Arch. Rational Mech. Anal. 201 (2011) 575–629
Digital Object Identifier (DOI) 10.1007/s00205-010-0395-6

Fracture Surfaces and the Regularity of Inverses for BV Deformations

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Communicated by S. Müller

Abstract

Motivated by nonlinear elasticity theory, we study deformations that are approximately differentiable, orientation-preserving and one-to-one almost everywhere, and in addition have finite *surface energy*. This surface energy $\mathcal{E}$ was used by the authors in a previous paper, and has connections with the theory of currents. In the present paper we prove that $\mathcal{E}$ measures exactly the area of the surface created by the deformation. This is done through a proper definition of *created surface*, which is related to the set of discontinuity points of the inverse of the deformation. In doing so, we also obtain an SBV regularity result for the inverse.

1. Introduction

In nonlinear elasticity theory, the total elastic energy of a deformation $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ of a body $\Omega \subset \mathbb{R}^n$ is given by

$$\int_\Omega W(\boldsymbol{x}, D\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}, \tag{1}$$

where n in the space dimension (which is usually assumed to be 3), and $W : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ is the elastic stored-energy function of the material. The seminal paper of Ball [6] proves existence of minimizers of (1) in a suitable class of Sobolev functions $\boldsymbol{u}$, under certain coercivity and polyconvexity assumptions on W. When cavitation or fracture are considered, the total energy of a deformation $\boldsymbol{u}$ will be the sum of the elastic energy (1) plus a term accounting for the energy needed to produce the cavitation or fracture. From the mathematical point of view, if one seeks minimizers of energy, that new term should enjoy the appropriate compactness and lower semicontinuity properties in order to make the direct method of the calculus of variations work.

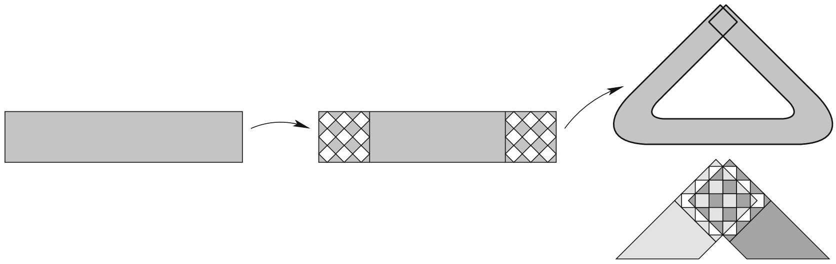

Fig. 1. Deformation that interlaces the cavities

In the quasi-static theory of fracture, the typical term accounting for the energy due to fracture is $\mathcal{H}^{n-1}(J_{\boldsymbol{u}})$, where $J_{\boldsymbol{u}}$ is the set of jump points of $\boldsymbol{u}$; see the pioneering papers [3,15] or the review paper [10] and the references therein. As for cavitation, the first variational model for cavitation that took into account the full n-dimensional case (as opposed to the radial case, which was studied earlier by BALL [7]) was due to MÜLLER AND SPECTOR [24]. This model has been influential in our work, and is explained in the next paragraph.

MÜLLER AND SPECTOR [24] proposed the term $\operatorname{Per} \boldsymbol{u}(\Omega)$ as an energy due to cavitation. Here Per denotes the perimeter of a set, and $\boldsymbol{u}(\Omega)$ is the image of Ω under $\boldsymbol{u}$ defined in a suitable way. Intuitively, $\operatorname{Per} \boldsymbol{u}(\Omega)$ measures the area of the cavities created by $\boldsymbol{u}$ together with the area of $\boldsymbol{u}(\partial\Omega)$. They pointed out, however, that, in some instances, the term $\operatorname{Per} \boldsymbol{u}(\Omega)$ fails to detect the area of the created cavities. Specifically, they constructed the deformation $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ depicted in Fig. 1. In that example, Ω is the rectangle $(-M, M) \times (-1, 1)$ for some $M \gg 1$. The deformation $\boldsymbol{u}$ is the composition of a first deformation that creates 9 cavities at each end of the rectangle, and then a deformation that bends the rectangle and interlaces the cavities in the way shown in the figure. From that figure, we can see why $\operatorname{Per} \boldsymbol{u}(\Omega)$ fails to detect the surface created: because two pieces of created surface have been put in contact, the common region of contact does not belong to the (reduced) boundary of $\boldsymbol{u}(\Omega)$. To overcome that inconvenience, they defined a topological condition which was called (INV) and is related to the invertibility of the deformation. Then they put the condition (INV) as a constraint in the admissible set of deformations, and, in this way, they excluded the deformation of Fig. 1 and other deformations with similar pathological behaviour.

In [18] we took a different approach. Instead of imposing a topological constraint, we replaced the term $\operatorname{Per} \boldsymbol{u}(\Omega)$ by $\mathcal{E}(\boldsymbol{u})$, which was defined as the supremum, when $\boldsymbol{f} \in C_c^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ and $\|\boldsymbol{f}\|_\infty \leqq 1$, of the quantity

$$\int_\Omega \left[\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}) \cdot D_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) + \det \nabla \boldsymbol{u}(\boldsymbol{x}) \operatorname{div}_{\boldsymbol{y}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\right] \mathrm{d}\boldsymbol{x}. \tag{2}$$

In [18] we motivated the definition of $\mathcal{E}$ as a surface energy, and proved existence of minimizers for the model

$$\int_\Omega W(\boldsymbol{x}, D\boldsymbol{u}(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} + \mathcal{E}(\boldsymbol{u}).$$

One of the main results of this paper is to provide a full proof of the fact that $\mathcal{E}(\boldsymbol{u})$ indeed provides the area of the created surface. Prior to that, it is necessary to have a definition of *created surface*. In this respect, the example of Fig. 1 shows that naive definitions such as 'the boundary of the image minus the image of the boundary' are inappropriate. In fact, Fig. 1 suggests that there seem to be two kinds of created surface: *visible* and *invisible*. Generically, at a point on a piece of visible created surface, there is matter on one side of the 'tangent hyperplane', and no matter on the other. In contrast, at a point on a piece of invisible created surface, there is matter on both sides, but the matter comes from two separated places in the reference configuration; thus, this piece of surface will not be detected by the term Per $\boldsymbol{u}(\Omega)$, and hence the name *invisible*. The example of Fig. 1 also indicates that the created surface has to do with the set of discontinuity points of the inverse of the deformation, and that the visible and the invisible surfaces correspond to two different kinds of discontinuity. Roughly speaking, at a point on an invisible surface, the inverse of $\boldsymbol{u}$ has a jump discontinuity, whereas at a point on a visible surface, it is the extension to $\mathbb{R}^n$ of the inverse of $\boldsymbol{u}$ by an arbitrary constant that has a jump discontinuity. Here the second topic of the paper appears: some regularity properties of the inverse are needed so that the set of its discontinuity points forms a 'surface'. We will see that an SBV regularity result for the inverse can be proved to be a consequence of the assumption $\mathcal{E}(\boldsymbol{u}) < \infty$.

Related results on the regularity of the inverse have appeared recently in [12, 19–23, 25]. The kind of results that they prove is that if a homeomorphism is Sobolev (or BV) and some extra condition holds, then the inverse is also Sobolev (or BV). Although similar in spirit, our result neither implies nor is implied by theirs, and the techniques are different.

In fact, the result on the SBV regularity of the inverse is natural once the following considerations have been made. The quantity $\mathcal{E}(\boldsymbol{u})$ is known in the theory of currents as the mass of the vertical part of the boundary of the current carried by the graph $\boldsymbol{u}$, and is denoted by $\mathbb{M}((\partial G_{\boldsymbol{u}})_{(n-1)})$. (For an exposition of the theory of currents, as well as for the notation and terminology used, we refer the reader to [16, 17]). On the other hand, it is easy to see that $\mathbb{M}((\partial G_{\boldsymbol{u}})_{(n-1)})$ coincides with the mass of the horizontal part of the boundary of the current carried by $\boldsymbol{u}^{-1}$, which is denoted by $\mathbb{M}((\partial G_{\boldsymbol{u}^{-1}})_{(0)})$. Now, Ambrosio [2] proved that a BV function is in SBV if and only if the mass of the horizontal part of the boundary of the current carried by its graph is finite. Putting these two things together would give (in principle) that if $\mathcal{E}(\boldsymbol{u}) < \infty$, then $\boldsymbol{u}^{-1}$ is in SBV. There are, however, two major difficulties that prevent our SBV regularity result from being just an immediate application of the two results mentioned above. The first is that the inverse of $\boldsymbol{u}$ is defined on (an appropriate definition of) $\boldsymbol{u}(\Omega)$, and this set does not necessarily coincide almost everywhere with an open set. Hence, the function spaces $BV(\boldsymbol{u}(\Omega), \mathbb{R}^n)$ or $SBV(\boldsymbol{u}(\Omega), \mathbb{R}^n)$ do not make sense. The second difficulty is, in fact, a preliminary step in the proof of the SBV regularity, and consists of previously showing that the inverse (defined in an appropriate way) is a function of bounded variation. The first difficulty is solved by truncating $\boldsymbol{u}$ in suitable open set U, and then extending the inverse of $\boldsymbol{u}|_U$ to $\mathbb{R}^n$ by an arbitrary constant.

Although we lack the needed notation, we state one of the main theorems of the paper in order to give an idea of the results proved here.

Theorem 1. *Let Ω be a bounded open set of $\mathbb{R}^n$ such that $\mathbf{0} \notin \bar{\Omega}$. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be a measurable map that is approximately differentiable in almost all Ω. Suppose that $\boldsymbol{u}$ is one-to-one almost everywhere,* $\det \nabla \boldsymbol{u} > 0$ *almost everywhere,* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n \times n})$ *and $\mathcal{E}(\boldsymbol{u}) < \infty$. Let $\Gamma_V(\boldsymbol{u})$ be the visible surface created by $\boldsymbol{u}$, and $\Gamma_I(\boldsymbol{u})$ the invisible surface created by $\boldsymbol{u}$, as defined in Definition 9. Then*

$$\mathcal{E}(\boldsymbol{u}) = \mathcal{H}^{n-1}(\Gamma_V(\boldsymbol{u})) + 2\mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u})). \tag{3}$$

Suppose, in addition, that $\det \nabla \boldsymbol{u} \in L^1_{\text{loc}}(\Omega)$*, and choose $\boldsymbol{x} \in \Omega$. Then, for almost every $r \in (0, \operatorname{dist}(\boldsymbol{x}, \partial\Omega))$, the function $\boldsymbol{u}^{-1}_{B(\boldsymbol{x},r)}$ defined in Definition 8 is in $SBV(\mathbb{R}^n, \mathbb{R}^n)$.*

The reason there is a factor 2 multiplying $\mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u}))$ in formula (3) is that the creation of invisible surface involves two pieces of created surface being put in contact with each other, so the area of the contact region must be accounted for twice. In particular, formula (3) meets our expectation for the deformation depicted in Fig. 1. The second result of Theorem 1 is an SBV regularity result for almost all truncations of the inverse of $\boldsymbol{u}$.

The assumption that $\boldsymbol{u}$ is one-to-one almost everywhere is not necessary for $\mathcal{E}(\boldsymbol{u})$ to be well defined or to have good lower semicontinuity and compactness properties. In contrast, it is essential in Theorem 1 in order to give $\mathcal{E}(\boldsymbol{u})$ the geometric interpretation of (3).

We now describe the contents of each section of the paper. In Section 2 we present the definitions and concepts that will be used throughout the paper, as well as some important preliminary results. In Section 3 we define a precise notion of the inverse $\boldsymbol{u}^{-1}$ and of the truncated inverse $\boldsymbol{u}^{-1}_U$ for any $U \subset \Omega$. Instead of working with the surface energy $\mathcal{E}$ defined above, we work with the related functional $\bar{\mathcal{E}}$, which satisfies $\bar{\mathcal{E}} \leqq \mathcal{E}$ and is such that the assumption $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$ is enough to carry out the analysis of Section 3. The main result of Section 3 is an SBV regularity result for $\boldsymbol{u}^{-1}_U$ under the assumption $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$. In Section 4 we establish the definitions of *created* surface $\Gamma(\boldsymbol{u})$, *visible* surface $\Gamma_V(\boldsymbol{u})$ and *invisible* surface $\Gamma_I(\boldsymbol{u})$ of a deformation $\boldsymbol{u}$, and show some important properties of these concepts.

From Section 4 onwards, the stronger assumption $\mathcal{E}(\boldsymbol{u}) < \infty$ is made (as opposed to $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$). This is essential in order to prove formula (3). We show a representation of the quantity $\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f})$, defined to be (2), as an integral over $\Gamma_I(\boldsymbol{u})$ and $\Gamma_V(\boldsymbol{u})$. That representation formula implies formula (3) and completes the proof of Theorem 1. Section 5 continues the study of the created surface initiated in Section 4. We prove some important relationships between several concepts of surface created by $\boldsymbol{u}$, the set of discontinuity points of the inverse, the image of the boundary, and the boundary of the image. In Section 6 we regard $\mathcal{E}(\boldsymbol{u})$ as a positive Radon measure $\mu_{\boldsymbol{u}}$, and show that it provides us with a natural generalization of the singular part of the distributional determinant. In addition, we prove that $\mu_{\boldsymbol{u}}$ satisfies the local counterpart of formula (3), so that $\mu_{\boldsymbol{u}}(U)$ measures the area of the surface created by U in $\boldsymbol{u}$, for 'good' open sets $U \subset \Omega$. Moreover, we find the set of singularities of $\boldsymbol{u}$ (in the reference configuration) that are responsible for the creation of surface. In Section 7, the following intuitive idea is made precise: the invisible surface can become visible if $\boldsymbol{u}$ is restricted to smaller parts of the body; for instance, the invisible surface created by the deformation of Fig. 1 becomes visible if we consider separately the deformation restricted to the left and to the right halves of the rectangle. In this way, we obtain a representation formula for $\mu_{\boldsymbol{u}}$ which states that for any open set $U \subset \Omega$, the quantity $\mu_{\boldsymbol{u}}(U)$ equals the supremum, among all families of 'good' disjoint balls B contained in U, of the sum of the area of the surface created by $\boldsymbol{u}$ in B. Finally, in Section 8 we derive the first variation for the surface energy $\mathcal{E}$, and, as a consequence, the equilibrium equations for the models of cavitation and fracture proposed in [18].

2. Notation and preliminaries

In this section we set the general notation of this paper, and state some important preliminary results. Most of our notation is standard and follows that of [5].

2.1. General notation

We will work in dimension n, and tacitly assume that $n \geqq 2$. Unless otherwise stated, expressions such as *measurable* or *almost everywhere* refer to the Lebesgue measure in $\mathbb{R}^n$, which is denoted by $\mathcal{L}^n$. The m-dimensional Hausdorff measure will be indicated by $\mathcal{H}^m$. Usually, m will be $n-1$. Our basic object will be the deformation, which is a measurable map $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ satisfying certain conditions, and where Ω is a bounded open set of $\mathbb{R}^n$ representing the reference configuration of a body. Vector-valued and matrix-valued quantities will be written in boldface. Coordinates in the reference configuration will generically be denoted by $\boldsymbol{x}$, while coordinates in the deformed configuration by $\boldsymbol{y}$. The divergence operator in the reference configuration (so with respect to the $\boldsymbol{x}$ coordinates) is denoted by Div, while div denotes the divergence operator in the deformed configuration (with respect to $\boldsymbol{y}$).

The closure of a set A is denoted by $\bar{A}$, its boundary by ∂A, and its interior by $\mathring{A}$. Given two open sets U, V of $\mathbb{R}^n$, we will say that U is compactly contained in V if $\bar{U} \subset V$; in this case, we will write $U \subset\subset V$. The open ball of radius $r > 0$ centred at $\boldsymbol{x} \in \mathbb{R}^n$ is denoted by $B(\boldsymbol{x}, r)$. Unless otherwise stated, a *ball* will always be an open ball. Half-spaces are denoted by

$$H^+(\boldsymbol{a}, \boldsymbol{\nu}) := \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x} - \boldsymbol{a}) \cdot \boldsymbol{\nu} \geqq 0\}, \quad H^-(\boldsymbol{a}, \boldsymbol{\nu}) := H^+(\boldsymbol{a}, -\boldsymbol{\nu}),$$

for a given $\boldsymbol{a} \in \mathbb{R}^n$ and a nonzero vector $\boldsymbol{\nu} \in \mathbb{R}^n$. The set of unit vectors in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$.

The identity matrix is denoted by $\mathbf{1}$. Given a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, its transpose is denoted by $\boldsymbol{A}^T$ and its determinant by $\det \boldsymbol{A}$. The cofactor matrix of $\boldsymbol{A}$, denoted by $\operatorname{cof} \boldsymbol{A}$, is the matrix that satisfies $(\det \boldsymbol{A})\mathbf{1} = \boldsymbol{A}^T \operatorname{cof} \boldsymbol{A}$. The transpose of $\operatorname{cof} \boldsymbol{A}$ is the adjoint matrix of $\boldsymbol{A}$, denoted by $\operatorname{adj} \boldsymbol{A}$. If $\boldsymbol{A}$ is invertible, its inverse is denoted by $\boldsymbol{A}^{-1}$, and the transpose of its inverse by $\boldsymbol{A}^{-T}$. The inner (dot) product of vectors will be denoted by $\cdot$, and the same notation will be used for the inner product of matrices. The tensor product of two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is denoted by $\boldsymbol{x} \otimes \boldsymbol{y}$, and is the matrix whose (i, j)th entry is $x_i y_j$. The Euclidean norm of a vector $\boldsymbol{x}$ is denoted by $|\boldsymbol{x}|$, and the associated matrix norm is also denoted by $|\cdot|$.

The identity function in $\mathbb{R}^n$ is denoted by **id**. We will denote by $\|\mathbf{id}\|_{L^\infty(\Omega,\mathbb{R}^n)}$ the norm of **id** as an element of $L^\infty(\Omega, \mathbb{R}^n)$, that is, $\sup_{\boldsymbol{x}\in\Omega} |\boldsymbol{x}|$.

Given a sequence of sets $\{V_k\}_{k\in\mathbb{N}}$, its inferior limit is defined as

$$\liminf_{k\to\infty} V_k := \bigcup_{p\in\mathbb{N}} \bigcap_{k\geqq p} V_k.$$

Given two sets A, B of $\mathbb{R}^n$, we will write $A \tilde{\subset} B$ if $\mathcal{H}^{n-1}(A\setminus B) = 0$. We will write $A \cong B$ if $A \tilde{\subset} B$ and $B \tilde{\subset} A$.

If μ is a measure on a set U, and V is a μ-measurable subset of U, then the restriction of μ to V, denoted by $\mu \llcorner V$, is the measure on U that satisfies $\mu \llcorner V(A) = \mu(A \cap V)$ for all μ-measurable sets A. The measure $|\mu|$ denotes the total variation of μ. The support of a measure μ or of a function f is denoted by $\operatorname{spt} \mu$ or $\operatorname{spt} f$, respectively. As usual, $⨍_A$ denotes $\int_A$ divided by the measure of A.

With $\langle \cdot, \cdot \rangle$ we will indicate the duality product, usually between a measure and a continuous function, although sometimes between a distribution and a smooth function.

A set E of $\mathbb{R}^n$ is said to be countably $\mathcal{H}^{n-1}$ rectifiable if for each $i \in \mathbb{N}$ there exists a Lipschitz map $\boldsymbol{f}_i : \mathbb{R}^{n-1} \to \mathbb{R}^n$ such that $\mathcal{H}^{n-1}(E\setminus \bigcup_{i\in\mathbb{N}} \boldsymbol{f}_i(\mathbb{R}^{n-1})) = 0$.

2.2. Approximate continuity and differentiability

Given a measurable set $A \subset \mathbb{R}^n$ and a point $\boldsymbol{x} \in \mathbb{R}^n$, the density of A at $\boldsymbol{x}$ is defined as

$$D(A, \boldsymbol{x}) := \lim_{r\searrow 0} \frac{\mathcal{L}^n(B(\boldsymbol{x}, r) \cap A)}{\mathcal{L}^n(B(\boldsymbol{x}, r))},$$

if that limit exists. The following notions are essentially due to FEDERER [14].

Definition 1. Let A be a measurable set in $\mathbb{R}^n$, and $\boldsymbol{u} : A \to \mathbb{R}^n$ a measurable function. Let $\boldsymbol{x}_0 \in \mathbb{R}^n$ satisfy $D(A, \boldsymbol{x}_0) = 1$, and let $\boldsymbol{y}_0 \in \mathbb{R}^n$.

(a) We will say that the approximate limit of $\boldsymbol{u}$ at $\boldsymbol{x}_0$ is $\boldsymbol{y}_0$ when

$$D\left(\{\boldsymbol{x} \in A : |\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{y}_0| \geqq \delta\}, \boldsymbol{x}_0\right) = 0 \qquad \text{for each } \delta > 0.$$

In this case, we will write $\operatorname{ap}\lim_{\boldsymbol{x}\to\boldsymbol{x}_0} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}_0$.

(b) Given a measurable set $F \subset A$ with $D(F, \boldsymbol{x}_0) > 0$, we will write

$$\operatorname{ap}\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0 \\ \boldsymbol{x}\in F}} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}_0$$

when $D\left(\{\boldsymbol{x} \in F : |\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{y}_0| \geqq \delta\}, \boldsymbol{x}_0\right) = 0$ for each $\delta > 0$.

(c) We will say that $\boldsymbol{u}$ is approximately continuous at $\boldsymbol{x}_0$ if $\boldsymbol{x}_0 \in A$ and $\operatorname{ap\,lim}_{\boldsymbol{x}\to\boldsymbol{x}_0} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}_0)$. The set $S_{\boldsymbol{u}}$ denotes the set of points at which A has density 1 and $\boldsymbol{u}$ is not approximately continuous.

(d) We will say that $\boldsymbol{x}_0$ is an approximate jump point of $\boldsymbol{u}$ if there exist $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{\nu} \in \mathbb{S}^{n-1}$ such that $\boldsymbol{a} \neq \boldsymbol{b}$, and

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0 \\ \boldsymbol{x}\in H^+(\boldsymbol{x}_0,\boldsymbol{\nu})}} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a}, \qquad \operatorname*{ap\,lim}_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0 \\ \boldsymbol{x}\in H^-(\boldsymbol{x}_0,\boldsymbol{\nu})}} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{b}.$$

The unit vector $\boldsymbol{\nu}$ is uniquely determined up to a sign. When a choice of the unit vector has been made, we say that it has been oriented, and the chosen unit vector is called the orientation vector and denoted by $\boldsymbol{\nu}_{\boldsymbol{u}}(\boldsymbol{x}_0)$. The points $\boldsymbol{a}$ and $\boldsymbol{b}$ are called the lateral traces of $\boldsymbol{u}$ at $\boldsymbol{x}_0$ with respect to the orientation $\boldsymbol{\nu}_{\boldsymbol{u}}(\boldsymbol{x}_0)$, and are denoted by $\boldsymbol{u}^+(\boldsymbol{x}_0)$ and $\boldsymbol{u}^-(\boldsymbol{x}_0)$, respectively. The set of approximate jump points of $\boldsymbol{u}$ is called the jump set of $\boldsymbol{u}$, and is denoted by $J_{\boldsymbol{u}}$.

(e) We will say that $\boldsymbol{u}$ is approximately differentiable at $\boldsymbol{x}_0$ if $\boldsymbol{x}_0 \in A$ and there exists $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ such that

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0 \\ \boldsymbol{x}\in A\setminus\{\boldsymbol{x}_0\}}} \frac{|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}_0) - \boldsymbol{L}(\boldsymbol{x}-\boldsymbol{x}_0)|}{|\boldsymbol{x}-\boldsymbol{x}_0|} = 0.$$

In this case, $\boldsymbol{L}$ (which is uniquely determined) is called the approximate differential of $\boldsymbol{u}$ at $\boldsymbol{x}_0$, and will be denoted by $\nabla\boldsymbol{u}(\boldsymbol{x}_0)$.

We will say that a map $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ is approximately differentiable in almost all Ω when it is measurable and approximately differentiable at almost each point of Ω. The set of approximate differentiability points of $\boldsymbol{u}$ is usually called Ω_d.

2.3. Function spaces, perimeter and boundary

If $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ is a function locally of bounded variation, $D\boldsymbol{u}$ denotes the distributional derivative of $\boldsymbol{u}$, which is a Radon measure in Ω. We will use the decomposition of $D\boldsymbol{u}$ in the absolutely continuous part $D^a\boldsymbol{u}$, the jump part $D^j\boldsymbol{u}$ and the Cantor part $D^c\boldsymbol{u}$; the singular part is denoted by $D^s\boldsymbol{u}$, and satisfies $D^s\boldsymbol{u} = D^c\boldsymbol{u} + D^j\boldsymbol{u}$. Our notation and definitions follow [5, Ch. 3]. By virtue of the Calderón–Zygmund theorem (see, for example, [5, Th. 3.83]), the density of $D^a\boldsymbol{u}$ with respect to $\mathcal{L}^n$ coincides almost everywhere with the approximate differential $\nabla\boldsymbol{u}$.

Note that we do not identify functions that coincide almost everywhere. The precise representative $\tilde{\boldsymbol{u}} : \mathbb{R}^n \to \mathbb{R}^n$ of the measurable function $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}_0) := \begin{cases} \operatorname{ap\,lim}_{\boldsymbol{x}\to\boldsymbol{x}_0} \boldsymbol{u}(\boldsymbol{x}) & \text{if } \operatorname{ap\,lim}_{\boldsymbol{x}\to\boldsymbol{x}_0} \boldsymbol{u}(\boldsymbol{x}) \text{ exists} \\ \boldsymbol{0} & \text{otherwise.} \end{cases}$$

The Lebesgue L^p and Sobolev $W^{1,p}$ spaces are defined in the usual way. So are the set of smooth functions C^∞, of bounded variation BV and of special bounded variation SBV; see, if necessary, [5] for the definitions. The set $C_c^\infty(\Omega, \mathbb{R}^n)$ denotes the space of C^∞ functions with compact support in Ω. We will always indicate the domain and target space, as in, for example, $L^1(\Omega, \mathbb{R}^n)$, except if the target space is $\mathbb{R}$, in which case we will simply write $L^1(\Omega)$. Sometimes we will use Lebesgue spaces in $n-1$ dimensional sets; for example, if Ω is a set with a Lipschitz boundary, then $L^1(\partial\Omega)$ denotes the Lebesgue L^1 space on $\partial\Omega$ with respect to the $\mathcal{H}^{n-1}$ measure. From the context it will be clear that these spaces are defined with respect to the $\mathcal{H}^{n-1}$ measure, and not to the $\mathcal{L}^n$ measure, so we will not indicate it explicitly.

The variation of $\boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ in the open set Ω is denoted by $V(\boldsymbol{u}, \Omega)$, and defined as

$$V(\boldsymbol{u}, \Omega) := \sup\left\{ \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \operatorname{Div} \boldsymbol{\phi}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} : \ \boldsymbol{\phi} \in C_c^\infty(\Omega, \mathbb{R}^{n\times n}), \ \|\boldsymbol{\phi}\|_\infty \leqq 1 \right\}.$$

Given a measurable set $A \subset \mathbb{R}^n$, its characteristic function will be denoted by χ_A, and its perimeter by $\operatorname{Per} A$ or $\operatorname{Per}(A)$, which is defined as

$$\operatorname{Per} A := \sup\left\{ \int_A \operatorname{div} \boldsymbol{g}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} : \ \boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n), \ \|\boldsymbol{g}\|_\infty \leqq 1 \right\}.$$

Definition 2. Let E be a measurable set of $\mathbb{R}^n$. We define the reduced boundary of E, and denote it by $\partial^* E$, as the set of points $\boldsymbol{y} \in \mathbb{R}^n$ for which a unit vector $\boldsymbol{\nu}_E(\boldsymbol{y})$ exists such that

$$D(E \cap H^-(\boldsymbol{y}, \boldsymbol{\nu}_E(\boldsymbol{y})), \boldsymbol{y}) = \frac{1}{2} \quad \text{and} \quad D(E \cap H^+(\boldsymbol{y}, \boldsymbol{\nu}_E(\boldsymbol{y})), \boldsymbol{y}) = 0.$$

This $\boldsymbol{\nu}_E(\boldsymbol{y})$ is uniquely determined and is called the unit outward normal to E.

This definition of reduced boundary, as well as its notation, may differ from other usual definitions, but thanks to FEDERER's [14] theorem (see also [5, Th. 3.61] or [28, Sect. 5.6]), it coincides $\mathcal{H}^{n-1}$-almost everywhere with all other definitions of reduced (or *essential* or *measure-theoretic*) boundary for sets of finite perimeter.

2.4. Geometric image

The following definition, due to CONTI AND DE LELLIS [11], is an adaptation of that of MÜLLER AND SPECTOR [24].

Definition 3. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω and suppose that $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ for almost every $\boldsymbol{x} \in \Omega$. Define Ω_0 as the set of $\boldsymbol{x} \in \Omega$ such that $\boldsymbol{u}$ is approximately differentiable at $\boldsymbol{x}$ with $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$, and there exist $\boldsymbol{w} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and a compact set $K \subset \Omega$ of density 1 at $\boldsymbol{x}$ such that $\boldsymbol{u}|_K = \boldsymbol{w}|_K$ and $\nabla \boldsymbol{u}|_K = D\boldsymbol{w}|_K$. For any measurable set A of Ω, we define the geometric image of A under $\boldsymbol{u}$ as $\boldsymbol{u}(A \cap \Omega_0)$, and denote it by $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, A)$.

