n\|^2_{H^1(\Gamma_{g_n})}. \tag{4.43}$$

By Theorem 3.2 and the definition of $T_{h_{n,t}}$ and $(\cdot,\cdot)_{\sim,h_{n,t}}$ we have

$$\begin{aligned} f_n''(t) = &-(T_{h_{n,t}}\psi_{n,t}, \psi_{n,t})_{\sim,h_{n,t}} + \|\psi_{n,t}\|^2_{\sim,h_{n,t}} \\ &- \int_{\Gamma_{h_{n,t}}} (Q(E(u_{n,t})) + k_{h_{n,t}})\partial_{\tau_{h_{n,t}}}\left((h'_{n,t}\circ\pi_1)\psi^2_{n,t}\right) d\mathcal{H}^1, \end{aligned} \tag{4.44}$$

where $\psi_{n,t} := \frac{g_n - h}{\sqrt{1+(h'_{n,t})^2}} \circ \pi_1$. By (4.40)

$$1 - \lambda_{1,h_{n,t}} > \frac{1-\lambda_{1,h}}{2} \tag{4.45}$$

for n large enough. Moreover, since $\sup_{t\in(0,1]} \|h_{n,t} - h\|_{C^2([0,b])} \to 0$, for n sufficiently large and for all $t \in (0,1]$, we also have

$$\frac{1}{2}\|\psi_n\|^2_{H^1(\Gamma_{g_n})} \leqq \|\psi_{n,t}\|^2_{H^1(\Gamma_{h_{n,t}})} \leqq 2\|\psi_n\|^2_{H^1(\Gamma_{g_n})}. \tag{4.46}$$

From the definition of $\lambda_{1,h_{n,t}}$ and using (4.39), (4.45), and (4.46), we deduce

$$\begin{aligned}-(T_{h_{n,t}}\psi_{n,t},\psi_{n,t})_{\sim,h_{n,t}}+\|\psi_{n,t}\|^2_{\sim,h_{n,t}} &\geqq (1-\lambda_{1,h_{n,t}})\|\psi_{n,t}\|^2_{\sim,h_{n,t}}\\ &> \frac{1-\lambda_{1,h}}{2}\|\psi_{n,t}\|^2_{\sim,h_{n,t}} \geqq \frac{C_1(1-\lambda_{1,h})}{6}\|\psi_{n,t}\|^2_{H^1(\Gamma_{h_{n,t}})}\\ &\geqq \frac{C_1(1-\lambda_{1,h})}{12}\|\psi_n\|^2_{H^1(\Gamma_{g_n})}. \end{aligned} \tag{4.47}$$

Recall that (h,u) is a critical point, and thus there exists a constant Λ such that $Q(E(u))+k_h\equiv\Lambda$ on Γ_h. Using (4.41) it is then easy to see that

$$\sup_{t\in(0,1]}\|Q(E(u_{n,t}))+k_{h_{n,t}}-\Lambda\|_{L^\infty(\Gamma_{h_{n,t}})}\to 0. \tag{4.48}$$

Hence,

$$\begin{aligned}&\int_{\Gamma_{h_{n,t}}}(Q(E(u_{n,t}))+k_{h_{n,t}})\partial_{\tau_{h_{n,t}}}\left((h'_{n,t}\circ\pi_1)\psi^2_{n,t}\right)\mathrm{d}\mathcal{H}^1\\ &=\int_{\Gamma_{h_{n,t}}}(Q(E(u_{n,t}))+k_{h_{n,t}}-\Lambda)\partial_{\tau_{h_{n,t}}}\left((h'_{n,t}\circ\pi_1)\psi^2_{n,t}\right)\mathrm{d}\mathcal{H}^1\\ &\geqq -C\|Q(E(u_{n,t}))+k_{h_{n,t}}-\Lambda\|_{L^\infty(\Gamma_{h_{n,t}})}\|\psi_{n,t}\|^2_{H^1(\Gamma_{h_{n,t}})}\\ &\geqq -2C\|Q(E(u_{n,t}))+k_{h_{n,t}}-\Lambda\|_{L^\infty(\Gamma_{h_{n,t}})}\|\psi_n\|^2_{H^1(\Gamma_{g_n})}, \end{aligned} \tag{4.49}$$

where $C>0$ is independent of n and the last inequality follows from (4.46). Combining (4.44), (4.47), (4.49), and taking into account (4.48), we obtain (4.43). Hence, since $f'_n(0)=0$, we have

$$\begin{aligned}F(h,u)=f_n(0)&=f_n(1)-\int_0^1(1-t)f''_n(t)\,\mathrm{d}t\\ &< f_n(1)-\frac{1}{24}C_1(1-\lambda_{1,h})\|\psi_n\|^2_{H^1(\Gamma_{g_n})}\int_0^1(1-t)\,\mathrm{d}t\\ &=F(g_n,v_n)-\frac{1}{48}C_1(1-\lambda_{1,h})\|\psi_n\|^2_{H^1(\Gamma_{g_n})}, \end{aligned}$$

that is, (4.12) with $C_2:=\frac{1}{48}C_1(1-\lambda_{1,h})$.

Theorem 4.6. *Let $(h,u)\in X(e_0,q;0,b)$ satisfy the assumptions of Proposition 4.5. Let (g_n,v_n) be any sequence in $X(e_0,q;0,b)$ such that $\int_0^b g_n\,\mathrm{d}x=\int_0^b h\,\mathrm{d}x$, $g_n\neq h$, and $\|g_n-h\|_{W^{2,\infty}(0,b)}\to 0$. Then, $F(h,u)<F(g_n,v_n)$ for n large enough.*

Proof. For every n let $\tilde{g}_n:=h+\rho_{\varepsilon_n}*(g_n-h)$, where $\rho_{\varepsilon_n}(x)=\frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon_n}\right)$ and ρ is a standard mollifier and let $\tilde{v}_n$ be the associated elastic equilibrium. Notice that $\int_0^b\tilde{g}_n\,\mathrm{d}x=\int_0^b h\,\mathrm{d}x$, $\tilde{g}_n\in C^\infty_\#([0,b])$, $\|\tilde{g}_n-h\|_{C^2([0,b])}\leqq\|g_n-h\|_{W^{2,\infty}(0,b)}$ and that ε_n can be chosen so small that

$$F(\tilde{g}_n,\tilde{v}_n)\leqq F(g_n,v_n)+\frac{C_2}{2}\|\tilde{\psi}_n\|^2_{H^1(\Gamma_{\tilde{g}_n})}, \tag{4.50}$$

where C_2 is the constant appearing in (4.12) and $\tilde{\psi}_n$ is defined as in Proposition 4.5, with g_n replaced by $\tilde{g}_n$. Since the sequence $(\tilde{g}_n, \tilde{v}_n)$ satisfies the assumptions of Proposition 4.5, by (4.12) we have

$$F(h, u) + C_2\|\tilde{\psi}_n\|^2_{H^1(\Gamma_{\tilde{g}_n})} \leqq F(\tilde{g}_n, \tilde{v}_n)$$

and the conclusion follows from (4.50).

We now prove a variant of Proposition 4.5 where (h, u) is replaced by a sequence of flat configurations $(d_n/b, u_{e_n}) \in X(e_n; 0, b)$, with $d_n \to d \in (0, +\infty]$ and $e_n \to e_0 > 0$. To this aim, define

$$\widetilde{H}^1_0(\Gamma_h) := \Big\{\varphi \in H^1(\Gamma_h) : \varphi(0, h(0)) = \varphi(b, h(b)) = 0, \int_{\Gamma_h} \varphi \, \mathrm{d}\mathcal{H}^1 = 0\Big\}. \tag{4.51}$$

Proposition 4.7. *Let $(d_n/b, u_{e_n}) \in X(e_n; 0, b)$ such that $d_n \to d \in (0, +\infty]$ and $e_n \to e_0 > 0$. Assume that*

$$\partial^2 F(d_n/b, u_{e_n})[\varphi] \geqq C_1\|\varphi\|^2_{H^1(\Gamma_{d_n/b})} \tag{4.52}$$

for all $\varphi \in \widetilde{H}^1_0(\Gamma_{d_n/b})$ and with C_1 independent of n. Let (g_n, v_n) be any sequence in $X(e_n; 0, b)$ such that $g_n \in C^\infty_\#([0, b])$, $\int_0^b g_n \, \mathrm{d}x = d_n$, $g_n(0) = g_n(b) = d_n/b$ and $\|g_n - d_n/b\|_{C^2([0,b])} \to 0$, and let $\psi_n \in \widetilde{H}^1_0(\Gamma_{g_n})$ be defined as $\psi_n := \frac{g_n - d_n/b}{\sqrt{1+g_n'^2}} \circ \pi_1$. Then, there exists a constant $C_2 > 0$ depending only on C_1, such that

$$F(d_n/b, u_{e_n}) + C_2\|\psi_n\|^2_{H^1(\Gamma_{g_n})} \leqq F(g_n, v_n)$$

for n large enough.

Proof. We deal only with the case $d = \infty$, since the other one is similar and, in fact, easier. Indeed, many technical difficulties in the unbounded case arise from the fact that we need uniform estimates on domains which become larger and larger. Moreover, since the proof of the present proposition is very similar to that of Proposition 4.5, we shall indicate only the main changes needed.

We will use the same notation introduced in the proof of Proposition 4.5, unless otherwise stated. In the rest of the proof, for simplicity, we will write h_n and u_n in place of d_n/b and u_{e_n}. As before, we may assume that $v_n \in X(e_n; 0, b)$ is the elastic equilibrium in Ω_{g_n}. In addition to the bilinear forms $(\cdot, \cdot)_{\sim, g_n}$, we also consider the bilinear forms corresponding to the functions h_n denoted by $(\cdot, \cdot)_{\sim, h_n} : \widetilde{H}^1_0(\Gamma_{h_n}) \times \widetilde{H}^1_0(\Gamma_{h_n}) \to \mathbb{R}$. Finally, let $\Phi_n : \Omega_{h_n} \to \Omega_{g_n}$ be a diffeomorphism of class $C^2(\overline{\Omega}_{h_n}; \mathbb{R}^2)$ such that $\Phi_n - Id$ is b-periodic in x together with its first and second derivatives, $\Phi_n \equiv Id$ in $\Omega_{h_n - 3}$, $\|\Phi_n - Id\|_{C^2(\overline{\Omega}_{h_n}; \mathbb{R}^2)} \leqq 2\|g_n - h_n\|_{C^2([0,b])}$, and $\Phi_n(x, y) := (x, y + g_n - h_n)$ in $[0, b] \times [d_n/b - 2, d_n/b]$.

As before, we split the proof of the proposition into several steps.

Step 1 We show that

$$\sup_n \int_{\Omega_{g_n} \setminus \Omega_{g_n - 3}} Q(E(v_n))\, \mathrm{d}z < +\infty. \tag{4.53}$$

Note that by the minimality of u_n in $(0, b) \times (0, \inf g_n - 3)$ and denoting the constant strain $E(u_n)$ by E_n, we have

$$\int_0^b \mathrm{d}x \int_0^{\inf g_n - 3} Q(E(v_n))\, dy \geqq Q(E_n) b (\inf g_n - 3). \tag{4.54}$$

On the other hand, by the minimality of v_n in Ω_{g_n} we also have

$$\int_{\Omega_{g_n}} Q(E(v_n))\, \mathrm{d}z \leqq Q(E_n) |\Omega_{g_n}|.$$

Combining with (4.54) we easily obtain

$$\int_{\Omega_{g_n} \setminus \Omega_{g_n - 3}} Q(E(v_n))\, \mathrm{d}z \leqq Q(E_n)(3 + \operatorname{osc} g_n) b,$$

which, in turn, gives (4.53).

We claim that for all $p > 1$

$$\| E_n - E(v_n) \circ \Phi_n \|_{W^{1,p}(\Omega_{h_n} \setminus \overline{\Omega}_{h_n - 1}; \mathbb{R}^2)} \to 0. \tag{4.55}$$

To this aim, note that by (4.53) and Lemma 4.1 (which holds uniformly in n since the C^1-norms of the functions g_n' are equibounded) we infer that for all $p > 1$

$$\sup_n \| E(v_n) \|_{W^{1,p}(\Omega_{g_n} \setminus \overline{\Omega}_{g_n - 2})} < +\infty.$$

Since the functions g_n have first and second derivatives equibounded in $L^\infty(0, b)$, for all n we may extend v_n to $\Omega_{g_n + 1}$ in such a way that the resulting functions, still denoted by v_n, are b-periodic in the x-variable and satisfy the estimate

$$\sup_n \| E(v_n) \|_{W^{1,p}(\Omega_{g_n + 1} \setminus \overline{\Omega}_{g_n - 2})} < +\infty. \tag{4.56}$$

Since $u_n - v_n \in A(\Omega_{h_n}) \cap A(\Omega_{g_n})$ and recalling that u_n and v_n are elastic equilibria in Ω_{h_n} and in Ω_{g_n}, respectively, we have

$$\int_{\Omega_{h_n}} \mathbb{C} E(u_n) : E(u_n - v_n)\, \mathrm{d}z = 0 \tag{4.57}$$

and

$$\int_{\Omega_{h_n}} \mathbb{C} E(v_n) : E(u_n - v_n)\, \mathrm{d}z = \int_{\Omega_{h_n} \setminus \Omega_{g_n}} \mathbb{C} E(v_n) : E(u_n - v_n)\, \mathrm{d}z - \int_{\Omega_{g_n} \setminus \Omega_{h_n}} \mathbb{C} E(v_n) : E(u_n - v_n)\, \mathrm{d}z. \tag{4.58}$$

Subtracting (4.58) from (4.57), by (4.56) and recalling that $|\Omega_{h_n} \triangle \Omega_{g_n}| \to 0$ we deduce

$$\int_{\Omega_{h_n}} Q(E(u_n - v_n))\,\mathrm{d}z \to 0.$$

Since $\|\Phi_n - Id\|_{C^2(\overline{\Omega}_{h_n};\mathbb{R}^2)} \to 0$ and $\Phi_n = Id$ in Ω_{h_n-3}, we conclude

$$\int_{\Omega_{h_n}} |E_n - E(v_n)\circ\Phi_n|^2\,\mathrm{d}z \to 0. \tag{4.59}$$

Having established this estimate, we obtain (4.55), arguing exactly as in Step 1 of Proposition 4.5. More precisely, the same estimate used there in Ω_{g_n} holds now, in a local version, in the domain $\Omega_{g_n} \setminus \Omega_{g_n-1}$ thanks to (4.59) and to the fact that the functions h_n are all constants and the gradients of u_n are uniformly bounded constants. Again, the same argument used to prove (4.17) gives now

$$\|(\nabla(Q\circ E(v_n))\cdot \nu_{g_n})\circ\Phi_n J_1\Phi_n - \partial_{\nu_{h_n}}[Q(E(u_n))]\|_{H_\#^{-\frac{1}{2}}(\Gamma_{h_n})} \to 0.$$

Step 2 For n large enough

$$\|\varphi\|^2_{H^1(\Gamma_{g_n})} \leqq 3C_1^{-1}\|\varphi\|^2_{\sim,g_n} \qquad \text{for all } \varphi \in \widetilde{H}^1_0(\Gamma_{g_n}) \tag{4.60}$$

(C_1 is the constant appearing in (4.52)). The proof of this estimate goes exactly as the proof of estimate (4.19).

Step 3 The proof of this step is very similar to the proof of Step 3 of Proposition 4.5, apart from a subtle point at the end of the argument. For the reader's convenience we give the whole proof in detail. Let C_3 be a positive constant such that

$$\|\varphi\|^2_{\sim,h_n} \leqq C_3\|\varphi\|^2_{H^1(\Gamma_{h_n})} \qquad \text{for all } \varphi \in \widetilde{H}^1_0(\Gamma_{h_n}), \tag{4.61}$$

where C_3 is independent of n. As in Step 3 of the proof of Proposition 4.5, we introduce T_{h_n} and T_{g_n}, the operators associated with h_n and g_n, respectively. Define the first eigenvalues of T_{h_n} and T_{g_n} on $\widetilde{H}^1_0(\Gamma_{h_n})$ and $\widetilde{H}^1_0(\Gamma_{g_n})$, respectively, as

$$\lambda_{1,h_n} = \max\{(T_{h_n}\varphi,\varphi)_{\sim,h_n} : \varphi \in \widetilde{H}^1_0(\Gamma_{h_n}),\ \|\varphi\|_{\sim,h_n} = 1\},$$
$$\lambda_{1,g_n} = \max\{(T_{g_n}\varphi,\varphi)_{\sim,g_n} : \varphi \in \widetilde{H}^1_0(\Gamma_{g_n}),\ \|\varphi\|_{\sim,g_n} = 1\}.$$

By (4.52) and (4.61), taking as φ an eigenfunction corresponding to λ_{1,h_n}, we have

$$\begin{aligned}\partial^2 F(h_n,u_n)[\varphi] &= \|\varphi\|^2_{\sim,h_n} - (T_{h_n}\varphi,\varphi)_{\sim,h_n}\\ &= (1-\lambda_{1,h_n})\|\varphi\|^2_{\sim,h_n} \geqq C_1C_3^{-1}\|\varphi\|^2_{\sim,h_n},\end{aligned}$$

which implies

$$0 < C_1C_3^{-1} < 1 \qquad \text{and} \qquad \lambda_{1,h_n} \leqq 1 - C_1C_3^{-1}. \tag{4.62}$$

for n large enough. We claim that

$$\limsup_{n\to\infty} \lambda_{1,g_n} \leqq 1 - C_1 C_3^{-1}. \tag{4.63}$$

Without loss of generality we assume that

$$\limsup_{n\to\infty} \lambda_{1,g_n} = \lim_{n\to\infty} \lambda_{1,g_n} =: \lambda_\infty.$$

Then there exist $\varphi_n \in C^\infty_\#(\Gamma_{g_n}) \cap \widetilde{H}^1_0(\Gamma_{g_n})$, with $\|\varphi_n\|_{\sim,g_n} = 1$, and $v_{\varphi_n,g_n} \in A(\Omega_{g_n})$, solution to

$$\int_{\Omega_{g_n}} \mathbb{C}E(v_{\varphi_n,g_n}) : E(w)\,\mathrm{d}z = \int_{\Gamma_{g_n}} \mathrm{div}_{\tau_{g_n}}(\varphi_n \mathbb{C}E(v_n)) \cdot w\,\mathrm{d}\mathcal{H}^1 \tag{4.64}$$

for all $w \in A(\Omega_{g_n})$, such that

$$(T_{g_n}\varphi_n, \varphi_n)_{\sim,g_n} = 2\int_{\Omega_{g_n}} Q(E(v_{\varphi_n,g_n}))\,\mathrm{d}z \to \lambda_\infty. \tag{4.65}$$

Note that by (4.60)

$$\sup_n \|\varphi_n\|_{H^1(\Gamma_{g_n})} < +\infty.$$

By the Imbedding Theorem 8.3 and the L^∞-equiboundedness of the functions g_n' it follows that $\sup_n \|\varphi_n\|_{W^{\frac{3}{4},4}(\Gamma_{g_n})} < +\infty$. From (4.55) it follows that $\sup_n \|\mathbb{C}E(v_n)\|_{C^{0,\alpha}(\Gamma_{g_n};\mathbb{M}^{2\times 2})} < \infty$ for all $\alpha \in (0,1)$. Then, using the definition of the Gagliardo seminorm of $W^{\frac{3}{4},4}(\Gamma_{g_n})$, it is easy to check that

$$\sup_n \|\varphi_n \mathbb{C}E(v_n)\|_{W^{\frac{3}{4},4}(\Gamma_{g_n};\mathbb{M}^{2\times 2})} < +\infty. \tag{4.66}$$

Using Lemma 4.4 we then have that

$$\begin{aligned} &\|E(v_{\varphi_n,g_n})\|_{L^{2+\delta}(\Omega_{g_n}\setminus\Omega_{g_n-1};\mathbb{M}^{2\times 2})} \\ &\quad< C\Big(\|E(v_{\varphi_n,g_n})\|_{L^2(\Omega_{g_n}\setminus\Omega_{g_n-2};\mathbb{M}^{2\times 2})} + \|\varphi_n \mathbb{C}E(v_n)\|_{W^{\frac{3}{4},4}(\Gamma_{g_n})}\Big) \end{aligned} \tag{4.67}$$

for some $\delta > 0$ and $C > 0$ independent of n. Set $\hat{v}_{\varphi_n,g_n} = v_{\varphi_n,g_n} + z_n$, where z_n is a suitable infinitesimal rigid motion such that

$$\begin{aligned} &\int_{\Omega_{g_n}\setminus\Omega_{g_n-1}} \left(\nabla\hat{v}_{\varphi_n,g_n} - \nabla^T\hat{v}_{\varphi_n,g_n}\right)\mathrm{d}z = 0, \\ &\int_{\Omega_{g_n}\setminus\Omega_{g_n-1}} \hat{v}_{\varphi_n,g_n}\,\mathrm{d}z = 0. \end{aligned} \tag{4.68}$$

