Arch. Rational Mech. Anal. 201 (2011) 209–249
Digital Object Identifier (DOI) 10.1007/s00205-010-0385-8

Microscopic Theory of Isothermal Elastodynamics

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Communicated by The Editors

Abstract

This paper examines random perturbations of the anharmonic chain of coupled oscillators. The microscopic system has two conservation laws, and its hyperbolic scaling limit results in the quasi-linear wave equation (p-system) of isothermal (isentropic) elasticity. In the shock regime, the compensated compactness method is used. Lastly results from J. W. Shearer and D. Serre are applied.

1. Introduction

The well-known p-system consists of a couple of conservation laws:

$$\partial_t \pi(t,x) = \partial_x S'(\rho(t,x)) \quad \text{and} \quad \partial_t \rho(t,x) = \partial_x \pi(t,x) \tag{1}$$

for $\pi, \rho \in \mathbb{R}, t \geqq 0$ and $x \in \mathbb{R}$, where the stress $S'(\rho)$ is the derivative of a smooth convex function $S(\rho)$. This system has a direct physical interpretation: in Lagrangian coordinates π and ρ are the momentum (velocity) and the deformation (strain) of an elastic medium in a thermal equilibrium; $\chi := \pi^2/2 + S(\rho)$ denotes the density of its entropy, while $J = -S'(\rho)$ is the flux of the momentum. The case of gas dynamics is more difficult because then S has a singularity at zero. Differentiating the second equation, if possible, we see that ρ satisfies a nonlinear sound equation: $\partial_t^2 \rho = \partial_x^2 S'(\rho)$. In general (1) does not have classical solutions because shock waves may emerge even if the initial values are smooth. *Weak solutions* are defined in a distributional sense, their existence and uniqueness are relevant, nontrivial issues. Our main purpose is to derive the p-system as the *hydrodynamic limit* (HDL) of a *Ginzburg–Landau* type microscopic lattice model.

The vanishing viscosity limit is a popular approximation scheme for hyperbolic systems of conservation laws. Introducing $u := (\pi, \rho)$ and $\Phi(u) := -(S'(\rho), \pi)$, one can rewrite (1) as $\partial_t u + \partial_x \Phi(u) = 0$. Its viscous approximation then reads as

$$\partial_t u_\sigma + \partial_x \Phi(u_\sigma) = \sigma \partial_x^2 u_\sigma, \tag{2}$$

where $\sigma \geqq 0$ may be a matrix, too. Although the standard energy bound controls the quadratic form $\partial_x u \cdot \sigma \partial_x u$, we send $\sigma \to 0$ in the limit; thus the method of parabolic energy inequalities does not work in such situations.[1]

Compensated compactness is an effective tool for avoiding difficulties related to the compactness of the sequence of approximate solutions. Extending the techniques of TARTAR [32] and MURAT [23], R. DiPerna managed, in his pioneering papers, to prove the existence of bounded weak solutions to (1) with general, measurable and bounded initial data, see for example [7] and also [5,22,28] with further references. Besides the strict convexity of S, the *genuine nonlinearity* of the flux is a crucial condition; this means that the equation $S'''(\rho) = 0$ has at most one root. Unfortunately, it is not easy to get uniform bounds for stochastic systems; therefore we follow SERRE [27], SHEARER [30] and SERRE–SHEARER [29] on the construction of weak solutions in L^p with $p < 2$.

Our basic system, the *anharmonic chain*, is certainly a fundamental physical model of one-dimensional elasticity, see for example [12,13,15] and [2,11]. The configurations of an infinite system of coupled oscillators on $\mathbb{Z}$ are denoted as $\omega := \{(p_k, q_k) : k \in \mathbb{Z}\}$, where $p_k, q_k \in \mathbb{R}$ are the momentum and the position of the oscillator at site $k \in \mathbb{Z}$, and the formal Hamiltonian of the anharmonic chain is an infinite sum

$$H(\omega) := \frac{1}{2} \sum_{k\in\mathbb{Z}} \left(p_k^2 + V(q_{k+1} - q_k) + V(q_k - q_{k-1}) \right).$$

In terms of the *deformation* $r_k := q_{k+1} - q_k$, the equations of motion read as

$$\dot{p_k} = V'(r_k) - V'(r_{k-1}) \quad \text{and} \quad \dot{r_k} = p_{k+1} - p_k \quad \text{for } k \in \mathbb{Z}. \tag{3}$$

In this context the interaction potential V is a priori symmetric, but this condition will not be exploited in our calculations. To ensure the existence of unique solutions in the space of configurations with a sub-exponential growth, we are assuming that V'' is bounded. The existence of stationary measures is an easy consequence of $\liminf V''(r) > 0$ as $|r| \to +\infty$; additional conditions are to be given later.

The equations of motion constitute a microscopic system of conservation laws, we have $\partial_t \sum p_k = 0$ and $\partial_t \sum r_k = 0$, moreover $\partial_t H = 0$, provided that these sums are finite. The formal generator, that is the *Liouville operator*

$$\mathscr{L}_0\varphi(\omega) := \sum_{k\in\mathbb{Z}} \left((V'(r_k) - V'(r_{k-1})) \frac{\partial \varphi}{\partial p_k} + (p_{k+1} - p_k) \frac{\partial \varphi}{\partial r_k} \right) \tag{4}$$

[1] Let us remark here that the structure of microscopic models of hydrodynamics resembles very much that of (2). These stochastic models are Markov processes generated by an operator $\mathscr{L} = \mathscr{L}_0 + \sigma\mathscr{G}$, where the basic process is defined by the asymmetric $\mathscr{L}_0$, while its perturbation $\mathscr{G}$ is symmetric with respect to the equilibrium states of the process. In cases of diffusive scaling, a direct strong compactness argument is based on the so-called two-blocks Lemma, see Theorem 4.6 and Theorem 4.7 of GUO–PAPANICOLAU–VARADHAN [19] for its original version. This powerful tool is a probabilistic reformulation of the parabolic energy inequality, and it is not available in the case of non-attractive hyperbolic models, see Remark 1 in Section 4.

is well defined for smooth functions φ of a finite number of variables, stationary measures λ are characterized by $\int \mathscr{L}_0\varphi \, \mathrm{d}\lambda = 0$ for such *local functions.*

Hyperbolic scaling means that the space and time are scaled in the same way, and we are interested in the limiting behavior of the scaled process u_ε when the scaling parameter $\varepsilon > 0$ goes to zero. More precisely, let $1_{\varepsilon,k}$ denote the indicator of the interval $(\varepsilon k - \varepsilon/2, \varepsilon k + \varepsilon/2)$, then

$$u_\varepsilon(t,x) := (\pi_\varepsilon(t,x), \rho_\varepsilon(t,x)) := \sum_{k\in\mathbb{Z}} 1_{\varepsilon,k}(x)\,(p_k(t/\varepsilon), r_k(t/\varepsilon))$$

is the simplest version of the underlying *empirical process.* We want to see $u_\varepsilon \to u$ in some sense, where u is a weak solution to (1), that is

$$\int_0^\infty \int_{-\infty}^\infty \left(\psi_t'(t,x)\cdot u(t,x) + \psi_x'(t,x)\cdot \Phi(u(t,x))\right) \mathrm{d}x\,\mathrm{d}t = 0 \tag{5}$$

for all $\psi \in C^1_{\mathrm{co}}(\mathbb{R}^2_+ \to \mathbb{R}^2)$. Here and also later on, $C^1_{\mathrm{co}}(\mathbb{R}^2_+ \to \mathbb{R}^d)$ denotes the space of continuously differentiable $\psi : \mathbb{R}^2_+ \mapsto \mathbb{R}^d$ such that ψ is compactly supported in the interior of $\mathbb{R}^2_+ := [0,+\infty) \times \mathbb{R}$. The uniform norm of the real real C spaces of continuous functions will be denoted by $\|\cdot\|$, while $\|\cdot\|_p$ is the norm of an L^p space. Local integrability of u is sufficient for the existence of the integrals above because Φ is linearly bounded in our case. The initial value of u has been omitted from the definition (5) of weak solutions because we are not in a position to discuss the uniqueness of the hydrodynamic limit.

Although (3) is a direct lattice approximation of the p-system

$$\partial_t\pi(t,x) = \partial_x V'(\rho(t,x)) \quad \text{and} \quad \partial_t\rho(t,x) = \partial_x\pi(t,x)$$

with flux V' in place of S', the convergence of the empirical process is rather problematic because of several reasons. In PDE theory, see for example [5,28], (3) is not considered as a stable numerical scheme for solving this system; thus we cannot believe in its convergence: the right way of its regularization is suggested by the small viscosity approach. Of course, one can stabilize (3) by adding viscid terms (second differences) to the equations of motion, but the convergence of certain approximate solutions is not the only point of view that we are having in mind. The theory of hydrodynamic limits [20,31] goes beyond numerical analysis, stationary measures of the microscopic system play a decisive role in the derivation of the macroscopic equations. The asymptotic evaluation of the microscopic currents should be based on these stationary distributions, and the associated *thermodynamical calculus* results in a modification of the flux function. Roughly speaking, we have to understand that the limiting expectation of a nonlinear function of ω can be identified with another function of the empirical process. In fact, the dynamics will be regularized by a *conservative noise* allowing us to replace V' by its equilibrium expectation S' as discussed below.

2. Stationary states and hydrodynamic limits

The anharmonic chain has three classical conservation laws, besides $P := \sum p_k$ and $R := \sum r_k$, total energy H is also preserved by the evolution; therefore a triplet, the compressible Euler equations are expected to govern its macroscopic behavior.

2.1. Stationary product measures

In association with these classical conservation laws, we have a three-parameter family $\lambda_{\beta,\pi,\gamma}$ of translation invariant stationary product measures such that the Lebesgue density of each couple (p_k, r_k) reads as

$$\Gamma_{\beta,\pi,\gamma}(p,r) := \exp\Big(-\beta(p-\pi)^2/2 - \beta V(r) + \gamma r - F(\beta,\gamma)\Big),$$

where $\beta > 0$ denotes the inverse temperature, $\pi, \gamma \in \mathbb{R}$, and F is the normalization (free energy). We see that p_k is a normal variable of mean π and variance $T = 1/\beta$ under $\lambda_{\beta,\pi,\gamma}$. In view of

$$F(\beta,\gamma) := \log \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\Big(-\beta(p-\pi)^2/2 - \beta V(r) + \gamma r\Big)\,\mathrm{d}p\,\mathrm{d}r \tag{6}$$

we see that

$$\kappa := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big((p-\pi)^2/2 + V(r)\Big)\,\Gamma_{\beta,\pi,\gamma}(p,r)\,\mathrm{d}p\,\mathrm{d}r = -F_\beta'(\beta,\gamma)$$

is the mean value of the *internal energy*, the equilibrium mean of the *total energy* $h_k := p_k^2/2 + V(r_k)$ at site $k \in \mathbb{Z}$ is just $\chi := \lambda_{\beta,\pi,\gamma}(h_k) = \kappa + \pi^2/2$, the mean deformation is $\rho = F_\gamma'(\beta,\gamma) = \lambda_{\beta,\pi,\gamma}(r_k)$, finally we obtain $\lambda_{\beta,\pi,\gamma}(V'(r_k)) = \gamma/\beta$ for the equilibrium expectation of V' by partial integration. The expectation of a function φ with respect to a measure μ will frequently be denoted as $\mu(\varphi) \equiv \int \varphi\,\mathrm{d}\mu$. The parameters β and γ can be expressed in terms of S,

$$S(\kappa,\rho) := \sup_{\beta,\gamma}\{\gamma\rho - \beta\kappa - F(\beta,\gamma) : \beta > 0, \gamma \in \mathbb{R}\}. \tag{7}$$

Indeed, as S is the convex conjugate of F, we have $\gamma = S_\rho'(\kappa,\rho)$ and $\beta = -S_\kappa'(\kappa,\rho)$ if $\rho = F_\gamma'(\beta,\gamma)$ and $\kappa = -F_\beta'(\beta,\gamma)$. In view of the *principle of local equilibrium*, see [20,31], mean values of functions of the scaled process should be calculated by means of a product measure of type $\lambda_{\beta,\pi,\gamma}$ with parameters depending on time and space. This means that the marginal density of the couple (p_k, q_k) reads as $\Gamma_{\beta,\pi,\gamma}$ with $\beta = \beta(t,\varepsilon k), \pi = \pi(t,\varepsilon k), \gamma = \gamma(t,\varepsilon k)$, and the evolved measure can be well approximated by such *local equilibrium distributions*. Since $\partial_t h_k = p_{k+1}V'(r_k) - p_k V'(r_{k-1})$, a formal calculation results in the set

$$\partial_t\pi = \partial_x J(\kappa,\rho), \quad \partial_t\rho = \partial_x\pi, \quad \partial_t\chi = \partial_x(\pi J(\kappa,\rho)) \tag{8}$$

of *compressible Euler equations*, where $J(\kappa,\rho) := \gamma/\beta = -S_\rho'(\kappa,\rho)/S_\kappa'(\kappa,\rho)$ and $\kappa = \chi - \pi^2/2$ denotes the internal energy per site, see [2,4,11,24].

2.2. The strong ergodic hypothesis

The heuristic derivation of the macroscopic equations via local equilibrium is based on the hypothesis that, apart from the classical H, P and R, the system (3) of anharmonic oscillators has no additional conservation laws, and a strong form of the ergodic hypothesis holds also true. In the case of infinite systems the notion "conservation law" is not quite obvious, there are several versions.[2] The *strong ergodic hypothesis* is exact; it means a description of all translation invariant stationary states as certain superpositions of product measures $\lambda_{\beta,\pi,\gamma}$; the mathematical treatment of hydrodynamic limits is usually based on this kind of ergodicity. In the paper [9] by DOBRUSHIN and coworkers, a full description of all stationary states, and that of the associated conserved quantities of the harmonic chain are given. It turned out that there is a huge class of additional stationary measures; therefore HDL of the harmonic chain results in a continuum of macroscopic equations involving all conservation laws. The anharmonic chain is much more difficult: there is no real hope to verify any version of the ergodic hypothesis in this case. Therefore we have to abandon the heuristic picture of local equilibrium.

2.3. Random perturbations of the harmonic chain

In the spirit of statistical physics, random perturbations of Hamiltonian dynamics result in a fully developed hydrodynamic behavior; more or less heuristic models allow one to derive correct conclusions. For example, we can perform random exchange of velocities between neighboring sites such that all actions are independent of each other, and the exchange rates are constant in space. This mechanism preserves the total energy, the momentum and the deformation; thus the product measures $\lambda_{\beta,\pi,\gamma}$ are all stationary states, and the converse [14] is also true.

A mathematical theory of local equilibrium has been initiated by YAU [34], see also [24] for the hyperbolic scaling limit of a Hamiltonian dynamics of particles with conservative noise. Assuming the strong ergodic hypothesis and also the smoothness of the macroscopic solution, one finds that this fairly general method yields conservation of local equilibrium in an asymptotic sense. The space-time dependent parameters $\beta(t, \varepsilon k)$, $\pi(t, \varepsilon k)$, $\gamma(t, \varepsilon k)$ of local equilibria are specified by the hydrodynamic equations, and the deviation of the true measure from the underlying local equilibrium distribution is measured by their *relative entropy.* If this deviation is "small" at time zero, then it remains small as long as the macroscopic solution is smooth, and this a priori bound implies that as the scaling parameter ε goes to zero, the evolved measure converges in the weak sense to the local equilibrium distribution predicted by the hydrodynamic equations. Using the characterization [14] of stationary states, one can easily extend this method to the anharmonic chain with random exchange of velocities; the resulting system of macroscopic equations is (8). We do not go into the details of this problem here, see the forthcoming notes [2] and the more recent paper [11] for further discussions.

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[2] In view of the KAM (Kolmogorov–Arnold–Moser) theory, finite subsystems of (3) may have many non-classical conservation laws, and due to the Liouville theorem, there is a correspondence between stationary measures and conservation laws.

2.4. The anharmonic chain with physical viscosity

A Ginzburg–Landau model with momentum preserving thermal noise is given by the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}p_k &= (V'(r_k) - V'(r_{k-1}))\,\mathrm{d}t + \sigma\,(p_{k+1} + p_{k-1} - 2p_k)\,\mathrm{d}t \\ &\quad + \sqrt{2\sigma}\,(\mathrm{d}w_k - \mathrm{d}w_{k-1}), \quad \mathrm{d}r_k = (p_{k+1} - p_k)\,\mathrm{d}t, \quad k \in \mathbb{Z}, \end{aligned} \tag{9}$$

where σ is a positive constant, and $\{w_k : k \in \mathbb{Z}\}$ is a family of independent Wiener processes. The existence and uniqueness of strong solutions to infinite systems like (9) is to be discussed a bit in the next section. The Markov process defined by (9) is generated by $\mathscr{L}_p := \mathscr{L}_0 + \sigma\mathscr{S}$, where $\partial_k := \partial/\partial p_k$,

$$\mathscr{S} := \sum_{k\in\mathbb{Z}} \mathscr{S}_k, \quad \mathscr{S}_k\varphi(\omega) := (\nabla_1\partial_k - \nabla_1 p_k)\,\nabla_1\partial_k\varphi(\omega), \tag{10}$$

finally $\nabla_l\eta_k := (1/l)(\eta_{k+l} - \eta_k)$ for sequences η_k indexed by $\mathbb{Z}$ if $l \in \mathbb{N}$.

The total energy is not preserved any more, and a thermal equilibrium of unit temperature is maintained by the noise; therefore the stationary product measures of this evolution law depend only on two parameters. Thus, we denote $\lambda_{\pi,\gamma} := \lambda_{1,\pi,\gamma}$. Under hyperbolic scaling the macroscopic behavior of (9) is governed by the p-system (1), where

$$\begin{aligned} S(\rho) &:= \sup_{\gamma}\{\gamma\rho - F(\gamma) : \gamma \in \mathbb{R}\}, \\ F(\gamma) &:= F(1,\gamma) = \log \int_{-\infty}^{\infty} \mathrm{e}^{\gamma r - V(r)}\,\mathrm{d}r. \end{aligned} \tag{11}$$

Note that $\lambda_{\pi,\gamma}(V'(r_k)) = \gamma = S'(\rho)$ if $\rho = \lambda_{\pi,\gamma}(r_k) = F'(\gamma)$, and $F''(\gamma)S''(\rho) = 1$ in this case, while

$$F''(\gamma) = \int (r_k - \rho)^2\,\mathrm{d}\lambda_{\pi,\gamma} = \int_{-\infty}^{\infty} (r-\rho)^2 \exp(\gamma r - V(r) - F(\gamma))\,\mathrm{d}r. \tag{12}$$

It is easy to check that both F'' and S'' are strictly positive and bounded, see Section 3.1. Let us remark that (1) is *strictly hyperbolic,* that is $S''(\rho) > 0$ because of $\|V''\| < +\infty$, while $\|S''\| < +\infty$ is a consequence of the condition that $\liminf V''(r) > 0$ as $|r| \to +\infty$.

By means of an entropy argument, it is not difficult to prove a strong version of the ergodic hypothesis; therefore the relative entropy argument of Yau applies. Assuming that the initial condition determines a smooth and, hence, the unique solution $u = (\pi, \rho)$ to (1) on $[0, T)$, we have

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \varphi(x)\cdot u_\varepsilon(t,x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)\cdot u(t,x)\,\mathrm{d}x$$

in probability for all compactly supported and continuous $\varphi : \mathbb{R} \mapsto \mathbb{R}^2$ and $t \in [0, T)$. The coefficient $\sigma > 0$ of microscopic viscosity can be fixed during the scaling procedure, but the initial distribution should be close to local equilibrium in the sense of relative entropy, see Theorem 14.1 in [15] for a precise formulation and proof. In contrast to the traditional approach to hydrodynamic limits as in [19,20] and [24,34], the initial distributions do not need to be periodic; later we also consider our microscopic model on the infinite line.

