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On the First Critical Field in Ginzburg–Landau Theory for Thin Shells and Manifolds

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Abstract

In this article, we investigate the response of a thin superconducting shell to an arbitrary external magnetic field. We identify the intensity of the applied field that forces the emergence of vortices in minimizers, the so-called first critical field H_{c1} in Ginzburg–Landau theory, for closed simply connected manifolds and arbitrary fields. In the case of a simply connected surface of revolution and vertical and constant field, we further determine the exact number of vortices in the sample as the intensity of the applied field is raised just above H_{c1}. Finally, we derive via Γ-convergence similar statements for three-dimensional domains of small thickness, where in this setting point vortices are replaced by vortex lines.

1. Introduction

In this article, we investigate the response of a thin superconducting shell to an arbitrary external magnetic field. The intensity of the applied field is taken of the order of the so-called first critical field H_{c1} in Ginzburg–Landau theory. The main goal is to identify the asymptotic value of H_{c1} as one lets the Ginzburg–Landau parameter κ go to infinity, when the thickness of the sample is sufficiently small. Once this is established, we specialize to shells constituting a neighborhood of a simply connected surface of revolution, and take the applied field to be constant and vertical. A second major thrust is then to determine, in this particular case, the exact number of vortex lines present in minimizers of the Ginzburg–Landau functional when the intensity of the external field is raised above H_{c1} by a lower order term. In addition, the asymptotic location of vortices is found analytically; vortex lines consist of two collections that concentrate near the poles. Finally, it is proved that the configurations of the limiting vortices in the manifold tend to minimize a renormalized energy.

We consider a sample occupying a neighborhood of a closed two dimensional manifold $\mathcal{M}$ in $\mathbb{R}^3$. More precisely, our object of study is the functional

$$G_{\varepsilon,\kappa}(\Psi, \mathbf{A}) = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left(|(\nabla - i\mathbf{A})\Psi|^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2\right) \mathrm{d}X \\ +\frac{1}{\varepsilon}\int_{\mathbb{R}^3} |\nabla \times \mathbf{A} - \mathbf{H}_{\mathrm{ext}}|^2 \, \mathrm{d}X, \tag{1.1}$$

where Ω_ε is a thin superconductor corresponding to an ε-neighborhood of $\mathcal{M}$, $\Psi : \Omega_\varepsilon \to \mathbb{C}$ is the order parameter, $\mathbf{H}_{\mathrm{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ is the external magnetic field, that is, a given smooth, divergence-free vector field, and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ corresponds to the induced magnetic potential. The functional (1.1) is the Ginzburg–Landau energy functional with a scaling factor of $1/\varepsilon$, that is normalized by the volume of the sample (up to a multiplicative factor). One reason for studying this functional stems from the fact that even though the literature available for the case of an infinite cylinder and constant applied field is extensive (see [25] and the references therein), much less is known for general three-dimensional domains and arbitrary applied fields. Unlike the case of an infinite cylinder where one considers a vertical applied field to reduce the problem to a two-dimensional one, the thin sample approach described below allows the possibility of studying features of the solutions arising from nontrivial geometries responding to general applied fields. Another reason that this setting is interesting is the fact that vortices cannot escape through the boundary. This also imposes the restriction that the total degree of the vortices must be zero.

One way to circumvent the difficulty of studying the full three-dimensional Ginzburg–Landau functional without losing the geometric and topological richness of generic domains is by considering a thin superconducting sample. This is the approach the author and Sternberg follow in [5], where we analyze the Ginzburg–Landau energy of a superconductor that occupies a neighborhood of a compact surface without boundary. We establish a relation between $G_{\varepsilon,\kappa}$ and a reduced model posed on the manifold in which the induced magnetic field is replaced by the tangential component of the applied one. More precisely, we prove that $G_{\varepsilon,\kappa}(\Psi, \mathbf{A})$ Γ-converges to $\mathcal{G}_{\mathcal{M},\kappa}(\psi)$, where

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi) = \int_{\mathcal{M}}\left(\left|(\nabla_{\mathcal{M}} - i(\mathbf{A}_{\mathrm{ext}})^\tau)\psi\right|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2\right) \mathrm{d}\mathcal{H}^2_{\mathcal{M}}(x). \tag{1.2}$$

Here $\mathbf{A}_{\mathrm{ext}}$ is a divergence free vector field satisfying $\nabla \times \mathbf{A}_{\mathrm{ext}} = \mathbf{H}_{\mathrm{ext}}$, and $(\mathbf{A}_{\mathrm{ext}})^\tau := \mathbf{A}_{\mathrm{ext}} - (\mathbf{A}_{\mathrm{ext}} \cdot \nu(x))\nu(x)$. The precise topology of convergence is presented in detail in [5] and in Section 2 below. In [5], we also obtain, for simply connected surfaces of revolution and vertical fields, the asymptotic value of the first critical field H_{c1}, that is, the minimum magnetic field strength that must be overcome in order to see vortices in minimizers when $\kappa \gg 1$. For the case of an infinite superconducting cylinder of constant cross-section, the authors of [23] carry out such an investigation and determine the critical coefficient of $\ln\kappa$, characterizing it in terms of a solution to a certain auxiliary problem related to the London equation. (See also [25] for much more detailed information about H_{c1} in this setting.) For the planar problem arising as a thin film limit, the authors of [6,7] determine this critical coefficient in terms of a different auxiliary problem. Rather remarkably, in the case of a surface of revolution and constant vertical field, Sternberg and the author show in [5] one has simply

$$H_{c1} \sim 4\pi/(\text{Area of } \mathcal{M}) \ln \kappa,$$

for $\kappa \gg 1$. Among other things, the author extends here this result of [5] to allow for arbitrary fields and general simply connected manifolds. In this paper we consider fields that are presented in the form $\mathbf{H}_{\text{ext}} = h(\kappa)\,\mathbf{H}^e$. Here the scalar $h(\kappa)$ denotes the intensity of the given external field and we assume $\|\mathbf{H}^e\|_\infty = 1$. We also write $\mathbf{H}^e = \nabla \times \mathbf{A}^e$, which we refer to as the normalized field and normalized potential, respectively. In Theorems 3.1 and 3.2, we prove that given a simply connected manifold $\mathcal{M}$, there are two kinds of applied fields, those that give rise to an infinite value of H_{c1} and those for which

$$H_{c1} = \frac{1}{\max_{\mathcal{M}} *F - \min_{\mathcal{M}} *F} \ln \kappa,$$

where $*F$ is a 0-form with F a solution of $d^\star F = (\mathbf{A}^e)^\tau$. Here $(\mathbf{A}^e)^\tau$ is the tangential component of the normalized applied potential $\mathbf{A}^e$ in some convenient gauge. The strategy of the proof is to first identify a "first critical field" for the reduced Ginzburg–Landau functional $\mathcal{G}_{\mathcal{M},\kappa}$ and then prove that this serves as the asymptotic value of H_{c1} for the full Ginzburg–Landau energy, provided the thickness is taken small enough. This is achieved through the Γ-convergence relation described above. To our knowledge, this is one of the first calculations of the first critical field for Ginzburg–Landau in a three-dimensional setting, preceded by [5], and by the determination of a candidate for H_{c1} for a solid ball in $\mathbb{R}^3$ in [1]. It also corresponds, to our knowledge, to one of the first calculations of H_{c1} for generic three-dimensional non-constant applied fields.

In the second half of the paper we try to understand how vortices emerge as one increases the strength of the field slightly above H_{c1}. We fix $\mathbf{H}^e = \hat{e}_z$ and let now $\check{\mathcal{M}}$ denote a simply connected surface of revolution obtained by rotating a C^∞-curve around the z-axis. The intensity that we consider here is

$$h(\kappa) = \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})} \ln \kappa + \sigma \ln \ln \kappa,$$

where $\sigma > 0$ is a fixed constant independent of κ. This intensity is just $o(\ln \kappa)$-above H_{c1}, and is within the regime where we expect the successive appearance of multiple vortices in the sample as σ increases. In [22], Serfaty proves that in a superconductor that is an infinite cylinder with cross section a disk $D^2 \subset \mathbb{R}^2$, subject to a constant vertical field, there exist locally minimizing solutions of Ginzburg–Landau exhibiting multiple vortices of degree one when the external magnetic field is raised above H_{c1} by an addition of a $\ln \ln \kappa$ term, whose coefficient determines exactly how many vortices there will be in the sample. She also proves that these vortices concentrate near the center of the disk and that their rescaled configuration tends to minimize a renormalized energy. This result was later extended in [25] to consider more general domains, and where the solutions thus obtained are shown to be global minimizers. With the insight gained from [22] in mind, it is natural to ask whether something of the sort holds in our setting. One big difference is that in our case the total degree zero restriction precludes the possibility of only degree $+1$ vortices. The concentration set of the vortices cannot be a singleton, either, for the same reason. If a renormalized energy is to be found, the way to account for the interaction of vortices is not clear a priori, since degree $+1$ and degree -1 are supposed to attract each other and they must both coexist in $\check{\mathcal{M}}$. We prove that if $\sigma \notin (4\pi/\mathcal{H}^2(\check{\mathcal{M}}))\mathbb{Z}$, then any global minimizer of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ possesses exactly

$$2n_0 := 2\left\lfloor \sigma \frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi} \right\rfloor + 2$$

vortices, where half of them are located near the north pole and have degree $+1$, while the rest lie close to the south pole and have an associated degree of -1. Here, $\lfloor \cdot \rfloor$ denotes the integer part of a real number. When $\sigma \in (4\pi/\mathcal{H}^2(\check{\mathcal{M}}))\mathbb{Z}$ a transition between consecutive integers occurs in the optimal number of vortices. The projections of these configurations of vortices onto the xy-plane, rescaled by a factor of $\sqrt{\ln \kappa}$, tend to minimize

$$\mathbf{R}^{n_0}(x_1, \ldots, x_n) := -\sum_{i \neq j} \ln |x_i - x_j| + \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})} \sum_{i=1}^{n_0} |x_i|^2 .$$

This happens for both sets of vortices independently. The leading order term forces the two configurations to be well separated and their interaction is fixed up to $o(1)$. These renormalized energies are thus decoupled and it could well be that both configurations converge to different minimizers as $\kappa \to \infty$. This result is later combined with the Γ-convergence result to obtain the same number of vortex-lines and similar locations for minimizers of $G_{\varepsilon,\kappa}$. The result on the number and asymptotic location of vortices constitutes an analogue of those in [22,25]. In these works the renormalized energy consists also of two components, a logarithmic interaction term and a quadratic one that confines the vortices near a preferred location.

While one of the reasons to study the reduced functional is to obtain new information about three-dimensional Ginzburg–Landau, the underlying problem, namely the pursuit of understanding salient features of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$, is an interesting problem in its own right. Within the physics community, there are numerous studies of the response of a spherical superconducting shell or thin film to a magnetic field, including the experimental study [27] and the theoretical studies [10,20,28], the latter being primarily computational. Within the applied mathematics community, we note the computational work in [11,12] on superconducting spheres in the presence of a vertical magnetic field. Here the authors capture various vortex patterns on the surface of the sphere as the magnetic field strength is varied. Note that all of the research cited above focuses solely on a spherical geometry and is largely computational. Thus, the result presented here on vortex location and multiplicity at the manifold level gives rigorous confirmation to the experiments in [11]. But it proves more; it holds for any simply connected connected surface of revolution, not only a 2-sphere, and it shows those solutions can be realized as global minimizers. We point out that in the general case (when the external field is not constant, and the surface is not of revolution), the derivation of the asymptotic location of vortices is more involved. First, the set where $\max_{\mathcal{M}} *F$ (resp. $\min_{\mathcal{M}} *F$) is achieved may not be a singleton and therefore the vortices carrying a positive degree (resp. negative) have multiple options regarding where to concentrate. This also makes the optimal number of vortices more difficult to derive. Another issue is that, depending on the external field and the manifold, the behavior of $*F$ near a concentration point may yield a weaker attraction of vortices towards it, affecting the renormalized energy in particular and making a particular concentration point more preferable than others. Also, when the symmetry is lost, the energy renormalization cannot be performed by simply projecting the vortices onto a single plane. All of these matters are currently being pursued by the author. Finally, in the case of higher genus, its effect on the first critical field and emergence of vortices, to our knowledge, remains unexplored in this manifold setting.

The article is organized as follows. In Section 2 we introduce the necessary notation and background. In Section 3 we obtain the value of H_{c1} for simply connected manifolds and arbitrary applied fields. We achieve this by first obtaining an upper bound for the energy of minimizers through a construction. Then, we obtain a lower bound based on an adaptation of the technology on energy concentration on balls developed in [15,23]. Section 4 may be regarded as a toolbox; it consists of several results that allow for the isolation of the singularities of minimizers and its consequences, such as lower bounds on the energy taking into account the location of the vortices. In these results we carefully adapt, when necessary, to our setting, the lower bounds based on ball construction techniques of [2,3,22], in the case of a bounded number of vortices. In this context the ball constructions are done using geodesic and isothermal balls and we employ the terminology of *pseudo-balls* indistinctly to refer to either type, to avoid confusion with Euclidean balls. In Section 5 we prove that the hypotheses of the technical propositions of Section 4 are satisfied in the cases we consider. We then use these tools to derive the results on multiplicity and location of vortices.