Choosing $w = v_{\varphi_n,g_n}$ in equation (4.64) and recalling that $\mathbb{C}E(v_n)[\nu_{g_n}] = 0$ and $\varphi_n(0, g_n(0)) = \varphi_n(b, g_n(b)) = 0$, also by Remark 3.1, we have

$$\begin{aligned}2\int_{\Omega_{g_n}} Q(E(v_{\varphi_n,g_n}))\,\mathrm{d}z &= \int_{\Gamma_{g_n}} \mathrm{div}_{\tau_{g_n}}(\varphi_n \mathbb{C}E(v_n)) \cdot v_{\varphi_n,g_n}\,\mathrm{d}\mathcal{H}^1 \\ &= \int_{\Gamma_{g_n}} \mathrm{div}_{\tau_{g_n}}(\varphi_n \mathbb{C}E(v_n)) \cdot \hat{v}_{\varphi_n,g_n}\,\mathrm{d}\mathcal{H}^1 \\ &\leqq \|\mathrm{div}_{\tau_{g_n}}(\varphi_n \mathbb{C}E(v_n))\|_{W_{\#}^{-\frac14,4}(\Gamma_{g_n};\mathbb{R}^2)} \|\hat{v}_{\varphi_n,g_n}\|_{W^{\frac14,\frac43}(\Gamma_{g_n};\mathbb{R}^2)} \\ &\leqq C\|\varphi_n \mathbb{C}E(v_n)\|_{W^{\frac34,4}(\Gamma_{g_n};\mathbb{R}^2)} \|\hat{v}_{\varphi_n,g_n}\|_{W^{1,\frac43}(\Omega_{g_n}\setminus\Omega_{g_n-1};\mathbb{R}^2)} \\ &\leqq C\|E(v_{\varphi_n,g_n})\|_{L^{\frac43}(\Omega_{g_n}\setminus\Omega_{g_n-1};\mathbb{R}^2)} \leqq C\|E(v_{\varphi_n,g_n})\|_{L^2(\Omega_{g_n};\mathbb{R}^2)}, \end{aligned} \tag{4.69}$$

where the third inequality follows from (4.66) and Korn's inequality with a constant independent of n. Note that Korn's inequality may be applied thanks to the normalization conditions imposed in (4.68). Note also that the second inequality follows from Corollary 8.7 and the trace Theorem 8.4. Moreover, the assumption that φ_n vanishes at the endpoints of Γ_{g_n} is crucial in order to get the second equality. Hence, from (4.67), we deduce that

$$\sup_n \|E(v_{\varphi_n,g_n})\|_{L^{2+\delta}(\Omega_{g_n}\setminus\Omega_{g_n-1};\mathbb{M}^{2\times2})} < \infty.$$

We may extend v_{φ_n,g_n} to a function in $A(\Omega_{g_n} \cup \Omega_{h_n})$ in such a way that

$$\sup_n \|E(v_{\varphi_n,g_n})\|_{L^{2+\delta}((\Omega_{g_n}\cup\Omega_h)\setminus\Omega_{g_n-1};\mathbb{M}^{2\times2})} < +\infty. \tag{4.70}$$

We finally set $\tilde{\varphi}_n := \ell_n(\varphi_n \circ \Phi_n) J_1\Phi_n$, where $\ell_n := \|(\varphi_n \circ \Phi_n) J_1\Phi_n\|^{-1}_{\sim,h_n}$. Arguing as in the proof of (4.33), it readily follows that

$$\ell_n \to 1.$$

Let $v_{\tilde{\varphi}_n,h_n}$ defined as in (3.18) with φ replaced by $\tilde{\varphi}_n$. Arguing as before, we may extend $v_{\tilde{\varphi}_n,h_n}$ to a function in $A(\Omega_{g_n} \cup \Omega_{h_n})$ in such a way that

$$\sup_n \|E(v_{\tilde{\varphi}_n,h_n})\|_{L^{2+\delta}((\Omega_{g_n}\cup\Omega_{h_n})\setminus\Omega_{g_n-1};\mathbb{M}^{2\times2})} < +\infty. \tag{4.71}$$

To conclude the proof of (4.63) it will be enough to show that

$$\lim_{n\to\infty} \int_{\Omega_{h_n}} Q(E(v_{\tilde{\varphi}_n,h_n} - v_{\varphi_n,g_n}))\,\mathrm{d}z = 0. \tag{4.72}$$

Indeed, by (4.62) and (4.65) this would imply

$$\begin{aligned}1 - C_1C_3^{-1} \geqq \lim_{n\to\infty} (T_{h_n}\tilde{\varphi}_n, \tilde{\varphi}_n)_{\sim,h_n} &= \lim_{n\to\infty} 2\int_{\Omega_{h_n}} Q(E(v_{\tilde{\varphi}_n,h_h}))\,\mathrm{d}z \\ &= \lim_{n\to\infty} 2\int_{\Omega_{h_n}} Q(E(v_{\varphi_n,g_n}))\,\mathrm{d}z \\ &= \lim_{n\to\infty} 2\int_{\Omega_{g_n}} Q(E(v_{\varphi_n,g_n}))\,\mathrm{d}z = \lambda_\infty\end{aligned}$$

where in the third equality we have used the equi-integrability of the functions $Q(E(v_{\varphi_n,g_n}))$ (implied by (4.70)) together with the fact that $|\Omega_{g_n}\triangle\Omega_{h_h}|\to 0$. In order to prove (4.72) we observe that $v_{\tilde\varphi_n,h_n}-v_{\varphi_n,g_n}\in A(\Omega_{h_n})\cap A(\Omega_{g_n})$ is an admissible test function for (4.64) and for the equation satisfied by $v_{\tilde\varphi_n,h_n}$. Using such a test function and subtracting the two equations, we obtain

$$
\begin{aligned}
&2\int_{\Omega_{h_n}} Q(E(v_{\tilde\varphi_n,h_n}-v_{\varphi_n,g_n}))\,\mathrm{d}z\\
&=-\int_{\Omega_{h_n}\setminus\Omega_{g_n}} \mathbb{C}E(v_{\varphi_n,g_n}):E(v_{\tilde\varphi_n,h_n}-v_{\varphi_n,g_n})\,\mathrm{d}z\\
&\quad+\int_{\Omega_{g_n}\setminus\Omega_{h_n}} \mathbb{C}E(v_{\varphi_n,g_n}):E(v_{\tilde\varphi_n,h_n}-v_{\varphi_n,g_n})\,\mathrm{d}z\\
&\quad+\int_{\Gamma_{h_n}} \mathrm{div}_{\tau_{h_n}}(\tilde\varphi_n\mathbb{C}E_n)\cdot(\hat v_{\tilde\varphi_n,h_n}-\hat v_{\varphi_n,g_n})\,\mathrm{d}\mathcal{H}^1\\
&\quad-\int_{\Gamma_{h_n}} \left[\mathrm{div}_{\tau_{g_n}}(\varphi_n\mathbb{C}E(v_n))\cdot(\hat v_{\tilde\varphi_n,h_n}-\hat v_{\varphi_n,g_n})\right]\circ\Phi_n J_1\Phi_n\,\mathrm{d}\mathcal{H}^1\\
&=: I_n^1+I_n^2+I_n^3+I_n^4,
\end{aligned}
$$

where $\hat v_{\tilde\varphi_n,h_n}$ are obtained by adding to $v_{\tilde\varphi_n,h_n}$ suitable infinitesimal rigid motions in such a way that $\sup_n\|\hat v_{\tilde\varphi_n,h_n}\|_{H^1((\Omega_{h_n}\cup\Omega_{g_n})\setminus\Omega_{g_n-1};\mathbb{R}^2)}<\infty$ and $\hat v_{\varphi_n,g_n}$ are defined as above. In particular, we also have

$$
\sup_n\|\hat v_{\varphi_n,g_n}\|_{H^1((\Omega_{h_n}\cup\Omega_{g_n})\setminus\Omega_{g_n-1};\mathbb{R}^2)}<\infty.
$$

Note that we have used, as in (4.69), the invariance of I_n^3 and I_n^4 under addition of infinitesimal rigid motions to the functions $v_{\tilde\varphi_n,h_n}$ and v_{φ_n,g_n}.

As in Step 3 of Proposition 4.5, $I_n^1+I_n^2\to 0$ due to (4.70), (4.71), and the fact that $|\Omega_{g_n}\triangle\Omega_{h_n}|\to 0$. The proof that $I_n^3+I_n^4\to 0$ can be obtained exactly as in the final part of Step 3 of Proposition 4.5, replacing $v_{\tilde\varphi_n,h}-v_{\varphi_n,g_n}$ by $\hat v_{\tilde\varphi_n,h_n}-\hat v_{\varphi_n,g_n}$.

Step 4 For $t\in[0,1]$ consider $(h_{n,t},u_{n,t})\in X(e_n;0,b)$, where $h_{n,t}:=h_n+t(g_n-h_n)$ and $u_{n,t}$ is the corresponding elastic equilibrium. Note that $(h_{n,1},u_{n,1})=(g_n,v_n)$. Let $(\cdot,\cdot)_{\sim,h_{n,t}}$, $T_{h_{n,t}}$, and $\lambda_{1,h_{n,t}}$ be the corresponding bilinear forms, operators, and first eigenvalues on $\widetilde H^1_0(\Gamma_{h_{n,t}})$. Set $\Phi_{n,t}(x,y):=(x,y+h_{n,t}-h_n)$ in $[0,b]\times[d_n/b-2,d_n/b]$. We claim that for n large enough and for all $t\in[0,1]$

$$
\|\varphi\|^2_{H^1(\Gamma_{h_{n,t}})}\leqq 3C_1^{-1}\|\varphi\|^2_{\sim,h_{n,t}}\qquad\text{for all }\varphi\in\widetilde H^1_\#(\Gamma_{h_{n,t}}),\tag{4.73}
$$

that

$$
\limsup_{n\to\infty}\sup_{t\in(0,1]}\lambda_{1,h_{n,t}}\leqq 1-C_1C_3^{-1},\tag{4.74}
$$

and that for all $p>1$

$$
\sup_{t\in[0,1]}\|E_n-E(u_{n,t})\circ\Phi_{n,t}\|_{W^{1,p}(\Omega_{h_n}\setminus\overline{\Omega}_{h_n-1};\mathbb{M}^{2\times 2})}.\tag{4.75}
$$

The claim can be proved by contradiction as in Step 4 of Proposition 4.5.

Step 5 We are now in a position to conclude the proof of the proposition. Let $f_n(t) := F(h_{n,t}, u_{n,t})$. We claim that

$$f_n''(t) > \frac{1}{24} C_1^2 C_3^{-1} \|\psi_n\|^2_{H^1(\Gamma_{g_n})}.$$

This can be proved using the estimates (4.73), (4.74) and (4.75) and arguing exactly as in Step 5 of Proposition 4.5. We only mention the fact that, since the curvatures k_{h_n} are all zero, equation (4.48) still holds with Λ replaced by $Q(E_n)$, which is bounded.

Remark 4.8. We remark that the conclusion of Proposition 4.7 also holds if the sequence $\{g_n\}$ does not satisfy the condition $g_n(0) = g_n(b) = d_n/b$. Indeed, if the sequence (g_n, v_n) satisfies all the remaining assumptions stated in Proposition 4.7, since $\int_0^b g_n \, dx = d_n$ for all n, there exists $a_n \in [0, b)$ such that $g_n(a_n) = d_n/b$. Consider the functions $\bar{g}_n(x) = g_n(x + a_n)$ and $\bar{v}_n(x, y) = v_n(x + a_n, y) - (e_n a_n, 0)$, and observe that $(\bar{g}_n, \bar{v}_n) \in X(e_n; 0, b)$, $\bar{g}_n(0) = \bar{g}_n(b) = d_n/b$, $F(g_n, v_n) = F(\bar{g}_n, \bar{v}_n)$ by periodicity, and $\|\bar{g}_n - d_n/b\|_{C^2([0,b])} = \|g_n - d_n/b\|_{C^2([0,b])} \to 0$.

From the previous remark and Proposition 4.7, arguing as in Theorem 4.6, we have the following result.

Theorem 4.9. *Let $(d_n/b, u_{e_n}) \in X(e_n; 0, b)$ be as in Proposition 4.7, and let $(g_n, v_n) \in X(e_n; 0, b)$ be such that $\int_0^b g_n \, dx = d_n$, $g_n \neq d_n/b$, and $\|g_n - d_n/b\|_{W^{2,\infty}(0,b)} \to 0$. Then $F(d_n/b, u_{e_n}) < F(g_n, v_n)$ for n large enough.*

5. Second Variation of the Flat Configuration

Using Theorem 3.2, we can now calculate the second variation of the flat configuration $(d/b, u_{e_0})$ evaluated at an admissible variation φ. In view of Remark 4.8 it is enough to consider variations φ belonging to the space $\widetilde{H}^1_0(\Gamma_{d/b})$ defined in (4.51). Clearly, such a space can be identified with

$$\widetilde{H}^1_0(0, b) = \left\{ \varphi \in H^1(0, b) : \varphi(0) = \varphi(b) = 0, \int_0^b \varphi \, dx = 0 \right\}.$$

Notice that for every admissible displacement u, by (2.6), we have

$$\mathbb{C}E(u) = \begin{pmatrix} (2\mu + \lambda)\frac{\partial u_1}{\partial x} + \lambda \frac{\partial u_2}{\partial y} & \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & (2\mu + \lambda)\frac{\partial u_2}{\partial y} + \lambda \frac{\partial u_1}{\partial x} \end{pmatrix}. \tag{5.1}$$

In particular, recalling (2.13),

$$\mathbb{C}E(u_{e_0}) = \begin{pmatrix} \tau & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad \tau := e_0 \frac{4\mu(\mu + \lambda)}{2\mu + \lambda}. \tag{5.2}$$

Observing that $\mathrm{div}_\tau(\mathbb{C}E(u_{e_0})\varphi) = \mathrm{div}(\mathbb{C}E(u_{e_0})\varphi) = \tau(\varphi', 0)$, we get

$$\partial^2 F\Big(\frac{d}{b}, u_{e_0}\Big)[\varphi] = -2\int_{\Omega_h} Q(E(v_\varphi))\,\mathrm{d}z + \int_0^b \varphi'^2\,\mathrm{d}x, \tag{5.3}$$

where $h \equiv d/b$, $\Omega_h = (0,b)\times(0,d/b)$ and $v_\varphi \in A(\Omega_h)$ is the unique solution to the equation

$$\int_{\Omega_h} \mathbb{C}E(v_\varphi) : E(w)\,\mathrm{d}z = \tau \int_0^b w_1(x, d/b)\varphi'(x)\,\mathrm{d}x \tag{5.4}$$

for all $w \in A(\Omega_h)$.

The next theorem deals with the positive definiteness of $\partial^2 F(d/b, u_{e_0})$. We follow the same argument used in [19] to express the second variation in terms of the Fourier coefficients of φ.

Theorem 5.1. *Let K be the function defined in* (2.14)*. Then,*

$$\partial^2 F\Big(\frac{d}{b}, u_{e_0}\Big) \text{ is positive definite } \iff K\Big(\frac{2\pi d}{b^2}\Big) < \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}\frac{1}{b}.$$

Conversely,

$$K\Big(\frac{2\pi d}{b^2}\Big) > \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}\frac{1}{b} \Rightarrow \partial^2 F\Big(\frac{d}{b}, u_{e_0}\Big)[\varphi] < 0 \text{ for some } \varphi \in \widetilde{H}^1_0(0,b).$$

Proof. We set

$$\tilde{v}_\varphi(x,y) := \frac{2\pi}{b} v_\varphi\Big(\frac{b}{2\pi}x, \frac{b}{2\pi}y\Big), \quad \tilde{\varphi}(x) := \frac{2\pi}{b}\varphi\Big(\frac{b}{2\pi}x\Big).$$

Then, $\tilde{v}_\varphi$ satisfies (5.4) in the interval $(0, 2\pi)$, with φ replaced by $\tilde{\varphi}$. Moreover, by (5.3), we get

$$\partial^2 F\Big(\frac{d}{b}, u_{e_0}\Big)[\varphi] = \frac{b^2}{4\pi^2}\Big[-2\int_{\Omega_{\tilde{h}}} Q(E(\tilde{v}_\varphi))\,\mathrm{d}z + \frac{2\pi}{b}\int_0^{2\pi} \tilde{\varphi}'^2\,\mathrm{d}x\Big], \tag{5.5}$$

where $\tilde{h}(x) \equiv 2\pi d/b^2$.

In order to compute the second variation, let us now solve equation (5.4) in $\Omega_{\tilde{h}} = (0, 2\pi)\times(0,\tilde{h})$ by considering the expansion in Fourier series of $\tilde{v}_\varphi(\cdot, y)$ for all $y \in (0,\tilde{h})$. To this aim, we set for all y and $n \in \mathbb{Z}$

$$a_n(y) := \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \mathrm{e}^{-inx}\,\tilde{v}_{\varphi\,1}(x,y)\,\mathrm{d}x, \quad b_n(y) := \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \mathrm{e}^{-inx}\,\tilde{v}_{\varphi\,2}(x,y)\,\mathrm{d}x,$$

and

$$\tilde{\varphi}_n := \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \mathrm{e}^{-inx}\tilde{\varphi}(x)\,\mathrm{d}x,$$

where $\tilde{v}_{\varphi\,1}, \tilde{v}_{\varphi\,2}$ are the components of $\tilde{v}_\varphi$. Recalling (5.1) and (5.2), equation (5.4) becomes

$$\begin{cases} (2\mu+\lambda)\dfrac{\partial^2 \tilde{v}_{\varphi\,1}}{\partial x^2} + \mu\dfrac{\partial^2 \tilde{v}_{\varphi\,1}}{\partial y^2} + (\lambda+\mu)\dfrac{\partial^2 \tilde{v}_{\varphi\,2}}{\partial x \partial y} = 0 & \text{in } \Omega_{\tilde{h}}, \\ \mu\dfrac{\partial^2 \tilde{v}_{\varphi\,2}}{\partial x^2} + (2\mu+\lambda)\dfrac{\partial^2 \tilde{v}_{\varphi\,2}}{\partial y^2} + (\lambda+\mu)\dfrac{\partial^2 \tilde{v}_{\varphi\,1}}{\partial x \partial y} = 0 & \text{in } \Omega_{\tilde{h}}, \\ \dfrac{\partial \tilde{v}_{\varphi\,1}}{\partial y} + \dfrac{\partial \tilde{v}_{\varphi\,2}}{\partial x} = \dfrac{\tau}{\mu}\tilde{\varphi}', \quad \lambda\dfrac{\partial \tilde{v}_{\varphi\,1}}{\partial x} + (2\mu+\lambda)\dfrac{\partial \tilde{v}_{\varphi\,2}}{\partial y} = 0 & \text{on } \{y = \tilde{h}\}, \\ \tilde{v}_\varphi = 0 & \text{on } \{y = 0\}. \end{cases}$$

Therefore, the Fourier coefficients a_n, b_n satisfy the following system of ODEs

$$\begin{cases} \mu a_n'' - in(\lambda+\mu)b_n' - (2\mu+\lambda)n^2 a_n = 0 & \text{in } (0,\tilde{h}), \\ (2\mu+\lambda)b_n'' - in(\lambda+\mu)a_n' - \mu n^2 b_n = 0 & \text{in } (0,\tilde{h}), \\ a_n'(\tilde{h}) - inb_n(\tilde{h}) = -\dfrac{\tau}{\mu} in\tilde{\varphi}_n, \\ -in\lambda a_n(\tilde{h}) + (2\mu+\lambda)b_n'(\tilde{h}) = 0, \\ a_n(0) = b_n(0) = 0. \end{cases} \tag{5.6}$$

From these equations it follows that there exist constants $C_i,\ K_i,\ i = 1, \ldots, 4$, such that

$$a_n(y) = C_1 e^{ny} + C_2 e^{-ny} + C_3 y e^{ny} + C_4 y e^{-ny},$$
$$b_n(y) = K_1 e^{ny} + K_2 e^{-ny} + K_3 y e^{ny} + K_4 y e^{-ny}.$$