2.5. Lax entropy pairs

As we have explained, the randomness in the above modifications of the anharmonic chain implies convergence to a classical solution of the macroscopic system (8) or (1) by the strong ergodic hypothesis, but in a regime of shocks much more regularity is needed to pass to the hydrodynamic limit. Effective coupling techniques [26] of *attractive models* are not available in the case of two-component systems, *compensated compactness* seems to be the only tool we could use. This argument is based on the existence and control of a class of *Lax entropy pairs* (h, J) consisting of a conserved quantity $h(\pi, \rho)$ and its flux $J(\pi, \rho)$ such that $\partial_t h + \partial_x J = 0$ along classical solutions. The p-system (1) has a fairly rich family of entropy pairs characterized by the equations

$$h'_\pi(\pi, \rho)S''(\rho) + J'_\rho(\pi, \rho) = h'_\rho(\pi, \rho) + J'_\pi(\pi, \rho) = 0.$$

These entropy pairs can be determined by solving $S''(\rho)h''_{\pi,\pi}(\pi, \rho) = h''_{\rho,\rho}(\pi, \rho)$, see for example [5,28]; the set of all entropy pairs of (1) is described in [25]. Because of its ergodicity, however, an underlying microscopic model cannot admit more conservation laws than the classical ones. The *entropy production* $X_\varepsilon := \partial_t h(u_\varepsilon) + \partial_x J(u_\varepsilon)$ is considered as a generalized function; it certainly does not vanish in a regime of shock waves as $\varepsilon \to 0$. Moreover, the fluctuations of X_ε might explode in the limit even if we define the empirical process in terms of *block averages*. The main principal difficulty consists of the identification of the macroscopic flux $J = -S'(\rho)$ in the microscopic expression of $\mathscr{L}_0 h$. It is then a technical question to show that the remainders really vanish in the limit.

3. Main result and some ideas of its proof

In order to suppress extreme fluctuations of Lax entropies, a strong artificial viscosity should be added to both components of the equations of motion. We consider a Ginzburg–Landau type stochastic system mimicking the viscous approximation:

$$\begin{aligned} \mathrm{d}p_k &= (V'(r_k) - V'(r_{k-1}))\,\mathrm{d}t + \sigma(\varepsilon)\,(p_{k+1} + p_{k-1} - 2p_k)\,\mathrm{d}t \\ &\qquad + \sqrt{2\sigma(\varepsilon)}\,(\mathrm{d}w_k - \mathrm{d}w_{k-1}), \quad k \in \mathbb{Z}, \\ \mathrm{d}r_k &= (p_{k+1} - p_k)\,\mathrm{d}t + \sigma(\varepsilon)\,(V'(r_{k+1}) + V'(r_{k-1}) - 2V'(r_k))\,\mathrm{d}t \\ &\qquad + \sqrt{2\sigma(\varepsilon)}\,(\mathrm{d}\tilde{w}_{k+1} - \mathrm{d}\tilde{w}_k), \quad k \in \mathbb{Z}, \end{aligned} \tag{13}$$

where $\{w_k : k \in \mathbb{Z}\}$ and $\{\tilde{w}_k : k \in \mathbb{Z}\}$ are independent families of independent Wiener processes, $1 \leqq \sigma(\varepsilon) \to +\infty$ such that $\varepsilon\sigma(\varepsilon) \to 0$ but $\varepsilon\sigma^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. The first assumption is quite natural in case of a hyperbolic scaling limit,

$\varepsilon\sigma(\varepsilon)$ is just the coefficient of the macroscopic viscosity. The less plausible second one will be used to control rapid oscillations emerging in the regime of shock waves. The interaction potential does not need to be symmetric. We are assuming that $V \in C^2(\mathbb{R})$ satisfies $0 < c_1 \leqq V''(r) \leqq c_2 < +\infty$ for all $r \in \mathbb{R}$. The strict convexity of V implies the Bakry–Emery criterion of logarithmic concavity for the canonical distributions of the equilibrium states $\lambda_{\pi,\gamma}$; the resulting logarithmic Sobolev inequality (LSI), see [8,21], is our main tool for the evaluation of entropy production. [3] LSI plays a crucial role here because it yields explicit bounds on the rate of convergence to local equilibrium. Replacing the microscopic current V' by its equilibrium expectation S', see Lemma 8, we obtain a random approximation scheme for (1). The structure of the deterministic part of this scheme satisfies the principles of the small viscosity limit; therefore, the rest of the proof turns out to be an adaptation of the argument for the existence of weak solutions to (1) via compensated compactness.

3.1. Conditions on the macroscopic flux

The small viscosity limit of the microscopic system to (1) will be materialized by the results of SHEARER [30] and SERRE–SHEARER [29].[4] The proofs in [29,30] presuppose the following severe conditions on the flux, concerning definitions and remarks we refer to Section 2.4, see (11) and (12) in particular. The third and fourth derivatives of a function f are denoted as f''' and $f^{(iv)}$, respectively.

(SH) Strict hyperbolicity: $\inf\{S''(\rho) : \rho \in \mathbb{R}\} > 0$.

(GN1) Genuine nonlinearity: The flux S' is strictly convex or concave in case of [30], that is either $S'''(\rho) > 0$ or $S'''(\rho) < 0$ for all $\rho \in \mathbb{R}$.

(GN2) Genuine nonlinearity apart from one point: The equation $S'''(\rho) = 0$ has exactly one root in the case of [29].

(LB) L^2 and L^∞ bounds: $S'''(\rho)(S''(\rho))^{-5/4}$ and $S^{(iv)}(\rho)(S''(\rho))^{-7/4}$ are in $L^2(\mathbb{R})$, while $S'''(\rho)(S''(\rho))^{-3/2}$ and $S^{(iv)}(\rho)(S''(\rho))^{-2}$ are bounded.

(GC) Growth conditions: $(S''(\rho))^q \leqq a + bS(\rho)$ for all $\rho \in \mathbb{R}$ with some constants $a, b > 0$ and $q > 1/2$, while $S'(\rho)/S(\rho) \to 0$ as $|\rho| \to +\infty$.

Both conditions (SH) and (GC) hold automatically true because S is infinitely differentiable, and S'' is strictly positive and bounded in our case. Observe now that the third and fourth derivatives of S and F are related by

$$\begin{aligned} S'''(\rho) &= -F'''(\gamma)\left(S''(\rho)\right)^3 \quad \text{and} \\ S^{(iv)}(\rho) &= 3\left(F''(\gamma)\right)^3 \left(S'''(\rho)\right)^2 - F^{(iv)}(\gamma)\left(S''(\rho)\right)^4 \end{aligned} \tag{14}$$

whenever $\gamma = S'(\rho)$, that is $\rho = F'(\gamma)$, where

$$F'''(\gamma) = \int (r_k - \rho)^3 \, \mathrm{d}\lambda_{\pi,\gamma} = \int_{-\infty}^{\infty} (r - \rho)^3 \exp(\gamma r - V(r) - F(\gamma)) \, \mathrm{d}r, \tag{15}$$

[3] The first application of a logarithmic Sobolev inequality in the context of conservative systems is due to CHANG–YAU [3].

[4] Let us remark that the vanishing viscosity of [29] is *physical* in the sense that it is added to the equation of momentum only, compare (9) and (13).

while the fourth derivative of F satisfies

$$F^{(iv)}(\gamma) + 3\left(F''(\gamma)\right)^2 = \int_{-\infty}^{\infty} (r-\rho)^4 \exp(\gamma r - V(r) - F(\gamma))\,\mathrm{d}r. \quad (16)$$

Therefore $S''' \in L^\infty$ and $S^{(iv)} \in L^\infty$ are direct consequences of $\liminf V''(r) > 0$ as $|r| \to +\infty$, see Section 4.1 for proofs.

In view of (15) and (16) the L^2 bound of the derivatives is related to the *asymptotic normality* of the equilibrium distribution $\lambda_{\pi,\gamma}$ of the coordinates r_k when $|\gamma| \to +\infty$. Indeed, the third centered moment of a Gaussian distribution is zero, while its fourth centered moment equals threefold the square of its variance. A convenient formulation of this assumption reads as

(AN) Asymptotic normality: There exist some positive constants V''_+, V''_-, α and R such that $|V''(r) - V''_+| < \mathrm{e}^{-\alpha r}$ if $r > R$, while $|V''(r) - V''_-| < \mathrm{e}^{\alpha r}$ if $r < -R$.

By means of the well-known Laplace method, we perform an asymptotic evaluation of the centered moments of r_k under $\lambda_{\pi,\gamma}$. The remainders turn out to be exponentially small, whence $F''' \in L^2$ and $F^{(iv)} \in L^2$ follow immediately; thus (16) and (14) imply the desired statement. Since V is strictly convex by assumption, there is a unique $\delta \in \mathbb{R}$ such that $\gamma = V'(\delta)$. This value plays a crucial role in our calculations because the maximum of the local density $\exp(\gamma r - V(r) - F(\gamma))$ of $\lambda_{\pi,\gamma}$ is attained there.

Let us consider now the conditions (GN1) and (GN2) of genuine nonlinearity. Suppose that $V(r) = r^2/2 + U(r)$, then

$$S'(\rho) = \gamma = \lambda_{\pi,\gamma}(V'(r_k)) = \rho + \int_{-\infty}^{\infty} U'(r) \exp(\gamma r - V(r) - F(\gamma))\,\mathrm{d}r$$

if $\rho = F'(\gamma)$; therefore we suspect that the global shape of $S'(\rho) - \rho$ follows that of U'. It is easy to see that S is symmetric if U is so; thus S' cannot be strictly convex or concave as $S'''(0) = 0$ in this case. Moreover, this kind of genuine nonlinearity of S' is also excluded if U' is bounded. We believe that the strict convexity/concavity of U' implies that of S' under some reasonable assumptions on V. In the second case when U is symmetric and $rU'''(r) \neq 0$ unless $r = 0$, the same property of $\rho S'''(\rho)$ might also hold true. Unfortunately we have no idea on the precise conditions of these conjectures. This is a difficult elementary problem of one-dimensional calculus; there are some simple, and also a bit more complex, generic examples.

Let us consider first the potential $V(r) := r^2/2 - \log\cosh(\kappa r)$ with some constant $\kappa > 0$. By an explicit computation

$$F(\gamma) = \log\sqrt{2\pi} + \gamma^2/2 + \kappa^2/2 + \log\cosh(\kappa\gamma),$$

whence $F'(\gamma) = \gamma + \kappa\tanh(\kappa\gamma)$; thus $\rho = 0$ is the only root of $S'''(\rho) = 0$ because $\gamma F'''(\gamma) > 0$ if $\gamma \neq 0$, see (14). The other conditions on S' including

(AN) are also valid in this case. There is a fairly big class $V = r^2/2 + U_\nu(r)$ of symmetric potentials such that

$$U_\nu(r) := -\log \sum_{k=1}^{n} \nu_k \cosh(\kappa_k r), \tag{17}$$

where ν_k and κ_k are positive numbers, whence we obtain

$$F(\gamma) = \log \sqrt{2\pi} + \gamma^2/2 + \log \sum_{k=1}^{n} \nu_k \exp(\kappa_k^2/2) \cosh(\kappa_k \gamma). \tag{18}$$

Of course, we are not able to check $\gamma F_\nu'''(\gamma) > 0$ for $\gamma \neq 0$ in this generality, but we have many simple examples such that $F(\gamma)$ turns into a nice sum; thus (GN2) is easily verified. Indeed, define $\Psi_\nu(\gamma) > 0$ by

$$\Psi_\nu(\gamma) := \prod_{j=1}^{m} (\beta \cosh \tilde{\kappa}_j \gamma)^{n_j}, \tag{19}$$

where $\beta > 0, n_j \in \mathbb{N}$ and $\tilde{\kappa}_j > 0$ are given numbers. Applying the identity

$$2 \cosh x \cosh y = \cosh(x + y) + \cosh(x - y)$$

several times, we get some new constants $\nu_k, \kappa_k > 0$ such that

$$\Psi_\nu(\gamma) = \sum_{k=1}^{n} \nu_k \exp\left(\kappa_k^2/2\right) \cosh(\kappa_k \gamma),$$

which determines a suitable U_ν via (17) and (18). The role of the factor β is to normalize Ψ_ν; it has been absorbed by the coefficients ν_k. The possible limits of such sequences are also good, at least if the numbers $\tilde{\kappa}_j$ are bounded, and they are bounded away from zero.

We have another explicitly solvable symmetric model, namely $V(r) := r^2/2 - \log \cosh(\kappa r^2)$. In this case the equation $S'''(\rho) = 0$ has three roots; thus the present strong form of (GN2) is not valid. Let us remark that $\partial_r^3 \log \cosh(\kappa r^2) = 0$ also has three roots.

In the asymmetric case $V(r) := (1 - \kappa) r^2/2 + \kappa r|r|/2$ is an exactly solvable model. The strict convexity of S', that is $S''' > 0$ follows from a direct computation if $\kappa \in (0, 1/2)$, while S' turns out to be strictly concave if $-1/2 < \kappa < 0$. Such a V is not twice differentiable, small local perturbations of exactly solvable models are to be discussed in Section 4.1.

3.2. The infinite dynamics

Since V'' is bounded by an assumption, it is not difficult to prove the existence of unique strong solutions in a space Ω_0 of configurations $\omega = \{(p_k, r_k) : k \in \mathbb{Z}\}$ with a sub-exponential growth. More precisely, $\Omega_0 = \cap\{\Omega_\alpha : \alpha > 0\}$, where Ω_α is the Hilbert space with norm $\|\omega\|_{2,\alpha}$ defined as

$$\|\omega\|_{2,\alpha}^2 := \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\alpha|k|}\big(p_k^2 + r_k^2\big).$$

Indeed, the right-hand side of (13) satisfies a Lipschitz condition in each Ω_α, thus a standard iteration procedure yields the statement, see for example [6]. The partial dynamics $\omega^{(n)}(t)$ are defined for $n \in \mathbb{N}$, for example by letting $\mathrm{d}p_k = \mathrm{d}q_k = 0$ if $|k| > n$ and having (13) in force otherwise. Again a standard argument shows that the infinite dynamics can be obtained as the limit of such finite subsystems as $n \to +\infty$. This approach is also useful when we have to verify that a given measure is really stationary under the time evolution. The formal generator of the process reads as $\mathscr{L} = \mathscr{L}_0 + \sigma(\varepsilon)\mathscr{G}$, where $\mathscr{G} := \mathscr{S} + \tilde{\mathscr{S}}$; $\mathscr{S}$ is the same as in (10), while $\tilde{\partial}_k := \partial/\partial r_k$ and

$$\tilde{\mathscr{S}} := \sum_{k\in\mathbb{Z}} \tilde{\mathscr{S}}_k, \quad \tilde{\mathscr{S}}_k\varphi(\omega) := (\nabla_1\tilde{\partial}_k - \nabla_1 V'(r_k))\,\nabla_1\tilde{\partial}_k\varphi(\omega). \tag{20}$$

It is not important that $\mathscr{S}$ and $\tilde{\mathscr{S}}$ are multiplied by identical factors σ; this is assumed for convenience only.

As in case of (9), the conservation of the total energy is violated by the noise; thus again $\{\lambda_{\pi,\gamma} : \pi, \gamma \in \mathbb{R}\}$ is the family of our stationary product measures. This follows easily from the usual finite volume approximation of the infinite dynamics. It is not necessary to refer to semigroup theory at this point. Indeed, the partial dynamics $\omega^{(n)}$, as defined above, preserve $\lambda_{\pi,\gamma}$, and this property remains in force after passing to the limit $n \to +\infty$. A converse statement is also true, but it will not be used in an explicit way because the consequences of our LSI are much stronger. The reversibility of $\mathscr{S}$ and $\tilde{\mathscr{S}}$ with respect to λ is found by integrating by parts. From $\lambda_{\pi,\gamma}(\partial_k\varphi) = \lambda_{\pi,\gamma}(\varphi p_k - \varphi\pi)$ and $\lambda_{\pi,\gamma}(\tilde{\partial}_k\varphi) = \lambda_{\pi,\gamma}(\varphi V_k' - \varphi\gamma)$, where the abbreviation $V_k' := V'(r_k)$ is also used, we get $\lambda_{\pi,\gamma}(\varphi\mathscr{L}_0\psi) = -\lambda_{\pi,\gamma}(\psi\mathscr{L}_0\varphi)$. Moreover

$$\begin{aligned} \int \varphi\mathscr{S}\psi\,\mathrm{d}\lambda_{\pi,\gamma} &= -\sum_{k\in\mathbb{Z}} \int (\nabla_1\partial_k\varphi)\,\nabla_1\partial_k\psi\,\mathrm{d}\lambda_{\pi,\gamma} \\ \int \varphi\tilde{\mathscr{S}}\psi\,\mathrm{d}\lambda_{\pi,\gamma} &= -\sum_{k\in\mathbb{Z}} \int \Big(\nabla_1\tilde{\partial}_k\varphi\Big)\,\nabla_1\tilde{\partial}_k\psi\,\mathrm{d}\lambda_{\pi,\gamma} \end{aligned} \tag{21}$$

for smooth functions φ and ψ of a finite number of variables. Let us remark that the operators

$$\mathscr{S}_{l,k} := \sum_{j=1}^{l-1} \mathscr{S}_{k-j} \quad \text{and} \quad \tilde{\mathscr{S}}_{l,k} := \sum_{j=1}^{l-1} \tilde{\mathscr{S}}_{k-j} \tag{22}$$

are *strictly elliptic* on the hyperplanes determined by $p_{k-l+1} + \cdots + p_k = l\pi$ and $r_{k-l+1} + \cdots + r_k = l\rho$, respectively.

At a level $\varepsilon > 0$ of scaling, $\mu_{t,\varepsilon}$ denotes the distribution of the evolved process, and $\mu_{t,n,\varepsilon}$ is the joint distribution of the variables $\{(p_k(t), r_k(t)) : |k| \leqq n\}$. Since we can not prove the uniqueness of the hydrodynamic limit, our only condition on the initial distribution $\mu_{0,\varepsilon}$ is that the entropy per site of $\mu_{0,n,\varepsilon}$ with respect to a reference measure λ is bounded. For simplicity choose $\lambda := \lambda_{0,0}$. We are assuming that $\mu_{0,n,\varepsilon} \ll \lambda$ and

$$S_n[\mu_{0,\varepsilon}|\lambda] := \int \log \frac{\mathrm{d}\mu_{0,n,\varepsilon}}{\mathrm{d}\lambda}\, \mathrm{d}\mu_{0,\varepsilon} \leqq Cn \tag{23}$$

for all $\varepsilon > 0$ and $n \in \mathbb{N}$ with the same constant C. Under this condition every translation invariant stationary state is a superposition of product measures $\lambda_{\pi,\gamma}$, see [13,15].