The condition $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ for almost everywhere $\boldsymbol{x} \in \Omega$ has been included in the hypotheses so that many pointwise properties of $\boldsymbol{u}$ (such as those of Lemma 1) hold for every $\boldsymbol{x} \in \Omega_0$. As for the remaining part of the definition, the motivation comes from the following properties of approximately differentiable maps. First, the set Ω_d of points of approximate differentiability can be written as the union of a countable family of measurable sets $\{A_j\}_{j\in\mathbb{N}}$ such that $\boldsymbol{u}|_{A_j}$ is Lipschitz continuous (Federer [14, Th. 3.1.8]). Combining this with Rademacher's theorem and Whitney's extension theorem, it is possible to find (see [14, Th. 3.1.16]) an increasing sequence of compact sets $\{K_j\}_{j\in\mathbb{N}}$ contained in Ω, and a sequence $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}$ of maps in $C^1(\mathbb{R}^n, \mathbb{R}^n)$, such that

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{j=1}^{\infty} K_j\right) = 0, \qquad \boldsymbol{u}|_{K_j} = \boldsymbol{w}_j|_{K_j}, \qquad (\nabla \boldsymbol{u})|_{K_j} = D\boldsymbol{w}_j|_{K_j}. \tag{4}$$

Letting K'_j denote the set of points of density 1 for K_j, it is easy to see that the set Ω_0 of Definition 3 contains $\bigcup_{j\in\mathbb{N}} K'_j$. Thus, the set Ω_0 is of full measure in Ω.

By Proposition 1 below we have that

$$\mathcal{L}^n(\boldsymbol{u}(N \cap \Omega_d)) = 0 \quad \text{whenever} \quad \mathcal{L}^n(N) = 0. \tag{5}$$

Consequently, it is equivalent (up to Lebesgue null sets) to define the geometric image of A as $\boldsymbol{u}(A\cap\Omega_d)$ or as $\boldsymbol{u}(A\cap\Omega_0)$. We have chosen to define it as $\boldsymbol{u}(A\cap\Omega_0)$ because there are definite advantages in working with the set of points at which $\boldsymbol{u}$ has a C^1 extension. In this paper, this will be manifest in the definition of the inverse $\boldsymbol{u}^{-1}$ of an approximately differentiable map $\boldsymbol{u}$, in the study of regularity properties for this inverse, and in the study of the notion of a fracture surface. We shall use, in particular, the following result due to Müller and Spector [24, Lemma 2.5].

Lemma 1. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω and suppose that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every $\boldsymbol{x} \in \Omega$. Let Ω_0 be as in Definition 3. Then for every $\boldsymbol{x} \in \Omega_0$ and every measurable set $A \subset \Omega$,*

$$D(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, A), \boldsymbol{u}(\boldsymbol{x})) = 1 \quad \textit{whenever} \quad D(A, \boldsymbol{x}) = 1.$$

Moreover, if $\boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and we define $\bar{\boldsymbol{v}} := (\operatorname{sgn} \det \nabla \boldsymbol{u}(\boldsymbol{x}))(\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}))\boldsymbol{v}$, *then for every $\boldsymbol{x} \in \Omega_0$,*

$$D(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, A) \cap H^+(\boldsymbol{u}(\boldsymbol{x}), \bar{\boldsymbol{v}}), \boldsymbol{u}(\boldsymbol{x})) = \frac{1}{2} \quad \textit{whenever} \quad D(A \cap H^+(\boldsymbol{x}, \boldsymbol{v}), \boldsymbol{x}) = \frac{1}{2}.$$

2.5. Change of variables in volume and surface integrals

We now recall the *area formula* of Federer [14, Thms. 3.1.8, 3.2.3 and 3.2.5], the formulation of which is taken from [24, Prop. 2.6]. In the statement below, $\mathbb{N}(\boldsymbol{u}, A, \boldsymbol{y})$ denotes the number of preimages under $\boldsymbol{u}$ of a point $\boldsymbol{y}$ in the set A.

Proposition 1. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, and call Ω_d the set of approximate differentiability points of $\boldsymbol{u}$. Then, for any measurable set $A \subset \Omega$ and any measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_A (\varphi \circ \boldsymbol{u}) |\det \nabla \boldsymbol{u}| \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \varphi(\boldsymbol{y}) \mathbb{N}(\boldsymbol{u}, \Omega_d \cap A, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}, \tag{6}$$

whenever either integral exists. Moreover, if $\psi : A \to \mathbb{R}$ *is measurable and* $\bar{\psi} : \boldsymbol{u}(\Omega_d \cap A) \to \mathbb{R}$ *is given by*

$$\bar{\psi}(\boldsymbol{y}) := \sum_{\substack{\boldsymbol{x} \in \Omega_d \cap A \\ \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}}} \psi(\boldsymbol{x}),$$

then $\bar{\psi}$ *is measurable and*

$$\int_A \psi(\boldsymbol{\varphi} \circ \boldsymbol{u}) |\det \nabla \boldsymbol{u}| \, \mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{u}(\Omega_d \cap A)} \bar{\psi} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{y} \tag{7}$$

whenever the integral on the left-hand side of (7) *exists.*

In the analysis of the created surface of Section 5, we will use the $(n-1)$-dimensional change of variables formula for approximately differentiable maps. In order to state this formula, we recall the notion of tangential approximate differentiability, and some notation from multilinear algebra. We restrict our attention to the case of maps defined on C^1 manifolds, although all what follows also holds, with a suitable notion of tangent space, for maps defined on countably $\mathcal{H}^{n-1}$ rectifiable sets. The following definition is due to FEDERER [14, Def. 3.2.16].

Definition 4. Let $S \subset \mathbb{R}^n$ be a C^1 differentiable manifold of dimension $n-1$, and let $\boldsymbol{x}_0 \in S$. Let $T_{\boldsymbol{x}_0} S$ be the linear tangent space of S at $\boldsymbol{x}_0$. A map $\boldsymbol{u} : S \to \mathbb{R}^n$ is said to be $\mathcal{H}^{n-1} \llcorner S$-approximately differentiable at $\boldsymbol{x}_0$ if there exists $\boldsymbol{L} \in \mathbb{R}^{n \times n}$ such that for all $\delta > 0$,

$$\lim_{r \searrow 0} r^{-(n-1)} \mathcal{H}^{n-1} \left(\left\{ \boldsymbol{x} \in S \cap B(\boldsymbol{x}_0, r) : \frac{|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}_0) - \boldsymbol{L}(\boldsymbol{x} - \boldsymbol{x}_0)|}{|\boldsymbol{x} - \boldsymbol{x}_0|} \geqq \delta \right\} \right) = 0.$$

The linear map $\boldsymbol{L}|_{T_{\boldsymbol{x}_0} S} : T_{\boldsymbol{x}_0} S \to \mathbb{R}^n$ is uniquely determined and called the tangential approximate derivative of $\boldsymbol{u}$ at $\boldsymbol{x}_0$. We denote it by $\nabla \boldsymbol{u}(\boldsymbol{x}_0)$.

If $\boldsymbol{L} : V \to \mathbb{R}^n$ is a linear transformation, where V is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, the transformation $\Lambda_{n-1} \boldsymbol{L} : \Lambda_{n-1} V \to \mathbb{R}^n$ is defined by

$$(\Lambda_{n-1} \boldsymbol{L})(\boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_{n-1}) := \boldsymbol{L}\boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{L}\boldsymbol{a}_{n-1}, \quad \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1} \in V.$$

Here $\wedge$ denotes the exterior product between vectors in $\mathbb{R}^n$. Since the subspace $\Lambda_{n-1} V$ can be identified with $\{\lambda \boldsymbol{v} : \lambda \in \mathbb{R}\}$, where $\boldsymbol{v}$ is a unit vector normal to V, the linear transformation $\Lambda_{n-1} \boldsymbol{L}$ is determined by the value $(\Lambda_{n-1} \boldsymbol{L}) \boldsymbol{v}$. This value can be computed as

$$(\Lambda_{n-1} \boldsymbol{L}) \boldsymbol{v} = (\operatorname{cof} \tilde{\boldsymbol{L}}) \boldsymbol{v}, \tag{8}$$

provided $\tilde{\boldsymbol{L}} : \mathbb{R}^n \to \mathbb{R}^n$ extends $\boldsymbol{L}$ linearly from V to $\mathbb{R}^n$. For a thorough exposition of the concepts and properties mentioned refer, for example, to [14, Ch. 1] or [26, Ch. 4].

The following area formula is due to FEDERER (for the the first part, see [14, Cor. 3.2.20]; for the second, use the standard technique of approximating non-negative functions by an increasing sequence of simple functions).

Proposition 2. *Let $S \subset \Omega$ be a C^1 differentiable manifold of dimension $n-1$.*

a) *Let $\boldsymbol{u} : S \to \mathbb{R}^n$ be $\mathcal{H}^{n-1} \llcorner S$-approximately differentiable in $\mathcal{H}^{n-1}$-almost all S. Denote the set of points of $\mathcal{H}^{n-1} \llcorner S$-approximate differentiability of $\boldsymbol{u}$ by S_d. Then, for any $\mathcal{H}^{n-1}$-measurable subset $A \subset S$,*

$$\int_A |\Lambda_{n-1} \nabla \boldsymbol{u}(\boldsymbol{x})| \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) = \int_{\mathbb{R}^n} \mathbb{N}(\boldsymbol{u}, S_d \cap A, \boldsymbol{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}),$$

whenever either integral exists.

b) *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, and such that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every $\boldsymbol{x} \in \Omega$. Let Ω_0 be the set of Definition 3. Suppose that a set $S_d \subset \Omega_0 \cap S$ exists such that $\mathcal{H}^{n-1}(S \setminus S_d) = 0$, and such that for every $\boldsymbol{x} \in S_d$ the restriction $\boldsymbol{u}|_S$ is $\mathcal{H}^{n-1} \llcorner S$-approximately differentiable at $\boldsymbol{x}$, and $\nabla(\boldsymbol{u}|_S)(\boldsymbol{x}) = \nabla \boldsymbol{u}(\boldsymbol{x})|_{T_{\boldsymbol{x}} S}$. Suppose, further, that* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1(S, \mathbb{R}^{n \times n})$. *Then, for every bounded and measurable $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$, and any $\mathcal{H}^{n-1}$-measurable subset $A \subset S$,*

$$\int_A \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) \boldsymbol{\nu}(\boldsymbol{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) = \int_{\boldsymbol{u}(S_d \cap A)} \boldsymbol{g}(\boldsymbol{y}) \cdot \tilde{\boldsymbol{\nu}}(\boldsymbol{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}), \tag{9}$$

where $\boldsymbol{\nu}$ denotes the unit normal to S, and

$$\tilde{\boldsymbol{\nu}}(\boldsymbol{y}) := \sum_{\substack{\boldsymbol{x} \in S_d \cap A \\ \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}}} \frac{(\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) \boldsymbol{\nu}(\boldsymbol{x})}{|(\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) \boldsymbol{\nu}(\boldsymbol{x})|}, \qquad \boldsymbol{y} \in \boldsymbol{u}(S_d \cap A).$$

Definition 5. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be measurable. For each open set $U \subset\subset \Omega$ with a C^1 boundary, we denote the set of $\mathcal{H}^{n-1} \llcorner \partial U$-approximate differentiability points of $\boldsymbol{u}|_{\partial U}$ by $\partial_d U$.

2.6. A class of 'good' open sets

The following well-known property (see for example, [13, Th. 16.25.2; 27, p. 112] or [24, p. 48]) allows us to parametrize a tubular neighbourhood of the boundary of C^2 open sets.

Proposition 3. *Let U be an open set compactly contained in Ω with a C^2 boundary. Let $\boldsymbol{\nu} : \partial U \to \mathbb{R}^n$ be the exterior unit normal. Then there exists $\delta > 0$ such that the map $\boldsymbol{w} : \partial U \times (-\delta, \delta) \to \Omega$ given by*

$$\boldsymbol{w}(\boldsymbol{x}, t) = \boldsymbol{x} - t \boldsymbol{\nu}(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial U, \ t \in \mathbb{R},$$

is a C^1 diffeomorphism between $\partial U \times (-\delta, \delta)$ and

$$N(\partial U, \delta) := \{\boldsymbol{x} \in \Omega : \ \operatorname{dist}(\boldsymbol{x}, \partial U) < \delta\}. \tag{10}$$

Moreover, the function $d : \Omega \to \mathbb{R}$ *given by*

$$d(\boldsymbol{x}) := \begin{cases} \operatorname{dist}(\boldsymbol{x}, \partial U) & \text{if } \boldsymbol{x} \in U \\ 0 & \text{if } \boldsymbol{x} \in \partial U \\ -\operatorname{dist}(\boldsymbol{x}, \partial U) & \text{if } \boldsymbol{x} \in \Omega \setminus \bar{U} \end{cases} \tag{11}$$

is continuous in Ω *and of class* C^2 *in* $N(\partial U, \delta)$, *and for every* $\boldsymbol{x} \in \partial U$ *and* $t \in (-\delta, \delta)$,

$$Dd(\boldsymbol{x} - t\boldsymbol{\nu}(\boldsymbol{x})) = -\boldsymbol{\nu}(\boldsymbol{x}) = -\boldsymbol{\nu}_t(\boldsymbol{x} - t\boldsymbol{\nu}(\boldsymbol{x})), \tag{12}$$

where $\boldsymbol{\nu}_t$ *is the unit exterior normal to* U_t. *Moreover, for every* $\boldsymbol{x} \in N(\partial U, \delta)$ *there exists a unique* $\boldsymbol{\xi}(\boldsymbol{x}) \in \partial U$ *such that* $|d(\boldsymbol{x})| = |\boldsymbol{x} - \boldsymbol{\xi}(\boldsymbol{x})|$. *Finally, for each* $\boldsymbol{x} \in \partial U$ *and* $t \in (-\delta, \delta)$, *we have that* $d(\boldsymbol{x} - t\boldsymbol{\nu}(\boldsymbol{x})) = t$ *and* $\boldsymbol{\xi}(\boldsymbol{x} - t\boldsymbol{\nu}(\boldsymbol{x})) = \boldsymbol{x}$.

For each $t \in (-\delta, \delta)$ we define the open set

$$U_t := \{\boldsymbol{x} \in \Omega : \ d(\boldsymbol{x}) > t\}. \tag{13}$$

Then, $\partial U_t = \{\boldsymbol{x} \in \Omega : \ d(\boldsymbol{x}) = t\}$, and $(U_t)_s = U_{t+s}$ for all $s \in (-\delta - t, \delta - t)$.

We shall study the behaviour of $\boldsymbol{u}$ in a suitable class of intermediate open sets, compactly contained in the domain. In Definition 6 we impose certain requirements that the trace of $\boldsymbol{u}$ on the boundary of those open sets must satisfy, in order to carry out this analysis. In Lemma 2 we show that the conditions are satisfied for open sets of the form U_t, for almost every t small enough, with U_t defined as in (13).

Definition 6. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω. We define $\mathcal{U}_{\boldsymbol{u}}$ as the class of nonempty open sets U that are compactly contained in Ω and satisfy the following conditions:

i) U has a C^2 boundary.
ii) $\operatorname{cof} \nabla \boldsymbol{u} \in L^1(\partial U, \mathbb{R}^{n \times n})$.
iii) $\lim_{j\to\infty} \fint_0^{\frac{1}{j}} \int_{\partial U_t} |\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})| \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) \, \mathrm{d}t = \|\operatorname{cof} \nabla \boldsymbol{u}\|_{L^1(\partial U, \mathbb{R}^{n\times n})}$, where the sets U_t are defined in (13).
iv) $\mathcal{H}^{n-1}(\partial U \setminus \Omega_0) = 0$, where Ω_0 is the set of Definition 3.
v) $\boldsymbol{u}|_{\partial U} : \partial U \to \mathbb{R}^n$ is $\mathcal{H}^{n-1} \llcorner \partial U$-approximately differentiable at $\boldsymbol{x}$, and $\nabla(\boldsymbol{u}|_{\partial U})(\boldsymbol{x}) = \nabla \boldsymbol{u}(\boldsymbol{x})|_{T_{\boldsymbol{x}}\partial U}$, for $\mathcal{H}^{n-1}$-almost every $\boldsymbol{x} \in \partial U$.
vi) For every $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$\lim_{j\to\infty} \fint_0^{\frac{1}{j}} \left| \int_{\partial U_t} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) \boldsymbol{\nu}_t(\boldsymbol{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) - \int_{\partial U} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) \boldsymbol{\nu}(\boldsymbol{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) \right| \mathrm{d}t = 0,$$

where $\boldsymbol{\nu}_t$ denotes the unit outward normal to U_t for each $t \in (-\delta, \delta)$, the sets U_t are defined in (13), the number δ is that of Proposition 3, and $\boldsymbol{\nu}$ denotes the unit outward normal to U.

Lemma 2. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, and such that* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$. *Let U be a nonempty open set compactly contained in Ω with a C^2 boundary. Let $\delta > 0$ and $d : \Omega \to \mathbb{R}$ be as in Proposition 3. For every $t \in (-\delta, \delta)$ define U_t as in (13). Then $U_t \in \mathcal{U}_{\boldsymbol{u}}$ for almost every $t \in (-\delta, \delta)$.*

Proof. Recall from (10) the definition of $N(\partial U, \delta)$. By assumption, $|\operatorname{cof} \nabla \boldsymbol{u}| \in L^1(N(\partial U, \delta))$. Therefore, as a consequence of the coarea formula (see, for example, [5, Eq. (2.74)]) and (12), we have that $|\operatorname{cof} \nabla \boldsymbol{u}| \in L^1(\partial U_t)$ for almost every $t \in (-\delta, \delta)$, and

$$\int_{N(\partial U,\delta)} |\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x} = \int_{-\delta}^{\delta} \int_{\partial U_t} |\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})| \,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) \,\mathrm{d}t.$$

Property iii) follows from an application of Lebesgue's differentiation theorem to the function

$$(-\delta, \delta) \ni t \mapsto \int_{\partial U_t} |\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})| \,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}).$$

Properties iv) and vi) are proved similarly; the latter by using a countable and dense (in the supremum norm) family of $C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$, and equality (12).

Now we show v). Let $\{K_j\}_{j\in\mathbb{N}}$ be the sequence of sets that appears in (4). As before, for almost every $t \in (-\delta, \delta)$, the set $\partial U_t \setminus \bigcup_{j\in\mathbb{N}} K_j$ has zero $\mathcal{H}^{n-1}$ measure, and

$$\lim_{r\searrow 0} r^{-(n-1)} \mathcal{H}^{n-1}((\partial U_t \setminus K_j) \cap B(\boldsymbol{x}, r)) = 0$$

for $\mathcal{H}^{n-1}$-almost every $\boldsymbol{x} \in K_j$ and all $j \in \mathbb{N}$. This, together with (4), shows property v). $\square$

Not all properties of $\mathcal{U}_{\boldsymbol{u}}$ are used throughout the paper; for instance, properties iv–vi) are used only in Sections 5–7. Note that, in the notation of Definition 6(v), and using (8), we have that

$$(\Lambda_{n-1} \nabla(\boldsymbol{u}|_{\partial U})(\boldsymbol{x}))\boldsymbol{\nu}_t(\boldsymbol{x}) = (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}))\boldsymbol{\nu}(\boldsymbol{x}),$$

where $\boldsymbol{\nu}(\boldsymbol{x})$ is the outward unit normal to U at $\boldsymbol{x}$.

3. *SBV* regularity of the inverse

Motivated by our analysis of cavitation and fracture, in [18] we considered the following as a tentative surface energy.

Definition 7. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω. Suppose that $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$. Assume that $\det \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega)$, or that $\boldsymbol{u}$ is one-to-one almost everywhere. For each $\phi \in C^1_c(\Omega)$ and $\boldsymbol{g} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, define

$$\bar{\mathcal{E}}_{\boldsymbol{u}}(\phi, \boldsymbol{g}) := \int_{\Omega} [\boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot \operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})\, D\phi(\boldsymbol{x}) + \det \nabla \boldsymbol{u}(\boldsymbol{x})\, \phi(\boldsymbol{x})\, \operatorname{div} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x}))] \,\mathrm{d}\boldsymbol{x} \tag{14}$$

and

$$\bar{\mathcal{E}}(\boldsymbol{u}) := \sup\left\{\bar{\mathcal{E}}_{\boldsymbol{u}}(\phi, \boldsymbol{g}) : \ \phi \in C_c^1(\Omega), \ \boldsymbol{g} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), \ \|\phi\|_\infty \leqq 1, \ \|\boldsymbol{g}\|_\infty \leqq 1\right\}.$$

Note that if $\boldsymbol{u}$ is one-to-one almost everywhere, then the integral of the second term in (14) is well defined, thanks to Proposition 1.

As explained in [18, Sect. 3], $\bar{\mathcal{E}}_{\boldsymbol{u}}$ is related to the phenomenon of creation of surface, whereby the equality $\boldsymbol{u}(\partial\Omega) = \partial\boldsymbol{u}(\Omega)$ is not satisfied. It was shown that its boundedness plays an important role in proving the weak continuity of the Jacobian determinants, and that the weak limit of one-to-one almost everywhere maps is also one-to-one almost everywhere. In this paper we prove that $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$ implies, in addition, the SBV regularity of the inverse, and we explore the geometrical significance of this regularity result in terms of the creation of surface.

Before entering into the technical details, we sketch the main idea of this section. We restrict our attention to an open set U compactly contained in the domain, for reasons to be clarified later (see the remarks after Theorem 2). By Proposition 1, the second term of the right-hand side of (14) can be written as

$$\int_{\mathrm{im_G}(\boldsymbol{u},U)} \phi(\boldsymbol{u}^{-1}(\boldsymbol{y}))\,\mathrm{div}\,\boldsymbol{g}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},$$

provided $\det \nabla\boldsymbol{u} > 0$ almost everywhere and $\boldsymbol{u}$ is one-to-one almost everywhere ($\boldsymbol{u}^{-1}$ is given a precise meaning in Definition 8). If $\mathrm{im_G}(\boldsymbol{u}, U)$ coincided almost everywhere with an open set, this would be the distributional derivative of $\phi \circ \boldsymbol{u}^{-1}$, acting on a test function $\boldsymbol{g}$. Since the first term in (14) contains no derivatives of $\boldsymbol{g}$, it is clear that $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$ would imply that $\phi \circ \boldsymbol{u}^{-1} \in BV(\mathrm{im_G}(\boldsymbol{u}, U))$ for all $\phi \in C_c^\infty(\Omega)$.

Since $\mathrm{im_G}(\boldsymbol{u}, U)$ need not coincide almost everywhere with an open set, instead of working with $(\boldsymbol{u}|_U)^{-1} : \mathrm{im_G}(\boldsymbol{u}, U) \to U$ directly, we consider an arbitrary extension of it, denoted by $\boldsymbol{u}_U^{-1}$, defined to be $\boldsymbol{0}$ in the rest of $\mathbb{R}^n$ (Definition 8). This creates an artificial jump across the reduced boundary of $\mathrm{im_G}(\boldsymbol{u}, U)$ (a set of finite perimeter, according to Theorem 2), which can be seen in its distributional derivative. Indeed, suppose that $\boldsymbol{u}$ is a diffeomorphism from $\bar{U}$ onto its image; then

$$\left\langle D\boldsymbol{u}_U^{-1}, \boldsymbol{G}\right\rangle = \int_{\boldsymbol{u}(U)} D\boldsymbol{u}^{-1}(\boldsymbol{y}) \cdot \boldsymbol{G}(\boldsymbol{y}) - \int_{\partial\boldsymbol{u}(U)} \boldsymbol{u}^{-1}(\boldsymbol{y}) \cdot \boldsymbol{G}(\boldsymbol{y})\boldsymbol{\nu}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}), \tag{15}$$

$\boldsymbol{G}$ being an $\mathbb{R}^{n\times n}$-valued test function. The above relation continues to hold, for example, if $\boldsymbol{u}$ is a diffeomorphism except for the opening of a finite number of cavities at points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M$ in U (cf. [18, Prop. 4]). In that case

$$J_{\boldsymbol{u}_U^{-1}} = \partial\boldsymbol{u}(U) = \boldsymbol{u}(\partial U) \cup \Gamma_1 \cup \cdots \cup \Gamma_M,$$

where $\Gamma_1, \ldots, \Gamma_M$ are the cavity surfaces. This shows that the jump set of the arbitrary extension is related to the created surface, but it is necessary to distinguish between the points of $J_{\boldsymbol{u}_U^{-1}}$ that come from the old boundary [those in $\boldsymbol{u}(\partial U)$] and those that truly correspond to the created surface.

Based on the previous ideas it will be possible to show that $\boldsymbol{u}_U^{-1}$ indeed belongs to $BV(\mathbb{R}^n, \mathbb{R}^n)$ (in fact, to $SBV(\mathbb{R}^n, \mathbb{R}^n)$, see Theorem 2), and, in particular, to define the lateral trace

$$\left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}_0) = \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in \mathrm{im_G}(\boldsymbol{u},U)}} \boldsymbol{u}_U^{-1}(\boldsymbol{y}).$$

This allows us to distinguish between $\boldsymbol{u}(\partial U)$ and the *created surface* by considering separately those points $\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}}$ with $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) \in U$ or with $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) \in \partial U$. A different way of making that distinction is to consider the singular part of the distributional derivative of $\phi \circ \boldsymbol{u}_U^{-1}$, with $\phi \in C_c^\infty(U)$, and its connection to the functional $\bar{\mathcal{E}}_{\boldsymbol{u}}$. Indeed, the absolutely continuous part of $D(\phi \circ \boldsymbol{u}_U^{-1})$, in the case of piecewise smooth functions $\boldsymbol{u}$, is given by

$$\nabla\left(\phi \circ \boldsymbol{u}_U^{-1}\right)(\boldsymbol{y}) = \chi_{\mathrm{im_G}(\boldsymbol{u},U)}(\boldsymbol{y})\nabla(\boldsymbol{u}^{-1}(\boldsymbol{y}))^{\mathrm{T}} D\phi(\boldsymbol{u}^{-1}(\boldsymbol{y})).$$

Testing against $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and changing variables we obtain

$$\begin{aligned}\left\langle D^a(\phi \circ \boldsymbol{u}_U^{-1}), \boldsymbol{g}\right\rangle &= \int_U \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\nabla \boldsymbol{u}(\boldsymbol{x}))^{-1} D\phi(\boldsymbol{x}) \det \nabla \boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &= \int_U \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot \operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})\, D\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Hence, by a computation similar to that leading to (15),

$$\begin{aligned}\bar{\mathcal{E}}_{\boldsymbol{u}}(\phi, \boldsymbol{g}) &= \int_U \left[\boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot \operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})\, D\phi(\boldsymbol{x}) + \det \nabla \boldsymbol{u}(\boldsymbol{x})\, \phi(\boldsymbol{x}) \operatorname{div} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x}))\right] \mathrm{d}\boldsymbol{x}\\ &= \left\langle D^a\left(\phi \circ \boldsymbol{u}_U^{-1}\right), \boldsymbol{g}\right\rangle - \left\langle D\left(\phi \circ \boldsymbol{u}_U^{-1}\right), \boldsymbol{g}\right\rangle = -\left\langle D^s\left(\phi \circ \boldsymbol{u}_U^{-1}\right), \boldsymbol{g}\right\rangle\\ &= \int_{\partial \boldsymbol{u}(U)} \phi(\boldsymbol{u}^{-1}(\boldsymbol{y}))\boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y})\\ &= \left[\int_{\boldsymbol{u}(\partial U)} + \int_\Gamma\right] \phi(\boldsymbol{u}^{-1}(\boldsymbol{y}))\boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}),\end{aligned} \tag{16}$$

where Γ denotes the created surface. Since ϕ is compactly supported in U, the integral on $\boldsymbol{u}(\partial U)$ vanishes.

The previous heuristic considerations will be made rigourous in Theorem 2. Our final goal is to prove that the two ways of defining the created surface, namely,

$$\Gamma := \left\{\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}} : \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in U\right\} \tag{17}$$

and $\Gamma := \partial^* \mathrm{im_G}(\boldsymbol{u}, U) \setminus \mathrm{im_G}(\boldsymbol{u}, \partial U)$, are essentially equivalent. This will follow from a representation result of the form (16), with Γ as defined in (17) (the invisible created surface must also be considered, but we postpone this discussion to the next section).

The following result, inspired by Müller and Spector [24, Lemma 3.4], allows us to define a precise notion of inverse.

Lemma 3. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and suppose that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every $\boldsymbol{x} \in \Omega$. Let Ω_0 be as in Definition 3. Then $\boldsymbol{u}|_{\Omega_0}$ is one-to-one.*

Proof. Let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be two different points in Ω_0. Let B_1, B_2 be two disjoint balls in Ω containing, respectively, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. By assumption, there exists a set Ω' of full measure in Ω such that $\boldsymbol{u}|_{\Omega'}$ is one-to-one almost everywhere. Then

$$\operatorname{im_G}(\boldsymbol{u}, B_1) \cap \operatorname{im_G}(\boldsymbol{u}, B_2) \subset \operatorname{im_G}(\boldsymbol{u}, \Omega \backslash \Omega'),$$

and, hence, by property (5),

$$\mathcal{L}^n(\operatorname{im_G}(\boldsymbol{u}, B_1) \cap \operatorname{im_G}(\boldsymbol{u}, B_2)) = 0. \tag{18}$$

Now, for each $i = 1, 2$, the set B_i has density 1 at $\boldsymbol{x}_i$, so by Lemma 1, the set $\operatorname{im_G}(\boldsymbol{u}, B_i)$ has density 1 at $\boldsymbol{u}(\boldsymbol{x}_i)$. Because of (18), we must have $\boldsymbol{u}(\boldsymbol{x}_1) \neq \boldsymbol{u}(\boldsymbol{x}_2)$. □

Definition 8. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and suppose that $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ for almost every $\boldsymbol{x} \in \Omega$. Let Ω_0 be as in Definition 3. The inverse $\boldsymbol{u}^{-1} : \operatorname{im_G}(\boldsymbol{u}, \Omega) \to \mathbb{R}^n$ of $\boldsymbol{u}$ is defined as the function that sends every $\boldsymbol{y} \in \operatorname{im_G}(\boldsymbol{u}, \Omega)$ to the only $\boldsymbol{x} \in \Omega_0$ such that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}$. Analogously, given any nonempty open subset U of Ω, we define

$$\boldsymbol{u}_U^{-1}(\boldsymbol{y}) := \begin{cases} \boldsymbol{u}^{-1}(\boldsymbol{y}) & \text{if } \boldsymbol{y} \in \operatorname{im_G}(\boldsymbol{u}, U) \\ \boldsymbol{0} & \text{if } \boldsymbol{y} \in \mathbb{R}^n \backslash \operatorname{im_G}(\boldsymbol{u}, U). \end{cases}$$

In the sequel, we will distinguish the map $\boldsymbol{u}^{-1} : \operatorname{im_G}(\boldsymbol{u}, \Omega) \to \mathbb{R}^n$ from the map $\boldsymbol{u}_\Omega^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. Note that, by Proposition 1, the maps $\boldsymbol{u}^{-1}$ and $\boldsymbol{u}_U^{-1}$ are measurable.