Inserting these expressions in (5.6), after some lengthy but straightforward computations, we obtain that

$$C_1 = -iK_1 - i\tfrac{K_3}{n}(3-4\nu_p), \quad C_2 = iK_2 - i\tfrac{K_4}{n}(3-4\nu_p),$$
$$C_3 = -iK_3, \quad \text{and} \quad C_4 = iK_4,$$

where ν_p is defined as in (2.15). Setting $\gamma_1 := K_1 + K_2$, $\gamma_2 := K_1 - K_2$, $\gamma_3 := K_3 + K_4$, $\gamma_4 := K_3 - K_4$, we may write

$$\begin{aligned} a_n(y) = &-i\gamma_1 \sinh(ny) - i\gamma_2 \cosh(ny) - i\gamma_3\Big[\frac{3-4\nu_p}{n}\cosh(ny) + y\sinh(ny)\Big] \\ &-i\gamma_4\Big[\frac{3-4\nu_p}{n}\sinh(ny) + y\cosh(ny)\Big] \\ b_n(y) = &\ \gamma_1\cosh(ny) + \gamma_2\sinh(ny) + \gamma_3 y\cosh(ny) + \gamma_4 y\sinh(ny). \end{aligned}$$

Imposing the condition $a_n(0) = b_n(0) = 0$, we get

$$\gamma_1 = 0, \quad \gamma_2 = -\frac{3-4\nu_p}{n}\gamma_3. \tag{5.7}$$

Enforcing $a_n'(\tilde{h}) - inb_n(\tilde{h}) = -\frac{\tau}{\mu} in\tilde{\varphi}_n$, we deduce

$$\gamma_3\Big[2\tilde{h}\cosh(n\tilde{h}) - \frac{2(1-2\nu_p)}{n}\sinh(n\tilde{h})\Big] + \gamma_4\Big[2\tilde{h}\sinh(n\tilde{h}) + \frac{4(1-\nu_p)}{n}\cosh(n\tilde{h})\Big] = \frac{\tau}{\mu}\tilde{\varphi}_n.$$

Finally, the condition $-in\lambda a_n(\tilde{h}) + (2\mu+\lambda)b_n'(\tilde{h}) = 0$ implies

$$\gamma_3\Big[-\frac{2(\lambda+2\mu)(1-2\nu_p)}{n}\cosh(n\tilde{h}) + 2\mu\tilde{h}\sinh(n\tilde{h})\Big] + \gamma_4\Big[\frac{2\mu^2}{n(\lambda+\mu)}\sinh(n\tilde{h}) + 2\mu\tilde{h}\cosh(n\tilde{h})\Big] = 0.$$

Solving the last two equations in γ_3, γ_4, we deduce

$$\gamma_3 = \frac{\tau}{2\mu} n\tilde{\varphi}_n \frac{n\tilde{h}\cosh(n\tilde{h}) + (1-2\nu_p)\sinh(n\tilde{h})}{n^2\tilde{h}^2 + 4(1-\nu_p)^2 + (3-4\nu_p)\sinh^2(n\tilde{h})},$$
$$\gamma_4 = \frac{\tau}{2\mu} n\tilde{\varphi}_n \frac{-n\tilde{h}\sinh(n\tilde{h}) + 2(1-\nu_p)\cosh(n\tilde{h})}{n^2\tilde{h}^2 + 4(1-\nu_p)^2 + (3-4\nu_p)\sinh^2(n\tilde{h})}.$$

From these equalities, (5.5), (5.7), and (5.4) we finally obtain

$$\begin{aligned}
\partial^2 F\Big(\frac{d}{b}, u_{e_0}\Big)[\varphi] &= \frac{b^2}{4\pi^2}\Big[-2\int_{\Omega_{\tilde{h}}} Q(E(\tilde{v}_\varphi))\,\mathrm{d}z + \frac{2\pi}{b}\int_0^{2\pi}\tilde{\varphi}'^2\,\mathrm{d}x\Big] \\
&= \frac{b^2}{4\pi^2}\int_0^{2\pi}\Big[-\tau\tilde{v}_{\varphi 1}(x,\tilde{h})\tilde{\varphi}'(x) + \frac{2\pi}{b}\tilde{\varphi}'^2\Big]\mathrm{d}x \\
&= \frac{b^2}{4\pi^2}\sum_{n\in\mathbb{Z}}\Big[-\tau ina_n(\tilde{h})\tilde{\varphi}_{-n} + \frac{2\pi}{b}n^2\tilde{\varphi}_n\tilde{\varphi}_{-n}\Big] \\
&= \frac{b}{2\pi}\sum_{n\in\mathbb{Z}} n^2\tilde{\varphi}_n\tilde{\varphi}_{-n}\Big[1 - \frac{\tau^2(1-\nu_p)bJ(n\tilde{h})}{2\pi\mu n}\Big]. \\
&= \sum_{n\in\mathbb{Z}} n^2\varphi_n\varphi_{-n}\Big[1 - \frac{\tau^2(1-\nu_p)bJ(2\pi nd/b^2)}{2\pi\mu n}\Big],
\end{aligned} \tag{5.8}$$

where the φ_n's are the Fourier coefficients of φ in the interval $(0, b)$ and J is the function introduced in (2.14). Also using (2.15) and (5.2) we have

$$\sup_{n\in\mathbb{Z}} \frac{\tau^2(1-\nu_p)bJ(2\pi nd/b^2)}{2\pi\mu n} \gtrless 1 \iff K\Big(\frac{2\pi d}{b^2}\Big) \gtrless \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}\frac{1}{b},$$

and the conclusion follows from (5.8).

Remark 5.2. From (5.8) it follows that

$$\partial^2 F\Big(\frac{d}{b}, u_{e_0}\Big)[\varphi] \geqq C_1 \|\varphi'\|^2_{L^2(0,b)},$$

with

$$C_1 := \Big[1 - \frac{4e_0^2\mu(\mu+\lambda)b}{\pi(2\mu+\lambda)} K\Big(\frac{2\pi d}{b^2}\Big)\Big].$$

Corollary 5.3. *The function K defined in (2.14) is strictly increasing and continuous, $K(y) \leqq Cy$ for some positive C, and $\lim\limits_{y\to+\infty} K(y) = 1$.*

Proof. The fact that K is strictly increasing is an easy consequence of Theorem 5.1, together with Proposition 3.6 and Corollary 3.8. The fact that $\lim_{y\to+\infty} K(y) = 1$ follows from the same property for J after observing that $K(y) = J(y)$ for y large. Also, the growth condition follows from the same property for J. Finally, the continuity of K is a consequence of the fact that if y varies in a set bounded from above and away from zero, then the maximum defining K can be restricted to a finite subset of $\mathbb{N}$.

6. Local Minimizers: Proofs

This section is mainly devoted to proving that $W^{2,\infty}$-local minimizers are, in fact, local minimizers in the sense of Definition 2.4. This fact, together with the results of Sections 4 and 5, will lead to the proof of Theorem 2.9. For the general strategy of the proof we refer to the introduction. We start with some technical lemmas. The first one is an approximation lemma proved in [6].

Lemma 6.1. *Given $(g, v) \in X(e_0, q; 0, b)$, there exists a sequence of Lipschitz b-periodic functions g_n, such that $g_n \uparrow g$ pointwise and $F(g_n, v) \to F(g, v)$. In particular,*

$$\mathcal{H}^1(\Gamma_{g_n}) \to \mathcal{H}^1(\Gamma_g) + 2\mathcal{H}^1(\Sigma_g).$$

The next lemma states a well-known approximation property for one-dimensional BV-functions.

Lemma 6.2. *Let $h : [0, b] \to \mathbb{R}$ be a lower semicontinuous function with finite total variation. Then, there exists a sequence of Lipschitz functions $g_n : [0, b] \to \mathbb{R}$ such that $g_n(0) = h(0)$, $g_n(b) = h(b)$, $g_n \to h$ in $L^1(0, b)$ such that*

$$\mathcal{H}^1(\Gamma_{g_n}) \to \mathcal{H}^1(\Gamma_h \cap ((0, b) \times \mathbb{R})) + |h(0+) - h(0)| + |h(b-) - h(b)|.$$

The next lemma and the subsequent corollary show that minimizing under the volume constraint is equivalent to minimizing with a sufficiently large penalization term.

Lemma 6.3. *Let $h_0 \in L^\infty(0,b)$ be nonnegative, let d, $e_0 > 0$, and let $(g,v) \in X(e_0,q;0,b)$ be a minimizer for the problem*

$$\min\Big\{F(k,w) + \Lambda\big||\Omega_k| - d\big| : (k,w) \in X(e_0,q;0,b),\ k \geqq h_0 \text{ in } [0,b]\Big\},$$

with $\Lambda > Q_0$, where

$$Q_0 := \frac{1}{b}\int_0^b Q(E_0 + E((q,0)))\,\mathrm{d}x, \quad \textit{and} \quad E_0 := E(u_{e_0}).$$

Then $|\Omega_g| \geqq d$. Moreover, if $h_0 \equiv c$, with c a nonnegative constant such that $cb \leqq d$, then $|\Omega_g| = d$.

Proof. We argue by contradiction. If $|\Omega_g| < d$, we set $\tilde{g} = g + (d - |\Omega_g|)/b$ and for all $(x,y) \in \Omega_{\tilde{g}}$

$$\tilde{v}(x,y) = \begin{cases} \Big(e_0 x, \dfrac{-\lambda e_0}{2\mu+\lambda} y\Big) + (q(x),0) & \text{if } 0 < y < \dfrac{d - |\Omega_g|}{b} \\ v\Big(x, y - \dfrac{d-|\Omega_g|}{b}\Big) + \Big(0, \dfrac{-\lambda e_0 (d - |\Omega_g|)}{b(2\mu+\lambda)}\Big) & \text{if } y \geqq \dfrac{d-|\Omega_g|}{b}. \end{cases}$$

Then

$$\begin{aligned} F(\tilde{g},\tilde{v}) + \Lambda\big||\Omega_{\tilde{g}}| - d\big| - F(g,v) - \Lambda\big||\Omega_g| - d\big| \\ = Q_0(d - |\Omega_g|) - \Lambda(d - |\Omega_g|) < 0, \end{aligned}$$

which is a contradiction to the minimality of (g,v).

Assume now that $h_0 \equiv c$. If $|\Omega_g| > d$, then we may truncate g in such a way that the resulting function $\tilde{g}$ satisfies the constraints $|\Omega_{\tilde{g}}| = d$ and $\tilde{g} \geqq h_0$. Then, we would get

$$F(\tilde{g},v) + \Lambda\big||\Omega_{\tilde{g}}| - d\big| < F(g,v) + \Lambda\big||\Omega_g| - d\big|,$$

which is again a contradiction to the minimality of (g,v).

An immediate consequence of the previous lemma is stated in the following corollary.

Corollary 6.4. *Let $(h,u) \in X(e_0,q;0,b)$ be a b-periodic global minimizer for the problem (2.9). Then, for all $\Lambda > Q_0$ we have that (h,u) is a minimizer for the problem*

$$\min\Big\{F(k,w) + \Lambda\big||\Omega_k| - d\big| : (k,w) \in X(e_0,q;0,b)\Big\}.$$

Proof. Apply Lemma 6.3 with $h_0 \equiv 0$.

The next lemma will be used to prove the isoperimetric inequality stated in Lemma 6.6 and the unilateral minimality property (1.4) stated in the introduction.

Lemma 6.5. *Let h be a function in $C^2_\#([0,b])$. Then,*

$$\mathcal{H}^1(\Gamma_k) + \Lambda_0 \int_0^b |k-h|\,\mathrm{d}x \geqq \mathcal{H}^1(\Gamma_h)$$

for all $k \in AP(0,b)$, where

$$\Lambda_0 := \Big\| \Big(\frac{h'}{\sqrt{1+h'^2}}\Big)' \Big\|_{L^\infty(0,b)}.$$

Proof. Assume first that $k \in \mathrm{Lip}([0,b]),\ k(0)=k(b)$. Then,

$$\begin{aligned}
\mathcal{H}^1(\Gamma_k) - \mathcal{H}^1(\Gamma_h) &= \int_0^b \big(\sqrt{1+k'^2} - \sqrt{1+h'^2}\big)\,\mathrm{d}x \\
&\geqq \int_0^b \frac{(k'-h')h'}{\sqrt{1+h'^2}}\,\mathrm{d}x \\
&= -\int_0^b |k-h| \Big(\frac{h'}{\sqrt{1+h'^2}}\Big)' \mathrm{sign}(k-h)\,\mathrm{d}x \qquad (6.1)\\
&\geqq -\Lambda_0 \int_0^b |k-h|\,\mathrm{d}x.
\end{aligned}$$

Assume now that $k \in AP(0,b)$. If $\Sigma_k = \emptyset$, then the result follows from the approximation Lemma 6.1. If $\Sigma_k \neq \emptyset$, one reduces to the previous case by replacing k by the function $k^- \in AP(0,b)$ defined in (2.1) for which $\Gamma_k = \Gamma_{k^-}$ and $\Sigma_{k^-} = \emptyset$.

The isoperimetric inequality proved in the following lemma is crucial to deduce the uniform inner ball condition stated in Lemma 6.7.

Lemma 6.6. *Let $k \in AP(0,b)$ be nonnegative, let $B_\rho(z_0)$ be a ball such that $B_\rho(z_0) \subset \{(x,y) : x \in (0,b) \text{ and } y < k(x)\}$, and let $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$ be points in $\partial B_\rho(z_0) \cap (\Gamma_k \cup \Sigma_k)$. Let γ be the shortest arc on $\partial B_\rho(z_0)$ connecting z_1 and z_2 (any of the two possible arcs if z_1 and z_2 are antipodal) and let γ' be the arc on $\Gamma_k \cup \Sigma_k$ connecting z_1 and z_2. Then*

$$\mathcal{H}^1(\gamma') - \mathcal{H}^1(\gamma) \geqq \frac{1}{\rho}|D|,$$

where D is the region enclosed by $\gamma \cup \gamma'$.

Proof. Denote by h the function whose graph coincides with γ. We observe that if $k \in \mathrm{Lip}([x_1,x_2])$, since $k(x_1) = h(x_1)$ and $k(x_2) = h(x_2)$, the same proof of (6.1), together with the fact that $-\Big(\frac{h'}{\sqrt{1+h'^2}}\Big)' = \frac{1}{\rho}$, yields

$$\mathcal{H}^1(\Gamma_k \cap [(x_1,x_2)\times\mathbb{R}]) - \mathcal{H}^1(\Gamma_h \cap [(x_1,x_2)\times\mathbb{R}]) \geqq \frac{1}{\rho}\int_{x_1}^{x_2} |k-h|\,\mathrm{d}x,$$

which is the conclusion when $k \in \mathrm{Lip}([x_1,x_2])$. The general case follows by approximating k in $[x_1,x_2]$ with a sequence k_n of Lipschitz functions according to Lemma 6.2 and by passing to the limit in the above formula.

Lemma 6.7. *Let h_0 be a nonnegative function in $C^2_\#([0,b])$, let $\Lambda > 0$, $d > 0$, and let $(g,v) \in Y(e_0;0,b)$ be a minimizer of the problem*

$$\min\Big\{F(k,w) + \Lambda\big||\Omega_k| - d\big| : (k,w) \in Y(e_0;0,b),$$
$$w(x,0) = v(x,0) \text{ for all } x \in (0,b)\ k \geqq h_0\Big\}.$$

Then $\rho < \min\{1/\Lambda, 1/\|h_0''\|_\infty\}$ implies that for all $z \in \Gamma_g \cup \Sigma_g$ there exists a ball $B_\rho(z_0) \subset \Omega_g^\# \cup \big(\mathbb{R} \times (-\infty,0]\big)$ such that $\partial B_\rho(z_0) \cap (\Gamma_g \cup \Sigma_g) = \{z\}$.

Proof. We start by observing that if $\rho < 1/\|h_0''\|_\infty$, $z_0 = (x_0,y_0)$ is the center of a ball of radius ρ and $S_\rho^+(z_0) := \partial B_\rho(z_0) \cap \{y \geqq y_0\}$, then an elementary argument shows that there cannot exist two points $z_1 = (x_1,y_1)$, $z_2 = (x_2,y_2) \in S_\rho^+(z_0) \cap \Gamma_{h_0}$ such that Γ_{h_0} lies above $S_\rho^+(z_0)$ in $(x_1,x_2)\times\mathbb{R}$. Now fix $\rho < 1/\|h_0''\|_\infty$ and assume that there exists a ball $B_\rho(z_0) \subset \Omega_g^\# \cup \big(\mathbb{R}\times(-\infty,0]\big)$ such that $S_\rho^+(z_0)$ intersects $\Gamma_g \cup \Sigma_g$ in two points $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$. From the above observations it follows that the arc γ on $S_\rho^+(z_0)$ connecting z_1 and z_2 lies above the graph of h_0. Hence we may modify g by replacing it with the function $\tilde{g}$, which coincides with g in $[0,b) \setminus (x_1,x_2)$ and whose graph on (x_1,x_2) is given by γ. Denote by γ' the arc on $\Gamma_g \cup \Sigma_g$ connecting z_1 and z_2, and by D the region enclosed by $\gamma' \cup \gamma$. Then we have

$$F(\tilde{g},v)+\Lambda\big||\Omega_{\tilde{g}}|-d\big|-F(g,v)-\Lambda\big||\Omega_g|-d\big| \leqq \mathcal{H}^1(\gamma)-\mathcal{H}^1(\gamma') + \Lambda|D| < 0,$$

where the last inequality is a consequence of Lemma 6.6 and the fact that $\rho < 1/\Lambda$. From this contradiction to the minimality of (g,v) the conclusion then follows arguing as in [9, Lemma 2] or [15, Proposition 3.3, Step 2].

Remark 6.8. Let $(g,v) \in X(e_0,q;0,b)$ be a b-periodic global minimizer and fix $z \in \Gamma_g^\#$. Then, by Corollary 6.4 and Lemma 6.7 (applied with $h_0 \equiv 0$), for $\rho < 1/Q_0$ there exists a ball $B_\rho(z_0) \subset \Omega_g^\# \cup (\mathbb{R}\times(-\infty,0])$ such that $z \in \partial B_\rho(z_0)$. Letting $\rho \uparrow \rho_0 := 1/Q_0$, we conclude that there exists a ball $B_{\rho_0}(z_0) \subset \Omega_g^\# \cup \big(\mathbb{R}\times(-\infty,0]\big)$ such that $z \in \partial B_{\rho_0}(z_0)$.

In the following theorem it is proved that the profiles g_n of solutions to the obstacle problems mentioned in the introduction (see (1.3)) converge to h in $W^{2,\infty}$, when (h,u) is a critical point. The theorem, in fact, deals with a slightly more general situation, which will be needed in the proof of statement (ii) of Theorem 2.11.

Theorem 6.9. *Let $h \in C^2_\#([0,b])$, $h > 0$ in $[0,b]$, and $\Lambda > \Lambda_0$, where Λ_0 is defined in Lemma 6.5. Let $(g_n,v_n) \in Y(e_0;0,b)$ be a solution to the following problem*:

$$\min\Big\{F(g,v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| : (g,v) \in Y(e_0;0,b),$$
$$v(x,0) = v_n(x,0) \text{ for all } x \in (0,b)\ g \geqq h - a_n\Big\},$$

where a_n is a sequence of positive numbers converging to zero. Also assume that $g_n \to h$ in $L^1(0,b)$,

$$\lim_{n\to\infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h) \quad \text{and} \quad \sup_n \int_{\Omega_{g_n}} Q(E(v_n))\,dz < +\infty.$$

Then for all $\alpha \in (0, \frac{1}{2})$ and for n large enough $g_n \in C^{1,\alpha}([0,b])$, the sequence $\{\nabla v_n\}$ is equibounded in $C^{0,\alpha}(\overline{U} \cap \overline{\Omega}_{g_n}; \mathbb{M}^{2\times 2})$, where U is any open set containing Γ_h such that $\overline{U} \cap \{y = 0\} = \emptyset$, and $g_n \to h$ in $C^{1,\alpha}([0,b])$. Moreover, if $\nabla v_n \rightharpoonup \nabla u$ in $L^2_{loc}(\Omega_h; \mathbb{M}^{2\times 2})$ for some $u \in H^1(\Omega_h; \mathbb{R}^2)$ such that $(h,u) \in X(e_0, q; 0, b)$ is a critical point, then $g_n \in W^{2,\infty}(0,b)$ for n large and $g_n \to h$ in $W^{2,\infty}(0,b)$.

Proof. Step 1 Up to a subsequence, we may assume that $\overline{\Gamma_{g_n} \cup \Sigma_{g_n}}$ converge in the Hausdorff metric to some compact connected set K. It can be easily seen that $\Gamma_h \subset K$. Hence, by Gołąb's theorem and observing that $\mathcal{H}^1(\overline{\Gamma_{g_n} \cup \Sigma_{g_n}}) = \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n})$, we have

$$\mathcal{H}^1(\Gamma_h) \leqq \mathcal{H}^1(K) \leqq \lim_{n\to\infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h).$$

Therefore, $\mathcal{H}^1(K \setminus \Gamma_h) = 0$. Since K is the Hausdorff limit of graphs, for all $x \in [0,b]$ the section $K \cap (\{x\} \times \mathbb{R})$ is connected. Hence, $K = \overline{\Gamma}_h$. From this equality, the definition of Hausdorff convergence and the continuity of h in $[0,b]$, we get that $\sup_{[0,b]} |g_n - h| \to 0$ as $n \to \infty$.