3.3. Block averages

We are going to develop compensated compactness at the microscopic level. This will be done in terms of block averages such that the size $l = l(\varepsilon)$ of blocks goes to infinity as the scaling parameter $\varepsilon > 0$ goes to 0. For any sequence η indexed by $\mathbb{Z}$, we define two kinds of block averages:

$$\bar{\eta}_{l,k} := \frac{1}{l}\sum_{j=0}^{l-1} \eta_{k-j} \quad \text{and} \quad \hat{\eta}_{l,k} := \frac{1}{l^2}\sum_{j=-l}^{l} \big||j| - l\big|\, \eta_{k+j}. \tag{24}$$

For example, $\bar{V}'_{l,k}$ denotes the corresponding averages of the sequence $V'_k = V'(r_k)$. Note that $\nabla_1 \hat{\eta}_{l,k} = (1/l)\nabla_l \bar{\eta}_{l,k} = (1/l)(\bar{\eta}_{l,k+l} - \bar{\eta}_{l,k})$. The arithmetic means $\bar{p}_{l,k}$ and $\bar{r}_{l,k}$ appear in canonical expectations. The more smooth averages $\hat{\eta}_{l,k}$ are used at the beginning of calculations. The empirical process, thus also the production of entropy, shall be defined by means of *mesoscopic blocks* of size $l = l(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. We are assuming that

$$\lim_{\varepsilon\to 0} \frac{\sigma(\varepsilon)}{\varepsilon l^3(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{l(\varepsilon)}{\sigma(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\sigma^2(\varepsilon)}{l^3(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon l^2(\varepsilon)} = 0, \tag{25}$$

where $\varepsilon l^3(\varepsilon) \geqq \sigma(\varepsilon) \geqq 1$ for convenience. Since $\varepsilon\sigma(\varepsilon) \to 0$ and $\varepsilon\sigma^2(\varepsilon) \to +\infty$, the last two relations of (25) are consequences of the first two conditions. The integer part of $\varepsilon^{-1/4}\sigma^{1/2}(\varepsilon)$ is an acceptable choice for $l = l(\varepsilon)$; in this case $\varepsilon l^3/\sigma \approx \sigma/l$.

Let $\hat{u}_\varepsilon$ denote the most frequently used *empirical process,* which is defined as

$$\hat{u}_\varepsilon(t,x) = \big(\hat{\pi}_\varepsilon(t,x), \hat{\rho}_\varepsilon(t,x)\big) := \sum_{k\in\mathbb{Z}} 1_{\varepsilon,k}(x)\, \big(\hat{p}_{l,k}(t/\varepsilon), \hat{r}_{l,k}(t/\varepsilon)\big)\,,$$

where the block size $l = l(\varepsilon)$ is specified according to (25). This $\hat{u}_\varepsilon$ is a process in space and time, while $\hat{\mathsf{P}}_\varepsilon$ denotes its distribution.

3.4. Young measure and measure-valued solutions

Several topologies can be introduced to study limit distributions of the empirical process. Besides the usual weak and strong local topologies of the L^p spaces, the notion of the *Young measure* is most useful because it leads to a fairly weak form of convergence of $\hat{\mathsf{P}}_\varepsilon$. Let $\mathscr{M}$ denote the space of locally finite Borel measures Θ on $\mathbb{R}^2_+ \times \mathbb{R}^2$ such that $\mathrm{d}\Theta = \mathrm{d}t\, \mathrm{d}x\, \theta_{t,x}(\mathrm{d}u)$, where $\theta = \{\theta_{t,x} : (t,x) \in \mathbb{R}^2_+\}$ is a measurable family of probability measures on $\mathbb{R}^2$, and equip $\mathscr{M}$ with the local weak topology of measures. Any realization of the empirical process can be represented as an element of $\mathscr{M}$ by choosing $\theta_{t,x}$ as the Dirac mass at $\hat{u}_\varepsilon(t,x)$. Thus $\hat{\mathsf{P}}_\varepsilon$ will be considered as a probability measure on $\mathscr{M}$. This family is obviously tight as $\varepsilon \to 0$, and $\mathscr{M}$ is of full measure with respect to any weak limit of $\hat{\mathsf{P}}_\varepsilon$, see Lemma 7 in Section 4. Of course, the Dirac property of the *Young family* $\theta_{t,x}$ may not remain in force after the limit. A Young family is called a *measure-valued solution* to (1), see (5) if

$$\int_0^\infty \int_{-\infty}^\infty \left(\psi_t'(t,x) \cdot \theta_{t,x}(u) + \psi_x'(t,x) \cdot \theta_{t,x}(\Phi(u))\right) \mathrm{d}x\, \mathrm{d}t = 0 \qquad (26)$$

for all $\psi \in C^1_{\mathrm{co}}(\mathbb{R}^2_+ \to \mathbb{R}^2)$; the initial value of θ is missing because the question of uniqueness is not discussed. A Young family θ represents a weak solution if, and only if $\theta_{t,x}$ is Dirac almost everywhere.

The existence of measure-valued solutions follows easily by the strong ergodicity of the microscopic dynamics. The crucial step consists of the replacement of block averages $\bar{V}'_{l,k}$ of the nonlinear part of the microscopic current induced by $\mathscr{L}_0$ with its *canonical equilibrium expectation* $S'(\bar{r}_{l,k})$. Since $\varepsilon\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$, the contribution of the random perturbation vanishes in the hydrodynamic limit. In the proof of Theorem 1, the following statement will be referred to, for a stronger version see Remark 3 after the proof in Section 6.

Proposition 1. *The family* $\{\hat{\mathsf{P}}_\varepsilon : \varepsilon > 0\}$ *is tight with respect to the local weak topology of the set* $\mathscr{M}$ *of Young measures, and any of its limit points, that is* $\hat{\mathsf{P}} := \lim \hat{\mathsf{P}}_{\varepsilon(n)}$ *as* $\varepsilon(n) \to 0$ *is concentrated on a set of measure-valued solutions.*

The main problem now is the Dirac property of the limiting Young measure. Since $\varepsilon\sigma(\varepsilon) \to 0$, we can not replace mesoscopic block averages with small macroscopic block averages of block size $l = \delta/\varepsilon$ with $\delta \to 0$ after $\varepsilon \to 0$, see Remark 1. That is why we say that a direct compactness argument is not available under hyperbolic scaling.

3.5. Compensated compactness

Due to this powerful theory, in some cases the Dirac property of the limiting Young measure can be derived from a "compensated" factorization property, the *Div–Curl Lemma* stating that

$$\theta_{t,x}(h_1 J_2) - \theta_{t,x}(h_2 J_1) = \theta_{t,x}(h_1)\theta_{t,x}(J_2) - \theta_{t,x}(h_2)\theta_{t,x}(J_1) \quad \text{a.s.} \qquad (27)$$

for certain couples (h_1, J_1) and (h_2, J_2) of entropy pairs. As a consequence of the Div–Curl Lemma, it was shown by DiPerna [7] that oscillations of uniformly bounded viscid solutions die out; thus pointwise convergence takes place along subsequences. In order to apply these techniques, we ought to show that $\hat{\mathsf{P}}$ of Proposition 1 is concentrated on a set of compactly supported Young measures $\theta_{t,x}$ with a common support. At this time, we have no answer to this question; therefore the approach of Shearer [30] and Serre–Shearer [29][5] is most useful for us. They managed to find some clever families of Lax entropy pairs, which work in the case of local L^p bounds, too.[6] As a consequence of the compensated compactness, we shall prove later that the weak convergence in Proposition 1 turns into a strong one.

The assumptions of this approach for S' have been listed and discussed in Section 3.1. Some sufficient conditions in terms of V are to be verified in Section 4.1. Genuine nonlinearity is the crucial, most problematic question. In fact we have presented explicit examples in Section 3.1, and we shall show in Section 4.1 that certain local perturbations of these concrete models also result in (GN1) or (GN2). Our main result reads as follows.

Theorem 1. *Suppose that our potential $V \in C^2(\mathbb{R})$ is strictly convex in the sense that $c_1 \leqq V''(r) \leqq c_2$ for all $r \in \mathbb{R}$, V satisfies (AN), and either the condition (GN1) or (GN2) of genuine nonlinearity holds true. Then the entropy bound* (23) *on the initial distribution of our system* (13) *implies that $\{\hat{\mathsf{P}}_\varepsilon : \varepsilon > 0\}$ is a tight family with respect to the local strong topology of $L^p(\mathbb{R}^2_+)$ if $1 \leqq p < 2$, and any limit distribution $\hat{\mathsf{P}}$ of $\hat{u}_\varepsilon$, which is obtained as $\varepsilon \to 0$ along some subsequence, is concentrated on a set of weak solutions to the p-system* (1).

The proof consists of several steps. First we treat Proposition 1 showing that limit distributions $\hat{\mathsf{P}}$ of the Young representation of the empirical process are all concentrated on a set of measure-valued solutions. Then we show that the Div–Curl Lemma, that is (27), holds true with probability one with respect to any limit distribution $\hat{\mathsf{P}}$. Finally, applying the Div–Curl Lemma with the clever entropy pairs of [29,30], the Dirac property of the limiting Young measure follows from the results of Shearer and Serre.

3.6. Entropy production

The proof of the Div–Curl Lemma is based on the evaluation of entropy production $X = \partial_t h + \partial_x J$ for Lax entropy pairs (h, J). Its *mesoscopic* version is

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[5] Although both Lu [22] and Serre [28] are surveying [29], and there are many other references on it, as far as I know, a complete proof has not been published yet.

[6] References [29,30] derive (27) from the energy inequalities for the viscid approximation (2), here we follow the robust method of [19]. Calculations are done in the space of probability measures on the configuration space Ω_0, and the a priori bounds for the proof of (27) are based on estimates using probabilistic entropy and its rate of production. Since we have local estimates of this kind, the finiteness of the initial energy (entropy) is not assumed.

defined as

$$X_\varepsilon(\psi, h) := -\int_0^\infty \int_{-\infty}^\infty \left(\psi_t'(t,x)h(\hat{u}_\varepsilon) + \psi_x'(t,x)J(\hat{u}_\varepsilon)\right) \mathrm{d}x\,\mathrm{d}t, \tag{28}$$

where $\psi \in C^1_{\mathrm{co}}(\mathbb{R}^2_+ \to \mathbb{R})$. In view of the Ito lemma, the stochastic differential equation $m_\varepsilon(\mathrm{d}t, x; h) := \mathrm{d}h(\hat{u}_\varepsilon) - (1/\varepsilon)\mathscr{L}h(\hat{u}_\varepsilon)\,\mathrm{d}t$ defines a martingale $m_\varepsilon(t, x, h)$ for each x such that

$$M_\varepsilon(\psi, h) := \int_{-\infty}^\infty \int_0^\infty \psi(t,x)\, m_\varepsilon(\mathrm{d}t, x, h)\,\mathrm{d}x \tag{29}$$

turns out to be the random component of entropy production:

$$\begin{aligned} X_\varepsilon(\psi, h) = \frac{1}{\varepsilon}\int_0^\infty \int_{-\infty}^\infty \psi(t,x)\mathscr{L}h(\hat{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + M_\varepsilon(\psi, h) \\ + \int_0^\infty \int_{-\infty}^\infty \psi(t,x)\left(J_\pi'(\hat{u}_\varepsilon)\nabla_\varepsilon\hat{\pi}_\varepsilon - J_\rho'(\hat{u}_\varepsilon)\nabla_\varepsilon^*\hat{\rho}_\varepsilon\right)\mathrm{d}x\,\mathrm{d}t + N_\varepsilon(\psi, h), \end{aligned} \tag{30}$$

where $\nabla_\varepsilon\varphi(x) := (1/\varepsilon)\left(\varphi(x+\varepsilon) - \varphi(x)\right)$, the adjoint ∇_ε^* of ∇_ε is defined as $\nabla_\varepsilon^*\varphi(x) := -\nabla_\varepsilon\varphi(x-\varepsilon)$, finally N_ε is the numerical error due to this discretization of the space derivative.

3.7. The stochastic Div–Curl Lemma

To prove (27), we are looking for a decomposition $X_\varepsilon = Y_\varepsilon + Z_\varepsilon$, where Y_ε vanishes in H^{-1}, while Z_ε is bounded in the space of measures, both in a local sense. More precisely, let $\mathscr{H}_1$ denote the set of entropy pairs (h, J) such that the first and second derivatives of h and J are all bounded, $(h, J) \in \mathscr{H}_0 \subset \mathscr{H}_1$ means that h and J are bounded, too. $H^{+1} \subset L^2(\mathbb{R}^2_+)$ is a Hilbert space with norm $\|\cdot\|_+$, $\|\psi\|_+^2 := \|\psi\|_2^2 + \|\psi_t'\|_2^2 + \|\psi_x'\|_2^2$, and H^{-1} is its dual with respect to L^2. We say that $(h, J) \in \mathscr{H}_1$ is a *well controlled entropy pair* if its entropy production decomposes as $X_\varepsilon(\psi, h) = Y_\varepsilon(\psi, h) + Z_\varepsilon(\psi, h)$, and we have two random functionals $A(\phi, h)$ and $B(\phi, h)$ such that

$$|Y_\varepsilon(\psi\phi, h)| \leqq A_\varepsilon(\phi)\|\psi\|_+ \quad \text{while} \quad |Z_\varepsilon(\psi\phi, h)| \leqq B_\varepsilon(\phi)\|\psi\|$$

for all $\psi, \phi \in C^1_{\mathrm{co}}(\mathbb{R}^2_+ \to \mathbb{R})$, $A_\varepsilon(\phi)$ and $B_\varepsilon(\phi)$ do not depend on ψ, finally

$$\lim_{\varepsilon\to 0} \mathsf{E}A_\varepsilon(\phi) = 0 \quad \text{and} \quad \limsup_{\varepsilon\to 0} \mathsf{E}B_\varepsilon(\phi) < +\infty.$$

The role of ϕ is to localize bounds. The stochastic version of the Div–Curl Lemma reads as follows:

Proposition 2. *Suppose that (h_1, J_1) and (h_2, J_2) are well-controlled entropy pairs of class $\mathscr{H}_0$ and that $\mathsf{E}\|\phi\hat{u}_\varepsilon\|_2^2$ is a bounded function of $\varepsilon > 0$ for each bounded $\phi : \mathbb{R}^2_+ \mapsto \mathbb{R}$ of compact support. Then the distributions of the Young measure are tight on $\mathscr{M}$, and (27) holds true almost surely with respect to any of the possible limit distributions $\hat{\mathsf{P}} = \lim \hat{\mathsf{P}}_\varepsilon$ obtained as $\varepsilon \to 0$ along some subsequence.*

The proof of this statement will be presented in Section 6. It is not difficult to find by means of the Skorohod embedding theorem, as it reduces to the original argument of Tartar and Murat. Our main task is to verify the conditions of Proposition 2. Several technical details of this argument go back to the papers [13,16]. These ideas are adapted to the present situation.[7]

Since a nontrivial entropy is not preserved at the microscopic level, its control requires some non-gradient tools. In contrast to parabolic systems, severe oscillations are generated also by the deterministic part of the evolution. In the scaling limit the microscopic time is accelerated as $t = \tau/\varepsilon$, where $\varepsilon \to 0$. Consequently the strong ergodic hypothesis is not sufficient to control the rapid oscillations of $(1/\varepsilon)\mathscr{L}_0 h(\hat{u}_\varepsilon)$; this is the most problematic term of X_ε. Nevertheless a full non-gradient analysis, see [20,33] for technical details, is not needed because the problem of entropy production is formulated in terms of mesoscopic block averages. As an example, the oscillations and their cancelation can be demonstrated in the case of the total energy as follows:

$$\begin{aligned}\mathscr{L}_0\left(\hat{p}_{l,k}^2/2 + S(\hat{r}_{l,k})\right) &= \hat{p}_{l,k}\left(\hat{V}'_{l,k} - \hat{V}'_{l,k-1}\right) + S'(\hat{r}_{l,k})(\hat{p}_{l,k+1} - \hat{p}_{l,k})\\ &= (\hat{p}_{l,k+1} - \hat{p}_{l,k})(S'(\hat{r}_{l,k}) - \hat{V}'_{l,k}) + \hat{p}_{l,k+1}\hat{V}'_{l,k} - \hat{p}_{l,k}\hat{V}'_{l,k-1},\end{aligned}$$

where $\hat{V}'_{l,k}$ is the microscopic flux of $\hat{p}_{l,k}$ induced by $\mathscr{L}_0$. Due to the logarithmic Sobolev inequality and $\varepsilon\sigma^2(\varepsilon) \to +\infty$, we have an explicit bound for the deviation of $\hat{V}'_{l,k}$ from its macroscopic counterpart $S'(\hat{r}_{l,k})$. The factor $\nabla_1\hat{p}_{l,k}$ can be controlled in a more direct way; thus it is possible to show that the space-time averages of the right-hand side above vanish in a mean sense. As a consequence, the standard energy inequality for the viscid approximation (2) of the p-system can be reproduced as a uniform bound:

$$\int_{-r}^{r} \mathsf{E}|\hat{u}_\varepsilon(\tau,x)|^2\,\mathrm{d}x + \varepsilon\sigma(\varepsilon)\int_0^\tau\int_{-r}^{r}\mathsf{E}|\nabla_\varepsilon\hat{u}_\varepsilon(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \leqq C(\tau,r) \tag{31}$$

for all $\varepsilon > 0$, where $C(\tau,r) < +\infty$ if $\tau, r < +\infty$. The oscillations of other entropies are controlled in a similar, but more complicated way.

Most ideas and many technical details of the forthcoming proofs in Sections 4–6 go back to [16–18]. In contrast to the GINZBURG–LANDAU model of [16], our system admits two conservation laws; thus the a priori bounds on the empirical process are more relevant than in [16]. Also the asymmetric component $\mathscr{L}_0$ of our generator is less trivial, but the main difference consists of the requirement of genuine nonlinearity, (GN1) or (GN2). Since [16] does not contain all technical details, for Reader's convenience all arguments are outlined, and complete proofs are presented if necessary.

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[7] The strict convexity of V is unnecessary for the derivation of the conditions of the stochastic Div–Curl Lemma above. As in [16], the logarithmic Sobolev inequality of LANDIM–PANIZO–YAU [21] works quite well at this point. However, V should be a bounded perturbation of the quadratic function in that case, which excludes the condition of genuine nonlinearity of [7,29,30], see Sections 3.1 and 4.1.

4. Evaluation of the macroscopic and microscopic flux

In this section we summarize information we need in developing the stochastic theory of compensated compactness for our model (13). Concerning definitions and some simple relations, we refer to Sections 2.4 and 3. Unless stated otherwise, $V \in C^2(\mathbb{R})$, and the condition $0 < c_1 \leqq V''(r) \leqq c_2$ is assumed throughout this section.

4.1. Basic properties of the macroscopic flux

Some facts on F and S are summarized, and we follow Section 3.1. Taking into account the explicit form $\exp(\gamma r - V(r) - F(\gamma))$ of the marginal density of r_k under $\lambda_{\pi,\gamma}$, we obtain a useful integration by parts formula:

$$\bar{\varphi}(\gamma) := \partial_\gamma \int \varphi(r_k)\,\mathrm{d}\lambda_{\pi,\gamma} = \int \varphi(r_k)(V_k' - \gamma)\,\mathrm{d}\lambda_{\pi,\gamma} = \int \varphi'(r_k)\,\mathrm{d}\lambda_{\pi,\gamma},$$

where $V_k' := V'(r_k)$. We see that $\bar{\varphi}$ is an increasing function of $\gamma \in \mathbb{R}$ if the right-hand side above is positive. As a first easy consequence, we prove

Lemma 1. *We have $c_2^{-1} \leqq F''(\gamma) \leqq c_1^{-1}$ for all $\gamma \in \mathbb{R}$, thus $c_1 \leqq S''(\rho) \leqq c_2$ for all $\rho \in \mathbb{R}$, while $G(\beta,\gamma) := \log \lambda_{\pi,\gamma}(\exp V_k') \leqq \gamma\beta + c_2\beta^2/2$.*

Proof. Recall that $\rho = F'(\gamma) = \lambda_{\pi,\gamma}(r_k)$; thus

$$1 = \int (r_k - \rho)(V'(r_k) - \gamma)\,\mathrm{d}\lambda_{\pi,\gamma} = \int (r_k - \rho)(V'(r_k) - V'(\rho))\,\mathrm{d}\lambda_{\pi,\gamma}.$$

Since $V'(r) - V'(\rho) = V''(\tilde{r}_k)(r - \rho)$ with some intermediate value $\tilde{r}_k$, while $F''(\gamma)$ is just the variance of r_k, this proves our bounds on F'' and on $S'' = 1/F''$, too.

