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Diffeomorphic Approximation of Sobolev Homeomorphisms

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Abstract

Every homeomorphism $h\colon \mathbb{X} \to \mathbb{Y}$ between planar open sets that belongs to the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, can be approximated in the Sobolev norm by $\mathscr{C}^\infty$-smooth diffeomorphisms.

1. Introduction

In a domain $\mathbb{X} \subset \mathbb{R}^n$, the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R})$, $1 \leqq p < \infty$, (also denoted $H^{1,p}$) is the completion of $\mathscr{C}^\infty$-smooth real functions having finite Sobolev norm

$$\|u\|_{\mathscr{W}^{1,p}(\mathbb{X})} = \|u\|_{\mathscr{L}^p(\mathbb{X})} + \|\nabla u\|_{\mathscr{L}^p(\mathbb{X})} < \infty.$$

The question of smooth approximation becomes more intricate for Sobolev mappings whose target is not a linear space, say a smooth manifold [12,20–22], or even for mappings between open subsets $\mathbb{X}, \mathbb{Y}$ of the Euclidean space $\mathbb{R}^n$. If a given homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ it is not obvious at all as to whether one can preserve the injectivity of the $\mathscr{C}^\infty$-smooth approximating mappings. It is rather surprising that this question remained unanswered after the global invertibility of Sobolev mappings became an issue in nonlinear elasticity [5,18,33,37]. It was formulated and promoted by John M. Ball in the following form.

Question 1. [7,8] If $h \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^n)$ is invertible, can h be approximated in $\mathscr{W}^{1,p}$ by piecewise affine invertible mappings?

J. Ball attributes this question to L.C. Evans and points out its relevance to the regularity of minimizers of neohookean energy functionals [6,10,15,17,36]. Partial results toward the Ball-Evans problem were obtained in [32] (for planar bi-Sobolev mappings that are smooth outside of a finite set) and in [11] (for planar bi-Hölder mappings, with approximation in the Hölder norm). The articles [7,35] illustrate the difficulty of preserving invertibility in the approximation process. In [26] we provided an affirmative answer to the Ball-Evans question in the planar case when $p = 2$. In the present paper we extend the result of [26] to all Sobolev classes $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ with $1 < p < \infty$. The case $p = 1$ still remains open.

Let $\mathbb{X}$ be a nonempty open set in $\mathbb{R}^2$. We study complex-valued functions $h = u + iv\colon \mathbb{X} \to \mathbb{C} \simeq \mathbb{R}^2$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$, $1 < p < \infty$. Their real and imaginary parts have well defined gradients in $\mathscr{L}^p(\mathbb{X}, \mathbb{R}^2)$

$$\nabla u\colon \mathbb{X} \to \mathbb{R}^2 \quad \text{and} \quad \nabla v\colon \mathbb{X} \to \mathbb{R}^2.$$

Next, we introduce the gradient mapping of h, by setting

$$\nabla h = (\nabla u, \nabla v)\colon \mathbb{X} \to \mathbb{R}^2 \times \mathbb{R}^2. \tag{1.1}$$

The $\mathscr{L}^p$-norm of the gradient mapping and the p-energy of h are defined by

$$\|\nabla h\|_{\mathscr{L}^p(\mathbb{X})} = \left[\int_{\mathbb{X}} \left(|\nabla u|^p + |\nabla v|^p\right)\right]^{\frac{1}{p}}, \quad \mathsf{E}_{\mathbb{X}}[h] = \mathsf{E}^p_{\mathbb{X}}[h] = \|\nabla h\|^p_{\mathscr{L}^p(\mathbb{X})}. \tag{1.2}$$

This norm is slightly different from that found in other texts in which the authors use the differential matrix of h instead of the gradient mapping, so

$$\|Dh\|_{\mathscr{L}^p(\mathbb{X})} = \left[\int_{\mathbb{X}} \left(|\nabla u|^2 + |\nabla v|^2\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}. \tag{1.3}$$

Thus our approach involves *coordinate-wise* p-harmonic mappings, which we still call p-harmonic for the sake of brevity. We shall take advantage of the gradient mapping on numerous occasions by exploring the associated *uncoupled* system of real p-harmonic equations for mappings with smallest p-energy. Our theorem reads as follows.

Theorem 1. *Let* $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ *be an orientation-preserving homeomorphism in the Sobolev space* $\mathscr{W}^{1,p}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, *defined for open sets* $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$. *Then there exist* $\mathscr{C}^\infty$*-diffeomorphisms* $h_\ell\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, $\ell = 1, 2, \dots$ *such that*

(i) $h_\ell - h \in \mathscr{W}^{1,p}_\circ(\mathbb{X}, \mathbb{R}^2)$, $\ell = 1, 2, \dots$
(ii) $\lim_{\ell\to\infty} (h_\ell - h) = 0$, *uniformly on* $\mathbb{X}$
(iii) $\lim_{\ell\to\infty} \|\nabla h_\ell - \nabla h\|_{\mathscr{L}^p(\mathbb{X})} = 0$
(iv) $\|\nabla h_\ell\|_{\mathscr{L}^p(\mathbb{X})} \leqq \|\nabla h\|_{\mathscr{L}^p(\mathbb{X})}$, *for* $\ell = 1, 2, \dots$
(v) *If h is a* $\mathscr{C}^\infty$*-diffeomorphism outside of a compact subset of* $\mathbb{X}$, *then there is a compact subset of* $\mathbb{X}$ *outside which* $h_\ell \equiv h$, *for all* $\ell = 1, 2, \dots$

A straightforward triangulation argument yields the following corollary.

Corollary 1. *Let* $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ *be an orientation-preserving homeomorphism in the Sobolev space* $\mathscr{W}^{1,p}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, *defined for open sets* $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$. *Then there exist piecewise affine homeomorphisms* $h_\ell\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, $\ell = 1, 2, \dots$ *such that*

(i) $h_\ell - h \in \mathscr{W}_\circ^{1,p}(\mathbb{X}, \mathbb{R}^2)$, $\ell = 1, 2, \dots$
(ii) $\lim_{\ell\to\infty} (h_\ell - h) = 0$, *uniformly on* $\mathbb{X}$
(iii) $\lim_{\ell\to\infty} \|\nabla h_\ell - \nabla h\|_{\mathscr{L}^p(\mathbb{X})} = 0$
(iv) *If h is affine outside of a compact subset of $\mathbb{X}$, then there is a compact subset of $\mathbb{X}$ outside which $h_\ell \equiv h$, for all $\ell = 1, 2, \dots$*

We conclude this introduction with a sketch of the proof. The construction of an approximating diffeomorphism involves five consecutive modifications of h. Steps 1, 2, and 4 are p-harmonic replacements based on the ALESSANDRINI–SIGALOTTI extension [4] of the Radó–Kneser–Choquet Theorem. The other steps involve an explicit smoothing procedure along crosscuts. For this, we have adopted some lines of arguments used in J. MUNKRES' work [34].

2. p-Harmonic mappings and preliminaries

Let Ω be a bounded domain in the complex plain $\mathbb{C} \simeq \mathbb{R}^2$. A function $u\colon \Omega \to \mathbb{R}$ in the Sobolev class $\mathscr{W}^{1,p}_{\text{loc}}(\Omega)$, $1 < p < \infty$, is called p-harmonic if

$$\Delta_p(u) := \operatorname{div} |\nabla u|^{p-2}\nabla u = 0 \tag{2.1}$$

meaning that

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle = 0 \qquad \text{for every } \varphi \in \mathscr{C}_\circ^\infty(\Omega). \tag{2.2}$$

The first observation is that the gradient map $f = \nabla u\colon \Omega \to \mathbb{R}^2$ is K-quasiregular with $1 \leqq K \leqq \max\{p-1, 1/(p-1)\}$, see [13]. Consequently $u \in \mathscr{C}^{1,\alpha}_{\text{loc}}(\Omega)$ with some $0 < \alpha = \alpha(p) \leqq 1$, cf. [38]. In fact [27] the foremost regularity of a p-harmonic function ($p \neq 2$) is $\mathscr{C}^{k,\alpha}_{\text{loc}}(\Omega)$, where the integer $k \geqq 1$ and the Hölder exponent $\alpha \in (0, 1]$ are determined by the equation

$$k + \alpha = \frac{7p - 6 + \sqrt{p^2 + 12p - 12}}{6p - 6} > 1 + \frac{1}{3}.$$

Thus, regardless of the exponent p, we have $u \in \mathscr{C}^{1,\alpha}_{\text{loc}}(\Omega)$ with $\alpha = 1/3$. Clearly, by elliptic regularity theory, outside the singular set

$$\mathcal{S} = \{z \in \Omega\colon \nabla u(z) = 0\},$$

we have $u \in \mathscr{C}^\infty(\Omega \setminus \mathcal{S})$. The singular set, being the set of zeros of a quasiregular mapping, consists of isolated points, unless $u \equiv$ const. Pertaining to regularity up to the boundary, we consider a domain Ω whose boundary near a point $z_\circ \in \partial\Omega$ is a $\mathscr{C}^\infty$-smooth arc, say $\Gamma \subset \partial\Omega$. Precisely, we assume that there exist a disk $D = D(z_\circ, \varepsilon)$ and a $\mathscr{C}^\infty$-smooth diffeomorphism $\varphi\colon D \xrightarrow{\text{onto}} \mathbb{C}$ such that

$$\varphi(D \cap \Omega) = \mathbb{C}_+ = \{z\colon \operatorname{Im} z > 0\}$$
$$\varphi(\Gamma) = \mathbb{R} = \{z\colon \operatorname{Im} z = 0\}$$
$$\varphi(D \setminus \overline{\Omega}) = \mathbb{C}_- = \{z\colon \operatorname{Im} z < 0\}.$$

Proposition 1. (Boundary Regularity) *Suppose $u \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ is p-harmonic in Ω and $\mathscr{C}^\infty$-smooth when restricted to Γ. Then u is $\mathscr{C}^{1,\alpha}$-regular up to Γ, meaning that u extends to D as a $\mathscr{C}^{1,\alpha}(D)$-regular function, where α depends only on p.*

The Dirichlet problem. There are two formulations of the Dirichlet boundary value problem for p-harmonic equations; both are essential for our investigation. We begin with the variational formulation.