Theorem 2. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u} > 0$ *almost everywhere,* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n \times n})$ *and $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$. Assume that $\boldsymbol{0} \notin \Omega$, and let $U \in \mathcal{U}_{\boldsymbol{u}}$. Then the following assertions hold:*

i) $\boldsymbol{u}_U^{-1} \in SBV_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ *and*

$$\begin{aligned} V\left(\boldsymbol{u}_U^{-1}, \mathbb{R}^n\right) &\leqq \|\operatorname{cof} \nabla \boldsymbol{u}\|_{L^1(U, \mathbb{R}^{n\times n})} + \|\mathbf{id}\|_{L^\infty(U,\mathbb{R}^n)} \|\operatorname{cof} \nabla \boldsymbol{u}\|_{L^1(\partial U, \mathbb{R}^{n\times n})} \\ &\quad + n\|\mathbf{id}\|_{L^\infty(U,\mathbb{R}^n)} \bar{\mathcal{E}}(\boldsymbol{u}) < \infty. \end{aligned} \tag{19}$$

If, in addition, $\det \nabla \boldsymbol{u} \in L^1(U)$ *then* $\boldsymbol{u}_U^{-1} \in SBV(\mathbb{R}^n, \mathbb{R}^n)$.

ii)

$$\operatorname{Per}(\operatorname{im_G}(\boldsymbol{u}, U)) \leqq \bar{\mathcal{E}}(\boldsymbol{u}) + \|\operatorname{cof} \nabla \boldsymbol{u}\|_{L^1(\partial U, \mathbb{R}^{n\times n})} < \infty. \tag{20}$$

iii) *For every $\boldsymbol{x}_0 \in \Omega_0$, the inverse $\boldsymbol{u}^{-1}$ is approximately differentiable at $\boldsymbol{u}(\boldsymbol{x}_0)$, and its approximate differential equals $(\nabla \boldsymbol{u}(\boldsymbol{x}_0))^{-1}$. In addition,*

$$\left\langle D^a \boldsymbol{u}_U^{-1}, \boldsymbol{G} \right\rangle = \int_U \operatorname{adj} \nabla \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{G}(\boldsymbol{u}(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}, \quad \textit{for all } \boldsymbol{G} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^{n\times n}).$$

iv) *For every* $\phi \in C_c^\infty(\Omega)$ *with* $\operatorname{spt}\phi \subset U$*, the map* $\psi : \mathbb{R}^n \to \mathbb{R}$*, defined as*

$$\psi(\boldsymbol{y}) := \begin{cases} \phi(\boldsymbol{u}^{-1}(\boldsymbol{y})) & \text{if } \boldsymbol{y} \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $SBV_{\mathrm{loc}}(\mathbb{R}^n)$ *(to* $SBV(\mathbb{R}^n)$ *if* $\det \nabla\boldsymbol{u} \in L^1_{\mathrm{loc}}(U)$*), with* $V(\psi, \mathbb{R}^n) < \infty$*. Moreover, for all* $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$*,*

$$\begin{aligned}\bar{\mathcal{E}}_{\boldsymbol{u}}(\phi, \boldsymbol{g}) &= -\langle D^s\psi, \boldsymbol{g}\rangle \\ &= \int_{J_{\boldsymbol{u}_U^{-1}}} \left[\phi(\left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y})) - \phi(\left(\boldsymbol{u}_U^{-1}\right)^+(\boldsymbol{y}))\right] \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}),\end{aligned}$$

where $J_{\boldsymbol{u}_U^{-1}}$ *denotes the jump set of* $\boldsymbol{u}_U^{-1}$*, and* $(\boldsymbol{u}_U^{-1})^+$*,* $(\boldsymbol{u}_U^{-1})^-$ *denote its lateral traces in* $J_{\boldsymbol{u}_U^{-1}}$ *with respect to the orientation given by* $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}$*.*

Proof. Fix $U \in \mathcal{U}_{\boldsymbol{u}}$. The proof is divided into five steps.

Step 1: BV regularity of $\boldsymbol{u}_U^{-1}$. Consider the number δ and the function d of Proposition 3, and, for each $t \in (-\delta, \delta)$, the set U_t defined in (13).

Fix $\varepsilon > 0$. Choose $\varphi \in C^\infty(\mathbb{R})$ satisfying $\varphi(t) = 0$ for $t \leqq 0$, $\varphi(t) = 1$ for $t \geqq 1$, and $0 \leqq \varphi' \leqq 1 + \varepsilon$. For each $j \in \mathbb{N}$, define the function $\eta_j : \Omega \to \mathbb{R}$ by

$$\eta_j(\boldsymbol{x}) := \varphi(j\, d(\boldsymbol{x})), \quad \boldsymbol{x} \in \Omega. \tag{21}$$

Note that $\eta_j \in C^1(\Omega)$ for sufficiently large j, and

$$\begin{aligned}&\lim_{j\to\infty} \eta_j(\boldsymbol{x}) = 1 \quad \text{for every } \boldsymbol{x} \in U,\\ &\eta_j(\boldsymbol{x}) = 0 \quad \text{for every } j \in \mathbb{N} \text{ and } \boldsymbol{x} \in \Omega\setminus U,\\ &D\eta_j(\boldsymbol{x}) = j\,\varphi'(j\, d(\boldsymbol{x}))\, Dd(\boldsymbol{x}) \quad \text{for every } j \in \mathbb{N} \text{ and } \boldsymbol{x} \in N(\partial U, \delta),\\ &\eta_j(\boldsymbol{x}) = 1 \quad \text{for every } j \in \mathbb{N} \text{ and } \boldsymbol{x} \in U_{\frac{1}{j}}.\end{aligned} \tag{22}$$

For $j \in \mathbb{N}$ large enough, define the functions $\boldsymbol{\phi}_j \in C_c^1(\Omega, \mathbb{R}^n)$ and $\boldsymbol{\psi}_j : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\boldsymbol{\phi}_j(\boldsymbol{x}) := \eta_j(\boldsymbol{x})\boldsymbol{x}, \quad \boldsymbol{x} \in \Omega, \tag{23}$$

and

$$\boldsymbol{\psi}_j(\boldsymbol{y}) := \begin{cases} \boldsymbol{\phi}_j(\boldsymbol{u}^{-1}(\boldsymbol{y})) & \text{if } \boldsymbol{y} \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) \\ \boldsymbol{0} & \text{otherwise.} \end{cases}$$

Clearly, $\boldsymbol{\psi}_j$ belongs to $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and, by Definition 7 and Proposition 1,

$$\langle D\boldsymbol{\psi}_j, \boldsymbol{G}\rangle = \int_\Omega \boldsymbol{G}(\boldsymbol{u}(\boldsymbol{x})) \cdot D\boldsymbol{\phi}_j(\boldsymbol{x}) \operatorname{adj} \nabla\boldsymbol{u}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \sum_{\alpha=1}^n \bar{\mathcal{E}}_{\boldsymbol{u}}(\phi_j^\alpha, \boldsymbol{g}_\alpha)$$

for all $\boldsymbol{G} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^{n\times n})$, where $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ correspond to the rows of $\boldsymbol{G}$, and $\phi_j^1, \ldots, \phi_j^n \in C_c^1(\Omega)$ are the components of $\boldsymbol{\phi}_j$. By (23) we have

$$D\boldsymbol{\phi}_j(\boldsymbol{x}) = \boldsymbol{x} \otimes D\eta_j(\boldsymbol{x}) + \eta_j(\boldsymbol{x})\boldsymbol{1}, \qquad \boldsymbol{x} \in \Omega,$$

hence

$$\langle D\boldsymbol{\psi}_j, \boldsymbol{G}\rangle = \int_\Omega \eta_j(\boldsymbol{x})\,\boldsymbol{G}(\boldsymbol{u}(\boldsymbol{x}))\cdot \operatorname{adj}\nabla\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{x}\cdot\boldsymbol{G}(\boldsymbol{u}(\boldsymbol{x}))(\operatorname{cof}\nabla\boldsymbol{u}(\boldsymbol{x}))D\eta_j(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \sum_{\alpha=1}^n \bar{\mathcal{E}}_{\boldsymbol{u}}(\phi_j^\alpha, \boldsymbol{g}_\alpha). \tag{24}$$

Since the functions η_j take values in $[0,1]$, by (22) we have

$$\left|\int_\Omega \eta_j(\boldsymbol{x})\,\boldsymbol{G}(\boldsymbol{u}(\boldsymbol{x}))\cdot\operatorname{adj}\nabla\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| \leqq \|\boldsymbol{G}\|_\infty \|\operatorname{cof}\nabla\boldsymbol{u}\|_{L^1(U,\mathbb{R}^{n\times n})} \tag{25}$$

and

$$\left|\sum_{\alpha=1}^n \bar{\mathcal{E}}_{\boldsymbol{u}}(\phi_j^\alpha, \boldsymbol{g}_\alpha)\right| \leqq n\|\mathbf{id}\|_{L^\infty(U,\mathbb{R}^n)}\|\boldsymbol{G}\|_\infty \bar{\mathcal{E}}(\boldsymbol{u}). \tag{26}$$

Using Equations (22) and (12), we find that

$$\left|\int_\Omega \boldsymbol{x}\cdot\boldsymbol{G}(\boldsymbol{u}(\boldsymbol{x}))(\operatorname{cof}\nabla\boldsymbol{u}(\boldsymbol{x}))D\eta_j(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| \leqq (1+\varepsilon)\|\mathbf{id}\|_{L^\infty(U,\mathbb{R}^n)}\|\boldsymbol{G}\|_\infty \fint_0^{\frac{1}{j}}\int_{\partial U_t} |\operatorname{cof}\nabla\boldsymbol{u}(\boldsymbol{x})|\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x})\,\mathrm{d}t. \tag{27}$$

By (24), (25), (26) and (27), we thus have that

$$V(\boldsymbol{\psi}_j, \mathbb{R}^n) \leqq \|\operatorname{cof}\nabla\boldsymbol{u}\|_{L^1(U,\mathbb{R}^{n\times n})} + (1+\varepsilon)\|\mathbf{id}\|_{L^\infty(U,\mathbb{R}^n)} \fint_0^{\frac{1}{j}}\int_{\partial U_t} |\operatorname{cof}\nabla\boldsymbol{u}(\boldsymbol{x})|\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x})\,\mathrm{d}t + n\|\mathbf{id}\|_{L^\infty(\Omega,\mathbb{R}^n)}\bar{\mathcal{E}}(\boldsymbol{u}). \tag{28}$$

Using condition iii) of Definition 6, we find that

$$\limsup_{j\to\infty} V(\boldsymbol{\psi}_j, \mathbb{R}^n) \leqq \|\operatorname{cof}\nabla\boldsymbol{u}\|_{L^1(U,\mathbb{R}^{n\times n})} + (1+\varepsilon)\|\mathbf{id}\|_{L^\infty(U,\mathbb{R}^n)}\|\operatorname{cof}\nabla\boldsymbol{u}\|_{L^1(\partial U,\mathbb{R}^{n\times n})} + n\|\mathbf{id}\|_{L^\infty(\Omega,\mathbb{R}^n)}\bar{\mathcal{E}}(\boldsymbol{u}) < \infty.$$

Thanks to (22) we have that $\boldsymbol{\psi}_j \to \boldsymbol{u}_U^{-1}$ pointwise in $\mathbb{R}^n$ as $j\to\infty$. With this, the Poincaré inequality (see, for example, [5, Th. 3.47]) and the embedding of BV into L^1, we conclude that there is a vector $\boldsymbol{a}\in\mathbb{R}^n$ such that $\boldsymbol{u}_U^{-1}+\boldsymbol{a}\in BV(\mathbb{R}^n,\mathbb{R}^n)$ and

$$V(\boldsymbol{u}_U^{-1}, \mathbb{R}^n) \leqq \|\operatorname{cof}\nabla\boldsymbol{u}\|_{L^1(U,\mathbb{R}^{n\times n})} + (1+\varepsilon)\|\mathbf{id}\|_{L^\infty(U,\mathbb{R}^n)}\|\operatorname{cof}\nabla\boldsymbol{u}\|_{L^1(\partial U,\mathbb{R}^{n\times n})} + n\|\mathbf{id}\|_{L^\infty(U,\mathbb{R}^n)}\bar{\mathcal{E}}(\boldsymbol{u}).$$

In particular, $\boldsymbol{u}_U^{-1}\in BV_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$.

Finally, if $\det \nabla \boldsymbol{u} \in L^1(U)$ then, thanks to Proposition 1, $\|\boldsymbol{u}_U^{-1}\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)} \leqq \|\mathbf{id}\|_{L^\infty(U,\mathbb{R}^n)} \|\det \nabla \boldsymbol{u}\|_{L^1(U)}$, so $\boldsymbol{u}_U^{-1} \in BV(\mathbb{R}^n, \mathbb{R}^n)$.

Step 2: Proof of (20). Let $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfy $\|\boldsymbol{g}\|_\infty \leqq 1$. Then, by (22) and dominated convergence,

$$\int_{\mathrm{im_G}(\boldsymbol{u},U)} \operatorname{div} \boldsymbol{g}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \lim_{j\to\infty} \int_{\mathrm{im_G}(\boldsymbol{u},U)} \eta_j(\boldsymbol{u}^{-1}(\boldsymbol{y}))\operatorname{div} \boldsymbol{g}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}. \tag{29}$$

Now, for each $j \in \mathbb{N}$, thanks to Proposition 1, (22) and Definition 7, we have that

$$\int_{\mathrm{im_G}(\boldsymbol{u},U)} \eta_j(\boldsymbol{u}^{-1}(\boldsymbol{y}))\operatorname{div} \boldsymbol{g}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \bar{\mathcal{E}}_{\boldsymbol{u}}(\eta_j, \boldsymbol{g}) \\ - \int_\Omega \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}))\, D\eta_j(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

Clearly, $\bar{\mathcal{E}}_{\boldsymbol{u}}(\eta_j, \boldsymbol{g}) \leqq \bar{\mathcal{E}}(\boldsymbol{u})$ and, as in (27), using Equations (22) and (12), we find that

$$\left| \int_\Omega \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) D\eta_j(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right| \\ \leqq (1+\varepsilon) \fint_0^{\frac{1}{j}} \int_{\partial U_t} |\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})|\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x})\, \mathrm{d}t.$$

Thus, by (29) and Definition 6 iii),

$$\int_{\mathrm{im_G}(\boldsymbol{u},U)} \operatorname{div} \boldsymbol{g}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \leqq \bar{\mathcal{E}}(\boldsymbol{u}) + (1+\varepsilon)\|\operatorname{cof} \nabla \boldsymbol{u}\|_{L^1(\partial U, \mathbb{R}^{n\times n})}.$$

As the left-hand side is independent of ε, and the right-hand side is independent of $\boldsymbol{g}$, the conclusion follows.

Step 3: Proof of iii). Let $\boldsymbol{x}_0 \in \Omega_0$, and define $\boldsymbol{y}_0 := \boldsymbol{u}(\boldsymbol{x}_0)$ and $\boldsymbol{F} := \nabla \boldsymbol{u}(\boldsymbol{x}_0)$. Note that $\boldsymbol{F}$ is invertible thanks to Definition 3, and that $D(\mathrm{im_G}(\boldsymbol{u}, \Omega), \boldsymbol{y}_0) = 1$ thanks to Lemma 1. Define, for each $\delta > 0$,

$$E_\delta := \left\{ \boldsymbol{x} \in \Omega_0 \setminus \{\boldsymbol{x}_0\} : \frac{|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}_0) - \boldsymbol{F}(\boldsymbol{x} - \boldsymbol{x}_0)|}{|\boldsymbol{x} - \boldsymbol{x}_0|} < \delta \right\}.$$

Since $\boldsymbol{u}$ is approximately differentiable at $\boldsymbol{x}_0$ and the set Ω_0 is of full measure in Ω, then $D(E_\delta, \boldsymbol{x}_0) = 1$ for all $\delta > 0$.

Call, for each $\varepsilon > 0$,

$$A_\varepsilon := \left\{ \boldsymbol{y} \in \mathrm{im_G}(\boldsymbol{u}, \Omega) \setminus \{\boldsymbol{u}(\boldsymbol{x}_0)\} : \frac{|\boldsymbol{u}^{-1}(\boldsymbol{y}) - \boldsymbol{x}_0 - (\nabla \boldsymbol{u}(\boldsymbol{x}_0))^{-1}(\boldsymbol{y} - \boldsymbol{u}(\boldsymbol{x}_0))|}{|\boldsymbol{y} - \boldsymbol{u}(\boldsymbol{x}_0)|} > \varepsilon \right\}.$$

Let $\boldsymbol{x} \in \Omega_0 \setminus \{\boldsymbol{x}_0\}$ and call $\boldsymbol{y} := \boldsymbol{u}(\boldsymbol{x})$. Thanks to Lemma 3, $\boldsymbol{y} \neq \boldsymbol{y}_0$. Set $\boldsymbol{r} := \boldsymbol{y} - \boldsymbol{y}_0 - \boldsymbol{F}(\boldsymbol{x} - \boldsymbol{x}_0)$. Then

$$\frac{|\boldsymbol{x} - \boldsymbol{x}_0 - \boldsymbol{F}^{-1}(\boldsymbol{y} - \boldsymbol{y}_0)|}{|\boldsymbol{y} - \boldsymbol{y}_0|} \leqq |\boldsymbol{F}^{-1}| \frac{|\boldsymbol{r}|}{|\boldsymbol{x} - \boldsymbol{x}_0|} \frac{|\boldsymbol{x} - \boldsymbol{x}_0|}{|\boldsymbol{y} - \boldsymbol{y}_0|} \\ \leqq |\boldsymbol{F}^{-1}| \frac{|\boldsymbol{r}|}{|\boldsymbol{x} - \boldsymbol{x}_0|} \frac{1}{\left| \boldsymbol{F}\left(\frac{\boldsymbol{x} - \boldsymbol{x}_0}{|\boldsymbol{x} - \boldsymbol{x}_0|} \right) \right| - \frac{|\boldsymbol{r}|}{|\boldsymbol{x} - \boldsymbol{x}_0|}}.$$

This shows that if $\boldsymbol{u}(E_\delta) \cap A_\varepsilon \neq \varnothing$ for some $\delta, \varepsilon > 0$, then

$$\varepsilon < |\boldsymbol{F}^{-1}| \frac{\delta}{\inf\{|\boldsymbol{F}\boldsymbol{v}| : |\boldsymbol{v}| = 1\} - \delta}. \tag{30}$$

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that (30) does not hold, and, hence, $\boldsymbol{u}(E_\delta) \cap A_\varepsilon = \varnothing$. As $D(E_\delta, \boldsymbol{x}_0) = 1$, then by Lemma 1, $D(\mathbb{R}^n \setminus \boldsymbol{u}(E_\delta), \boldsymbol{y}_0) = 0$, and, hence, $D(A_\varepsilon, \boldsymbol{y}_0) = 0$. This proves that $\nabla \boldsymbol{u}^{-1}(\boldsymbol{u}(\boldsymbol{x}_0)) = (\nabla \boldsymbol{u}(\boldsymbol{x}_0))^{-1}$.

Thanks to the Calderón–Zygmund theorem, $\boldsymbol{u}_U^{-1}$ is approximately differentiable almost everywhere, and $D^a \boldsymbol{u}_U^{-1} = \nabla \boldsymbol{u}_U^{-1} \mathcal{L}^n$. As $\boldsymbol{u}_U^{-1}$ and $\boldsymbol{u}^{-1}$ coincide in $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)$, then $\nabla \boldsymbol{u}_U^{-1}(\boldsymbol{y}) = \nabla \boldsymbol{u}^{-1}(\boldsymbol{y})$ for every approximate differentiability point $\boldsymbol{y} \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)$ of $\boldsymbol{u}^{-1}$ such that $D(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U), \boldsymbol{y}) = 1$. Consequently,

$$\nabla \boldsymbol{u}_U^{-1}(\boldsymbol{y}) = \left(\nabla \boldsymbol{u}(\boldsymbol{u}^{-1}(\boldsymbol{y}))\right)^{-1} = \frac{\operatorname{adj} \nabla \boldsymbol{u}(\boldsymbol{u}^{-1}(\boldsymbol{y}))}{\det \nabla \boldsymbol{u}(\boldsymbol{u}^{-1}(\boldsymbol{y}))} \tag{31}$$

for almost every $\boldsymbol{y} \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)$, by Lebesgue's theorem. On the other hand, as $\boldsymbol{u}_U^{-1} = \boldsymbol{0}$ in $\mathbb{R}^n \setminus \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)$ then $\nabla \boldsymbol{u}_U^{-1}(\boldsymbol{y}) = \boldsymbol{0}$ for every $\boldsymbol{y} \in \mathbb{R}^n \setminus \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)$ such that $D(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U), \boldsymbol{y}) = 0$. Hence, $\nabla \boldsymbol{u}_U^{-1} = \boldsymbol{0}$ almost everywhere in $\mathbb{R}^n \setminus \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)$. This and Proposition 1 show that

$$\left\langle D^a \boldsymbol{u}_U^{-1}, \boldsymbol{G} \right\rangle = \int_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)} \frac{\operatorname{adj} \nabla \boldsymbol{u}(\boldsymbol{u}^{-1}(\boldsymbol{y}))}{\det \nabla \boldsymbol{u}(\boldsymbol{u}^{-1}(\boldsymbol{y}))} \cdot \boldsymbol{G}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = \int_U \operatorname{adj} \nabla \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{G}(\boldsymbol{u}(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x}$$

for every $\boldsymbol{G} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^{n \times n})$, concluding the proof.

Step 4: Characterization of $\bar{\mathcal{E}}$. Let $\phi \in C_c^\infty(\Omega)$ satisfy $\operatorname{spt} \phi \subset U$. Extending ϕ by 0 outside Ω we have that $\psi = \phi \circ \boldsymbol{u}_U^{-1}$. By the chain rule in BV (see [4, Th. 2.1], or [5, Th. 3.96]) we obtain that $\psi \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ with $V(\psi, \mathbb{R}^n) < \infty$ and

$$\begin{aligned}
D^a \psi &= (\nabla \boldsymbol{u}_U^{-1})^T \, D\phi \left(\boldsymbol{u}_U^{-1}\right) \mathcal{L}^n, \\
D^j \psi &= \left[\phi\left(\left(\boldsymbol{u}_U^{-1}\right)^+\right) - \phi\left(\left(\boldsymbol{u}_U^{-1}\right)^-\right)\right] \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}} \, \mathcal{H}^{n-1} \llcorner J_{\boldsymbol{u}_U^{-1}}, \\
D^c \psi &= D\phi(\tilde{\boldsymbol{u}}_U^{-1}) \, D^c \boldsymbol{u}_U^{-1}.
\end{aligned} \tag{32}$$

As in Step 1, $\psi \in L^1(\mathbb{R}^n)$ if $\det \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(U)$.

Fix $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Since $\nabla \boldsymbol{u}_U^{-1}$ and $\phi \circ \boldsymbol{u}_U^{-1}$ vanish almost everywhere in $\mathbb{R}^n \setminus \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)$, by Proposition 1, (31), and (32) we obtain that

$$\begin{aligned}
\bar{\mathcal{E}}_{\boldsymbol{u}}(\phi, \boldsymbol{g}) &= \int_U [\boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot \operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}) \, D\phi(\boldsymbol{x}) + \phi(\boldsymbol{x}) \operatorname{div} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \det \nabla \boldsymbol{u}(\boldsymbol{x})] \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)} \left[\boldsymbol{g}(\boldsymbol{y}) \cdot (\nabla \boldsymbol{u}_U^{-1}(\boldsymbol{y}))^T D\phi(\boldsymbol{u}^{-1}(\boldsymbol{y})) + \phi(\boldsymbol{u}^{-1}(\boldsymbol{y})) \operatorname{div} \boldsymbol{g}(\boldsymbol{y})\right] \mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbb{R}^n} \left[\boldsymbol{g}(\boldsymbol{y}) \cdot (\nabla \boldsymbol{u}_U^{-1}(\boldsymbol{y}))^T D\phi(\boldsymbol{u}_U^{-1}(\boldsymbol{y})) + \phi(\boldsymbol{u}_U^{-1}(\boldsymbol{y})) \operatorname{div} \boldsymbol{g}(\boldsymbol{y})\right] \mathrm{d}\boldsymbol{y}
\end{aligned}$$

$$
\begin{aligned}
&= \langle D^a\psi, \boldsymbol{g}\rangle - \langle D\psi, \boldsymbol{g}\rangle = -\langle D^s\psi, \boldsymbol{g}\rangle \\
&= \int_{J_{\boldsymbol{u}_U^{-1}}} \left[\phi\left(\left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y})\right) - \phi\left((\boldsymbol{u}_U^{-1})^+(\boldsymbol{y})\right)\right] \boldsymbol{g}(\boldsymbol{y})\cdot \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) \\
&\quad - \int_{\mathbb{R}^n} D\phi(\tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}))\otimes \boldsymbol{g}(\boldsymbol{y})\cdot \mathrm{d}D^c\boldsymbol{u}_U^{-1}(\boldsymbol{y}).
\end{aligned}
$$

Step 5: SBV regularity of $\boldsymbol{u}_U^{-1}$ and ψ. By Step 4, it only remains to show that $D^c\boldsymbol{u}_U^{-1} = \mathbf{0}$. This essentially follows from the proof of Ambrosio [2, Th. 2.3] (see also [1; 5, Prop. 4.12]). We show here how to adapt his proof to our case.

From the above characterization of $\bar{\mathcal{E}}$, it follows that

$$
\begin{aligned}
&\left|\int_{J_{\boldsymbol{u}_U^{-1}}} \left[\phi\left(\left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y})\right) - \phi\left(\left(\boldsymbol{u}_U^{-1}\right)^+(\boldsymbol{y})\right)\right] \boldsymbol{g}(\boldsymbol{y})\cdot \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y})\right. \\
&\quad \left. - \int_{\mathbb{R}^n} D\phi(\tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}))\otimes \boldsymbol{g}(\boldsymbol{y})\cdot \mathrm{d}D^c\boldsymbol{u}_U^{-1}(\boldsymbol{y})\right| \leqq \|\phi\|_\infty \|\boldsymbol{g}\|_\infty \bar{\mathcal{E}}(\boldsymbol{u})
\end{aligned}
\tag{33}
$$

for all $\phi \in C_c^\infty(\Omega)$ and $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{spt}\phi \subset U$. Moreover, (32) tells us that

$$
\left[\phi\left(\left(\boldsymbol{u}_U^{-1}\right)^+\right) - \phi\left(\left(\boldsymbol{u}_U^{-1}\right)^-\right)\right] \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}\,\mathcal{H}^{n-1} \llcorner J_{\boldsymbol{u}_U^{-1}} \quad \text{and} \quad D\phi(\tilde{\boldsymbol{u}}_U^{-1})\, D^c\boldsymbol{u}_U^{-1}
$$

are finite Borel measures. Since (33) does not contain derivatives of $\boldsymbol{g}$, it is valid for all bounded Borel functions $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$ (by Lusin's theorem, approximating $\boldsymbol{g}$ by a sequence $\{\boldsymbol{g}_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\boldsymbol{g}_j\|_\infty \leqq \|\boldsymbol{g}\|_\infty$). Using the fact that $\mathcal{H}^{n-1}\llcorner J_{\boldsymbol{u}_U^{-1}}$ and $D^c\boldsymbol{u}_U^{-1}$ are mutually singular (see, for example, [2, Prop. 1.1] or [5, Prop. 3.92(c)]), we obtain that

$$
\left|\int_{\mathbb{R}^n} D\phi(\tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}))\otimes \boldsymbol{g}(\boldsymbol{y})\cdot \mathrm{d}D^c\boldsymbol{u}_U^{-1}(\boldsymbol{y})\right| \leqq \|\phi\|_\infty \|\boldsymbol{g}\|_\infty \bar{\mathcal{E}}(\boldsymbol{u}) \tag{34}
$$

for all bounded Borel functions $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$ and all $\phi \in C_c^\infty(\Omega)$ with $\operatorname{spt}\phi \subset U$.