In the case of G, again by partial integration, we get

$$\begin{aligned} G_\beta'(\beta,\gamma) &= \int_{-\infty}^{\infty} V'(r)\exp(\beta V'(r) + \gamma r - V(r) - G(\beta,\gamma) - F(\gamma))\,\mathrm{d}r \\ &= \int_{-\infty}^{\infty} (\beta V''(r) + \gamma)\exp(\beta V'(r) + \gamma r - V(r) - G(\beta,\gamma) - F(\gamma))\,\mathrm{d}r; \end{aligned}$$

thus $G_\beta'(\beta,\gamma) \leqq \gamma + c_2\beta$ if $\beta > 0$, while $G_\beta'(\beta,\gamma) \geqq \gamma + c_2\beta$ if $\beta < 0$, which imply the desired bound of G because $G(0,\gamma) = 0$. □

Observe now that ρ and δ are close to each other and recall that δ is defined by $\gamma = V'(\delta)$, that is the exponent $\gamma r - V(r)$ is maximal if $r = \delta$. Integrating by parts we get

$$1 = \int (r_k - \delta)(V_k' - \gamma)\,\mathrm{d}\lambda_{\pi,\gamma} = \int V''(\tilde{r}_k)(r_k - \delta)^2\,\mathrm{d}\lambda_{\pi,\gamma}$$

with some intermediate value $\tilde{r}_k$, whence

$$\int (r_k - \delta)^2\,\mathrm{d}\lambda_{\pi,\gamma} = F''(\gamma) + (\rho - \delta)^2 \leqq 1/c_1,$$

that is $|\rho - \delta|$ is a bounded function of γ.

Lemma 2. *Both S''' and $S^{(iv)}$ are bounded functions of $\rho \in \mathbb{R}$.*

Proof. By partial integration, we obtain that

$$3\int (r_k-\delta)^2\,\mathrm{d}\lambda_{\pi,\gamma} = \int (r_k-\delta)^3(V_k'-\gamma)\,\mathrm{d}\lambda_{\pi,\gamma} = \int V''(\tilde{r}_k)(r_k-\delta)^4\,\mathrm{d}\lambda_{\pi,\gamma}$$

with some $\tilde{r}_k \in \mathbb{R}$, while $(r-\rho)^4 \leqq 8(r-\delta)^4 + 8(\rho-\delta)^4$, thus $\lambda_{\pi,\gamma}(r_k-\rho)^4$ is a bounded function of $\gamma \in \mathbb{R}$, whence $F''' \in L^\infty(\mathbb{R})$ follows from Schwarz. The proof is completed by (14) and Lemma 1. □

To see the square integrability of S''' and $S^{(iv)}$, an asymptotic calculation is needed. In view of (14) the corresponding derivatives of F should be evaluated, and from (AN) we get exponential bounds when $\gamma \to \pm\infty$. Indeed, following Laplace we expand V around δ to conclude

$$\exp\left(F(\gamma) - \gamma\delta + V(\delta)\right) = \int_{-\infty}^{\infty} \exp(-V''(\tilde{r})(r-\delta)^2/2)\,\mathrm{d}r$$

with some intermediate value $\tilde{r}$. This implies immediately that the left-hand side is bounded, and it is bounded away from zero. The evaluation of F and its derivatives requires an asymptotic expansion of the first four centered moments of the r_k variables as $|\gamma| \to +\infty$.

Lemma 3. *Assume (AN), then S''' and $S^{(iv)}$ are both square integrable.*

Proof. We may assume that $\gamma \to +\infty$, then $\delta \to +\infty$, because $\rho = F'(\gamma)$ and $|\rho - \delta|$ are bounded. Splitting the integral above into three parts: for $r < \delta - a$, $r > \delta + a$ and $|r-\delta| \leqq a$, by a direct computation, it follows that

$$\left|\exp\left(F(\gamma) - \gamma\delta + V(\delta)\right) - \sqrt{2\pi/V_+''}\right| \leqq \mathrm{e}^{-\tilde{\alpha}\gamma}$$

with some positive $\tilde{\alpha} < \alpha$, provided that γ; thus also δ are large enough. Let $a \in (0,\gamma)$ be a multiple of γ, for $r > \delta + a$ we can use the standard Gaussian bound:

$$a\beta \int_{\delta+a}^{\infty} \exp(-\beta^2(r-\delta)^2/2)\,\mathrm{d}r \leqq \exp(-\beta^2 a^2/2) \quad \text{if } a, \beta > 0\,; \qquad (32)$$

the case of $r < \delta - a$ is the same. In view of (AN), on the interval $|r - \delta| \leqq a$, we have $|V''(\tilde{r}) - V_+''|(r-\delta)^2 \leqq \mathrm{e}^{-\tilde{\alpha}\gamma}$ with a new constant $\tilde{\alpha} > 0$ if necessary, which completes this part of the proof. The same decomposition is applied in the forthcoming computation of various centered moments of r_k. There is no problem if $r > \delta + a$ or $r < \delta - a$ because for $m \leqq 4$ we can find some positive constants $\bar{R}$ and β such that $\beta^2 < c_1$ and

$$|r-\rho|^m \exp\left(-V''(\tilde{r})(r-\delta)^2/2\right) \leqq \exp\left(-\beta^2(r-\delta)^2/2\right) \quad \text{if } \gamma > \bar{R}.$$

Indeed, remember that $|\rho - \delta|$ is bounded, while $\rho = F'(\gamma)$ admits linear upper and lower bounds in terms of γ. On the central interval we must be more careful.

Next by treating the integral

$$\rho = \int_{-\infty}^{\infty} r\ \exp\left(\gamma r - V(r) - F(\gamma)\right)\,\mathrm{d}r$$
$$= \int_{-\infty}^{\infty} r\ \exp\left(\gamma\delta - V(\delta) - F(\gamma) - V''(\tilde r)(r-\delta)^2/2\right)\,\mathrm{d}r$$

in the same way as above, we obtain that $|\rho - \delta| \to 0$ at an exponential rate as $\gamma \to +\infty$. Similarly, from

$$F''(\gamma) + (\rho-\delta)^2 = \int_{-\infty}^{\infty} (r-\delta)^2 \exp\left(\gamma r - V(r) - F(\gamma)\right)\,\mathrm{d}r$$

we get $F''(\gamma) \to (1/V''_+)\sqrt{2\pi}$ as $\gamma \to +\infty$, and the rate of convergence is exponential again. Since $F'''(\gamma)$ is just the third centered moment of r_k under $\lambda_{\pi,\gamma}$, which is zero in the Gaussian case, we have $F'''(\gamma) \to 0$ at an exponential rate if $\gamma \to +\infty$. On the central interval $(\delta - a, \delta + a)$, we write

$$(r-\rho)^3 = (r-\delta)^3 + (\delta-\rho)\left((r-\rho)^2 + (r-\rho)(r-\delta) + (r-\delta)^2\right),$$

and check that the contribution of each term goes to zero at an exponential rate.

The case of $F^{(iv)}$ is a bit more complicated because it is the difference of two non-vanishing terms, see (16). Since $F''(\gamma) \approx (1/V''_+)\sqrt{2\pi}$, we have to compute

$$M_4 := \int_{-\infty}^{\infty} (r-\rho)^4 \exp\left(\gamma r - V(r) - F(\gamma)\right)\,\mathrm{d}r.$$

$(r-\rho)^4 = (r-\delta)^4 + (\delta-\rho)(2r-\rho-\delta)(r-\rho)^2 + (\delta-\rho)(2r-\rho-\delta)(r-\delta)^2$; thus we see that $M_4 \to 6\pi(V''_+)^{-2}$ at an exponential rate as $\gamma \to +\infty$. This is the contribution of the first term on the central interval. The contribution of the rest does vanish faster, which completes the proof via (16). □

Let us consider now the problem of genuine nonlinearity for small perturbations of explicitly tractable models. First we describe situations when S' happens to be strictly convex or concave. If $b > 0$, $b+c > 0$ and $V(r) = br^2/2 + W(r) + cr^2/2$, then by the usual integration by parts trick from

$$\int (r_k-\rho)^2 (V'(r_k) - \gamma)\,\mathrm{d}\lambda_{\pi,\gamma} = 2\int (r_k-\rho)\,\mathrm{d}\lambda_{\pi,\gamma} = 0$$

we get

$$-\,bF'''(\gamma) = \int (r_k-\rho)^2 (W'(r_k) + cr - \bar W'(\gamma) - c\rho)\,\mathrm{d}\lambda_{\pi,\gamma}, \tag{33}$$

where $\bar W'(\gamma) = \gamma - (b+c)\rho$ denotes the expectation of $W'(r_k)$ under $\lambda_{\pi,\gamma}$. Note that in this setting $U(r) = W(r) + cr^2/2$ is considered as the perturbation of $br^2/2$. The nonlinear term W may contain a quadratic component, $c < 0$ is also allowed, and we can redistribute the quadratic parts by changing c in such a way that the value of $b+c$ is not altered. This remark will be used to simplify the next coming trivial but tedious computations.

Lemma 4. *Suppose that* $V(r) = r^2/2 + \vartheta U(r)$, *where* U' *is convex,* $\|U'''\| < +\infty$, *and we have an* $R > 0$ *such that* $U'''(r) = 0$ *if* $|r| > R$, *while* $U''(R) \neq U''(-R)$. *Then there exists a constant* $\vartheta_u > 0$ *depending on* U *such that* $0 < |\vartheta| < \vartheta_u$ *implies* (*GN1*).

Proof. As the statement is a bit technical, we are having in mind that U' is a smooth and convex modification of the function $y = \kappa|r|_+$, where $|r|_+ := \max\{0, r\}$ denotes the positive part of r, and $|\kappa| < 1/2$. In view of the previous identity and remark, we may, and do assume that $U'(r) = 0$ if $r < -R$ because U can be replaced by $U(r) - U''(-R)r^2/2$.

Suppose now for a moment that $V(r) = (1-\kappa)r^2/2 + \kappa r|r|_+/2$ with $0 < \kappa < 1/2$; the associated free energy is denoted by $F_1(\gamma)$ in this case, and $\rho_1 := F_1'(\gamma)$. Since

$$\int_0^\infty r e^{\gamma r - r^2/2}\,\mathrm{d}r = 1 + \gamma\,\Phi(\gamma)e^{\gamma^2/2},$$

where $\Phi(\gamma) := \int_{-\gamma}^\infty \exp(-x^2/2)\,\mathrm{d}x$, and

$$\int_0^\infty r(r-\rho)^2 \exp(\gamma r - r^2/2)\,\mathrm{d}r = 2 + (\gamma - \rho_1)^2 + \left(3\gamma - 2\rho_1 + \gamma(\gamma - \rho_1)^2\right)\Phi(\gamma)e^{\gamma^2/2},$$

we obtain

$$-F_1'''(\gamma) = \kappa\,\exp(-F_1(\gamma))\left(2 - F''(\gamma) + (\gamma - \rho_1)^2\right) + \kappa\,\exp(-F_1(\gamma))\left(\gamma(3 - F_1''(\gamma)) - 2\rho + \gamma(\gamma - \rho_1)^2\right)\Phi(\gamma)e^{\gamma^2/2}$$

by a direct computation. Observe that (32) applies if $\gamma < 0$; thus from $(1-\kappa\rho_1) < \gamma$ and $1 < F_1''(\gamma) < 1/(1-\kappa) < 2$ we get a constant $\eta_\kappa > 0$ such that $-F_1'''(\gamma) \geqq \eta_\kappa \exp(-F_1(\gamma))$. The case of $\gamma > 0$ seems to be more complicated, but this unpleasant task can be avoided. In view of our remark on the redistribution of the quadratic components of V, we have the same lower bound for $-F_1'''(\gamma)$ even if $\gamma > 0$.

The treatment of the general case consists of two parts. Assuming $V'(r) = (1-\kappa)r + \kappa|r|_+ + \kappa W'(r)$, first we do an asymptotic calculation, similar to that in the proof of Lemma 3. We have

$$-(1-\kappa)F'''(\gamma) = \kappa \int_{-\infty}^\infty (r-\rho)^2\,(|r|_+ - \bar{r}(\gamma))\exp(\gamma r - V(r) - F(\gamma))\,\mathrm{d}r + \kappa \int_{-\infty}^\infty (r-\rho)^2\left(W(r) - \bar{W}(\gamma)\right)\exp(\gamma r - V(r) - F(\gamma))\,\mathrm{d}r,$$

where $\bar{r}$ and $\bar{W}$ denote the expectations of r_k and $W(r_k)$ with respect to $\lambda_{\pi,\gamma}$, whence $F'''(\gamma) \leqq (\eta_1 + \eta_2/|\gamma|)\exp(-\alpha\gamma^2)$ follows if $|\gamma|$ is large enough, where $\alpha > 0$ and $\eta_1 < 0$. Indeed, as W is compactly supported, $(1/\gamma)\exp(-\alpha\gamma^2)$ is the order of the second integral. The substitution of the first integral by a multiple of

$$\int_{-\infty}^\infty (r-\rho_1)^2\,(|r|_+ - \bar{r}_1(\gamma))\exp(\gamma r - (1-\kappa)r^2/2 - \kappa r|r|_+/2 - F_1(\gamma))\,\mathrm{d}r$$

results in an error of the same order, consequently $\eta_1 < 0$ and $\alpha > 0$.

In terms of the original formulation this means that we have a constant Γ_U depending on $\|U''\|$ such that $F'''(\gamma) < 0$ if $|\gamma| > \Gamma_U$ and $\vartheta > 0$, while $F'''(\gamma) > 0$ if $|\gamma| > \Gamma_U$ and $\vartheta < 0$. To complete the proof we show that $F''(\gamma)$ is strictly decreasing on the interval $(-\Gamma_U, \Gamma_U)$ if $\vartheta > 0$, while it is strictly increasing if $\vartheta < 0$. By the usual integration by parts trick, it follows that

$$F''(\gamma) = 1 + \vartheta^2 \int (U'(r_k) - \bar{U}'(\gamma))^2 \,\mathrm{d}\lambda_{\pi,\gamma} + \vartheta \int U''(r_k)\,\mathrm{d}\lambda_{\pi,\gamma},$$

where $\bar{U}'(\gamma)$ is the mean value of $U'(r_k)$. Since U' is convex by assumption, the second integral on the right-hand side is an increasing function of γ. On the other hand, the variance of U' is bounded; therefore F'' is really a strictly monotonic function on the interval $(-\Gamma_U, \Gamma_U)$ if $\vartheta \neq 0$ is small. $\square$

In the symmetric case we choose $V(r) := r^2 + U_\nu(r) + \vartheta W(r)$, U_ν is as in (17) and (19), and W is a symmetric and compactly supported C^2 function. We have

Lemma 5. *Suppose that $\tilde{\kappa}_j \geqq \alpha > 0$ in the definition of U_ν, then there exists a constant $\vartheta_w > 0$ such that $|\vartheta| < \vartheta_w$ implies (GU2).*

Proof. It is easy because if $|\gamma|$ is large then the effect of W can be neglected, while on a fixed interval the contribution of U_ν dominates that of W. $\square$

4.2. Relative entropy and its Dirichlet form

This section summarizes some estimates based on measure-theoretical entropy and its rate of production. The basic ideas go back to [13,19], see also [16]. Since those models are different from our, and here we need fairly explicit bounds, we are presenting the main steps of the arguments. The relative entropy of a probability measure μ on a probability space $(\Omega, \mathscr{F}, \lambda)$ is defined as

$$S[\mu|\lambda, \mathscr{F}] := \sup_{\varphi} \{\mu(\varphi) - \log \lambda(\mathrm{e}^{\varphi}) : \varphi \sim \mathscr{F}, \lambda(\mathrm{e}^{\varphi}) < +\infty\},$$

$\varphi \sim \mathscr{F}$ indicates measurability of φ with respect to $\mathscr{F}$. It is easy to see that $S[\mu|\lambda, \mathscr{F}] < +\infty$ implies $\mu \ll \lambda$ and $S[\mu|\lambda, \mathscr{F}] = \mu(\log f)$, where $f := \mathrm{d}\mu/\mathrm{d}\lambda$. Another variational formula,

$$D[\mu|\mathscr{G}, \mathscr{F}] := \sup_{\psi > 0} \left\{ - \int \frac{\mathscr{G}\psi}{\psi} \,\mathrm{d}\mu : \psi \sim \mathscr{F}, \psi \in \mathrm{Dom}(\mathscr{G}) \right\}$$

defines the *rate function* of Donsker and Varadhan, where $\mathscr{G}$ is the generator of a Markov process in $(\Omega, \mathscr{F})$. Concerning basic facts and technical details, we refer to [8] and [20] with further references.

Both S and D are nonnegative, convex and lower semi-continuous functionals of μ, thus $S[\mu|\lambda, \mathscr{F}] \leqq S[\mu|\lambda, \mathscr{A}]$ and $D[\mu|\mathscr{G}, \mathscr{F}] \leqq D[\mu|\mathscr{G}, \mathscr{A}]$ if $\mathscr{F} \subset \mathscr{A}$. Subadditivity is again a direct consequence of the definition: $S[\mu_1|\lambda, \mathscr{F}_1] + S[\mu_2|\lambda, \mathscr{F}_2] \leqq S[\mu|\lambda, \mathscr{F}]$ if λ is a product measure on $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$ and $\mu_i, i = 1, 2$ denotes the restriction of μ to $\mathscr{F}_i$. Again from the definition, we have

$D[\mu|\mathscr{G}_1,\mathscr{F}_1]+D[\mu|\mathscr{G}_2,\mathscr{F}_2]\leqq D[\mu|\mathscr{G}_1+\mathscr{G}_2,\mathscr{F}]$ in the same situation. It is less obvious that if $\mathscr{G}$ is self-adjoint in $L^2(\lambda,\mathscr{F})$, $\mathrm{d}\mu = f\,\mathrm{d}\lambda$ and $D[\mu|\mathscr{G},\mathscr{F}]<+\infty$, then $\sqrt{f}\in\mathrm{Dom}(\mathscr{G})$ and D is a *Dirichlet form:* $D[\mu|\mathscr{G},\mathscr{F}] := -\lambda(\sqrt{f}\,\mathscr{G}\sqrt{f})$, that is $\psi=\sqrt{f}$ is the optimal choice in the definition of D. Moreover, if λ denotes a stationary state of the process generated by $\mathscr{G}$ in $(\Omega,\mathscr{F})$ and μ_t is the evolved measure, then $S[\mu_t|\lambda,\mathscr{F}]$ is a non-increasing function of time, and its rate of change can be estimated by $D[\mu_t|\mathscr{G},\mathscr{F}]$ in many cases. For example, if $\mathscr{G}$ generates a partial dynamics of (13), then $\partial_t S+4\sigma(\varepsilon)D\leqq 0$, see the forthcoming calculations.

The *entropy inequality:* $\mu(\varphi)\leqq S[\mu|\lambda,\mathscr{F}]+\log\lambda(\mathrm{e}^{\varphi})$ follows immediately from the definition of S if $\varphi\sim\mathscr{F}$, and this universal bound extends to conditional distributions, too. This is important because entropy of the conditional distribution given the block average $\bar{r}_{l,k}$ is well controlled via LSI for the *canonical measure* as follows. Due to $\sigma(\varepsilon)\to+\infty$ as $\varepsilon\to 0$, these bounds are more effective than those we can get directly from the entropy inequality. From now on the notation of Section 3 is used, too.