Lemma 1. *Let $u_\circ \in \mathscr{W}^{1,p}(\Omega)$ be a given Dirichlet data. There exists precisely one function $u \in u_\circ + \mathscr{W}_\circ^{1,p}(\Omega)$ which minimizes the p-harmonic energy:*

$$\mathcal{E}_p[u] = \inf \left\{ \int_\Omega |\nabla w|^p : w \in u_\circ + \mathscr{W}_\circ^{1,p}(\Omega) \right\}.$$

The solution u is certainly a p-harmonic function, so $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$-regular. However, more useful to us will be the following classical formulation of the Dirichlet problem.

Problem 1. Given $u_\circ \in \mathscr{C}(\partial\Omega)$ find a p-harmonic function u in Ω which extends continuously to $\overline{\Omega}$ such that $u_{|\partial\Omega} = u_\circ$.

It is not difficult to see that such a solution (if one exists) is unique. However, the existence poses rather delicate conditions on $\partial\Omega$ and the data $u_\circ \in \mathscr{C}(\overline{\Omega})$. We shall confine ourselves to Jordan domains $\Omega \subset \mathbb{C}$ and the Dirichlet data $u_\circ \in \mathscr{C}(\overline{\Omega})$ of finite p-harmonic energy. In this case both formulations are valid and lead to the same solution. Indeed, the variational solution is continuous up to the boundary because each boundary point of a planar Jordan domain is a regular point for the p-Laplace operator Δ_p [19, p. 418]. See [23, 6.16] for the discussion of boundary regularity and relevant capacities and [29, Lemma 2] for a capacity estimate that applies to simply connected domains.

Proposition 2. (Existence) *Let $\Omega \subset \mathbb{C}$ be a bounded Jordan domain and $u_\circ \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$. There exists, unique, p-harmonic function $u \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ such that $u_{|\partial\Omega} = u_{\circ|\partial\Omega}$.*

Radó–Kneser–Choquet Theorem. Let $h = u + iv$ be a complex harmonic mapping in a Jordan domain $\mathbb{U}$ that is continuous on $\overline{\mathbb{U}}$. Assume that the boundary mapping $h : \partial\mathbb{U} \xrightarrow{\text{onto}} \gamma$ is an orientation-preserving homeomorphism onto a convex Jordan curve. Then h is a $\mathscr{C}^\infty$-smooth diffeomorphism of $\mathbb{U}$ onto the bounded component of $\mathbb{C} \setminus \gamma$. Thus, in particular, the Jacobian determinant $J(z, h) = |h_z|^2 - |h_{\bar z}|^2$ is strictly positive in $\mathbb{U}$, see [16, p. 20] (where $\mathbb{U}$ is assumed to be a disk with no loss of generality due to the Riemann mapping theorem). Suppose, in addition, that $\partial\mathbb{U}$ contains a $\mathscr{C}^\infty$-smooth arc $\Gamma \subset \partial\mathbb{U}$, and h restricts to a $\mathscr{C}^\infty$-smooth diffeomorphism of Γ onto a subarc in γ. Then h is $\mathscr{C}^\infty$-smooth up to Γ and its Jacobian determinant is positive on Γ as well, see [16, p. 116]. Numerous presentations of the proof of Radó–Kneser–Choquet Theorem can be found, some of which appear in [16], see also [3]. The idea that goes back to KNESER [28] and CHOQUET [14] is to look at the structure of the level curves of the coordinate functions $u = \operatorname{Re} h$, $v = \operatorname{Im} h$ and their linear combinations. These ideas have been applied to more general linear and nonlinear elliptic systems of PDEs in the complex plane [9], see also [1,2,30,31] for related problems concerning critical points. In the present paper we shall explore a result due to ALESSANDRINI and SIGALOTTI [4] for a nonlinear system that consists of two p-harmonic equations

$$\begin{cases} \operatorname{div} |\nabla u|^{p-2} \nabla u = 0 \\ \operatorname{div} |\nabla v|^{p-2} \nabla v = 0 \end{cases}, \qquad 1 < p < \infty, \quad h = u + iv.$$

We call this an *uncoupled p-harmonic system*. The novelty and key element in [4] is the associated single linear elliptic PDE of divergence type (with variable coefficients) for a linear combination of u and v. Such combination represents a real part of a quasiregular mapping and, therefore, admits only isolated critical points. We shall not go into their arguments in detail, but instead extract the following p-harmonic analogue of the Radó–Kneser–Choquet Theorem.

Theorem 2. (G. ALESSANDRINI and M. SIGALOTTI) *Let $\mathbb{U}$ be a bounded Jordan domain and $h = u + iv \colon \overline{\mathbb{U}} \to \mathbb{C}$ be a continuous mapping whose coordinate functions $u, v \in \mathscr{W}^{1,p}(\mathbb{U})$, $1 < p < \infty$, are p-harmonic. Suppose that $h \colon \partial \mathbb{U} \xrightarrow{\text{onto}} \gamma$ is an orientation-preserving homeomorphism onto a convex Jordan curve γ. Then*

(i) *h is a $\mathscr{C}^\infty$-diffeomorphism from $\mathbb{U}$ onto the bounded component of $\mathbb{C} \setminus \gamma$. In particular,*

$$J(z, h) = |h_z|^2 - |h_{\bar{z}}|^2 > 0 \quad \textit{in } \mathbb{U}.$$

(ii) *If, in addition, $\partial \mathbb{U}$ contains a $\mathscr{C}^\infty$-smooth arc $\Gamma \subset \partial \mathbb{U}$ and h restricts to a $\mathscr{C}^\infty$-smooth diffeomorphism of Γ onto a subarc of γ, then h is $\mathscr{C}^{1,\alpha}$-regular up to Γ, for some $0 < \alpha = \alpha(p) < 1$ (actually $\mathscr{C}^\infty$). Moreover $J(z, h) > 0$ on Γ as well.*

This theorem is a straightforward corollary of Theorem 5.1 in [4]. However, three remarks are in order.

1. In their Theorem 5.1, the authors of [4] assume that $\mathbb{U}$ satisfies an exterior cone condition. This is needed only insofar as to ensure the existence of a continuous extension of a given homeomorphism $\Phi \colon \partial \mathbb{U} \to \gamma$ into $\mathbb{U}$ whose coordinate functions are p-harmonic in $\mathbb{U}$. Obviously, such an extension is unique, though the p-harmonic energy need not be finite. Once we have such a mapping, the exterior cone condition on $\mathbb{U}$ for the conclusion of Theorem 5.1 is redundant, see Remark 3.2 in [4]. This is exactly the case we are dealing with in Theorem 2.
2. In regard to the statement (ii) we point out that in Theorem 5.1 of [4] the authors work with the mappings that are smooth up to the entire boundary of $\mathbb{U}$. Nonetheless, their proof that $J(z, h) > 0$ on $\partial \mathbb{U}$ is local, so applies without any change to our case (ii).
3. Since $J(z, h) > 0$ in $\mathbb{U}$ up to the arc $\Gamma \subset \partial \mathbb{U}$ the coordinate functions of h have nonvanishing gradients. This means that p-harmonic equation is uniformly elliptic up to Γ. Consequently, h is $\mathscr{C}^\infty$-smooth on $\mathbb{U}$ up to Γ.

The p-harmonic replacement. Let Ω be a bounded domain in $\mathbb{R}^2 \simeq \mathbb{C}$. We consider the class $\mathcal{A}(\Omega) = \mathcal{A}^p(\Omega)$, $1 < p < \infty$, of uniformly continuous functions $h = u + iv \colon \Omega \to \mathbb{C}$ having finite p-harmonic energy and furnish it with the norm

$$\|h\|_{\mathcal{A}^p(\Omega)} = \|h\|_{\mathscr{C}(\Omega)} + \|\nabla h\|_{\mathscr{L}^p(\Omega)}.$$

The closure of $\mathscr{C}^\infty_\circ(\Omega)$ in $\mathcal{A}^p(\Omega)$ will be denoted by $\mathcal{A}^p_\circ(\Omega)$.