2. Notation

Let $\mathcal{M}$ be C^2 orientable and a closed simply connected 2-dimensional manifold without boundary in $\mathbb{R}^3$. In this paper X will typically be a point in $\mathbb{R}^3$, while x or p will usually represent points on $\mathcal{M}$. In addition, we write $\nu(x)$ for the outer unit normal to the manifold at a given point $x \in \mathcal{M}$ and denote by $\mathbf{V}^\nu(x) := (\mathbf{V}(x) \cdot \nu(x))\nu(x)$ and $\mathbf{V}^\tau(x) := \mathbf{V}(x) - \mathbf{V}^\nu(x)$ the components, normal and tangential to the manifold, of a vector field $\mathbf{V}$ in $\mathbb{R}^3$. Finally $\mathcal{H}^2_{\mathcal{M}}$ will denote the two-dimensional Hausdorff measure restricted to $\mathcal{M}$. The map

$$T_\varepsilon : \mathcal{M} \times (0, 1) \to \mathbb{R}^3 \text{ given by } X = T_\varepsilon(x, t) := x + \varepsilon t \nu(x), \tag{2.1}$$

is smoothly invertible for ε small, in light of the regularity assumed on $\mathcal{M}$. Our purpose will be to study certain properties of minimizers of the Ginzburg–Landau functional

$$G_{\varepsilon,\kappa}(\Psi, \mathbf{A}) = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left(|(\nabla - i\mathbf{A})\Psi|^2 + \frac{\kappa^2}{2}\left(|\Psi|^2 - 1\right)^2\right) \mathrm{d}X + \frac{1}{\varepsilon}\int_{\mathbb{R}^3} |\nabla \times \mathbf{A} - \mathbf{H}_{\text{ext}}|^2 \,\mathrm{d}X, \tag{2.2}$$

where Ω_ε is a thin superconductor corresponding to an ε-neighborhood of $\mathcal{M}$. More precisely,

$$\Omega_\varepsilon := \{X \in \mathbb{R}^3 : \ X = x + \varepsilon t \nu(x) \text{ for } x \in \mathcal{M}, \ t \in (0, 1)\}.$$

In the functional (2.2) the constant $\kappa > 0$ is the Ginzburg–Landau parameter, $\mathbf{H}_{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ is the applied magnetic field, that is, a given smooth, divergence-free vector field, and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ corresponds to the induced magnetic potential. As is natural, we take $G_{\varepsilon,\kappa}$ to be defined for $\Psi \in H^1(\Omega_\varepsilon; \mathbb{C})$. Regarding the domain of definition of the potential $\mathbf{A}$, we introduce

$$\mathcal{H} := \overline{\{\mathbf{A} \in C^\infty(\mathbb{R}^3; \mathbb{R}^3) : \ \mathbf{A} \text{ compactly supported}\}}, \tag{2.3}$$

where the closure above is with respect to the norm

$$\|\nabla \mathbf{A}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 \,\mathrm{d}x\right)^{1/2}.$$

Then we set $\mathcal{H}_0 = \{\mathbf{A} \in \mathcal{H} : \operatorname{div} \mathbf{A} = 0\}$. Consider $\mathbf{A}_{\text{ext}}$ a magnetic potential corresponding to the given external magnetic field $\mathbf{H}_{\text{ext}}$ to be any vector field satisfying the requirements

$$\nabla \times \mathbf{A}_{\text{ext}} = \mathbf{H}_{\text{ext}} \quad \text{and} \quad \operatorname{div} \mathbf{A}_{\text{ext}} = 0 \text{ in } \mathbb{R}^3. \tag{2.4}$$

These conditions determine $\mathbf{A}_{\text{ext}}$ up to the gradient of a harmonic function. Thus, $G_{\varepsilon,\kappa}$ will take pairs $(\psi, \mathbf{A}) \in H^1(\mathcal{M}; \mathbb{C}) \times (\{\mathbf{A}_{\text{ext}}\} + \mathcal{H}_0)$.

In [5], the Γ-limit of $G_{\varepsilon,\kappa}$ as $\varepsilon \to 0$ is obtained. We introduce the topology of this convergence; given $(\Psi^\varepsilon, \mathbf{A}^\varepsilon) \subset H^1(\Omega_\varepsilon; \mathbb{C}) \times (\{\mathbf{A}_{\text{ext}}\} + \mathcal{H}_0)$ and $(\psi, \mathbf{A}) \in H^1(\mathcal{M} \times (0, 1); \mathbb{C}) \times (\{\mathbf{A}_{\text{ext}}\} + \mathcal{H}_0)$ we will write $(\Psi^\varepsilon, \mathbf{A}^\varepsilon) \xrightarrow{Y} (\psi, \mathbf{A})$ provided

$$\psi^\varepsilon \rightharpoonup \psi \text{ weakly in } H^1(\mathcal{M} \times (0, 1); \mathbb{C}) \quad \text{and} \quad \mathbf{A}^\varepsilon - \mathbf{A} \to 0 \ \text{ strongly in } \mathcal{H}_0, \tag{2.5}$$

where $\psi^\varepsilon = \Psi^\varepsilon \circ T_\varepsilon$.

Then for $(\psi, \mathbf{A}) \in H^1(\mathcal{M}; \mathbb{C}) \times (\{\mathbf{A}_{\text{ext}}\} + \mathcal{H}_0)$ we define

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi) = \int_{\mathcal{M}}\left(\left|(\nabla_{\mathcal{M}} - i(\mathbf{A}_{\text{ext}})^\tau)\psi\right|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2\right) \mathrm{d}\mathcal{H}^2_{\mathcal{M}}(x), \tag{2.6}$$

and for $(\psi, \mathbf{A}) \in H^1(\mathcal{M} \times (0, 1); \mathbb{C}) \times (\{\mathbf{A}_{\text{ext}}\} + \mathcal{H}_0)$ we define

$$G_{\mathcal{M},\kappa}(\psi, \mathbf{A}) = \begin{cases} \mathcal{G}_{\mathcal{M},\kappa}(\psi) & \text{if } \psi_t = 0 \text{ almost everywhere in } \mathcal{M} \times (0, 1), \ \mathbf{A} = \mathbf{A}_{\text{ext}}, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.7}$$

where $\psi_t := \frac{\partial \psi}{\partial t}$. We point out that in (2.7) we have made the obvious identification between elements ψ of $H^1(\mathcal{M} \times (0,1); \mathbb{C})$ satisfying the condition $\psi_t = 0$ almost everywhere and elements of $H^1(\mathcal{M}; \mathbb{C})$.

Theorem 2.1. (cf. [5], Theorem 3.1 and Proposition 3.4) *The sequence of functionals $G_{\varepsilon,\kappa}$ Γ-converges as $\varepsilon \to 0$ to $G_{\mathcal{M},\kappa}$ in the Y-topology. In addition, given any sequence $\{(\Psi^\varepsilon, \mathbf{A}^\varepsilon)\} \subset H^1(\Omega_\varepsilon; \mathbb{C}) \times (\{\mathbf{A}^e\} + \mathcal{H}_0)$, satisfying a uniform energy bound*

$$G_{\varepsilon,\kappa}(\Psi^\varepsilon, \mathbf{A}^\varepsilon) \leqq C,$$

there exists a function $\psi \in H^1(\mathcal{M}; \mathbb{C})$ such that after passing to a subsequence one has

$$\begin{aligned} \psi_\varepsilon := \Psi^\varepsilon(T_\varepsilon) \rightharpoonup \psi \text{ weakly in } H^1(\mathcal{M} \times (0,1); \mathbb{C}) \\ \text{and } \ (\psi_\varepsilon)_t \to 0 \text{ strongly in } L^2(\mathcal{M} \times (0,1); \mathbb{C}), \end{aligned} \tag{2.8}$$

while

$$\mathbf{A}^\varepsilon - \mathbf{A}_{\text{ext}} \to 0 \text{ strongly in } \mathcal{H}_0. \tag{2.9}$$

The following is an improvement on a proposition in [5], which can be easily obtained via a bootstrap argument, when enough regularity of $\mathcal{M}$ (also $\partial\mathcal{M}$ when the manifold has boundary) is assumed.

Proposition 2.1. (cf. [5], Proposition 3.5) *Fix any $\kappa > 0$. For any $\varepsilon > 0$, let $\Psi_{\varepsilon,\kappa} : \Omega_\varepsilon \to \mathbb{C}$ and $\mathbf{A}_{\varepsilon,\kappa} : \mathbb{R}^3 \to \mathbb{R}^3$ denote a minimizing pair for $G_{\varepsilon,\kappa}$ with $\psi_{\varepsilon,\kappa} : \mathcal{M} \times (0,1) \to \mathbb{C}$ associated with $\Psi_{\varepsilon,\kappa}$ via $\psi_{\varepsilon,\kappa}(x,t) := \Psi_{\varepsilon,\kappa}(x + t\varepsilon\nu(x))$. Then there exists a subsequence $\{\varepsilon_j\} \to 0$ and a minimizer ψ_κ of $\mathcal{G}_{\mathcal{M},\kappa}$ such that $\psi_{\varepsilon_j,\kappa} \to \psi_\kappa$ in $C^{1,\alpha}(\mathcal{M} \times (0,1))$ for any positive $\alpha < 1$.*

Through Theorem 2.1 it is possible to establish a correspondence between properties of minimizers of $G_{\varepsilon,\kappa}$ and $G_{\mathcal{M},\kappa}$, provided ε is small, and in principle also between local minimizers (see [18]) and even non-degenerate critical points (see [17]). In light of this we study the limiting behavior of minimizers of $G_{\mathcal{M},\kappa}$, as $\kappa \to \infty$.

3. H_{c1} of a simply connected manifold and its associated thin shell

In this section we will take $\mathbf{H}_{\text{ext}}$ to depend on κ with the aim of determining the asymptotic value $\lim_{\kappa\to\infty} \mathbf{H}_{\text{ext}}(\kappa)$, above which the global minimizers of $G_{\mathcal{M},\kappa}$ and $G_{\varepsilon,\kappa}$ exhibit vortices. This is done in Theorems 3.1 and 3.2 below. These results extend those of [5] where the surface is taken to be of revolution and the applied field is constant and vertical. In the present work the surface is any simply connected smooth two-dimensional manifold without boundary and the field is arbitrary. To be more precise, let $\mathbf{H}^e$, $\mathbf{A}^e$ be smooth vector fields such that

$$\mathbf{H}^e = \nabla \times \mathbf{A}^e, \text{ for some } \mathbf{A}^e \text{ with } \operatorname{div} \mathbf{A}^e = 0 \text{ in } \mathbb{R}^3, \quad \text{and} \quad \left\|\mathbf{H}^e\right\|_{L^\infty} = 1. \tag{3.1}$$

Thus, given $\mathbf{H}^e$, $\mathbf{A}^e$ satisfying (3.1), we study asymptotically the response of a superconductor subject to external fields $\mathbf{H}_{\text{ext}} = \mathbf{H}_{\text{ext}}(\kappa)$, with applied potentials $\mathbf{A}_{\text{ext}} = \mathbf{A}_{\text{ext}}(\kappa)$, of the form:

$$\mathbf{H}_{\text{ext}}(\kappa) := h(\kappa)\,\mathbf{H}^e, \quad \mathbf{A}_{\text{ext}}(\kappa) := h(\kappa)\,\mathbf{A}^e. \tag{3.2}$$

We call $h(\kappa)$ the intensity or strength of $\mathbf{H}_{\text{ext}}$. The most commonly studied case corresponds to the family of fields arising from $\mathbf{H}^e = \hat{e}_z$, and our definition of strength is consistent with the one utilized in that situation. In this terminology, the first critical field, or H_{c1}, for $G_{\mathcal{M},\kappa}$ (resp. $G_{\varepsilon,\kappa}$) is the minimum value $h(\kappa)$ such that any global minimizer has at least one vortex (resp. vortex line). In order to describe how the value H_{c1} depends on $\mathbf{H}^e$ and the manifold, it is first necessary to divide the vector fields $\mathbf{H}^e$ according to:

(H_1) We say that $\mathbf{H}^e$ satisfies (H_1) if $\mathbf{H}^e$ is s.t. there exists $\phi \in \mathcal{C}^\infty(\mathcal{M};\mathbb{R})$, satisfying $\nabla_{\mathcal{M}}\phi = (\mathbf{A}^e)^\tau$ restricted to $\mathcal{M}$.

(H_2) We say that $\mathbf{H}^e$ satisfies (H_2) if $\mathbf{H}^e$ does not satisfy (H_1).

It is worth mentioning that neither of the above conditions is void. The next proposition implies in particular that non-vanishing vector fields satisfy (H_2) (see Remark 3.1 below). As an example of a vector field $\mathbf{H}^e$ satisfying (H_1), consider $\mathcal{M} = \mathbb{S}^2$ and $\mathbf{H}^e := \nabla \times \mathbf{A}^e$, where

$$\mathbf{A}^e = r^2 \sin\theta\,\hat{\theta} + \left(-\frac{r^2}{2}\cos\theta\right)\hat{r} + 0\cdot\hat{\phi},$$

and $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are the unit vectors for the spherical coordinates. One readily checks that $\mathbf{H}^e = \nabla \times \mathbf{A}^e = (\nabla \times \mathbf{A}^e)^\tau$, and $\nabla \cdot \mathbf{A}^e = 0$. Thus, since for a smooth function f we have

$$\nabla_{\mathbb{S}^2} f = \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi},$$

we get that for $f = -\cos\theta\, r^3$, $\nabla_{\mathbb{S}^2} f = (\mathbf{A}^e)^\tau = r^2\sin\theta\,\hat{\theta}$.

Proposition 3.1. *Given a manifold $\mathcal{M}$, assume $\mathbf{H}^e$ is a given smooth vector field satisfying (H_1). Then $\mathbf{H}^e = (\mathbf{H}^e)^\tau$ on $\mathcal{M}$.*

Proof. Let $\mathcal{S} \subset \mathcal{M}$ be any open simply connected subset of the manifold with boundary Γ. Hypothesis (H_1) guarantees the existence of a smooth function ϕ defined in a neighborhood in $\mathbb{R}^3$ of the manifold $\mathcal{M}$ such that its restriction to $\mathcal{M}$ satisfies: $\nabla_{\mathcal{M}}\phi = (\mathbf{A}^e)^\tau$. It then follows that

$$\begin{aligned} 0 &= \int_{\mathcal{S}} (\nabla \times (\nabla\phi))\cdot \nu = \int_{\Gamma} \nabla\phi \cdot \tau_\Gamma = \int_\Gamma \nabla_{\mathcal{M}}\phi\cdot\tau_\Gamma \\ &= \int_\Gamma (\mathbf{A}^e)^\tau \cdot \tau_\Gamma = \int_\Gamma \mathbf{A}^e\cdot\tau_\Gamma = \int_{\mathcal{S}} \mathbf{H}^e \cdot \nu. \end{aligned}$$

Since $\mathcal{S}$ is arbitrary, this implies $\mathbf{H}^e = (\mathbf{H}^e)^\tau$ on $\mathcal{M}$. □

Remark 3.1. Proposition 3.1 guarantees that if $\mathbf{H}^e$ does not vanish on $\mathcal{M}$, then $\mathbf{H}^e$ satisfies (H_2). Indeed $(\mathbf{H}^e)^\tau$ is a smooth vector field on $\mathcal{M}$, so by the Poincare–Hopf theorem it must vanish at some point p in $\mathcal{M}$. But if $\mathbf{H}^e(p) \neq \mathbf{0}$, then $\mathbf{H}^e(p) \neq (\mathbf{H}^e)^\tau(p)$ and hence $\mathbf{H}^e$ satisfies (H_2).