Step 2 We claim $g_n \in C^0([0,b])$ and $\Sigma_{g_n,c} = \emptyset$ (see (2.12)) for n large enough.

Fix $\rho < \min\{1/\Lambda, 1/\|h''\|_\infty\}$. By Lemma 6.7 for all n and for all $z \in \Gamma_{g_n} \cup \Sigma_{g_n}$ there exists a ball $B_\rho(z_0) \subset \Omega^\#_{g_n} \cup (\mathbb{R} \times (-\infty, 0])$ such that $\partial B_\rho(z_0) \cap (\Gamma_{g_n} \cup \Sigma_{g_n}) = \{z\}$. We show that for n large enough g_n is continuous, by proving that its extended graph does not contain vertical segments. Indeed, assume by contradiction that there exists $x \in [0,b)$ such that $g_n(x) < g_n^+(x)$ and take $z = (x, g_n(x))$. Then we may find a ball $B_\rho(z_0) \subset \Omega^\#_{g_n} \cup (\mathbb{R} \times (-\infty, 0])$, which is tangent at z to the vertical segment connecting z and $(x, g_n^+(x))$. Without loss of generality we may assume $z_0 = z + (\rho, 0)$. Let us now set $M := \|h'\|_\infty$. Then,

$$z + \rho\left(1 - \frac{M}{\sqrt{1+M^2}}, \frac{1}{\sqrt{1+M^2}}\right) =: z + \rho(w_1, w_2) \in \partial B^+_\rho(z_0).$$

Therefore, $g_n(x + \rho w_1) \geqq g_n(x) + \rho w_2$. On the other hand, $h(x + \rho w_1) \leqq h(x) + \rho M w_1 \leqq g_n(x) + \varepsilon_n + M\rho w_1$, where $\varepsilon_n := \sup_{[0,b)} |h - g_n| \to 0$. Hence,

$$g_n(x+\rho w_1) - h(x+\rho w_1) \geqq \rho w_2 - M\rho w_1 - \varepsilon_n = \rho(\sqrt{1+M^2} - M) - \varepsilon_n > \varepsilon_n,$$

where the last inequality, which holds for n large enough, gives a contradiction and proves the continuity of g_n. An entirely similar argument shows that $\Sigma_{g_n,c} = \emptyset$ for n sufficiently large, say $n \geqq n_0$.

Step 3 We now claim that $g_n \in C^1([0, b])$ for all $n > n_0$.

To this aim, observe that the uniform inner ball condition recalled in the previous step, the result proved in [15, Proposition 3.5], and the fact that $\Sigma_{g_n} \cup \Sigma_{g_n,c} = \emptyset$, imply that g_n is a Lipschitz function such that its right and left derivatives exist everywhere and are right and left continuous, respectively. From this it follows that for all $z_0 \in \Gamma_{g_n}$ there exist $c_n > 0$, a radius r_n, and an exponent $\alpha_n \in (1/2, 1)$ (depending possibly also on z_0) such that

$$\int_{B_r(z_0)\cap\Omega_{g_n}} |\nabla v_n|^2 \,\mathrm{d}z \leqq c_n r^{2\alpha_n}$$

for all $r \in (0, r_n)$. The proof of this decay estimate relies only on the fact that v_n minimizes the elastic energy in Ω_{g_n} and on the fact that Γ_{g_n} admits right and left tangents at z_0 and can be obtained repeating word for word the proof of [15, Theorem 3.13] (see also the proof of Theorem 6.10 in the present paper). Notice that actually, in our case, the proof is even simpler: indeed, by taking n sufficiently large, we may assume that $g_n > 0$ so that Step 5 of the same proof is not needed.

Let us now prove that g_n is of class C^1. To this aim, notice that since g_n is Lipschitz we may extend v_n outside Ω_{g_n} in such a way that, denoting this extension by $\tilde{v}_n$, we have

$$\int_{B_r(z_0)} |\nabla \tilde{v}_n|^2 \,\mathrm{d}z \leqq c_n r^{2\alpha_n}. \tag{6.2}$$

For $r < r_n$ denote by z'_r and $z''_r \in \Gamma_{g_n} \cap \partial B_r(z_0)$ two points such that the open subarcs of Γ_{g_n}, γ'_r and γ''_r, of endpoints z'_r, z_0 and z''_r, z_0, respectively, are contained in $\Gamma_{g_n} \cap B_r(z_0)$. Setting $z'_r := (x'_r, g_n(x'_r))$, $z''_r := (x''_r, g_n(x''_r))$, define $\tilde{g}_n$ as

$$\tilde{g}_n(x) := \begin{cases} g_n(x) & x \in [0, b) \setminus (x'_r, x''_r), \\ \max\{h(x) - a_n, s(x)\} & t \in [x'_r, x''_r], \end{cases}$$

where s is the affine function whose graph connects z'_r and z''_r. By (6.2) and the minimality of (g_n, v_n), we then obtain

$$\begin{aligned} 0 &\geqq F(g_n, v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| - F(\tilde{g}_n, \tilde{v}_n) - \Lambda\big||\Omega_{\tilde{g}_n}| - |\Omega_h|\big| \\ &\geqq \mathcal{H}^1(\gamma'_r \cup \gamma''_r) - \int_{x'_r}^{x''_r} \sqrt{1 + (\tilde{g}'_n)^2}\,\mathrm{d}x \\ &\quad - \int_{B_r(z_0)} Q(E(\tilde{v}_n))\,\mathrm{d}z - \Lambda|\Omega_{g_n} \triangle \Omega_{\tilde{g}_n}| \\ &\geqq |z'_r - z_0| + |z''_r - z_0| - |z'_r - z''_r| \\ &\quad - \int_{x'_r}^{x''_r} \big(\sqrt{1 + (\tilde{g}'_n)^2} - \sqrt{1 + (s')^2}\big)\,\mathrm{d}x - c_n r^{2\alpha_n} - \Lambda\pi r^2. \end{aligned}$$

Then, from the previous chain of inequalities we obtain that

$$2r - |z'_r - z''_r| \leqq \int_{(x'_r, x''_r)\cap\{h > s + a_n\}} \big(\sqrt{1 + (h')^2} - \sqrt{1 + (s')^2}\big)\,\mathrm{d}x + c'_n r^{2\alpha_n}.$$

Since $h(x_r') \leqq s(x_r') + a_n$, $h(x_r'') \leqq s(x_r'') + a_n$, either the set $(x_r', x_r'') \cap \{h > s + a_n\}$ is empty or, by Lagrange Theorem, there exists $\bar{x}_r \in (x_r', x_r'')$ such that $h'(\bar{x}_r) = s'$. Therefore, dividing both sides of the inequality above by r, we deduce

$$2 - \frac{|z_r' - z_r''|}{r} \leqq 2 \operatorname*{osc}_{(x_r', x_r'')} h' + c_n' r^{2\alpha_n - 1}.$$

Letting $r \to 0$, since h' is continuous and $\alpha_n > 1/2$, we obtain

$$\lim_{r \to 0^+} \frac{|z_r' - z_r''|}{r} = 2,$$

thus showing that the left and right tangent lines at z_0 coincide. This concludes the proof of the C^1-regularity of g_n.

Step 4 We claim that $g_n \to h$ in $C^1([0, b])$. To this aim fix $\varepsilon > 0$ and find $\delta > 0$ so small that

$$\begin{aligned} &\|h''\|_\infty \delta < \varepsilon, \\ &\frac{z' \cdot z''}{\rho^2} > 1 - \varepsilon \quad \text{for all } z',\ z'' \in \partial B_\rho(0) \text{ with } |z' - z''| < C_0\sqrt{\delta}, \end{aligned} \tag{6.3}$$

where ρ, $C_0 = C_0(\rho) > 0$ will be fixed later. Let us now consider the δ-tubular neighborhood $\mathcal{N}_\delta(\Gamma_{h-a_n})$ of Γ_{h-a_n}. By Step 1 we have $\Gamma_{g_n} \subset \mathcal{N}_\delta(\Gamma_{h-a_n})$ if n is sufficiently large. Now take $z = (x, g_n(x))$ and the corresponding ball $B_\rho(z_0) \subset \Omega^\#_{g_n} \cup (\mathbb{R} \times (-\infty, 0])$ touching Γ_{g_n} tangentially at z, with ρ chosen in such a way that $2\rho < \min\{1/\Lambda, 1/\|h''\|_\infty\}$. If $h(x) - a_n = g_n(x)$, then $h'(x) = g_n'(x)$ since $g_n \geqq h - a_n$. Otherwise, arguing as in the proof of Lemma 6.7 and recalling that $2\rho < 1/\|h''\|_\infty$, we infer that the set $\partial B_\rho(z_0) \cap \{(x, y) : y \geqq h(x) - a_n\}$ is connected. Let us denote this subarc by γ. Note that $\gamma \subset \mathcal{N}_\delta(\Gamma_{h-a_n})$. Let $z_1 \in \gamma$ be a such that

$$\operatorname{dist}(z_1, \Gamma_{h-a_n}) = \max_{w \in \gamma} \operatorname{dist}(w, \Gamma_{h-a_n}).$$

Since z_1 belongs to the relative interior of γ, the normal to $\partial B_\rho(z_0)$ at z_1 coincides with the normal to Γ_{h-a_n} at the point $z_2 = (x_2, y_2)$ such that $|z_2 - z_1| = \operatorname{dist}(z_1, \Gamma_{h-a_n})$, that is,

$$\frac{z_1 - z_0}{\rho} = \left(-\frac{h'(x_2)}{\sqrt{1 + (h'(x_2))^2}}, \frac{1}{\sqrt{1 + (h'(x_2))^2}} \right). \tag{6.4}$$

We claim that $\mathcal{H}^1(\gamma) \leqq C_0\sqrt{\delta}$, for some $C_0 > 0$ depending only on ρ. To prove this, consider the ball $B_{2\rho}(\tilde{z}_0)$ tangent to $\partial B_\rho(z_0)$ at z_1 and such that $B_\rho(z_0) \subset B_{2\rho}(\tilde{z}_0)$. Then the ball $B_{2\rho}(\tilde{z}_0 + z_2 - z_1)$ is tangent to Γ_{h-a_n} at z_2 and is contained in $\Omega^\#_{h-a_n} \cup (\mathbb{R} \times (-\infty, 0])$ since $2\rho < 1/\|h''\|_\infty$. Denote by γ' the smallest arc on $\partial B_\rho(z_0)$ whose endpoints are given by the intersection of $\partial B_\rho(z_0)$ with $\partial B_{2\rho}(\tilde{z}_0 + z_2 - z_1)$. Since $B_{2\rho}(\tilde{z}_0 + z_2 - z_1) \subset \Omega^\#_{h-a_n}$, we clearly have $\gamma \subset \gamma'$.

Moreover, as $|z_2 - z_1| \leqq \delta$, an elementary calculation shows that $\mathcal{H}^1(\gamma') \leqq C_0\sqrt{\delta}$, for a suitable $C_0 > 0$ depending only on ρ. Hence, the claim follows. In particular,

$$|z - z_1| \leqq \mathcal{H}^1(\gamma) \leqq C_0\sqrt{\delta}.$$

Thus, observing that

$$\frac{z - z_0}{\rho} = \Big(-\frac{g_n'(x)}{\sqrt{1 + (g_n'(x))^2}}, \frac{1}{\sqrt{1 + (g_n'(x))^2}}\Big),$$

from the the second inequality in (6.3) and from (6.4) we obtain

$$\frac{(1 + g_n'(x)h'(x_2))^2}{(1 + (g_n'(x))^2)(1 + (h'(x_2))^2)} > (1 - \varepsilon)^2 > 1 - 2\varepsilon,$$

or, equivalently,

$$(h'(x_2) - g_n'(x))^2 < 2\varepsilon(1 + (h'(x_2))^2)(1 + (g_n'(x))^2) < 2\varepsilon(1 + M^2)(1 + (g_n'(x))^2),$$

where $M = \|h'\|_\infty$. Taking ε sufficiently small, we first deduce that $|g_n'(x)|$ is bounded by a constant M_1 independent of x and n for n sufficiently large. In turn, recalling the first inequality in (6.3), we infer

$$\begin{aligned} |g_n'(x) - h'(x)| &\leqq |g_n'(x) - h'(x_2)| + |h'(x_2) - h'(x)| \\ &< \sqrt{2\varepsilon(1 + M^2)(1 + M_1^2)} + \varepsilon, \end{aligned}$$

for n sufficiently large. By the arbitrariness of ε the claim stated at the beginning of the step follows.

Step 5 We claim that for all $\alpha \in (0, 1/2)$, the sequence g_n converges to h in $C^{1,\alpha}([0, b])$, $v_n \in C^{1,\alpha}(\overline{U} \cap \overline{\Omega}_{g_n}; \mathbb{R}^2)$ for all $\alpha \in (0, 1/2)$ and for any open set U containing Γ_h and such that $\overline{U} \cap \{y = 0\} = \emptyset$, and $\sup_n \|v_n\|_{C^{1,\alpha}(\overline{U} \cap \overline{\Omega}_{g_n}; \mathbb{R}^2)} < \infty$.

To prove the first claim, it will be enough to show that for all $\sigma \in (1/2, 1)$ the sequence g_n' is equibounded in $C^{0,\sigma-\frac{1}{2}}([0, b])$. Fix any such σ. Then by Step 4 and Theorem 6.10 below, we know that for all $n > n_0$ and for all $z_0 \in \Gamma_{g_n}$

$$\int_{B_r(z_0) \cap \Omega_{g_n}} |\nabla v_n|^2 \, dz \leqq c_0 r^{2\sigma}, \tag{6.5}$$

for all $r \in (0, r_0)$, where c_0 and r_0 are independent of n. Let $0 < r < \frac{r_0}{M_1}$, where $M_1 := \sup_{n > n_0} \|g_n'\|_\infty$. Fix any point $x_0 \in [0, b)$ and any n. As in Step 3, using (6.5), we may extend v_n outside Ω_{g_n} in such a way that, denoting by $\tilde{v}_n$ this extension, we have

$$\int_{B_{2M_1 r}(z_0)} |\nabla \tilde{v}_n|^2 \, dz \leqq c_1 r^{2\sigma}, \tag{6.6}$$

where $z_0 := (x_0, g_n(x_0))$ and c_1 is independent of n. Denote by γ_r the open arc contained in Γ_{g_n} of endpoints $(x_0 + r, g_n(x_0 + r))$ and z_0, and define $\tilde{g}_n$ as

$$\tilde{g}_n(x) := \begin{cases} g_n(x) & x \in [0, b) \setminus (x_0, x_0 + r), \\ \max\{h(x) - a_n, s(x)\} & x \in [x_0, x_0 + r], \end{cases}$$

where s is the affine function connecting z_0 and $(x_0 + r, g_n(x_0 + r))$. Using the decay estimate (6.6), the minimality of (g_n, v_n), and the fact that the graph of g_n over $(x_0, x_0 + r)$ is contained in $B_{2M_1 r}(z_0)$, we obtain, arguing as in Step 3

$$\int_{x_0}^{x_0+r} \sqrt{1 + g_n'^2}\, \mathrm{d}x \leqq c r^{2\sigma} + \int_{x_0}^{x_0+r} \sqrt{1 + \tilde{g}_n'^2}\, \mathrm{d}x,$$

for some constant c depending only on c_1, Λ, and M_1. This inequality can be equivalently written as

$$\begin{aligned} &\int_{x_0}^{x_0+r} \sqrt{1 + g_n'^2}\, \mathrm{d}x - \sqrt{(g_n(x_0 + r) - g_n(x_0))^2 + r^2} \\ &\leqq c r^{2\sigma} + \int_{x_0}^{x_0+r} \left(\sqrt{1 + \tilde{g}_n'^2} - \sqrt{1 + s'^2}\right) \mathrm{d}x \\ &= c r^{2\sigma} + \int_{(x_0, x_0+r)\cap\{h - a_n > s\}} \left(\sqrt{1 + h'^2} - \sqrt{1 + s'^2}\right) \mathrm{d}x \\ &= c r^{2\sigma} + \int_{(x_0, x_0+r)\cap\{h - a_n > s\}} \left(\sqrt{1 + h'^2} - \sqrt{1 + h'^2(\bar{x})}\right) \mathrm{d}x \leqq c' r^{2\sigma}. \end{aligned} \tag{6.7}$$

Note that in the second equality we used the Lagrange theorem to find $\bar{x} \in (x_0, x_0 + r) \cap \{h - a_n > s\}$ such that $h'(\bar{x}) = s'$, while in the last one we used the fact that h' is Lipschitz. On the other hand, using the elementary inequality

$$\sqrt{1 + b^2} - \sqrt{1 + a^2} \geqq \frac{a(b - a)}{\sqrt{1 + a^2}} + \frac{(b - a)^2}{2(1 + \max\{a^2, b^2\})^{3/2}}$$

with $a := \fint_{x_0}^{x_0+r} g_n'\, \mathrm{d}x$ and $b := g_n'(x)$, and integrating the result in $(x_0, x_0 + r)$, we get

$$\begin{aligned} &\frac{1}{2(1 + M_1^2)^{3/2}} \fint_{x_0}^{x_0+r} \left(g_n'(x) - \fint_{x_0}^{x_0+r} g_n'\, \mathrm{d}s\right)^2 \mathrm{d}x \\ &\qquad \leqq \frac{1}{r} \int_{x_0}^{x_0+r} \sqrt{1 + g_n'^2}\, \mathrm{d}x - \frac{1}{r}\sqrt{(g_n(x_0 + r) - g_n(x_0))^2 + r^2} \leqq c' r^{2\sigma - 1}, \end{aligned}$$

where we also used (6.7). Thus, in particular,

$$\fint_{x_0}^{x_0+r} \left| g_n'(x) - \fint_{x_0}^{x_0+r} g_n'\, \mathrm{d}s \right| \mathrm{d}x \leqq c'' r^{\sigma - \frac{1}{2}}.$$

A similar inequality also holds in the interval $(x_0 - r, x_0)$. Hence, by [4, Theorem 7.51] we conclude that the sequence g_n is bounded in $C^{1,\sigma - \frac{1}{2}}([0, b])$ for all $\sigma \in (1/2, 1)$, as claimed.

To show the second claim, recall that v_n is a solution to the Lamé system (2.11) in Ω_{g_n}. Therefore, from what we have just proved and from the elliptic estimates proved in Proposition 8.9 we conclude that for n sufficiently large, the sequence ∇v_n is uniformly bounded in $C^{0,\alpha}(\overline{U} \cap \overline{\Omega}_{g_n}; \mathbb{M}^{2\times 2})$ for all $\alpha \in (0, 1/2)$ and for any open set U containing Γ_h and such that $\overline{U} \cap \{y = 0\} = \emptyset$.

Step 6 Let us now prove the second part of the statement. As $g_n \to h$ in $C^{1,\alpha}([0, b])$ by Step 5, for any $n > n_0$ there exists a $C^{1,\alpha}$-diffeomorphism $\Phi_n : \Omega_{g_n} \to \Omega_h$ such that $\Phi_n \to Id$ in $C^{1,\alpha}$. From the local weak convergence of ∇v_n to ∇u and by Step 5, we have that

$$\nabla v_n \circ \Phi_n^{-1} \to \nabla u \qquad \text{in } C^{0,\alpha}(\overline{U} \cap \overline{\Omega}_h; \mathbb{M}^{2\times 2}) \text{ for all } \alpha \in \left(0, \frac{1}{2}\right) \tag{6.8}$$

for any open set U containing Γ_h and such that $\overline{U} \cap \{y = 0\} = \emptyset$. We now set

$$K_n := \left\{x \in [0, b] : \, g_n(x) = h(x) - a_n\right\},$$

and we assume without loss of generality that $A_n := (0, b) \setminus K_n$ is not empty. Notice that, since g_n' and h_n' are continuous functions and $g_n \geqq h - a_n$, from the definition of K_n it follows that

$$g_n'(x) = h'(x) \qquad \text{for all } x \in K_n. \tag{6.9}$$

By the minimality of (g_n, v_n) we have that for all $x \in A_n$

$$\left(\frac{g_n'(x)}{\sqrt{1 + g_n'^2(x)}}\right)' = Q(E(v_n)(x, g_n(x)) + \lambda_n, \tag{6.10}$$

for some Lagrange multiplier $\lambda_n \in \mathbb{R}$ (see the last equation in (2.11)). Note that λ_n is the same for all connected components of A_n. Since (h, u) is a critical point, a similar equation holds for h, that is, for all $x \in [0, b]$

$$\left(\frac{h'(x)}{\sqrt{1 + h'^2(x)}}\right)' = Q(E(u)(x, h(x)) + \lambda, \tag{6.11}$$

for a suitable Lagrange multiplier λ. We claim that $\lambda_n \to \lambda$. Indeed, splitting each open A_n into the union of its connected components $(\alpha_{i,n}, \beta_{i,n})$, integrating (6.10), and using (6.11), we obtain

$$\begin{aligned}
\lambda_n |A_n| + \int_{A_n} Q(E(v_n)(x, g_n(x))\, dx &= \sum_i \int_{\alpha_{i,n}}^{\beta_{i,n}} \left(\frac{g_n'}{\sqrt{1 + g_n'^2}}\right)' dx \\
&= \sum_i \left(\frac{g_n'(\beta_{i,n})}{\sqrt{1 + g_n'^2(\beta_{i,n})}} - \frac{g_n'(\alpha_{i,n})}{\sqrt{1 + g_n'^2(\alpha_{i,n})}}\right) \\
&= \sum_i \left(\frac{h'(\beta_{i,n})}{\sqrt{1 + h'^2(\beta_{i,n})}} - \frac{h'(\alpha_{i,n})}{\sqrt{1 + h'^2(\alpha_{i,n})}}\right) \\
&= \int_{A_n} \left(\frac{h'}{\sqrt{1 + h'^2}}\right)' dx \\
&= \lambda |A_n| + \int_{A_n} Q(E(u)(x, h(x))\, dx,
\end{aligned}$$

which, in turn, gives

$$\lambda_n - \lambda = \fint_{A_n} \left[Q(E(u)(x, h(x)) - Q(E(v_n)(x, g_n(x)) \right] \mathrm{d}x.$$

From (6.8) one easily deduces

$$Q(E(v_n)(\cdot, g_n(\cdot)) \to Q(E(u)(\cdot, h(\cdot)) \qquad \text{uniformly in } [0, b].$$

Hence the convergence of λ_n to λ follows.