Proposition 3. *Let $\mathbb{U} \Subset \Omega$ be a Jordan subdomain of Ω. There exists a unique operator*

$$\mathbf{R}_{\mathbb{U}} \colon \mathcal{A}^p(\Omega) \to \mathcal{A}^p(\Omega)$$

(*nonlinear if $p \neq 2$*) *such that for every $h \in \mathcal{A}^p(\Omega)$*

$$\mathbf{R}_{\mathbb{U}} h = h \qquad \text{in } \Omega \setminus \mathbb{U} \tag{2.3}$$

$$\mathbf{R}_{\mathbb{U}} h \in h + \mathscr{W}^{1,p}_\circ(\mathbb{U}) \tag{2.4}$$

$$\Delta_p \mathbf{R}_{\mathbb{U}} h = 0 \qquad \text{in } \mathbb{U} \tag{2.5}$$

$$\mathsf{E}_\Omega[\mathbf{R}_{\mathbb{U}} h] \leqq \mathsf{E}_\Omega[h] \tag{2.6}$$

Equality occurs in (2.6) *if and only if h is p-harmonic in $\mathbb{U}$.*

Proof. For $h = u + iv$ we define

$$\mathbf{R}_{\mathbb{U}} h = \mathbf{R}_{\mathbb{U}} u + i\, \mathbf{R}_{\mathbb{U}} v.$$

It is therefore enough to construct the replacement for real-valued functions. For $u \in \mathcal{A}^p(\Omega)$ real, we define

$$\mathbf{R}_{\mathbb{U}} u = \begin{cases} u & \text{in } \Omega \setminus \mathbb{U} \\ \tilde{u} & \text{in } \mathbb{U} \end{cases}$$

where $\tilde{u}$ is determined uniquely as a solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} = 0 & \text{in } \mathbb{U} \\ \tilde{u} \in u + \mathscr{W}^{1,p}_\circ(\mathbb{U}) \end{cases}$$

so conditions (2.3) are fulfilled. That $\mathbf{R}_{\mathbb{U}} u$ is continuous in Ω is guaranteed by Proposition 2. The solution $\tilde{u}$ is found as the minimizer of the p-harmonic energy in the class $u + \mathscr{W}^{1,p}_\circ(\mathbb{U})$, so we certainly have

$$\mathsf{E}_\Omega[\mathbf{R}_{\mathbb{U}} u] \leqq \mathsf{E}_\Omega[u]$$

The same estimate holds for the imaginary part of h, so adding them up yields

$$\mathsf{E}_\Omega[\mathbf{R}_{\mathbb{U}} h] \leqq \mathsf{E}_\Omega[h].$$

Remark 1. The reader may wish to know that the operator $\mathbf{R}_{\mathbb{U}} \colon \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is continuous, though we do not appeal to this fact.

Smoothing along a crosscut. Consider a bounded Jordan domain $\mathbb{U}$ and a $\mathscr{C}^\infty$-smooth crosscut $\Gamma \subset \mathbb{U}$ with two distinct endpoints in $\partial\mathbb{U}$. By definition, this means that there is a $\mathscr{C}^\infty$-diffeomorphism $\varphi\colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{U}$ such that $\Gamma = \varphi(\mathbb{R})$, and its distinct endpoints are given by

$$\lim_{x\to-\infty} \varphi(x) \in \partial\mathbb{U}$$
$$\lim_{x\to\infty} \varphi(x) \in \partial\mathbb{U}$$

This Γ splits $\mathbb{U}$ into two Jordan subdomains

$$\mathbb{U}_+ = \varphi(\mathbb{C}_+), \quad \mathbb{C}_+ = \{z\colon \operatorname{Im} z > 0\}$$
$$\mathbb{U}_- = \varphi(\mathbb{C}_-), \quad \mathbb{C}_- = \{z\colon \operatorname{Im} z < 0\}.$$

Suppose we are given a homeomorphism $f\colon \overline{\mathbb{U}} \to \mathbb{C}$ such that each of two mappings

$$f\colon \mathbb{U}_+ \to \mathbb{R}^2 \quad \text{and} \quad f\colon \mathbb{U}_- \to \mathbb{R}^2$$

is $\mathscr{C}^\infty$-smooth up to Γ. Assume that for some constant $0 < m < \infty$ we have

$$|Df(z)| \leqq m \quad \text{and} \quad \det Df(z) \geqq \frac{1}{m}$$

on $\mathbb{U}_+$ and on $\mathbb{U}_-$. Thus $f\colon \mathbb{U} \to \mathbb{R}^2$ is, in fact, locally bi-Lipschitz.

Proposition 4. *Under the above conditions there is a constant $0 < M < \infty$ such that for every open set $\mathbb{V} \subset \mathbb{U}$ containing Γ one can find a homeomorphism $g\colon \overline{\mathbb{U}} \xrightarrow{\text{onto}} f(\overline{\mathbb{U}})$ which is a $\mathscr{C}^\infty$-diffeomorphism in $\mathbb{U}$, with the following properties:*

$$g(z) = f(z), \quad \textit{for } z \in (\overline{\mathbb{U}} \setminus \mathbb{V}) \cup \Gamma \tag{2.7}$$
$$|Dg(z)| \leqq M \quad \textit{and} \quad \det Dg(z) > \frac{1}{M} \ \textit{on } \mathbb{U}. \tag{2.8}$$

The key element of this smoothing device is that the constant M is independent of the neighborhood $\mathbb{V}$ of Γ, see Fig. 1. The proof is given in [26] following the ideas of [34].

We shall recall a similar smoothing device for cuts along Jordan curves. Let $\mathbb{U}$ be a simply connected domain with $\mathscr{C}^\infty$-regular cut along a Jordan curve $\Gamma \subset \mathbb{U}$. This means there is a diffeomorphism $\varphi\colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{U}$ such that $\Gamma = \varphi(\mathbb{S}^1)$, $\mathbb{S}^1 = \{z \in \mathbb{C}\colon |z| = 1\}$. As before Γ splits $\mathbb{U}$ into

$$\mathbb{U}_+ = \varphi(\mathbb{D}_+), \quad \mathbb{D}_+ = \{z\colon |z| < 1\}$$
$$\mathbb{U}_- = \varphi(\mathbb{D}_-), \quad \mathbb{D}_- = \{z\colon |z| > 1\}.$$

Suppose we are given a homeomorphism $f\colon \mathbb{U} \to \mathbb{R}^2$ such that each of two mappings

$$f\colon \mathbb{U}_+ \to \mathbb{R}^2 \quad \text{and} \quad f\colon \mathbb{U}_- \to \mathbb{R}^2$$

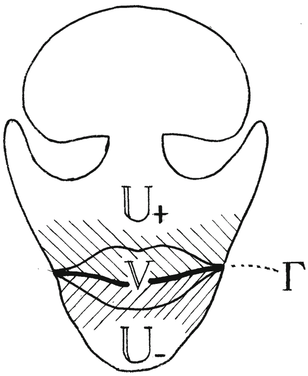

Fig. 1. Jordan domain with a crosscut Γ and its neighborhood $\mathbb{V}$

is $\mathscr{C}^\infty$-smooth up to Γ. Assume that for some constant $0 < m < \infty$ we have

$$|Df(z)| \leqq m \quad \text{and} \quad \det Df(z) \geqq \frac{1}{m}$$

on $\mathbb{U}_+$ and $\mathbb{U}_-$.

Proposition 5. ([26]) *Under the above conditions, there is a constant* $0 < M < \infty$ *such that for every open set* $\mathbb{V} \subset \mathbb{U}$ *containing* Γ *one can find a* $\mathscr{C}^\infty$*-diffeomorphism* $g\colon \mathbb{U} \xrightarrow{\text{onto}} f(\mathbb{U})$ *with the following properties*

$$g(z) = f(z), \; \textit{for } z \in (\mathbb{U} \setminus \mathbb{V}) \cup \Gamma \tag{2.9}$$

$$|Dg(z)| \leqq M \quad \textit{and} \quad \det Dg(z) > \frac{1}{M} \; \textit{on } \mathbb{U}. \tag{2.10}$$

Having disposed of the above preliminaries we shall now proceed to the construction of the approximating sequence of diffeomorphisms.

3. The proof

Scheme of the proof. Let us begin with a convention. We will often suppress the explicit dependence on the Sobolev exponent $1 < p < \infty$ in the notation, whenever it becomes self explanatory. For every $\varepsilon > 0$ we shall construct a $\mathscr{C}^\infty$-diffeomorphism $\hbar\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ such that

(A) $\hbar - h \in \mathscr{A}_\circ(\mathbb{X})$
(B) $\|\hbar - h\|_{\mathscr{C}(\mathbb{X})} \leqq \varepsilon$
(C) $\|\nabla\hbar - \nabla h\|_{\mathscr{L}^p(\mathbb{X})} \leqq \varepsilon$
(D) $\mathsf{E}_{\mathbb{X}}[\hbar] \leqq \mathsf{E}_{\mathbb{X}}[h]$
(E) If h is a $\mathscr{C}^\infty$-diffeomorphism outside of a compact subset of $\mathbb{X}$, then there exists a compact subset of $\mathbb{X}$ outside of which we have $\hbar \equiv h$.