Our first result shows that for external fields defined in (3.2), where $\mathbf{H}^e$ satisfies (H_1), there is no first critical field for $G_{\mathcal{M},\kappa}$; that is, global minimizers never vanish, regardless of the strength of the applied field. Surprisingly, merging this with Theorem 2.1, we also obtain that for $G_{\varepsilon,\kappa}$ the value of H_{c1} is infinite for ε sufficiently small.

Theorem 3.1. *Let $\kappa > 0$ be a given positive number. Let $G_{\varepsilon,\kappa}$ and $\mathcal{G}_{\mathcal{M},\kappa}$ be the functionals defined in* (2.2) *and* (2.6)*, respectively, where $\mathbf{H}_{\text{ext}} = \mathbf{H}_{\text{ext}}(h) = h\,\mathbf{H}^e$, and $\mathbf{A}_{\text{ext}} = \mathbf{A}_{\text{ext}}(h) = h\,\mathbf{A}^e$, for $\mathbf{H}^e$, $\mathbf{A}^e$ satisfying* (3.1).

If $\mathbf{H}^e$ satisfies (H_1), then global minimizers of $\mathcal{G}_{\mathcal{M},\kappa}$ never vanish, independent of the intensity h of the external field. Furthermore, there is an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, any global minimizer Ψ^ε, of $G_{\varepsilon,\kappa}$ satisfies $|\Psi^\varepsilon| \geqq \frac{3}{4}$.

We point out that the hypothesis of Theorem 3.1 that $\mathbf{H}^e$ satisfies (H_1) is actually very sensitive even to arbitrarily small C^1 perturbations of $\mathcal{M}$. Thus, in general we expect to be in the case where $\mathbf{H}^e$ satisfies (H_2). As we will see in Theorem 3.2 below, such a perturbation would have the effect of lowering the value of H_{c1}, from effectively ∞, to $\mathcal{O}(\ln\kappa)$.

Proof. The proof is remarkably easy. First note that even though the limiting functional $\mathcal{G}_{\mathcal{M},\kappa}$ does not enjoy the gauge invariance of $G_{\varepsilon,\kappa}$, one still has

$$\begin{aligned}\mathcal{P}_1 &:= \min_{\psi} \int_{\mathcal{M}} \left\{ \left|(\nabla_{\mathcal{M}} - i\,h(\mathbf{A}^e)^\tau)\psi\right|^2 + \frac{\kappa^2}{2}(|\psi|^2-1)^2 \right\} \mathrm{d}\mathcal{H}^2_{\mathcal{M}} \\ &= \mathcal{P}_2 := \min_{\eta} \int_{\mathcal{M}} \left\{ \left|(\nabla_{\mathcal{M}} - i\,h((\mathbf{A}^e)^\tau + \nabla_{\mathcal{M}}\phi))\eta\right|^2 + \frac{\kappa^2}{2}(|\eta|^2-1)^2 \right\} \mathrm{d}\mathcal{H}^2_{\mathcal{M}},\end{aligned} \tag{3.3}$$

with the minimizers related via $\eta \sim \psi e^{i\phi}$. Note also that the vortex structure is preserved under this transformation. Since $\mathbf{H}^e$ satisfies (H_1), we can choose ϕ above to erase the contribution of the applied potential $h\,\mathbf{A}^e$ completely from the energy. Clearly then, the global minimizers of the resulting functional are simply constants of modulus 1. The last statement of the theorem follows immediately from the uniform convergence of minimizers provided by Proposition 2.1. □

We have now seen that fields $\mathbf{H}^e$ that satisfy (H_1) yield $H_{c1} = \infty$. In the rest of the section we compute the leading order term of the first critical field in the case in which $\mathbf{H}^e$ satisfies (H_2), and we find it is of order $\mathcal{O}(\ln\kappa)$.

Recall the definition of intensity $h(\kappa)$ of an external field $\mathbf{H}_{\text{ext}}(\kappa)$ given by (3.2). We assume the intensity obeys

$$\lim_{\kappa\to\infty}\frac{h(\kappa)}{\ln\kappa}=C_0, \tag{3.4}$$

for some non-negative constant C_0.

In what comes, we will at times view $(\mathbf{A}^e)^\tau$ as a 1-form, and whenever we do so it will be clear from the context. Let ϕ be a 0-form (or a function) on $\mathcal{M}$ satisfying

$$-\Delta_{\mathcal{M}}\phi=d^\star\left((\mathbf{A}^e)^\tau\right), \tag{3.5}$$

where $d^\star = *d*$ is the Hodge differential and $*$ is the Hodge star operator on forms. Equation (3.5) is always solvable since the kernel of $\Delta_{\mathcal{M}}$ consists only of constant functions while $\int_{\mathcal{M}} d*f=0$ for any 1-form f. Now let ϕ be a solution of (3.5) and extend it smoothly to all of $\mathbb{R}^3$. Call this extension $\tilde{\phi}$ and let

$$\tilde{\mathbf{A}^e}:=\nabla\tilde{\phi}+\mathbf{A}^e. \tag{3.6}$$

Notice that $d^\star((\tilde{\mathbf{A}^e})^\tau)=0$ on $\mathcal{M}$, where we are again making the identification of $(\mathbf{A}^e)^\tau$ with a 1-form. In light of (3.3), without loss of generality, we assume $\tilde{\mathbf{A}^e}=\mathbf{A}^e$. Since in our case $H^1_{dR}(\mathcal{M})=H^1_{dR}(\mathbb{S}^2)=0$, this implies the existence of a 2-form, F such that

$$d^\star F=(\mathbf{A}^e)^\tau. \tag{3.7}$$

Remark 3.2. Note that F is determined up to a constant. Recall that $\mathbf{H}^e$ satisfies (H_2) which implies $(\mathbf{A}^e)^\tau=d^\star F$ is not identically zero, so $*F$ is not constant.

We now present the main theorem of this section. The second part provides an equivalent of the main result of [23] in our setting. In our case, the role of ξ_0 in [23] is played by $*F$.

Theorem 3.2. *Let $\mathcal{G}_{\mathcal{M},\kappa}$ be the functional defined in (2.6) where the parameters are defined in (3.1), (3.2) and $\mathbf{H}^e$ satisfies (H_2). Then, if the intensity $h(\kappa)$ obeys (3.4) with*

$$\frac{1}{\max_{\mathcal{M}}*F-\min_{\mathcal{M}}*F}<C_0, \tag{3.8}$$

where F is any solution of (3.7), there exists a value κ_0 such that for all $\kappa\geqq\kappa_0$, any global minimizer ψ_κ of $\mathcal{G}_{\mathcal{M},\kappa}$ has at least two vortices of nonzero degree. If, instead the external field satisfies (3.4) with

$$\frac{1}{\max_{\mathcal{M}}*F-\min_{\mathcal{M}}*F}>C_0, \tag{3.9}$$

then there exists a value κ_0 such that for all $\kappa\geqq\kappa_0$, any global minimizer of $\mathcal{G}_{\mathcal{M},\kappa}$ does not vanish.

Theorem 2.1 allows us also to assert in this section that the value $C_0\ln\kappa$ serves as an asymptotic value for H_{c1} for the 3d Ginzburg–Landau energy $G_{\varepsilon,\kappa}$ as well, when ε is sufficiently small. To that end, for any $t\in(0,1)$ and $\varepsilon\in(0,\varepsilon_0)$, we introduce the manifold

$$\mathcal{M}_{\varepsilon,t}:=\{x+\varepsilon t\nu(x):\ x\in\mathcal{M}\}. \tag{3.10}$$

The following holds:

Theorem 3.3. *Let $G_{\varepsilon,\kappa}$ be the functional defined in (2.2) where the parameters are defined in (3.1), (3.2) and $\mathbf{H}^e$ satisfies (H_1). Fix any value $\kappa \geqq \kappa_0$ where κ_0 is the value arising in Theorem 3.2. Then, there exists a value $\varepsilon_0 = \varepsilon_0(\kappa)$ such that for all positive $\varepsilon < \varepsilon_0$, if (3.8) holds, any global minimizer $\Psi_{\varepsilon,\kappa}$ of $G_{\varepsilon,\kappa}$ vanishes at least twice on each manifold $\mathcal{M}_{\varepsilon,t}$, for $0 < t < 1$. On the other hand, if (3.9) holds, $\Psi_{\varepsilon,\kappa}$ does not vanish in Ω_ε.*

In the proof of Theorem 3.2 we will make use of a result that requires some background. We will denote by $\exp_p$ the exponential map for $\mathcal{M}$ at p, cf. [8]. It is well known that for r small enough, $\exp_p$ provides a local diffeomorphism from $T_p\mathcal{M}$ onto its image in $\mathcal{M}$. In this section a pseudo-ball will be the diffeomorphic image of a Euclidean ball under the exponential map, that is $\hat{B}(p,r) := \exp_p[B(0,r)]$ for $B(0,r) \subset T_p\mathcal{M}$. We state without proof the following proposition (see [5] for additional comments) which is nothing but the translation to our setting of a vortex-ball construction technique developed in [15,23].

Proposition 3.2. (cf. [5], Proposition 5.7) *Let ψ_κ be a sequence of smooth functions defined on $\mathcal{M}$, satisfying $|\nabla_{\mathcal{M}}\psi_\kappa| \leqq C \cdot \kappa$, with*

$$\int_{\mathcal{M}} |\nabla_{\mathcal{M}}\psi_\kappa|^2 + \frac{\kappa^2}{2}\left(|\psi_\kappa|^2 - 1\right)^2 \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}} \leqq C \cdot (\ln \kappa)^2. \tag{3.11}$$

Then, there exists a family $\hat{B}_j := \hat{B}(p_j, r_j)$ of disjoint pseudo-balls, with $p_j \in \mathcal{M}$ for $j = 1, \ldots, N_\kappa$, such that for κ sufficiently large

1. $\{|\psi_\kappa|^{-1}[0, 3/4)\} \subset \bigcup_{j \in I} \hat{B}_j$
2. $N_\kappa \leqq C \cdot (\ln \kappa)^2$
3. $r_j \leqq C \cdot (\ln \kappa)^{-6}$
4. $\int_{\hat{B}_j} |\nabla_{\mathcal{M}}\psi_\kappa|^2 + \frac{\kappa^2}{2}(|\psi_\kappa|^2 - 1)^2 \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}} \geqq 2\pi \left|d_j^\kappa\right| (\ln \kappa - O(\ln \ln \kappa))$,

where we have defined $d_j^{(\kappa)} := \deg(\psi_\kappa, \partial \hat{B}_j)$.

We will apply this proposition to global minimizers of $\mathcal{G}_{\mathcal{M},\kappa}$. The hypotheses will be satisfied since under assumption (3.4), we can compare the energy of a minimizer to the energy of $\psi \equiv 1$ to get the energy bound (3.11). Then the needed hypothesis $|\nabla_{\mathcal{M}}\psi_\kappa| \leqq C \cdot \kappa$ follows from elliptic regularity by working in local coordinates, rescaling these by $\frac{1}{\kappa}$ and applying standard Schauder theory, cf. [13]. By the compactness of $\mathcal{M}$, one constant C can be obtained such that the estimate holds along the entire manifold.

Prior to proving Theorems 3.2 and 3.3, we proceed to prove a few lemmas that will be of relevance.

We begin by constructing a comparison map that will give us more accurate control on one term of the energy that, as we will see, forces the emergence of vortices in minimizers. To that end, let $p_1 \in (*F)^{-1}(\max_{\mathcal{M}} *F)$ and

$p_2 \in (*F)^{-1}(\min_{\mathcal{M}} *F)$. Fix δ small and let $\hat{B}(p_1, \delta)$ and $\hat{B}(p_2, \delta)$ be two disjoint pseudo-balls. Define $f_\kappa : [0, \delta] \to \mathbb{R}$ by

$$f_\kappa(r) := \begin{cases} 0, & r \in \left[0, \frac{1}{2\kappa}\right) \\ 2\kappa\left(r - \frac{1}{2\kappa}\right), & r \in \left[\frac{1}{2\kappa}, \frac{1}{\kappa}\right) \\ 1, & r \in \left[\frac{1}{\kappa}, \delta\right]. \end{cases} \tag{3.12}$$

For $\ell = 1, 2$, let $\beta_\kappa^\ell(r, \theta) = f_\kappa(r) e^{(-1)^\ell i\theta}$. By definition, $\hat{B}(p_1, \delta)$and $\hat{B}(p_2, \delta)$ are diffeomorphic images of neighborhoods in $T_{p_1}\mathcal{M}$ and in $T_{p_2}\mathcal{M}$ under the exponential maps $\exp_{p_1}$ and $\exp_{p_2}$ respectively. We will parametrize each of these neighborhoods using polar coordinates (r_1, θ_1) and (r_2, θ_2) accordingly, where for $\ell = 1, 2$, θ_ℓ is measured clockwise, fitting with the orientation of $T_{p_\ell}\mathcal{M}$ for $\ell = 1, 2$ corresponding to the outer normal of $\mathcal{M}$ at each p_ℓ. Now, for each $x \in \hat{B}(p_\ell, \delta)$ there exists a unique (r_ℓ, θ_ℓ) s.t. $\exp_{p_\ell}(r_\ell, \theta_\ell) = x$ and we define

$$\tilde{\psi}_\kappa^\ell(x) := \beta_\kappa^\ell(r_\ell, \theta_\ell). \tag{3.13}$$

One readily checks

$$\left|\nabla_{\mathcal{M}} \tilde{\psi}_\kappa^\ell(x(r_\ell, \theta_\ell))\right|^2 \leqq (f_\kappa^{'})^2(r_\ell) + \frac{1}{r_\ell^2} f_\kappa^2(r_\ell) + C, \tag{3.14}$$

where C is independent of κ. Now $\mathcal{C} := \mathcal{M} \setminus (\hat{B}(p_1, \delta) \cup \hat{B}(p_2, \delta))$ is diffeomorphic to a cylinder and therefore we can find a function $\tilde{\psi} : \mathcal{C} \to \mathbb{S}^1 \subset \mathbb{C}$ such that $\tilde{\psi}|_{\partial \hat{B}(p_\ell, \delta)} = \tilde{\psi}_\kappa^\ell$, for $\ell = 1, 2$.