Let $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be the function corresponding to the polar decomposition of $D^c\boldsymbol{u}_U^{-1}$, that is, $\boldsymbol{A}$ satisfies $|\boldsymbol{A}(\boldsymbol{y})| = 1$ for $|D^c\boldsymbol{u}_U^{-1}|$-almost every $\boldsymbol{y} \in \mathbb{R}^n$, and $D^c\boldsymbol{u}_U^{-1} = \boldsymbol{A}|D^c\boldsymbol{u}_U^{-1}|$. Let Q be a closed cube contained in U, say $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Fix $\alpha, \beta \in \{1, \ldots, n\}$, and let $\pi_\alpha : \mathbb{R}^n \to \mathbb{R}$ be the projection onto the α-th coordinate, and $\pi_{\alpha\beta} : \mathbb{R}^{n\times n} \to \mathbb{R}$ the projection onto the (α, β)-th entry. Let $\psi \in C^\infty([a_\alpha, b_\alpha])$ and choose $\phi \in C_c^\infty(\Omega)$ such that $\operatorname{spt}\phi \subset U$, $\phi|_Q = \psi \circ \pi_\alpha|_Q$, and $\|\phi\|_\infty \leqq 1 + \|\psi\|_\infty$. Define the function $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$
\pi_\beta \circ \boldsymbol{g} = \chi_{\{\boldsymbol{y}\in\mathbb{R}^n : \tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y})\in Q\}} \operatorname{sgn}\left(\psi' \circ \pi_\alpha \circ \tilde{\boldsymbol{u}}_U^{-1}\right) \operatorname{sgn}(\pi_{\alpha\beta} \circ \boldsymbol{A})
$$

and $\pi_\gamma \circ \boldsymbol{g} = 0$ for all $\gamma \in \{1, \ldots, n\} \setminus \{\beta\}$. We apply (34) to this choice, and obtain that

$$\int_{\{\boldsymbol{y} \in \mathbb{R}^n : \tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}) \in Q\}} |\psi'(\pi_\alpha(\tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y})))| \, |\pi_{\alpha\beta}(\boldsymbol{A}(\boldsymbol{y}))| \, \mathrm{d}|D^c \boldsymbol{u}_U^{-1}|(\boldsymbol{y}) \leqq (1 + \|\psi\|_\infty) \bar{\mathcal{E}}(\boldsymbol{u}). \tag{35}$$

for every $\alpha, \beta \in \{1, \ldots, n\}$, every $\psi \in C^\infty([a_\alpha, b_\alpha])$, and every cube $Q \subset U$ such that $\pi_\alpha(Q) = [a_\alpha, b_\alpha]$.

Fix $\epsilon > 0$. In (35), first we choose $\psi(t) := \sin(t/\epsilon)$ and then $\psi(t) := \cos(t/\epsilon)$, for $t \in [a_\alpha, b_\alpha]$. Then we sum the resulting expressions and use that $|\sin t| + |\cos t| \geqq 1$ for any $t \in \mathbb{R}$, to obtain that

$$\frac{1}{\epsilon} \int_{\{\boldsymbol{y} \in \mathbb{R}^n : \tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}) \in Q\}} |\pi_{\alpha\beta}(\boldsymbol{A}(\boldsymbol{y}))| \, \mathrm{d}|D^c \boldsymbol{u}_U^{-1}|(\boldsymbol{y}) \leqq 4\bar{\mathcal{E}}(\boldsymbol{u}).$$

for every $\alpha, \beta \in \{1, \ldots, n\}$ and every closed cube $Q \subset U$. Now we sum in $\alpha, \beta \in \{1, \ldots, n\}$ and obtain that

$$\left| D^c \boldsymbol{u}_U^{-1} \right| \left(\{ \boldsymbol{y} \in \mathbb{R}^n : \tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}) \in Q \} \right) \leqq C \epsilon \bar{\mathcal{E}}(\boldsymbol{u})$$

for every closed cube $Q \subset \mathbb{R}^n$, and some constant $C > 0$ depending only on n. As ϵ is arbitrary, this shows that $|D^c \boldsymbol{u}_U^{-1}|(\{\boldsymbol{y} \in \mathbb{R}^n : \tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}) \in Q\}) = 0$. Since U can be expressed as a countable union of closed cubes contained in U, then

$$\left| D^c \boldsymbol{u}_U^{-1} \right| \left(\left\{ \boldsymbol{y} \in \mathbb{R}^n : \tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}) \in U \right\} \right) = 0. \tag{36}$$

Now let $V \in \mathcal{U}_{\boldsymbol{u}}$ satisfy $U \subset\subset V$. Let S be the set of points $\boldsymbol{y}_0 \in \mathbb{R}^n$ such that $D(\mathrm{im_G}(\boldsymbol{u}, U), \boldsymbol{y}_0) = 1$ and $\operatorname{ap\,lim}_{\boldsymbol{y} \to \boldsymbol{y}_0} \boldsymbol{u}_U^{-1}(\boldsymbol{y})$ exists. Clearly,

$$\operatorname*{ap\,lim}_{\boldsymbol{y} \to \boldsymbol{y}_0} \boldsymbol{u}_V^{-1}(\boldsymbol{y}) = \operatorname*{ap\,lim}_{\boldsymbol{y} \to \boldsymbol{y}_0} \boldsymbol{u}_U^{-1}(\boldsymbol{y}), \quad \boldsymbol{y}_0 \in S.$$

Consequently, $\tilde{\boldsymbol{u}}_U^{-1}$ and $\tilde{\boldsymbol{u}}_V^{-1}$ coincide in S. Therefore, by (36) and the locality of the distributional derivative (see, for example, [5, Rk. 3.93]),

$$\left| D^c \boldsymbol{u}_U^{-1} \right| \left(\left\{ \boldsymbol{y} \in S : \tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}) \in V \right\} \right) = \left| D^c \boldsymbol{u}_V^{-1} \right| \left(\left\{ \boldsymbol{y} \in S : \tilde{\boldsymbol{u}}_V^{-1}(\boldsymbol{y}) \in V \right\} \right) = 0. \tag{37}$$

If $\mathrm{im_G}(\boldsymbol{u}, U)$ has density zero at some $\boldsymbol{y} \in \mathbb{R}^n$, then $\tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}) = \boldsymbol{0}$ (since $\boldsymbol{u}_U^{-1}(\boldsymbol{y}) = \boldsymbol{0}$ for $\boldsymbol{y} \in \mathbb{R}^n \setminus \mathrm{im_G}(\boldsymbol{u}, U)$). By the locality of the distributional derivative, this implies that

$$\left| D^c \boldsymbol{u}_U^{-1} \right| \left(\{ \boldsymbol{y} \in \mathbb{R}^n : D(\mathrm{im_G}(\boldsymbol{u}, U), \boldsymbol{y}) = 0 \} \right) = 0. \tag{38}$$

We now have all the ingredients to prove that the Cantor part of the derivative of $\boldsymbol{u}_U^{-1}$ vanishes. By Theorem 2 and standard continuity properties of BV functions (see, for example, [28, Th. 5.9.6]), for all $\boldsymbol{y}_0 \in \mathbb{R}^n$ except a set that is σ-finite with respect to $\mathcal{H}^{n-1}$, the limit ap $\lim_{y\to y_0} \boldsymbol{u}_U^{-1}(\boldsymbol{y})$ exists, and, by definition of the precise representative, it coincides with $\tilde{\boldsymbol{u}}_U^{-1}(\boldsymbol{y}_0)$, which for simplicity we call $\boldsymbol{x}_0$. By definition of approximate limit, for any $r > 0$

$$D\left(\left\{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{u}_U^{-1}(\boldsymbol{y}) \in B(\boldsymbol{x}_0, r)\right\}, \boldsymbol{y}_0\right) = 1.$$

Therefore, if $D(\mathrm{im_G}(\boldsymbol{u}, U), \boldsymbol{y}_0) > 0$, there exists a sequence $\{\boldsymbol{y}_k\}_{k\in\mathbb{N}}$ in $\mathrm{im_G}(\boldsymbol{u}, U)$ such that $\boldsymbol{u}_U^{-1}(\boldsymbol{y}_k)$ converges to $\boldsymbol{x}_0$. In particular, $\boldsymbol{x}_0$ belongs to $\bar{U}$. Analogously, if $D(\mathbb{R}^n \setminus \mathrm{im_G}(\boldsymbol{u}, U), \boldsymbol{y}_0) > 0$, then $\boldsymbol{x}_0 = \boldsymbol{0}$. Since $\boldsymbol{0} \notin \Omega$, only one of the two possibilities occur. The case $\boldsymbol{x}_0 = \boldsymbol{0}$ is covered by (38). The case $\boldsymbol{x}_0 \in \bar{U}$ is covered by (37) (the fact that $\boldsymbol{x}_0$ may lie on ∂U explains the necessity to extend (36) to an open set V strictly larger than U). Since $D^c \boldsymbol{u}_U^{-1}$ (which coincides with $D^c \tilde{\boldsymbol{u}}_U^{-1}$) neglects sets that are σ-finite with respect to $\mathcal{H}^{n-1}$ (see, for example, [5, Prop. 3.92]), the proof is completed. □

We finish this section with some comments on Theorem 2. Its conclusion is close to having that $\boldsymbol{u}^{-1}$ is locally SBV in $\mathrm{im_G}(\boldsymbol{u}, \Omega)$, the problem being that $\mathrm{im_G}(\boldsymbol{u}, \Omega)$ need not coincide almost everywhere with an open set.

Each of the terms of the right-hand side of (19) has a natural interpretation: the first is related to the absolutely continuous part of $D\boldsymbol{u}_U^{-1}$, while the second and third correspond to the artificial jump provoked at $\partial^* \mathrm{im_G}(\boldsymbol{u}, U)$ by extending $\boldsymbol{u}^{-1}$ arbitrarily by a constant outside $\mathrm{im_G}(\boldsymbol{u}, U)$. The second term corresponds to the image of the old boundary [the $L^1(\partial U, \mathbb{R}^{n\times n})$ norm of cof $D\boldsymbol{u}$ controls the area of $\mathrm{im_G}(\boldsymbol{u}, \partial U)$], while the third is due to the created surface.

Finally, note that the BV regularity of functions of the form $\phi \circ \boldsymbol{u}_U^{-1}$ is actually established before the BV regularity of $\boldsymbol{u}_U^{-1}$. A sequence $\{\boldsymbol{\phi}_j\}_{j\in\mathbb{N}}$ approximating the identity is then taken in order to obtain the result for $\boldsymbol{u}_U^{-1}$ itself. This explains why it is necessary to consider open sets U compactly contained in the domain. If it were possible to obtain property iii) of Definition 6 for $\partial\Omega$ itself, we would be able to obtain the global regularity result for $\boldsymbol{u}^{-1}$ (to be precise, for $\boldsymbol{u}_\Omega^{-1}$), and not only for its truncation to smaller sets.

4. Created surface and a characterization of $\mathcal{E}(\boldsymbol{u})$

4.1. Definitions of visible and invisible created surface

As explained in [18], the surface energy $\bar{\mathcal{E}}$ and the study of the notion of created surface were motivated by the example of Müller and Spector [24, Sect. 11] in which cavities are created and then filled with material from elsewhere in the body (see Fig. 1). In that example, $\mathrm{Per}(\mathrm{im_G}(\boldsymbol{u}, \Omega))$ fails to compute the area of all the created surface, because a great part of it is surrounded by material at both sides. This makes the surface 'invisible', in the sense of not belonging to the reduced boundary of $\mathrm{im_G}(\boldsymbol{u}, \Omega)$.

Based on the example of Müller and Spector, and on the analysis of the previous section, we define the notions of *invisible*, *visible* and *created surface* in terms of the discontinuities of the inverse.

Definition 9. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost every, and suppose that $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ for almost everywhere $\boldsymbol{x} \in \Omega$.

a) We define the invisible surface created by $\boldsymbol{u}$, denoted $\Gamma_I(\boldsymbol{u})$, as the set

$$\Gamma_I(\boldsymbol{u}) := \left\{ \boldsymbol{y} \in J_{\boldsymbol{u}^{-1}} : (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in \Omega \text{ and } (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in \Omega \right\}.$$

Analogously, for every open set $U \subset \Omega$ we define the invisible surface created by $\boldsymbol{u}$ in U, and denote it by $\Gamma_I(\boldsymbol{u}, U)$, as

$$\Gamma_I(\boldsymbol{u}, U) := \left\{ \boldsymbol{y} \in J_{\boldsymbol{u}^{-1}} : (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in U \text{ and } (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in U \right\}.$$

b) We define the visible surface created by $\boldsymbol{u}$, and denote it by $\Gamma_V(\boldsymbol{u})$, as the set of points $\boldsymbol{y}_0 \in \mathbb{R}^n$ for which there exists $\boldsymbol{\nu} \in \mathbb{S}^{n-1}$ satisfying the following conditions:
  i) $D(\mathrm{im_G}(\boldsymbol{u}, \Omega) \cap H^-(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0) = \frac{1}{2}$.
  ii) The lateral trace

$$(\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0) = \operatorname*{ap\,lim}_{\substack{\boldsymbol{y} \to \boldsymbol{y}_0 \\ \boldsymbol{y} \in H^-(\boldsymbol{y}_0, \boldsymbol{\nu})}} \boldsymbol{u}^{-1}(\boldsymbol{y})$$

  exists and is in Ω.
  iii) $D(\mathrm{im_G}(\boldsymbol{u}, U) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0) = 0$ for every open set $U \subset\subset \Omega$.

  The vector $\boldsymbol{\nu}$ is denoted $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}_0)$.

c) We define the surface created by $\boldsymbol{u}$, and denote it by $\Gamma(\boldsymbol{u})$, as the set

$$\Gamma(\boldsymbol{u}) := \Gamma_V(\boldsymbol{u}) \cup \Gamma_I(\boldsymbol{u}).$$

Obviously, $\Gamma_I(\boldsymbol{u}) = \Gamma_I(\boldsymbol{u}, \Omega)$, and $\Gamma_I(\boldsymbol{u}, U_1) \subset \Gamma_I(\boldsymbol{u}, U_2)$ if $U_1 \subset U_2 \subset \Omega$. Also, in order for a point $\boldsymbol{y}$ to be in $\Gamma_I(\boldsymbol{u})$, the set $\mathrm{im_G}(\boldsymbol{u}, \Omega)$ must have density 1 at $\boldsymbol{y}$ (see Definition 1).

We will see in Proposition 5(vi) that the vector $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}_0)$ of Definition 9 b) is uniquely determined. In Lemma 5 we will see that $\Gamma_V(\boldsymbol{u})$ and $\Gamma_I(\boldsymbol{u})$ are Borel sets, and that the map $(\boldsymbol{u}^{-1})^- : \Gamma_V(\boldsymbol{u}) \to \mathbb{S}^{n-1}$ is Borel.

Definition 9 is illustrated through the following examples.

(a) Let Ω be the rectangle $(1, 2) \times (0, 2\pi)$ in $\mathbb{R}^2$, and $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ the deformation given by $\boldsymbol{u}(x_1, x_2) := (x_1 \cos x_2, x_1 \sin x_2)$. This deformation transforms Ω into an annular region, as shown in Fig. 2a, and produces a self-contact. It is easy to check that $\Gamma(\boldsymbol{u}) = \varnothing$ and $\bar{\mathcal{E}}(\boldsymbol{u}) = 0$, which corresponds to our intuition, since $\boldsymbol{u}$ does not create any surface because it is smooth. Note also that $D(\mathrm{im_G}(\boldsymbol{u}, \Omega), \boldsymbol{y}) = 1$ for all $\boldsymbol{y} \in (1, 2) \times \{0\}$, despite $D(\mathrm{im_G}(\boldsymbol{u}, U), \boldsymbol{y}) = 0$ for every open set $U \subset\subset \Omega$.

(b) Let Ω be the square $(1, 3) \times (-1, 1)$ in $\mathbb{R}^2$, and let $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ be given by

$$\boldsymbol{u}(\boldsymbol{x}) := \begin{cases} \boldsymbol{x} & \text{if } \boldsymbol{x} \in (1, 2) \times (-1, 1) \\ (5, 0) - \boldsymbol{x} & \text{if } \boldsymbol{x} \in [2, 3) \times (-1, 1). \end{cases}$$

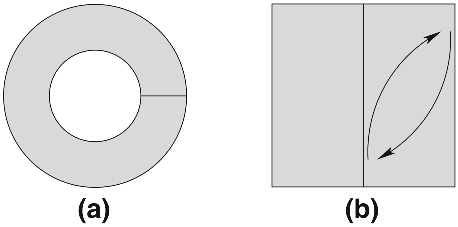

Fig. 2. Deformations of the examples **a** and **b**

This deformation produces a fracture with self-contact, as shown in Fig. 2b. It is easy to check that $\Gamma(\boldsymbol{u}) = \Gamma_V(\boldsymbol{u}) = \{2, 3\} \times (-1, 1)$ and $\Gamma_I(\boldsymbol{u}) = \varnothing$. The reason $\{2\} \times (-1, 1)$ forms part of the visible surface, and not of the invisible one (perhaps against what the names suggest), is that we have reserved the term *invisible* for the case in which the pieces of surface put together are both parts of the created surface. In this example, one of the two surfaces put in contact ($\boldsymbol{u}(\{3\} \times [-1, 1])$) was not created, it already existed (it is part of $\boldsymbol{u}(\partial\Omega)$).
Note also that $\operatorname{im_G}(\boldsymbol{u}, \Omega)$ has density 1 at each point of $\{2\} \times (-1, 1)$. This example shows that our definition of visible created surface is not equivalent to $\{\boldsymbol{y} \in J_{\boldsymbol{u}_\Omega^{-1}} \cap \partial^* \operatorname{im_G}(\boldsymbol{u}, \Omega) : (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in \Omega\}$. That is to say, condition b(iii) of Definition 9 does not imply that $D(\operatorname{im_G}(\boldsymbol{u}, \Omega) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0) = 0$.

(c) Let $\boldsymbol{u}$ be as in [18, Prop. 4]. Then, following the notation there, we have that $\Gamma(\boldsymbol{u}) = \Gamma_V(\boldsymbol{u}) = \bigcup_{i=1}^M \Gamma_i$ and $\Gamma_I(\boldsymbol{u}) = \varnothing$.

(d) For each $j \in \mathbb{N}$, consider the deformation $\boldsymbol{u}_j$ constructed in [24, pp. 51–53], which we have represented in Fig. 1 for the case $j = 3$. Then, following the notation of [18, Sect. 3], we have that

$$\Gamma(\boldsymbol{u}_j) \cong \left(\bigcup_{z \in A_j^-} C_{j,z}^-\right) \cup \left(\bigcup_{z \in A_j^+} C_{j,z}^+\right), \quad \Gamma_I(\boldsymbol{u}_j) \cong \left(\bigcup_{z \in A_j^-} C_{j,z}^-\right) \cap \left(\bigcup_{z \in A_j^+} C_{j,z}^+\right),$$
$$\Gamma_V(\boldsymbol{u}_j) \cong \Gamma(\boldsymbol{u}_j) \setminus \Gamma_I(\boldsymbol{u}_j).$$

We note that $\bar{\mathcal{E}}$ cannot 'see' the jumps of $\boldsymbol{u}^{-1}$ across the image of $\partial\Omega$, since the test functions ϕ of Definition 7 are compactly supported in Ω. Therefore, it does not detect, for example, the phenomenon of cavitation at the boundary [24, Sect. 11]. Neither can the functional $\mathcal{E}$, to be defined in Subsection 4.3.

4.2. A characterization of $\bar{\mathcal{E}}$

One of the main goals of this section is to refine the characterization of $\bar{\mathcal{E}}$ obtained in Theorem 2(iv), so that it is not necessary to restrict our discussion to an open set U compactly contained in the domain. Proposition 4 shows that, in fact, $\bar{\mathcal{E}}$ is supported on the created surface.

Proposition 4. *Let* $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ *be approximately differentiable in almost all* Ω, *one-to-one almost every, and such that* $\det \nabla \boldsymbol{u} > 0$ *almost everywhere and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n \times n})$. *Assume that* $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$ *and* $\mathbf{0} \notin \bar{\Omega}$. *Let* $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}} : \Gamma_I(\boldsymbol{u}) \to$

$\mathbb{S}^{n-1}$ be a Borel orientation of $\Gamma_I(\boldsymbol{u})$, and denote the lateral traces of $\boldsymbol{u}^{-1}$ with respect to $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}|_{\Gamma_I(\boldsymbol{u})}$ by $(\boldsymbol{u}^{-1})^{\pm}$. Then, for every $\phi \in C_c^\infty(\Omega)$ and $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$\bar{\mathcal{E}}_{\boldsymbol{u}}(\phi, \boldsymbol{g}) = \int_{\Gamma_V(\boldsymbol{u})} \phi((\boldsymbol{u}^{-1})^-(\boldsymbol{y}))\, \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) + \int_{\Gamma_I(\boldsymbol{u})} \Big[\phi((\boldsymbol{u}^{-1})^-(\boldsymbol{y})) - \phi((\boldsymbol{u}^{-1})^+(\boldsymbol{y}))\Big]\, \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) \tag{39}$$

The fact that $\Gamma_I(\boldsymbol{u})$ admits a Borel orientation and that the traces $(\boldsymbol{u}^{-1})^{\pm}|_{\Gamma_I(\boldsymbol{u})}$ are Borel will be established in Lemma 5.

Some technical lemmas are necessary for the proof of Proposition 4, which is given at the end of this subsection. We begin with the following straightforward observations. Recall that $J_{\boldsymbol{u}_U^{-1}}$ denotes the jump set of $\boldsymbol{u}_U^{-1}$, and $(\boldsymbol{u}_U^{-1})^{\pm}$ its lateral traces in $J_{\boldsymbol{u}_U^{-1}}$ with respect to the orientation $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}$.

Lemma 4. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost every, and suppose that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost everywhere $\boldsymbol{x} \in \Omega$. Assume $\boldsymbol{0} \notin \bar{\Omega}$ and let U be any nonempty open subset of Ω. Let $\boldsymbol{x}_0, \boldsymbol{y}_0 \in \mathbb{R}^n$ and $\boldsymbol{\nu} \in \mathbb{S}^{n-1}$.*

i) *Suppose that*

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu})}} \boldsymbol{u}_U^{-1}(\boldsymbol{y}) = \boldsymbol{x}_0. \tag{40}$$

Then $\boldsymbol{x}_0 \in \bar{U} \cup \{\boldsymbol{0}\}$, and

$$\boldsymbol{x}_0 \in \bar{U} \quad \textit{if and only if} \quad D\left(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0\right) = \frac{1}{2};$$
$$\boldsymbol{x}_0 = \boldsymbol{0} \quad \textit{if and only if} \quad D\left(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0\right) = 0.$$

If $\boldsymbol{x}_0 \in \bar{U}$ we also have that

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu})}} \boldsymbol{u}_\Omega^{-1}(\boldsymbol{y}) = \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu})}} \boldsymbol{u}^{-1}(\boldsymbol{y}) = \boldsymbol{x}_0. \tag{41}$$

ii) *If*

$$D\left(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0\right) = \frac{1}{2} \tag{42}$$

and (41) *holds, then $\boldsymbol{x}_0 \in \bar{U}$ and* (40) *is satisfied.*

iii) *If*

$$D\left(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0\right) = 0. \tag{43}$$

then

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu})}} \boldsymbol{u}_U^{-1}(\boldsymbol{y}) = \boldsymbol{0}.$$

iv) *Suppose that* $D(\mathrm{im_G}(\boldsymbol{u},\Omega)\cap H^+(\boldsymbol{y}_0,\boldsymbol{\nu}),\boldsymbol{y}_0)=\frac{1}{2}$ *and*

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu})}} \boldsymbol{u}^{-1}(\boldsymbol{y})=\boldsymbol{x}_0$$

with $\boldsymbol{x}_0\in U$. *Then* $D(\mathrm{im_G}(\boldsymbol{u},U)\cap H^+(\boldsymbol{y}_0,\boldsymbol{\nu}),\boldsymbol{y}_0)=\frac{1}{2}$ *and*

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu})}} \boldsymbol{u}_U^{-1}(\boldsymbol{y})=\boldsymbol{x}_0.$$

v) *Suppose that* $\boldsymbol{y}_0\in J_{\boldsymbol{u}_U^{-1}}$. *Then, at least one of the following options occur:*

a) $D(\mathrm{im_G}(\boldsymbol{u},U)\cap H^+(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)),\boldsymbol{y}_0)=\frac{1}{2}$ *and*

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}_\Omega^{-1}(\boldsymbol{y})=\left(\boldsymbol{u}_U^{-1}\right)^+(\boldsymbol{y}_0).$$

b) $D(\mathrm{im_G}(\boldsymbol{u},U)\cap H^-(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)),\boldsymbol{y}_0)=\frac{1}{2}$ *and*

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^-(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}_\Omega^{-1}(\boldsymbol{y})=\left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y}_0).$$

Proof. Suppose first that (40) holds. Then, for any $r>0$,

$$D\left(\left\{\boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu}):\ \boldsymbol{u}_U^{-1}(\boldsymbol{y})\in B(\boldsymbol{x}_0,r)\right\},\boldsymbol{y}_0\right)=\frac{1}{2}. \tag{44}$$

In particular, there exists a sequence $\{\boldsymbol{y}_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $\boldsymbol{u}_U^{-1}(\boldsymbol{y}_k)$ converges to $\boldsymbol{x}_0$. As $\boldsymbol{u}_U^{-1}(\boldsymbol{y})\in U\cup\{\boldsymbol{0}\}$ for every $\boldsymbol{y}\in\mathbb{R}^n$, then $\boldsymbol{x}_0\in\bar{U}\cup\{\boldsymbol{0}\}$. If $\boldsymbol{x}_0=\boldsymbol{0}$, by choosing $r>0$ such that $B(\boldsymbol{0},r)\cap\bar{\Omega}=\varnothing$, we infer from (44) that (43) holds. Analogously, if $\boldsymbol{x}_0\in\bar{U}$, by choosing $r>0$ such that $\boldsymbol{0}\notin B(\boldsymbol{x}_0,r)$, we infer from (44) that (42) holds; moreover, since $\boldsymbol{u}_U^{-1}=\boldsymbol{u}_\Omega^{-1}=\boldsymbol{u}^{-1}$ in $\mathrm{im_G}(\boldsymbol{u},U)$, obviously (41) follows from (40) and (42).

Conversely, (42) and (41) trivially imply (40), and, thanks to i), $\boldsymbol{x}_0\in\bar{U}$.

To prove iii) we note that $\boldsymbol{u}_U^{-1}$ is identically $\boldsymbol{0}$ outside $\mathrm{im_G}(\boldsymbol{u},U)$, so if (43) holds it is clear that (40) holds too with $\boldsymbol{x}_0=\boldsymbol{0}$.

We now prove iv). For any $r>0$ we have that $D(\mathrm{im_G}(\boldsymbol{u},B(\boldsymbol{x}_0,r))\cap H^+(\boldsymbol{y}_0,\boldsymbol{\nu}))=\frac{1}{2}$, so if r is small enough we have in particular that $D(\mathrm{im_G}(\boldsymbol{u},U)\cap H^+(\boldsymbol{y}_0,\boldsymbol{\nu}))=\frac{1}{2}$. As (41) holds, the conclusion follows from ii).

We show v). By Definition 1(d), the vectors $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}_0)$ and $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}_0)$ are different, so one of them is not $\boldsymbol{0}$. The conclusion follows from i). □

As is well known, the orientation vector at a jump point is uniquely determined up to a sign, and the choice of that sign is somewhat arbitrary [see Definition 1(d)]. In the following definition, we establish a convention on the sign choice of the vector corresponding to a jump point of the inverse. This convention is based on Lemma 4, and in particular, it favours option v(b). Recall from Definition 2 the definition of reduced boundary.

Definition 10. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and suppose that $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ for almost every $\boldsymbol{x} \in \Omega$. Assume $\boldsymbol{0} \notin \bar{\Omega}$ and let U be any nonempty open subset of Ω. Suppose that $\boldsymbol{y}_0 \in J_{\boldsymbol{u}_U^{-1}}$. The orientation of $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)$ [corresponding to Definition 1(d)] is chosen so that the following properties are satisfied:

i) $D(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^-(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)), \boldsymbol{y}_0) = \frac{1}{2}$, $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}_0) \in \bar{U}$, and

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^-(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}_\Omega^{-1}(\boldsymbol{y}) = \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^-(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}^{-1}(\boldsymbol{y}) = \left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y}_0).$$

ii) If $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}_0) = \boldsymbol{0}$ then

$$D(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)), \boldsymbol{y}_0) = 0 \quad \text{and} \quad \boldsymbol{y}_0 \in \partial^* \operatorname{im_G}(\boldsymbol{u}, U).$$

If $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}_0) \in \bar{U}$ then

$$\begin{gathered} D\left(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)), \boldsymbol{y}_0\right) = \frac{1}{2}, \\ D(\operatorname{im_G}(\boldsymbol{u}, U), \boldsymbol{y}_0) = D(\operatorname{im_G}(\boldsymbol{u}, \Omega), \boldsymbol{y}_0) = 1, \quad \text{and} \\ \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}_\Omega^{-1}(\boldsymbol{y}) = \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}^{-1}(\boldsymbol{y}) = \left(\boldsymbol{u}_U^{-1}\right)^+(\boldsymbol{y}_0). \end{gathered}$$

iii) If both $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}_0)$ and $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}_0)$ are in $\bar{U}$, but one is in U and the other is in ∂U, then $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}_0) \in U$.

As an immediate consequence of Definition 10, we have that

$$\text{if } \left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y}_0) \in \partial U \text{ then } \left(\boldsymbol{u}_U^{-1}\right)^+(\boldsymbol{y}_0) \notin U. \tag{45}$$

Thus, the following consequence of Lemma 4(v) holds.

Corollary 1. *Let* $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ *be approximately differentiable in almost all* Ω*, one-to-one almost everywhere, and suppose that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every* $\boldsymbol{x} \in \Omega$*. Assume* $\boldsymbol{0} \notin \bar{\Omega}$ *and let* U *be any nonempty open subset of* Ω*. Suppose that* $\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}}$*. Let* $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y})$ *have the orientation according to Definition 10. Then:*

i) $\boldsymbol{y} \in \partial^* \operatorname{im_G}(\boldsymbol{u}, U)$ *if and only if* $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}) = \boldsymbol{0}$*. Moreover, in this case,* $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y})$ *equals the unit outward normal to* $\operatorname{im_G}(\boldsymbol{u}, U)$ *(according to Definition 2).*

ii) *If both* $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y})$ *and* $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y})$ *are in* $\bar{U}$*, then* $\boldsymbol{y} \in J_{\boldsymbol{u}_\Omega^{-1}} \cap J_{\boldsymbol{u}^{-1}}$ *and there exist* $s_1, s_2 \in \{-1, 1\}$ *such that* $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}) = s_1 \boldsymbol{\nu}_{\boldsymbol{u}_\Omega^{-1}}(\boldsymbol{y}) = s_2 \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y})$*.*

Note that in Corollary 1(ii), no specific orientation of $\boldsymbol{\nu}_{\boldsymbol{u}_{\Omega}^{-1}}(\boldsymbol{y})$ or $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y})$ was chosen.