We may and do assume that h is not a $\mathscr{C}^\infty$-diffeomorphism, since otherwise $\hbar = h$ satisfies the desired properties. Let $x_\circ \in \mathbb{X}$ be a point such that h fails to be $\mathscr{C}^\infty$-diffeomorphism in any neighborhood of $x_\circ$.

We shall consider dyadic squares in $\mathbb{Y}$ with respect to a selected rectangular coordinate system in $\mathbb{R}^2$. By choosing the origin of the system we ensure that the preimage under h of the boundary of each dyadic square has zero area and does not contain $x_\circ$.

Let us fix $\varepsilon > 0$. The construction of $\hbar$ proceeds in 5 steps, each of which gives a homeomorphism $\hbar_k : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, $k = 0, 1, \dots, 5$, in the Sobolev class $\mathscr{W}^{1,p}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ such that $\hbar_0 = h$, $\hbar_k \in \hbar_{k-1} + \mathcal{A}_\circ(\mathbb{X})$, $k = 1, \dots, 5$ and $\hbar_5 = \hbar$ is the desired diffeomorphism. For each $k = 1, 2, \dots, 5$ we will secure conditions analogous to (A)–(E). Namely,

(A_k) $\hbar_k - \hbar_{k-1} \in \mathcal{A}_\circ(\mathbb{X})$

(B_k) $\|\hbar_k - \hbar_{k-1}\|_{\mathscr{C}(\mathbb{X})} \leqq \varepsilon/5$

(C_k) $\|\nabla\hbar_k - \nabla\hbar_{k-1}\|_{\mathscr{L}^p(\mathbb{X})} \leqq \varepsilon/5$

(D_k) $\|\nabla\hbar_1\|_{\mathscr{L}^p(\mathbb{X})} \leqq \|\nabla\hbar_0\|_{\mathscr{L}^p(\mathbb{X})} - 2\delta$, for some $\delta > 0$;
$\|\nabla\hbar_k\|_{\mathscr{L}^p(\mathbb{X})} \leqq \|\nabla\hbar_{k-1}\|_{\mathscr{L}^p(\mathbb{X})}$, for $k = 2, 4$;
$\|\nabla\hbar_k\|_{\mathscr{L}^p(\mathbb{X})} \leqq \|\nabla\hbar_{k-1}\|_{\mathscr{L}^p(\mathbb{X})} + \delta$, for $k = 3, 5$

(E_k) If h_{k-1} is a $\mathscr{C}^\infty$-diffeomorphism outside of a compact subset of $\mathbb{X}$, then there exists a compact subset in $\mathbb{X}$ outside which we have $\hbar_k \equiv \hbar_{k-1}$.

Partition of $\mathbb{X}$ into cells. Let us distinguish one particular Whitney type partition of $\mathbb{Y}$ and keep it fixed for the rest of our arguments.

$$\mathbb{Y} = \bigcup_{\nu=1}^{\infty} \overline{\mathbb{Y}_\nu},$$

where $\mathbb{Y}_\nu$ are mutually disjoint open dyadic squares such that

$$\operatorname{diam} \mathbb{Y}_\nu \leqq \operatorname{dist}(\mathbb{Y}_\nu, \partial\mathbb{Y}) \leqq 3 \operatorname{diam} \mathbb{Y}_\nu \quad \text{for } \nu = 1, 2, \dots$$

unless $\mathbb{Y} = \mathbb{R}^2$, in which case $\mathbb{Y}_\nu$ are unit squares. Thus the cover of $\mathbb{Y}$ by $\overline{\mathbb{Y}_\nu}$ is locally finite. The preimages

$$\mathbb{X}_\nu = h^{-1}(\mathbb{Y}_\nu), \qquad \nu = 1, 2, \dots$$

are Jordan domains which we call *cells* in $\mathbb{X}$. In the forthcoming Step 1 we shall need to further divide each cell into a finite number of *daughter cells* in $\mathbb{X}$. Note that all but a finite number of cells $\mathbb{X}_\nu$, $\nu = 1, 2, \dots$ lie outside a given compact subset of $\mathbb{X}$.

Step 1

To avoid undue indexing in the forthcoming division of cells, we shall argue in two substeps.

Step 1a. Examine one of the cells in $\mathbb{X}$, say $\mathfrak{X} = \mathbb{X}_\nu$, for some fixed $\nu = 1, 2, \dots$. Call it a *parent cell*. Thus $h(\mathfrak{X}) = \Upsilon$ is the corresponding Whitney square $\Upsilon = \mathbb{Y}_\nu \subset \mathbb{Y}$. To every $n = 1, 2, \dots$, there corresponds a partition of Υ into 4^n-dyadic congruent squares Υ_i, $i = 1, \dots, 4^n$

$$\overline{\Upsilon} = \overline{\Upsilon_1} \cup \cdots \cup \overline{\Upsilon_{4^n}}.$$

This gives rise to a division of $\mathfrak{X}$ into daughter cells $\mathfrak{X}_i = h^{-1}(\Upsilon_i)$

$$\overline{\mathfrak{X}} = \overline{\mathfrak{X}_1} \cup \overline{\mathfrak{X}_2} \cup \cdots \cup \overline{\mathfrak{X}_{4^n}}.$$

We look at the homeomorphisms

$$h\colon \overline{\mathfrak{X}_i} \xrightarrow{\text{onto}} \overline{\Upsilon_i}, \qquad i = 1, 2, \dots 4^n$$

By virtue of Proposition 3 we may replace them with p-harmonic homeomorphisms

$$\widetilde{h}_i = \mathbf{R}_{\mathfrak{X}_i} h\colon \overline{\mathfrak{X}_i} \xrightarrow{\text{onto}} \overline{\Upsilon_i}, \qquad i = 1, 2, \dots, 4^n$$

which coincide with h on $\partial \mathfrak{X}_i$. This procedure may not be necessary if $h\colon \mathfrak{X}_i \to \Upsilon_i$ is already a $\mathscr{C}^\infty$-diffeomorphism. In such cases we always use the *trivial replacement* $\widetilde{h}_i = h$. After all such replacements are made, we arrive at a homeomorphism

$$\widetilde{h}\colon \overline{\mathfrak{X}} \xrightarrow{\text{onto}} \overline{\Upsilon}$$

which is a $\mathscr{C}^\infty$-diffeomorphism in each cell $\mathfrak{X}_i$ and coincides with h on $\partial \mathfrak{X}_i$. Obviously,

$$\widetilde{h} = h + \sum_{i=1}^{4^n} [\widetilde{h}_i - h]_\circ \in h + \mathcal{A}_\circ(\mathfrak{X})$$

where $[\widetilde{h}_i - h]_\circ$ stands for zero extension of $\widetilde{h}_i - h$ outside $\mathfrak{X}_i$ and, therefore, belongs to $\mathcal{A}_\circ(\mathfrak{X}_i)$. Furthermore, by principle of minimal p-harmonic energy, we have

$$\mathsf{E}_{\mathfrak{X}}[\widetilde{h}] = \sum_{i=1}^{4^n} \mathsf{E}_{\mathfrak{X}_i}[\widetilde{h}_i] \leqq \sum_{i=1}^{4^n} \mathsf{E}_{\mathfrak{X}_i}[h] = \mathsf{E}_{\mathfrak{X}}[h].$$

The eventual aim is to fix the number of daughter cells in $\mathfrak{X}$. For this we vary n and look closely at the resulting homeomorphisms, denoted by f_n. This sequence of mappings is bounded in $\mathcal{A}(\mathfrak{X})$. It actually converges to h uniformly on $\overline{\mathfrak{X}}$. Indeed, given any point $x \in \overline{\mathfrak{X}}$, say $x \in \overline{\mathfrak{X}_i}$, for some $i = 1, 2, \dots, 4^n$, we have

$$|f_n(x) - h(x)| = |\widetilde{h}_i(x) - h(x)| \leqq \operatorname{diam} \Upsilon_i = 2^{-n} \operatorname{diam} \Upsilon.$$

Thus

$$\lim_{n\to\infty} f_n = h, \quad \text{uniformly in } \overline{\mathfrak{X}}.$$

On the other hand, the mappings f_n are bounded in the Sobolev space $\mathscr{W}^{1,p}(\mathfrak{X})$, and so converge to h weakly in $\mathscr{W}^{1,p}(\mathfrak{X})$. The key observation now is that

$$\|\nabla h\|_{\mathscr{L}^p(\mathfrak{X})} \leqq \liminf_{n\to\infty} \|\nabla f_n\|_{\mathscr{L}^p(\mathfrak{X})} \leqq \|\nabla h\|_{\mathscr{L}^p(\mathfrak{X})}$$

because of convexity of the energy functional. This gives

$$\lim_{n\to\infty} \|\nabla f_n\|_{\mathscr{L}^p(\mathfrak{X})} = \|\nabla h\|_{\mathscr{L}^p(\mathfrak{X})}$$

Then, the usual application of Clarkson's inequalities in $\mathscr{L}^p$-spaces, $1 < p < \infty$, yields

$$\lim_{n\to\infty} \|\nabla f_n - \nabla h\|_{\mathscr{L}^p(\mathfrak{X})} = 0$$

meaning that $f_n - h \to 0$ in the norm topology of $\mathcal{A}(\mathfrak{X})$. We can now determine the number $n = n_\nu = n(\mathfrak{X})$, simply requiring the division of $\mathfrak{X}$ be fine enough to satisfy two conditions.

$$\begin{cases} \operatorname{diam} \Upsilon_i = 2^{-n} \operatorname{diam} \Upsilon \leqq \varepsilon/5, \quad i = 1, \dots, 4^n \\ \|\nabla f_n - \nabla h\|_{\mathscr{L}^p(\mathfrak{X})} \leqq \frac{\varepsilon}{5\cdot 2^\nu} \end{cases} \tag{3.1}$$

where we recall that $\mathfrak{X}$ stands for $\mathbb{X}_\nu$.