Finally define $\tilde{\psi}_\kappa : \mathcal{M} \to \mathbb{C}$ by

$$\tilde{\psi}_\kappa(x) := \begin{cases} \tilde{\psi}_\kappa^\ell(x) & \text{for } x \in \hat{B}(p_1, \delta) \cup \hat{B}(p_2, \delta) \\ \tilde{\psi}(x) & \text{otherwise.} \end{cases} \tag{3.15}$$

Lemma 3.1. *Assume $h(\kappa)$ satisfies (3.4) and that (3.7) holds. Let $\tilde{\psi}_\kappa$ be the function defined in (3.15). Then:*

$$\mathcal{G}_{\mathcal{M},\kappa}(\tilde{\psi}_\kappa) \leqq (h(\kappa))^2 \left\|(\mathbf{A}^e)^\tau\right\|_{L^2(\mathcal{M})}^2 + 4\pi \left(\ln \kappa - (\max_{\mathcal{M}} *F - \min_{\mathcal{M}} *F) h(\kappa)\right) + \mathcal{O}(1). \tag{3.16}$$

The next lemma gives a bound that contains crucial information about any minimizer. To that end we first need to introduce for any smooth $\mathbf{A} : \mathbb{R}^\mathbf{3} \to \mathbb{R}^\mathbf{3}$, and $\psi \in H^1(\mathcal{M}; \mathbb{C})$,

$$\Lambda(\mathbf{A}, \psi) := i \int_{\mathcal{M}} \mathbf{A}^\tau \cdot (\psi \nabla_{\mathcal{M}} \psi^* - \psi^* \nabla_{\mathcal{M}} \psi) \, d\mathcal{H}_{\mathcal{M}}^2, \tag{3.17}$$

where as before, $\mathbf{A}^\tau := \mathbf{A} - (\mathbf{A} \cdot \nu)\, \nu$. The superscript “ $*$ ” in formula (3.17) means complex conjugation and is not to be confused with expressions of the form $*g$, which denote the application of the star operation on forms.

Lemma 3.2. *Assume $h(\kappa)$ obeys (3.4) and that C_0 satisfies (3.8). Then any minimizer ψ_κ of $\mathcal{G}_{\mathcal{M},\kappa}$ satisfies*

$$h(\kappa)\Lambda(\mathbf{A}^e, \psi_\kappa) \geqq 4\pi \left((\max_{\mathcal{M}} *F - \min_{\mathcal{M}} *F)\, h(\kappa) - \ln\kappa\right) - \mathcal{O}(1).$$

Proof of Lemma 3.1. First observe that for any ψ:

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi) = \int_{\mathcal{M}} |\nabla_{\mathcal{M}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}} - h(\kappa)\Lambda(\mathbf{A}^e, \psi) \\ + (h(\kappa))^2 \int_{\mathcal{M}} \left|(\mathbf{A}^e)^\tau\right|^2 |\psi|^2 \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}}. \tag{3.18}$$

We compute, using the fact that $\left|\tilde{\psi}_\kappa\right| = 1$ outside $\hat{B}(p_1, \delta) \cup \hat{B}(p_2, \delta)$, definition (3.12) and estimate (3.14):

$$\begin{aligned} &\int_{\mathcal{M}} \left|\nabla_{\mathcal{M}}\tilde{\psi}_\kappa\right|^2 + \frac{\kappa^2}{2}\left(\left|\tilde{\psi}_\kappa\right|^2 - 1\right)^2 \mathrm{d}\mathcal{H}^2_{\mathcal{M}} \\ &= \int_{\mathcal{M}\setminus\hat{B}(p_1,\delta)\cup\hat{B}(p_2,\delta)} \left|\nabla_{\mathcal{M}}\tilde{\psi}_\kappa\right|^2 \mathrm{d}\mathcal{H}^2_{\mathcal{M}} + \sum_{\ell=1}^{2}\int_{\hat{B}(p_\ell,\delta)} \left|\nabla_{\mathcal{M}}\tilde{\psi}^\ell_\kappa\right|^2 \\ &\quad + \frac{\kappa^2}{2}\left(\left|\tilde{\psi}^\ell_\kappa\right|^2 - 1\right)^2 \mathrm{d}\mathcal{H}^2_{\mathcal{M}} \\ &\leqq C + \sum_{\ell=1}^{2}\int_0^{2\pi}\int_0^{\delta} \frac{1}{r^2}\,|f_\kappa|^2\,(1 + \mathcal{O}(r))r\,\mathrm{d}r\,\mathrm{d}\theta + \frac{\kappa^2}{2}\mathcal{H}^2_{\mathcal{M}}(\{f_\kappa \leqq 1\}) \\ &\leqq 4\pi\,\ln\kappa + \mathcal{O}(1). \end{aligned} \tag{3.19}$$

We next turn our attention to

$$\begin{aligned} -h(\kappa)\Lambda(\mathbf{A}^e, \tilde{\psi}_\kappa) &= ih(\kappa)\int_{\mathcal{M}\setminus\hat{B}(p_1,\frac{1}{\kappa})\cup\hat{B}(p_2,\frac{1}{\kappa})} \mathrm{d} * F \wedge \left(\tilde{\psi}^*_\kappa d\tilde{\psi}_\kappa - \tilde{\psi}_\kappa d\tilde{\psi}^*_\kappa\right) \\ &+ ih(\kappa)\sum_{\ell=1}^{2}\int_{\hat{B}(p_\ell,\frac{1}{\kappa})} (\mathbf{A}^e)^\tau \cdot \left(\left(\tilde{\psi}^\ell_\kappa\right)^* \nabla_{\mathcal{M}}\tilde{\psi}^\ell_\kappa - \tilde{\psi}^\ell_\kappa \nabla_{\mathcal{M}}\left(\tilde{\psi}^\ell_\kappa\right)^*\right) \\ &=: I^1_{\tilde{\psi}_\kappa} + I^2_{\tilde{\psi}_\kappa}. \end{aligned} \tag{3.20}$$

The quantity $I^2_{\tilde{\psi}_\kappa}$ is negligible. Using Hölder's inequality together with $h(\kappa) = \mathcal{O}(\ln\kappa)$ and $\max_{\ell=1,2}\mathcal{H}^2_{\mathcal{M}}(\hat{B}(p_\ell, \delta)) = \mathcal{O}\left(\frac{1}{\kappa^2}\right)$, we see

$$\left|I^2_{\tilde{\psi}_\kappa}\right| \leqq 2h(\kappa)\cdot\left\|\mathbf{A}^e\right\|_{L^\infty}\cdot\max_{\ell=1,2}\left\{\left\|\nabla_{\mathcal{M}}\tilde{\psi}^\ell_\kappa\right\|_{L^2(\hat{B}(p_\ell,\delta))}\right\}\cdot 1\cdot\frac{1}{\kappa} \leqq C\cdot\frac{(\ln\kappa)^2}{\kappa}. \tag{3.21}$$

As for the first term, we have

$$I^1_{\tilde{\psi}_\kappa} = ih(\kappa)\int_{\mathcal{M}\backslash \hat{B}\left(p_1,\frac{1}{\kappa}\right)\cup \hat{B}\left(p_2,\frac{1}{\kappa}\right)} d\left[*F\cdot\left(\tilde{\psi}^*_\kappa d\tilde{\psi}_\kappa - \tilde{\psi}_\kappa d\tilde{\psi}^*_\kappa\right)\right]$$
$$+ih(\kappa)\int_{\mathcal{M}\backslash \hat{B}\left(p_1,\frac{1}{\kappa}\right)\cup \hat{B}\left(p_2,\frac{1}{\kappa}\right)} *F\cdot(d\tilde{\psi}_\kappa\wedge d\tilde{\psi}^*_\kappa - d\tilde{\psi}_\kappa\wedge d\tilde{\psi}^*_\kappa).$$

But $(d\tilde{\psi}_\kappa\wedge d\tilde{\psi}^*_\kappa - d\tilde{\psi}_\kappa\wedge d\tilde{\psi}^*_\kappa) = 0$ on $\mathcal{M}\backslash\hat{B}(p_1,\frac{1}{\kappa})\cup\hat{B}(p_2,\frac{1}{\kappa})$, since $f_\kappa \equiv 1$ there. Thus, integration by parts yields $I^1_{\tilde{\psi}_\kappa} = ih(\kappa)\sum_{\ell=1}^2\int_{\partial\hat{B}(p_\ell,\frac{1}{\kappa})} *F((\tilde{\psi}^\ell_\kappa)^* d\tilde{\psi}^\ell_\kappa - \tilde{\psi}^\ell_\kappa\,\mathrm{d}(\tilde{\psi}^\ell_\kappa)^*)$, where the boundaries $\partial\hat{B}(p_\ell,\frac{1}{\kappa})$ adopt the induced orientation by $\mathcal{M}$. It follows then that:

$$I^1_{\tilde{\psi}_\kappa} = i\,h(\kappa)\left[\int_0^{2\pi} *F\left(x\left(\frac{1}{\kappa},\theta_1\right)\right)\left(2i+\mathcal{O}\left(\frac{1}{\kappa}\right)\right)\mathrm{d}\theta_1\right.$$
$$\left.+\int_0^{2\pi} *F\left(x\left(\frac{1}{\kappa},\theta_2\right)\right)\left(-2i+\mathcal{O}\left(\frac{1}{\kappa}\right)\right)\mathrm{d}\theta_2\right]. \tag{3.22}$$

On the other hand $*F(x(\frac{1}{\kappa},\theta_\ell)) = *F(p_\ell)+\mathcal{O}(\frac{(\nabla_{\mathcal{M}}*F)}{\kappa})$. Plugging this into (3.22) and replacing (3.21) and (3.22) in (3.20), yields

$$-h(\kappa)\Lambda(\mathbf{A}^e,\tilde{\psi}_\kappa) = 4\pi\, h(\kappa)\,[*F(p_2) - *F(p_1)] + o(1). \tag{3.23}$$

Finally the last term in (3.18) can be computed rather easily using (3.15). One has,

$$(h(\kappa))^2\int_{\mathcal{M}} \left|(\mathbf{A}^e)^\tau\right|^2\cdot\left|\tilde{\psi}_\kappa\right|^2 \mathrm{d}\mathcal{H}^2_{\mathcal{M}} = (h(\kappa))^2\int_{\mathcal{M}}\left|(\mathbf{A}^e)^\tau\right|^2 \mathrm{d}\mathcal{H}^2_{\mathcal{M}} + o(1). \tag{3.24}$$

Estimates (3.19), (3.23) and (3.24) applied to (3.18), allow us to conclude (3.16). □

Proof of Lemma 3.2. Simply by considering the function às a competitor, we observe that any global minimizer ψ_κ must satisfy the bound

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi_\kappa) \leqq (h(\kappa))^2\left\|(\mathbf{A}^e)^\tau\right\|^2_{L^2(\mathcal{M})}. \tag{3.25}$$

Likewise, any global minimizer ψ_κ satisfies

$$(h(\kappa))^2\int_{\mathcal{M}}\left|(\mathbf{A}^e)^\tau\right|^2|\psi_\kappa|^2\,\mathrm{d}\mathcal{H}^2_{\mathcal{M}} - h(\kappa)\Lambda(\tilde{\mathbf{A}}^e,\psi_\kappa) \leqq \mathcal{G}_{\mathcal{M},\kappa}(\psi_\kappa)\leqq\mathcal{G}_{\mathcal{M},\kappa}(\tilde{\psi}_\kappa). \tag{3.26}$$

Writing $(h(\kappa))^2\int_{\mathcal{M}}|(\mathbf{A}^e)^\tau|^2|\psi_\kappa|^2\,\mathrm{d}\mathcal{H}^2_{\mathcal{M}} = (h(\kappa))^2\int_{\mathcal{M}}|(\mathbf{A}^e)^\tau|^2\,\mathrm{d}\mathcal{H}^2_{\mathcal{M}} + I$, and appealing to estimate (3.25), we know by (3.4) that

$$|I| \leqq (h(\kappa))^2\left\|(\mathbf{A}^e)^\tau\right\|^2_{L^4(\mathcal{M})}\cdot\left(\frac{\ln\kappa}{\kappa}\right)\leqq C\frac{(\ln\kappa)^3}{\kappa}. \tag{3.27}$$

Hence,

$$(h(\kappa))^2 \int_{\mathcal{M}} \left|(\mathbf{A}^e)^\tau\right|^2 |\psi_\kappa|^2 \, d\mathcal{H}^2_{\mathcal{M}} = (h(\kappa))^2 \left\|(\mathbf{A}^e)^\tau\right\|^2_{L^2(\mathcal{M})} + o(1). \tag{3.28}$$

Thus, as Lemma 3.1 provides us with an upper bound for $\mathcal{G}_{\mathcal{M},\kappa}(\tilde{\psi}_\kappa)$, we can rearrange the terms in (3.18) to obtain the desired conclusion. □

We are now able to present

Proof of Theorem 3.2. We divide the proof in two parts. First:

Upper Bound for the first critical field We assume C_0 satisfies (3.8). Since $\{\psi_\kappa\}$ are global minimizers and their energy satisfies the energy bound (3.25), we can appeal to Proposition 3.2 to obtain up to $o(1)$ the value of the quantity:

$$\begin{aligned} h(\kappa)\Lambda(\mathbf{A}^e, \psi_\kappa) &= h(\kappa) i \int_{\bigcup_{j\in I} \hat{B}_j} (\mathbf{A}^e)^\tau \cdot \left(\psi_\kappa \nabla_{\mathcal{M}} \psi_\kappa^* - \psi_\kappa^* \nabla_{\mathcal{M}} \psi_\kappa\right) d\mathcal{H}^2_{\mathcal{M}} \\ &\quad + h(\kappa) i \int_{\mathcal{M}\setminus\bigcup_{j\in I} \hat{B}_j} (\mathbf{A}^e)^\tau \cdot \left(\psi_\kappa \nabla_{\mathcal{M}} \psi_\kappa^* - \psi_\kappa^* \nabla_{\mathcal{M}} \psi_\kappa\right) d\mathcal{H}^2_{\mathcal{M}} \\ &= II + III. \end{aligned}$$

This will be achieved by performing, following Sandier and Serfaty (cf. [25]):

Jacobian estimates on a manifold First, Hölder's inequality with the aid of Proposition 3.2 yields

$$|II| \leqq C \cdot h(\kappa) \left\|\mathbf{A}^e\right\|_{L^\infty} \|\nabla_{\mathcal{M}} \psi_\kappa\|_{L^2(\mathcal{M})} \cdot \frac{N_\kappa}{(\ln \kappa)^6} \leqq \frac{C}{(\ln \kappa)^2}. \tag{3.29}$$