4.3. The logarithmic Sobolev inequality

For $m\in\mathbb{N}$ and $k\in\mathbb{Z}$ let $\mathscr{F}_{m,k}$ denote the σ-field of Ω_0 generated by the variables $\{(p_j,r_j): k-m<j\leqq k\}$, and $\mu_{m,k}$ is the restriction of a probability measure μ to $\mathscr{F}_{m,k}$. Given $\bar{p}_{m,k}=p$ and $\bar{r}_{m,k}=r$, the conditional distributions of $\lambda=\lambda_{0,0}$ and μ on $\mathscr{F}_{m,k}$ are denoted as $\bar{\lambda}^{p,r}_{m,k}$ and $\bar{\mu}^{p,r}_{m,k}$, respectively. These are the so-called *canonical distributions.* Note that the canonical distributions of the stationary product measures $\lambda_{\pi,\rho}$ do not depend on the parameters (π,ρ). Since V is strictly convex, the Bakry–Emery criterion implies the following LSI, see the monograph [8] by Deuschel and Stroock with references on the original sources.

Proposition 3. *There exists a universal constant C_{lsi} depending only on V such that for $\mathscr{G}_{m,k}:=\mathscr{S}_{m,k}+\tilde{\mathscr{S}}_{m,k}$, we have*

$$\int S\left[\bar{\mu}^{\pi,\rho}_{m,k}\middle|\bar{\lambda}^{\pi,\rho}_{m,k},\mathscr{F}_{m,k}\right]\mathrm{d}\mu\leqq m^2\,C_{lsi}\,D\left[\mu\middle|\mathscr{G}_{m,k},\mathscr{F}_{m,k}\right]$$

for any probability measure μ on $\mathscr{F}_{m,k}$ and $m>2$, $k\in\mathbb{Z}$, $\pi,\rho\in\mathbb{R}$.

Proof. Since the exponent of the corresponding local density of λ is strictly concave also on the hyperplane specified by $\bar{p}_{m,k}=\pi$ and $\bar{r}_{m,k}=\rho$, the Bakry–Emery criterion yields LSI for the canonical distributions with a universal constant C_{lsi} depending only on V. In fact we have

$$\int S\left[\bar{\mu}^{\pi,\rho}_{m,k}|\bar{\lambda}^{\pi,\rho}_{m,k},\mathscr{F}_{m,k}\right]\leqq C_{\mathrm{lsi}}\,D\left[\bar{\mu}^{\pi,\rho}_{m,k}|\mathscr{C}_{m,k},\mathscr{F}_{m,k}\right],$$

where $\mathscr{C}_{m,k}$ denotes the generator of a non-conservative Ginzburg–Landau model preserving λ, its Dirichlet form reads as

$$-\int\varphi\,\mathscr{C}_{m,k}\varphi\,\mathrm{d}\lambda=\sum_{j=0}^{m-1}\int\left((\partial_{k-j}\varphi)^2+(\tilde{\partial}_{k-j}\varphi)^2\right)\mathrm{d}\lambda.$$

On the other hand, $\tilde{\partial}_k - \tilde{\partial}_i = \nabla_1 \tilde{\partial}_i + \nabla_1 \tilde{\partial}_{i+1} + \cdots + \nabla_1 \tilde{\partial}_{k-1}$ if $i < k$, and we have the same formula for $\partial_k - \partial_i$, while

$$-\int \varphi \mathscr{G}_{m,k} \varphi = \sum_{j=1}^{m-1} \int \left((\nabla_1 \partial_{k-j} \varphi)^2 + (\nabla_1 \tilde{\partial}_{k-j} \varphi)^2 \right) \mathrm{d}\lambda.$$

Since D is convex, this completes the proof by the Cauchy inequality, see Equations (3.20)–(3.21) in the paper [3] by Chang and Yau. □

It will also be very important later on that C_{lsi} is universal, since it does not depend on π, ρ and μ.

4.4. LSI in action

In the forthcoming calculations Proposition 3 is used in the following way: Let $\phi : \mathbb{R}^{2m} \mapsto \mathbb{R}$ and set $\phi_k(\omega) := \phi(p_{k-m+1}, r_{k-m+1}, \ldots, p_{k-1}, r_{k-1}, p_k, r_k)$, then for $\beta > 0$ from the entropy inequality

$$\beta \sum_{|k|<n} \int \phi_k \, \mathrm{d}\mu \leqq \sum_{|k|<n} \int S\left[\bar{\mu}^{\pi,\rho}_{m,k} | \bar{\lambda}^{\pi,\rho}_{m,k}, \mathscr{F}_{m,k}\right] \mathrm{d}\mu + 2n \, \log \int \mathrm{e}^{\beta \phi_0} \, \mathrm{d}\bar{\lambda}^{\pi,\rho}_{m,0},$$

provided that the canonical moment generating function on the right-hand side, as it will be true in our cases, does not depend on π and ρ. Due to the subadditivity of D,

$$\sum_{|k|<n} \int S\left[\bar{\mu}^{\pi,\rho}_{m,k} | \bar{\lambda}^{\pi,\rho}_{m,k}, \mathscr{F}_{m,k}\right] \mathrm{d}\mu \leqq m^3 C_{\mathrm{lsi}} D\left[\mu | \mathscr{G}_{n+m,n}, \mathscr{F}_{n+m,n}\right];$$

thus the problem has been reduced to the calculation of the canonical moment generating function. It is well known that the canonical distributions $\bar{\lambda}^{\pi,\rho}_{m,k}$ do not depend on k, and they converge weakly to $\lambda_{\pi,\rho}$ when $m \to +\infty$, but here we need explicit bounds. In fact, the marginal density of $\bar{\lambda}^{\pi,\rho}_{m,k}$ on $\mathscr{F}_{l,j}$ is dominated by the corresponding local density of $\lambda_{\pi,\rho}$, at least if $l \leqq m/2$, $\mathscr{F}_{l,j} \subset \mathscr{F}_{m,k}$ and m is large enough.

Lemma 6. *There exists a universal constant $C' < +\infty$ and $l_0 \in \mathbb{N}$ such that*

$$\int \mathrm{e}^{\beta\phi} \, \mathrm{d}\bar{\lambda}^{\pi,\rho}_{m,k} \leqq C' \int \mathrm{e}^{\beta\phi} \, \mathrm{d}\lambda_{\pi,\gamma} \quad \textit{if} \quad \gamma = S'(\rho)$$

whenever $\phi(\omega) \sim \mathscr{F}_{l,j} \subset \mathscr{F}_{m,k}$, and $m \geqq 2l \geqq 2l_0$.

Proof. It is standard, similar arguments can be found in [19–21]. The canonical density of the r_k variables in the box $(0, m]$ is a ratio $Q_m(r_1, r_2, \ldots, r_m) := \Gamma_m / \bar{\Gamma}_m(x)$, where

$$\Gamma_m(r_1, r_2, \ldots, r_m) := \exp \sum_{j=1}^{m} \left(\gamma r_j - V(r_j) - F(\gamma)\right),$$

while $\bar{\Gamma}_m(x)$ is just the integral of Γ_m over the hyperplane $\bar{r}_{m,m} = x$. The marginal canonical density $Q_{m,l}$ of l variables can be written as

$$\bar{Q}_{m,l}(r_1, r_2, \dots, r_l) = \frac{\Gamma_l(r_1, r_2, \dots, r_l)\bar{\Gamma}_{m-l}(mx - r_1 - r_2 - \cdots - r_l)}{\bar{\Gamma}_m(x)};$$

thus the *local central limit theorem* with remainders (the Edgeworth expansion) is most useful in its evaluation.

Let $\varphi_n(x)$ denote the density function of $\sqrt{n}(\bar{r}_{n,n} - \rho)(F''(\gamma))^{-1/2}$ with respect to $\lambda_{\pi,\gamma}$, while $\varphi(x) = \lim \varphi_n(x)$ is the standard normal density, then

$$\varphi_n(x) = \varphi(x) + \frac{\varphi(x)(x^3 - 3x)F'''(\gamma)}{6F''(\gamma)^{3/2}n^{1/2}} + \varepsilon_n(x),$$

where $\varepsilon_n(x)\sqrt{n} \to 0$ uniformly in $x, \gamma \in \mathbb{R}$ as $n \to +\infty$. Since F''' is bounded and

$$\bar{\Gamma}_n(x) = \frac{\varphi_n(y)}{\sqrt{nF''(\gamma)}} \quad \text{with} \quad y = (x - n\rho)/\sqrt{nF''(\gamma)},$$

we see that Q_m/Γ_l is bounded, which completes the proof. $\square$

In our cases ϕ is a sum; thus the computation can be completed by Schwarz. For example, if $\psi_k := \psi(p_k, r_k)$ and $\gamma = S'(\rho)$ then

$$\log \int \left(\exp \sum_{k=1}^{m} \beta_k \psi_k \right) \mathrm{d}\bar{\lambda}^{\pi,\rho}_{m,m} \leqq \log C' + \frac{1}{2} \log \int \exp\left(\sum_{k=1}^{m} 2\beta_k \psi_k \right) \mathrm{d}\lambda_{\pi,\gamma}.$$

4.5. The local bound of relative entropy

To launch LSI, an a priori bound on entropy and its Dirichlet form is required. We consider HDL on the infinite line, and finiteness of the initial entropy is not assumed, compare to the uniform energy bound of [29,30]. Therefore the Hamiltonian part $\mathscr{L}_0$ of the evolution induces entropy transport, while the symmetric Ginzburg–Landau generators $\mathscr{S}$ and $\tilde{\mathscr{S}}$ induce diffusion. For convenience, as in (23) set $S_n[\mu_{t,\varepsilon}|\lambda] := S[\mu_{t,\varepsilon}|\lambda, \mathscr{F}_{2n+1,n}]$, $D_n[\mu_{t,\varepsilon}|\lambda] := D[\mu_{t,\varepsilon}|\mathscr{S}_{2n+1,n}, \mathscr{F}_{2n+1,n}]$ and $\tilde{D}_n[\mu_{t,\varepsilon}|\lambda] := D[\mu_{t,\varepsilon}|\tilde{\mathscr{S}}_{2n,n}, \mathscr{F}_{2n+1,n}]$. Here, and also in the next section $\varepsilon > 0$ and $\sigma = \sigma(\varepsilon) \geqq 1$ are arbitrarily fixed numbers.

Proposition 4. *Assume* (23)*, then for all* $t, \varepsilon > 0$ *and* $n \in \mathbb{N}$*, we have*

$$S_n[\mu_{t,\varepsilon}|\lambda] + 2\sigma(\varepsilon) \int_0^t \left(D_n[\mu_{s,\varepsilon}|\lambda] + \tilde{D}_n[\mu_{s,\varepsilon}|\lambda] \right) \mathrm{d}s \leqq C_0 \left(t + \sqrt{n^2 + \sigma(\varepsilon)t} \right)$$

with the same constant C_0 *depending only on* C *and* V.

Proof. The basic ideas go back to [13], and similar arguments are used also in [15] and [16,18]. Assume first that the evolved measure, $\mu_{t,\varepsilon}$ has smooth local densities; let $f_n(t,\omega) := \mathrm{d}\mu_{t,n,\varepsilon}/\mathrm{d}\lambda$, $S_n(t) := S_n[\mu_{t,\varepsilon}|\lambda]$, $D_n(t) := D_n[\mu_{t,\varepsilon}|\lambda]$ and $\tilde{D}_n(t) := \tilde{D}_n[\mu_{t,\varepsilon}|\lambda]$ for brevity. A formal application of the Ito lemma yields

$$\partial_t S_n = \int (\partial_t + \mathscr{L}) \log f_n(t,\omega)\, \mu_{t,\varepsilon}(d\omega) = \int (f_{n+1} - f_n)\mathscr{L}_0 \log f_n \,\mathrm{d}\lambda$$
$$-\sigma(\varepsilon)\sum_{k\in\mathbb{Z}} \int (\nabla_1 \partial_k f_{n+1}) \frac{\nabla_1 \partial_k f_n}{f_n}\,\mathrm{d}\lambda - \sigma(\varepsilon)\sum_{k\in\mathbb{Z}} \int (\nabla_1 \tilde{\partial}_k f_{n+1}) \frac{\nabla_1 \tilde{\partial}_k f_n}{f_n}\,\mathrm{d}\lambda$$

because $\lambda(\partial_t f_n) = 0$ and $\lambda(\mathscr{L}_0 f_n) = 0$ follow from the definition of f_n; the second line has been obtained via integration by parts, see (21). Since

$$D_n(t) = \sum_{k=-n}^{n-1} \int (\nabla_1 \partial_k \sqrt{f_n})^2 \,\mathrm{d}\lambda = \sum_{k=-n}^{n-1} \int \frac{1}{4f_n} (\nabla_1 \partial_k f_n)^2 \,\mathrm{d}\lambda,$$
$$\tilde{D}_n(t) = \sum_{k=-n}^{n-1} \int (\nabla_1 \tilde{\partial}_k \sqrt{f_n})^2 \,\mathrm{d}\lambda = \sum_{k=-n}^{n-1} \int \frac{1}{4f_n} (\nabla_1 \tilde{\partial}_k f_n)^2 \,\mathrm{d}\lambda,$$

by a direct calculation we obtain that

$$\partial_t S_n + 4\sigma(\varepsilon) D_n(t) + 4\sigma(\varepsilon)\tilde{D}_n(t) = B_n^o(t) + \sigma(\varepsilon) B_n(t) + \sigma(\varepsilon)\tilde{B}_n(t),$$

where the boundary terms read as

$$B_n^o(t) := \int (p_{n+1}\tilde{\partial}_n f_n + V'_{-n-1}\partial_{-n} f_n) \frac{f_{n+1}}{f_n}\,\mathrm{d}\lambda,$$
$$B_n(t) := \int \left((\nabla_1 \partial_n f_{n+1}) \frac{\partial_n f_n}{f_n} - (\nabla_1 \partial_{-n-1} f_{n+1}) \frac{\partial_{-n} f_n}{f_n} \right) \mathrm{d}\lambda$$
$$= \int \left((p_{n+1} f_{n+1} - \partial_n f_n) \frac{\partial_n f_n}{f_n} + (V'_{n+1} f_{n+1} - \partial_{-n} f_n) \frac{\partial_{-n} f_n}{f_n} \right) \mathrm{d}\lambda,$$
$$\tilde{B}_n(t) := \int \left((\nabla_1 \tilde{\partial}_n f_{n+1}) \frac{\tilde{\partial}_{n-1} f_n}{f_n} - (\nabla_1 \tilde{\partial}_{-n-1} f_{n+1}) \frac{\tilde{\partial}_{-n} f_n}{f_n} \right) \mathrm{d}\lambda$$
$$= \int \left((V'_{n+1} f_{n+1} - \tilde{\partial}_n f_n) \frac{\tilde{\partial}_n f_n}{f_n} + (V'_{-n-1} f_{n+1} - \tilde{\partial}_{-n} f_n) \frac{\tilde{\partial}_{-n} f_n}{f_n} \right) \mathrm{d}\lambda,$$

where integration by parts has been used to rewrite B_n and $\tilde{B}_n$.

From the first lines by the Cauchy–Schwarz inequality

$$B_n(t) + \tilde{B}_n(t) \leqq \sqrt{\Delta_{n+1}(t) - \Delta_n(t)}\sqrt{D_{\partial n}(t)}, \tag{34}$$

where $\Delta_n(t) := D_n(t) + \tilde{D}_n(t)$ and

$$D_{\partial n}(t) := \int \frac{1}{f_n} \left((\partial_n f_n)^2 + (\partial_{-n} f_n)^2 + (\tilde{\partial}_n f_n)^2 + (\tilde{\partial}_{-n} f_n)^2 \right) \mathrm{d}\lambda.$$

The basic entropy inequality provides another way to estimate the boundary terms. Let $\nu := \mu_{t,\varepsilon}(\cdot|\mathscr{F}_{2n+1,n})$, and $k = n+1$ or $k = -n-1$, then from Lemma 1

$$\beta\nu(V_k') \leqq S[\nu|\lambda] + G(\beta, 0) \leqq S[\nu|\lambda] + \|V''\|\beta^2/2,$$

whence by choosing β in the optimal way, that is $\beta = \sqrt{2S/\|V''\|}$, and applying the same trick to $-V_k'$, we obtain that $\nu^2(V_k') \leqq 2\|V''\|S[\nu|\lambda]$; $\nu^2(p_k) \leqq 2S[\nu|\lambda]$ follows in the same way. Therefore

$$B_n^o(t) \leqq K_0\sqrt{S_{n+1}(t) - S_n(t)}\sqrt{D_{\partial n}(t)} \quad \text{and} \tag{35}$$

$$B_n(t) + \tilde{B}_n(t) \leqq -D_{\partial n}(t) + K_0\sqrt{S_{n+1}(t) - S_n(t)}\sqrt{D_{\partial n}(t)} \tag{36}$$

with some universal constant K_0.

Observe now that $B_n + \tilde{B}_n \geqq 0$ implies $D_{\partial n} \leqq K_0^2\,(S_{n+1} - S_n)$, thus (34) turns into

$$B_n(t) + \tilde{B}_n(t) \leqq K_0\sqrt{S_{n+1}(t) - S_n(t)}\sqrt{\Delta_{n+1}(t) - \Delta_n(t)}. \tag{37}$$

Therefore we have a closed system of differential inequalities for S_n and Δ_n, namely

$$\partial_t S_n + 2\sigma(\varepsilon)\Delta_n \leqq K\left(S_{n+1}(t) - S_n(t) + \sigma(\varepsilon)\sqrt{S_{n+1} - S_n}\sqrt{\Delta_{n+1} - \Delta_n}\right)$$

for all $n \in \mathbb{N}, t \geqq 0$ and $\sigma \geqq 1$ with the same K depending only on C and U. Indeed, in view of (35), there is nothing to prove if $D_{\partial n} \leqq K_1\,(S_{n+1} - S_n)$, but the right-hand side of (36) becomes negative when $D_{\partial n} \geqq K_1\,(S_{n+1} - S_n)$ and K_1 is large. In fact, in this case

$$\begin{aligned} B_n^o + \sigma B_n + \sigma\tilde{B}_n &\leqq K_0\sqrt{(S_{n+1} - S_n)D_{\partial n}} \\ &\quad + \sigma K_0\sqrt{S_{n+1} - S_n}\sqrt{D_{\partial n}} - \sigma D_{\partial n} \leqq K_0^2\,(S_{n+1} - S_n) \end{aligned}$$

if $\sigma K_0\sqrt{(S_{n+1} - S_n)D_{\partial n}} - (\sigma - 1/4)D_{\partial n} \leqq 0$, which completes the proof of our differential inequalities. This system can explicitly be solved, see Lemma 3 in [13], the result is just the desired bound on S and Δ. Since the final statement does not depend on the smoothness of f_n any more, this restriction can be removed by a standard regularization argument. For example, the partial dynamics are smooth, and as they have been defined after (13), λ is preserved; thus our calculations are correct. Therefore the proof can be completed by removing the cutoff of the dynamics because both S and D are lower semi-continuous. □

4.6. The one block and two blocks estimates

In the rest of this section, we summarize some a priori bounds that we need for the evaluation of entropy production. As a first consequence of Proposition 4, from the entropy inequality we get a moment condition.