Step 1b. Now, having $n = n_\nu$ fixed for each cell $\mathfrak{X}_\nu$, we construct our first approximating mapping

$$\hbar_1 : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$$

by setting

$$\hbar_1 := h + \sum_{\nu=1}^{\infty} \left[f_{n_\nu} - h \right]_\circ \in h + \mathcal{A}_\circ(\mathbb{X})$$

where, as always, $[f_{n_\nu} - h]_\circ$ stands for the zero extension of $f_{n_\nu} - h$ outside $\mathbb{X}_\nu$. This mapping is a $\mathscr{C}^\infty$-diffeomorphism in every daughter cell. Clearly, we have the condition

$$\hbar_1 - h \in \mathcal{A}_\circ(\mathbb{X}). \tag{A_1}$$

Moreover, by the condition in (3.1) imposed on every n_ν,

$$\|\hbar_1 - h\|_{\mathscr{C}(\mathbb{X})} \leqq \sup_{\nu=1,2,\dots} \{\operatorname{diam} \Upsilon_i : \Upsilon_i \subset \mathbb{Y}_\nu, i = 1, \dots, 4^{n_\nu}\} \leqq \frac{\varepsilon}{5} \tag{B_1}$$

and

$$\|\nabla \hbar_1 - \nabla h\|^p_{\mathscr{L}^p(\mathbb{X})} = \sum_{\nu=1}^{\infty} \|\nabla \hbar_1 - \nabla h\|^p_{\mathscr{L}^p(\mathbb{X}_\nu)} \leqq \left(\frac{\varepsilon}{5}\right)^p \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} < \left(\frac{\varepsilon}{5}\right)^p. \tag{C_1}$$

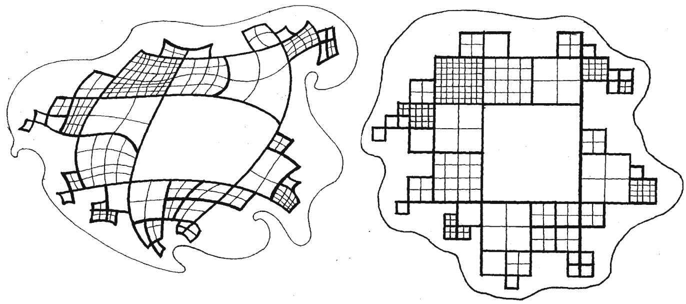

Fig. 2. $\hbar_1$ is a $\mathscr{C}^\infty$-diffeomorphism in each cell $\mathfrak{X}^\alpha \subset \mathbb{X}$

Regarding condition (D_1), we observe that summing up the energies over all daughter cells $\mathfrak{X}_i \subset \mathbb{X}_\nu$, $i = 1, 2, \dots 4^{n_\nu}$ and $\nu = 1, 2, \dots$, gives the total energy of $\hbar_1$ not larger than that of h. Even more, since h fails to be a $\mathscr{C}^\infty$-diffeomorphism in at least one of these cells, the p-harmonic replacement takes place in this cell and, consequently, $\hbar_1$ has strictly smaller energy. Hence

$$\|\nabla \hbar_1\|_{\mathscr{L}^p(\mathbb{X})} \leqq \|\nabla h\|_{\mathscr{L}^p(\mathbb{X})} - 2\delta, \quad \text{for some } \delta > 0. \qquad (D_1)$$

Regarding condition (E_1), we note that under the assumption therein we made only a finite number of nontrivial (p-harmonic) replacements. The same remark will apply to the subsequent steps and will not be mentioned again. Step 1 is complete.

Before proceeding to Step 2, let us put all daughter cells in $\mathbb{X}$ in a single sequence

$$\mathfrak{X}^1, \mathfrak{X}^2, \dots \subset \mathbb{X}.$$

Thus, from now on, the daughter cells from different parents are indistinguishable as far as the mapping $\hbar_1$ is concerned. The point is that $\hbar_1$ is a $\mathscr{C}^\infty$-diffeomorphism in every such cell, a property that will be pertinent to all new cells coming later either by splitting or merging the existing cells. Note that the images $\Upsilon^\alpha = h(\mathfrak{X}^\alpha)$, $\alpha = 1, 2, \dots$, form a partition of $\mathbb{Y}$ into dyadic squares, see Fig. 2.

$$\mathbb{Y} = \bigcup_{\alpha=1}^{\infty} \overline{\Upsilon^\alpha}, \qquad \text{where} \quad \operatorname{diam} \Upsilon^\alpha \leqq \frac{\varepsilon}{5}$$

Step 2

Step 2a. (Adjacent cells) Let $\mathcal{C}(\mathbb{Y}) \subset \mathbb{Y}$ be the collection of all corners of dyadic squares Υ^α, $\alpha = 1, 2, \dots$, and $\mathcal{V}(\mathbb{X}) \subset \mathbb{X}$ denote the set of their preimages under h, called *vertices of cells*. Whenever two closed cells $\overline{\mathfrak{X}^\alpha}$ and $\overline{\mathfrak{X}^\beta}$, $\alpha \neq \beta$, intersect, their common part is either a point in $\mathcal{V}(\mathbb{X})$ or an edge, that is, a closed Jordan arc with endpoints in $\mathcal{V}(\mathbb{X})$. In this latter case we say that $\mathfrak{X}^\alpha$ and $\mathfrak{X}^\beta$ are adjacent cells with common edge

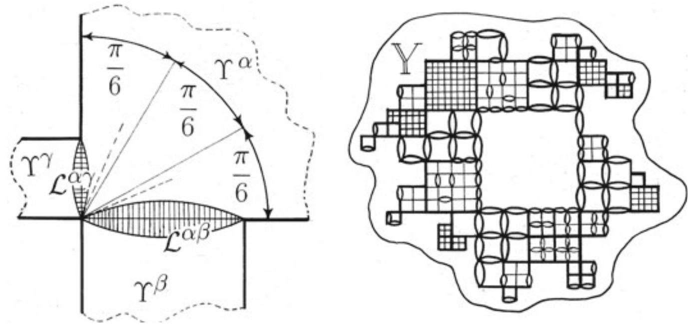

Fig. 3. Lenses

$$\overline{C^{\alpha\,\beta}} = \overline{\mathfrak{X}^\alpha} \cap \overline{\mathfrak{X}^\beta}.$$

This is the closure of a Jordan open arc $C^{\alpha\,\beta} = \overline{C^{\alpha\,\beta}} \setminus \mathcal{V}(\mathbb{X})$. The mappings

$$\hbar_1 : \mathfrak{X}^\alpha \xrightarrow{\text{onto}} \Upsilon^\alpha \quad \text{and} \quad \hbar_1 : \mathfrak{X}^\beta \xrightarrow{\text{onto}} \Upsilon^\beta$$

are $\mathscr{C}^\infty$-diffeomorphisms, but they do not necessarily match smoothly along the edges. We shall now produce a new cell $\mathfrak{X}^{\alpha\,\beta}$, a daughter of the adjacent cells $\mathfrak{X}^\alpha$ and $\mathfrak{X}^\beta$, such that

$$C^{\alpha\,\beta} \subset \mathfrak{X}^{\alpha\,\beta} \subset \mathfrak{X}^\alpha \cup C^{\alpha\,\beta} \cup \mathfrak{X}^\beta.$$

To construct $\mathfrak{X}^{\alpha\,\beta}$ we look at the adjacent dyadic squares $\overline{\Upsilon^\alpha}$ and $\overline{\Upsilon^\beta}$ in $\mathbb{Y}$. The intersection $\overline{\Upsilon^\alpha} \cap \overline{\Upsilon^\beta} = h(\overline{C^{\alpha\,\beta}})$ is a closed interval. Let R be a number greater than the length of $h(C^{\alpha\,\beta})$ to be chosen sufficiently large later on. There exist exactly two open disks of radius R for which $h(C^{\alpha\,\beta})$ is a chord. Their intersection, denoted by $\mathcal{L}^{\alpha\,\beta}$, is a symmetric doubly convex lens of curvature R^{-1}. Thus $\mathcal{L}^{\alpha\,\beta}$ is enclosed between two open circular arcs $\gamma^{\alpha\,\beta} = \Upsilon^\alpha \cap \partial\mathcal{L}^{\alpha\,\beta} \subset \Upsilon^\alpha$ and $\gamma^{\beta\,\alpha} = \Upsilon^\beta \cap \partial\mathcal{L}^{\alpha\,\beta} \subset \Upsilon^\beta$. Note that $\mathcal{L}^{\alpha\,\beta} = \mathcal{L}^{\beta\,\alpha}$, but $\gamma^{\alpha\,\beta} \neq \gamma^{\beta\,\alpha}$. We call

$$\mathfrak{X}^{\alpha\,\beta} = \hbar_1^{-1}(\mathcal{L}^{\alpha\,\beta}), \quad \text{a daughter of the adjacent cells } \mathfrak{X}^\alpha \text{ and } \mathfrak{X}^\beta. \tag{3.2}$$