An important step in the proof of Proposition 4 is to establish the connection between the created surface $\Gamma(\boldsymbol{u})$ and the jump set of truncated inverses of the form $\boldsymbol{u}_{V_k}^{-1}$, for a suitable increasing sequence $\{V_k\}_{k\in N}$ of open sets compactly contained in Ω.

Definition 11. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and suppose that $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ for almost every $\boldsymbol{x} \in \Omega$, and $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$. Fix a sequence $\{V_k\}_{k\in\mathbb{N}}$ in $\mathcal{U}_{\boldsymbol{u}}$ such that $\bar{V}_k \subset V_{k+1}$ for all $k \in \mathbb{N}$, and $\Omega = \bigcup_{k=1}^{\infty} V_k$. For each $k \in \mathbb{N}$, the function $\boldsymbol{u}_{V_k}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ of Definition 8 will be denoted by $\boldsymbol{v}_k$.

The existence of $\{V_k\}_{k\in\mathbb{N}}$ can be easily obtained by using Uryshon functions, Sard's lemma and Lemma 2. Throughout the rest of the paper, the sets V_k and the functions $\boldsymbol{v}_k$ of Definition 11 will be fixed.

In the following proposition, we list some interesting properties of the visible and invisible surfaces, and relate them to the sequence $\{J_{\boldsymbol{v}_k}\}_{k\in\mathbb{N}}$. The main idea is the following: for a fixed $k \in \mathbb{N}$, the jump set of $\boldsymbol{v}_k$ has two parts, one corresponding to the surface created in V_k, the other corresponding to the image of ∂V_k. The first part will appear in the jump set of $\boldsymbol{v}_\ell$, for all $\ell > k$. In contrast, the second part, which is due to having defined $\boldsymbol{v}_k$ by $\boldsymbol{0}$ outside $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, V_k)$ (see Definitions 8 and 11), will no longer be contained in $J_{\boldsymbol{v}_\ell}$ for $\ell > k$. Thus, the created surface $\Gamma(\boldsymbol{u})$ can be obtained as the set of points that are in all but finitely many of the $J_{\boldsymbol{v}_k}$.

Proposition 5. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every $\boldsymbol{x} \in \Omega$. Assume that* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$ *and $\boldsymbol{0} \notin \bar{\Omega}$. Consider the sets V_k and the functions $\boldsymbol{v}_k$ of Definition 11. Then the following statements hold:*

i) $\Gamma_I(\boldsymbol{u}, U) \subset J_{\boldsymbol{u}_U^{-1}}$ *for any open subset U of Ω.*
ii) $\Gamma(\boldsymbol{u}) = \liminf_{k\to\infty} J_{\boldsymbol{v}_k}$.
iii) $\Gamma_V(\boldsymbol{u}) = \Gamma(\boldsymbol{u}) \cap \liminf_{k\to\infty} \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, V_k)$.
iv) $\Gamma_I(\boldsymbol{u}) = \bigcup_{k\in\mathbb{N}} \Gamma_I(\boldsymbol{u}, V_k) = \liminf_{k\to\infty} \Gamma_I(\boldsymbol{u}, V_k)$.
v) $\Gamma_V(\boldsymbol{u}) \cap \Gamma_I(\boldsymbol{u}) = \varnothing$.
vi) *For each $\boldsymbol{y} \in \Gamma_V(\boldsymbol{u})$ there exists $p \in \mathbb{N}$ such that for all $k \geqq p$, the vector $\boldsymbol{\nu}$ of Definition 9*b*) coincides with the orientation vector $\boldsymbol{\nu}_{\boldsymbol{v}_k}(\boldsymbol{y})$ of $J_{\boldsymbol{v}_k}$ according to Definition 10, and with the outward normal to* $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, V_k)$ *according to Definition 2.*

Proof. Statement iv) is obvious, and statement i) follows from Lemma 4(iv). For the rest of the statements, apply the convention of Definition 10 to the functions $\boldsymbol{v}_k$. Fix $\boldsymbol{y} \in \liminf_{k\to\infty} J_{\boldsymbol{v}_k}$, and let $p \in \mathbb{N}$ be such that $\boldsymbol{y} \in J_{\boldsymbol{v}_k}$ for all $k \geqq p$. If

$$\boldsymbol{v}_k^+(\boldsymbol{y}) = \boldsymbol{0} \quad \text{for all } k \geqq p \tag{46}$$

then $\boldsymbol{y} \in \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, V_k)$ for all $k \geqq p$, according to Corollary 1(i), and we obtain that $\boldsymbol{y} \in \liminf_{k\to\infty} \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, V_k)$ as a consequence. If (46) does not hold, then we must have, by Definition 10(ii), that $D(\operatorname{im_G}(\boldsymbol{u}, V_k), \boldsymbol{y}) = 1$ and $\boldsymbol{y} \in J_{\boldsymbol{u}^{-1}}$, with $(\boldsymbol{u}^{-1})^+(\boldsymbol{y})$ and $(\boldsymbol{u}^{-1})^-(\boldsymbol{y})$ both in $\bar{V}_k \subset \Omega$, for some $k \geqq p$. Thus

$$\liminf_{k\to\infty} J_{\boldsymbol{v}_k} \subset \left(\liminf_{k\to\infty} \partial^* \operatorname{im_G}(\boldsymbol{u}, V_k)\right) \cup \Gamma_I(\boldsymbol{u}). \tag{47}$$

Now suppose that $\boldsymbol{y} \in \Gamma_I(\boldsymbol{u})$. Since $(\boldsymbol{u}^{-1})^+(\boldsymbol{y})$ and $(\boldsymbol{u}^{-1})^-(\boldsymbol{y})$ are in Ω, there exists $p \in \mathbb{N}$ such that for all $k \geqq p$ the traces $(\boldsymbol{u}^{-1})^+(\boldsymbol{y})$ and $(\boldsymbol{u}^{-1})^-(\boldsymbol{y})$ are in V_k. Using i) we obtain that

$$\boldsymbol{y} \in \liminf_{k\to\infty} \Gamma_I(\boldsymbol{u}, V_k) \subset \liminf_{k\to\infty} J_{\boldsymbol{v}_k}.$$

Moreover, thanks to (47) we obtain that

$$\liminf_{k\to\infty} J_{\boldsymbol{v}_k} = \liminf_{k\to\infty} \left(J_{\boldsymbol{v}_k} \cap \partial^* \operatorname{im_G}(\boldsymbol{u}, V_k)\right) \cup \Gamma_I(\boldsymbol{u}). \tag{48}$$

For each $k \in \mathbb{N}$, by Lemma 4(iv), $D(\operatorname{im_G}(\boldsymbol{u}, V_k), \boldsymbol{y}) = 1$ for every $\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}, V_k)$, and, hence, $\Gamma_I(\boldsymbol{u}, V_k) \cap \partial^* \operatorname{im_G}(\boldsymbol{u}, V_k) = \varnothing$. Thanks to iv), this shows that

$$\Gamma_I(\boldsymbol{u}) \cap \liminf_{k\to\infty} \partial^* \operatorname{im_G}(\boldsymbol{u}, V_k) = \varnothing. \tag{49}$$

Hence, it only remains to prove that

$$\Gamma_V(\boldsymbol{u}) = \liminf_{k\to\infty} \left(J_{\boldsymbol{v}_k} \cap \partial^* \operatorname{im_G}(\boldsymbol{u}, V_k)\right),$$

and to verify that vi) is satisfied.

Suppose $\boldsymbol{y} \in \Gamma_V(\boldsymbol{u})$, let $p \in \mathbb{N}$ be such that $(\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in V_p$, and let $k \geqq p$. Then, by Lemma 4(i) we have that $\boldsymbol{v}_k^-(\boldsymbol{y}) = (\boldsymbol{u}^{-1})^-(\boldsymbol{y})$, whereas by Lemma 4(iii) we have that $\boldsymbol{v}_k^+(\boldsymbol{y}) = \boldsymbol{0}$, whence $\boldsymbol{y} \in J_{\boldsymbol{v}_k}$. Here we have taken the lateral traces with respect to the vector $\boldsymbol{\nu}(\boldsymbol{y})$ of Definition 9(b). This also shows that $\boldsymbol{\nu}(\boldsymbol{y}) = \boldsymbol{\nu}_{J_{\boldsymbol{v}_k}}(\boldsymbol{y})$, according to Definition 10. Now, by Corollary 1(i), we have that $\boldsymbol{y} \in \partial^* \operatorname{im_G}(\boldsymbol{u}, V_k)$ and $\boldsymbol{\nu}(\boldsymbol{y}) = \boldsymbol{\nu}_{\operatorname{im_G}(\boldsymbol{u}, V_k)}(\boldsymbol{y})$, according to Definition 2. In total, we have proved vi) and showed that $\boldsymbol{y} \in \liminf_{k\to\infty}(J_{\boldsymbol{v}_k} \cap \partial^* \operatorname{im_G}(\boldsymbol{u}, V_k))$.

Conversely, suppose that $\boldsymbol{y}_0 \in \liminf_{k\to\infty}(J_{\boldsymbol{v}_k} \cap \partial^* \operatorname{im_G}(\boldsymbol{u}, V_k))$. Then, by Corollary 1(i) and Lemma 4, a natural number p exists such that

a) $\boldsymbol{y}_0 \in J_{\boldsymbol{v}_k}$ for all $k \geqq p$;
b) $D(\operatorname{im_G}(\boldsymbol{u}, \Omega) \cap H^-(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{v}_p}(\boldsymbol{y}_0)), \boldsymbol{y}_0) = D(\operatorname{im_G}(\boldsymbol{u}, V_p) \cap H^-(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{v}_p}(\boldsymbol{y}_0)), \boldsymbol{y}_0) = \frac{1}{2}$;
c) the lateral trace

$$(\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0) = \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^-(\boldsymbol{y}, \boldsymbol{\nu}_{\boldsymbol{v}_p}(\boldsymbol{y}_0))}} \boldsymbol{u}^{-1}(\boldsymbol{y})$$

exists and coincides with $\boldsymbol{v}_p^-(\boldsymbol{y}_0)$, which is in $\bar{V}_p \subset \Omega$; and
d) $D(\operatorname{im_G}(\boldsymbol{u}, V_k) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{v}_p}(\boldsymbol{y}_0)), \boldsymbol{y}_0) = 0$ for all $k \geqq p$,

which implies that $\boldsymbol{y}_0 \in \Gamma_V(\boldsymbol{u})$. $\square$

In the following lemma, we establish a convention on the orientation of $\Gamma(\boldsymbol{u})$ analogous to the convention of Definition 10.

Lemma 5. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every $\boldsymbol{x} \in \Omega$, and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$. *Assume $\boldsymbol{0} \notin \bar{\Omega}$. Then the sets $\Gamma(\boldsymbol{u})$, $\Gamma_I(\boldsymbol{u})$ and $\Gamma_V(\boldsymbol{u})$ are Borel.*

Moreover, consider the sets V_k and the functions $\boldsymbol{v}_k$ of Definition 11. Then for each $k \in \mathbb{N}$ there exists an orientation of $J_{\boldsymbol{v}_k}$ satisfying the convention of Definition 10 and such that the resulting maps

$$\boldsymbol{\nu}_{\boldsymbol{v}_k} : J_{\boldsymbol{v}_k} \to \mathbb{S}^{n-1}, \quad \boldsymbol{v}_k^- : J_{\boldsymbol{v}_k} \to \bar{V}_k, \quad \boldsymbol{v}_k^+ : J_{\boldsymbol{v}_k} \to \bar{V}_k \cup \{\boldsymbol{0}\}$$

are Borel, and there exist Borel maps

$$\boldsymbol{\nu}_{\boldsymbol{u}^{-1}} : \Gamma(\boldsymbol{u}) \to \mathbb{S}^{n-1}, \quad (\boldsymbol{u}^{-1})^- : \Gamma(\boldsymbol{u}) \to \Omega, \quad (\boldsymbol{u}^{-1})^+ : \Gamma_I(\boldsymbol{u}) \to \Omega$$

with the following properties:

i) *For every $\boldsymbol{y}_0 \in \Gamma_I(\boldsymbol{u})$ and every $k \in \mathbb{N}$ such that $\boldsymbol{y}_0 \in J_{\boldsymbol{v}_k}$, we have that*

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}^{-1}(\boldsymbol{y}) = (\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0),$$

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^-(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}^{-1}(\boldsymbol{y}) = (\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0) = \boldsymbol{v}_k^-(\boldsymbol{y}_0)$$

and $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}_0) = \boldsymbol{\nu}_{\boldsymbol{v}_k}(\boldsymbol{y}_0)$.

ii) *For every $\boldsymbol{y}_0 \in \Gamma_V(\boldsymbol{u})$, we have that*

a) $D(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) \cap H^-(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}_0)), \boldsymbol{y}_0) = \frac{1}{2}$.

b) $\displaystyle\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^-(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}^{-1}(\boldsymbol{y}) = (\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0).$

c) *For every open set $U \subset\subset \Omega$ containing $(\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0)$,*

$$\boldsymbol{y}_0 \in \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) \cap J_{\boldsymbol{u}_U^{-1}}, \quad \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}_0) = \boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)}(\boldsymbol{y}_0) = \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)$$

$$\text{and } \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}_0) = (\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0),$$

where $J_{\boldsymbol{u}_U^{-1}}$ has been oriented according to Definition 10.

Proof. As $J_{\boldsymbol{v}_k}$ is a Borel set for every $k \in \mathbb{N}$ (see, for example, [5, Prop. 3.69]), then $\Gamma(\boldsymbol{u})$ is a Borel set. Analogously, $J_{\boldsymbol{u}^{-1}}$ is a Borel set, and, hence, $\Gamma_I(\boldsymbol{u})$ is a Borel set. Finally, $\Gamma_V(\boldsymbol{u})$ is a Borel set since, by Proposition 5, it coincides with $\Gamma(\boldsymbol{u})\setminus\Gamma_I(\boldsymbol{u})$.

Now, for each $k \in \mathbb{N}$, fix a Borel orientation $\boldsymbol{\nu}_{\boldsymbol{v}_k} : J_{\boldsymbol{v}_k} \to \mathbb{S}^{n-1}$ for the jump set $J_{\boldsymbol{v}_k}$ such that the convention of Definition 10 is respected, and that whenever $\boldsymbol{y} \in J_{\boldsymbol{v}_{k_1}} \cap J_{\boldsymbol{v}_{k_2}}$ for some $k_1, k_2 \in \mathbb{N}$, the orientating vectors $\boldsymbol{\nu}_{\boldsymbol{v}_{k_1}}(\boldsymbol{y})$ and $\boldsymbol{\nu}_{\boldsymbol{v}_{k_2}}(\boldsymbol{y})$ coincide. Let $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}} : \Gamma(\boldsymbol{u}) \to \mathbb{S}^{n-1}$ be defined by $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}) := \boldsymbol{\nu}_{k(\boldsymbol{y})}(\boldsymbol{y})$, for each $\boldsymbol{y} \in \Gamma(\boldsymbol{u})$, where $k(\boldsymbol{y})$ is the first integer $k \in \mathbb{N}$ such that $\boldsymbol{y} \in J_{\boldsymbol{v}_k}$. The resulting maps are Borel because so are $\boldsymbol{\nu}_{\boldsymbol{v}_k}$, and it is clear that i) and ii) hold. □

We note that if $\det \nabla \boldsymbol{u} > 0$ almost everywhere and $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$, then $\Gamma_V(\boldsymbol{u})$ and $\Gamma_I(\boldsymbol{u})$ are not only Borel, but also countably $\mathcal{H}^{n-1}$ rectifiable. This follows from Theorem 2, Proposition 5, and the rectifiability properties of jump sets (see, for example, [5, Th. 3.78]).

Suppose $U \subset \Omega$ is an open set and ϕ is a $C_c^\infty(\Omega)$ function with support in U. Recall that we are assuming, by the convention of Definition 10, that (45) holds. Consequently, if

$$\phi\left(\left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y})\right) - \phi\left(\left(\boldsymbol{u}_U^{-1}\right)^+(\boldsymbol{y})\right) \neq 0$$

then $\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}}$ and $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) \in U$. Based on this idea, we compare in the following lemmas the set $\Gamma(\boldsymbol{u})$ with the set of $\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}}$ such that $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) \in U$. We also compare the lateral traces of $\boldsymbol{u}_U^{-1}$ with the traces of $\boldsymbol{u}^{-1}$, and the orientation vector $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}$ with $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}$. Recall from Definition 1(c) the notation $S_{\boldsymbol{v}}$ for the approximate discontinuity set of the function $\boldsymbol{v}$, and from Section 2.1 the notation $\widetilde{\subset}$.

Lemma 6. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and suppose that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every $\boldsymbol{x} \in \Omega$. Let U be an open set compactly contained in Ω. Assume $\boldsymbol{0} \notin \bar{\Omega}$. Then*

$$\begin{aligned}\Gamma(\boldsymbol{u}) \setminus J_{\boldsymbol{u}_U^{-1}} \subset (S_{\boldsymbol{u}_U^{-1}} \setminus J_{\boldsymbol{u}_U^{-1}}) \cup \{\boldsymbol{y} \in \Gamma_V(\boldsymbol{u}) :\ (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in \Omega \setminus U\} \\ \cup \{\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}) :\ (\boldsymbol{u}^{-1})^+(\boldsymbol{y}),\ (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in \Omega \setminus U\}.\end{aligned} \tag{50}$$

Now apply to $\boldsymbol{u}_U^{-1}$ the convention of Definition 10. If, in addition,

$$\det \nabla \boldsymbol{u} > 0 \textit{ almost everywhere}, \quad \operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n}), \quad \bar{\mathcal{E}}(\boldsymbol{u}) < \infty, \quad U \in \mathcal{U}_{\boldsymbol{u}}, \tag{51}$$

and $\phi \in C_c^\infty(\Omega)$ satisfies $\operatorname{spt}\phi \subset U$, then $\mathcal{H}^{n-1}$-almost every $\boldsymbol{y}$ in

$$\Gamma(\boldsymbol{u}) \setminus \left\{\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}} :\ \left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y}) \in U\right\} \tag{52}$$

is such that $\phi((\boldsymbol{u}^{-1})^+(\boldsymbol{y})) = \phi((\boldsymbol{u}^{-1})^-(\boldsymbol{y})) = 0$.

Proof. Let $\boldsymbol{y}_0 \in \Gamma(\boldsymbol{u}) \setminus J_{\boldsymbol{u}_U^{-1}}$ satisfy that one of $(\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0)$, $(\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0)$ belongs to U. In order to prove (50), it suffices to show that $\boldsymbol{u}_U^{-1}$ is not approximately continuous at $\boldsymbol{y}_0$. By Lemma 4(iv) we have that

$$D\,(\operatorname{im_G}(\boldsymbol{u}, U), \boldsymbol{y}_0) \neq 0. \tag{53}$$

Let $p \in \mathbb{N}$ be such that $\boldsymbol{y}_0 \in J_{\boldsymbol{v}_k}$ and $U \subset V_k$ for all $k \geqq p$. If $\boldsymbol{y}_0$ were an approximate continuity point for $\boldsymbol{u}_U^{-1}$ with $\operatorname{ap\,lim}_{\boldsymbol{y}\to\boldsymbol{y}_0} \boldsymbol{u}_U^{-1}(\boldsymbol{y}) \neq \boldsymbol{0}$, then, by Lemma 4, $\boldsymbol{y}_0$ would be a point of approximate continuity for $\boldsymbol{v}_k$, a contradiction, whereas if $\operatorname{ap\,lim}_{\boldsymbol{y}\to\boldsymbol{y}_0} \boldsymbol{u}_U^{-1}(\boldsymbol{y}) = \boldsymbol{0}$ then, by Lemma 4(i), $D(\operatorname{im_G}(\boldsymbol{u}, U), \boldsymbol{y}_0) = 0$, a contradiction with (53). This shows (50).

Now note that, thanks to Definition 10(i), $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) \in \bar{U}$ for every $\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}}$, and, hence

$$\Gamma(\boldsymbol{u})\setminus\left\{\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}} : \left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y}) \in U\right\}$$
$$= (\Gamma(\boldsymbol{u})\setminus J_{\boldsymbol{u}_U^{-1}}) \cup \left\{\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}} \cap \Gamma(\boldsymbol{u}) : \left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y}) \in \partial U\right\}.$$

Thus, by (50) and (45), every $\boldsymbol{y}$ in (52) satisfies that $\boldsymbol{y} \in S_{\boldsymbol{u}_U^{-1}}\setminus J_{\boldsymbol{u}_U^{-1}}$ or both $(\boldsymbol{u}^{-1})^+(\boldsymbol{y})$, $(\boldsymbol{u}^{-1})^-(\boldsymbol{y})$ are in $\mathbb{R}^n\setminus U$. Therefore, if, in addition, the assumptions (51) hold, then $\boldsymbol{u}_U^{-1} \in SBV_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ by Theorem 2, and, hence, by the Federer–Vol'pert theorem (see, for example, [5, Th. 3.78]), $\mathcal{H}^{n-1}(S_{\boldsymbol{u}_U^{-1}}\setminus J_{\boldsymbol{u}_U^{-1}}) = 0$. □

Lemma 7. *Let* $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ *be approximately differentiable in almost all* Ω*, and such that* $\operatorname{cof}\nabla\boldsymbol{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$*. Suppose that* $\det\nabla\boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every* $\boldsymbol{x} \in \Omega$*, and* $\boldsymbol{u}$ *is one-to-one almost everywhere. Let* U *be an open set compactly contained in* Ω*. Assume* $\boldsymbol{0} \notin \bar{\Omega}$*, consider a point* $\boldsymbol{y}_0 \in J_{\boldsymbol{u}_U^{-1}}$ *and orient the normal vector at* $\boldsymbol{y}_0$ *according to Definition 10. Suppose that* $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}_0) \in U$*. Consider the sets* V_k *and the functions* $\boldsymbol{v}_k$ *of Definition 11. Then the following properties hold:*

i) *If* $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}_0) \in U$ *then* $\boldsymbol{y}_0 \in \Gamma_I(\boldsymbol{u}, U) \subset \Gamma_I(\boldsymbol{u})$.
ii) *If* $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}_0) \in \partial U$ *then* $\boldsymbol{y}_0 \in \Gamma_I(\boldsymbol{u})\setminus\Gamma_I(\boldsymbol{u}, U)$.
iii) *If* $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}_0) = \boldsymbol{0}$ *then* $\boldsymbol{y}_0 \in \Gamma(\boldsymbol{u}) \cup \bigcup_{k\in\mathbb{N}}(S_{\boldsymbol{v}_k}\setminus J_{\boldsymbol{v}_k})$.

If, in addition, $\det\nabla\boldsymbol{u} > 0$ *almost everywhere,* $U \in \mathcal{U}_{\boldsymbol{u}}$ *and* $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$ *then*

$$\left\{\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}} : \left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y}) \in U\right\} \mathrel{\widetilde{\subset}} \Gamma(\boldsymbol{u}).$$

Proof. Statements i) and ii) clearly follow from Lemma 4(i).

Let $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}_0) = \boldsymbol{0}$. By Corollary 1(i), $\boldsymbol{y}_0 \in \partial^* \operatorname{im_G}(\boldsymbol{u}, U)$ and $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0) = \boldsymbol{\nu}_{\operatorname{im_G}(\boldsymbol{u},U)}(\boldsymbol{y}_0)$. There are two possibilities: either

$$D(\operatorname{im_G}(\boldsymbol{u}, V_k) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)), \boldsymbol{y}_0) = 0 \tag{54}$$

for all k such that $U \subset\subset V_k$, or there exists $k \in \mathbb{N}$ such that $U \subset\subset V_k$ and Equation (54) does not hold. In the first case we have that $\boldsymbol{y}_0 \in \Gamma_V(\boldsymbol{u})$, and, in particular, $\boldsymbol{y}_0 \in \Gamma(\boldsymbol{u})$. In the second case we have to consider two further possible scenarios. Note first that if the approximate limit

$$\boldsymbol{v}_k^+(\boldsymbol{y}_0) = \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{v}_k(\boldsymbol{y})$$

exists, it cannot be $\boldsymbol{0}$, since in that case we would have, thanks to Definition 10(ii), that equality (54) holds. Therefore, if it exists, by Lemma 4(i), we must have that

$$D(\operatorname{im_G}(\boldsymbol{u}, \Omega) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)), \boldsymbol{y}_0) = \frac{1}{2} \tag{55}$$

and that

$$\boldsymbol{v}_k^+(\boldsymbol{y}_0) = \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{v}_k(\boldsymbol{y}) = (\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0) = \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0 \\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0))}} \boldsymbol{u}^{-1}(\boldsymbol{y}).$$

Furthermore, as $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}_0) = \boldsymbol{0}$, we have, by Lemma 4(i), that

$$D(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0)), \boldsymbol{y}_0) = 0. \tag{56}$$

Now, Equations (55), (56) and Lemma 4(iv) imply that $\boldsymbol{v}_k^+(\boldsymbol{y}_0) \notin U$. Since, on the other hand, $\boldsymbol{v}_k^-(\boldsymbol{y}_0) = (\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}_0) \in U$ (by Lemma 4) we conclude that, if (54) does not hold then $\boldsymbol{y}_0$ cannot be a point of approximate continuity for $\boldsymbol{v}_k$. Therefore, if (54) does not hold then either $\boldsymbol{y}_0 \in S_{\boldsymbol{v}_k} \setminus J_{\boldsymbol{v}_k}$ or

$$\boldsymbol{y}_0 \in J_{\boldsymbol{v}_k}, \quad \boldsymbol{v}_k^+(\boldsymbol{y}_0) = (\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0) \in \bar{V}_k \subset \Omega \quad \text{and} \quad \boldsymbol{y}_0 \in \Gamma_I(\boldsymbol{u}) \subset \Gamma(\boldsymbol{u}),$$

the last relation being due to Corollary 1(ii). This finishes the proof of iii).

In particular, $\{\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}} : (\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) \in U\} \subset \Gamma(\boldsymbol{u}) \cup \bigcup_{k\in\mathbb{N}} (S_{\boldsymbol{v}_k} \setminus J_{\boldsymbol{v}_k})$. Therefore, if, in addition, $\det \nabla \boldsymbol{u} > 0$ almost everywhere and $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$, then, by Theorem 2 and the Federer–Vol'pert theorem (see, for example, [5, Th. 3.78]), $\mathcal{H}^{n-1}(S_{\boldsymbol{v}_k} \setminus J_{\boldsymbol{v}_k}) = 0$ for each $k \in \mathbb{N}$. This concludes the proof. □

Proof (of Proposition 4). Apply to $\boldsymbol{u}_U^{-1}$ the convention of Definition 10. Call, for simplicity,

$$F_U(\boldsymbol{y}) := \left[\phi\left(\left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y})\right) - \phi\left(\left(\boldsymbol{u}_U^{-1}\right)^+(\boldsymbol{y})\right)\right] \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}),$$
$$G_U(\boldsymbol{y}) := \phi\left(\left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y})\right) \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}),$$

for $\mathcal{H}^{n-1}$-almost every $\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}}$.

Since $\phi(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \notin U$, by virtue of (45) and Lemma 7, we have that

$$\int_{J_{\boldsymbol{u}_U^{-1}}} F_U(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) = \int_{\{\boldsymbol{y}\in J_{\boldsymbol{u}_U^{-1}} \cap \Gamma(\boldsymbol{u}):\ \left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y})\in U\}} F_U(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).$$

Moreover, thanks to Proposition 5 and Lemma 4(i), we have that

$$\begin{aligned}\int_{J_{\boldsymbol{u}_U^{-1}}} F_U(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) &= \int_{\{\boldsymbol{y}\in J_{\boldsymbol{u}_U^{-1}} \cap \Gamma_V(\boldsymbol{u}):\ \left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y})\in U\}} G_U(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) \\ &\quad + \int_{\{\boldsymbol{y}\in J_{\boldsymbol{u}_U^{-1}} \cap \Gamma_I(\boldsymbol{u}):\ \left(\boldsymbol{u}_U^{-1}\right)^-(\boldsymbol{y})\in U\}} F_U(\boldsymbol{y})\,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).\end{aligned}$$

Now call, for $\mathcal{H}^{n-1}$-almost every $\boldsymbol{y} \in J_{\boldsymbol{u}^{-1}}$,

$$F(\boldsymbol{y}) := \left[\phi((\boldsymbol{u}^{-1})^-(\boldsymbol{y})) - \phi((\boldsymbol{u}^{-1})^+(\boldsymbol{y}))\right] \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}),$$
$$G(\boldsymbol{y}) := \phi((\boldsymbol{u}^{-1})^-(\boldsymbol{y}))\, \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}).$$

Due to Lemma 5 we have that

$$\int_{\{y\in J_{u_U^{-1}}\cap\Gamma_V(u):\ \left(u_U^{-1}\right)^-(y)\in U\}}[G_U(y)-G(y)]\,\mathrm{d}\mathcal{H}^{n-1}(y)$$
$$=\int_{\{y\in J_{u_U^{-1}}\cap\Gamma_I(u):\ \left(u_U^{-1}\right)^-(y)\in U\}}[F_U(y)-F(y)]\,\mathrm{d}\mathcal{H}^{n-1}(y)=0.$$

Moreover, by Lemma 6, F vanishes $\mathcal{H}^{n-1}$-almost everywhere in $\Gamma(\boldsymbol{u})\setminus\{\boldsymbol{y}\in J_{\boldsymbol{u}_U^{-1}}:(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y})\in U\}$, so we obtain that

$$\int_{J_{u_U^{-1}}} F_U(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)=\int_{\Gamma_V(u)} G(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)+\int_{\Gamma_I(u)} F(y)\,\mathrm{d}\mathcal{H}^{n-1}(y).$$

The conclusion then follows from Theorem 2(iv). □

4.3. A characterization of $\mathcal{E}$

As discussed in [18, Sect. 3], $\bar{\mathcal{E}}(\boldsymbol{u})$ measures correctly, in many cases, the area of the created surface. However, examples were given of one-to-one almost everywhere deformations at which $\bar{\mathcal{E}}$ behaves unexpectedly [18, Sect. 6]. For those examples, it was shown that a more suitable definition of the area of the created surface was the following.