Then writing $\alpha := \frac{\psi_\kappa}{|\psi_\kappa|}$, III can be computed by

$$\begin{aligned} III &= h(\kappa) i \int_{\mathcal{M}\setminus\bigcup_{j\in I} \hat{B}_j} (\mathbf{A}^e)^\tau \cdot \left(\alpha \nabla_{\check{\mathcal{M}}} \alpha^* - \alpha^* \nabla_{\mathcal{M}} \alpha\right) d\mathcal{H}^2_{\mathcal{M}} \\ &\quad + h(\kappa) i \int_{\mathcal{M}\setminus\bigcup_{j\in I} \hat{B}_j} (|\psi_\kappa|^2 - 1)(\mathbf{A}^e)^\tau \cdot \left(\alpha \nabla_{\check{\mathcal{M}}} \alpha^* - \alpha^* \nabla_{\mathcal{M}} \alpha\right) d\mathcal{H}^2_{\mathcal{M}} \\ &= IV + V. \end{aligned} \tag{3.30}$$

The term V is actually harmless. Estimate (3.25) together with the fact that $|\psi| \geqq 3/4$ on $\mathcal{M} \setminus \bigcup_{i\in I} \hat{B}_j$ imply:

$$\begin{aligned} |V| &\leqq 2h(\kappa) \left\|\mathbf{A}^e\right\|_{L^\infty} \left(\int_{\mathcal{M}\setminus\bigcup_{j\in I} \hat{B}_j} (|\psi_\kappa|^2 - 1)^2 \, d\mathcal{H}^2_{\mathcal{M}}\right)^{1/2} \\ &\quad \times \left(\int_{\mathcal{M}\setminus\bigcup_{j\in I} \hat{B}_j} |\nabla_{\mathcal{M}} \alpha|^2 \, d\mathcal{H}^2_{\mathcal{M}}\right)^{1/2} \\ &\leqq C\,(\ln \kappa) \left(\frac{\ln \kappa}{\kappa}\right) \left((4/3)^2 \int_{\mathcal{M}\setminus\bigcup_{j\in I} \hat{B}_j} |\nabla_{\mathcal{M}} \psi_\kappa|^2 \, d\mathcal{H}^2_{\mathcal{M}}\right)^{1/2} \\ &\leqq C \left(\frac{(\ln \kappa)^3}{\kappa}\right). \end{aligned} \tag{3.31}$$

We now turn to IV. Recall F satisfies (3.7), thus

$$IV = ih(\kappa)\int_{\mathcal{M}\setminus\bigcup_{j\in I}\hat{B}_j}(\alpha\,\mathrm{d}*F\wedge d\alpha^* - \alpha^*\,\mathrm{d}*F\wedge d\alpha).$$

Then

$$IV = ih(\kappa)\int_{\mathcal{M}\setminus\bigcup_{j\in I}\hat{B}_j}\mathrm{d}\left(*F(\alpha d\alpha^* - \alpha^* d\alpha)\right) \\ +ih(\kappa)\int_{\mathcal{M}\setminus\bigcup_{j\in I}\hat{B}_j}*F(d\alpha^*\wedge d\alpha - d\alpha\wedge d\alpha^*). \tag{3.32}$$

The last integral is zero because $|\alpha| = 1$. We integrate by parts to obtain

$$IV = -4\pi h(\kappa)\sum_{j=1}^{N_\kappa}*F(p_j)d_j{}^{(\kappa)} \\ +h(\kappa)\sum_{j=1}^{N_\kappa}\int_{\partial\hat{B}_j}(*F - *F(p_j))i(\alpha d\alpha^* - \alpha^* d\alpha). \tag{3.33}$$

We will argue that the last sum above is $o(1)$. Indeed, define

$$\hat{\psi} := \begin{cases}\psi_\kappa & \text{if } |\psi_\kappa| \leqq 3/4,\\ \frac{3}{4}\frac{\psi_\kappa}{|\psi_\kappa|} & \text{if } |\psi_\kappa| > 3/4,\end{cases}$$

and $\hat{\alpha} := \frac{\hat{\psi}}{\left|\hat{\psi}\right|}$. Then the sum becomes, letting $\hat{R} = \sum_{j=1}^{N_\kappa}\int_{\partial\hat{B}_j}\left(*F - *F(p_j)\right)$ $i\,(\alpha d\alpha^* - \alpha^* d\alpha)$

$$\begin{aligned}\hat{R} &= \sum_{j=1}^{N_\kappa}\int_{\partial\hat{B}_j}\left(*F - *F(p_j)\right)i\,(\hat{\alpha}d\hat{\alpha}^* - \hat{\alpha}^* d\hat{\alpha})\\ &= \frac{16}{9}\sum_{j=1}^{N_\kappa}\int_{\partial\hat{B}_j}\left(*F - *F(p_j)\right)i\,(\hat{\psi}d\hat{\psi}^* - \hat{\psi}^* d\hat{\psi})\\ &= \frac{16}{9}\sum_{j=1}^{N_\kappa}\int_{\hat{B}_j}d\left(\left(*F - *F(p_j)\right)i\,(\hat{\psi}d\hat{\psi}^* - \hat{\psi}^* d\hat{\psi})\right)\\ &= \frac{16}{9}\sum_{j=1}^{N_\kappa}\int_{\hat{B}_j}d*F\wedge i\left(\hat{\psi}d\hat{\psi}^* - \hat{\psi}^* d\hat{\psi}\right)\\ &\quad+\frac{32}{9}\sum_{j=1}^{N_\kappa}\int_{\hat{B}_j}\left(*F - *F(p_j)\right)d\,\hat{\psi}\wedge d\,\hat{\psi}^* = R_1 + R_2.\end{aligned} \tag{3.34}$$

But since the gradient of $*F$ is bounded on $\mathcal{M}$ and the norm of the gradient of $\hat{\psi}$ is bounded by the norm of the gradient of ψ_κ, we can invoke Proposition 3.2 to find that

$$\begin{aligned} h(\kappa)\,|R_1| &\leqq Ch(\kappa)\sum_{j=1}^{N_\kappa} \|\nabla_{\mathcal{M}}\psi_\kappa\|_{L^2(\hat{B}_j)}\ \|1\|_{L^2(\hat{B}_j)} \\ &\leqq C\ (\ln\kappa)^2\ \frac{N_\kappa}{(\ln\kappa)^6} \\ &\leqq C\frac{(\ln\kappa)^4}{(\ln\kappa)^6}. \end{aligned} \tag{3.35}$$

To estimate R_2, note that inside each pseudo-ball $\hat{B}_j$ we have $\left|*F - *F(p_j)\right| \leqq \frac{C}{|\ln\kappa|^6}$. In this way we see that

$$h(\kappa)\,|R_2| \leqq C\ (\ln\kappa)\ \|\nabla_{\mathcal{M}}\psi_\kappa\|^2_{L^2(\mathcal{M})}\ \frac{N_\kappa}{(\ln\kappa)^6} \leqq C\ \frac{(\ln\kappa)^5}{(\ln\kappa)^6}. \tag{3.36}$$

So we have

$$h(\kappa)\Lambda(\mathbf{A}^e,\psi_\kappa) = -4\pi h(\kappa)\sum_{j=1}^{N_\kappa} *F(p_j)d_j^{(\kappa)} + o(1), \tag{3.37}$$

thanks to (3.29), (3.30), (3.31), (3.33), (3.34), (3.35) and (3.36).

Thus, we conclude that if either $N_\kappa = 0$ or if $d_j^{(\kappa)} = 0$ for all j, then $h(\kappa)\Lambda(\mathbf{A}^e,\psi_\kappa) = o(1)$, and this would conflict with Lemma 3.2. To finish the proof, simply take $0 \leqq j_\kappa \leqq N_\kappa$ such that $d_{j_\kappa}^{(\kappa)} \neq 0$. Now $\partial\hat{B}_{j_\kappa}$ divides $\mathcal{M}$ into two submanifolds, each of them homeomorphic to a disk. But $\mathcal{M}$ is simply connected and it then follows that the zeros of ψ_κ are isolated, whence each of the submanifolds contains a vortex of nonzero degree.

Lower Bound for the first critical field In this part, we assume C_0 satisfies (3.9). To establish this we first claim that in this case

$$\begin{aligned} \mathcal{G}_{\mathcal{M},\kappa}(\psi_\kappa) &\geqq 2\pi\sum_{j=1}^{N_\kappa}\left|d_j^{(\kappa)}\right|(\ln\kappa - \mathcal{O}(\ln\ln\kappa)) + h(\kappa)^2\left\|(\mathbf{A}^e)^\tau\right\|^2_{L^2(\mathcal{M})} \\ &\quad +4\pi h(\kappa)\sum_{j=1}^{N_\kappa} *F(p_j)d_j^{(\kappa)} - o(1). \end{aligned} \tag{3.38}$$

Indeed, through an appeal to Proposition 3.2

$$\int_{\mathcal{M}} |\nabla_{\mathcal{M}}\psi_\kappa|^2 + \frac{\kappa^2}{2}\left(|\psi_\kappa|^2 - 1\right)^2\,\mathrm{d}\mathcal{H}^2_{\mathcal{M}} \geqq 2\pi\sum_{j=1}^{N_\kappa}\left|d_j^{(\kappa)}\right|(\ln\kappa - \mathcal{O}(\ln\ln\kappa)). \tag{3.39}$$

Also, because (3.28) and (3.37) are still valid in the present situation, we have

$$(h(\kappa))^2 \int_{\mathcal{M}} \left|(\mathbf{A}^e)^\tau\right|^2 |\psi_\kappa|^2 \, d\mathcal{H}^2_{\mathcal{M}} - h(\kappa)\Lambda(\mathbf{A}^e, \psi_\kappa)$$
$$= (h(\kappa))^2 \left\|(\mathbf{A}^e)^\tau\right\|^2_{L^2_{(\mathcal{M})}} + 4\pi h(\kappa) \sum_{j=1}^{N_\kappa} *F(p_j) d_j^{(\kappa)} + o(1). \quad (3.40)$$

Adding up (3.39) and (3.40), yields (3.38). Recall that $\alpha := \frac{\psi_\kappa}{|\psi_\kappa|}$. We note that

$$4\pi \sum_{j\in I} d_j^{(\kappa)} = i \sum_{j=1}^{N_\kappa} \int_{\partial \hat{B}_j} \alpha d\alpha^* - \alpha^* d\alpha = i \int_{\mathcal{M}\setminus \bigcup_{j\in I} \hat{B}_j} d(\alpha d\alpha^* - \alpha^* d\alpha) = 0. \quad (3.41)$$

Denoting by N_κ^+ the number of pseudo-balls out of the total of N_κ that carry a positive degree and assuming, without any loss of generality, that the pseudo-balls are ordered so that the ones with positive degree are listed first, we can express (3.41) as

$$\sum_{j=1}^{N_\kappa^+} d_j^{(\kappa)} + \sum_{j=N_\kappa^+ +1}^{N_\kappa} d_j^{(\kappa)} = 0 \quad \text{or equivalently,} \quad \sum_{j=1}^{N_\kappa} \left|d_j^{(\kappa)}\right| = 2 \sum_{j=1}^{N_\kappa^+} d_j^{(\kappa)}. \quad (3.42)$$

We then invoke (3.38) and the inequality $\mathcal{G}_{\mathcal{M},\kappa}(\psi_\kappa) \leqq \mathcal{G}_{\mathcal{M},\kappa}(1)$ to obtain

$$(h(\kappa))^2 \left\|(\mathbf{A}^e)^\tau\right\|_{L^2(\mathcal{M})} \geqq 2\pi \sum_{j=1}^{N_\kappa} \left|d_j^{(\kappa)}\right| (\ln \kappa - \mathcal{O}(\ln \ln \kappa))$$
$$+ (h(\kappa))^2 \left\|(\mathbf{A}^e)^\tau\right\|_{L^2(\mathcal{M})}$$
$$+ 4\pi h(\kappa) \sum_{j=1}^{N_\kappa} *F(p_j) d_j^{(\kappa)} - o(1). \quad (3.43)$$

This implies

$$(1 + o(1)) \ln \kappa \sum_{j=1}^{N_\kappa} \left|d_j^{(\kappa)}\right| \leqq \left(\max_{\mathcal{M}} *F - \min_{\mathcal{M}} *F\right) h(\kappa) \sum_{j=1}^{N_\kappa} \left|d_j^{(\kappa)}\right| + o(1).$$

But in view of (3.4) and the assumption $C_0 < \frac{1}{\max_{\mathcal{M}} *F - \min_{\mathcal{M}} *F}$, this cannot hold for κ sufficiently large unless

$$\sum_{j=1}^{N_\kappa} \left|d_j^{(\kappa)}\right| = 0, \quad (3.44)$$

that is, unless the zeros (if any) of the minimizer ψ_κ all have zero degree. Pursuing this possibility, however, we note that (3.37) would then imply that $\Lambda(\mathbf{A}^e, \psi_\kappa)$

$= o(1)$ and so in view of the fact that ψ_κ is a minimizer, we would find $\int_{\mathcal{M}} |\nabla_{\mathcal{M}}\psi_\kappa|^2 + \frac{\kappa^2}{2}\left(|\psi_\kappa|^2 - 1\right)^2 \, d\mathcal{H}^2_{\mathcal{M}} = o(1)$. But if there exists even one zero of ψ of zero degree, say at $x = p \in \mathcal{M}$, then the estimate $|\nabla_{\mathcal{M}}\psi| \leqq C \cdot \kappa$ implies that $|\psi| \leqq 1/2$ on a pseudo-ball $\hat{B}(p, r)$ for a radius $r \geqq \frac{C_1}{\kappa}$ for some C_1 independent of κ. Hence, we can rule out the possibility of (3.44) since we would then have

$$\int_{\mathcal{M}} |\nabla_{\mathcal{M}}\psi_\kappa|^2 + \frac{\kappa^2}{2}\left(|\psi_\kappa|^2 - 1\right)^2 \, d\mathcal{H}^2_{\mathcal{M}} \geqq \int_{\hat{B}(p,r)} |\nabla_{\mathcal{M}}\psi_\kappa|^2 + \frac{\kappa^2}{2}\left(|\psi_\kappa|^2 - 1\right)^2 \, d\mathcal{H}^2_{\mathcal{M}} \geqq C_2,$$

for some positive constant C_2 independent of κ, a contradiction. The theorem is proved. □

We conclude this section with the extension of Theorem 3.2 to the small thickness setting.