Lemma 7. *We have a universal constant C_1 such that*

$$\sum_{|k|<n}\int (p_k^2 + r_k^2)\,\mathrm{d}\mu_{t,\varepsilon} \leqq C_1\left(t + \sqrt{n^2 + \sigma(\varepsilon)t}\right).$$

Proof. For any $\beta > 0$ we have

$$\beta \sum_{|k|<n} \int (p_k^2 + r_k^2)\,\mathrm{d}\mu_{t,\varepsilon} \leqq S_n(t) + 2n \log \int \exp(\beta p_0^2 + \beta r_0^2)\,\mathrm{d}\lambda,$$

and $\log \lambda(\mathrm{e}^{\beta p_0^2}) = -\log\sqrt{1-2\beta} \leqq 2\beta$ if $4\beta \leqq 1$, while

$$\begin{aligned}\log \int \mathrm{e}^{\beta r_0^2}\,\mathrm{d}\lambda &\leqq -F(0) + \log \int_{-\infty}^{\infty} \exp(\beta r^2 - c_1 r^2/2)\,\mathrm{d}r \\ &= -F(0) + \log\sqrt{2\pi/(c_1 - 2\beta)},\end{aligned}$$

which completes the proof via Proposition 4 by letting $\beta = \min\{1/4, c_1/4\}$, say. These obvious bounds shall be used several times also later on. □

A bound of $\sup\{\mathsf{E}\|\phi\hat{u}_\varepsilon\|_2^2 : \varepsilon > 0\}$ is derived by integrating with respect to time when ϕ is compactly supported, which implies the first part of the energy inequality (31). The One-Block Lemma on the rate of convergence to local equilibrium is a consequence of Propositions 3 and 4, see Lemma 3.4 of [16] for a first version. This result goes back to Theorem 4.2 of [19], where an explicit bound has not yet been given.

Lemma 8. *There exists a universal constant C_2 such that for $l_0 < l < n$*

$$\sum_{|k|<n} \int_0^t \int \left(\bar{V}_{l,k}' - S'(\bar{r}_{l,k})\right)^2 \mathrm{d}\mu_{s,\varepsilon}\,\mathrm{d}s \leqq C_2\left(\frac{nt}{l} + \frac{l^2\sqrt{n^2 + \sigma(\varepsilon)t}}{\sigma(\varepsilon)}\right).$$

Proof. The canonical expectation of each term on the left-hand side is estimated first by the entropy inequality, and LSI is used then to bound the expectation of the conditional entropy. To estimate the canonical moment generating function, let η denote a standard normal variable, then

$$\int \exp\left(\beta(\bar{V}_{l,k}' - S'(\rho))^2\right)\mathrm{d}\bar{\lambda}_{l,k}^{\pi,\rho} = \mathsf{E}_\eta \int \exp\left(\eta\sqrt{2\beta}(\bar{V}_{l,k}' - S'(\rho))\right)\mathrm{d}\bar{\lambda}_{l,k}^{\pi,\rho},$$

where E_η denotes the expectation with respect to η. As we have explained after Proposition 3, an upper bound is obtained if we replace β by 2β and $\bar{\lambda}_{l,k}^{\pi,\rho}$ by $\lambda_{\pi,\gamma}$ such that $\gamma = S'(\rho)$. On the other hand,

$$\int \exp\left(\eta\sqrt{8\beta_0/l}\,(V_k' - \gamma)\right)\mathrm{d}\lambda_{\pi,\gamma} = \exp\left(G(\eta\sqrt{8\beta_0/l}, \gamma) - \eta\gamma\sqrt{8\beta_0/l}\right),$$

where $\beta_0 := \beta/l$ and $G(\eta\sqrt{8\beta_0/l}, \gamma) \leqq \gamma\eta\sqrt{8\beta_0/l} + 4\beta_0 c_2 \eta^2 l^{-1}$, see Lemma 1; the Gaussian integral is computed as before. Following the instructions we have added after Lemma 6, the statement follows from a direct calculation. In fact, $O(nt/l)$ is the contribution of the large deviation part (canonical expectation), the rest comes from our bound on the time integral of the Dirichlet form. □

For the comparison of remote blocks the integration by parts formula

$$\int (V'_{k+m} - V'_k)\varphi f \, \mathrm{d}\lambda = \int (\tilde{\partial}_{k+m}\varphi - \tilde{\partial}_k\varphi) f \, \mathrm{d}\lambda + \int \varphi \, (\tilde{\partial}_{k+m} f - \tilde{\partial}_k f) \, \mathrm{d}\lambda$$

is more effective than LSI, see Lemma 5 and Lemma 7 of [13], or Lemma 3.4 of [18].

Lemma 9. *We have a universal C_3 such that for $l_0 < l \leqq m < n$*

$$\sum_{|k|<n} \int_0^t \int (\eta_{l,k+m} - \eta_{l,k})^2 \, \mathrm{d}\mu_{s,\varepsilon} \, \mathrm{d}s \leqq C_3 \left(\frac{nt}{l} + \frac{m^2\sqrt{n^2 + \sigma(\varepsilon)t}}{\sigma(\varepsilon)} \right)$$

whenever $\eta_{l,k} = \bar{p}_{l,k}$, $\eta_{l,k} = \bar{V}'_{l,k}$, $\eta_{l,k} = S'(\bar{r}_{l,k})$ or $\eta_{l,k} = \bar{r}_{l,k}$.

Proof. Suppose first that $f = \mathrm{d}\mu/\mathrm{d}\lambda$ is smooth enough, then

$$Q := \int (\bar{V}'_{l,k+m} - \bar{V}'_{l,k})^2 \, \mathrm{d}\mu = \frac{1}{l} \int (\bar{V}''_{l,k+m} + \bar{V}''_{l,k}) \, \mathrm{d}\mu$$
$$+ \int (\bar{V}'_{l,k+m} - \bar{V}'_{l,k})(\bar{\tilde{\partial}}_{l,k+m} f - \bar{\tilde{\partial}}_{l,k} f) \, \mathrm{d}\lambda \leqq \frac{2\|V''\|}{l} + \sqrt{QR},$$

that is $Q \leqq (4/l)\|V''\| + 2R$ as $Q \leqq a + \sqrt{QR}$ implies $Q \leqq 2a + 2R$, where

$$R := 4 \int (\bar{\tilde{\partial}}_{l,k+m}\sqrt{f} - \bar{\tilde{\partial}}_{l,k}\sqrt{f})^2 \, \mathrm{d}\lambda$$
$$\leqq \frac{4}{l} \sum_{j=k-l+1}^{k} \int (\tilde{\partial}_{j+m}\sqrt{f} - \tilde{\partial}_j\sqrt{f})^2 \, \mathrm{d}\lambda$$
$$\leqq 4(l+m) \sum_{j=k-l+1}^{k+m-1} \int (\nabla_1 \tilde{\partial}_j \sqrt{f})^2 \, \mathrm{d}\lambda,$$

the conclusion has been obtained by the Cauchy inequality. Now we are in a position to apply Proposition 4. Exploiting the subadditivity and lower semi-continuity of D, the statement for $\eta_{l,k} = \bar{V}'_{l,k}$ follows from a direct computation; the case of $\eta_{l,k} = \bar{p}_{l,k}$ is similar.

Finally, $S''(\rho) \geqq c_1$ in view of Lemma 1; thus

$$c_1 \, (\bar{r}_{l,k+l} - \bar{r}_{k,l})^2 \leqq \left(S'(\bar{r}_{l,k+l}) - S'(\bar{r}_{l,k})\right)^2 \leqq 3 \left(\bar{V}'_{l,k+l} - \bar{V}'_{l,k}\right)^2$$
$$+3 \left(S'(\bar{r}_{l,k}) - \bar{V}'_{l,k}\right)^2 + 3 \left(S'(\bar{r}_{l,k+l}) - \bar{V}'_{l,k+l}\right)^2,$$

which completes the proof by Lemma 8. □

As a direct consequence we obtain the missing part of the energy inequality (31): for all $\tau, r > 0$ we have

$$\limsup_{\varepsilon \to 0} \, \varepsilon\sigma(\varepsilon) \int_0^\tau \int_{-r}^r \mathsf{E}|\nabla_\varepsilon \hat{u}_\varepsilon(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t < +\infty.$$

Remark 1. By means of the Cauchy inequality, Lemma 9 yields a bound for the mean square deviation of block averages of different size. The allowed microscopic length is $m \approx \sqrt{\sigma(\varepsilon)/\varepsilon}$, which can not reach a macroscopic value because $\varepsilon\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The block averages of type $\bar{\eta}_{l,k}$ and $\hat{\eta}_{l,k}$ can also be compared.

Lemma 10. *For $l > l_0$ and $\eta_j = p_j$ or $\eta_j = r_j$, we have*

$$\sum_{|k|<n} \int_0^t \int (\hat{\eta}_{l,k} - \bar{\eta}_{l,k})^2 \,\mathrm{d}\mu_{s,\varepsilon}\,\mathrm{d}s \leqq C_4 \left(\frac{nt}{l} + \frac{l^2\sqrt{n^2 + \sigma(\varepsilon)t}}{\sigma(\varepsilon)} \right)$$

with some universal constant C_4.

Proof. Since integration by parts does not work in the case of $\eta_j = r_j$, we apply LSI in much the same way as in the proof of Lemma 8. We have

$$\int \exp\left(\beta(\hat{r}_{l,k} - \bar{r}_{l,k})^2\right) \mathrm{d}\bar{\lambda}^{\pi,\rho}_{2l,2l} = \mathsf{E}_\eta \int \exp\left(\sqrt{2\beta}\,\eta(\hat{r}_{l,k} - \bar{r}_{l,k})\right) \mathrm{d}\bar{\lambda}^{\pi,\rho}_{2l,2l},$$

where η is again a standard normal variable, and the canonical expectation will be estimated by means of $\lambda_{\pi,\gamma}$ with $\gamma = S'(\rho)$. Observe that there is a sequence $\{\alpha_j : j \in \mathbb{Z}\}$ such that

$$\hat{r}_{l,k} - \bar{r}_{l,k} = \sum_{j\in\mathbb{Z}} \alpha_j r_{k-j}, \quad \sum_{j\in\mathbb{Z}} \alpha_j = 0, \quad \sum_{j\in\mathbb{Z}} \alpha_j^2 \leqq 2/l$$

and $\alpha_j = 0$ if $|j| \geqq l$. Therefore we calculate

$$\begin{aligned} \log \int \exp(\eta\sqrt{8\beta}\alpha_j r_j)\,\mathrm{d}\lambda_{\pi,\gamma} &= F(\alpha_j\eta\sqrt{8\beta} + \gamma) - F(\gamma) \\ &\leqq F'(\gamma)\alpha_j\eta\sqrt{8\beta} + 4\beta\alpha_j^2\|F''\|\eta^2, \end{aligned}$$

where $\|F''\| < 1/c_1$ in view of Lemma 1, consequently

$$\log \mathsf{E}_\eta \int \exp\left(\eta\sqrt{8\beta} \sum_{j\in\mathbb{Z}} \alpha_j r_j\right) \mathrm{d}\lambda_{\pi,\gamma} \leqq \mathsf{E}_\eta \exp(8c_1^{-1}\beta\eta^2 l^{-2}),$$

whence the statement for $\eta_j = r_j$ follows from the usual calculation. The case of $\eta_j = p_j$ is essentially the same, though it is even a bit simpler. □

5. Evaluation of entropy production

Now we are in a position to show that all entropy pairs $(h, J) \in \mathscr{H}_1$ are well controlled. We investigate the decomposition $X_\varepsilon = X_{a,\varepsilon} + X_{s,\varepsilon} + \tilde{X}_{s,\varepsilon} + M_\varepsilon + N_\varepsilon$ of entropy production (28), see also (29) and (30).

5.1. The components of entropy production

The integrator m_ε of M_ε reads as

$$m_\varepsilon(t,x;h) = \sqrt{2\varepsilon\sigma(\varepsilon)}\Big(h'_\rho(\hat u_\varepsilon)\,\nabla_\varepsilon\hat\zeta_\varepsilon(t,x) - h'_\pi(\hat u_\varepsilon)\,\nabla^*_\varepsilon\hat\eta_\varepsilon(t,x)\Big), \tag{38}$$

where η and ζ denote the scaled Wiener processes, that is

$$\hat\eta_\varepsilon(t,x) := \sqrt{\varepsilon}\sum_{k\in\mathbb{Z}} 1_{\varepsilon,k}(x)\hat w_{l,k}(t/\varepsilon), \quad \hat\zeta_\varepsilon(t,x) := \sqrt{\varepsilon}\sum_{k\in\mathbb{Z}} 1_{\varepsilon,k}(x)\hat{\tilde w}_{l,k}(t/\varepsilon).$$

Since $h'_\rho(\hat u_\varepsilon)\nabla_\varepsilon\hat\pi_\varepsilon = -J'_\pi(\hat u_\varepsilon)\nabla_\varepsilon\hat\pi_\varepsilon$, it is convenient to define N_ε as

$$\begin{aligned} N_\varepsilon(\psi,h) := &-\int_0^\infty\int_{-\infty}^\infty \psi'_x(t,x)J(\hat u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &-\int_0^\infty\int_{-\infty}^\infty \psi(t,x)\left(J'_\pi(\hat u_\varepsilon)\nabla_\varepsilon\hat\pi_\varepsilon - J'_\rho(\hat u_\varepsilon)\nabla^*_\varepsilon\hat\rho_\varepsilon\right)\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{39}$$

Therefore the asymmetric part of X_ε becomes

$$\begin{aligned} X_{a,\varepsilon}(\psi,h) := &\frac{1}{\varepsilon}\int_0^\infty\int_{-\infty}^\infty \psi(t,x)\mathscr{L}_0 h(\hat u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_0^\infty\int_{-\infty}^\infty \psi(t,x)\left(J'_\pi(\hat u_\varepsilon)\nabla_\varepsilon\hat\pi_\varepsilon - J'_\rho(\hat u_\varepsilon)\nabla^*_\varepsilon\hat\rho_\varepsilon\right)\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{40}$$

while the contributions of the symmetric components $\mathscr{S}$ and $\tilde{\mathscr{S}}$ of the generator read as

$$X_{s,\varepsilon}(\psi,h) := \frac{\sigma(\varepsilon)}{\varepsilon}\int_0^\infty\int_{-\infty}^\infty \psi(t,x)\mathscr{S}h(\hat u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t, \tag{41}$$

$$\tilde X_{s,\varepsilon}(\psi,h) := \frac{\sigma(\varepsilon)}{\varepsilon}\int_0^\infty\int_{-\infty}^\infty \psi(t,x)\tilde{\mathscr{S}}h(\hat u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t. \tag{42}$$

To get $X_\varepsilon = Y_\varepsilon + Z_\varepsilon$ as needed in Proposition 2, we split some terms into new ones, and each of them will be cast into one of two categories named by Y and Z, according to the bound it satisfies. More precisely, quantities of type Y or Z should be estimated by $A_\varepsilon(\phi)\|\psi\|_+$ and $B_\varepsilon(\phi)\|\psi\|$, respectively, such that $\mathsf{E}A_\varepsilon \to 0$ while $\mathsf{E}B_\varepsilon$ must remain bounded as $\varepsilon\to 0$. Throughout this section we consider an entropy pair $(h,J)\in\mathscr{H}_1$, remember that ϕ and ψ are compactly supported smooth functions.

5.2. The martingale

It is treated in the dual space H^{-1} of H^{+1}; the norm of H^{-1} is defined as $\|\psi\|_- := \sup\{\langle\psi,\varphi\rangle : \|\varphi\|_+ \leqq 1\}$.

Lemma 11. *The stochastic integral M_ε is of type Y.*

Proof. Since $\phi\partial_t m_\varepsilon = \partial_t(\phi m_\varepsilon) - \phi_t' m_\varepsilon$ in H^{-1},

$$|M_\varepsilon(\psi\phi, h)| \leqq \|\psi\|_+\|\phi\partial_t m_\varepsilon\|_- \leqq \|\psi\|_+(\|\phi m_\varepsilon\|_2 + \|\phi_t' m_\varepsilon\|_2),$$

thus we have to estimate $\mathsf{E}(m_\varepsilon^2)$. Since

$$\mathsf{E}((\nabla_\varepsilon^* \hat{\eta}_\varepsilon(t,x))^2) = \mathsf{E}((\nabla_\varepsilon \hat{\zeta}_\varepsilon(t,x))^2) = 2t\varepsilon^{-2}l^{-3}(\varepsilon),$$

because η and ζ are independent families of Wiener processes,

$$\mathsf{E}\left(m_\varepsilon^2(t,x;h)\right) = \frac{4\sigma(\varepsilon)}{\varepsilon l^3(\varepsilon)}\left((h_\pi'(\hat{u}_\varepsilon(t,x)))^2 + (h_\rho'(\hat{u}_\varepsilon(t,x)))^2\right) \to 0$$

as $\varepsilon \to 0$, and the convergence is uniform in $x \in \mathbb{R}$ and $t \leqq T$, thus $\mathsf{E}\|\phi\partial_t m_\varepsilon\|_-^2 = O(\sigma(\varepsilon)/\varepsilon l^3(\varepsilon)) \to 0$. □

5.3. The numerical error

The treatment of N_ε is based on the two-blocks estimates, Lemma 9 and Lemma 10 of Section 4.

Lemma 12. *The numerical error decomposes as $N_\varepsilon = Y_{n,\varepsilon} + Z_{n,\varepsilon}$, where $Y_{n,\varepsilon}$ is of type Y, while $Z_{n,\varepsilon}$ is of type Z with a vanishing bound.*

Proof. Due to localization by ϕ, $N_\varepsilon(\phi\psi, h)$ should be evaluated; remember that $\|\psi'\|$ can not appear in the bounds. Set $\varphi := \phi\psi$ and

$$\varphi_\varepsilon(t,x) := \sum_{k\in\mathbb{Z}} 1_{\varepsilon,k}(x)\varphi(t, \varepsilon k + \varepsilon/2).$$

Since $\hat{u}_\varepsilon$ is a step function of x, the first line of (39) becomes

$$\begin{aligned}
L_1 &:= -\int_0^\infty \int_{-\infty}^\infty \varphi_x'(t,x) J(\hat{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&= \sum_{k\in\mathbb{Z}} \int_0^\infty \varphi(t, \varepsilon k + \varepsilon/2)\left(J(\hat{u}_\varepsilon(t, \varepsilon k + \varepsilon)) - J(\hat{u}_\varepsilon(t, \varepsilon k))\right)\mathrm{d}t \\
&= \int_0^\infty \int_{-\infty}^\infty \varphi_\varepsilon(t,x)\left(J_\pi'(\hat{v}_\varepsilon)\nabla_\varepsilon\hat{\pi}_\varepsilon(t,x) + J_\rho'(\hat{v}_\varepsilon)\nabla_\varepsilon\hat{\rho}_\varepsilon(t,x)\right)\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where $\hat{v}_\varepsilon$ is an intermediate point between $\hat{u}_\varepsilon(t,x)$ and $\hat{u}_\varepsilon(t, x+\varepsilon)$. Fitting the second line of (39) to this formula, we obtain $N_\varepsilon(\phi\psi, h) = I_1 + I_2$, where

$$\begin{aligned}
I_1 &:= \int_0^\infty \int_{-\infty}^\infty \left(\varphi_\varepsilon(t,x)J_\pi'(\hat{v}_\varepsilon) - \varphi(t,x)J_\pi'(\hat{u}_\varepsilon(t,x))\right)\nabla_\varepsilon\hat{\pi}_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t, \\
I_2 &:= \int_0^\infty \int_{-\infty}^\infty \left(\varphi_\varepsilon(t,x)J_\rho'(\hat{v}_\varepsilon) - \varphi(t,x+\varepsilon)J_\rho'(\hat{u}_\varepsilon(t,x+\varepsilon))\right)\nabla_\varepsilon\hat{\rho}_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

In the case of I_1 we write $\varphi_\varepsilon J'_\pi(\hat{v}) - \varphi J'_\pi(\hat{u}_\varepsilon) = \delta_{12} + \delta_{12} + \delta_{13}$, where

$$\delta_{11} := \varphi_\varepsilon(t,x)\left(J'_\pi(\hat{v}_\varepsilon) - J'_\pi(\hat{u}_\varepsilon(t,x))\right),$$
$$\delta_{12} := \sum_{k\in\mathbb{Z}} 1_{\varepsilon,k}(x)\phi(t,x)\left(\psi(t,\varepsilon k+\varepsilon/2) - \psi(t,x)\right) J'_\pi(\hat{v}_\varepsilon),$$
$$\delta_{13} := \sum_{k\in\mathbb{Z}} 1_{\varepsilon,k}(x)\psi(t,\varepsilon k+\varepsilon/2)\left(\phi(t,\varepsilon k+\varepsilon/2) - \phi(t,x)\right) J'_\pi(\hat{v}_\varepsilon),$$

and the integrand of I_2 decomposes in a similar way. By Schwarz

$$1_{\varepsilon,k}(x)\left(\psi(t,\varepsilon k+\varepsilon/2) - \psi(t,x)\right)^2 \leqq \varepsilon \|1_{\varepsilon,k}\psi'_x\|_2^2,$$

while $1_{\varepsilon,k}(x)(\phi(t,\varepsilon k+\varepsilon/2) - \phi(t,x))^2 = O(\varepsilon^2)$ is a uniform bound. On the other hand,

$$\left(J'_\pi(\hat{v}_\varepsilon) - J'_\pi(\hat{u}_\varepsilon(t,x))\right)^2 + \left(J'_\rho(\hat{v}_\varepsilon) - J'_\rho(\hat{u}_\varepsilon(t,x+\varepsilon))\right)^2 = \varepsilon^2\, O(|\nabla_\varepsilon \hat{u}_\varepsilon|^2)$$

because the second derivatives of J are bounded, and $\hat{v}_\varepsilon$ is a convex combination; thus

$$|\hat{v}_\varepsilon(t,x) - \hat{u}_\varepsilon(t,x)| + |\hat{v}_\varepsilon(t,x) - \hat{u}_\varepsilon(t,x+\varepsilon)| \leqq |\hat{u}_\varepsilon(t,x+\varepsilon) - \hat{u}_\varepsilon(t,x)|.$$

Therefore we can use the two-blocks estimates to evaluate N_ε as follows.