Definition 12. Let $\boldsymbol{u}:\Omega\to\mathbb{R}^n$ be approximately differentiable in almost all Ω. Suppose that $\operatorname{cof}\nabla\boldsymbol{u}\in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^{n\times n})$. Assume that $\det\nabla\boldsymbol{u}\in L^1_{\mathrm{loc}}$, or that $\boldsymbol{u}$ is one-to-one almost everywhere. For every $\boldsymbol{f}\in C^1_c(\Omega\times\mathbb{R}^n,\mathbb{R}^n)$, define

$$\mathcal{E}_u(f):=\int_\Omega\left[\nabla_x f(x,u(x))\cdot\operatorname{cof}\nabla u(x)+\operatorname{div}_y f(x,u(x))\det\nabla u(x)\right]\mathrm{d}x$$

and

$$\mathcal{E}(u):=\sup\left\{\mathcal{E}_u(f):\ f\in C^1_c(\Omega\times\mathbb{R}^n,\mathbb{R}^n),\ \|f\|_\infty\leqq 1\right\}.$$

The notation $D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ refers to the derivative of the map $\boldsymbol{f}(\cdot,\boldsymbol{y})$ evaluated at $\boldsymbol{x}$, while $\operatorname{div}_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ denotes the divergence of the map $\boldsymbol{f}(\boldsymbol{x},\cdot)$ evaluated at $\boldsymbol{y}$. As in Definition 7, if $\boldsymbol{u}$ is one-to-one almost everywhere then $\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f})$ is well defined by Proposition 1.

In the language of the theory of Cartesian currents, we are defining $\mathcal{E}(\boldsymbol{u})$ to be the mass $\mathbb{M}((\partial G_{\boldsymbol{u}})_{(n-1)})$ of the vertical part of the boundary of the current $G_{\boldsymbol{u}}$ carried by the graph of $\boldsymbol{u}$. We refer the reader to [16,17] for an exposition of the theory of currents.

The functionals $\bar{\mathcal{E}}_{\boldsymbol{u}}$ and $\mathcal{E}_{\boldsymbol{u}}$ are related in the following manner: if $\phi\in C^1_c(\Omega)$, $\boldsymbol{g}\in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$, and we define $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}):=\phi(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{y})$, then $\bar{\mathcal{E}}_{\boldsymbol{u}}(\phi,\boldsymbol{g})=\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f})$. Furthermore, $\bar{\mathcal{E}}\leqq\mathcal{E}$, and if $\bar{\mathcal{E}}(\boldsymbol{u})=0$ then $\mathcal{E}(\boldsymbol{u})=0$. Nevertheless, as shown in [18, Sect. 6], $\bar{\mathcal{E}}\neq\mathcal{E}$.

Theorem 3 gives a characterization of $\mathcal{E}(\boldsymbol{u})$ in terms of our notion of created surface. This justifies the use of this functional in the existence theory of [18], finishes the proof of Theorem 1, and constitutes one of the main results of this paper.

Theorem 3. *Let* $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ *be approximately differentiable in almost all* Ω, *one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u} > 0$ *almost everywhere and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$. *Assume* $\mathcal{E}(\boldsymbol{u}) < \infty$ *and* $\boldsymbol{0} \notin \bar{\Omega}$. *Let* $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}} : \Gamma_I(\boldsymbol{u}) \to \mathbb{S}^{n-1}$ *be a Borel orientation of* $\Gamma_I(\boldsymbol{u})$, *and denote the lateral traces of* $\boldsymbol{u}^{-1}$ *with respect to* $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}|_{\Gamma_I(\boldsymbol{u})}$ *by* $(\boldsymbol{u}^{-1})^{\pm}$. *Then*

$$\begin{aligned}\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f}) = &\int_{\Gamma_V(\boldsymbol{u})} \boldsymbol{f}((\boldsymbol{u}^{-1})^{-}(\boldsymbol{y}), \boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}) \,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) \\ &+ \int_{\Gamma_I(\boldsymbol{u})} \Big[\boldsymbol{f}((\boldsymbol{u}^{-1})^{-}(\boldsymbol{y}), \boldsymbol{y}) - \boldsymbol{f}((\boldsymbol{u}^{-1})^{+}(\boldsymbol{y}), \boldsymbol{y})\Big] \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}) \,\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).\end{aligned} \tag{57}$$

for all $\boldsymbol{f} \in C^\infty_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$. *Furthermore,*

$$\mathcal{E}(\boldsymbol{u}) = \mathcal{H}^{n-1}(\Gamma_V(\boldsymbol{u})) + 2\mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u})). \tag{58}$$

Proof. As $\mathcal{E}(\boldsymbol{u}) < \infty$, by Riesz' representation theorem, there exists an $\mathbb{R}^n$-valued Borel measure $\boldsymbol{\Lambda}$ in $\Omega \times \mathbb{R}^n$ such that $|\boldsymbol{\Lambda}|(\Omega \times \mathbb{R}^n) = \mathcal{E}(\boldsymbol{u})$ and

$$\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f}) = \int_{\Omega\times\mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{\Lambda}(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{f} \in C^\infty_c(\Omega \times \mathbb{R}^n). \tag{59}$$

Using (59), (33), and the fact that $D\boldsymbol{u}_U^{-1}$ has no Cantor part, we obtain that

$$\int_{J_{\boldsymbol{u}_U^{-1}}} \left[\phi \circ \left(\boldsymbol{u}_U^{-1}\right)^{-} - \phi \circ \left(\boldsymbol{u}_U^{-1}\right)^{+}\right] \boldsymbol{g} \cdot \boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}} \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Omega\times\mathbb{R}^n} \phi(\boldsymbol{x})\, \boldsymbol{g}(\boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{\Lambda}(\boldsymbol{x}, \boldsymbol{y}) \tag{60}$$

for all bounded Borel functions $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$ and all $\phi \in C^\infty_c(\Omega)$ with $\operatorname{spt}\phi \subset U$. Recall that the orientation of $J_{\boldsymbol{u}_U^{-1}}$ is subjected to the convention of Definition 10.

Let $F \subset J_{\boldsymbol{u}_U^{-1}}$ be a Borel set such that $\mathcal{H}^{n-1}(F) < \infty$, and consider the $\mathbb{R}^n$-valued measure

$$\boldsymbol{\lambda}_F := \left[\left(\left(\boldsymbol{u}_U^{-1}\right)^{-} \bowtie \mathbf{id}\right)_{\sharp} - \left(\left(\boldsymbol{u}_U^{-1}\right)^{+} \bowtie \mathbf{id}\right)_{\sharp}\right] \left(\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}} \mathcal{H}^{n-1} \llcorner F\right).$$

Here, the operator $\sharp$ denotes the push-forward of a measure (see, for example, [5, Def. 1.70]), and the function $(\boldsymbol{u}_U^{-1})^{\pm} \bowtie \mathbf{id} : J_{\boldsymbol{u}_U^{-1}} \to \mathbb{R}^n \times \mathbb{R}^n$ is defined by

$$\left(\left(\boldsymbol{u}_U^{-1}\right)^{\pm} \bowtie \mathbf{id}\right)(\boldsymbol{y}) = \left(\left(\boldsymbol{u}_U^{-1}\right)^{\pm}(\boldsymbol{y}), \boldsymbol{y}\right), \quad \boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}}.$$

By (60) and the definition of push-forward we thus have that

$$\int_{\Omega\times\mathbb{R}^n} \phi(\boldsymbol{x})\, \boldsymbol{g}(\boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{\lambda}_F(\boldsymbol{x},\boldsymbol{y}) = \int_{\Omega\times\mathbb{R}^n} \phi(\boldsymbol{x})\, \boldsymbol{g}(\boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{\Lambda}(\boldsymbol{x},\boldsymbol{y}).$$

for all bounded Borel functions $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$ and all $\phi \in C_c^\infty(\Omega)$ with spt $\phi \subset U$. The measure $\boldsymbol{\lambda}_F$ is finite; therefore, by the density in $C_c(U \times \mathbb{R}^n, \mathbb{R}^n)$ of the set of sums of functions having the form $\phi(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{y})$, we have that

$$\int_{\Omega\times\mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{\lambda}_F(\boldsymbol{x},\boldsymbol{y}) = \int_{\Omega\times\mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{\Lambda}(\boldsymbol{x},\boldsymbol{y}) \tag{61}$$

for all $\boldsymbol{f} \in C_c(U \times \mathbb{R}^n, \mathbb{R}^n)$. By virtue of the Riesz representation theorem, this implies that $\boldsymbol{\Lambda} \llcorner (U \times F) = \boldsymbol{\lambda}_F \llcorner (U \times F)$. Using that the images of the functions $(\boldsymbol{u}_U^{-1})^- \bowtie \mathbf{id}$ and $(\boldsymbol{u}_U^{-1})^+ \bowtie \mathbf{id}$ are disjoint, it is easy to check, by the definition of total variation of a measure (see, for example, [5, Def. 1.4]) that

$$|\boldsymbol{\lambda}_F| = \left| \left(\left(\boldsymbol{u}_U^{-1}\right)^- \bowtie \mathbf{id} \right)_\sharp \left(\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}} \mathcal{H}^{n-1} \llcorner F \right) \right| + \left| \left(\left(\boldsymbol{u}_U^{-1}\right)^+ \bowtie \mathbf{id} \right)_\sharp \left(\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}} \mathcal{H}^{n-1} \llcorner F \right) \right|.$$

This, together with [2, Lemma 1.3; 5, Prop. 1.23], and Lemmas 6 and 7, yields that

$$\begin{aligned} &|\boldsymbol{\Lambda}|(U \times F) \\ &\quad = \mathcal{H}^{n-1}\left(\left\{ \boldsymbol{y} \in F : \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in U \right\} \right) + \mathcal{H}^{n-1}\left(\left\{ \boldsymbol{y} \in F : \left(\boldsymbol{u}_U^{-1}\right)^+ (\boldsymbol{y}) \in U \right\} \right) \\ &\quad = \mathcal{H}^{n-1}\left(\left\{ \boldsymbol{y} \in F : \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in U \right\} \right) + \mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u}, U) \cap F), \end{aligned} \tag{62}$$

for all Borel sets $F \subset J_{\boldsymbol{u}_U^{-1}}$ such that $\mathcal{H}^{n-1}(F) < \infty$. Furthermore, since $J_{\boldsymbol{u}_U^{-1}}$ is σ-finite with respect to $\mathcal{H}^{n-1}$, the assumption $\mathcal{H}^{n-1}(F) < \infty$ can be neglected.

Consider now the sets V_k and the functions $\boldsymbol{v}_k$ of Definition 11, as well as the orientation of the jump sets $J_{\boldsymbol{v}_k}$ specified in Lemma 5. For each $p \in \mathbb{N}$, define $F_p := \bigcap_{k \geqq p} J_{\boldsymbol{v}_k}$. Applying (62) to $U = V_p$ and $F = F_p$ we obtain that

$$\mathcal{H}^{n-1}(\{\boldsymbol{y} \in F_p : (\boldsymbol{v}_p)^-(\boldsymbol{y}) \in V_p\}) + \mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u}, V_p) \cap F_p) = |\boldsymbol{\Lambda}|(V_p \times F_p).$$

Having in mind that the three sequences of sets

$$\left\{ \left\{ \boldsymbol{y} \in F_p : \boldsymbol{v}_p^-(\boldsymbol{y}) \in V_p \right\} \right\}_{p\in\mathbb{N}}, \quad \left\{ \Gamma_I(\boldsymbol{u}, V_p) \cap F_p \right\}_{p\in\mathbb{N}}, \quad \left\{ V_p \times F_p \right\}_{p\in\mathbb{N}},$$

are increasing, and that

$$\Gamma_I(\boldsymbol{u}) = \bigcup_{p\in\mathbb{N}} \left[\Gamma_I(\boldsymbol{u}, V_p) \cap F_p \right], \quad \Omega \times \Gamma(\boldsymbol{u}) = \bigcup_{p\in\mathbb{N}} V_p \times F_p,$$

it follows that

$$\mathcal{H}^{n-1}\left(\bigcup_{p\in\mathbb{N}} \{\boldsymbol{y} \in F_p : (\boldsymbol{v}_p)^-(\boldsymbol{y}) \in V_p\} \right) + \mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u})) = |\boldsymbol{\Lambda}|(\Omega \times \Gamma(\boldsymbol{u})). \tag{63}$$

From Proposition 5(ii) we know that $\Gamma(\boldsymbol{u}) = \bigcup_{p\in\mathbb{N}} F_p$. Moreover, if $\boldsymbol{y} \in F_p$ for some $p \in \mathbb{N}$, then, by Lemma 5, we have that $\boldsymbol{v}_k^-(\boldsymbol{y}) = \boldsymbol{v}_p^-(\boldsymbol{y}) \in \bar{V}_p \subset V_k$ for all $k \geqq p+1$. Therefore,

$$\Gamma(\boldsymbol{u}) = \bigcup_{p\in\mathbb{N}} \{\boldsymbol{y} \in F_p : (\boldsymbol{v}_p)^-(\boldsymbol{y}) \in V_p\}, \tag{64}$$

and since $\mathcal{E}(\boldsymbol{u}) = |\boldsymbol{\Lambda}|(\Omega\times\mathbb{R}^n)$ and $\Gamma(\boldsymbol{u}) = \Gamma_V(\boldsymbol{u}) \cup \Gamma_I(\boldsymbol{u})$, with disjoint union, it is clear that (63) imples (58), provided we can prove that $\boldsymbol{\Lambda}$ is supported in $\Omega \times \Gamma(\boldsymbol{u})$, that is, that

$$|\boldsymbol{\Lambda}|(\Omega \times (\mathbb{R}^n \setminus \Gamma(\boldsymbol{u}))) = 0. \tag{65}$$

In order to show (65), apply (60), for each $k \in \mathbb{N}$, with V_k and $\boldsymbol{g}\chi_{\mathbb{R}^n\setminus J_{\boldsymbol{v}_k}}$, to obtain that

$$\int_{\Omega\times(\mathbb{R}^n\setminus J_{\boldsymbol{v}_k})} \phi(\boldsymbol{x})\, \boldsymbol{g}(\boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{\Lambda}(\boldsymbol{x}, \boldsymbol{y}) = 0$$

for every bounded Borel function $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$, and every $\phi \in C_c^\infty(\Omega)$ supported in V_k. Since $\boldsymbol{\Lambda}$ is a finite measure, by virtue of the density in $C_c(U \times \mathbb{R}^n, \mathbb{R}^n)$ of the set of sums of functions of the form $\phi(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{y})$, we find that $|\boldsymbol{\Lambda}|(V_k \times (\mathbb{R}^n\setminus J_{\boldsymbol{v}_k})) = 0$ for every $k \in \mathbb{N}$. This, together with the fact that $\Omega \times (\mathbb{R}^n\setminus\Gamma(\boldsymbol{u})) \subset \bigcup_{k\in\mathbb{N}}[V_k \times (\mathbb{R}^n\setminus J_{\boldsymbol{v}_k})]$, yields the desired result.

Let $\boldsymbol{f} \in C_c^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ and choose $k \in \mathbb{N}$ such that $\operatorname{spt} \boldsymbol{f} \subset V_k \times \mathbb{R}^n$. By (59), (65), and (61) applied to $U = V_k$ and $F = J_{\boldsymbol{v}_k} \cap \Gamma(\boldsymbol{u})$, we have that

$$\begin{aligned} \mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f}) &= \int_{\Omega\times\mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{\Lambda}(\boldsymbol{x}, \boldsymbol{y}) = \int_{V_k\times(J_{\boldsymbol{v}_k}\cap\Gamma(\boldsymbol{u}))} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{\Lambda}(\boldsymbol{x}, \boldsymbol{y}) \\ &= \int_{J_{\boldsymbol{v}_k}\cap\Gamma(\boldsymbol{u})} \left[\boldsymbol{f}\left(\boldsymbol{v}_k^-(\boldsymbol{y}), \boldsymbol{y}\right) - \boldsymbol{f}\left(\boldsymbol{v}_k^+(\boldsymbol{y}), \boldsymbol{y}\right)\right] \cdot \boldsymbol{\nu}_{\boldsymbol{v}_k}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}). \end{aligned}$$

Proceeding as in the proof of Proposition 4, this yields (57), completing the proof. □

5. Boundary of the image versus image of the boundary

In Section 4 we gave a definition of created surface based on lateral traces and jump discontinuities of the inverse (Definition 9). However, it is more intuitive to think of the phenomenon of creation of surface in terms of the inequality $\partial\boldsymbol{u}(\Omega) \neq \boldsymbol{u}(\partial\Omega)$. In this section, we prove that $\partial^* \operatorname{im_G}(\boldsymbol{u}, U)\setminus \operatorname{im_G}(\boldsymbol{u}, \partial U)$ is $\mathcal{H}^{n-1}$-equivalent, for all $U \in \mathcal{U}_{\boldsymbol{u}}$, to the set of $\boldsymbol{y} \in \mathbb{R}^n$ such that

i) $\boldsymbol{y} \in \Gamma_V(\boldsymbol{u})$ and $(\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in U$, or
ii) $\boldsymbol{y} \in \Gamma_I(\boldsymbol{u})$ and one of the traces $(\boldsymbol{u}^{-1})^\pm(\boldsymbol{y})$ belongs to U, while the other belongs to $\Omega\setminus U$.

By so doing, we endow our definition of $\Gamma(\boldsymbol{u})$ with a richer geometric content, and continue to make rigourous the informal discussion of Sections 1 and 3.

We begin by showing that no part of the boundary ∂U is lost under the deformation (that is, $\operatorname{im_G}(\boldsymbol{u}, \partial U) \subset \partial^* \operatorname{im_G}(\boldsymbol{u}, U)$), for any given subset U of Ω.

Proposition 6. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ *for almost every $\boldsymbol{x} \in \Omega$. Let Ω_0 be as in Definition 3. Then, for any measurable set $U \subset\subset \Omega$ and every $\boldsymbol{x}_0 \in \partial^* U \cap \Omega_0$, the following are satisfied:*

i) $\boldsymbol{u}(\boldsymbol{x}_0) \in \partial^* \operatorname{im_G}(\boldsymbol{u}, U)$.
ii) *For any open set $V \subset \Omega$ such that $U \subset\subset V$, we have that* $D(\operatorname{im_G}(\boldsymbol{u}, V), \boldsymbol{u}(\boldsymbol{x}_0)) = 1$ *and* $\boldsymbol{u}(\boldsymbol{x}_0) \in \partial^* \operatorname{im_G}(\boldsymbol{u}, V \setminus U)$.
iii) $\boldsymbol{u}^{-1}$ *is approximately continuous at $\boldsymbol{u}(\boldsymbol{x}_0)$, and* $\operatorname{ap\,lim}_{\boldsymbol{y} \to \boldsymbol{u}(\boldsymbol{x}_0)} \boldsymbol{u}^{-1}(\boldsymbol{y}) = \boldsymbol{x}_0$.
iv) $\boldsymbol{u}(\boldsymbol{x}_0) \notin \Gamma(\boldsymbol{u})$.
v) *If $\boldsymbol{\nu}_U$ and $\boldsymbol{\nu}_{\operatorname{im_G}(\boldsymbol{u}, U)}$ denote the unit outward normal (according to Definition 2) to U and to $\operatorname{im_G}(\boldsymbol{u}, U)$, respectively, then*

$$\boldsymbol{\nu}_{\operatorname{im_G}(\boldsymbol{u},U)}(\boldsymbol{u}(\boldsymbol{x}_0)) = (\operatorname{sgn} \det \nabla \boldsymbol{u}(\boldsymbol{x}_0)) \frac{(\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}_0)) \boldsymbol{\nu}_U(\boldsymbol{x}_0)}{|(\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}_0)) \boldsymbol{\nu}_U(\boldsymbol{x}_0)|}.$$

Proof. Let $\boldsymbol{x}_0 \in \partial^* U \cap \Omega_0$, and set $\boldsymbol{y}_0 := \boldsymbol{u}(\boldsymbol{x}_0)$ and

$$\bar{\boldsymbol{\nu}} := (\operatorname{sgn} \det \nabla \boldsymbol{u}(\boldsymbol{x}_0)) \frac{(\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}_0)) \boldsymbol{\nu}_U(\boldsymbol{x}_0)}{|(\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}_0)) \boldsymbol{\nu}_U(\boldsymbol{x}_0)|}.$$

Let $V \subset \Omega$ be any open set such that $U \subset\subset V$. By Lemma 1 we have that

$$D(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^-(\boldsymbol{y}_0, \bar{\boldsymbol{\nu}}), \boldsymbol{y}_0) = D(\operatorname{im_G}(\boldsymbol{u}, V \setminus U) \cap H^+(\boldsymbol{y}_0, \bar{\boldsymbol{\nu}}), \boldsymbol{y}_0) = \frac{1}{2}.$$

By Lemma 3, $\operatorname{im_G}(\boldsymbol{u}, U) \cap \operatorname{im_G}(\boldsymbol{u}, V \setminus U) = \varnothing$, hence

$$D(\operatorname{im_G}(\boldsymbol{u}, U) \cap H^+(\boldsymbol{y}_0, \bar{\boldsymbol{\nu}}), \boldsymbol{y}_0) = D(\operatorname{im_G}(\boldsymbol{u}, V \setminus U) \cap H^-(\boldsymbol{y}_0, \bar{\boldsymbol{\nu}}), \boldsymbol{y}_0) = 0.$$

This shows i), ii) and v). The approximate continuity of $\boldsymbol{u}^{-1}$ at $\boldsymbol{y}_0$ clearly follows from Lemma 1 (or from Theorem 2(iii)). By iii) and Lemma 4(iv), $\boldsymbol{y}_0 \notin \Gamma(\boldsymbol{u})$. This finishes the proof. □

Due to the role that it will play in Sections 6 and 7, we introduce a special notation for the part of $\partial^* \operatorname{im_G}(\boldsymbol{u}, U)$ that is newly created (as opposed to $\operatorname{im_G}(\boldsymbol{u}, \partial U)$, which corresponds to the part of the boundary that existed already).

Definition 13. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ for almost every $\boldsymbol{x} \in \Omega$. Assume $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n \times n})$, and let $U \subset \Omega$ be a measurable set. We define the created boundary of U under $\boldsymbol{u}$, and denote it by $\Gamma_C(\boldsymbol{u}, U)$, as the set $\partial^* \operatorname{im_G}(\boldsymbol{u}, U) \setminus \operatorname{im_G}(\boldsymbol{u}, \partial U)$.

The following lemma shows that at $\mathcal{H}^{n-1}$-almost every point on $\partial^* \operatorname{im_G}(\boldsymbol{u}, U)$, the inverse $\boldsymbol{u}^{-1}$ has a well-defined lateral trace.

Lemma 8. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u} > 0$ *almost everywhere and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n \times n})$. *Assume $\bar{\mathcal{E}}(\boldsymbol{u}) < \infty$ and $\boldsymbol{0} \notin \bar{\Omega}$. Let $U \in \mathcal{U}_{\boldsymbol{u}}$, and let $J_{\boldsymbol{u}_U^{-1}}$ have the orientation according to Definition 10. Then the following can be said of $\mathcal{H}^{n-1}$-almost every $\boldsymbol{y}_0 \in \partial^* \operatorname{im_G}(\boldsymbol{u}, U)$:*

i) $\boldsymbol{y}_0 \in J_{\boldsymbol{u}_U^{-1}}$ *and* $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0) = \boldsymbol{\nu}_{\operatorname{im_G}(\boldsymbol{u},U)}(\boldsymbol{y}_0)$.

ii) *The limit* $\operatorname{ap\,lim}_{\substack{\boldsymbol{y} \to \boldsymbol{y}_0 \\ \boldsymbol{y} \in \operatorname{im_G}(\boldsymbol{u},U)}} \boldsymbol{u}^{-1}(\boldsymbol{y})$ *exists, coincides with* $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}_0)$, *and belongs to* $\bar{U}$.

iii) *Either* $\boldsymbol{y}_0 \in \Gamma(\boldsymbol{u})$, *or*

$$D(\operatorname{im_G}(\boldsymbol{u}, \Omega), \boldsymbol{y}_0) = 1 \text{ and the limit } \operatorname{ap\,lim}_{\boldsymbol{y} \to \boldsymbol{y}_0} \boldsymbol{u}^{-1}(\boldsymbol{y}) \text{ exists.}$$

Proof. We first prove i) and ii). Let $\boldsymbol{y}_0 \in \partial^* \operatorname{im_G}(\boldsymbol{u}, U)$. By Lemma 4 iii),

$$\operatorname{ap\,lim}_{\substack{\boldsymbol{y} \to \boldsymbol{y}_0 \\ \boldsymbol{y} \in H^+(\boldsymbol{y}_0, \boldsymbol{\nu}_{\operatorname{im_G}(\boldsymbol{u},U)})}} \boldsymbol{u}_U^{-1}(\boldsymbol{y}) = \boldsymbol{0}.$$

Now, by Theorem 2 and the Federer–Vol'pert theorem (see, for example, [5, Th. 3.78]), for $\mathcal{H}^{n-1}$-almost every $\boldsymbol{y}_0 \in \partial^* \operatorname{im_G}(\boldsymbol{u}, U)$ the limit

$$\operatorname{ap\,lim}_{\substack{\boldsymbol{y} \to \boldsymbol{y}_0 \\ \boldsymbol{y} \in H^-(\boldsymbol{y}_0, \boldsymbol{\nu}_{\operatorname{im_G}(\boldsymbol{u},U)})}} \boldsymbol{u}_U^{-1}(\boldsymbol{y})$$

exists, and, by Lemma 4(i), coincides with

$$\operatorname{ap\,lim}_{\substack{\boldsymbol{y} \to \boldsymbol{y}_0 \\ \boldsymbol{y} \in \operatorname{im_G}(\boldsymbol{u},U)}} \boldsymbol{u}^{-1}(\boldsymbol{y})$$

and belongs to $\bar{U}$. In addition, $\boldsymbol{\nu}_{\boldsymbol{u}_U^{-1}}(\boldsymbol{y}_0) = \boldsymbol{\nu}_{\operatorname{im_G}(\boldsymbol{u},U)}(\boldsymbol{y}_0)$ because of Corollary 1(i).

Now we show iii). Consider the sets V_k and the functions $\boldsymbol{v}_k$ of Definition 11. By virtue of Theorem 2, and the Federer-Vol'pert theorem, for all $k \in \mathbb{N}$ and $\mathcal{H}^{n-1}$-almost every $\boldsymbol{y}_0 \in \mathbb{R}^n$ there exists $\boldsymbol{\nu}_k(\boldsymbol{y}_0) \in \mathbb{S}^{n-1}$ for which the limits

$$\boldsymbol{x}_0^- := \operatorname{ap\,lim}_{\substack{\boldsymbol{y} \to \boldsymbol{y}_0 \\ \boldsymbol{y} \in H^-(\boldsymbol{y}_0, \boldsymbol{\nu}_k(\boldsymbol{y}_0))}} \boldsymbol{v}_k(\boldsymbol{y}), \quad \boldsymbol{x}_0^+ := \operatorname{ap\,lim}_{\substack{\boldsymbol{y} \to \boldsymbol{y}_0 \\ \boldsymbol{y} \in H^+(\boldsymbol{y}_0, \boldsymbol{\nu}_k(\boldsymbol{y}_0))}} \boldsymbol{v}_k(\boldsymbol{y})$$

exist, and $D(\operatorname{im_G}(\boldsymbol{u}, V_k), \boldsymbol{y}_0) \in \{0, \frac{1}{2}, 1\}$. Fix such a $\boldsymbol{y}_0$ and suppose, in addition, that it belongs to $\partial^* \operatorname{im_G}(\boldsymbol{u}, U)$. Fix also $p \in \mathbb{N}$ such that $U \subset V_p$.