Proof of Theorem 3.3. First we prove that under the assumption that C_0 satisfies (3.8) and for fixed $\kappa > \kappa_0$, global minimizers of $G_{\varepsilon,\kappa}$ must vanish at least twice on each $\mathcal{M}_{\varepsilon,t}$, for all $t \in (0, 1)$, provided ε is sufficiently small. We argue by contradiction. Suppose for some $t \in (0, 1)$ that there is a sequence $\{\varepsilon_j\} \to 0$, and a sequence of global minimizers $\Psi_{\varepsilon_j,\kappa}$ that do not vanish on $\mathcal{M}_{\varepsilon_j,t}$. After perhaps passing to a further subsequence (still denoted ε_j), we may apply Proposition 2.1 to establish that $\psi_{\varepsilon_j,\kappa} \to \psi_\kappa$ in $C^{0,\alpha}$, where, ψ_κ is a global minimizer of $\mathcal{G}_{\mathcal{M},\kappa}$. Associated with this minimizer there is a pseudo-ball $\hat{B}$ guaranteed by Theorem 3.2 and Proposition 3.2 with an associated degree $d^{(\kappa)} \neq 0$. Since ψ_κ is independent of t, the degree $\deg(\psi_\kappa, \{x + \varepsilon_j t\nu(x) : x \in \partial\hat{B}\})$, must be different from zero for all $t \in (0, 1)$ as well. But then $\deg(\psi_{\varepsilon_j,\kappa}, \{x + \varepsilon_j t\nu(x) : x \in \partial\hat{B}\}) \neq 0$, must be valid in light of uniform convergence. Since the set $\{x + \varepsilon_j t\nu(x) : x \in \partial\hat{B}\}$ is diffeomorphic to a circle, it divides the manifold into two disjoint components, each of which is diffeomorphic to a disk, whence each contains a zero, and a contradiction is reached.

Now, to prove the second statement in Theorem 3.3 we simply note that it is a straightforward consequence of the uniform convergence of minimizer of $G_{\varepsilon,\kappa}$ guaranteed by Proposition 2.1, coupled with the non-vanishing property of minimizers of the Γ-limit provided by the second part of Theorem 3.2. □

4. Energy estimates for critical points when there is a bounded number vortices

The results presented in this section comprise several propositions that are basically drawn from [2,3], where a similar functional is studied in a planar setting. When needed, we carefully present the necessary adjustments to those results to fit our purposes. Here, we assume that $\sum_{j=1}^{N_\kappa} \left|d_j^{(\kappa)}\right|$ is bounded independent of κ which allows us to isolate the singularities of ψ_κ in a bounded (independent of κ)

number of pseudo-balls. From this, the lower bound on the energy of a minimizer obtained in Section 3 is improved by adding a sum of terms that accounts for the vortex interaction. This is not possible if we employ the pseudo-balls provided by Proposition 3.2; they are too large in the sense that they may contain many vortices. That is the reason why we need a smaller scale concentration construction.

In this section suitable competitors are also constructed, providing us with an almost matching upper bound for the energy of ψ_κ. In this section we assume that the manifold $\mathcal{M}$ is analytic in addition to being simply connected. We denote by $d_{\mathcal{M}}(x, y)$, the geodesic distance between points $x, y \in \mathcal{M}$ whenever it makes sense.

At this point, it will be more convenient to work with isothermal balls rather than with geodesic balls as we did in the first part of the paper. The reason behind this is the simple form that the Laplace–Beltrami operator takes in these coordinates, which allows us to write a Pohozaev's identity that is the basis for a small scale concentration construction, as in [2,3]. As we point out in the introduction, we use the term pseudo-ball indistinctly when referring either to a geodesic ball or an isothermal ball, with the only purpose of avoiding confusion with Euclidean space terminology. We fix notation that we use until the end of this paper. First, let r_0 denote the injectivity radius. For each point $p \in \mathcal{M}$, let $(U_p, \mathcal{I}_p)$ be an isothermal coordinate chart (which always exists since we are in dimension 2), that is, $\mathcal{I}_p$ is a conformal map from U_p onto $\mathbb{R}^2$. We define $\mathcal{B}(p, r) := \mathcal{I}_p^{-1}(B(\mathcal{I}_p(p), r))$. In these coordinates, we can write the metric near p as

$$\lambda^2(dx^2+dy^2) \text{ where } \lambda \text{ is a smooth function with the property } \lambda(\mathcal{I}_p(p))=1, \tag{4.1}$$

and the Laplace–Beltrami operator takes the form

$$\Delta_{\mathcal{M}} = \frac{1}{\lambda^2}\Delta, \tag{4.2}$$

where Δ denotes the Euclidean Laplacian.

First we prove a lemma that gives a Pohozaev identity bound at the level of pseudo-balls of radius $\frac{1}{\kappa^\alpha}$, for $0 < \alpha < 1$.

Lemma 4.1. *Fix $0 < \alpha < 1$. Let $\{\psi_\kappa\}$ be a sequence of critical points of $\mathcal{G}_{\mathcal{M},\kappa}$. Assume the intensity $h(\kappa)$ satisfies (3.4). Assume also the uniform bound*

$$\int_{\mathcal{M}} |\nabla_{\mathcal{M}}\psi_\kappa|^2 + \frac{\kappa^2}{2}\left(|\psi_\kappa|^2 - 1\right)^2 \, d\mathcal{H}^2_{\mathcal{M}} \leqq C \ln \kappa. \tag{4.3}$$

Then, for all κ such that $\frac{1}{\kappa^\alpha} < \frac{r_0}{2}$, one has

$$\kappa^2 \int_{\mathcal{B}\left(p, \frac{1}{\kappa^\alpha}\right)} (1 - |\psi_\kappa|^2)^2 < C_\alpha, \tag{4.4}$$

where C_α depends on α, but not on the point $p \in \mathcal{M}$.

Proof of Lemma 4.1. Since ψ_κ is a critical point and $d^\star(\mathbf{A}^e)^\tau = 0$, we have the Euler-Lagrange equation

$$- \Delta_{\mathcal{M}}\psi_\kappa = \kappa^2\psi_\kappa(1 - |\psi_\kappa|^2) - 2ih(\kappa)[(\mathbf{A}^e)^\tau \cdot \nabla_{\mathcal{M}}]\psi_\kappa \\ -(h(\kappa))^2 \left|(\mathbf{A}^e)^\tau\right|^2 \psi_\kappa \quad \text{on } \mathcal{M}. \tag{4.5}$$

In the proof we use the notation $\psi_\kappa^{\mathbf{euc}} := \psi_\kappa \circ (\mathcal{I}_p)^{-1}$. Let (x_1, x_2) denote the canonical Euclidean coordinates in $\mathbb{R}^2$. We identify a complex valued function $\psi = \mathfrak{Re}\psi + i\mathfrak{Im}\psi$ with $(\mathfrak{Re}\psi, \mathfrak{Im}\psi)$, and consistently $i\psi$ with $\psi^\perp := (-\mathfrak{Im}\psi, \mathfrak{Re}\psi)$. We denote by $\langle\cdot,\cdot\rangle$, the scalar product in $\mathbb{R}^2$. Define $P_{p,r} := \kappa^2 \int_{B(\mathcal{I}_p(p),r)} \left(1 - |\psi_\kappa^{\mathbf{euc}}|^2\right)^2 \left[\lambda^2 + 2\lambda\langle\nabla\lambda, ((x_1, x_2) - \mathcal{I}_p(p))\rangle\right] \mathrm{d}x$. Phrasing (4.5) in isothermal coordinates, multiplying (4.5) by

$$S := \lambda^2 \left[x_1 \frac{\partial\psi_\kappa^{\mathbf{euc}}}{\partial x_1} + x_2 \frac{\partial\psi_\kappa^{\mathbf{euc}}}{\partial x_2}\right] \tag{4.6}$$

and integrating by parts on $B(\mathcal{I}_p(p), r) \subseteq \mathbb{R}^2$, where $r \in \left[\frac{1}{\kappa^\alpha}, \frac{1}{\kappa^{\frac{\alpha}{2}}}\right]$, yields

$$\begin{aligned} P_{p,r} = \frac{\kappa^2}{2} &\int_{\partial B(\mathcal{I}_p(p),r)} \lambda^2 \left(1 - |\psi_\kappa^{\mathbf{euc}}|^2\right)^2 \langle((x_1, x_2) - \mathcal{I}_p(p)), \nu\rangle \\ &+ \int_{\partial B(\mathcal{I}_p(p),r)} \left(\left|\frac{\partial\psi_\kappa^{\mathbf{euc}}}{\partial\tau}\right|^2 - \left|\frac{\partial\psi_\kappa^{\mathbf{euc}}}{\partial\nu}\right|^2\right)\langle((x_1, x_2) - \mathcal{I}_p(p)), \nu\rangle \\ &- \int_{\partial B(\mathcal{I}_p(p),r)} \left\langle\left(\frac{\partial\psi_\kappa^{\mathbf{euc}}}{\partial\tau}\right), \left(\frac{\partial\psi_\kappa^{\mathbf{euc}}}{\partial\nu}\right)\right\rangle\langle((x_1, x_2) - \mathcal{I}_p(p)), \tau\rangle \\ &+2h(\kappa) \int_{\mathcal{B}(p,r)} \langle[(\mathbf{A}^e)^\tau \cdot \nabla_{\mathcal{M}}](\psi_\kappa^\perp), \hat{S}\rangle \,\mathrm{d}\mathcal{H}^2_{\mathcal{M}} \\ &+(h(\kappa))^2 \int_{\mathcal{B}(p,r)} \left|(\mathbf{A}^e)^\tau\right|^2 \langle\psi_\kappa, \hat{S}\rangle \,\mathrm{d}\mathcal{H}^2_{\mathcal{M}}, \end{aligned} \tag{4.7}$$

where $\hat{S}$ denotes the pullback of S via $\mathcal{I}_p$. The penultimate term on the right-hand side can be controlled rather easily. Indeed,

$$\begin{aligned} h(\kappa)\left|\int_{\mathcal{B}(p,r)} \langle[(\mathbf{A}^e)^\tau \cdot \nabla_{\mathcal{M}}]\psi_\kappa^\perp, \hat{S}\rangle\right| \mathrm{d}\mathcal{H}^2_{\mathcal{M}} &\leqq c \cdot \ln\kappa \cdot r \int_{\mathcal{B}(p,r)} |\nabla_{\mathcal{M}}\psi_\kappa|^2 \,\mathrm{d}\mathcal{H}^2_{\mathcal{M}} \\ &\leqq \frac{c(\ln\kappa)^2}{\kappa^\alpha} = o(1). \end{aligned} \tag{4.8}$$

The last term is also of small order. From (4.3) and Hölder:

$$\begin{aligned} &(h(\kappa))^2 \left|\int_{\mathcal{B}(p,r)} \left|(\mathbf{A}^e)^\tau\right|^2 \langle\psi_\kappa, \hat{S}\rangle \,\mathrm{d}\mathcal{H}^2_{\mathcal{M}}\right| \\ &\leqq c \cdot (\ln\kappa)^2 \int_{\mathcal{B}(p,r)} |\nabla_{\mathcal{M}}\psi_\kappa| \, d_{\mathcal{M}}(x, p) \\ &\leqq c \cdot (\ln\kappa)^2 \cdot \|\nabla_{\mathcal{M}}\psi_\kappa\|_{L^2} \cdot \|d_{\mathcal{M}}(x, p)\|_{L^2(\mathcal{B}(p,r))} \\ &\leqq c \cdot (\ln\kappa)^2 \cdot \sqrt{\ln\kappa} \cdot \frac{1}{\kappa^\alpha} = o(1). \end{aligned} \tag{4.9}$$

The result will follow if we can show that for some $r > \frac{1}{\kappa^\alpha}$ the rest of the terms on the right-hand side of (4.7) are bounded by a constant. In turn, this can be obtained after noticing $\langle((x_1, x_2) - \mathcal{I}_p(p)), \nu\rangle = r$, $((x_1, x_2) - \mathcal{I}_p(p)) \perp \tau$, $\left|\frac{\partial\psi_\kappa}{\partial\tau}\right|^2 + \left|\frac{\partial\psi_\kappa}{\partial\nu}\right|^2 = \mathcal{O}(|\nabla_{\mathcal{M}}\psi_\kappa|^2)$ and the fact that for some $r \in [\frac{1}{\kappa^\alpha}, \frac{1}{\kappa^{\frac{\alpha}{2}}}]$

$$\int_{\partial\mathcal{B}(p,r)} |\nabla_{\mathcal{M}}\psi_\kappa|^2 + \frac{\kappa^2}{2}\left(|\psi_\kappa|^2 - 1\right)^2 \mathrm{d}\mathcal{H}^1_{\mathcal{M}} \leqq \frac{c_\alpha}{r}, \tag{4.10}$$

where the constant c_α does not depend on the point p. This last statement is a consequence of (4.3) as in [3]. The left-hand side of (4.7) is bounded from below by a quantity comparable to $\kappa^2 \int_{\mathcal{B}(p,\frac{1}{\kappa^\alpha})} \left(|\psi_\kappa|^2 - 1\right)^2 \mathrm{d}\mathcal{H}^2_{\mathcal{M}}$. Equations (4.8)–(4.10) yield (4.4). □

As anticipated, the purpose of the derivation of (4.4) is the concentration result presented immediately below:

Proposition 4.1. *Let ψ_κ be a sequence of global minimizers of $\mathcal{G}_{\mathcal{M},\kappa}$ Borrowing the notation from Proposition 3.2, assume*

$$\sum_{i=1}^{N_\kappa} \left|d_j^{(\kappa)}\right| < C, \tag{4.11}$$

for a non-negative constant C independent of κ. Then there exist $N_0 \in \mathbb{N}$, a constant $\lambda_0 > 0$ and points $p_1^\kappa, \ldots, p_{m_\kappa}^\kappa$ in $\mathcal{M}$ with $m_\kappa \leqq N_0$, such that $|\psi_\kappa| \geqq \frac{1}{2}$ on $\mathcal{M} \setminus \bigcup_{i=1,\ldots,m_\kappa} \mathcal{B}(p_i^\kappa, \frac{\lambda_0}{\kappa})$, where the pseudo-balls $\mathcal{B}(p_i^\kappa, \frac{\lambda_0}{\kappa})$, $i = 1, \ldots, m_\kappa$ are disjoint and $|\psi_\kappa|(p_i^\kappa) < \frac{1}{2}$. In addition if κ is large enough, for any $0 < \alpha < \frac{1}{2}$ there exists a number $0 < \alpha_0 < \alpha$ and points $a_1^\kappa, \ldots, a_{n_\kappa}^\kappa$ in $\mathcal{M}$ with $n_\kappa \leqq N_0$, such that $\mathcal{B}(a_i^\kappa, \frac{1}{\kappa^{\alpha_0}}) \bigcap \mathcal{B}(a_j^\kappa, \frac{1}{\kappa^{\alpha_0}}) = \emptyset$, for $i \neq j$, and $|\psi_\kappa| \geqq \frac{1}{2}$ on $\mathcal{M} \setminus \bigcup_{i=1}^{n_\kappa} \mathcal{B}\left(a_i^\kappa, \frac{1}{\kappa^{\alpha_0}}\right)$.