To conclude the proof, notice that (6.9) and (6.10) imply that g_n' is a Lipschitz function for all n. Then, the same equations, together with the convergence of λ_n to λ and the uniform convergence of $Q(E(v_n)(\cdot, g_n(\cdot))$ to $Q(E(u)(\cdot, h(\cdot))$ also yield that

$$\left(\frac{g_n'(\cdot)}{\sqrt{1 + g_n'^2(\cdot)}} \right)' \to \left(\frac{h'(\cdot)}{\sqrt{1 + h'^2(\cdot)}} \right)' \qquad \text{uniformly in } [0, b].$$

Hence, $g_n'' \to h''$ in $L^\infty(0, b)$ and this concludes the proof of the theorem.

We are now in a position to prove Theorem 2.10.

Proof of Theorem 2.10. Recall that by Theorem 4.6, (h, u) is an isolated local minimizer with respect to sufficiently small $W^{2,\infty}$-perturbations of h. Hence, it is enough to show that $W^{2,\infty}$-local minimality of (h, u) implies the local minimality in the sense of Definition 2.4. We prove this by contradiction assuming that for every n there exists $(\tilde{g}_n, \tilde{v}_n) \in X(e_0, q; 0, b)$, with $|\Omega_{\tilde{g}_n}| = |\Omega_h|$, such that $F(\tilde{g}_n, \tilde{v}_n) \leqq F(h, u)$ and $0 < \sup_{[0,b]} |\tilde{g}_n - h| \leqq 1/n$. Let $(g_n, v_n) \in X(e_0, q; 0, b)$ be solutions of the following problems

$$\min \left\{ F(g, v) + \Lambda \big| |\Omega_g| - |\Omega_h| \big| : (g, v) \in X(e_0, q; 0, b),\ g \geqq h - 1/n \text{ in } [0, b] \right\},$$

with $\Lambda > \max\{\Lambda_0, Q_0\}$ and Λ_0 defined as in Lemma 6.5.

Assume first that (up to a non-relabeled subsequence) $F(g_n, v_n) < F(\tilde{g}_n, \tilde{v}_n) \leqq F(h, u)$. By the compactness Theorem 2.2 we may assume that (g_n, v_n) converges in $X(e_0, q; 0, b)$ to some pair (k, v).

Fix $(g, w) \in X(e_0, q; 0, b)$, with $g \geqq h$. By lower semicontinuity and the minimality of (g_n, u_n), we get

$$\begin{aligned} F(k, v) + \Lambda \big| |\Omega_k| - |\Omega_h| \big| &\leqq \liminf_{h \to \infty} \Big[F(g_n, v_n) + \Lambda \big| |\Omega_{g_n}| - |\Omega_h| \big| \Big] \\ &\leqq F(g, w) + \Lambda \big| |\Omega_g| - |\Omega_h| \big|. \end{aligned} \tag{6.12}$$

Since $k \geqq h$, by applying the above inequality with $(g, w) = (h, v)$ we obtain, in particular, that

$$\mathcal{H}^1(\Gamma_k) + \Lambda \int_0^b |k - h| \leqq \mathcal{H}^1(\Gamma_h).$$

Recalling that $\Lambda > \Lambda_0$, by Lemma 6.5 it follows that $k = h$. Notice that, in particular, we have just proved that $g_n \to h$ in $L^1(0, b)$ and that (h, v) minimizes F in the class of all $(g, w) \in X(e_0, q; 0, b)$ such that $g \geqq h$. In particular, v must coincide with the elastic equilibrium u. By the lower semicontinuity of $g \mapsto \mathcal{H}^1(\Gamma_g)$ with respect to the L^1-convergence and the lower semicontinuity of the elastic energy with respect to the weak $H^1_{\rm loc}$- convergence, applying (6.12) with $(g, w) = (h, u)$ again, we deduce

$$\mathcal{H}^1(\Gamma_h) = \lim_{n\to\infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}), \quad \int_{\Omega_h} Q(E(u))\,dz = \lim_{n\to\infty} \int_{\Omega_{g_n}} Q(E(v_n))\,dz.$$

From Theorem 6.9 we obtain that $g_n \to h$ in $W^{2,\infty}(0, b)$. By the choice of Λ and by Lemma 6.3 it follows that $|\Omega_{g_n}| \geqq |\Omega_h|$ for all n. Therefore, by replacing g_n with $\hat{g}_n = g_n - \big(|\Omega_{g_n}| - |\Omega_h|\big)/b$, we have

$$|\Omega_{\hat{g}_n}| = |\Omega_h|, \; F(\hat{g}_n, v_n) < F(h, u) \text{ for all } n, \text{ and } \hat{g}_n \to h \text{ in } W^{2,\infty}(0, b),$$

a contradiction to the strict $W^{2,\infty}$-local minimality of (h, u).

If, instead, $F(g_n, v_n) = F(\tilde{g}_n, \tilde{v}_n)$, we may reproduce the same argument with (g_n, v_n) replaced by $(\tilde{g}_n, \tilde{v}_n)$ to deduce that $\hat{g}_n := \tilde{g}_n - \big(|\Omega_{\tilde{g}_n}| - |\Omega_h|\big)/b$ converge to h in $W^{2,\infty}(0, b)$. Note that either $\hat{g}_n = \tilde{g}_n$ or $F(\hat{g}_n, \tilde{v}_n) < F(\tilde{g}_n, \tilde{v}_n) \leqq F(h, u)$. In all cases $\hat{g}_n \neq h$, thus giving a contradiction.

The proof of Theorem 2.9 is now immediate.

Proof of Theorem 2.9. The conclusion of the theorem is an easy consequence of Theorems 2.10 and 5.1.

We conclude this section by proving the following regularity theorem, which was used in the proof of Theorem 6.9.

Theorem 6.10. *Let $g_n \in C^1_\#([0, b])$ such that $g_n \to h$ in $C^1_\#([0, b])$, where $h > 0$, and let v_n be the solution to*

$$\min\Big\{\int_{\Omega_{g_n}} Q(E(v))\,dz : (g_n, v) \in Y(e_0; 0, b), \\ v(x, 0) = v_n(x, 0) \text{ for all } x \in (0, b)\Big\}.$$

Assume also that

$$\sup_n \int_{\Omega_{g_n}} Q(E(v_n))\,dz < \infty$$

and let $\sigma \in (1/2, 1)$. Then there exist $c_0, r_0 > 0$, independent of n, such that for all $r \in (0, r_0)$ and for all $z_0 \in \Gamma_{g_n}$

$$\int_{B_r(z_0)\cap\Omega_{g_n}} |\nabla v_n|^2\,dz \leqq c_0 r^{2\sigma}. \tag{6.13}$$

Proof. The proof is very similar to [15, Theorem 3.16]. For the reader's convenience we give the details in order to show that the estimates can be made independent of n.

We begin by showing that there exists $c_1 > 0$ such that, for all $\tau \in (0, 1)$, there exists a radius $r_\tau > 0$ such that whenever $r \in (0, r_\tau)$ and $z_0 \in \Gamma_{g_n}$

$$\int_{B_{\tau r}(z_0)\cap\Omega_{g_n}} |\nabla v_n|^2 \,\mathrm{d}z \leqq c_1\tau^2 \int_{B_r(z_0)\cap\Omega_{g_n}} |\nabla v_n|^2 \,\mathrm{d}z. \tag{6.14}$$

We argue by contradiction, assuming that (6.14) is false for some $\tau \in (0, 1)$. Hence, we may find a (not relabeled) subsequence of g_n, a sequence of radii $r_n \to 0$, $z_n = (x_n, g_n(x_n))$ converging to some $z_0 = (x_0, h(x_0))$ such that

$$\int_{B_{\tau r_n}(z_n)\cap\Omega_{g_n}} |\nabla v_n|^2 \,\mathrm{d}z > c_1\tau^2 \int_{B_{r_n}(z_n)\cap\Omega_{g_n}} |\nabla v_n|^2 \,\mathrm{d}z. \tag{6.15}$$

Define the sets

$$B_n := \frac{1}{r_n}\Big[-z_n + B_{r_n}(z_n) \cap \Omega_{g_n}\Big]$$

and observe that $\chi_{B_n} \to \chi_{B_\infty}$ in $L^2(\mathbb{R}^2)$, where $B_\infty := \{(x, y) \in B_1(0) : y < h'(x_0)x\}$. This is true since $g_n \to h$ in $C^1([0, b])$. We now also rescale the function v_n by setting for $z \in B_n$

$$w_n(z) := \frac{v_n(z_n + r_n z) - a_n}{\lambda_n r_n},$$

where

$$a_n := \fint_{B_{r_n}(z_n)\cap\Omega_{g_n}} v_n \,\mathrm{d}z, \qquad \lambda_n^2 := \fint_{B_{r_n}(z_n)\cap\Omega_{g_n}} |\nabla v_n|^2 \,\mathrm{d}z.$$

Note that

$$\fint_{B_n} |\nabla w_n|^2 \,\mathrm{d}z = 1, \qquad \int_{B_n} w_n \,\mathrm{d}z = 0.$$

By Poincaré Inequality and a standard extension argument we may extend each function w_n to the ball $B_1(0)$ in such a way that the resulting function, still denoted w_n, satisfies $\|w_n\|_{H^1(B_1(0))} \leqq C$, with C independent of n. Without loss of generality we may assume that the sequence $w_n \rightharpoonup w_\infty \in H^1(B_1(0); \mathbb{R}^2)$. Moreover, it is easy to see that the functions w_n satisfy the equation

$$\int_{B_n} \mathbb{C}(E(w_n)) : E(\varphi)\,\mathrm{d}z = 0 \tag{6.16}$$

for every $\varphi \in C_0^1(B_1(0); \mathbb{R}^2)$.

We claim that for all functions $\psi \in C_0^1\,(B_1(0))$we have

$$\lim_{n\to\infty} \int_{B_n} \psi^2\,|\nabla w_n - \nabla w_\infty|^2 \,\mathrm{d}z = 0. \tag{6.17}$$

From (6.16), and the fact that $\chi_{B_n} \to \chi_{B_\infty}$ in $L^2(B_1(0))$, $w_n \rightharpoonup w_\infty$ in $H^1\left(B_1(0);\mathbb{R}^2\right)$, we get that

$$\int_{B_\infty} \mathbb{C}E(w_\infty) : E(\varphi)\, \mathrm{d}z = 0 \tag{6.18}$$

for all $\varphi \in C_0^1(B_1(0);\mathbb{R}^2)$. Fix $\psi \in C_0^1(B_1(0))$ and choose $\varphi := \psi^2 w_n$ in (6.16) ($\varphi := \psi^2 w_\infty$ in (6.18) respectively), thus obtaining

$$\begin{aligned}\int_{B_n} \psi^2 \mathbb{C}E(w_n) : E(w_n)\, \mathrm{d}z \\ = -\int_{B_n} \psi\, \mathbb{C}E(w_n) : (w_n \otimes \nabla\psi + (w_n \otimes \nabla\psi)^T)\, \mathrm{d}z\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}\int_{B_\infty} \psi^2 \mathbb{C}E(w_\infty) : E(w_\infty)\, \mathrm{d}z \\ = -\int_{B_\infty} \psi\, \mathbb{C}E(w_\infty) : (w_n \otimes \nabla\psi + (w_\infty \otimes \nabla\psi)^T)\, \mathrm{d}z.\end{aligned} \tag{6.20}$$

Letting $n \to \infty$ in (6.19) and using the fact that the right-hand side converges to the right-hand side of (6.20), we obtain that

$$\lim_{n\to\infty} \int_{B_n} \psi^2 \mathbb{C}E(w_n) : E(w_n)\, \mathrm{d}z = \int_{B_\infty} \psi^2 \mathbb{C}E(w_\infty) : E(w_\infty)\, \mathrm{d}z,$$

from which we easily get

$$\lim_{n\to\infty} \int_{B_n} \mathbb{C}E(\psi(w_n - w_\infty)) : E(\psi(w_n - w_\infty))\, \mathrm{d}z = 0.$$

Hence the claim follows from the Korn's inequality stated in [15, Theorem 4.2].

It follows from (6.18) that w_∞ is a weak solution of the problem

$$\begin{aligned} \mu\Delta w_\infty + (\lambda+\mu)\nabla(\operatorname{div} w_\infty) = 0 \quad &\text{in } B_\infty \\ \left[\mu\left(\nabla w_\infty + \nabla w_\infty^T\right) + \lambda(\operatorname{div} w_\infty) I\right]\nu = 0 \quad &\text{on } \Gamma_{g_\infty} \cap B_1(0), \end{aligned}$$

where $g_\infty(x) := h'(x_0)x$. By [15, Theorem 3.7] it follows that there exists $c > 0$ such that

$$\sup_{\frac{1}{2}B_\infty} |\nabla w_\infty|^2\, \mathrm{d}z \leqq c \int_{B_\infty} |\nabla w_\infty|^2\, \mathrm{d}z.$$

Hence, we have

$$\int_{\tau B_\infty} |\nabla w_\infty|^2\, \mathrm{d}z \leqq c\tau^2 \int_{B_\infty} |\nabla w_\infty|^2\, \mathrm{d}z \leqq c_2\tau^2,$$

where we have used the fact that $\int_{B_\infty} |\nabla w_\infty|^2 \, dz \leqq |B_\infty|$. By (6.17) we then have that

$$\lim_{n\to\infty} \int_{\tau B_n} |\nabla w_n|^2 \, dz = \int_{\tau B_\infty} |\nabla w_\infty|^2 \, dz$$

and so

$$\lim_{n\to\infty} \frac{\int_{B_{\tau r_n}(z_n)\cap\Omega_{g_n}} |\nabla v_n|^2 \, dz}{\int_{B_{r_n}(z_n)\cap\Omega_{g_n}} |\nabla v_n|^2 \, dz} = \frac{1}{|B_\infty|} \lim_{n\to\infty} \int_{\tau B_n} |\nabla w_n|^2 \, dz \leqq \frac{c_2}{|B_\infty|}\tau^2$$

which contradicts (6.15), provided we take

$$c_1 \geqq 2\frac{c_2}{|B_\infty|}.$$

This proves (6.14).

We are now in a position to prove (6.13). Fix $\sigma \in (1/2, 1)$ and choose τ such that

$$c_1\tau^2 \leqq \tau^{2\sigma}.$$

Fix $0 < r < r_\tau$ and find $k \in \mathbb{N}$ such that $\tau^{k+1} r_\tau \leqq r \leqq \tau^k r_\tau$. By iterating (6.14) and by the choice of τ, for every n and for every $z_0 \in \Gamma_{g_n}$ we have

$$\begin{aligned}
\int_{B_r(z_0)\cap\Omega_{g_n}} |\nabla v_n|^2 \, dz &\leqq \int_{B_{\tau^k r_\tau}(z_0)\cap\Omega_{g_n}} |\nabla v_n|^2 \, dz \\
&\leqq \tau^{2k\sigma} \int_{B_{r_\tau}(z_0)\cap\Omega_{g_n}} |\nabla v_n|^2 \, dz \\
&\leqq \frac{r^{2\sigma}}{(\tau r_\tau)^{2\sigma}} \int_{\Omega_{g_n}} |\nabla v_n|^2 \, dz
\end{aligned}$$

and this concludes the proof of the theorem.

7. Global Minimizers: Proofs

This section is devoted to the proof of all results concerning global minimizers. Here we will often work with sequences (h_n, u_n) of global minimizers having different periods b_n; for this reason we shall consider the energy F over varying intervals. When needed, we will underline the dependence on the interval $(0, b_n)$ by writing F_{b_n} instead of F. More precisely, $F_{b_n}(g, v)$ will denote the energy defined in (2.8) with Ω_g, Γ_g, and Σ_g as in (2.2), (2.3), and (2.4), respectively, with $(0, b)$ replaced by $(0, b_n)$. Moreover, throughout this section the function q appearing in the Dirichlet datum is assumed to be zero.

We start by stating the following simple generalization of the Compactness Theorem 2.2.

Theorem 7.1. *Let* $(h_n, u_n) \in X(e_n; 0, b_n)$ *be such that*

$$\sup_n \left\{ \int_{\Omega_{h_n}} |E(u_n)|^2 \, dz + \mathrm{Var}(h_n; 0, b_n) + |\Omega_{h_n}| \right\} < +\infty,$$

where $\Omega_{h_n} := \{(x, y) : x \in (0, b_n), 0 < y < h_n(x)\}$. *Assume that* $b_n \to b_0 > 0$ *and* $e_n \to e_0 \geqq 0$. *Then there exist* $(h, u) \in X(e_0, q; 0, b)$ *and a subsequence (not relabeled) such that*

$$d_H(\mathbb{R}^2_+ \setminus \Omega^\#_{h_n}, \mathbb{R}^2_+ \setminus \Omega^\#_h) \to 0 \quad \text{and} \quad u_n \rightharpoonup u \text{ in } H^1_{\mathrm{loc}}(\Omega^\#_n; \mathbb{R}^2). \tag{7.1}$$

Moreover,

$$F_b(h, u) \leqq \liminf_{n\to\infty} F_{b_n}(h_n, u_n). \tag{7.2}$$

The compactness part follows from the same argument used in [6] to prove Theorem 2.2, hence we omit its proof. Concerning the lower semicontinuity, it easily follows from the fact that for any positive δ the functional $F_{b-\delta}$ is lower semicontinuous with respect to the convergence given in (7.1).

As a consequence of the previous compactness result, we have the following lemma showing that global minimizers with possibly different periodicities converge to a global minimizer.

Lemma 7.2. *Let* $b_n \to b > 0$, $e_n \to e_0$, $d_n \to d > 0$. *Let* $(h_n, u_n) \in X(e_n; 0, b_n)$ *be a sequence of* b_n*-periodic global minimizers with* $\int_0^{b_n} h_n \, dx = d_n$. *Then there exist a b-periodic global minimizer* $(h, u) \in X(e_0; 0, b)$ *and a subsequence* (h_{n_k}, u_{n_k}) *such that* (7.1) *holds.*

Proof. By Theorem 7.1 there exist $(h, u) \in X(e_0; 0, b)$ and a subsequence (not relabeled) such that (7.1) holds. Let $(g, v) \in X(e_0; 0, b)$, with g Lipschitz and $\int_0^b g \, dx = d$, and define

$$g_n(x) := \frac{bd_n}{db_n} g\left(\frac{b}{b_n} x\right), \quad v_n(x, y) := \frac{b_n e_n}{b e_0} v\left(\frac{b}{b_n} x, \frac{db_n}{bd_n} y\right).$$

Then, $(g_n, v_n) \in X(e_n; 0, b_n)$, $\int_0^{b_n} g_n \, dx = d_n$, and $F_{b_n}(g_n, v_n) \to F_b(v, g)$. Thus, by the minimality of (h_n, u_n) and by (7.2), we have

$$F_b(h, u) \leqq \liminf_{n\to\infty} F_{b_n}(h_n, u_n) \leqq \lim_{n\to\infty} F_{b_n}(g_n, v_n) = F_b(g, v).$$

We conclude by applying the approximation Lemma 6.1.

The following lemma deals with the uniqueness part of Theorem 2.11.