As in Lemma 5, for all $k \geqq p$ we can choose the orientation of $\boldsymbol{\nu}_k(\boldsymbol{y}_0)$ such that $\boldsymbol{\nu}_k(\boldsymbol{y}_0) = \boldsymbol{\nu}_{\operatorname{im_G}(\boldsymbol{u},U)}(\boldsymbol{y}_0)$ and

$$D(\operatorname{im_G}(\boldsymbol{u}, V_k) \cap H^-(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0) = \frac{1}{2},$$

where $\boldsymbol{\nu}$ denotes, henceforth, the vector $\boldsymbol{\nu}_{\mathrm{im_G}(\boldsymbol{u},U)}(\boldsymbol{y}_0)$. There are two possibilities. First, suppose that $D(\mathrm{im_G}(\boldsymbol{u}, V_k) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0) = 0$ for all $k \geqq p$. In this case $\boldsymbol{y}_0 \in \Gamma_V(\boldsymbol{u})$, since by Lemma 4(i),

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^-(\boldsymbol{y}_0,\boldsymbol{\nu})}} \boldsymbol{u}^{-1}(\boldsymbol{y}) = \boldsymbol{x}_0^- \in \bar{V}_p \subset \Omega.$$

Alternatively, there may exist $k > p$ such that

$$D(\mathrm{im_G}(\boldsymbol{u}, V_k) \cap H^+(\boldsymbol{y}_0, \boldsymbol{\nu}), \boldsymbol{y}_0) = \frac{1}{2}.$$

In this case, by Lemma 4(i),

$$\operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^-(\boldsymbol{y}_0,\boldsymbol{\nu})}} \boldsymbol{u}^{-1}(\boldsymbol{y}) = \boldsymbol{x}_0^- \in \bar{U} \subset \Omega \quad \text{and} \quad \operatorname*{ap\,lim}_{\substack{\boldsymbol{y}\to\boldsymbol{y}_0\\ \boldsymbol{y}\in H^+(\boldsymbol{y}_0,\boldsymbol{\nu})}} \boldsymbol{u}^{-1}(\boldsymbol{y}) = \boldsymbol{x}_0^+ \in \bar{V}_k \subset \Omega.$$

If $\boldsymbol{x}_0^- = \boldsymbol{x}_0^+$ then $\operatorname{ap\,lim}_{\boldsymbol{y}\to\boldsymbol{y}_0} \boldsymbol{u}^{-1}(\boldsymbol{y})$ exists, whereas if $\boldsymbol{x}_0^- \neq \boldsymbol{x}_0^+$, then $\boldsymbol{y}_0 \in \Gamma_I(\boldsymbol{u})$. This concludes the proof. □

We now present the main result of this section, the principal ingredients of which are the characterization of $\mathcal{E}$ of Theorem 3, and the change of variables for surface integrals of Proposition 2. Recall from Section 2.5 the definition of the set $\partial_d U$ of tangential approximate differentiability points.

Theorem 4. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u} > 0$ *almost every and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$. *Assume that $\mathcal{E}(\boldsymbol{u}) < \infty$ and $\boldsymbol{0} \notin \bar{\Omega}$. Let $U \in \mathcal{U}_{\boldsymbol{u}}$, and consider the functions $(\boldsymbol{u}^{-1})^{\pm}$ and $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}}$ of Lemma 5, and the function $(\boldsymbol{u}_U^{-1})^-$ of Definition 10. Define the sets*

$$\begin{aligned} A &:= \{\boldsymbol{y} \in \Gamma_V(\boldsymbol{u}) : \ (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in U\}, \\ B_1 &:= \{\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}) : \ (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in U \ \textit{and} \ (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in \Omega\setminus U\}, \\ B_2 &:= \{\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}) : \ (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in \Omega\setminus U \ \textit{and} \ (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in U\}. \end{aligned}$$

Then

$$\begin{aligned} \Gamma_C(\boldsymbol{u}, U) \cong A \cup B_1 \cup B_2 &\cong \left\{\boldsymbol{y} \in \partial^* \mathrm{im_G}(\boldsymbol{u}, U) : \ \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in U\right\} \\ &\cong \partial^* \mathrm{im_G}(\boldsymbol{u}, U) \cap \Gamma(\boldsymbol{u}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{im_G}(\boldsymbol{u}, \partial U) \cong \mathrm{im_G}(\boldsymbol{u}, \partial_d U) &\cong \left\{\boldsymbol{y} \in \partial^* \mathrm{im_G}(\boldsymbol{u}, U) : \ \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in \partial U\right\} \\ &\cong \partial^* \mathrm{im_G}(\boldsymbol{u}, U)\setminus\Gamma(\boldsymbol{u}). \end{aligned}$$

Moreover,

$$J_{\boldsymbol{u}_U^{-1}} \cong \mathrm{im_G}(\boldsymbol{u}, \partial U) \cup \Gamma_C(\boldsymbol{u}, U) \cup \Gamma_I(\boldsymbol{u}, U) \cup C,$$

with disjoint union, where

$$C := \left\{ \boldsymbol{y} \in \Gamma_I(\boldsymbol{u}) \cap J_{\boldsymbol{u}_U^{-1}} : \ (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \text{ and } (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \text{ are both in } \partial U \right\}.$$

Proof. By Definition 9(b) and Lemma 4(iv) we have that $A \subset \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)$, whereas, by Proposition 6, $A \cap \partial \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U) = \varnothing$. Therefore,

$$A \subset \Gamma_C(\boldsymbol{u}, U), \quad \text{with} \quad \boldsymbol{\nu}_{\boldsymbol{u}^{-1}} = \boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)} \text{ in } A. \tag{66}$$

Moreover, by Proposition 6 and Lemma 4(iv), we have that

$$B_i \cap \Gamma_C(\boldsymbol{u}, U) = B_i \cap \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U), \quad \text{for } i = 1, 2, \tag{67}$$

with $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}} = \boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)}$ in $B_1 \cap \Gamma_C(\boldsymbol{u}, U)$, and $\boldsymbol{\nu}_{\boldsymbol{u}^{-1}} = -\boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)}$ in $B_2 \cap \Gamma_C(\boldsymbol{u}, U)$, where $\boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)}$ is oriented according to Definition 2.

Now we prove that

$$\Gamma_C(\boldsymbol{u}, U) \cong A \cup B_1 \cup B_2 \quad \text{and} \quad \mathcal{H}^{n-1}(C_1) = 0, \tag{68}$$

with $C_1 := \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U \setminus \partial_d U)$. Consider the function d of Proposition 3, and choose $\varphi \in C^\infty(\mathbb{R})$ satisfying $\varphi(t) = 0$ for $t \leqq 0$, $\varphi(t) = 1$ for $t \geqq 1$, and $\varphi' \geqq 0$. For each $j \in \mathbb{N}$, consider the function $\eta_j : \Omega \to \mathbb{R}$ defined in (21). Let $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then, by (22), (12), and the coarea formula,

$$\begin{aligned}
\int_\Omega \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) D\eta_j(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} &= \int_{\partial U} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) Dd(\boldsymbol{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) \\
&+ \fint_0^{\frac{1}{j}} \varphi'(jt) \bigg[\int_{\partial U_t} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) Dd(\boldsymbol{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) \\
&\qquad - \int_{\partial U} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) Dd(\boldsymbol{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) \bigg] \, \mathrm{d}t.
\end{aligned}$$

Now, by (12), Definitions 5 and 6, and Propositions 2 and 6,

$$\begin{aligned}
\lim_{j \to \infty} \int_\Omega & \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) D\eta_j(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\
&= - \int_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial_d U)} \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)}(\boldsymbol{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).
\end{aligned}$$

On the other hand, by (22), dominated convergence, Proposition 1, (20), and the divergence theorem for sets of finite perimeter (see, for example, [5, Th. 3.36]),

$$\begin{aligned}
\lim_{j \to \infty} \int_\Omega \eta_j(\boldsymbol{x}) (\operatorname{div} \boldsymbol{g})(\boldsymbol{u}(\boldsymbol{x})) \det \nabla \boldsymbol{u}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} &= \int_U (\operatorname{div} \boldsymbol{g})(\boldsymbol{u}(\boldsymbol{x})) \det \nabla \boldsymbol{u}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\
= \int_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)} \operatorname{div} \boldsymbol{g}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} &= \int_{\partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)} \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)}(\boldsymbol{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).
\end{aligned}$$

In total, we obtain that

$$\begin{aligned}
\lim_{j \to \infty} \bar{\mathcal{E}}_{\boldsymbol{u}}(\eta_j, \boldsymbol{g}) &= \int_{\partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U) \setminus \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial_d U)} \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)}(\boldsymbol{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) \\
&= \int_{\Gamma_C(\boldsymbol{u},U) \cup C_1} \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u},U)}(\boldsymbol{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).
\end{aligned} \tag{69}$$

The next step is to calculate $\lim_{j\to\infty} \bar{\mathcal{E}}_{\boldsymbol{u}}(\eta_j, \boldsymbol{g})$ in a different way. Using Proposition 4, (22), and dominated convergence, we obtain

$$\lim_{j\to\infty} \bar{\mathcal{E}}_{\boldsymbol{u}}(\eta_j, \boldsymbol{g}) = \int_{\Gamma_V(\boldsymbol{u})} \chi_U((\boldsymbol{u}^{-1})^-(\boldsymbol{y}))\, \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y})$$
$$+ \int_{\Gamma_I(\boldsymbol{u})} \left[\chi_U((\boldsymbol{u}^{-1})^-(\boldsymbol{y})) - \chi_U((\boldsymbol{u}^{-1})^+(\boldsymbol{y}))\right] \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).$$

Because of the definition of A, B_1 and B_2, this equality reads

$$\lim_{j\to\infty} \bar{\mathcal{E}}_{\boldsymbol{u}}(\eta_j, \boldsymbol{g}) = \int_{A\cup B_1} \boldsymbol{g} \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}\, \mathrm{d}\mathcal{H}^{n-1} - \int_{B_2} \boldsymbol{g} \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}\, \mathrm{d}\mathcal{H}^{n-1}. \tag{70}$$

Comparing (69) and (70), and using (66) and (67), we obtain that

$$\int_{C_1\cup\Gamma_C(\boldsymbol{u},U)\setminus(A\cup B_1\cup B_2)} \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\mathrm{im_G}(\boldsymbol{u},U)}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y})$$
$$= \int_{B_1\setminus\Gamma_C(\boldsymbol{u},U)} \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) - \int_{B_2\setminus\Gamma_C(\boldsymbol{u},U)} \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).$$

This equality holds for all $\boldsymbol{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. By density, it holds for every $\boldsymbol{g} \in C_c(\mathbb{R}^n, \mathbb{R}^n)$ as well. As the sets

$$C_1, \quad \Gamma_C(\boldsymbol{u}, U)\setminus(A \cup B_1 \cup B_2), \quad B_1\setminus\Gamma_C(\boldsymbol{u}, U), \quad B_2\setminus\Gamma_C(\boldsymbol{u}, U)$$

are disjoint, by Lusin's theorem this implies that

$$\mathcal{H}^{n-1}(C_1) = \mathcal{H}^{n-1}(\Gamma_C(\boldsymbol{u}, U)\setminus(A \cup B_1 \cup B_2))$$
$$= \mathcal{H}^{n-1}((B_1 \cup B_2)\setminus\Gamma_C(\boldsymbol{u}, U)) = 0,$$

showing (68).

The equivalence

$$\Gamma_C(\boldsymbol{u}, U) \cong \partial^* \mathrm{im_G}(\boldsymbol{u}, U) \cap \Gamma(\boldsymbol{u}). \tag{71}$$

is immediate, considering that $\mathrm{im_G}(\boldsymbol{u}, \partial U) \cap \Gamma(\boldsymbol{u}) = \varnothing$ (by Proposition 6) and $\Gamma_C(\boldsymbol{u}, U) \tilde{\subset} \Gamma(\boldsymbol{u})$ (by (68)). Also, Lemma 4(iv), Proposition 6, and (68) imply that

$$\left\{\boldsymbol{y} \in \partial^* \mathrm{im_G}(\boldsymbol{u}, U) : \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in U\right\} \cong \Gamma_C(\boldsymbol{u}, U). \tag{72}$$

By Theorem 2, the Federer–Vol'pert theorem (see, for example, [5, Th. 3.78]), and Definition 10,

$$\left\{\boldsymbol{y} \in \partial^* \mathrm{im_G}(\boldsymbol{u}, U) : \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in \partial U\right\}$$
$$\cong \partial^* \mathrm{im_G}(\boldsymbol{u}, U)\setminus \left\{\boldsymbol{y} \in \partial^* \mathrm{im_G}(\boldsymbol{u}, U) : \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in U\right\},$$

so, by (71) and (72), we obtain that

$$\left\{ \boldsymbol{y} \in \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) : \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in \partial U \right\} \cong \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) \setminus \Gamma(\boldsymbol{u}).$$

By (71) and the equality $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U) \cap \Gamma(\boldsymbol{u}) = \varnothing$, we obtain that

$$\partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) \setminus \Gamma(\boldsymbol{u}) \cong \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U).$$

Finally, by (68) and Definition 6, $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial_d U) \cong \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U)$. This proves the first part of the theorem.

For the second part, observe that thanks to Proposition 6(i) and Lemma 8(i),

$$\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U) \cup \Gamma_C(\boldsymbol{u}, U) = \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) \,\widetilde{\subset}\, J_{\boldsymbol{u}_U^{-1}}.$$

By Proposition 5(i), $\Gamma_I(\boldsymbol{u}, U) \subset J_{\boldsymbol{u}_U^{-1}}$. Clearly, $C \subset J_{\boldsymbol{u}_U^{-1}}$.

By Proposition 6, both $\Gamma_I(\boldsymbol{u}, U)$ and C are disjoint with $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U)$, since they are contained in $\Gamma(\boldsymbol{u})$. By definition, $\Gamma_C(\boldsymbol{u}, U)$ is also disjoint with $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U)$. Now, $\Gamma_C(\boldsymbol{u}, U)$ is disjoint with $\Gamma_I(\boldsymbol{u}, U)$ and C by Lemma 4(i). That $\Gamma_I(\boldsymbol{u}, U)$ is disjoint with C is obvious.

It remains only to prove that

$$J_{\boldsymbol{u}_U^{-1}} \,\widetilde{\subset}\, \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U) \cup \Gamma_C(\boldsymbol{u}, U) \cup \Gamma_I(\boldsymbol{u}, U) \cup C.$$

Let $\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}}$. By Definition 10, $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) \in \bar{U}$. Suppose first that $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) \in U$. By virtue of Lemma 7, we may assume that $\boldsymbol{y} \in \Gamma(\boldsymbol{u})$. Moreover, by Lemmas 4(i) and 5, we have that $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) = (\boldsymbol{u}^{-1})^-(\boldsymbol{y})$ and

$$\left(\boldsymbol{u}_U^{-1}\right)^+ (\boldsymbol{y}) = (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \quad \text{if } \boldsymbol{y} \in \Gamma_I(\boldsymbol{u}).$$

Therefore, we have only the following possibilities:

a) $\boldsymbol{y} \in \Gamma_V(\boldsymbol{u})$, in which case $\boldsymbol{y} \in A$.
b) $\boldsymbol{y} \in \Gamma_I(\boldsymbol{u})$ and both traces $(\boldsymbol{u}^{-1})^{\pm}(\boldsymbol{y})$ are in U. In this case, $\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}, U)$.
c) $\boldsymbol{y} \in \Gamma_I(\boldsymbol{u})$ and only one of the traces $(\boldsymbol{u}^{-1})^{\pm}(\boldsymbol{y})$ is in U. In this case $\boldsymbol{y} \in B_1 \cup B_2$.

By (68), we may conclude that $\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}, U) \cup \Gamma_C(\boldsymbol{u}, U)$.

Suppose now that $\boldsymbol{y} \in J_{\boldsymbol{u}_U^{-1}}$ and $(\boldsymbol{u}_U^{-1})^-(\boldsymbol{y}) \in \partial U$. By (45), there are only the two following possibilities:

a) $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}) \notin \bar{U}$. In this case, by Lemma 4(i), we have in fact $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}) = \boldsymbol{0}$, and by Corollary 1(i), $\boldsymbol{y} \in \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)$. Therefore, $\boldsymbol{y} \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial U) \cup \Gamma_C(\boldsymbol{u}, U)$.
b) $(\boldsymbol{u}_U^{-1})^+(\boldsymbol{y}) \in \partial U$, in which case $\boldsymbol{y} \in C$.

□

As a consequence of Theorem 4, we have that

$$\mathcal{H}^{n-1}\left(\left\{ \boldsymbol{y} \in \partial^* \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) : \left(\boldsymbol{u}_U^{-1}\right)^- (\boldsymbol{y}) \in \partial U \right\} \setminus \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \partial_d U)\right) = 0.$$

This is an $(n-1)$-dimensional Lusin's condition, stating that the restriction of $\boldsymbol{u}$ to $\partial U \setminus \partial_d U$ creates no surface.

6. $\mathcal{E}$ as a generalization of the distributional determinant

We start this section by regarding $\mathcal{E}_{\boldsymbol{u}}$ as a Radon measure in Ω.

Definition 14. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω. Suppose that $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$. For each open subset U of Ω, we define

$$\mu_{\boldsymbol{u}}(U) := \sup\{\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f}) : \ \boldsymbol{f} \in C^\infty_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n), \ \|\boldsymbol{f}\|_\infty \leqq 1, \ \operatorname{spt} \boldsymbol{f} \subset U \times \mathbb{R}^n\}. \tag{73}$$

If, moreover, $\mathcal{E}(\boldsymbol{u}) < \infty$, we define $\mu_{\boldsymbol{u}}$ as the unique positive Radon measure in Ω that extends (73).

The motivation of the definition of $\mu_{\boldsymbol{u}}$ is the following. Müller and Spector [24, Th. 8.4] proved that if $U \subset \Omega$ is an open set, $p > n - 1$ and $\boldsymbol{u}$ is a $W^{1,p}$ function satisfying some invertibility conditions, then $(\operatorname{Det} \nabla \boldsymbol{u})(U)$ equals $\mathcal{L}^n(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U))$ plus the volume of the cavities originated in U. In other words, the measure $\operatorname{Det} \nabla \boldsymbol{u} - (\det \nabla \boldsymbol{u})\mathcal{L}^n$ acting on U provides the volume of the cavities originated in U. Here $\operatorname{Det} \nabla \boldsymbol{u}$ denotes the distributional Jacobian determinant of $\boldsymbol{u}$. In this section we will prove that $\mu_{\boldsymbol{u}}(U)$ equals the area of the surface created in U, so in this sense $\mu_{\boldsymbol{u}}$ is a generalization of $\operatorname{Det} D\boldsymbol{u} - (\det D\boldsymbol{u})\mathcal{L}^n$. It is also a generalization because $\operatorname{Det} \nabla \boldsymbol{u} - (\det \nabla \boldsymbol{u})\mathcal{L}^n$ can be obtained analytically as the supremum of $\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f})$ among functions $\boldsymbol{f}$ having the special form $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = -\phi(\boldsymbol{x})\frac{\boldsymbol{y}}{n}$ with $\|\phi\|_\infty \leqq 1$. Indeed, this function $\boldsymbol{f}$ satisfies

$$\begin{aligned}\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f}) &= -\frac{1}{n}\int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x})) D\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\Omega \phi(\boldsymbol{x}) \det \nabla \boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \langle \operatorname{Det} \nabla \boldsymbol{u} - (\det \nabla \boldsymbol{u})\mathcal{L}^n, \phi\rangle.\end{aligned}$$

A further justification that $\mu_{\boldsymbol{u}}$ is a generalization of the distributional determinant is the following. The motivation of the definition of the distributional determinant comes from the Laplace formula

$$\det \nabla \boldsymbol{u}(\boldsymbol{x}) = \frac{1}{n} \operatorname{Div}\left[(\operatorname{adj} \nabla \boldsymbol{u}(\boldsymbol{x}))\boldsymbol{u}(\boldsymbol{x})\right], \tag{74}$$

valid for smooth functions $\boldsymbol{u}$. Similarly, the motivation for the expression in Definition 7 comes from the equation

$$(\operatorname{div} \boldsymbol{g})(\boldsymbol{u}(\boldsymbol{x})) \det \nabla \boldsymbol{u}(\boldsymbol{x}) = \operatorname{Div}\left[(\operatorname{adj} \nabla \boldsymbol{u}(\boldsymbol{x}))\boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x}))\right], \tag{75}$$

which can be considered a generalized Laplace formula. Analogously, the expression for $\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f})$ (Definition 12) corresponds to

$$\begin{aligned}&\operatorname{div}_{\boldsymbol{y}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \det \nabla \boldsymbol{u}(\boldsymbol{x}) + D_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \cdot \operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}) \\ &\quad = \operatorname{Div}\left[(\operatorname{adj} \nabla \boldsymbol{u}(\boldsymbol{x}))\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\right],\end{aligned}$$

which is a generalization of (74) and (75).

The main goal of this section is to obtain a characterization of the support of $\mu_{\boldsymbol{u}}$ analogous to the characterization of the support of the singular part of the distributional determinant due to MÜLLER AND SPECTOR [24, Sect. 8] for the case of cavitation. In our case, this involves being able to trace the fracture surfaces back to the reference configuration, that is, to find the singularities of the deformation that are responsible for the creation of surface.

Definition 15. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ for almost every $\boldsymbol{x} \in \Omega$. Assume $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$ and $\boldsymbol{0} \notin \bar{\Omega}$. Consider the functions $(\boldsymbol{u}^{-1})^+$ and $(\boldsymbol{u}^{-1})^-$ of Lemma 5. For every Borel subset E of Ω, define

$$\mu_{\boldsymbol{u}}^+(E) := \mathcal{H}^{n-1}\Big(\{\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}) : \ (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in E\}\Big),$$
$$\mu_{\boldsymbol{u}}^-(E) := \mathcal{H}^{n-1}\Big(\{\boldsymbol{y} \in \Gamma(\boldsymbol{u}) : \ (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in E\}\Big).$$

It is obvious that $\mu_{\boldsymbol{u}}^+$ and $\mu_{\boldsymbol{u}}^-$ are Borel measures. Of course, the notation $\mu_{\boldsymbol{u}}^\pm$ does not refer to the positive (or negative) part of the measure $\mu_{\boldsymbol{u}}$.

The following result is the local counterpart of Theorem 3.

Proposition 7. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u} > 0$ *almost everywhere and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. *Assume $\mathcal{E}(\boldsymbol{u}) < \infty$ and $\boldsymbol{0} \notin \bar{\Omega}$. Then, $\mu_{\boldsymbol{u}} = \mu_{\boldsymbol{u}}^+ + \mu_{\boldsymbol{u}}^-$, and for every $U \in \mathcal{U}_{\boldsymbol{u}}$,*

$$\mu_{\boldsymbol{u}}(U) = \mathcal{H}^{n-1}(\Gamma_C(\boldsymbol{u}, U)) + 2\mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u}, U)).$$

Proof. By Theorem 3, the measures $\mathcal{H}^{n-1} \llcorner \Gamma_I(\boldsymbol{u})$ and $\mathcal{H}^{n-1} \llcorner \Gamma(\boldsymbol{u})$ are finite. It then follows that $\mu_{\boldsymbol{u}}^+$ and $\mu_{\boldsymbol{u}}^-$ are finite positive Radon measures. Therefore, the proof of the identity $\mu_{\boldsymbol{u}} = \mu_{\boldsymbol{u}}^+ + \mu_{\boldsymbol{u}}^-$ will be finished as soon as we show that $\mu_{\boldsymbol{u}}(U) = \mu_{\boldsymbol{u}}^+(U) + \mu_{\boldsymbol{u}}^-(U)$ for every open set $U \subset \Omega$.

Let U be an open subset of Ω, and let A, B_1 and B_2 be defined as in Theorem 4. By Proposition 5(v), the sets A, B_1, B_2 and $\Gamma_I(\boldsymbol{u}, U)$ are disjoint, and

$$\mu_{\boldsymbol{u}}^+(U) = \mathcal{H}^{n-1}(B_2) + \mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u}, U)),$$
$$\mu_{\boldsymbol{u}}^-(U) = \mathcal{H}^{n-1}(A) + \mathcal{H}^{n-1}(B_1) + \mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u}, U)).$$

On the other hand, by Theorem 3 we have that for any $\boldsymbol{f} \in C_c^\infty(U \times \mathbb{R}^n, \mathbb{R}^n)$,

$$\begin{aligned}\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f}) = &\left[\int_A + \int_{B_1}\right] \boldsymbol{f}((\boldsymbol{u}^{-1})^-(\boldsymbol{y}), \boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) \\ &- \int_{B_2} \boldsymbol{f}((\boldsymbol{u}^{-1})^+(\boldsymbol{y}), \boldsymbol{y}) \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) \\ &+ \int_{\Gamma_I(\boldsymbol{u}, U)} [\boldsymbol{f}((\boldsymbol{u}^{-1})^-(\boldsymbol{y}), \boldsymbol{y}) - \boldsymbol{f}((\boldsymbol{u}^{-1})^+(\boldsymbol{y}), \boldsymbol{y})] \cdot \boldsymbol{\nu}_{\boldsymbol{u}^{-1}}(\boldsymbol{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).\end{aligned}$$

As $(\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \neq (\boldsymbol{u}^{-1})^+(\boldsymbol{y})$ for every $\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}, U)$, arguing as in the proof of Theorem 3 we can conclude that

$$\mu_{\boldsymbol{u}}(U) = \mathcal{H}^{n-1}(A) + \mathcal{H}^{n-1}(B_1) + \mathcal{H}^{n-1}(B_2) + 2\mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u}, U)).$$

This shows that $\mu_{\boldsymbol{u}} = \mu_{\boldsymbol{u}}^+ + \mu_{\boldsymbol{u}}^-$.

Finally, if, in addition, $U \in \mathcal{U}_{\boldsymbol{u}}$ then, by (68),

$$\mathcal{H}^{n-1}(A) + \mathcal{H}^{n-1}(B_1) + \mathcal{H}^{n-1}(B_2) = \mathcal{H}^{n-1}(\Gamma_C(\boldsymbol{u}, U)),$$

which concludes the proof. □

Definition 16. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that $\det \nabla \boldsymbol{u}(\boldsymbol{x}) \neq 0$ for almost every $\boldsymbol{x} \in \Omega$, and $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. Assume $\boldsymbol{0} \notin \bar{\Omega}$. Consider the functions $(\boldsymbol{u}^{-1})^+$ and $(\boldsymbol{u}^{-1})^-$ of Lemma 5. For each $U \subset \Omega$, we define the set

$$\begin{aligned}\Lambda(\boldsymbol{u}, U) := U \cap \Big[&(\boldsymbol{u}^{-1})^+(\{\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}) : (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in U \text{ or } (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in U\}) \\ &\cup (\boldsymbol{u}^{-1})^-(\{\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}) : (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in U \text{ or } (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in U\}) \\ &\cup (\boldsymbol{u}^{-1})^-(\{\boldsymbol{y} \in \Gamma_V(\boldsymbol{u}) : (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in U\})\Big].\end{aligned}$$

The following theorem, which is the main result of this section, shows that $\Lambda(\boldsymbol{u}, U)$ corresponds to the set in the reference configuration responsible for the surface created in U by $\boldsymbol{u}$.

Theorem 5. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u} > 0$ *almost everywhere and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. *Assume $\mathcal{E}(\boldsymbol{u}) < \infty$ and $\boldsymbol{0} \notin \bar{\Omega}$. Let U be a Borel subset of Ω. Then $\Lambda(\boldsymbol{u}, U)$ is $\mu_{\boldsymbol{u}}$-measurable and $\mu_{\boldsymbol{u}} \llcorner U = \mu_{\boldsymbol{u}} \llcorner \Lambda(\boldsymbol{u}, U)$.*

Proof. For simplicity, the function $(\boldsymbol{u}^{-1})^+ : \Gamma_I(\boldsymbol{u}) \to \mathbb{R}^n$ of Lemma 5 is denoted by $\boldsymbol{v}$. Recall that $\mathcal{H}^{n-1}$ is an outer measure and that $\boldsymbol{v}$ is a Borel function (Lemma 5), so $\boldsymbol{v}^{-1}(U)$ is $\mathcal{H}^{n-1}$-measurable for each Borel subset U of Ω.

For every $E \subset \Omega$ we define $\mu^+_{\boldsymbol{u}}(E) := \mathcal{H}^{n-1}(\boldsymbol{v}^{-1}(E))$. Clearly, $\mu^+_{\boldsymbol{u}}$ is an outer measure in Ω that, restricted to the Borel sets of Ω, coincides with the Borel measure $\mu^+_{\boldsymbol{u}}$ of Definition 15. Now fix $E \subset \Omega$ and a Borel subset U of Ω. Since

$$\boldsymbol{v}^{-1}(E \cap \Lambda(\boldsymbol{u}, U)) = \boldsymbol{v}^{-1}(E) \cap \boldsymbol{v}^{-1}(\Lambda(\boldsymbol{u}, U)),$$
$$\boldsymbol{v}^{-1}(E \cap U) = \boldsymbol{v}^{-1}(E) \cap \boldsymbol{v}^{-1}(U)$$

and $\boldsymbol{v}^{-1}(\Lambda(\boldsymbol{u}, U)) = \boldsymbol{v}^{-1}(U)$ then

$$\mu^+_{\boldsymbol{u}}(E \cap \Lambda(\boldsymbol{u}, U)) = \mu^+_{\boldsymbol{u}}(E \cap U). \tag{76}$$

Analogously,

$$\boldsymbol{v}^{-1}(E \setminus \Lambda(\boldsymbol{u}, U)) = \boldsymbol{v}^{-1}(E) \setminus \boldsymbol{v}^{-1}(\Lambda(\boldsymbol{u}, U)) = \boldsymbol{v}^{-1}(E) \setminus \boldsymbol{v}^{-1}(U) = \boldsymbol{v}^{-1}(E \setminus U)$$

and $\mu^+_{\boldsymbol{u}}(E \setminus \Lambda(\boldsymbol{u}, U)) = \mu^+_{\boldsymbol{u}}(E \setminus U)$. Therefore, as U is a Borel set and $\mu^+_{\boldsymbol{u}}$ is a Borel measure,

$$\mu^+_{\boldsymbol{u}}(E) = \mu^+_{\boldsymbol{u}}(E \cap U) + \mu^+_{\boldsymbol{u}}(E \setminus U) = \mu^+_{\boldsymbol{u}}(E \cap \Lambda(\boldsymbol{u}, U)) + \mu^+_{\boldsymbol{u}}(E \setminus \Lambda(\boldsymbol{u}, U)).$$

This shows that $\Lambda(\boldsymbol{u}, U)$ is $\mu^+_{\boldsymbol{u}}$-measurable, and, by virtue of (76), that $\mu^+_{\boldsymbol{u}} \llcorner U = \mu^+_{\boldsymbol{u}} \llcorner \Lambda(\boldsymbol{u}, U)$. Since the same argument is valid for $\mu^-_{\boldsymbol{u}}$, Proposition 7 concludes the proof. □

7. Making the invisible boundary visible

A great part of the present work has been inspired by the example by Müller and Spector [24, Sect. 11] of a deformation that exhibits the creation and subsequent filling of cavities. This example (see Fig. 1) motivated the definition of *invisible created surface* as a created surface that does not form part of the reduced boundary of the deformed body. In their example, however, it is clear that the created surface may become 'visible' (that is, it can be detected as part of the reduced boundary of the image of the deformation) if we restrict our attention to smaller parts of the body (for example, to the left half and to the right half of the rectangle representing the reference configuration) in which the cavitating singularities can be studied separately. That is the idea behind the main result of this section.