Proof of Proposition 4.1. First observe that (3.37) and assumption (4.11) yield

$$\left|h(\kappa)\Lambda(\mathbf{A}^e, \psi_\kappa)\right| \leqq C \ln \kappa. \tag{4.12}$$

We plug (4.12), (3.24) and (3.25) in (3.18) to obtain (4.3). Lemma 4.1 is applicable here. The proof of the existence of the p_i^κ's and their associated pseudo-balls then proceeds exactly as in Theorem IV.1 in [3], so we omit it. From this, the larger pseudo-balls can be obtained by a merging procedure. □

Since we are now dealing with pseudo-balls of different sizes, we fix some notation to avoid confusion. Consider a family of global minimizers $\{\psi_\kappa\}$ satisfying the hypotheses of Proposition 4.1. Borrowing the notation contained there, we write

$$\begin{aligned} d^\kappa_{\alpha,i} &:= \deg\left(\psi_\kappa; \partial\mathcal{B}\left(a_i^\kappa, \frac{1}{\kappa^\alpha}\right)\right), \text{ and similarly } d_i^\kappa \\ &= \deg\left(\psi_\kappa, \partial\mathcal{B}\left(p_i^\kappa, \frac{\lambda_0}{\kappa}\right)\right). \end{aligned} \tag{4.13}$$

These correspond to the degrees in smaller pseudo-balls as opposed to the $d_j^{(\kappa)}$'s defined in Proposition 3.2. Note that also in this case, necessarily

$$\sum_{i=1}^{n_\kappa} d^\kappa_{\alpha,i} = \sum_{i=1}^{m_\kappa} d^\kappa_i = 0.$$

We see that equation (4.10) should hold for some $r \in (\frac{1}{\kappa^{\alpha_0}}, \frac{1}{\kappa^{\frac{\alpha_0}{2}}})$, with $\alpha = \alpha_0$, $p = a_i^\kappa$, otherwise we would reach a contradiction with (4.3), after integration with respect to r. This implies

$$\left|d^\kappa_{\alpha_0,i}\right| \leqq \frac{2}{\pi} c_{\alpha_0}. \tag{4.14}$$

We focus now on estimating the energy of certain functions that will yield lower bounds for minimizers of Ginzburg–Landau in the next section. To that end, consider points $b_1, \ldots, b_{\ell(\kappa)}$ in $\mathcal{M}$, and numbers $d_1, \ldots, d_{\ell(\kappa)}$. Let N_0 be the integer obtained in Proposition 4.1, and α_0 the number found in Proposition 4.1. From now on we assume, whenever we use the letter r, that

$$r \in \left(\frac{1}{\kappa^{\alpha_0}}, \frac{1}{\kappa^{\alpha_0^{N_0+1}}}\right). \tag{4.15}$$

We will only be interested in collections satisfying the conditions

$$\ell(\kappa) \leqq N_0, \quad \sum_{i=1}^{\ell(\kappa)} d_i = 0, \quad |d_i| \leqq \frac{2}{\pi} c_{\alpha_0} \quad \text{for all } i = 1, \ldots, \ell(\kappa),$$
$$\text{and the pseudo-balls } \{\mathcal{B}(b_i, r)\}_{i=1}^{\ell(\kappa)} \text{ are pairwise disjoint.} \tag{4.16}$$

The condition (4.15) may seem strange, but its meaning will become apparent in Proposition 4.5 below. Next, consider Φ_r satisfying

$$\begin{cases} \Delta_{\mathcal{M}} \Phi_r = 0 \text{ on } \mathcal{M} \setminus \bigcup_{i=1}^{\ell(\kappa)} \mathcal{B}(b_i, r), \\ \Phi_r = c_i \text{ on } \partial\mathcal{B}(b_i, r), \\ \int_{\partial\mathcal{B}(b_i,r)} \frac{\partial \Phi_r}{\partial \nu} = 2\pi d_i, \\ \int_{\mathcal{M}\setminus\bigcup_{i=1}^{\ell(\kappa)} \mathcal{B}(b_i,r)} \Phi_r = 0. \end{cases} \tag{4.17}$$

Such a Φ_r can be obtained as a minimizer of

$$\min_{\mathcal{C}} \int_{\mathcal{M}\setminus\bigcup_{i=1}^{\ell(\kappa)} \mathcal{B}(b_i,r)} |\nabla\phi|^2 + 2\pi \sum_{i=1}^{\ell(\kappa)} d_i \, \phi|_{\partial\mathcal{B}(b_i,r)},$$

where

$$\mathcal{C} = \left\{ \phi \in H^1(\mathcal{M} \setminus \bigcup_{i=1}^{\ell(\kappa)} \mathcal{B}(b_i, r); \mathbb{R}) \; s.t. \, \phi \text{ is a constant on each } \partial\mathcal{B}(b_i, r) \right\}.$$

Consider also Φ as a solution of

$$\begin{cases} \Delta_{\mathcal{M}}\Phi = 2\pi \sum_{i=1}^{\ell(\kappa)} d_i\, \delta_{b_i} \\ \int_{\mathcal{M}} \Phi = 0. \end{cases} \tag{4.18}$$

Let G be the Green's function of $\mathcal{M}$; that is, G satisfies

$$\begin{cases} \Delta_{\mathcal{M}}G(\cdot\,, p) = \delta_p - \frac{1}{\mathcal{H}^2(\mathcal{M})} \\ \int G(x, p)\, \mathrm{d}\mathcal{H}^2_{\mathcal{M}}(x) = 0. \end{cases} \tag{4.19}$$

Note that

$$\Phi(x) = \sum_{i=1}^{\ell(\kappa)} 2\pi d_i\, G(b_i, x). \tag{4.20}$$

The following energy decomposition is an analogue to that in [2] for a planar model.

Proposition 4.2. *Let* $\{\mathcal{B}(b_i, r)\}_{i=1}^{\ell(\kappa)}$ *be a family of pseudo-balls and* $\{d_i\}_{i=1}^{\ell(\kappa)}$ *integers satisfying the conditions* (4.16). *Then the following expansion holds*

$$\begin{aligned} \int_{\mathcal{M}\setminus\bigcup_{i=1}^{\ell(\kappa)} \mathcal{B}(b_i,r)} |\nabla_{\mathcal{M}}\Phi_r|^2\, \mathrm{d}\mathcal{H}^2_{\mathcal{M}} = &-4\pi^2 \sum_{i\neq j} d_i\, d_j\, G(b_i, b_j) \\ &-4\pi^2 \sum_{i=1}^{\ell(\kappa)} d_i^2\, G(b_i, x_i) + \mathcal{O}(1), \end{aligned} \tag{4.21}$$

as $r \to 0$, *where* x_i *is any point in* $\partial\mathcal{B}(b_i, r)$.

Proof of Proposition 4.2. The proof is as in [2]. The proof of Lemma I.3 of [2] works in this setting yielding

$$\sup_{\mathcal{M}\setminus\bigcup_{i=1}^{\ell(\kappa)} \mathcal{B}(b_i,r)} (\Phi - \Phi_r) - \inf_{\mathcal{M}\setminus\bigcup_{i=1}^{\ell(\kappa)} \mathcal{B}(b_i,r)} (\Phi - \Phi_r) \leqq \sum_{i=1}^{\ell(\kappa)} \sup_{\partial\mathcal{B}(b_i,r)} \Phi - \inf_{\partial\mathcal{B}(b_i,r)} \Phi.$$

Then, the claim follows if

$$\|\Phi_r - \Phi\|_{L^\infty} = \mathcal{O}(1). \tag{4.22}$$

From now on we write, for real valued functions f, g, $f \lesssim g$ to mean there is a uniform constant C, throughout the manifold, independent of κ, such that $f \leqq C\cdot g$. Observe now that since $|G(p, x)| \lesssim (1 + \ln d_{\mathcal{M}}(p, x))$, for any p, x in $\mathcal{M}$, it follows that

$$\begin{aligned} \left| \sup_{\partial\mathcal{B}(b_i,r)} \Phi - \inf_{\partial\mathcal{B}(b_i,r)} \Phi \right| \lesssim &\sup_{j=1,\dots,\ell_\kappa,\, x, y\in\partial\mathcal{B}(b_i,r)} \left|\ln d_{\mathcal{M}}(b_j, x) - \ln d_{\mathcal{M}}(b_j, y)\right| \\ &+\mathcal{O}(r). \end{aligned} \tag{4.23}$$

But,

$$\sup_{x,y\in\partial\mathcal{B}(b_i,r)} |\ln d_{\mathcal{M}}(b_i,x) - \ln d_{\mathcal{M}}(b_i,y)| = \left|\ln \frac{\mathcal{O}(r)}{r}\right| = \mathcal{O}(1), \quad (4.24)$$

and similarly for $j \neq i$ and $x, y \in \partial\mathcal{B}(b_i, r)$, using that the pseudo-balls are disjoint and of radius r we obtain:

$$\begin{aligned}\left|\ln d_{\mathcal{M}}(b_j,x) - \ln d_{\mathcal{M}}(b_j,y)\right| &\lesssim \left|\ln\left(\frac{d_{\mathcal{M}}(b_j,y)+d_{\mathcal{M}}(y,x)}{d_{\mathcal{M}}(b_j,y)}\right)\right| \\ &\lesssim \left|\ln\left(1+\frac{\mathcal{O}(r)}{d_{\mathcal{M}}(b_j,y)}\right)\right| = \mathcal{O}(1). \quad (4.25)\end{aligned}$$

In this way, (4.22) is a consequence of (4.23)–(4.25). We integrate by parts the left-hand side of (4.21) to obtain

$$\int_{\mathcal{M}\setminus\bigcup_{i=1}^{\ell(\kappa)}\mathcal{B}(b_i,r)} |\nabla_{\mathcal{M}}\Phi_r|^2 \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}} = -\sum_{i=1}^{\ell(\kappa)} 2\pi d_i \Phi(x_i) + \mathcal{O}(1),$$

where $x_i \in \partial\mathcal{B}(b_i, r)$, thanks to (4.22). Finally, for $i \neq j$ and $x_j \in \partial\mathcal{B}(b_j, r)$, we can deduce

$$G(b_i, x_j) = G(b_i, b_j) + \mathcal{O}(1), \quad (4.26)$$

using the same argument utilized in (4.25). We can substitute (4.20) and (4.26) on the left-hand side of (4.21), and the result follows. □

One of the implications of Proposition 4.2 is a lower bound on the energy of minimizers that is optimal up to $\mathcal{O}(1)$. To this end, define the map $S_\kappa : C^1(\mathcal{M};\mathbb{C}) \mapsto \mathbb{R}$ by

$$S_\kappa(\psi) := |\nabla_{\mathcal{M}}\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2-1)^2, \quad (4.27)$$

and establish:

Proposition 4.3. *Let ψ_κ be a global minimizer of $\mathcal{G}_{\mathcal{M},\kappa}$ satisfying the hypotheses of Proposition 4.1. Using the notation in Proposition 4.1, assume in addition that $\{\mathcal{B}(a_i^\kappa, r)\}_{i=1}^{n_\kappa}$ is a disjoint family. Then for any $x_i \in \partial\mathcal{B}(a_i^\kappa, r)$, $i = 1, \ldots, n_\kappa$, the lower bound*

$$\begin{aligned}\mathcal{G}_{\mathcal{M},\kappa}(\psi_\kappa) \geqq &-4\pi^2 \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right|^2 G\left(a_i^\kappa, x_i\right) - 4\pi^2 \sum_{i\neq j} d^\kappa_{\alpha_0,i}\, d^\kappa_{\alpha_0,j} G\left(a_i^\kappa, a_j^\kappa\right) \\ &+2\pi \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right| \ln(\kappa\cdot r) + h(\kappa)^2 \left\|(\mathbf{A}^e)^\tau\right\|^2_{L^2(\mathcal{M})} \\ &+4\pi h(\kappa) \sum_{i=1}^{n_\kappa} *F\left(a_i^\kappa\right) d^\kappa_{\alpha_0,i} + \mathcal{O}(1) \quad (4.28)\end{aligned}$$

holds for κ large.