As the curvature of the lens $\mathcal{L}^{\alpha\,\beta}$ approaches zero, the area of $\mathfrak{X}^{\alpha\,\beta}$ tends to 0 because $C^{\alpha\,\beta}$ has zero area. This allows us to choose R so that

$$\|\nabla \hbar_1\|_{\mathscr{L}^p(\mathfrak{X}^{\alpha\,\beta})} \leqq \frac{\varepsilon}{5 \cdot 2^{\alpha+\beta}}. \tag{3.3}$$

The lenses $\mathcal{L}^{\alpha\,\beta}$ are disjoint because the opening angle of each lens (the angle between arcs at their common endpoints) is at most $\pi/3$ and their long axes are either parallel or orthogonal, see Fig. 3. Therefore, the cells $\mathfrak{X}^{\alpha\,\beta} = \hbar_1^{-1}(\mathcal{L}^{\alpha\,\beta})$ are also disjoint. However, their closures may have a common point that lies in $\mathcal{V}(\mathbb{X})$. The boundary of $\mathfrak{X}^{\alpha\,\beta}$ consists of two open arcs

$$\Gamma^{\alpha\,\beta} = \mathfrak{X}^\alpha \cap \partial\mathfrak{X}^{\alpha\,\beta} \quad \text{and} \quad \Gamma^{\beta\,\alpha} = \mathfrak{X}^\beta \cap \partial\mathfrak{X}^{\alpha\,\beta}$$

plus their endpoints. These open arcs are $\mathscr{C}^\infty$-smooth because they come as images of the circular arcs enclosing the lens $\mathcal{L}^{\alpha\,\beta}$ under a $\mathscr{C}^\infty$-diffeomorphism.

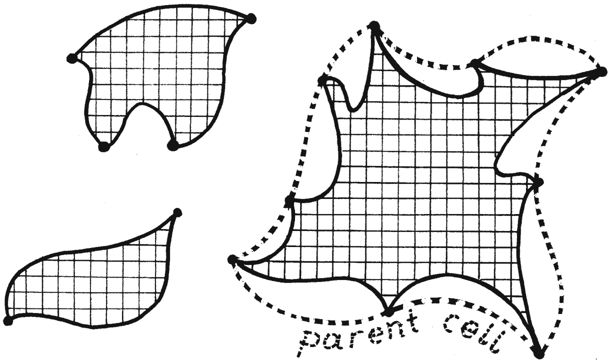

Fig. 4. Three types of cells

Remark 2. In what follows we shall consider only the pairs (α, β) of indices $\alpha = 1, 2, \ldots$ and $\beta = 1, 2, \ldots$ which correspond to adjacent cells. Such pairs will be designated the symbol $\alpha\beta$.

Step 2b. (Replacements in $\mathfrak{X}^{\alpha\,\beta}$) The lenses $\mathcal{L}^{\alpha\,\beta} \subset \mathbb{Y}$ are convex, so with the aid of Proposition 3 and Theorem 2, we may replace $\hbar_1 \colon \mathfrak{X}^{\alpha\,\beta} \to \mathcal{L}^{\alpha\,\beta}$ with the p-harmonic extension of $\hbar_1 \colon \partial\mathfrak{X}^{\alpha\,\beta} \to \partial\mathcal{L}^{\alpha\,\beta}$. We do this, and denote the result by $\hbar_2^{\alpha\beta} \colon \mathfrak{X}^{\alpha\beta} \to \mathcal{L}^{\alpha\beta}$, only on the cells in which $\hbar_1 \colon \mathfrak{X}^{\alpha} \cup \mathfrak{X}^{\beta} \cup \mathfrak{X}^{\alpha\beta} \to \mathbb{R}^2$ is not a $\mathscr{C}^\infty$-diffeomorphism. In other cells we set $\hbar_2^{\alpha\beta} = \hbar_1$. In either case $\hbar_2^{\alpha\beta} \in \hbar_1 + \mathcal{A}_\circ(\mathfrak{X}^{\alpha\,\beta})$ so we define

$$\hbar_2 = \hbar_1 + \sum_{\alpha\beta} [\hbar_2^{\alpha\,\beta} - \hbar_1]_\circ.$$

Thus we have

$$\hbar_2 - \hbar_1 \in \mathcal{A}_\circ(\mathbb{X}). \tag{A_2}$$

The advantage of using $\hbar_2$ in the next step lies in the fact that it is not only a $\mathscr{C}^\infty$-diffeomorphism in every cell, but also is $\mathscr{C}^\infty$-smooth with positive Jacobian determinant, up to each edge of the cells created here. These edges are $\mathscr{C}^\infty$-smooth open arcs. By cells created here, we mean not only $\mathfrak{X}^{\alpha\,\beta}$ but also those obtained from the parent cell $\mathfrak{X}^{\alpha}$ by removing the adjacent daughters; that is,

$$\mathfrak{X}^{\alpha} \setminus \bigcup_{\alpha\beta} \mathfrak{X}^{\alpha\,\beta}, \qquad \alpha = 1, 2, \ldots$$

See Fig. 4. The estimates of $\hbar_2$ run as follows. By (3.1) we have,

$$\|\hbar_2 - \hbar_1\|_{\mathscr{C}(\mathbb{X})} \leqq \sup_{\alpha\beta}\{\operatorname{diam} \mathcal{L}^{\alpha\,\beta}\} \leqq \sup_{\alpha}\{\operatorname{diam} \mathbb{Y}^{\alpha}\} \leqq \frac{\varepsilon}{5}. \tag{B_2}$$

In view of the minimum p-harmonic energy principle, we have

$$\begin{aligned}\|\nabla \hbar_2 - \nabla \hbar_1\|_{\mathcal{L}^p(\mathbb{X})} &= \sum_{\alpha\beta} \|\nabla \hbar_2 - \nabla \hbar_1\|_{\mathcal{L}^p(\cup \mathfrak{X}^{\alpha\beta})} \\ &\leqq \sum_{\alpha\beta} \left[\|\nabla \hbar_2\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} + \|\nabla \hbar_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})}\right] \\ &\leqq 2\sum_{\alpha\beta} \|\nabla \hbar_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} \leqq \frac{2\varepsilon}{5} \sum_{\alpha\beta} 2^{-\alpha-\beta}.\end{aligned}$$

by (3.3). Hence

$$\|\nabla \hbar_2 - \nabla \hbar_1\|_{\mathcal{L}^p(\mathbb{X})} \leqq \frac{\varepsilon}{5}. \tag{C_2}$$

The minimum energy principle also yields estimate

$$\begin{aligned}\|\nabla \hbar_2\|^p_{\mathcal{L}^p(\mathbb{X})} &= \|\nabla \hbar_2\|^p_{\mathcal{L}^p(\cup \mathfrak{X}^{\alpha\beta})} + \|\nabla \hbar_1\|^p_{\mathcal{L}^p(\mathbb{X}\setminus \cup \mathfrak{X}^{\alpha\beta})} \\ &\leqq \|\nabla \hbar_1\|^p_{\mathcal{L}^p(\cup \mathfrak{X}^{\alpha\beta})} + \|\nabla \hbar_1\|^p_{\mathcal{L}^p(\mathbb{X}\setminus \cup \mathfrak{X}^{\alpha\beta})} = \|\nabla \hbar_1\|^p_{\mathcal{L}^p(\mathbb{X})}.\end{aligned}$$

In particular

$$\|\nabla \hbar_2\|_{\mathcal{L}^p(\mathbb{X})} \leqq \|\nabla \hbar_1\|_{\mathcal{L}^p(\mathbb{X})}, \tag{D_2}$$

completing the proof of Step 2.

Note that $\hbar_2$ is locally bi-Lipschitz in $\mathbb{X} \setminus \mathcal{V}(\mathbb{X})$. The exceptional set $\mathcal{V}(\mathbb{X})$ is discrete.

Step 3

We shall now merge all the adjacent cells together by smoothing $\hbar_2$ around the edges $\Gamma^{\alpha\beta} \subset \mathfrak{X}^\alpha$. To achieve proper estimates, we need to remove small neighborhoods of all vertices, outside of which $\hbar_2$ is certainly locally bi-Lipschitz.

Step 3a. First we cover the set $\mathcal{C}(\mathbb{Y})$ of corners of dyadic squares by disks $\mathbb{D}_c$ centered at $c \in \mathcal{C}(\mathbb{Y})$. These disks will be chosen small enough to satisfy all the conditions listed below.