We start by singling out the families of balls that are suitable for our analysis.

Definition 17. Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, and suppose that $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. Consider the measure $\mu_{\boldsymbol{u}}$ of Definition 14. For every open set U in Ω, we define $\mathcal{F}_U$ as the family of closed balls B contained in U such that $\mathring{B} \in \mathcal{U}_{\boldsymbol{u}}$ and $\mu_{\boldsymbol{u}}(\partial B) = 0$. We define $\mathfrak{C}_U$ as the set of families $\mathcal{B} \subset \mathcal{F}_U$ of balls such that $A \cap B = \varnothing$ for every $A, B \in \mathcal{B}$ with $A \neq B$.

Of course, every element of $\mathfrak{C}_U$ is at most countable. Note that, thanks to Lemma 2, if $\mathcal{E}(\boldsymbol{u}) < \infty$ then for each $\boldsymbol{x} \in U$, we have that almost every $r \in (0, \operatorname{dist}(\boldsymbol{x}, \partial U))$ satisfies $\bar{B}(\boldsymbol{x}, r) \in \mathcal{F}_U$.

The following is the main result of this section.

Theorem 6. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u} > 0$ *almost everywhere and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$*. Assume $\mathcal{E}(\boldsymbol{u}) < \infty$ and $\boldsymbol{0} \notin \bar{\Omega}$. Let U be a nonempty open subset of Ω. Then*

$$\mu_{\boldsymbol{u}}(U) = \sup_{\mathcal{B} \in \mathfrak{C}_U} \sum_{B \in \mathcal{B}} \mathcal{H}^{n-1}(\Gamma_C(\boldsymbol{u}, \mathring{B}))$$

This representation formula for $\mu_{\boldsymbol{u}}$ is of a different nature to that of Proposition 7. The conclusion of Theorem 6 states that the area of the surface created in U can be calculated by summing the area of the created (visible) surface of 'good' disjoint balls covering U. This formula explains the title of this section: the invisible surface becomes visible when the deformation is restricted to suitable balls. The rest of the section is devoted to the proof of Theorem 6.

Lemma 9. *Let $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable in almost all Ω, one-to-one almost everywhere, and such that* $\det \nabla \boldsymbol{u} > 0$ *almost everywhere and* $\operatorname{cof} \nabla \boldsymbol{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$*. Assume $\mathcal{E}(\boldsymbol{u}) < \infty$ and $\boldsymbol{0} \notin \bar{\Omega}$. Let $\mathcal{B} \in \mathfrak{C}_U$ satisfy $\mu_{\boldsymbol{u}}(U) = \mu_{\boldsymbol{u}}(\bigcup \mathcal{B})$. Then there exists $\mathcal{B}' \in \mathfrak{C}_U$ such that*

i) *for every $B' \in \mathcal{B}'$ there exists a unique $B \in \mathcal{B}$ such that $B' \subset B$,*
ii) *$\mu_{\boldsymbol{u}}(U) = \mu_{\boldsymbol{u}}(\bigcup \mathcal{B}')$, and*
iii) *$\mathcal{H}^{n-1}(\bigcup_{B' \in \mathcal{B}'} \Gamma_I(\boldsymbol{u}, \mathring{B}')) \leqq \frac{3}{4}\mathcal{H}^{n-1}(\bigcup_{B \in \mathcal{B}} \Gamma_I(\boldsymbol{u}, \mathring{B}))$.*

Proof. Let $\Gamma := \bigcup_{B\in\mathcal{B}} \Gamma_I(\boldsymbol{u}, \mathring{B})$. Then, thanks to Definition 9(a), for each $\boldsymbol{y} \in \Gamma$ there exists $r = r(\boldsymbol{y}) > 0$ such that $r < \frac{1}{3}|(\boldsymbol{u}^{-1})^+(\boldsymbol{y}) - (\boldsymbol{u}^{-1})^-(\boldsymbol{y})|$ and the balls $B((\boldsymbol{u}^{-1})^\pm(\boldsymbol{y}), r)$ are each contained in a ball of the family $\mathcal{B}$.

By the Lebesgue–Besicovitch differentiation theorem (see, for example, [5, Cor. 2.23]), both $(\boldsymbol{u}^{-1})^\pm|_\Gamma$ are approximately continuous with respect to $\mathcal{H}^{n-1} \llcorner \Gamma$ at $\mathcal{H}^{n-1}$-almost every point in Γ. In particular, for $\mathcal{H}^{n-1}$-almost every $\boldsymbol{y}_0 \in \Gamma$ and every $r > 0$, there exists $\rho_0 > 0$ such that for every $0 < \rho < \rho_0$,

$$\frac{\mathcal{H}^{n-1}(\{\boldsymbol{y} \in \Gamma \cap B(\boldsymbol{y}_0, \rho) : (\boldsymbol{u}^{-1})^\pm(\boldsymbol{y}) \in B((\boldsymbol{u}^{-1})^\pm(\boldsymbol{y}_0), r)\})}{\mathcal{H}^{n-1}(\Gamma \cap B(\boldsymbol{y}_0, \rho))} \geqq \frac{3}{4},$$

and, hence,

$$\mathcal{H}^{n-1}\left(\left\{\boldsymbol{y} \in \Gamma \cap B(\boldsymbol{y}_0, \rho) : (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in B((\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0), r) \quad \text{and} \right.\right.$$
$$\left.\left.(\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in B((\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0), r)\right\}\right) \geqq \frac{1}{2}\mathcal{H}^{n-1}(\Gamma \cap B(\boldsymbol{y}_0, \rho)).$$

We then apply Besicovitch's covering theorem (see, for example, [28, Th. 1.3.6]) to $\mathcal{H}^{n-1} \llcorner \Gamma$ to obtain a finite number M of disjoint balls $B(\boldsymbol{y}_j, \rho_j)$, for $j = 1, \ldots, M$, such that

$$\sum_{j=1}^{M} \mathcal{H}^{n-1}(\Gamma \cap B(\boldsymbol{y}_j, \rho_j)) \geqq \frac{1}{2}\mathcal{H}^{n-1}(\Gamma), \tag{77}$$

and that for each $1 \leqq j \leqq M$,

$$\mathcal{H}^{n-1}(\Gamma_j) \geqq \frac{1}{2}\mathcal{H}^{n-1}(\Gamma \cap B(\boldsymbol{y}_j, \rho_j)), \tag{78}$$

where

$$\Gamma_j := \{\boldsymbol{y} \in \Gamma \cap B(\boldsymbol{y}_j, \rho_j) : (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in B(\boldsymbol{x}_j^+, r_j) \quad \text{and} \quad (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in B(\boldsymbol{x}_j^-, r_j)\},$$
$$\boldsymbol{x}_j^\pm := (\boldsymbol{u}^{-1})^\pm(\boldsymbol{y}_j) \quad \text{and} \quad r_j := r(\boldsymbol{y}_j).$$

Let $A := \bigcup_{B\in\mathcal{B}} \Lambda(\boldsymbol{u}, \mathring{B})$. Apply the Besicovitch covering theorem to $\mu_{\boldsymbol{u}}$ and a fine covering of A with balls in $\mathcal{F}_U$ of diameter less than $\min_{1\leqq j\leqq M} r_j$, each contained in a ball in $\mathcal{B}$: we thus obtain a $\mathcal{B}' \in \mathfrak{C}_U$ such that every $B' \in \mathcal{B}'$ has a diameter less than $\min_{1\leqq j\leqq M} r_j$ and is contained in a ball in $\mathcal{B}$, and

$$\mu_{\boldsymbol{u}}\left(A \setminus \bigcup \mathcal{B}'\right) = 0. \tag{79}$$

In particular, i) is satisfied. As the family $\mathcal{B}$ is disjoint and $\Lambda(\boldsymbol{u}, V) \subset V$ for any open set $V \subset \Omega$, then the sets $\Lambda(\boldsymbol{u}, \mathring{B})$ are disjoint for $B \in \mathcal{B}$. Therefore, by Theorem 5 and Definition 17 we have that

$$\mu_{\boldsymbol{u}}(A) = \sum_{B\in\mathcal{B}} \mu_{\boldsymbol{u}}(\Lambda(\boldsymbol{u}, \mathring{B})) = \sum_{B\in\mathcal{B}} \mu_{\boldsymbol{u}}(B) = \mu_{\boldsymbol{u}}\left(\bigcup \mathcal{B}\right) = \mu_{\boldsymbol{u}}(U).$$

With (79) we obtain that

$$\mu_{\boldsymbol{u}}(U) = \mu_{\boldsymbol{u}}\left(A \cap \bigcup \mathcal{B}'\right) \leqq \mu_{\boldsymbol{u}}\left(\bigcup \mathcal{B}'\right) \leqq \mu_{\boldsymbol{u}}(U),$$

so ii) is satisfied.

Now we prove that the sets

$$\Gamma_j, \quad 1 \leqq j \leqq M; \qquad \Gamma_I(\boldsymbol{u}, \mathring{B}'), \quad B' \in \mathcal{B}' \tag{80}$$

are disjoint. By construction, the sets Γ_j are disjoint for $1 \leqq j \leqq M$, and so are the sets $\Gamma_I(\boldsymbol{u}, \mathring{B}')$ for $B' \in \mathcal{B}'$. Suppose, looking for a contradiction, that $\boldsymbol{x}^+ = (\boldsymbol{u}^{-1})^+(\boldsymbol{y}) \in B(\boldsymbol{x}_j^+, r_j)$ and $\boldsymbol{x}^- = (\boldsymbol{u}^{-1})^-(\boldsymbol{y}) \in B(\boldsymbol{x}_j^-, r_j)$ for some $\boldsymbol{y} \in \Gamma_I(\boldsymbol{u}, \mathring{B}')$, some $1 \leqq j \leqq M$, and some $B' \in \mathcal{B}'$. Then

$$3r_j < |\boldsymbol{x}_j^+ - \boldsymbol{x}_j^-| \leqq |\boldsymbol{x}_j^+ - \boldsymbol{x}^+| + |\boldsymbol{x}^+ - \boldsymbol{x}^-| + |\boldsymbol{x}^- - \boldsymbol{x}_j^-| \leqq 2r_j + \min_{1 \leqq i \leqq M} r_i,$$

a contradiction.

As the sets in (80) are disjoint and contained in Γ [because of i], we have that

$$\sum_{j=1}^{M} \mathcal{H}^{n-1}(\Gamma_j) + \mathcal{H}^{n-1}\left(\bigcup_{B' \in \mathcal{B}'} \Gamma_I(\boldsymbol{u}, \mathring{B}')\right) \leqq \mathcal{H}^{n-1}(\Gamma).$$

The proof is concluded by noting that $\sum_{j=1}^{M} \mathcal{H}^{n-1}(\Gamma_j) \geqq \frac{1}{4}\mathcal{H}^{n-1}(\Gamma)$, thanks to (77) and (78). $\square$

Proof (of Theorem 6). Let $\mathcal{B} \in \mathfrak{C}_U$. By Proposition 7, $\mathcal{H}^{n-1}(\Gamma_C(\boldsymbol{u}, \mathring{B})) \leqq \mu_{\boldsymbol{u}}(B)$ for every $B \in \mathcal{B}$. Hence

$$\sum_{B \in \mathcal{B}} \mathcal{H}^{n-1}(\Gamma_C(\boldsymbol{u}, \mathring{B})) \leqq \sum_{B \in \mathcal{B}} \mu_{\boldsymbol{u}}(B) = \mu_{\boldsymbol{u}}\left(\bigcup \mathcal{B}\right) \leqq \mu_{\boldsymbol{u}}(U).$$

We now prove that for every $\delta > 0$ there is a $\mathcal{B} \in \mathfrak{C}_U$ such that

$$\sum_{B \in \mathcal{B}} \mathcal{H}^{n-1}(\Gamma_C(\boldsymbol{u}, \mathring{B})) \geqq \mu_{\boldsymbol{u}}(U) - 2\delta.$$

Since, by Proposition 7, $\mu_{\boldsymbol{u}}(B) = \mu_{\boldsymbol{u}}(\mathring{B}) = \mathcal{H}^{n-1}(\Gamma_C(\boldsymbol{u}, \mathring{B})) + 2\mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u}, \mathring{B}))$ for all $B \in \mathcal{B}$ and all $\mathcal{B} \in \mathfrak{C}_U$, it suffices to find a $\mathcal{B} \in \mathfrak{C}_U$ such that $\mu_{\boldsymbol{u}}(U) = \mu_{\boldsymbol{u}}(\bigcup \mathcal{B})$ and

$$\sum_{B \in \mathcal{B}} \mathcal{H}^{n-1}(\Gamma_I(\boldsymbol{u}, \mathring{B})) \leqq \delta.$$

That $\mathcal{B}$ will be found by an iterative process. First, obtain a $\mathcal{B}_0 \in \mathfrak{C}_U$ such that $\mu_{\boldsymbol{u}}(U) = \mu_{\boldsymbol{u}}(\bigcup \mathcal{B}_0)$ by applying the Besicovitch theorem to $\mu_{\boldsymbol{u}}$. Then apply Lemma 9 to $\mathcal{B}_0$ to obtain a $\mathcal{B}_1 \in \mathfrak{C}_U$ such that $\mu_{\boldsymbol{u}}(U) = \mu_{\boldsymbol{u}}(\bigcup \mathcal{B}_1)$ and

$$\mathcal{H}^{n-1}\left(\bigcup_{B \in \mathcal{B}_1} \Gamma_I(\boldsymbol{u}, \mathring{B})\right) \leqq \frac{3}{4}\mathcal{H}^{n-1}\left(\bigcup_{B \in \mathcal{B}_0} \Gamma_I(\boldsymbol{u}, \mathring{B})\right).$$

By induction, for each $N \in \mathbb{N}$, we obtain a $\mathcal{B}_N \in \mathfrak{C}_U$ such that $\mu_{\boldsymbol{u}}(U) = \mu_{\boldsymbol{u}}(\bigcup \mathcal{B}_N)$ and

$$\mathcal{H}^{n-1}\left(\bigcup_{B\in\mathcal{B}_N} \Gamma_I(\boldsymbol{u}, \mathring{B})\right) \leqq \left(\frac{3}{4}\right)^N \mathcal{H}^{n-1}\left(\bigcup_{B\in\mathcal{B}_0} \Gamma_I(\boldsymbol{u}, \mathring{B})\right).$$

Choose, then, $N \in \mathbb{N}$ such that

$$\left(\frac{3}{4}\right)^N \mathcal{H}^{n-1}\left(\bigcup_{B\in\mathcal{B}_0} \Gamma_I(\boldsymbol{u}, \mathring{B})\right) \leqq \delta,$$

and $\mathcal{B}_N$ as the desired family. This completes the proof. □

Note that we have actually proved that

$$\inf_{\mathcal{B}\in\mathfrak{C}_U} \mathcal{H}^{n-1}\left((\Gamma_C(\boldsymbol{u}, U) \cup \Gamma_I(\boldsymbol{u}, U)) \setminus \bigcup_{B\in\mathcal{B}} \Gamma_C(\boldsymbol{u}, \mathring{B})\right) = 0.$$

8. Equilibrium equations for the surface energy

In [18] we proposed a model for cavitation in nonlinear elasticity based on the minimization of

$$\begin{aligned} I_1(\boldsymbol{u}) &:= \int_\Omega W(D\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} + \lambda_1 \mathcal{E}(\boldsymbol{u}), \quad \boldsymbol{u} \in \mathcal{A}_1, \\ \mathcal{A}_1 &:= \{\boldsymbol{u} \in W^{1,p}(\Omega, \mathbb{R}^n) : \det D\boldsymbol{u} > 0 \text{ almost everywhere}, \\ &\qquad \boldsymbol{u} \text{ is one-to-one almost everywhere}, \ \boldsymbol{u}|_{\Gamma_D} = \boldsymbol{b}_1\}, \end{aligned}$$

and a model for cavitation and fracture based on the minimization of

$$\begin{aligned} I_2(\boldsymbol{u}) &:= \int_\Omega W(\nabla \boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} + \lambda_1 \mathcal{E}(\boldsymbol{u}) + \lambda_2 \mathcal{H}^{n-1}(J_{\boldsymbol{u}}) \\ &\quad + \lambda_2 \mathcal{H}^{n-1}(\{\boldsymbol{x} \in \Gamma_D : \boldsymbol{b}_2^+(\boldsymbol{x}) \neq \boldsymbol{u}(\boldsymbol{x})\}), \qquad \boldsymbol{u} \in \mathcal{A}_2, \\ \mathcal{A}_2 &:= \big\{\boldsymbol{u} \in SBV(\Omega, \mathbb{R}^n) : \nabla \boldsymbol{u} \in L^p(\Omega, \mathbb{R}^{n\times n}), \ \det \nabla \boldsymbol{u} > 0 \text{ almost everywhere}, \\ &\qquad \boldsymbol{u} \text{ is one-to-one almost everywhere}, \ \boldsymbol{u}(\boldsymbol{x}) \in K \text{ for almost every } \boldsymbol{x} \in \Omega\big\}. \end{aligned}$$

The equality on Γ_D is in the sense of traces; see Theorem 7 below or [18, Thms. 4 and 5] for the precise assumptions. In this final section we obtain, as an application of Theorem 3, the equilibrium equations of those variational models.

As is well known, the invertibility and orientation-preserving constraints make it impossible to carry out the standard proof of the Euler–Lagrange equations in nonlinear elasticity. Nevertheless, it was observed by Ball [8] that one can still do outer variations of the form $\boldsymbol{h}_s \circ \boldsymbol{u}$ and inner variations of the form $\boldsymbol{u} \circ \boldsymbol{h}_s$, where $\{\boldsymbol{h}_s\}_{s\in\mathbb{R}}$ is a family of diffeomorphisms such that $\boldsymbol{h}_0 = \mathbf{id}$.

The equilibrium equations involve the operator div^M of tangential divergence over a countably $\mathcal{H}^{n-1}$ rectifiable set M, the definition of which can be found, for example, in [5, Sect. 7.3].

Theorem 7. *Let Ω be a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary. Let $\Gamma_D \subset \partial\Omega$ be a countably $\mathcal{H}^{n-1}$ rectifiable set, let $K \subset \mathbb{R}^n$ be compact, and let $\lambda_1, \lambda_2 > 0$. Let $\boldsymbol{b}_1 : \Gamma_D \to \mathbb{R}^n$ a measurable map, and $\boldsymbol{b}_2 \in SBV(\Omega', \mathbb{R}^n)$, where $\Omega' \subset \mathbb{R}^n$ is an open set containing $\Omega \cup \Gamma_D$. Denote by $\boldsymbol{b}_2^+$ the lateral trace of $\boldsymbol{b}_2$ on Γ_D coming from $\Omega' \setminus (\Omega \cup \Gamma_D)$. Let $W : \{\boldsymbol{F} \in \mathbb{R}^{n\times n} : \det \boldsymbol{F} > 0\} \to \mathbb{R}$ be a C^1 function such that there exists $c > 0$ satisfying*

$$\left| DW(\boldsymbol{F})\boldsymbol{F}^T \right| \leqq c\,(W(\boldsymbol{F}) + 1) \quad \textit{for all } \boldsymbol{F} \in \mathbb{R}^{n\times n} \textit{ with } \det \boldsymbol{F} > 0.$$

Let $p \in [1, \infty)$. If $\boldsymbol{u}$ is a minimizer of I_1 in $\mathcal{A}_1$, or a minimizer of I_2 in $\mathcal{A}_2$ then, for all $\boldsymbol{\varphi} \in C^1(\mathbb{R}^n, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ such that $\boldsymbol{\varphi} \circ \boldsymbol{u}|_{\Gamma_D} = \boldsymbol{0}$ in the sense of traces, we have that

$$\int_\Omega \left[DW(\nabla \boldsymbol{u}(\boldsymbol{x}))\, \nabla \boldsymbol{u}(\boldsymbol{x})^T \right] \cdot D\boldsymbol{\varphi}(\boldsymbol{u}(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_V(\boldsymbol{u})} \operatorname{div}^{\Gamma_V(\boldsymbol{u})} \boldsymbol{\varphi}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y})$$
$$+2 \int_{\Gamma_I(\boldsymbol{u})} \operatorname{div}^{\Gamma_I(\boldsymbol{u})} \boldsymbol{\varphi}(\boldsymbol{y})\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}) = 0.$$

Proof. We will do the proof for I_2, the proof for I_1 being analogous.

Let $\boldsymbol{u}$ be a minimizer of I_2 in $\mathcal{A}_2$, and let $\boldsymbol{\varphi} \in C^1(\mathbb{R}^n, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ satisfy that $\boldsymbol{\varphi} \circ \boldsymbol{u}|_{\Gamma_D} = \boldsymbol{0}$ in the sense of traces. It is clear that, for all $\tau \in \mathbb{R}$ with $|\tau|$ small, the function $\mathbf{id} + \tau\boldsymbol{\varphi}$ is a C^1 diffeomorphism from $\mathbb{R}^n$ onto itself such that $\det D(\mathbf{id} + \tau\boldsymbol{\varphi}) > 0$. Thanks to the chain rule for BV functions (see, for example, [5, Th. 3.96]), this implies that if $\boldsymbol{u} \in \mathcal{A}_2$ then $(\mathbf{id} + \tau\boldsymbol{\varphi}) \circ \boldsymbol{u} \in \mathcal{A}_2$. It is also clear that

$$\left\{\boldsymbol{x} \in \Gamma_D : \boldsymbol{b}_2^+(\boldsymbol{x}) \neq (\mathbf{id} + \tau\boldsymbol{\varphi})(\boldsymbol{u}(\boldsymbol{x}))\right\} = \left\{\boldsymbol{x} \in \Gamma_D : \boldsymbol{b}_2^+(\boldsymbol{x}) \neq \boldsymbol{u}(\boldsymbol{x})\right\}.$$

Next we prove that

$$\Gamma_I((\mathbf{id} + \tau\boldsymbol{\varphi}) \circ \boldsymbol{u}) = (\mathbf{id} + \tau\boldsymbol{\varphi})\,(\Gamma_I(\boldsymbol{u}))\,, \quad \Gamma_V((\mathbf{id} + \tau\boldsymbol{\varphi}) \circ \boldsymbol{u}) = (\mathbf{id} + \tau\boldsymbol{\varphi})\,(\Gamma_V(\boldsymbol{u}))\,. \tag{81}$$

Let $\boldsymbol{h} : \Omega \to \mathbb{R}^n$ be a C^1 diffeomorphism from $\mathbb{R}^n$ onto itself. In order to prove (81), it suffices to show that $\boldsymbol{h}(\Gamma_I(\boldsymbol{u})) \subset \Gamma_I(\boldsymbol{h} \circ \boldsymbol{u})$ and $\boldsymbol{h}(\Gamma_V(\boldsymbol{u})) \subset \Gamma_V(\boldsymbol{h} \circ \boldsymbol{u})$. First, it is easy to check that

$$\operatorname{im}_{\mathrm{G}}(\boldsymbol{h} \circ \boldsymbol{u}, \Omega) = \boldsymbol{h}\,(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega)) = \operatorname{im}_{\mathrm{G}}(\boldsymbol{h}, \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega)).$$

This and Definition 8 imply that $(\boldsymbol{h} \circ \boldsymbol{u})^{-1} = \boldsymbol{u}^{-1} \circ \boldsymbol{h}^{-1}$. As a consequence, and using also Lemma 1, we find that $J_{(\boldsymbol{h}\circ\boldsymbol{u})^{-1}} = \boldsymbol{h}(J_{\boldsymbol{u}^{-1}})$. Moreover, with this formula and Lemma 1, we can define in $J_{(\boldsymbol{h}\circ\boldsymbol{u})^{-1}}$ a natural orientation induced from $J_{\boldsymbol{u}^{-1}}$. With that orientation, the lateral traces satisfy $((\boldsymbol{h} \circ \boldsymbol{u})^{-1})^\pm = (\boldsymbol{u}^{-1})^\pm \circ \boldsymbol{h}^{-1}$. This shows that $\boldsymbol{h}(\Gamma_I(\boldsymbol{u})) \subset \Gamma_I(\boldsymbol{h} \circ \boldsymbol{u})$.

Now, checking that $\boldsymbol{h}(\Gamma_V(\boldsymbol{u})) \subset \Gamma_V(\boldsymbol{h} \circ \boldsymbol{u})$ is a routine application of Lemma 1 and the equalities $(\boldsymbol{h} \circ \boldsymbol{u})^{-1} = \boldsymbol{u}^{-1} \circ \boldsymbol{h}^{-1}$ and $((\boldsymbol{h} \circ \boldsymbol{u})^{-1})^- = (\boldsymbol{u}^{-1})^- \circ \boldsymbol{h}^{-1}$. Thus, (81) is proved. Moreover, it is also immediate that $J_{\boldsymbol{u}} = J_{\boldsymbol{h}\circ\boldsymbol{u}}$.

As $\boldsymbol{u}$ is a minimizer of I_2 in $\mathcal{A}_2$, and $(\mathbf{id} + \tau\boldsymbol{\varphi}) \circ \boldsymbol{u} \in \mathcal{A}_2$ for all $\tau \in \mathbb{R}$ with $|\tau|$ small, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} I_2((\mathbf{id} + \tau\boldsymbol{\varphi}) \circ \boldsymbol{u})\bigg|_{\tau=0} = 0,$$

provided that the left-hand side exists, which will be proved in the next paragraph.

It was shown by Ball [8] (see also [9,24]) that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_\Omega W(\nabla((\mathbf{id} + \tau\boldsymbol{\varphi}) \circ \boldsymbol{u}))\, \mathrm{d}\boldsymbol{x}\bigg|_{\tau=0} = \int_\Omega \left[DW(\nabla\boldsymbol{u})\, \nabla\boldsymbol{u}^T\right] \cdot D\boldsymbol{\varphi}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{x}.$$

Now, using (81) and the formula for the first variation of the area (see, for example, [5, Th. 7.31]), we have that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{H}^{n-1}(\Gamma_I((\mathbf{id} + \tau\boldsymbol{\varphi}) \circ \boldsymbol{u}))\bigg|_{\tau=0} = \int_{\Gamma_I(\boldsymbol{u})} \operatorname{div}^{\Gamma_I(\boldsymbol{u})} \boldsymbol{\varphi}\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}),$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{H}^{n-1}(\Gamma_V((\mathbf{id} + \tau\boldsymbol{\varphi}) \circ \boldsymbol{u}))\bigg|_{\tau=0} = \int_{\Gamma_V(\boldsymbol{u})} \operatorname{div}^{\Gamma_V(\boldsymbol{u})} \boldsymbol{\varphi}\, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{y}).$$

In addition, we obviously have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{H}^{n-1}\left(\{\boldsymbol{x} \in \Gamma_D : \boldsymbol{b}_2^+(\boldsymbol{x}) \neq (\mathbf{id} + \tau\boldsymbol{\varphi})(\boldsymbol{u}(\boldsymbol{x}))\}\right)\bigg|_{\tau=0} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{H}^{n-1}(J_{(\mathbf{id}+\tau\boldsymbol{\varphi})\circ\boldsymbol{u}})\bigg|_{\tau=0} = 0.$$

The proof is concluded by using Proposition 5 and Theorem 3. □

In the proof of Theorem 7 we have worked with outer variations. We could have worked with inner variations, too, but they provide no information regarding the term $\mathcal{E}(\boldsymbol{u})$. Indeed, if $\boldsymbol{\varphi} \in C_c^1(\Omega, \mathbb{R}^n)$ and $|\tau|$ is small, then the map $\theta_\tau := \mathbf{id} + \tau\boldsymbol{\varphi}$ is a C^1 diffeomorphism from Ω onto itself. By virtue of the the chain rule, for each $i = 1, 2$, if $\boldsymbol{u} \in \mathcal{A}_i$ then $\boldsymbol{u} \circ \theta_\tau \in \mathcal{A}_i$ and $\mathcal{E}_{\boldsymbol{u}\circ\theta_\tau}(\boldsymbol{f}) = \mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f}_\tau)$, where for each $\boldsymbol{f} \in C_c^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, the function $\boldsymbol{f}_\tau \in C_c^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ is defined as $\boldsymbol{f}_\tau(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{f}(\theta_\tau^{-1}(\boldsymbol{x}), \boldsymbol{y})$. It then follows that $\mathcal{E}(\boldsymbol{u} \circ \theta_\tau) = \mathcal{E}(\boldsymbol{u})$. Therefore, $\mathcal{E}$ is invariant under this kind of variation.

It is instructive to notice the similarities of the equilibrium equations of Theorem 7 to those of the model of Müller and Spector [24, Sect. 6]. As explained in [18], the term $\mathcal{E}(\boldsymbol{u})$ in the model of Theorem 7 is the counterpart of the term $\operatorname{Per} \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega)$ in the model of [24].

Acknowledgments. We wish to thank Prof. J.M. Ball, who introduced us to the problem of formulating a unified model for cavitation and fracture, and has provided us with his guidance and constant support. D.H. gratefully acknowledges financial support from the Catholic University of Chile, the Chilean Ministry of Planning, and the Clarendon Fund of the Oxford University Press; C.M.-C. from Projects MTM2009-07662 of the Spanish Ministry of Science and Innovation, and PI2010-01 of the Basque Department of Education, Universities and Research. We are indebted to the anonymous referee for his careful revision and valuable suggestions, which helped to improve the exposition of the paper.

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(*Received July 21, 2009 / Accepted December 21, 2010*)
Published online February 1, 2011 – © Springer-Verlag (2011)