We have decomposed N_ε into six integrals, and each integrand is a product of three factors. Two of these factors are differences, the third one is bounded. The differences can be separated by means of the Schwarz inequality, the domains of integration are determined by the support of ϕ. We have $I_1 = I_{11} + I_{12} + I_{13}$ and $I_2 = I_{21} + I_{22} + I_{23}$, where $|I_{11}| \leqq K_{11}\|\psi\|\, Q_\varepsilon$,

$$Q_\varepsilon(\tau,r) := \int_0^\tau \int_{-r}^r \varepsilon\, |\nabla_\varepsilon \hat{u}_\varepsilon(t,x)|^2\, \mathrm{d}x\, \mathrm{d}t, \tag{43}$$

$|I_{12}| \leqq K_{12}\|\psi'\|_2\sqrt{Q_\varepsilon}$, finally $|I_{13}| \leqq K_{13}\|\psi\|\sqrt{\varepsilon Q_\varepsilon}$, where $\tau \geqq 0$, $r \geqq 1$, and the constants K_{11}, K_{12}, K_{13} depend only on ϕ and J. The case of I_2 is quite similar. Since

$$1_{\varepsilon,k}(x)\nabla_\varepsilon \hat{u}_\varepsilon(t,x) = \frac{1}{\varepsilon l(\varepsilon)}\left(\bar{p}_{l,k+l}(t/\varepsilon) - \bar{p}_{l,k}(t/\varepsilon), \bar{r}_{k,k+l}(t/\varepsilon) - \bar{p}_{k,k+l}(t/\varepsilon)\right),$$

Lemma 10 for $\tau \geqq 0$ and $r \geqq 1$ yields

$$Q_\varepsilon(\tau,r) \leqq C_4\left(\frac{r\tau}{\varepsilon l^3(\varepsilon)} + \frac{\sqrt{r^2 + \varepsilon\sigma(\varepsilon)\tau}}{\sigma(\varepsilon)}\right), \tag{44}$$

thus $Y_\varepsilon = I_{12} + I_{22}$, while $Z_\varepsilon = I_{11} + I_{13} + I_{21} + I_{23}$. As we have shown, all bounds do vanish. $\square$

5.4. The asymmetric component

In the following calculations, we have to refer to results of the previous section very frequently. Thus the empirical process will be rewritten in terms of block averages of the microscopic configuration. Space integrals turn into sums, and the integral mean

$$\varphi_k(t) := \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} \psi(t\varepsilon, k\varepsilon + x)\phi(t\varepsilon, k\varepsilon + x)\,\mathrm{d}x \tag{45}$$

appears several times. For functions like h'_π, J'_ρ, S'' and so on, a short-hand notation as $H_k(t) := H(\hat{u}_\varepsilon(\varepsilon t, \varepsilon k)) = H(\hat{p}_{l,k}(t), \hat{r}_{l,k}(t))$ shall be used.

Lemma 13. *The asymmetric functional $X_{a,\varepsilon}$ reads as $X_{a,\varepsilon} = Y_{a,\varepsilon} + Z_{a,\varepsilon}$, where $Z_{a,\varepsilon}$ is of type Z with a vanishing bound, $Y_{a,\varepsilon}$ is of type Y.*

Proof. Since $h'_\rho + J'_\pi = 0$, the corresponding terms of $X_{a,\varepsilon}$ cancel, thus substituting $-J'_\rho = h'_\pi S''$ we get

$$X_{a,\varepsilon}(\phi\psi, h) = \varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty \varphi_k(t) h'_{\pi,k}(t)\left(\nabla_l \bar{V}'_{l,k-l}(t) + S''_k(t)\nabla_l \bar{r}_{l,k-l}(t)\right)\mathrm{d}t$$

because $\nabla_1 \hat{\eta}_{l,k} = \nabla_l \bar{\eta}_{l,k-l}$ is an identity. The crucial step consists of the replacement of the microscopic current $\bar{V}'_{l,k}$ with its canonical expectation $S'(\bar{r}_{l,k})$ via Lemma 8. Indeed, $S'(\bar{r}_{l,k}) - S'(\bar{r}_{l,k-l}) = S''(\tilde{r}_k)(\bar{r}_{l,k} - \bar{r}_{l,k-l})$ with some intermediate value $\tilde{r}_k$, and

$$|S''(\hat{r}_{l,k}) - S''(\tilde{r}_k)| \leqq \|S'''\|\left(|\hat{r}_{l,k} - \bar{r}_{l,k}| + |\hat{r}_{l,k} - \bar{r}_{l,k-l}|\right),$$

where $\|S'''\| < +\infty$ in view of Lemma 2. Consequently we obtain $X_{a,\varepsilon} = Q_{a,\varepsilon} + X^*_{a,\varepsilon}$ by a direct computation, where

$$X^*_{a,\varepsilon}(\phi\psi, h) := \frac{\varepsilon}{l(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty (\varphi_{k+l}(t)h'_{\pi,k+l} - \varphi_k(t)h'_{\pi,k})\left(\bar{V}'_{l,k} - S'(\bar{r}_{l,k})\right)\mathrm{d}t,$$

while $Q_{a,\varepsilon}(\phi\psi)$ is a bilinear form of $\bar{r}_{l,k} - \bar{r}_{l,k-l}$ and $\hat{r}_{l,k} - \tilde{r}_k$,

$$Q_{a,\varepsilon} := \frac{\varepsilon}{l(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty \varphi_k(t)h'_{\pi,k}(t)\left(S''(\tilde{r}_k) - S''(\hat{r}_{l,k})\right)(\bar{r}_{l,k} - \bar{r}_{l,k-l})\,\mathrm{d}t,$$

This expression is a bit more complicated than that of $Q_\varepsilon(\tau, r)$ in the previous proof, nevertheless Lemma 10 applies in a direct way. We see that $Q_{a,\varepsilon}$ is of type Z with a vanishing bound of the same order as that of $Q_\varepsilon(\tau, r)$, see (44).

From $\varphi_{k+l}h'_{\pi,k+l} - \varphi_k h'_{\pi,k} = (\varphi_{k+l} - \varphi_k)h'_{\pi,k+l} + \varphi_k\,(h'_{\pi,k+l} - h'_{\pi,k})$ we obtain a decomposition $X^*_{a,\varepsilon} = X^*_{a1,\varepsilon} + X^*_{a2,\varepsilon}$ such that the Cauchy–Schwarz inequality results in $|X^*_{a1,\varepsilon}|^2 \leqq \|h'_\pi\|^2 R_{a,\varepsilon}\, Q^*_{a,\varepsilon}$ and $|X^*_{a2,\varepsilon}|^2 \leqq Q_{a2,\varepsilon}\, Q^*_{a,\varepsilon}$, where

$$\begin{aligned} R_{a,\varepsilon}(\phi\psi) &:= \frac{\varepsilon}{l(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty (\varphi_{k+l}(t) - \varphi_k(t))^2 \,\mathrm{d}t \\ &\leqq \frac{1}{\varepsilon l(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty \int_{\varepsilon k - \varepsilon l/2}^{\varepsilon k + \varepsilon l/2} \varphi'^2(t,y)\,\mathrm{d}y\,\mathrm{d}t \\ &= \|\varphi'_x\|_2^2 \leqq 2\|\phi\|^2\|\psi'_x\|_2^2 + 2\|\phi'_x\|^2\|\psi\|_2^2, \end{aligned} \tag{46}$$

that is $X^*_{a1,\varepsilon}(\phi\psi, h)$ has been split into two parts: $X^*_{a1,\varepsilon} = Y^*_{a1,\varepsilon} + Z^*_{a1,\varepsilon}$ such that $Y^*_{a1,\varepsilon}$ is of type Y, while $Z^*_{a1,\varepsilon}$ is of type Z with a vanishing bound. Moreover,

$$\begin{aligned} Q_{a2,\varepsilon}(\phi\psi) &:= \frac{\varepsilon}{l(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty \varphi_k^2(t)(h'_{\pi,k+l} - h'_{\pi,k})^2 \,\mathrm{d}t \\ &\leqq 2\|\phi\|^2\|\psi\|^2(\|h''_{\pi\pi}\|^2 + \|h''_{\pi\rho}\|^2)\, Q_\varepsilon(\tau, r) \end{aligned}$$

if ϕ is supported in $[0,\tau]\times[-r,r]$ because

$$h'_{\pi,k+l} - h'_{\pi,k} = h''_{\pi\pi}(\tilde u_\varepsilon)\left(\hat p_{l,k+l}(t) - \hat p_{l,k}(t)\right) + h''_{\rho\rho}(\tilde u_\varepsilon)\left(\hat r_{l,k+l}(t) - \hat r_{l,k}(t)\right)$$

with some intermediate value $\tilde u_\varepsilon$ between $\hat u_\varepsilon(\varepsilon t, \varepsilon k)$ and $\hat u_\varepsilon(\varepsilon t, \varepsilon k + \varepsilon l)$, and

$$\hat\eta_{l,k+l} - \hat\eta_{l,k} = \sum_{j=0}^{l-1} \nabla_1 \hat\eta_{l,k+j} = \frac{1}{l}\sum_{j=0}^{l-1}(\bar\eta_{l,k+j+l} - \bar\eta_{l,k+j}),$$

see (43) for the definition of $Q_\varepsilon(\tau, r)$. Finally

$$Q^*_{a,\varepsilon}(\tau, r) := \frac{\varepsilon}{l(\varepsilon)} \sum_{|k|<r/\varepsilon} \int_0^{\tau/\varepsilon} (\bar V'_{l,k} - S'(\bar r_{l,k}))^2 \,\mathrm{d}t$$

for $\tau \geqq 0$ and $r \geqq 1$ depending on the support of ϕ. In view of Lemma 8

$$Q^*_{a,\varepsilon}(\tau, r) \leqq C_3 \left(\frac{r\tau}{\varepsilon l^2(\varepsilon)} + \frac{l(\varepsilon)\sqrt{r^2 + \varepsilon\sigma(\varepsilon)\tau}}{\sigma(\varepsilon)} \right).$$

Comparing the calculations above, we see that the decomposition of $X_{a,\varepsilon}$ is complete, each bound does vanish. □

5.5. The symmetric component

The contribution of $\tilde{\mathscr{S}}$ decomposes as $\tilde X_{s,\varepsilon} = \tilde X_{s1,\varepsilon} + \tilde Z_{s1,\varepsilon}$, where by the Ito lemma

$$\begin{aligned} \tilde X_{s1,\varepsilon}(\phi\psi, h) &:= \frac{\varepsilon\sigma(\varepsilon)}{l(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty (\varphi_k h'_{\rho,k} - \varphi_{k+1} h'_{\rho,k+1})\left(\bar V'_{l,k+l} - \bar V'_{l,k}\right) \mathrm{d}t, \\ \tilde Z_{s1,\varepsilon}(\phi\psi, h) &:= \frac{\varepsilon\sigma(\varepsilon)}{l^2(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty \varphi_k h''_{\pi\pi,k}\,(\mathrm{d}\bar{\tilde w}_{l,k+l} - \mathrm{d}\bar{\tilde w}_{l,k})^2 \\ &= \frac{2\varepsilon\sigma(\varepsilon)}{l^3(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty \varphi_k h''_{\pi\pi,k}(\hat u_\varepsilon)\,\mathrm{d}t. \end{aligned}$$

As there is no problem with $\tilde{Z}_{s1,\varepsilon}$ because $h''_{\pi\pi}$ is bounded, it is of the type Z with a vanishing bound:

$$|\tilde{Z}_{s1,\varepsilon}(\phi\psi, h)| \leqq \frac{2\sigma(\varepsilon)}{\varepsilon l^3(\varepsilon)} \|h''_{\pi\pi}\| \, \|\psi\| \, \|\phi\|_1.$$

The evaluation of $\tilde{X}_{s1,\varepsilon}$ is almost the same as that of $X^*_{a,\varepsilon}$ in the proof of Lemma 13 above, just that $\bar{V}'_{l,k+l} - \bar{V}'_{l,k}$ is playing the role that $\bar{V}'_{l,k} - S'(\bar{r}_{l,k})$ did earlier. Indeed, $\tilde{X}_{s1,\varepsilon} = \tilde{X}_{s2,\varepsilon} - \tilde{Z}^*_{s1,\varepsilon}$, where

$$\tilde{X}_{s2,\varepsilon}(\phi\psi, h) := \frac{\varepsilon\sigma(\varepsilon)}{l(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty (\varphi_k - \varphi_{k+1}) h'_{\rho,k+1} \left(\bar{V}'_{l,k+l} - \bar{V}'_{l,k}\right) \mathrm{d}t,$$

$$\tilde{Z}^*_{s1,\varepsilon}(\phi\psi, h) := \frac{\varepsilon\sigma(\varepsilon)}{l(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_0^\infty \varphi_k(t)(h'_{\rho,k+1} - h'_{\rho,k})(\bar{V}'_{l,k+l} - \bar{V}'_{l,k}) \, \mathrm{d}t.$$

From $\partial_x \varphi = \phi'_x \psi + \phi \psi'_x$ we get $\tilde{X}_{s2,\varepsilon} = \tilde{Y}_{s2,\varepsilon} + \tilde{Z}_{s2,\varepsilon}$; thus the Cauchy–Schwarz inequality can be used to separate products, see the lines before (46). On the other hand, $h'_{\rho,k+1} - h'_{\rho,k} = h''_{\pi\rho}(\tilde{u})\nabla_l \bar{p}_{l,k} + h''_{\rho\rho}(\tilde{u})\nabla_l \bar{r}_{l,k}$ with some intermediate value $\tilde{u}$; thus the evaluation of $\tilde{X}_{s1,\varepsilon}$ reduces to the estimation of quadratic forms of type

$$Q_{s,\varepsilon}(\tau, r) := \frac{\varepsilon\sigma(\varepsilon)}{l^2(\varepsilon)} \sum_{|k|<r/\varepsilon} \int_0^{\tau/\varepsilon} (\eta_{k+l} - \eta_{k,l})^2 \, \mathrm{d}t,$$

where $\eta_k = \bar{V}'_{l,k}$, $\eta_k = \bar{p}_{l,k}$ or $\eta_k = \bar{r}_{l,k}$; τ and r depend on the support of ϕ. Lemma 9 and Lemma 10 imply now the existence of a universal constant C_5 such that

$$Q_{s,\varepsilon}(\tau, r) \leqq C_5 \left(\frac{\tau r \sigma(\varepsilon)}{\varepsilon l^3(\varepsilon)} + \sqrt{r^2 + \varepsilon\sigma(\varepsilon)\tau} \right)$$

for all $\varepsilon > 0$, $\tau > 0$ and $r \geqq 1$. The case of $X_{s,\varepsilon}(\phi\psi, h)$ is quite similar; thus we have got the desired decomposition.

Lemma 14. *We have $X_{s,\varepsilon} + \tilde{X}_{s,\varepsilon} = Y_{s,\varepsilon} + Z_{s,\varepsilon} - Z^*_{s,\varepsilon}$, where $Y_{s,\varepsilon}$ is of type Y, $Z_{s,\varepsilon}$ is of type Z with a vanishing bound. $Z^*_{s,\varepsilon}$ is also of type Z, but its bound never vanishes.*

Proof. We have got $\tilde{X}_{s,\varepsilon} = \tilde{Z}_{s1,\varepsilon} + \tilde{Y}_{s2\varepsilon} + \tilde{Z}_{s2,\varepsilon} - \tilde{Z}^*_{s1,\varepsilon}$, and $X_{s,\varepsilon} = Z_{s1,\varepsilon} + Y_{s2\varepsilon} + Z_{s2,\varepsilon} - Z^*_{s1,\varepsilon}$ in the same way, whence $Y_{s,\varepsilon} = Y_{s2,\varepsilon} + \tilde{Y}_{s2,\varepsilon}$, $Z_{s,\varepsilon} = Z_{s1,\varepsilon} + \tilde{Z}_{s1,\varepsilon} + Z_{s2,\varepsilon} + \tilde{Z}_{s2,\varepsilon}$, while $Z^*_{s,\varepsilon} = Z^*_{s1,\varepsilon} + \tilde{Z}^*_{s1,\varepsilon}$. □

Remark 2. If h is convex and $\psi \geqq 0$ then we expect $\limsup X_\varepsilon(\psi, h) \leqq 0$ in probability as $\varepsilon \to 0$. Our present tools are not sufficient for the proof of this *Lax inequality*, a more careful large deviation analysis would be required. We do not go into details because the Lax inequality is not known to imply the uniqueness of weak solutions to the p-system.

6. Completion of the Proofs

Since $\mathscr{L} p_k$ and $\mathscr{L} r_k$ are differences of the corresponding microscopic fluxes, the convergence of the empirical process in distribution along subsequences to a set of measure-valued solutions is almost immediate, we can easily replace microscopic currents with their canonical expectations by a one-block lemma as follows.