(i) $\operatorname{diam} \mathbb{D}_c < \varepsilon/5$ for every $c \in \mathcal{C}(\mathbb{Y})$,

(ii) $\displaystyle\sum_{v\in\mathcal{V}(\mathbb{X})} \int_{\mathbb{F}_v} |\nabla \hbar_2|^p \leqq \left(\frac{\varepsilon}{20}\right)^p$, where $\mathbb{F}_v = \hbar_2^{-1}(\mathbb{D}_c)$, $c = \hbar_2(v) = h(v)$.

Denote by $\mathbb{X}_\circ = \mathbb{X} \setminus \bigcup \overline{\mathbb{F}_v}$. We truncate each edge $\Gamma^{\alpha\beta}$ near the endpoints by setting

$$\Gamma^{\alpha\beta}_\circ = \Gamma^{\alpha\beta} \cap \mathbb{X}_\circ. \tag{3.4}$$

These are mutually disjoint open arcs; their closures are isolated continua in $\mathbb{X} \setminus \mathcal{V}(\mathbb{X})$. This means that there are disjoint neighborhoods of them. We are actually interested in neighborhoods $\mathbb{U}^{\alpha\beta} \subset \mathfrak{X}^\alpha$ of $\Gamma^{\alpha\beta}_\circ$ that are Jordan domains in which $\Gamma^{\alpha\beta}_\circ \subset \mathbb{U}^{\alpha\beta}$ are $\mathscr{C}^\infty$-smooth crosscuts with two endpoints in $\partial\mathbb{U}^{\alpha\beta}$, see Section 2. It is geometrically clear that such mutually disjoint neighborhoods exist. Now the stage for the next substep is established.

Step 3b. ($\mathscr{C}^\infty$-replacement within $\mathbb{U}^{\alpha\beta}$) It is at this stage that we will improve $\hbar_2$ in $\mathbb{U}^{\alpha\beta}$ to a $\mathscr{C}^\infty$-smooth diffeomorphism with no harm to the previously established estimates for $\hbar_2$. The tool is Proposition 4. As always, we shall make no replacement of $\hbar_2 \colon \mathbb{U}^{\alpha\beta} \to \Upsilon^\alpha$ if it is already $\mathscr{C}^\infty$-diffeomorphism. Recall that we have a bi-Lipschitz mapping $\hbar_2 \colon \mathbb{U}^{\alpha\beta} \to \hbar_2(\mathfrak{X}^\alpha) = \Upsilon^\alpha$ that takes the crosscut $\Gamma_\circ^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$ onto a circular arc. Denote the components $\mathbb{U}_+^{\alpha\beta} = \mathbb{U}^{\alpha\beta} \setminus \overline{\mathfrak{X}^{\alpha\beta}}$ and $\mathbb{U}_-^{\alpha\beta} = \mathbb{U}^{\alpha\beta} \cap \mathfrak{X}^{\alpha\beta}$. Furthermore, we have

$$|D\hbar_2| \leqq m_{\alpha\beta} \quad \text{and} \quad \det D\hbar_2 \geqq \frac{1}{m_{\alpha\beta}}, \quad \text{for some } m_{\alpha\beta} > 0$$

on each component. The mappings $\hbar_2 \colon \mathbb{U}_+^{\alpha\beta} \to \Upsilon^\alpha$ and $\hbar_2 \colon \mathbb{U}_-^{\alpha\beta} \to \Upsilon^\alpha$ are $\mathscr{C}^\infty$-diffeomorphisms up to $\Gamma_\circ^{\alpha\beta}$. In accordance with Proposition 4 we find a constant $M_{\alpha\beta}$ such that whenever open set $\mathbb{V}^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$ contains the crosscut $\Gamma_\circ^{\alpha\beta}$, there exists a homeomorphism $\hbar_3^{\alpha\beta} \colon \overline{\mathbb{U}^{\alpha\beta}} \xrightarrow{\text{onto}} \hbar_2(\overline{\mathbb{U}^{\alpha\beta}})$ which is a $\mathscr{C}^\infty$-diffeomorphism in $\mathbb{U}^{\alpha\beta}$, with the following properties

- $\hbar_3^{\alpha\beta} \equiv \hbar_2$ on $(\overline{\mathbb{U}^{\alpha\beta}} \setminus \mathbb{V}^{\alpha\beta}) \cup \Gamma_\circ^{\alpha\beta}$;
- $|\nabla \hbar_3^{\alpha\beta}| \leqq M_{\alpha\beta}$ and $\det \nabla \hbar_3^{\alpha\beta} \geqq \frac{1}{M_{\alpha\beta}}$ in $\mathbb{U}^{\alpha\beta}$.

Since $M_{\alpha\beta}$ does not depend on $\mathbb{V}^{\alpha\beta}$ it will be advantageous to take neighborhoods $\mathbb{V}^{\alpha\beta}$ of $\Gamma_\circ^{\alpha\beta}$ thin enough to satisfy

- $\overline{\mathbb{V}^{\alpha\beta}} \subset \mathbb{U}^{\alpha\beta} \cup \overline{\Gamma_\circ^{\alpha\beta}}$;
- $|\mathbb{V}^{\alpha\beta}| \leqq \frac{1}{5^p \cdot 2^{\alpha+\beta}} \left[\frac{\varepsilon}{m_{\alpha\beta}+M_{\alpha\beta}}\right]^p$ and also $|\mathbb{V}^{\alpha\beta}| \leqq \frac{\delta}{2^{\alpha+\beta} M_{\alpha\beta}}$.

Note that $\hbar_3^{\alpha\beta}, \hbar_2 \in \mathscr{W}^{1,\infty}(\mathbb{U}^{\alpha\beta}) \subset \mathscr{W}^{1,p}(\mathbb{U}^{\alpha\beta})$ and $\hbar_3^{\alpha\beta} = \hbar_2$ on $\partial\mathbb{U}^{\alpha\beta}$, so we have

$$\hbar_3^{\alpha\beta} - \hbar_2 \in \mathscr{W}_\circ^{1,p}(\mathbb{U}^{\alpha\beta}).$$

Step 3c We now define a homeomorphism $\hbar_3 \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by the rule

$$\hbar_3 = \begin{cases} \hbar_3^{\alpha\beta} & \text{in } \mathbb{U}^{\alpha\beta} \\ \hbar_2 & \text{in } \mathbb{X} \setminus \bigcup_{\alpha\beta} \mathbb{U}^{\alpha\beta}. \end{cases}$$

Obviously, $\hbar_3$ is a $\mathscr{C}^\infty$-diffeomorphism in $\mathbb{X}_\circ$ and $\hbar_3 - \hbar_2 \in \mathscr{W}_\circ^{1,p}(\mathbb{X}_\circ)$. Since $\hbar_3$ coincides with $\hbar_2$ outside $\mathbb{X}_\circ$, we have $\hbar_3 = \hbar_2 + [\hbar_3 - \hbar_2]_\circ$. Hence

$$\hbar_3 - \hbar_2 \in \mathcal{A}_\circ(\mathbb{X}). \tag{A_3}$$

Then, for every $x \in \mathbb{X}$,

$$|\hbar_3(x) - \hbar_2(x)| \leqq \begin{cases} \operatorname{diam} \hbar_2(\mathbb{U}^{\alpha\beta}), & \text{for } x \in \mathbb{U}^{\alpha\beta} \\ 0, & \text{otherwise} \end{cases} \leqq \operatorname{diam} \Upsilon^\alpha \leqq \frac{\varepsilon}{5}$$

meaning that

$$\|\hbar_3 - \hbar_2\|_{\mathscr{C}(\mathbb{X})} \leqq \frac{\varepsilon}{5}. \tag{B_3}$$

The computation of p-norms goes as follows

$$\begin{aligned}\|\nabla \hbar_3 - \nabla \hbar_2\|^p_{\mathscr{L}^p(X)} &= \sum_{\alpha\beta} \int_{\mathbb{V}^{\alpha\beta}} |\nabla \hbar_3 - \nabla \hbar_2|^p \\ &\leqq \sum_{\alpha\beta} |\mathbb{V}^{\alpha\beta}| \left[\|\nabla \hbar_3\|_{\mathscr{C}(\mathbb{V}^{\alpha\beta})} + \|\nabla \hbar_2\|_{\mathscr{C}(\mathbb{V}^{\alpha\beta})} \right]^p \\ &\leqq \sum_{\alpha\beta} |\mathbb{V}^{\alpha\beta}| \left(m_{\alpha\beta} + M_{\alpha\beta}\right)^p \leqq \sum_{\alpha\beta} \frac{\varepsilon^p}{5^p\, 2^{\alpha+\beta}} \leqq \left(\frac{\varepsilon}{5}\right)^p.\end{aligned}$$

Hence

$$\|\nabla \hbar_3 - \nabla \hbar_2\|_{\mathscr{L}^p(X)} \leqq \frac{\varepsilon}{5}. \tag{C_3}$$

In the finite energy case, when $\|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} < \infty$, we observe that

$$\|\nabla \hbar_3\|_{\mathscr{L}^p(\mathbb{X}\setminus \cup \mathbb{V}^{\alpha\beta})} = \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X}\setminus \cup \mathbb{V}^{\alpha\beta})} \leqq \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})}.$$

Therefore, by the triangle inequality,

$$\begin{aligned}\|\nabla \hbar_3\|_{\mathscr{L}^p(\mathbb{X})} &\leqq \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} + \sum_{\alpha\beta} \|\nabla \hbar_3\|_{\mathscr{L}^p(\mathbb{V}^{\alpha\beta})} \\ &\leqq \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} + \sum_{\alpha\beta} |\mathbb{V}^{\alpha\beta}| \cdot \|\nabla \hbar_3\|_{\mathscr{C}(\mathbb{V}^{\alpha\beta})} \\ &\leqq \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} + \sum_{\alpha\beta} \frac{\delta}{2^{\alpha+\beta} M_{\alpha\beta}} \cdot M_{\alpha\beta}\end{aligned}$$

which yields

$$\|\nabla \hbar_3\|_{\mathscr{L}^p(\mathbb{X})} \leqq \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} + \delta. \tag{D_3}$$

The third step is completed.