Lemma 7.3. *Let* $b, d > 0$ *such that the minimum problem*

$$\min\left\{ F(g, v) : (g, v) \in X(e_0; 0, b), \ |\Omega_g| = d \right\} \tag{7.3}$$

has a non-flat solution. Then, for all $d' > d$ *the flat configuration is not a global minimizer for the minimum problem*

$$\min\left\{ F(g, v) : (g, v) \in X(e_0; 0, b), \ |\Omega_g| = d' \right\}.$$

Proof. Let $d' > d$ and let (g, v) be a non-flat minimizer for (7.3). We set $\tilde{g} := g + (d' - d)/b$ and

$$\tilde{v}(x, y) := \begin{cases} \left(e_0 x, \dfrac{-\lambda e_0}{2\mu + \lambda} y\right) & \text{if } 0 \leqq y \leqq \dfrac{d' - d}{b} \\ v\left(x, y - \dfrac{d' - d}{b}\right) + \left(0, \dfrac{-\lambda e_0}{2\mu + \lambda} \dfrac{d' - d}{b}\right) & \text{if } y \geqq \dfrac{d' - d}{b}. \end{cases}$$

Let $(d'/b, u_{e_0})$ be the flat configuration and denote by $\tilde{u}$ the minimizer of the elastic energy in $\Omega_{\tilde{g}}$ under the usual periodicity and boundary condition. Note that

$$\int_{\Omega_{\tilde{g}}} Q(E(\tilde{u}))\,\mathrm{d}z < \int_{\Omega_{\tilde{g}}} Q(E(\tilde{v}))\,\mathrm{d}z. \tag{7.4}$$

Indeed, if not, then $\tilde{v}$ would be a solution of the Lamé system coinciding in an open set with $\left(e_0 x, \frac{-\lambda e_0}{2\mu+\lambda} y\right)$, and therefore, by analyticity, it would coincide with this function everywhere. Hence, by the minimality of (g, v) we would have

$$Q_0 d + \mathcal{H}^1(\Gamma_g) + 2\mathcal{H}^1(\Sigma_g) = F(g, v) \leqq F(d/b, u_{e_0}) = Q_0 d + b,$$

where, we recall, $Q_0 = Q(E_0)$ and $E_0 = E(u_{e_0})$ (see Lemma 6.3). The above inequality would then imply that $g = d/b$, which is impossible. Thus, from (7.4), we conclude that

$$F(\tilde{g}, \tilde{u}) < F(\tilde{g}, \tilde{v}) = F(g, v) + Q_0(d' - d) \leqq F(d'/b, u_{e_0}),$$

thus showing that the flat configuration $(d'/b, u_{e_0})$ cannot be a global minimizer.

We are now in a position to prove part (i) of Theorem 2.11.

Proof of Theorem 2.11(i). We fix b and we first show that there exists $d_0 > 0$ such that if $0 < d < d_0$ then $(d/b, u_{e_0})$ is a b-periodic global minimizer. To this aim, we argue by contradiction by assuming that there exist a sequence $d_n \downarrow 0$, a sequence $(k_n, w_n) \in X(e_0; 0, b)$ minimizing F under the constraint $|\Omega_{k_n}| = d_n$, such that $F(k_n, w_n) < F(d_n/b, u_{e_0})$.

Let $\varepsilon > 0$ be such that $\partial^2 F(\varepsilon, u_{e_0})$ is positive definite. This is possible thanks to Theorem 5.1. Then, by Theorem 2.10 the flat configuration (ε, u_{e_0}) is a an isolated local minimizer for F. We set

$$\varepsilon_n := \varepsilon - \frac{d_n}{b}$$

and we denote by (g_n, v_n) a sequence of minimizers of the following problems

$$\min\left\{F(g, v) + \Lambda\big||\Omega_g| - \varepsilon b\big| : (g, v) \in X(e_0; 0, b),\ g \geqq \varepsilon_n\right\},$$

where $\Lambda > Q_0$. Arguing as in the second part of the proof of Theorem 2.10, we deduce that $g_n \to \varepsilon$ in $L^1(0, b)$ and $\lim_{n\to\infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = b$. This, in turn, implies $\sup_{[0,b]} |g_n - \varepsilon| \to 0$.

Still arguing as in the second part of the proof of Theorem 2.10, we may conclude that $|\Omega_{g_n}| \geqq \varepsilon b$ and the functions $\tilde{g}_n := g_n - (|\Omega_{g_n}| - \varepsilon b)/b$ satisfy $\sup_{[0,b]} |\tilde{g}_n - \varepsilon| \to 0$ and $F(\tilde{g}_n, v_n) \leqq F(g_n, v_n)$.

Therefore, setting $\tilde{k}_n := k_n + \varepsilon_n$ and

$$\tilde{w}_n(x, y) := \begin{cases} \Big(e_0 x, \dfrac{-\lambda e_0}{2\mu + \lambda} y\Big) & \text{if } 0 \leqq y \leqq \varepsilon_n \\ w_n(x, y - \varepsilon_n) + \Big(0, \dfrac{-\lambda e_0}{2\mu + \lambda} \varepsilon_n\Big) & \text{if } y \geqq \varepsilon_n, \end{cases}$$

we have

$$\begin{aligned} F(\tilde{g}_n, v_n) &\leqq F(\tilde{k}_n, \tilde{w}_n) = F(k_n, w_n) + F(\varepsilon_n, u_{e_0}) \\ &< F(d_n/b, u_{e_0}) + F(\varepsilon_n, u_{e_0}) = F(\varepsilon, u_{e_0}). \end{aligned}$$

Since $\sup_{[0,b]} |\tilde{g}_n - \varepsilon| \to 0$, the inequality above contradicts the local minimality of ε, thus proving that there exists $d_0 > 0$ such that if $0 < d < d_0$ the flat configuration is an absolute minimizer. Then, setting

$$d_{\text{glob}}(b) := \sup\Big\{d > 0 : \Big(\frac{d}{b}, u_{e_0}\Big) \text{ is a minimizer of (2.9)} \\ \text{in the interval } (0, b)\Big\},$$

from Lemma 7.3 we also obtain that for $d \in (0, d_{\text{glob}}(b))$ the flat configuration is the unique minimizer. Finally, $(d_{\text{glob}}(b)/b, u_{e_0})$ is also a global minimizer since it is the limit of global minimizers (see Lemma 7.2).

Before proving part (ii) of Theorem 2.11, we need to show that the map $b \mapsto d_{\text{glob}}(b)$ is upper semicontinuous.

Lemma 7.4. *The function* $d_{\text{glob}} : (0, +\infty) \to (0, +\infty]$ *is upper semicontinuous.*

Proof. Let b_n a sequence of positive numbers converging to $b > 0$. It is enough to show that if $d < \limsup_n d_{\text{glob}}(b_n)$ then $d_{\text{glob}}(b) \geqq d$. Without loss of generality we may assume that the limsup is in fact a limit. By Theorem 2.11(i) we then have that $(\frac{d}{b_n}, u_{e_0})$ is a b_n-global minimizer of F for n large. By Lemma 7.2 it follows that $(d/b, u_{e_0})$ is a b-periodic global minimizer, thus proving $d_{\text{glob}}(b) \geqq d$.

Proof of Theorem 2.11(ii). By Theorem 4.9 and Remark 5.2, if $0 < b \leqq b_0 := \frac{\pi}{16} \frac{2\mu+\lambda}{e_0^2 \mu(\mu+\lambda)}$, there exists $\delta > 0$ (possibly depending on b) such that for all $d > 0$, $(g, v) \in X(te_0; 0, b)$, and $t \in (\frac{1}{2}, 2)$, we have

$$0 < \Big\|g - \frac{d}{b}\Big\|_{W^{2,\infty}(0,b)} < \delta \quad \text{and} \quad \int_0^b g \, dx = d \\ \Longrightarrow F_b(g, v) > F_b\Big(\frac{d}{b}, t u_{e_0}\Big). \tag{7.5}$$

Step 1 We claim that there exists $\bar{\delta} > 0$ such that for all $b \in [b_0/2, b_0]$ and all $(g, v) \in X(e_0; 0, b)$ we have

$$0 < \sup_{[0,b]} \left| g - \frac{d}{b} \right| < \bar{\delta}, \text{ and } \int_0^b g \, dx = d \implies F_b(g, v) > F_b\Big(\frac{d}{b}, u_{e_0}\Big). \quad (7.6)$$

To prove this claim we argue by contradiction, assuming that there exist $\delta_n \downarrow 0$, $b_n \to b \in [b_0/2, b_0]$, $d_n \to \overline{d} \in [0, \infty]$, $(\tilde{g}_n, \tilde{v}_n) \in X(e_0; 0, b_n)$ such that

$$0 < \sup_{[0,b_n]} \left| \tilde{g}_n - \frac{d_n}{b_n} \right| < \delta_n, \quad \int_0^{b_n} \tilde{g}_n = d_n, \quad \text{and} \quad F_{b_n}(\tilde{g}_n, \tilde{v}_n) \leqq F_{b_n}\Big(\frac{d_n}{b_n}, u_{e_0}\Big). \quad (7.7)$$

We will provide a proof only in the case $\overline{d} = \infty$, the other cases being similar and easier. We start by rescaling all the functions to the fixed interval $(0, b)$, namely, we set

$$\hat{g}_n(x) := \frac{b}{b_n} \tilde{g}_n\Big(\frac{x b_n}{b}\Big), \qquad \hat{v}_n(x, y) := \sqrt{\frac{b}{b_n}} \tilde{v}_n\Big(\frac{x b_n}{b}, \frac{y b_n}{b}\Big).$$

Notice that from (7.7) we get

$$0 < \sup_{[0,b]} \left| \hat{g}_n - \frac{b d_n}{b_n^2} \right| < \frac{b \delta_n}{b_n}, \qquad \int_0^b \hat{g}_n = \frac{b^2 d_n}{b_n^2}, \quad (7.8)$$

$$F_b(\hat{g}_n, \hat{v}_n) = \frac{b}{b_n} F_{b_n}(\tilde{g}_n, \tilde{v}_n) \leqq \frac{b}{b_n} F_{b_n}\Big(\frac{d_n}{b_n}, u_{e_0}\Big)$$
$$= F_b\Big(\frac{b d_n}{b_n^2}, \sqrt{\frac{b_n}{b}} u_{e_0}\Big). \quad (7.9)$$

Choose $\Lambda > Q_0$ and denote by $(g_n, v_n) \in X(e_0; 0, b)$ a minimizer to the problem

$$\min\Big\{ F_b(g, v) + \Lambda \Big| |\Omega_g| - \frac{b^2 d_n}{b_n^2} \Big| :$$
$$(g, v) \in X\Big(\sqrt{\frac{b_n}{b}} e_0; 0, b\Big), g \geqq \frac{b d_n}{b_n^2} - \frac{b \delta_n}{b_n} \Big\}.$$

We start by assuming that, up to a not-relabeled subsequence,

$$F_b(g_n, v_n) < F_b(\hat{g}_n, \hat{v}_n) \leqq F_b\Big(\frac{b d_n}{b_n^2}, \sqrt{\frac{b_n}{b}} u_{e_0}\Big).$$

By Lemma 6.3 we have that $|\Omega_{g_n}| = (b^2 d_n)/b_n^2$, where

$$\Omega_{g_n} := \{(x, y) : x \in (0, b), \ 0 < y < g_n(x)\}.$$

Thus, by the minimality of v_n in Ω_{g_n}

$$\int_{\Omega_{g_n}} Q(E(v_n)) \, dz \leqq \int_{\Omega_{g_n}} Q\Big(E\Big(\sqrt{\frac{b_n}{b}} u_{e_0}\Big)\Big) \, dz = \frac{b d_n}{b_n} Q_0. \quad (7.10)$$

By the minimality of $\sqrt{\frac{b_n}{b}}u_{e_0}$ in $(0,b)\times(0,\frac{bd_n}{b_n^2}-\frac{b\delta_n}{b_n})$ we have

$$\int_{\Omega_{g_n}} Q(E(v_n))\,\mathrm{d}z \geqq \int_0^b \mathrm{d}x \int_0^{\frac{bd_n}{b_n^2}-\frac{b\delta_n}{b_n}} Q(E(v_n))\,\mathrm{d}y$$
$$\geqq \frac{bd_n}{b_n}Q_0 - \delta_n b Q_0.$$

Using this inequality and the fact that, by (7.8) and (7.9),

$$F_b(g_n,v_n) < F_b(\hat{g}_n,\hat{v}_n) \leqq F_b\Big(\frac{bd_n}{b_n^2},\sqrt{\frac{b_n}{b}}u_{e_0}\Big),$$

we also obtain

$$\mathcal{H}^1(\Gamma_{g_n}) + 2\mathcal{H}^1(\Sigma_{g_n}) - \delta_n b Q_0 \leqq b. \tag{7.11}$$

From (7.11) and the trivial inequality $b \leqq \mathcal{H}^1(\Gamma_{g_n}) + 2\mathcal{H}^1(\Sigma_{g_n})$, we deduce

$$\lim_{h\to\infty}\big(\mathcal{H}^1(\Gamma_{g_n}) + 2\mathcal{H}^1(\Sigma_{g_n})\big) = b.$$

Since $\|g_n - \frac{bd_n}{b_n^2} + \frac{1}{b}\|_{L^1(0,b)} = 1$, from the above equality we easily get that $g_n - \frac{bd_n}{b_n^2} + \frac{1}{b} \to \frac{1}{b}$ in $L^1(0,b)$.

We now translate the functions $g_n,\ v_n$ by setting

$$h_n(x) := g_n(x) - \frac{bd_n}{b_n^2} + \frac{1}{b}, \qquad w_n(x,y) = v_n\Big(x, y + \frac{bd_n}{b_n^2} - \frac{1}{b}\Big).$$

Notice that (h_n, w_n) solves the minimum problem

$$\min\Big\{F_b(g,v) + \Lambda\big||\Omega_g| - 1\big| : (g,v) \in Y(e_0;0,b),$$
$$v(x,0) = w_n(x,0) \text{ for all } x \in (0,b),\ g \geqq \frac{1}{b} - \frac{b\delta_n}{b_n}\Big\}.$$

We are now going to apply Theorem 6.9 to (h_n, w_n). To this aim, notice that by what we have proved above, we have that $h_n \to \frac{1}{b}$ in $L^1(0,b)$ and $\mathcal{H}^1(\Gamma_{h_n}\cup\Sigma_{h_n})$ converges to b. On the other hand, since by the minimality of $\sqrt{\frac{b_n}{b}}u_{e_0}$ in $(0,b)\times(0,\frac{bd_n}{b_n^2}-\frac{1}{b})$ we have

$$\Big(\frac{bd_n}{b_n^2} - \frac{1}{b}\Big)b_n Q_0 \leqq \int_0^b \mathrm{d}x \int_0^{\frac{bd_n}{b_n^2}-\frac{1}{b}} Q(E(v_n))\,\mathrm{d}y,$$

recalling also (7.10), we infer

$$\int_{\Omega_{h_n}} Q(E(w_n))\,\mathrm{d}z \leqq \frac{b_n}{b}Q_0.$$

Thus we may apply the first part of Theorem 6.9 to deduce that for all $\alpha \in (0, \frac{1}{2})$, $h_n \to \frac{1}{b}$ in $C^{1,\alpha}([0, b])$ and ∇w_n is equibounded in $C^{0,\alpha}(\overline{\Omega}_{h_n})$. Let us now show that $\nabla w_n \to \nabla u_{e_0}$ in $L^2_{\mathrm{loc}}((0, b) \times (0, \frac{1}{b}); \mathbb{M}^{2\times 2})$ and that

$$\lim_{h\to\infty} \int_{\Omega_{h_n}} Q(E(w_n))\, \mathrm{d}z = Q_0. \tag{7.12}$$

This will enable us to use also the second part of the statement of Theorem 6.9. To this aim, notice that we may extend v_n in $(0, b_0) \times (\frac{d_n}{b_0} + 2\delta_n)$ in such a way that the the gradients of these extensions are equibounded in $C^{0,\alpha}([0, b_0] \times [\frac{d_n}{b_0} - 2\delta_n, \frac{d_n}{b_0} + 2\delta_n])$. We shall still denote by v_n the resulting extensions. By the minimality of v_n in Ω_{g_n} and of $\sqrt{\frac{b_n}{b}} u_{e_0}$ in $(0, b) \times (\frac{bd_n}{b_n^2} - \frac{b\delta_n}{b_n})$, we get

$$\int_{\Omega_{g_n}} \mathbb{C}E(v_n) : E\Big(v_n - \sqrt{\frac{b_n}{b}} u_{e_0}\Big)\, \mathrm{d}z = 0,$$
$$\int_0^b \mathrm{d}x \int_0^{\frac{bd_n}{b_n^2} - \frac{b\delta_n}{b_n}} \mathbb{C}\Big[\sqrt{\frac{b_n}{b}} E_0\Big] : E\Big(v_n - \sqrt{\frac{b_n}{b}} u_{e_0}\Big)\, \mathrm{d}y = 0$$

and thus, subtracting the two equations,

$$\int_0^b \mathrm{d}x \int_0^{\frac{bd_n}{b_n^2} - \frac{b\delta_n}{b_n}} Q\Big(E(v_n) - \sqrt{\frac{b_n}{b}} E_0\Big)\, \mathrm{d}y$$
$$= -\int_0^b \mathrm{d}x \int_{\frac{bd_n}{b_n^2} - \frac{b\delta_n}{b_n}}^{g_n(x)} \mathbb{C}E(v_n) : E\Big(v_n - \sqrt{\frac{b_n}{b}} u_{e_0}\Big)\, \mathrm{d}y.$$

By using the uniform bounds on the $C^{0,\alpha}$-norm of ∇v_n we first deduce that

$$\int_0^b \mathrm{d}x \int_0^{\frac{1}{b} - \frac{b\delta_n}{b_n}} Q\Big(E(w_n) - \sqrt{\frac{b_n}{b}} E_0\Big)\, \mathrm{d}y \to 0.$$

Again using the uniform bounds on the $C^{0,\alpha}$-norm of ∇w_n and the fact that $h_n \to \frac{1}{b}$ in $L^1(0, b)$, we conclude

$$\lim_{n\to\infty} \int_{\Omega_{h_n}} Q(E(w_n) - E_0)\, \mathrm{d}z = 0,$$

which, in turn, implies (7.12) and, by Korn's inequality, that $\nabla w_n \to \nabla u_{e_0}$ in $L^2_{\mathrm{loc}}((0, b) \times (0, \frac{1}{b}); \mathbb{M}^{2\times 2})$. Therefore, from the second part of Theorem 6.9 we may deduce that $h_n \to \frac{1}{b}$ in $W^{2,\infty}(0, b)$, or, equivalently, $\|g_n - \frac{bd_n}{b_n^2}\|_{W^{2,\infty}(0,b)} \to 0$. Recalling that $F_b(g_n, v_n) < F_b(\frac{bd_n}{b_n^2}, \sqrt{\frac{b_n}{b}} u_{e_0})$, we have a contradiction to (7.5). If instead $F_b(g_n, v_n) = F_b(\hat{g}_n, \hat{v}_n)$, we may reproduce the same argument as before, with g_n replaced by $\hat{g}_n$, to deduce that $\|\hat{g}_n - \frac{bd_n}{b_n^2}\|_{W^{2,\infty}(0,b)} \to 0$. Again, we have reached a contradiction to (7.5), since $\hat{g}_n \neq \frac{bd_n}{b_n^2}$.

Step 2 We now prove that for all $b \in (0, b_0]$ and all $(g, v) \in X(e_0; 0, b)$ the implication (7.6) holds with the same $\bar{\delta}$. To this aim, let us fix $b \in (0, b_0/2)$, $(g, v) \in X(e_0; 0, b)$ with $\sup_{[0,b]} |g - d/b| < \bar{\delta}$ and let m the smallest integer such that $mb \in [b_0/2, b_0]$. Let us now extend g by periodicity to the interval $(0, mb)$ and denote this extension by $\tilde{g}$. Similarly, let us extend v to $\{(x, y) : x \in (0, mb), 0 < y < \tilde{g}(x)\}$, by setting

$$\tilde{v}(x, y) := u_{e_0}(x, y) + \Big(v\Big(x - \Big[\frac{x}{b}\Big]b, y\Big) - u_{e_0}\Big(x - \Big[\frac{x}{b}\Big]b, y\Big)\Big),$$

where $\left[\frac{x}{b}\right]$ denotes the integer part of $\frac{x}{b}$. Clearly $\sup_{[0,mb]} |\tilde{g} - d/b| < \bar{\delta}$ and

$$F_{mb}(\tilde{g}, \tilde{v}) = m F_b(g, v).$$

Therefore, since by Step 1 $F_{mb}(\tilde{g}, \tilde{v}) > F_{mb}(d/b, u_{e_0}) = m F_b(d/b, u_{e_0})$, we conclude that $F_b(g, v) > F_b(d/b, u_{e_0})$, as claimed.