6.1. Proof of Proposition 1

We have to evaluate $X_\varepsilon(\psi, h)$ if $h(\pi, \rho) = \pi$ or $h(\pi, \rho) = \rho$. As these calculations are much easier than in the general case, a non-gradient analysis is not needed at all. Moreover, the second derivative of h are missing from the decomposition of X_ε. The stochastic equations of the momenta can be rewritten as

$$\hat{\pi}_\varepsilon(\mathrm{d}t, x) = -\nabla^*_{\varepsilon l} \bar{V}'_{\varepsilon,l}(t, x)\,\mathrm{d}t + \varepsilon\sigma(\varepsilon)\Delta_\varepsilon \hat{\pi}_\varepsilon(t, x)\,\mathrm{d}t - \sqrt{2\varepsilon\sigma(\varepsilon)}\nabla^*_\varepsilon \hat{\eta}_\varepsilon(\mathrm{d}t, x),$$

where $\Delta_\varepsilon := -\nabla^*_\varepsilon \nabla_\varepsilon$, $l = l(\varepsilon)$ and $\bar{V}'_{\varepsilon,l}(t, x) := \bar{V}'_{l,k}(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$; for the definition of the scaled Wiener process $\hat{\eta}_\varepsilon$ see Section 5.1. Therefore

$$\begin{aligned}
0 = \int_{-\infty}^{\infty} \psi(t, x)\hat{\pi}_\varepsilon(t, x)\,\mathrm{d}x &= \int_0^\infty \int_{-\infty}^{\infty} \psi'_t(t, x)\hat{\pi}_\varepsilon(t, x)\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\infty \int_{-\infty}^{\infty} (\nabla_{l\varepsilon}\psi(t, x))\,\bar{V}'_{\varepsilon,l}(t, x)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \varepsilon\sigma(\varepsilon) \int_0^\infty \int_{-\infty}^{\infty} (\Delta_\varepsilon \psi(t, x))\,\hat{\pi}_\varepsilon(t, x)\,\mathrm{d}x\,\mathrm{d}t \\
&- \sqrt{2\varepsilon\sigma(\varepsilon)} \int_{-\infty}^{\infty} \int_0^\infty (\nabla_\varepsilon \psi(t, x))\,\hat{\eta}_\varepsilon(\mathrm{d}t, x)\,\mathrm{d}x
\end{aligned}$$

follows from a direct computation for $\psi \in C^2_{\mathrm{co}}(\mathbb{R}^2_+ \to \mathbb{R})$. Observe now that Lemma 8 and Lemma 10 allow us to replace $\bar{V}'_\varepsilon$ with $S'(\hat{\rho}_\varepsilon)$, while Lemma 7 and $\mathsf{E}\hat{\eta}^2_\varepsilon(t, x) = O(t/l)$ imply that $\mathsf{E}|X_\varepsilon(\psi, \pi)| \to 0$ as $\varepsilon \to 0$ because $\varepsilon\sigma(\varepsilon) \to 0$ and $\nabla_{\varepsilon l}\psi \to \psi'_x$. The case of $X_\varepsilon(\psi, \rho)$ is similar. Since the tightness of the distributions of the Young measure follows from Lemma 7, we can directly pass to (26). $\square$

Remark 3. The convergence of the empirical process to a set of measure-valued solutions holds true under fairly general conditions. It is valid even for the model (9) with a fixed value of $\sigma > 0$. Moreover, the interaction V does not need to be convex because LSI is not needed at all. It is not necessary to introduce mesoscopic block averages, and as in [19] we can send $l \to +\infty$ after $\varepsilon \to 0$, and the Young measure and its limit distribution should also be defined in this way. Indeed, the strong ergodic hypothesis is sufficient for the replacement of the microscopic current $\bar{V}'_{l,k}$ with its canonical expectation $S'(\bar{r}_{l,k})$. Instead of Lemma 8 we only need

$$\lim_{l\to\infty} \limsup_{\varepsilon\to 0} \int_0^{\tau/\varepsilon} \sum_{|k|<r/\varepsilon} \varepsilon^2\, \mathsf{E}\,|\bar{V}'_{l,k}(t) - S'(\bar{r}_{l,k}(t))|\,\mathrm{d}t = 0$$

for all $\tau, r > 0$, which implies the statement.

Remark 4. A comparison of the microscopic and mesoscopic block averages is also possible. Let us consider $\bar{u}_{\varepsilon,l} = (\bar{\pi}_{\varepsilon,l}, \bar{\rho}_{\varepsilon,l})$, where $\bar{\pi}_{\varepsilon,l}(t,x) := \bar{p}_{l,k}(t/\varepsilon)$ and $\bar{\rho}_{\varepsilon,l}(t,x) := \bar{r}_{l,k}(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$. In view of Lemma 10, we can control the differences of block averages of type $\hat{\eta}_{l,k}$ and $\bar{\eta}_{l,k}$, while Lemma 9 and the Cauchy inequality allow us to replace all microscopic block averages by their mesoscopic counterparts. We get

$$\lim_{l\to\infty} \limsup_{\varepsilon\to 0} \int_0^\tau \int_{-r}^r \mathsf{E}|\bar{u}_{\varepsilon,l}(t,x) - \hat{u}_\varepsilon(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $\tau, r > 0$. In view of Remark 1, however, without the theory of compensated compactness, we can not tell anything more on the limiting behavior of the empirical process.

The main conditions of the stochastic Div–Curl Lemma have been verified in the previous section, so it is the time to prove it. The first version of this extension of the original results by Tartar [32] and Murat [23] is Theorem 10.2 in [15], see also Proposition 2.1 of [16] and Lemma 7, Lemma 8 in [17].

6.2. *Proof of Proposition 2*

The strategy is to reduce the statement to the deterministic one by means of the Skorohod Representation Theorem, see Theorem 1.8 in Chapter 3 of [10].

Skorohod's Theorem: *Suppose that* P *and* $\mathsf{P}_n, n \in \mathbb{N}$ *are probability measures on a separable metric space* $(\mathbb{X}, d)$, *and* $P_n \to P$ *in the weak sense. Then there exist a probability space* $(\Omega, \mathscr{A}, \mathbb{P})$ *on which we have* $\mathbb{X}$*-valued random variables* X *and* X_n *with distributions* P *and* P_n, *respectively, such that* $\mathbb{P}[X_n \to X] = 1$.

Before applying this theorem, usually we have the tightness of certain families of probability distributions allowing us to select weakly convergent subsequences; the selection of such subsequences shall be done in several, consecutive steps. The local metric topologies are also metrizable, and the diagonal procedure can be used for example in the case of the local weak topology of measures. Let us emphasize that both $\mathscr{M}$ and the local L^p spaces are separable metric spaces, but the weak topology of L^2 is not a metric one. Our first a priori bound is the following moment condition. Let $\|u_\varepsilon\|_{2,\tau,r}$ denote the local L^2 norm of the scaled process $u_\varepsilon \in \mathbb{R}^2$,

$$\|u_\varepsilon\|_{2,\tau,r}^2 := \int_0^\tau \int_{-r}^r |u_\varepsilon(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t$$

for $\tau, r > 0$. In view of Lemma 7, we have

$$\limsup_{\varepsilon\to 0} \mathsf{E}\|u_\varepsilon\|_{2,\tau,r}^2 \leqq C_1(\tau^2/2 + \tau r) < +\infty, \tag{47}$$

and the same bound holds true also for the empirical process $\hat{u}_\varepsilon$ by convexity. Therefore the distributions of the real variables $\|u_\varepsilon\|_{2,\tau,r}^2$ form a tight family, and we have the weak convergence of distributions along a subsequence $\varepsilon(n) \to 0$ such that, by embedding,

$$\limsup_{n\to\infty} \|u_{\varepsilon(n)}\|_{2,\tau,r} < +\infty \tag{48}$$

with $\mathbb{P}$-probability one, simultaneously for all $\tau, r > 0$. This relation replaces the uniform energy bound of [29,30] in the forthcoming argument, where subsequences of this $\varepsilon(n)$ shall be denoted again by $\varepsilon(n)$.

Skorohod's theorem applies also in the case of the Young measure of $\hat{u}_\varepsilon$, see [1] and [30] for the L^p version of the theory of Young measures. There exists a (t, x) defined random family $\theta_{t,x}$ of probability measures on $\mathbb{R}^2$ such that

$$\lim_{n\to\infty} \int_0^\infty \int_{-\infty}^\infty \psi(t,x) g(\hat{u}_{\varepsilon(n)}(t,x))\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_{-\infty}^\infty \psi(t,x)\theta_{t,x}(g)\,\mathrm{d}x\,\mathrm{d}t$$

with probability one, simultaneously for all continuous, compactly supported $\psi : \mathbb{R}^2_+ \to \mathbb{R}$, and for all linearly bounded, continuous $g : \mathbb{R}^2 \mapsto \mathbb{R}$. We get the statement for bounded g first, the extension via uniform integrability is due to (48).

Now we are in a position to deal with an entropy pair $(h, J) \in \mathscr{H}_0$. Since the test function ϕ of Proposition 2 is merely used to localize the problem, it suffices to have the a.s. convergence of the functionals $A_\varepsilon(\phi)$ and $B_\varepsilon(\phi)$ for a countable set of ϕ by Skorohod's representation. Since h and J are bounded by assumption, all conditions of the Murat Lemma are fulfilled, see Lemma 16.2.1 in [5]. Therefore the entropy production $\partial_t h(\hat{u}_{\varepsilon(n)}) + \partial_x J(\hat{u}_{\varepsilon(n)})$ lies in a strongly compact set of the local H^{-1} space; thus Tartar's theorem, see for example Theorem 16.2.1 of [5] applies. Indeed, passing to a refined subsequence we get the additional conditions of Tartar's Theorem: both $\phi h(\hat{u}_{\varepsilon(n)})$ and $\phi J(\hat{u}_{\varepsilon(n)})$ are weakly convergent in $L^2(\mathbb{R}^2_+)$. $\square$

Remark 5. Via (48) the proof extends to entropy pairs $(h, J) \in \mathscr{H}_1$ such that $h(u) = O(|u|^q)$ and $J(u) = O(|u|^q)$ with some $0 < q < 1$ as $|u| \to +\infty$. Indeed, then we have some $r > 2$ such that $|h|^r$ and $|J|^r$ are locally integrable.

6.3. *Proof of Theorem 1*

It is now a direct application of known results from PDE theory, an L^p theory of compensated compactness is applied. It has been initiated by Ball [1] and developed further by Shearer [30], essential parts of the argument are discussed by Serre [28], too. In fact, Serre–Shearer [29] have constructed some families of entropy pairs by solving the Goursat problem for the linear wave equation

$$S''(\rho)h''_{\pi\pi}(\pi, \rho) = h''_{\rho\rho}(\pi, \rho) \tag{49}$$

in terms of the *Riemann invariants* W_1 and W_2 of (1); $W_1(\pi, \rho) := \pi + A(\rho)$ and $W_2(\pi, \rho) := \pi - A(\rho)$, where $A(0) = 0$, and $A'(\rho) = (S''(\rho))^{1/2}$. Choosing continuous and compactly supported Goursat data, one obtains half-plane supported entropies of class $\mathscr{H}_0$ (called entropies of type East or West in Chapter 9 of [28]) in this way. The assumptions (SH),(LB) and (GC) of such constructions have been verified in Section 4.1 under reasonable conditions on our interaction potential V. The crucial conditions (GN1) or (GN2) of genuine nonlinearity are more problematic, interesting examples are discussed in Section 3.1, see also Lemma 4 and Lemma 5 on small local perturbations of these examples. Therefore the conclusions of Lemma 3 in [30] and Lemma 3 in [29] are in force, consequently Proposition 2 implies Tartar's equation (27) on the compensated factorization of the limiting Young measure. The Dirac property of the limiting Young measure follows from the above-mentioned clever couples of entropy pairs, see Lemma 6 and Lemma 7 of [30], Lemmas 3–8 in [29], or Lemma 9.3.2 and Lemma 9.4.1 in [28].

So far we have only proven a weak form of Theorem 1, namely that all limit distributions $\hat{\mathsf{P}} := \lim \hat{\mathsf{P}}_{\varepsilon(n)}$ in $\mathscr{M}$ as $\varepsilon(n) \to 0$ are concentrated on a set of weak solutions. We could not prove the tightness of $\{\hat{\mathsf{P}}_\varepsilon : \varepsilon > 0\}$ with respect to the strong local topology of $L^p_{\mathrm{loc}}(\mathbb{R}^2_+ \to \mathbb{R}^2)$ in a direct way, but this follows from the present version via Skorohod's representation and the L^p theory of the Young measure. Indeed, we can refer to Skorohod once more, and in the Skorohod picture the Young measure converges almost surely to a function, while the uniform integrability of $\|\hat{u}_\varepsilon\|^p_{2,\tau,r}$ for $p < 2$ is a consequence of Lemma 7; thus the family $\hat{\mathsf{P}}_\varepsilon$ is tight also with respect to the strong local topology of $L^p_{\mathrm{loc}}(\mathbb{R}^2_+ \to \mathbb{R}^2)$ if $p < 2$. □

Remark 6. If $p < 2$ is fixed then the Skorohod representation implies the existence of some subsequences $\varepsilon(n) \to 0$ such that

$$\lim_{n\to\infty} \int_0^\tau \int_{-r}^r \mathsf{E}\, |\hat{u}_{\varepsilon(n)}(t,x) - u(t,x)|^p \,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $\tau, r > 0$, where $u = (\pi, \rho)$ is a randomly selected weak solution depending also on the subsequence. In view of Remark 4, this conclusion can be reformulated as

$$\lim_{l\to\infty} \limsup_{n\to\infty} \int_0^\tau \int_{-r}^r \mathsf{E}\, |\bar{u}_{\varepsilon(n),l}(t,x) - u(t,x)|^p \,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $r > 0$ and $\tau > 0$, see [18].

Acknowledgements. Work supported in parts by: Hungarian Science Foundation Grant K-60708, Foundation Sciences Mathématiques de Paris, Hungarian-French Grant HP:FR-8/08 and Institute Henri Poincaré-Centre Borel. I am deeply indebted to the anonymous referees for valuable criticism, remarks and comments.

References

1. Ball, J.M.: A version of the fundamental theorem for Young measures. In: *Partial Differential Equations and Continuum Models of Phase Transitions* (Eds. Rascle, M., Serre, D., Slemrod, M.). Lecture Notes in Physics, Vol. 344. Springer-Verlag, New York, 207–215, 1988
2. Bernardin, C., Olla, S.: Non-equilibrium macroscopic dynamics of chains of anharmonic oscillators. IHP Preprint. http://www.ceremade.dauphine.fr/olla (2010)
3. Chang, C.C., Yau, H.-T.: Fluctuations of one dimensional Ginzburg-Landau models in nonequilibrium. *Commun. Math. Phys.* **145**, 209–245 (1992)
4. Chen, G.-Q., Dafermos, C.M.: The vanishing viscosity method in one-dimensional thermoelasticity. *Trans. AMS* **347**, 531–541 (1995)

5. Dafermos, C.M.: *Hyperbolic Conservation Laws in Continuum Physics*. Grundlehren der Mathematishen Wissenschaften **325**, 2nd edn. Springer-Verlag, Berlin, 2005
6. Dalecky, Yu.L., Fomin, S.V.: *Measures and Differential Equations in Infinite-Dimensional Spaces*. Kluwer, Dordrecht, 1991
7. DiPerna, R.: Measure-valued solutions to conservation laws. *Arch. Rational Mech. Anal.* **88**, 223–270 (1985)
8. Deuschel, J.-D., Stroock, D.W.: *Large Deviations*. Academic Press, Boston, 1989
9. Dobrushin, R.L., Pellegrinotti, A., Suhov, Yu.M., Triolo, L.: One-dimensional harmonic lattice caricature of hydrodynamics. *J. Stat. Phys.* **38**, 473–507 (1985)
10. Ethier, S.N., Kurtz, T.G.: *Markov Processes: Characterization and Convergence*. Wiley, New York, 1986
11. Even, N., Olla, S.: Hydrodynamic limit for an Hamiltonian system with boundary conditions and conservative noise. *Arch. Rational Mech. Anal.* (2010, submitted)
12. Fritz, J., Maes, Ch.: Derivation of hydrodynamic equations for Ginzburg-Landau models in an external field. *J. Stat. Phys.* **53**, 79–106 (1988)
13. Fritz, J.: On the diffusive nature of entropy flow in infinite systems: remarks to a paper by Guo, Papanicolau and Varadhan. *Commun. Math. Phys.* **133**, 331–352 (1990)
14. Fritz, J., Funaki, T., Lebowitz, J.L.: Stationary states of random Hamiltonian systems. *Probab. Theory Relat. Fields* **99**, 211–236 (1994)
15. Fritz, J.: *An Introduction to the Theory of Hydrodynamic Limits*. Lectures in Mathematical Sciences, Vol. 18. The University of Tokyo, Tokyo, 2001. ISSN 0919-8180
16. Fritz, J.: Entropy pairs and compensated compactness for weakly asymmetric systems. *Adv. Stud. Pure Math.* **39**, 143–171 (2004)
17. Fritz, J., Tóth, B.: Derivation of the Leroux system as the hydrodynamic limit of a two-component lattice gas. *Commun. Math. Phys.* **249**, 1–27 (2004)
18. Fritz, J., Nagy, K.: On uniqueness of the Euler limit of one-component lattice gas models. *Latin Am. J. Probab.* **1**, 367–392 (2006)
19. Guo, G.C., Papanicolau, G., Varadhan, S.R.S.: Nonlinear diffusion limit for a system with nearest neighbor interactions. *Commun. Math. Phys.* **8**, 31–59 (1988)
20. Kipnis, C., Landim, C.: *Scaling Limit of Interacting Particle Systems*. Grundlehren der Mathematishen Wissenschaften **320**. Springer-Verlag, Berlin, 1999
21. Landim, C., Panizo, G., Yau, H.T.: Spectral gap and logarithmic Sobolev inequality for unbounded conservative spin systems. *Ann. Inst. H. Poincaré Probab. Stat.* **38**, 739–777 (2002)
22. Lu, Y.-G.: *Conservation Laws and the Compensated Compactness Method*. Chapman & Hall/CRC, Boca Raton, 2003
23. Murat, F.: Compacité par compensation. *Ann. Sci. Norm. Sup. Pisa* **5**, 489–507 (1978)
24. Olla, S., Varadhan, S.R.S., Yau, H.-T.: Hydrodynamical limit for a Hamiltonian system with weak noise. *Commun. Math. Phys.* **155**, 523–560 (1993)
25. Perthame, B., Tzavaras, A.: Kinetic formulation for systems of two conservation laws and elastodynamics. *Arch. Rational Mech. Anal.* **155**, 1–48 (2000)
26. Rezakhanlou, F.: Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$. *Commun. Math. Phys.* **140**, 417–448 (1991)
27. Serre, D.: La compacité par compensation pour les systemes hyperboliques non linéaires de deux éqations a une dimension d'space. *J. Maths. Pures et Appl.* **65**, 423–468 (1986)
28. Serre, D.: *Systems of Conservation Laws*, Vols. 1–2. Cambridge University Press, Cambridge, 2000
29. Serre, D., Shearer, J.W.: *Convergence with Physical Viscosity for Nonlinear Elasticity*. Preprint (1994)
30. Shearer, J.W.: Global existence and compactness in L^p for the quasi-linear wave equation. *Commun. Partial Differ. Equ.* **19**, 1829–1877 (1994)
31. Spohn, H.: *Large Scale Dynamics of Interacting Particle Systems*. Springer-Verlag, Berlin, 1991

32. Tartar, L.: Compensated compactness and applications to partial differential equations. In: *Nonlinear Analysis and Mechanics*. Heriot-Watt Symposium, Vol. IV. Pitman, London, 136–212, 1979
33. Varadhan, S.R.S.: Nonlinear diffusion limit for a system with nearest neighbor interactions II. In: *Asymptotic Problems in Probability Theory: Stochastic Models and Diffusion on Fractals* (Eds. Elworthy, K.D., Ikeda, N.). Pitman Research Notes in Mathematics, Vol. 283. Wiley, New York, 75–128, 1994
34. Yau, H.-T.: Relative entropy and hydrodynamics of Ginzburg–Landau models. *Lett. Math. Phys.* **22**, 63–80 (1991)

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(*Received May 2, 2009 / Accepted October 4, 2010*)
Published online January 5, 2011 –