Step 4

We have already upgraded the mapping h to a homeomorphism $\hbar_3 \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, that is, a $\mathscr{C}^\infty$-diffeomorphism in $\mathbb{X}_\circ = \mathbb{X} \setminus \bigcup_{v\in\mathcal{V}(\mathbb{X})} \overline{\mathbb{F}_v}$, where $\mathbb{F}_v$ are small surroundings of the vertices of cells. Their images $\hbar_3(\mathbb{F}_v) = \hbar_2(\mathbb{F}_v) = \mathbb{D}_c$ are small disks centered at $c = h(v)$. In Step 3a, one of the preconditions on those disks was that $\operatorname{diam} \mathbb{D}_c < \varepsilon/5$. Furthermore, the closed disks $\overline{\mathbb{D}_c}$ are isolated continua in $\mathbb{Y}$ for all $c \in \mathcal{C}(\mathbb{Y})$, as are the sets $\overline{\mathbb{F}_v}$ in $\mathbb{X}$. We shall now consider slightly larger concentric open disks $\mathbb{D}'_c \supset \overline{\mathbb{D}_c}$, $c \in \mathcal{C}(\mathbb{Y})$, and their preimages $\mathbb{F}'_v = h_3^{-1}(\mathbb{D}'_c) \subset \mathbb{X}$, $v = h^{-1}(c) \in \mathcal{V}(\mathbb{X})$. The annulus $\mathbb{D}'_c \setminus \overline{\mathbb{D}_c}$ will be thin enough to ensure that $\mathbb{D}'_c$ are still disjoint,

$$\operatorname{diam} \mathbb{D}'_c < \frac{\varepsilon}{5} \quad \text{for all } c \in \mathcal{C}(\mathbb{Y})$$

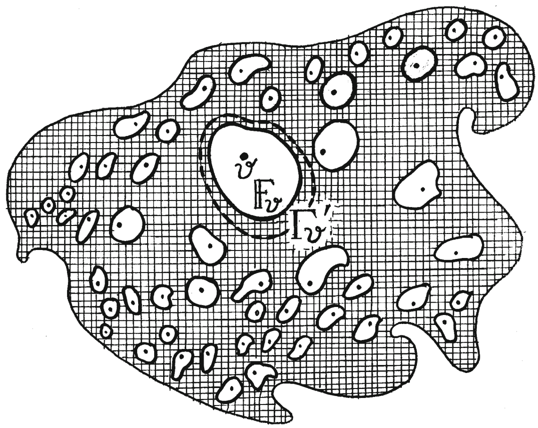

Fig. 5. Neighborhoods of vertices

and

$$\sum_{v\in\mathcal{V}(\mathbb{X})} \|\nabla \hbar_3\|^p_{\mathcal{L}^p(\mathbb{F}_v'\setminus\mathbb{F}_v)} \leqq \left(\frac{\varepsilon}{20}\right)^p .$$

Let Γ_v', $v\in\mathcal{V}(\mathbb{X})$ denote the boundary of $\mathbb{F}_v'$. These are $\mathscr{C}^\infty$-smooth Jordan curves, see Fig. 5. We now define a homeomorphism $\hbar_4\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by performing p-harmonic replacement of mappings $\hbar_3\colon \mathbb{F}_v' \xrightarrow{\text{onto}} \mathbb{D}_c'$, whenever such a mapping fails to be a $\mathscr{C}^\infty$-diffeomorphism. Thus every $\hbar_4\colon \mathbb{F}_v' \xrightarrow{\text{onto}} \mathbb{D}_c'$ is a $\mathscr{C}^\infty$-diffeomorphism up to Γ_v'. Moreover $\hbar_4 \in \hbar_3 + \mathscr{W}_\circ^{1,p}(\mathbb{F}_c')$, so

$$\hbar_4 - \hbar_3 \in \mathcal{A}_\circ(\mathbb{X}). \tag{A_4}$$

For every $x\in\mathbb{X}$, we have

$$|\hbar_4(x) - \hbar_3(x)| \leqq \begin{cases} \operatorname{diam}\mathbb{D}_c' & \text{in } \mathbb{F}_v',\ c = h(v) \\ 0 & \text{otherwise} \end{cases} \leqq \frac{\varepsilon}{5}.$$

Hence

$$\|\hbar_4 - \hbar_3\|_{\mathscr{C}(\mathbb{X})} \leqq \frac{\varepsilon}{5}. \tag{B_4}$$

By virtue of the minimum energy principle, we compute the p-norms

$$\begin{aligned} \|\hbar_4 - \hbar_3\|^p_{\mathscr{L}^p(\mathbb{X})} &= \sum_{v\in\mathcal{V}(\mathbb{X})} \|\hbar_4 - \hbar_3\|^p_{\mathscr{L}^p(\mathbb{F}_v')} \\ &\leqq \sum_{v\in\mathcal{V}(\mathbb{X})} \left[\|\hbar_4\|_{\mathscr{L}^p(\mathbb{F}_v')} + \|\hbar_3\|_{\mathscr{L}^p(\mathbb{F}_v')}\right]^p \\ &\leqq 2^p \sum_{v\in\mathcal{V}(\mathbb{X})} \|\hbar_3\|^p_{\mathscr{L}^p(\mathbb{F}_v')} \end{aligned}$$

$$\leqq 2^{2p-1} \sum_{v \in \mathcal{V}(\mathbb{X})} \left[\|\hbar_3\|^p_{\mathscr{L}^p(\mathbb{F}'_v \setminus \mathbb{F}_v)} + \|\hbar_3\|^p_{\mathscr{L}^p(\mathbb{F}_v)} \right]$$

$$\leqq 2^{2p-1} \left[\left(\frac{\varepsilon}{20}\right)^p + \sum_{v \in \mathcal{V}(\mathbb{X})} \|\hbar_2\|^p_{\mathscr{L}^p(\mathbb{F}_v)} \right]$$

$$\leqq 2^{2p} \left(\frac{\varepsilon}{20}\right)^p = \left(\frac{\varepsilon}{5}\right)^p .$$

Hence

$$\|\hbar_4 - \hbar_3\|_{\mathscr{L}^p(\mathbb{X})} \leqq \frac{\varepsilon}{5}. \tag{C_4}$$

Again, by the minimum energy principle, we find that

$$\|\hbar_4\|^p_{\mathscr{L}^p(\mathbb{X})} \leqq \|\hbar_3\|^p_{\mathscr{L}^p(\mathbb{X})}. \tag{D_4}$$

Just as in the previous steps, condition (E_4) remains valid, finishing Step 4.

Step 5

The final step consists of smoothing $\hbar_4$ in a neighborhood of each smooth Jordan curve Γ'_v, $v \in \mathcal{V}(\mathbb{X})$. We argue in much the same way as in Step 3, but this time we appeal to Proposition 5 instead of Proposition 4. By smoothing $\hbar_4$ in a sufficiently thin neighborhood of each Γ'_v we obtain a $\mathscr{C}^\infty$-diffeomorphism $\hbar_5 \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$,

$$\hbar_5 - \hbar_4 \in \mathcal{A}_\circ(\mathbb{X}). \tag{A_5}$$

$$\|\hbar_5 - \hbar_4\|_{\mathscr{C}(\mathbb{X})} \leqq \frac{\varepsilon}{5}. \tag{B_5}$$

$$\|\hbar_5 - \hbar_4\|_{\mathscr{L}^p(\mathbb{X})} \leqq \frac{\varepsilon}{5}. \tag{C_5}$$

$$\|\hbar_5\|_{\mathscr{L}^p(\mathbb{X})} \leqq \|\hbar_4\|_{\mathscr{L}^p(\mathbb{X})} + \delta. \tag{D_5}$$

□

4. Open questions

Question 3. *Does Theorem 1 extend to $n = 3$?*

Given the recent interest in bi-Sobolev homeomorphisms [24,25], it is natural to raise the following question.

Question 4. *A bi-Sobolev homeomorphism $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a mapping of class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 \leqq p < \infty$, whose inverse $h^{-1} \colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ belongs to a Sobolev class $\mathscr{W}^{1,q}(\mathbb{Y}, \mathbb{X})$, $1 \leqq q < \infty$. Can h be approximated by bi-Sobolev diffeomorphisms $\{h_\ell\}$ so that $h_\ell \to h$ in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ and $h_\ell^{-1} \to h^{-1}$ in $\mathscr{W}^{1,q}(\mathbb{Y}, \mathbb{X})$?*

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