Step 3 Let us now prove that for $b < b_0$ sufficiently small and for all $d > 0$ the flat configuration $(d/b, u_{e_0})$ is a global minimizer for F_b. To this aim, let us fix $\rho := 1/Q_0$. Without loss of generality we may assume $\bar{\delta}$ in (7.6) to be smaller than ρ.

We argue by contradiction, by assuming that there exists a non-flat minimal configuration $(g, v) \in X(e_0; 0, b)$ such that $\int_0^b g = d$. By Steps 1 and 2 it then follows that $\sup_{[0,b]} |g - d/b| \geqq \bar{\delta}$. Moreover, since $\int_0^b g \, dx = d$, there exists $x \in [0, b)$ such that $g(x) = d/b$. Hence,

$$\operatorname*{osc}_{[0,b)} g \geqq \bar{\delta}. \tag{7.13}$$

Let $(x_0, y_0) \in \Gamma_g$ be such that $y_0 = \max\{y : (x, y) \in \Gamma_g,\ x \in [0, b)\}$. By translation invariance of F, we may assume that $x_0 = b/2$. By Remark 6.8 and the choice of ρ, we know that there exists a ball $B_\rho(z) \subset \Omega_g^\#$ such that $z_0 \in \partial B_\rho(z)$. Thus $B_\rho(z)$ touches z_0 at its highest point, that is, $z = (b/2, y_0 - \rho)$. Call $\tilde{g}$ the function whose graph represents the upper half of $\partial B_\rho(z)$. If $b < 2\rho$ then $\tilde{g} \leqq g \leqq y_0 = \tilde{g}(b/2)$ and, in turn, an elementary calculation

$$\operatorname*{osc}_{[0,b)} g \leqq \operatorname*{osc}_{[0,b)} \tilde{g} = \rho - \sqrt{\rho^2 - \frac{b^2}{4}} < \bar{\delta},$$

where the last inequality holds provided that $b < 2\sqrt{2\rho\bar{\delta} - \bar{\delta}^2}$. Since this contradicts (7.13), the claim is proved.

Step 4 Set

$$b_{\text{crit}} := \sup\{b > 0 : d_{\text{glob}}(b) = +\infty\}.$$

By Step 3 and Theorem 2.9 we have $0 < b_{\text{crit}} \leqq \frac{\pi}{4} \frac{2\mu+\lambda}{e_0^2 \mu(\mu+\lambda)}$. To conclude the proof of the theorem it is enough to show that

$$d_{\text{glob}}(b) = +\infty \qquad \text{for all } 0 < b \leqq b_{\text{crit}}. \tag{7.14}$$

We introduce the following temporary notation. Given $b > 0$ and $\gamma > 0$, we let $F_{b,\gamma}$ be the functional defined on $X(e_0; 0, b)$ as

$$F_{b,\gamma}(g, v) := \int_{\Omega_g} Q(E(v))\,dz + \gamma\mathcal{H}^1(\Gamma_g) + 2\gamma\mathcal{H}^1(\Sigma_g),$$

with Ω_g, Γ_g, and Σ_g defined as in (2.2), (2.3), and (2.4), respectively.

Note now that by Lemma 7.4 and from the definition of b_{crit} we have

$$d_{\text{glob}}(b_{\text{crit}}) = +\infty. \tag{7.15}$$

Let $\gamma \in (0, 1)$, and for all $(g, v) \in X(e_0; 0, b_{\text{crit}})$ let $(g_\gamma, v_\gamma) \in X(e_0; 0, \gamma b_{\text{crit}})$ be defined as

$$g_\gamma(x) := \gamma g\Big(\frac{x}{\gamma}\Big) \qquad v_\gamma(x, y) := \gamma v\Big(\frac{x}{\gamma}, \frac{y}{\gamma}\Big).$$

Fix $d > 0$ and note that $((d/b_{\text{crit}})_\gamma, (u_{e_0})_\gamma) = (\gamma d/b_{\text{crit}}, u_{e_0})$. As $(d/b_{\text{crit}}, u_{e_0})$ is a b_{crit}-periodic global minimizer for $F_{b_{\text{crit}},1}$ thanks to (7.15), a rescaling argument yields

$$F_{\gamma b_{\text{crit}},\gamma}(\gamma d/b_{\text{crit}}, u_{e_0}) \leqq F_{\gamma b_{\text{crit}},\gamma}(g_\gamma, v_\gamma) \tag{7.16}$$

for all $(g, v) \in X(e_0; 0, b_{\text{crit}})$ with $\int_0^{b_{\text{crit}}} g\,dx = d$. Since

$$F_{\gamma b_{\text{crit}},\gamma}(\gamma d/b_{\text{crit}}, u_{e_0}) = Q_0\gamma^2 d + \gamma^2 b_{\text{crit}},$$

it follows from (7.16) and the trivial inequality

$$\gamma b_{\text{crit}} \leqq \mathcal{H}^1(\Gamma_{g_\gamma}) + 2\mathcal{H}^1(\Sigma_{g_\gamma})$$

that

$$F_{\gamma b_{\text{crit}},1}(\gamma d/b_{\text{crit}}, u_{e_0}) \leqq F_{\gamma b_{\text{crit}},1}(g_\gamma, v_\gamma)$$

for all $(g, v) \in X(e_0; 0, b_{\text{crit}})$, with $\int_0^{b_{\text{crit}}} g\,dx = d$; that is, $(\gamma d/b_{\text{crit}}, u_{e_0})$ is a global minimizer for $F_{\gamma b_{\text{crit}},1}$ among all pairs $(k, w) \in X(e_0; 0, \gamma b_{\text{crit}})$ such that $\int_0^{\gamma b_{\text{crit}}} k\,dx = \gamma^2 d$. From the arbitrariness of $d > 0$ it follows that $d_{\text{glob}}(\gamma b_{\text{crit}}) = +\infty$. As this is true for all $\gamma > 0$, we have established (7.14) and we have concluded the proof of the theorem.

Let us now prove that the critical thickness for the global minimality of the flat configuration $d_{\text{glob}}(b)/b$ tends to zero as $b \to \infty$, while the critical thickness for the local minimality $d_{\text{loc}}(b)/b$ is always bounded away from zero.

Proof of Proposition 2.12. From Corollary 5.3, there exists a constant $c(\lambda, \mu)$ such that $K(y) \leqq c(\lambda, \mu)y$. Therefore, from (2.16) we get that if $b > \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$, then

$$\frac{d_{\text{loc}}(b)}{b^2} \geqq \frac{1}{2\pi c(\lambda, \mu)} K\Big(\frac{2\pi d_{\text{loc}}(b)}{b^2}\Big) = \frac{c_0(\lambda, \mu)}{be_0^2},$$

thus proving (2.18).

Let us now show that the critical thickness $d_{\text{glob}}(b)/b$ for the global minimality tends to zero as $b \to +\infty$. Given a constant $s > 0$, we have to prove that the flat configuration (s, u_{e_0}) is not a global minimizer over $X(e_0; 0, b)$ if b is large enough. For $\alpha \in (0, 1)$ (to be chosen later) we consider the competitor $(g_b, u_b) \in X(e_0; 0, b)$ defined as

$$g_b(x) := \begin{cases} 0 & \text{for } x = 0, \\ \dfrac{s}{\alpha} & \text{if } 0 < x < \alpha b, \\ 0 & \text{if } \alpha b \leqq x < b \end{cases} \quad \text{and} \quad u_b(x, y) := (e_0 w_b(x, y), 0) + u_{e_0}(x, y),$$

where $w_b(x, y) := -\eta \frac{s}{\alpha \cosh\left(\frac{\alpha^2 b}{s}\right)} \sinh\left(\frac{\alpha x}{s}\right) \sin\left(\frac{\alpha \pi y}{s}\right)$, with $\eta \in (0, 1)$ to be chosen later. Note that

$$F_b(g_b, u_b) = \int_0^{\frac{s}{\alpha}} \int_0^{\alpha b} \left(\mu |E(u_b)|^2 + \frac{\lambda}{2} (\operatorname{div} u_b)^2\right) \mathrm{d}x \, \mathrm{d}y + \frac{2s}{\alpha} + b,$$

while

$$F_b(s, u_{e_0}) = \left(\mu |E_0|^2 + \frac{\lambda}{2} (\operatorname{div} u_{e_0})^2\right) sb + b.$$

Therefore, to prove that $F_b(g_b, u_b) < F_b(s, u_{e_0})$ for b large enough it suffices to show that

$$\begin{aligned} \operatorname{Gap}(b) &:= \int_0^{\frac{s}{\alpha}} \int_0^{\alpha b} \left(\mu\left(|E(u_b)|^2 - |E_0|^2\right) + \frac{\lambda}{2}\left((\operatorname{div} u_b)^2 - (\operatorname{div} u_{e_0})^2\right)\right) \mathrm{d}x \, \mathrm{d}y \\ &< -\frac{2s}{\alpha}. \end{aligned} \tag{7.17}$$

Indeed, using the definition of w_b, after some lengthy but straightforward computations, we get

$$\begin{aligned} \frac{\operatorname{Gap}(b)}{e_0^2} &= \left(\mu + \frac{\lambda}{2}\right) \int_0^{\frac{s}{\alpha}} \int_0^{\alpha b} \left(\frac{\partial w_b}{\partial x}\right)^2 \mathrm{d}x \, \mathrm{d}y + \frac{\mu}{2} \int_0^{\frac{s}{\alpha}} \int_0^{\alpha b} \left(\frac{\partial w_b}{\partial y}\right)^2 \mathrm{d}x \, \mathrm{d}y \\ &\quad + \frac{4\mu(\mu + \lambda)}{2\mu + \lambda} \int_0^{\frac{s}{\alpha}} \int_0^{\alpha b} \frac{\partial w_b}{\partial x} \, \mathrm{d}x \, \mathrm{d}y \\ &= \left(\mu + \frac{\lambda}{2}\right) \frac{\eta^2}{\cosh^2\left(\frac{\alpha^2 b}{s}\right)} \int_0^{\alpha b} \cosh^2\left(\frac{\alpha x}{s}\right) \mathrm{d}x \int_0^{\frac{s}{\alpha}} \sin^2\left(\frac{\alpha \pi y}{s}\right) \mathrm{d}y \\ &\quad + \frac{\mu}{2} \frac{\eta^2 \pi^2}{\cosh^2\left(\frac{\alpha^2 b}{s}\right)} \int_0^{\alpha b} \sinh^2\left(\frac{\alpha x}{s}\right) \mathrm{d}x \int_0^{\frac{s}{\alpha}} \cos^2\left(\frac{\alpha \pi y}{s}\right) \mathrm{d}y \\ &\quad - \frac{4\mu(\mu + \lambda)}{2\mu + \lambda} \frac{\eta}{\cosh\left(\frac{\alpha^2 b}{s}\right)} \int_0^{\alpha b} \cosh\left(\frac{\alpha x}{s}\right) \mathrm{d}x \int_0^{\frac{s}{\alpha}} \sin\left(\frac{\alpha \pi y}{s}\right) \mathrm{d}y \\ &= \left(\mu + \frac{\lambda}{2}\right) \frac{\eta^2}{\cosh^2\left(\frac{\alpha^2 b}{s}\right)} \left[\frac{s}{2\alpha} \sinh\left(\frac{\alpha^2 b}{s}\right) \cosh\left(\frac{\alpha^2 b}{s}\right) + \frac{\alpha b}{2}\right] \frac{s}{2\alpha} \end{aligned}$$

$$+\frac{\mu}{2}\frac{\eta^2\pi^2}{\cosh^2\left(\frac{\alpha^2 b}{s}\right)}\left[\frac{s}{2\alpha}\sinh\left(\frac{\alpha^2 b}{s}\right)\cosh\left(\frac{\alpha^2 b}{s}\right)-\frac{\alpha b}{2}\right]\frac{s}{2\alpha}$$
$$-\frac{8\mu(\mu+\lambda)}{2\mu+\lambda}\frac{\eta}{\cosh\left(\frac{\alpha^2 b}{s}\right)}\frac{s^2}{\alpha^2\pi}\sinh\left(\frac{\alpha^2 b}{s}\right).$$

Letting $b\to+\infty$ we obtain

$$\lim_{b\to+\infty}\mathrm{Gap}(b)=\frac{\eta e_0^2 s^2}{\alpha^2}\left[\left(\mu+\frac{\lambda}{2}\right)\frac{\eta}{4}+\frac{\mu\pi^2\eta}{8}-\frac{8\mu(\mu+\lambda)}{2\mu+\lambda}\frac{1}{\pi}\right]. \tag{7.18}$$

We choose $\eta\in(0,1)$ so small that

$$c_\eta:=-\left[\left(\mu+\frac{\lambda}{2}\right)\frac{\eta}{4}+\frac{\mu\pi^2\eta}{8}-\frac{8\mu(\mu+\lambda)}{2\mu+\lambda}\frac{1}{\pi}\right]>0.$$

It is now clear that if we also choose $\alpha<\frac{\eta e_0^2 c_\eta s}{2}$, by (7.18) we obtain (7.17).

The next lemma is needed in the proof of Theorem 2.14. This is the only point of the paper where we assume the condition $\lambda\geqq-\frac{17}{18}\mu$.

Lemma 7.5. *Assume* $\lambda\geqq-\frac{17}{18}\mu$ *and set* $b_0:=\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$. *Then*

$$d_{\mathrm{loc}}(b_0)<\frac{\pi}{8}b_0^2.$$

Proof. By Theorem 2.9

$$K\left(\frac{2\pi\, d_{\mathrm{loc}}(b_0)}{b_0^2}\right)=\frac{\pi}{4}.$$

Since the function K is strictly increasing by Corollary 5.3, it is enough to show that

$$K\left(\frac{2\pi}{b_0^2}\frac{\pi}{8}b_0^2\right)=K\left(\frac{\pi^2}{4}\right)>\frac{\pi}{4}.$$

By the definition of K (see (2.14)), this inequality is true if $J(\pi^2/4)>\pi/4$. In turn, since under our assumptions $\nu_p\in[-8.5,1/2)$, this amounts to proving that

$$\psi(\nu_p):=\pi(1-\nu_p)^2$$
$$+\left[\frac{\pi}{4}\sinh^2\left(\frac{\pi^2}{4}\right)-\sinh\left(\frac{\pi^2}{4}\right)\cosh\left(\frac{\pi^2}{4}\right)\right](3-4\nu_p)+\frac{\pi^5}{64}-\frac{\pi^2}{4}<0$$

for all $\nu_p\in[-8.5,1/2)$. Since ψ is convex, the conclusion follows from the fact that both $\psi(-8.5)$ and $\psi(1/2)$ are negative, as can be checked by elementary calculations.

Proof of Theorem 2.14. We start by proving part (i). Recall that by Remark 6.8, if (h, u) is a b-global minimizer for some $b > 0$, then $\Omega_h^\# \cup (\mathbb{R} \times (-\infty, 0])$ satisfies an interior ball condition with radius $\rho = \frac{1}{Q_0}$, that is, for every $z \in \widetilde{\Gamma}_h^\#$ there exists a ball $B_\rho(z_0) \subset \Omega_h^\# \cup (\mathbb{R} \times (-\infty, 0])$ such that $z \in \partial B_\rho(z_0)$. Notice that the assumption on b is equivalent to $Q_0 = 2e_0^2 \frac{\mu(\mu+\lambda)}{2\mu+\lambda} < \frac{2}{b}$. Assume now that either $z \in \Sigma_h$ or $z \in \Gamma_h$ with a vertical tangent. In both cases the tangent inner ball can be chosen with $z_0 = z \pm (\rho, 0)$. Without loss of generality, $z_0 = z + (\rho, 0)$. Since $\rho > \frac{b}{2}$, the point $z + (b, 0)$ belongs to the interior of $B_\rho(z_0)$, a contradiction to the fact that by periodicity $z + (b, 0) \in \Gamma_h^\#$. This contradiction shows that $\widetilde{\Gamma}_h$ does not contain vertical segments. Due to Theorem 2.7(ii), part (i) of the statement follows.

To prove part (ii) we argue by contradiction. Assume that there exist $b_n \to \frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$ with $b_n \geqq \frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$, $d_n \in [d_{\mathrm{glob}}(b_n), d_{\mathrm{glob}}(b_n) + \frac{1}{n})$, b_n-periodic global minimizers (h_n, u_n) with $|\Omega_{h_n}| = d_n$, and balls $B_\rho(z_n) \subset \Omega_{h_n}^\# \cup (\mathbb{R} \times (-\infty, 0])$ with $\rho = \frac{1}{Q_0} = \frac{2\mu+\lambda}{2e_0^2\mu(\mu+\lambda)}$, such that at least one of the points $z_n + (\rho, 0)$ and $z_n - (\rho, 0)$ belongs to $\widetilde{\Gamma}_{h_n}^\#$. By Lemma 7.2 we may assume that there exists a b_0-global minimizer $(h, u) \in X(e_0; 0, b_0)$, with $b_0 = \frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$, such that (7.1) holds. Moreover, there exists a ball $B_\rho(z_0) \subset \Omega_h^\# \cup (\mathbb{R} \times (-\infty, 0])$ such that both points $z_0 + (\rho, 0)$ and $z_0 - (\rho, 0)$ belong to $\widetilde{\Gamma}_h^\#$ (here we are using the fact that $2\rho = b_0$ and the b_0-periodicity). Without loss of generality, by translation invariance we may assume that $z_0 = (\frac{b_0}{2}, y_0)$. Hence, in particular, the upper half ball $B_\rho^+(z_0) \subset \Omega_h$. On the other hand, by Lemmas 7.4 and 7.5 we have that $\lim_n d_n = |\Omega_h| \leqq d_{\mathrm{glob}}(b_0) \leqq d_{\mathrm{loc}}(b_0) < \frac{\pi}{8} b_0^2 = |B_\rho^+(z_0)|$, a contradiction.

Next we show the existence of nontrivial analytic configurations.

Proof of Theorem 2.16. We start by proving that

$$\lim_{b \to b_{\mathrm{crit}}^+} d_{\mathrm{glob}}(b) = +\infty. \tag{7.19}$$

Recall that since $b_{\mathrm{crit}} \leqq \frac{\pi}{4} \frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$, by Remark 5.2 and using the fact that $0 < K < 1$ (see Corollary 5.3), for $M > 0$ there exists $c_M > 0$ such that for all $d \in [0, M]$ and $\varphi \in \widetilde{H}^1_\#(0, b_{\mathrm{crit}})$

$$\partial^2 F\Big(\frac{d}{b_{\mathrm{crit}}}, u_{e_0}\Big)[\varphi] \geqq c_M \|\varphi\|^2_{H^1(0, b_{\mathrm{crit}})}. \tag{7.20}$$

Assume by contradiction that there exist a sequence $b_n \to b_{\mathrm{crit}}^+$ such that $d_{\mathrm{glob}}(b_n) < M/2$ for all n and for some $M > 0$. Then, Remark 5.2, together with (7.20), implies

$$\partial^2 F\Big(\frac{d}{b_n}, u_{e_0}\Big)[\varphi] \geqq \frac{c_M}{2} \|\varphi\|^2_{H^1(0, b_n)}$$

for all $d \in [0, M]$ and $\varphi \in \widetilde{H}^1_\#(0, b_n)$, provided that n is large enough. Arguing as in the proof of Theorem 2.11(ii), we may conclude that there exists $\delta > 0$ such that if $d \in [0, M]$ and $(h, u) \in X(e_0; 0, b_n)$ with $0 < \sup_{[0,b_n]} |h - d/b_n| < \delta$, then $F(d/b_n, u_{e_0}) < F(h, u)$, provided n is sufficiently large. Fix $M/2 < d < M$ and let $(h_n, u_n) \in X(e_0; 0, b_n)$ be a non-flat minimal configuration such that $\int_0^{b_n} h_n \, dx = d$. In particular, $\sup_{[0,b_n]} |h_n - d/b_n| \geqq \delta$ for n large enough. By Lemma 7.2, we deduce the existence of a b_{crit}-periodic global minimizer (h, u) such that (up to a subsequence) (7.1) holds and $\sup_{[0,b_{\text{crit}}]} |h - d/b_{\text{crit}}| \geqq \delta$. This contradicts the fact that $d_{\text{glob}}(b_{\text{crit}}) = +\infty$ and concludes the proof of (7.19).

To prove the theorem choose $b_1 \in (b_{\text{crit}}, b_0)$, where $b_0 := \frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$, such that

$$\frac{d_{\text{glob}}(b)b_0}{\sqrt{(d_{\text{glob}}(b))^2 + b_0^4}} > b \quad \text{for all } b \in (b_{\text{crit}}, b_1). \tag{7.21}$$

Note that this is possible thanks to (7.19). Now fix any such b and let (h, u) be a non-flat b-periodic global minimizer, with $|\Omega_h| =: d > d_{\text{glob}}(b)$. If $h > 0$ then h is analytic by Theorems 2.14(i) and 2.7. If $h(x) = 0$ for some $x \in [0, b)$, since $\max_{[0,b]} h \geqq d/b$ and h is of class C^1 by Theorem 2.14, there exists $\bar{x} \in [0, b)$ such that $h'(\bar{x}) = d/b^2$. By Remark 6.8, there exists a ball $B_\rho(z_0) \in \Omega_h^\# \cup (\mathbb{R} \times (-\infty, 0])$ such that $(\bar{x}, h(\bar{x})) \in \partial B_\rho(z_0)$ with $\rho = b_0/2$. Since Γ_h and $\partial B_\rho(z_0)$ share the same tangent at $(\bar{x}, h(\bar{x}))$, an elementary calculation shows that the horizontal chord of $B_\rho(z_0)$ starting from $(\bar{x}, h(\bar{x}))$ has length equal to $\frac{db_0}{\sqrt{d^2+b^4}}$. As

$$\frac{db_0}{\sqrt{d^2 + b^4}} > \frac{d_{\text{glob}}(b)b_0}{\sqrt{(d_{\text{glob}}(b))^2 + b_0^4}} > b$$

by (7.21), we obtain a contradiction to the fact that the chord must lie inside $\Omega_h^\#$.

Finally, we prove the nonuniqueness result stated in Theorem 2.13.

Proof of Theorem 2.13. The argument is similar to the one used in the second part of the previous proof. Since under our assumption the second variation $\partial^2 F(d/b, u_{e_0})$ is uniformly positive definite as d varies in a sufficiently small neighborhood of $d_{\text{glob}}(b)$, choosing $d_n := d_{\text{glob}}(b) + \frac{1}{n}$ any non-flat b-periodic global minimizer (h_n, u_n), with $|\Omega_{h_n}| = d_n$, satisfies $\sup_{[0,b]} |h_n - d_n/b| \geqq \delta$ for some $\delta > 0$ and for n large enough. Therefore, up to a subsequence, by Lemma 7.2 the sequence (h_n, u_n) converges to a non-flat b-periodic global minimizer (h, u) with $|\Omega_h| = d_{\text{glob}}(b)$.

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8. Appendix

8.1. Fractional Sobolev Spaces and Trace Theorems

Throughout this section h denotes a strictly positive b-periodic function belonging to $C^1(\mathbb{R})$. To simplify the notation introduced in Section 2.1, we write Γ and $\Gamma^\#$ in place of Γ_h and $\Gamma_h^\#$, respectively. For $p \geqq 1$ we let $L^p(\Gamma)$ denote the space of all functions $u : \Gamma \to \mathbb{R}$ such that $\int_\Gamma |u|^p \, d\mathcal{H}^1 < \infty$.

Definition 8.1. Let $0 < s < 1$ and $1 < p < \infty$. We denote by $W^{s,p}(\Gamma)$ the fractional Sobolev space consisting of all functions $u \in L^p(\Gamma)$ such that

$$[u]_{s,p,\Gamma} := \left(\int_\Gamma \int_\Gamma \frac{|u(z) - u(w)|^p}{|z - w|^{1+sp}} \, d\mathcal{H}^1(w) \, d\mathcal{H}^1(z) \right)^{\frac{1}{p}} < \infty, \tag{8.1}$$

and we set $\|u\|_{W^{s,p}(\Gamma)} := \|u\|_{L^p(\Gamma)} + [u]_{s,p,\Gamma}$. We shall refer to $[u]_{s,p,\Gamma}$ as the Gagliardo seminorm of u.

By $W^{s,p}_\#(\Gamma)$, we denote the subspace (endowed with the same norm) of all functions in $W^{s,p}(\Gamma)$ whose b-periodic extension to $\Gamma^\#$ belongs to $W^{s,p}_{\text{loc}}(\Gamma^\#)$.

The spaces $W^{-s,\frac{p}{p-1}}(\Gamma)$ and $W_\#^{-s,\frac{p}{p-1}}(\Gamma)$ are defined as the dual spaces of $W^{s,p}(\Gamma)$ and $W^{s,p}_\#(\Gamma)$, respectively.

Remark 8.2. Notice that from the very definition of fractional spaces it follows that if $-1 \leqq t \leqq s < 1$ and $p > 1$, then $W^{s,p}(\Gamma)$ is continuously imbedded in $W^{t,p}(\Gamma)$.

When $p = 2$ we often write $H^s(\Gamma)$ and $H^s_\#(\Gamma)$ instead of $W^{s,2}(\Gamma)$ and $W^{s,2}_\#(\Gamma)$. A similar notation will be used for their dual spaces. We recall the following classical imbedding theorem. Notice that we use the convention $W^{0,p}(\Gamma) := L^p(\Gamma)$.

Theorem 8.3. *Let $-1 \leqq t \leqq s \leqq 1$, $q \geqq p$ such that $s - 1/p \geqq t - 1/q$. Then $W^{s,p}(\Gamma)$ is continuously imbedded in $W^{t,q}(\Gamma)$. Moreover, the imbedding constant depends only on s, t, p, q, b and on the C^1-norm of h.*

This theorem can be easily deduced from [20, Theorem 1.4.4.1], with a simple change of variable, using Remark 8.2. From [20, Theorem 1.5.1.2] we also have the following trace theorem.

Theorem 8.4. *Let c_0 be a positive constant such that $\min_{[0,b]} h \geqq c_0$. If $p > 1$, then there exists a continuous linear operator $T : W^{1,p}(\Omega_h) \mapsto W^{1-\frac{1}{p},p}(\Gamma)$ such that $Tu = u|_\Gamma$ whenever u is continuous on $\overline{\Omega}_h$. Moreover, the norm of T is bounded by a constant depending only on c_0, $\|h\|_{C^1([0,b])}$, b and p.*

As an immediate consequence of the previous theorem we have that if $u \in W^{1,p}(\Omega_h)$ is b-periodic in the x-variable, then $Tu \in W_\#^{1-\frac{1}{p},p}(\Gamma)$. Conversely, any function in $W_\#^{1-\frac{1}{p},p}(\Gamma)$ is the trace of a b-periodic function in $W^{1,p}(\Omega_h)$. More precisely, we have the following result.

Theorem 8.5. *Let c_0 be as in Theorem 8.4 and $p > 1$. Then, for all $\varphi \in W_{\#}^{1-\frac{1}{p},p}(\Gamma)$ there exists $u \in W^{1,p}(\Omega_h)$, b-periodic in the x-variable, such that $Tu = \varphi$ and*

$$\|u\|_{W^{1,p}(\Omega_h)} \leqq C\|\varphi\|_{W^{1-\frac{1}{p},p}(\Gamma)}$$

for some constant C depending only on c_0, $\|h\|_{C^1([0,b])}$, b, and p.

This result is more or less standard except for the fact that here we have to guarantee the periodicity of the lifting u. This follows from the fact that u can be explicitly defined as

$$u(x,y) = \frac{1}{h(x)-y}\int_{\mathbb{R}} \rho\Big(\frac{t-x}{h(x)-y}\Big)\varphi(t,h(t))\,\mathrm{d}t,$$

where ρ is a standard mollifier. The estimate of $\|u\|_{W^{1,p}(\Omega_h)}$ can be obtained arguing as in [20, Lemma 1.4.1.4].

The next result is a simple consequence of Theorem 8.5.

Theorem 8.6. *Let c_0 and p be as above. Then, for all $u \in W^{1,p}(\Omega_h)$, b-periodic in the x-variable,*

$$\|\partial_\tau u\|_{W_{\#}^{-\frac{1}{p},p}(\Gamma)} \leqq C\|\nabla u\|_{L^p(\Omega_h;\mathbb{R}^2)}$$

for some constant C depending only on c_0, $\|h\|_{C^1([0,b])}$, b, and p.

Proof. By a density argument we may assume $u \in C^2(\overline{\Omega}_h)$. Fix $\varphi \in W_{\#}^{\frac{1}{p},\frac{p}{p-1}}(\Gamma)$. By Theorem 8.5 we may find a lifting in $W^{1,\frac{p}{p-1}}(\Omega_h)$ (still denoted by φ), b-periodic in the x-variable, such that $\|\varphi\|_{W^{1,\frac{p}{p-1}}(\Omega_h)} \leqq C\|\varphi\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma)}$. Moreover, by replacing C with a possibly larger constant, we may also assume that $\varphi(x,0)=0$. Hence,

$$\begin{aligned}
\int_\Gamma \partial_\tau u\varphi\,\mathrm{d}\mathcal{H}^1 &= \int_\Gamma \varphi\Big(-\frac{\partial u}{\partial y},\frac{\partial u}{\partial x}\Big)\cdot\nu\,\mathrm{d}\mathcal{H}^1\\
&= \int_{\Omega_h} \operatorname{div}\Big(-\varphi\frac{\partial u}{\partial y},\varphi\frac{\partial u}{\partial x}\Big)\,\mathrm{d}z = \int_{\Omega_h} \nabla u\cdot\Big(\frac{\partial\varphi}{\partial y},-\frac{\partial\varphi}{\partial x}\Big)\,\mathrm{d}z\\
&\leqq \|\nabla u\|_{L^p(\Omega_h;\mathbb{R}^2)}\|\nabla\varphi\|_{L^{\frac{p}{p-1}}(\Omega_h;\mathbb{R}^2)}\\
&\leqq C\|\nabla u\|_{L^p(\Omega_h;\mathbb{R}^2)}\|\varphi\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma)},
\end{aligned}$$

where in the second equality we have used the b-periodicity of φ, u and h. This concludes the proof.

Combining the previous result with the lifting Theorem 8.5, we deduce the following corollary.

Corollary 8.7. *Let $p > 1$. Then, for all $u \in W_{\#}^{1-\frac{1}{p},p}(\Gamma)$ we have*

$$\|\partial_\tau u\|_{W_{\#}^{-\frac{1}{p},p}(\Gamma)} \leqq C\|u\|_{W^{1-\frac{1}{p},p}(\Gamma)},$$

for some constant C depending only on $\|h\|_{C^1([0,b])}$, b, and p.

We conclude this subsection by stating a lemma that is used several times in Section 4.

Lemma 8.8. *Let* $-1 < s < 1$, $p > 1$, *and let* u *be a smooth function. The following two statements hold.*

(i) *If* $a \in C^1(\Gamma)$, *then*

$$\|ua\|_{W^{s,p}(\Gamma)} \leqq C\|a\|_{C^1(\Gamma)}\|u\|_{W^{s,p}(\Gamma)},$$

for some constant C *depending only on* s *and* p.

(ii) *If* $\Phi : \Gamma \to \Phi(\Gamma)$ *is a* C^1*-diffeomorphism, then*

$$\|u \circ \Phi^{-1}\|_{W^{s,p}(\Phi(\Gamma))} \leqq C\|u\|_{W^{s,p}(\Gamma)},$$

for some constant C *depending only on the* C^1*-norms of* Φ *and* Φ^{-1}, *on* s, *and on* p.

If s is positive, the two statements follow from the definition (8.1) of the Gagliardo seminorm. If s is negative, they follow from the previous case by duality. We leave the easy details to the reader.

8.2. A Regularity Result for the Lamé System

We prove here a regularity result for the Lamé system with homogeneous Neumann boundary conditions, which is used in the proof of Theorem 6.9.

Proposition 8.9. *Let* $(h, u) \in Y(e_0; 0, b)$ *satisfy the first three equations in* (2.11). *Assume that there exist* $z_0 = (x_0, h(x_0)) \in \Gamma_h$ *and* $r_0 \in (0, h(x_0)/2)$ *such that* $h \in C^{1,\alpha}([x_0 - 2r_0, x_0 + 2r_0])$ *for some* $0 < \alpha < 1$. *Then, there exists a constant* C *depending only on* λ, μ, *the* $C^{1,\alpha}$*-norm of* h *in* $[x_0 - 2r_0, x_0 + 2r_0]$, *on* r_0, *and on the* L^2*-norm of* $E(u)$ *in* $\Omega_h \cap B_{2r_0}(z_0)$, *such that*

$$\|\nabla u\|_{C^{0,\alpha}(\overline{\Omega}_h \cap \overline{B}_{r_0}(z_0))} \leqq C.$$

Proof. Let Φ be a $C^{1,\alpha}$ diffeomorphism from $\overline{\Omega}_h \cap \overline{B}_{2r_0}(z_0)$ onto $\overline{U}$, where U is an open set contained in $\mathbb{R} \times (0, \infty)$, such that $\Phi(\Gamma_h \cap B_{2r_0}(z_0)) = S$ is an open segment contained in $\{y = 0\}$ and the $C^{1,\alpha}$-norms of Φ and Φ^{-1} are controlled from above by the $C^{1,\alpha}$-norm of h and r_0. Setting $\tilde{u} := u \circ \Phi^{-1}$, then it is easily checked that $\tilde{u}$ solves a linear system of the type

$$\int_U \mathbb{A}(z)\nabla\tilde{u} : \nabla w \, \mathrm{d}z = 0, \tag{8.2}$$

for all $w \in H^1(U; \mathbb{R}^2)$ vanishing in a neighborhood of $\partial U \cap \{y > 0\}$, where the fourth order tensor $\mathbb{A}$ is of class $C^{0,\alpha}$ in $\overline{U}$, with the $C^{0,\alpha}$-norm controlled by the $C^{1,\alpha}$-norm of h.

Let us now fix $\overline{z} \in S$ and $R > 0$ such that the half ball $B^+_{2R}(\overline{z}) \subset U$.

For any point $z_0 \in \overline{B}^+_R(\bar{z})$ and any $0 < r < R$, denote by v the unique solution to the constant coefficients system

$$\int_{B_r^+(z_0)} \mathbb{A}(z_0)\nabla v : \nabla w \, \mathrm{d}z = 0 \tag{8.3}$$

for all $w \in H^1(B_r^+(z_0); \mathbb{R}^2)$, with $w = 0$ on $\partial B_r^+(z_0) \setminus S$, such that $v = \tilde{u}$ on $\partial B_r^+(z_0) \setminus S$. Subtracting the two equations (8.2), (8.3), choosing $w = \tilde{u} - v$ and using the fact that $\mathbb{A}$ is $C^{0,\alpha}$, one easily gets that

$$\int_{B_r^+(z_0)} |\nabla\tilde{u} - \nabla v|^2 \, \mathrm{d}z \leqq cr^{2\alpha} \int_{B_r^+(z_0)} |\nabla\tilde{u}|^2 \, \mathrm{d}z, \tag{8.4}$$

for some positive constant c. Moreover, since v solves a linear system with constant coefficients, standard elliptic estimates yield that for all $0 < \rho < r$,

$$\begin{gathered}\int_{B_\rho^+(z_0)} |\nabla v|^2 \, \mathrm{d}z \leqq c\left(\frac{\rho}{r}\right)^2 \int_{B_r^+(z_0)} |\nabla v|^2 \, \mathrm{d}z, \\ \int_{B_\rho^+(z_0)} |\nabla v - (\nabla v)_{\rho,z_0}|^2 \, \mathrm{d}z \leqq c\left(\frac{\rho}{r}\right)^4 \int_{B_r^+(z_0)} |\nabla v - (\nabla v)_{r,z_0}|^2 \, \mathrm{d}z.\end{gathered} \tag{8.5}$$

Thus, from the first equation in (8.5), recalling (8.4), we get that for all $0 < \rho < r$

$$\begin{aligned}\int_{B_\rho^+(z_0)} |\nabla\tilde{u}|^2 \mathrm{d}z &\leqq 2\int_{B_\rho^+(z_0)} |\nabla v|^2 \mathrm{d}z + 2\int_{B_\rho^+(z_0)} |\nabla\tilde{u} - \nabla v|^2 \mathrm{d}z \\ &\leqq c\left(\frac{\rho}{r}\right)^2 \int_{B_r^+(z_0)} |\nabla v|^2 \mathrm{d}z + cr^{2\alpha} \int_{B_r^+(z_0)} |\nabla\tilde{u}|^2 \mathrm{d}z \\ &\leqq c'\left(\frac{\rho}{r}\right)^2 \int_{B_r^+(z_0)} |\nabla\tilde{u}|^2 \mathrm{d}z + c' r^{2\alpha} \int_{B_{2R}^+(\bar{z})} |\nabla\tilde{u}|^2 \mathrm{d}z,\end{aligned} \tag{8.6}$$

for some constant c' depending ultimately only on R,λ, μ, and the $C^{1,\alpha}$-norm of h. From this estimate, a standard iteration argument (see for instance [4, Lemma 7.54]) yields that for any $\delta > 0$ there exists a constant c depending only on c', α, δ, such that for all $0 < r < R$ one has

$$\int_{B_r^+(z_0)} |\nabla\tilde{u}|^2 \, \mathrm{d}z \leqq cr^{2-\delta} \int_{B_{2R}^+(\bar{z})} |\nabla\tilde{u}|^2 \mathrm{d}z. \tag{8.7}$$

Using this estimate and the second inequality in (8.5), and arguing as in the proof of (8.6), we then get that for all $z_0 \in \overline{B}^+_R(\bar{z})$ and all $0 < \rho < r < R$

$$\begin{aligned}&\int_{B_\rho^+(z_0)} |\nabla\tilde{u} - (\nabla\tilde{u})_{\rho,z_0}|^2 \mathrm{d}z \\ &\leqq c\int_{B_\rho^+(z_0)} |\nabla v - (\nabla v)_{\rho,z_0}|^2 \mathrm{d}z + c\int_{B_\rho^+(z_0)} |\nabla\tilde{u} - \nabla v|^2 \mathrm{d}z \\ &\leqq c\left(\frac{\rho}{r}\right)^4 \int_{B_r^+(z_0)} |\nabla v - (\nabla v)_{r,z_0}|^2 \mathrm{d}z + cr^{2\alpha} \int_{B_r^+(z_0)} |\nabla\tilde{u}|^2 \mathrm{d}z \\ &\leqq c\left(\frac{\rho}{r}\right)^4 \int_{B_r^+(z_0)} |\nabla\tilde{u} - (\nabla\tilde{u})_{r,z_0}|^2 \mathrm{d}z + cr^{2+2\alpha-\delta} \int_{B_{2R}^+(\bar{z})} |\nabla\tilde{u}|^2 \mathrm{d}z.\end{aligned} \tag{8.8}$$

From this estimate, the iteration lemma used above yields that

$$\int_{B_r^+(z_0)} |\nabla\tilde{u} - (\nabla\tilde{u})_{r,z_0}|^2 \mathrm{d}z \leqq \; cr^{2+2\alpha-\delta} \int_{B_{2R}^+(\bar{z})} |\nabla\tilde{u}|^2 \mathrm{d}z$$

for all $0 < r < R$. From this inequality, another standard iteration argument (see [4, Theorem 7.51]) implies that $\tilde{u} \in C^{1,\alpha-\delta/2}(\overline{B}_R^+(\bar{z}))$ and that

$$\|\nabla\tilde{u}\|_{C^{0,\alpha-\delta/2}(\overline{B}_R^+(\bar{z}))} \leqq c\left(\int_{B_{2R}^+(\bar{z})} |\nabla\tilde{u}|^2 \mathrm{d}z\right)^{\frac{1}{2}}.$$

Finally, from this inequality it is clear that (8.7) holds in a stronger form, namely that for all $z_0 \in \overline{B}_R^+(\bar{z})$ and and all $0 < r < R$

$$\int_{B_r^+(z_0)} |\nabla\tilde{u}|^2 \, \mathrm{d}z \leqq cr^2 \int_{B_{2R}^+(\bar{z})} |\nabla\tilde{u}|^2 \mathrm{d}z,$$

which, in turn, implies (see (8.8)) that for all $0 < \rho < r < R$

$$\int_{B_\rho^+(z_0)} |\nabla\tilde{u} - (\nabla\tilde{u})_{\rho,z_0}|^2 \mathrm{d}z$$
$$\leqq c\left(\frac{\rho}{r}\right)^4 \int_{B_r^+(z_0)} |\nabla\tilde{u} - (\nabla\tilde{u})_{r,z_0}|^2 \mathrm{d}z + cr^{2+2\alpha} \int_{B_{2R}^+(\bar{z})} |\nabla\tilde{u}|^2 \mathrm{d}z.$$

This last inequality implies the $C^{0,\alpha}$ regularity of $\nabla\tilde{u}$, since [4, Theorem 7.51] gives that

$$\|\nabla\tilde{u}\|_{C^{1,\alpha}(\overline{B}_R^+(\bar{z}))} \leqq c\left(\int_{B_{2R}^+(\bar{z})} |\nabla\tilde{u}|^2 \mathrm{d}z\right)^{\frac{1}{2}}.$$

Hence, the proof of the proposition is concluded.

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