Proof of Proposition 4.3. First, note that all the hypotheses of Proposition 4.2 are met here. Now, the estimate

$$h(\kappa)\Lambda(\mathbf{A}^e, \psi_\kappa) = -4\pi h(\kappa) \sum_{j=1}^{n_\kappa} *F(a_i^\kappa) d_{\alpha_0,i}^\kappa + o(1), \tag{4.29}$$

can be obtained in a similar fashion to what we did for the larger pseudo-balls in (3.29)–(3.36). The fact that in this case n_κ is bounded independently of κ only makes the calculation simpler. Secondly, (3.28) also holds here, so we need only to estimate $\int_{\mathcal{M}} |\nabla_{\mathcal{M}}\psi_\kappa|^2 + \frac{\kappa^2}{2}(|\psi_\kappa|^2-1)^2 \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}} = \int_{\mathcal{M}} S_\kappa(\psi_\kappa) \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}}$. Writing $\mathcal{B}_{i,r} := \mathcal{B}(a_i^\kappa, r)$, one has

$$\begin{aligned}\int_{\bigcup_{i=1}^{n_\kappa} \mathcal{B}_{i,r}} S_\kappa(\psi_\kappa) \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}} &\geqq \int_{\bigcup_{i=1}^{n_\kappa} B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa), r)} \left| \nabla \left(\psi_\kappa \circ (\mathcal{I}_{a_i^\kappa})^{-1} \right) \right|^2 \mathrm{d}x \\ &\geqq \sum_{i=1}^{n_\kappa} \left| d_{\alpha_0,i}^\kappa \right| \ln(\kappa \cdot r) - C. \end{aligned} \tag{4.30}$$

which follows as in V.II of [3] without modification, for κ large, thanks to (4.14) and invariance of degree in the annulus $\mathcal{B}(a_i^\kappa, r) \setminus \mathcal{B}(a_i^\kappa, \frac{1}{\kappa^{\alpha_0}})$. In turn, defining $f_\kappa = \frac{\psi_\kappa}{|\psi_\kappa|}$, we notice

$$\int_{\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa} \mathcal{B}_{i,r}} S_\kappa(\psi_\kappa) \geqq \int_{\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa} \mathcal{B}_{i,r}} |\nabla_{\mathcal{M}} f_\kappa|^2 \, |\psi_\kappa|^2 \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}}. \tag{4.31}$$

Introducing H through the usual Hodge-de-Rham decomposition, $i(f_\kappa \wedge df_\kappa^* - f_\kappa^* \wedge df_\kappa) = *d\,\Phi_r + dH$, where Φ_r satisfies (4.17), thanks to (4.31), it holds that

$$\begin{aligned}\int_{\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa} \mathcal{B}_{i,r}} S_\kappa(\psi_\kappa) &\geqq \int_{\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa} \mathcal{B}_{i,r}} |\psi_\kappa|^2 \, |\nabla_{\mathcal{M}}\Phi_r|^2 \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}} \\ &\quad +2\int_{\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa} \mathcal{B}_{i,r}} |\psi_\kappa|^2 \, \mathrm{d}\Phi_r \wedge dH \\ &=: I + II. \end{aligned} \tag{4.32}$$

Below, we make use of the pointwise estimates $|\nabla_{\mathcal{M}}\Phi_r| \leqq \frac{c}{r}$, $|\nabla_{\mathcal{M}} H| \leqq c\kappa$, that follow from elliptic regularity. We examine

$$\begin{aligned}&\left| I - \int_{\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa} \mathcal{B}_{i,r}} |\nabla_{\mathcal{M}}\Phi_r|^2 \, \mathrm{d}\mathcal{H}^2_{\mathcal{M}} \right| \\ &\quad \leqq c\,\frac{1}{r^2} \left(\int_{\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa} \mathcal{B}_{i,r}} (|\psi_\kappa|^2 - 1)^2 \right)^{\frac{1}{2}} \cdot [H^2(\mathcal{M})]^{\frac{1}{2}} \\ &\quad \leqq c \cdot \kappa^{2\alpha_0} \cdot \frac{\sqrt{\ln\kappa}}{\kappa} = o(1). \end{aligned} \tag{4.33}$$

In the last inequality we used $\alpha_0 < \frac{1}{2}$ and (4.3). On the other hand, $d\Phi_r \wedge dH$ is closed, therefore its integral reduces to boundary terms. These terms vanish since Φ_r is constant on each component. One has

$$\begin{aligned}\left|\frac{II}{2}\right| &\leqq \int_{\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa}\mathcal{B}_{i,r}} (1-|\psi_\kappa|^2)\,|\nabla_{\mathcal{M}}\Phi_r|\,|\nabla_{\mathcal{M}}H| \\ &= \int_{(\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa}\mathcal{B}_{i,r})\bigcap\{|\psi_\kappa|>1-\frac{1}{(\ln\kappa)^2}\}}\cdot + \int_{(\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa}\mathcal{B}_{i,r})\bigcap\{|\psi_\kappa|\leqq 1-\frac{1}{(\ln\kappa)^2}\}}\cdot \\ &=: III + IV. \end{aligned} \tag{4.34}$$

We see

$$\begin{aligned} III &\leqq 4\frac{c}{(\ln\kappa)^2}\int_{\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa}\mathcal{B}_{i,r}} |\nabla_{\mathcal{M}}\psi_\kappa|^2\,\mathrm{d}\mathcal{H}^2_{\mathcal{M}} \\ &\leqq \frac{c}{(\ln\kappa)^2}\ln\kappa = o(1), \end{aligned} \tag{4.35}$$

where the last inequality comes from (4.3). Again invoking (4.3), one readily checks $\mathcal{H}^2_{\mathcal{M}}((\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa}\mathcal{B}_{i,r})\bigcap\{|\psi_\kappa|\leqq 1-\frac{1}{(\ln\kappa)^2}\}) \leqq c\cdot\frac{(\ln\kappa)^5}{\kappa^2}$. Thus

$$\begin{aligned} IV &\leqq \int_{(\mathcal{M}\setminus\bigcup_{i=1}^{n_\kappa}\mathcal{B}_{i,r})\bigcap\{|\psi_\kappa|\leqq 1-\frac{1}{(\ln\kappa)^2}\}} (1-|\psi_\kappa|^2)\cdot\frac{1}{r}\cdot\kappa\,\mathrm{d}\mathcal{H}^2_{\mathcal{M}} \\ &\leqq c\kappa^{\alpha_0+1}\cdot\frac{(\ln\kappa)^{\frac{1}{2}}}{\kappa}\cdot\frac{(\ln\kappa)^{\frac{5}{2}}}{\kappa} = o(1). \end{aligned} \tag{4.36}$$

The result now follows from (3.28), (4.29), (4.30), (4.32), (4.33), (4.35), (4.36), and Proposition 4.2. □

To conclude, we state a couple of propositions that are required in Section 5 when the energy of $\mathcal{G}_{\mathcal{M},\kappa}(\psi_\kappa)$ is determined up to $o(1)$. This time, we consider any collection of points $q_1,\ldots,q_{2n_0}$ in $\mathcal{M}$, where n_0 is a given non-negative integer. We use polar coordinates (r,θ) to parametrize the Euclidean ball $B(\mathcal{I}_p(p),r)$, where p is a point in $\mathcal{M}$. We define the family of sets

$$\begin{aligned} &\mathcal{F}_{q_1,\ldots,q_{2n_0}}(r) \\ &= \Big\{\psi\in\mathcal{H}^1\left(\mathcal{M}\setminus\cup\mathcal{B}(q_i,r);\mathbb{S}^1\right),\ \text{s.t. for each } j \text{ there is a constant } \theta_j \text{ with} \\ &\quad \psi\circ(\mathcal{I}_{q_i})^{-1}(r,\theta) = e^{i(\theta+\theta_j)} \text{ on } \partial B(\mathcal{I}_{q_i}(q_i),r) \text{ for } i=2,4,\ldots,2n_0, \text{ and} \\ &\quad \psi\circ(\mathcal{I}_{q_i})^{-1}(r,\theta) = e^{i(-\theta+\theta_j)} \text{ on } \partial B(\mathcal{I}_{q_i}(q_i),r) \text{ for } i=1,3,\ldots,2n_0-1.\Big\} \end{aligned} \tag{4.37}$$

Also associated to a pseudo-ball centered at $p\in\mathcal{M}$ carrying a degree $+1$ (resp. -1), we define the set $\mathcal{F}^+_{p,r}$ (resp. $\mathcal{F}^-_{p,r}$), by

$$\mathcal{F}^\pm_{p,r} = \left\{\psi\in\mathcal{H}^1\left(\mathcal{B}(p,r);\mathbb{C}\right) \text{ such that } \psi\circ(\mathcal{I}_p)^{-1}|_{B(\mathcal{I}_p(p),r)} = e^{\pm i\theta}\right\}. \tag{4.38}$$

We recall that in (4.37) and in (4.38), the radius r is understood to satisfy (4.15) while the family $\{\mathcal{B}(q_i, r)\}_{i=1}^{2n_0}$ is disjoint with n_0 bounded independent of κ.

Proposition 4.4. *Let $\mathcal{F}_{q_1,\ldots,q_{2n_0}}(r)$ be as in (4.37). Assume that for all $i \neq j$, one has*

$$d_{\mathcal{M}}(q_i, q_j) \cdot \kappa^{\alpha_0^{N_0+1}} \to \infty, \text{ as } \kappa \to \infty. \tag{4.39}$$

Then there exists $\psi_{out}^r \in \mathcal{F}_{q_1,\ldots,q_{2n_0}}(r)$ and a real valued $\tilde{\phi}$ such that

$$\begin{aligned}\int_{\mathcal{M}\setminus\bigcup_{i=1}^{2n_0}\mathcal{B}(q_i,r)} \left|\nabla_{\mathcal{M}}\psi_{out}^r\right|^2 &= \inf_{\psi\in\mathcal{F}_{q_1,\ldots,q_{2n_0}}(r)} \int_{\mathcal{M}\setminus\bigcup_{i=1}^{2n_0}\mathcal{B}(q_i,r)} |\nabla_{\mathcal{M}}\psi|^2 \\ &= \int_{\mathcal{M}\setminus\cup\mathcal{B}(q_i,r)} \left|\nabla_{\mathcal{M}}\tilde{\phi}\right|^2 \\ &= -4\pi^2\sum_{i=1}^{2n_0} G(q_i, x_i) - 4\pi^2\sum_{i\neq j} G(q_i, q_j)(-1)^{i+j} \\ &\quad + o(1). \end{aligned} \tag{4.40}$$

Here x_i is any point in $\partial\mathcal{B}(q_i, r)$ and $\tilde{\phi}$ is a solution of

$$\begin{cases} \Delta_{\mathcal{M}}\tilde{\phi} = 0 & \text{in } \mathcal{M}\setminus\bigcup_{i=1}^{2n_0}\mathcal{B}(q_i, r) \\ \frac{\partial\tilde{\phi}}{\partial\nu} = \frac{(-1)^i}{r} & \text{on } \partial\mathcal{B}(q_i, r). \end{cases} \tag{4.41}$$

Proof of Proposition 4.4. The proof can be carried out as the one of Theorem I.9 in [2], modulo obvious modifications, so we omit it. □

Remark 4.1. The improvement in the order of magnitude of the error in the equation (4.40) with respect to (4.21) stems from the assumption that the distance between the centers of the pseudo-balls is much greater than

$$\frac{1}{\kappa^{\alpha_0^{N_0+1}}},$$

which allows one to sharpen (4.22), (4.24), (4.25) and (4.26) in this case.

Following [2], we now let $B(q, R)$ be a ball in two-dimensional Euclidean space and let $f : B(q, r_0) \mapsto \mathbb{R}$, where r_0 is the injectivity radius. Define

$$\mathcal{C}_R^{\pm} := \left\{\psi \in H^1(B(q, R); \mathbb{C}) \text{ s.t. } \psi|_{\partial B(q,R)} = e^{\pm i\theta}\right\}. \tag{4.42}$$

For any $f : B \to \mathbb{R}$, we now write

$$I_{\pm}^f(\kappa, R) := \min_{\psi\in\mathcal{C}_R^{\pm}} \int_{B(q,R)} |\nabla\psi|^2 + f^2\frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 dx, \tag{4.43}$$

where dx is the Lebesgue measure in $\mathbb{R}^2$. When $f \equiv 1$ we simply write $I_{\pm}(\kappa, R) := I_{\pm}^1(\kappa, R)$. We also set

$$I_{\pm}(\kappa) := I_{\pm}(\kappa, 1), \tag{4.44}$$

and note that

$$I_{\pm}(\kappa, R) = I_{\pm}\left(\frac{1}{\kappa R}\right). \tag{4.45}$$

From Lemma III.1 in [2] it follows that there exists a constant $\mathbf{c}_0$ independent of whether the minimum in (4.43) is taken over $\mathcal{C}_R^+$ or $\mathcal{C}_R^-$, such that

$$(I_{\pm}(s) + 2\pi \ln s) \searrow \mathbf{c}_0, \text{ as } s \to 0. \tag{4.46}$$

We relate the previous to our problem in the following proposition:

Proposition 4.5. *Let ψ_κ be a global minimizer of $\mathcal{G}_{\mathcal{M},\kappa}$ satisfying the conditions of Proposition 4.1. Assume that for all $i = 1, \ldots, n_\kappa$, one has $\left|d^\kappa_{\alpha,i}\right| = 1$. Then there exists a radius $r = r(\kappa)$ satisfying (4.15) and a function $\psi^{\kappa,\pm}_{\mathbf{in},i} \in \mathcal{F}^{\pm}_{a_i^\kappa,r}$ such that for any $i = 1, \ldots, n_\kappa$ the following asymptotic bound holds for κ large*

$$\begin{aligned}\int_{\mathcal{B}_{i,r}} S_\kappa(\psi_\kappa) d\mathcal{H}^2_{\mathcal{M}} &\geqq 2\pi \ln(\kappa \cdot r) + \mathbf{c}_0 + o(1) = \min_{\psi \in \mathcal{F}^{\pm}_{a_i^\kappa,r}} \int_{\mathcal{B}_{i,r}} S_\kappa(\psi) d\mathcal{H}^2_{\mathcal{M}} \\ &= \int_{\mathcal{B}_{i,r}} S_\kappa(\psi^{\kappa,\pm}_{\mathbf{in},i}) d\mathcal{H}^2_{\mathcal{M}} + o(1).\end{aligned} \tag{4.47}$$

Proof of Proposition 4.5. Without loss of generality, assume $d^\kappa_{\alpha_0,i} = 1$. First one finds a radius $r \in (\frac{1}{\kappa^{\alpha_0}}, \frac{1}{\kappa^{\alpha_0^{N_0+1}}})$ such that

$$|\psi_\kappa| \geqq 1 - \frac{1}{(\ln \kappa)^2} \text{ on } \partial\mathcal{B}_{i,r}. \tag{4.48}$$

This can be done in the same way as in the proof of Proposition 3.1 in [22], where a similar result is obtained for a complex-valued function u defined on an open set $\Omega \subset \mathbb{R}^2$, based solely on the assumption

$$\kappa^2 \int_\Omega \left(|u|^2 - 1\right)^2 \leqq C \ln \kappa. \tag{4.49}$$

Such a bound is also true in our case thanks to (4.3). The construction is iterative, which is why the exponent $N_0 + 1$ appears. Next we write

$$\psi_\kappa^{\mathbf{euc}} := \psi_\kappa \circ (\mathcal{I}_{a_i^\kappa})^{-1}. \tag{4.50}$$

Inequality (4.48) allows one to extend $\psi_\kappa^{\mathbf{euc}}$ to the annulus $B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa), 2r) \setminus B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa), r)$ exactly as in the proof of Proposition 5.2 in [22], to a function $\psi_\kappa^{\mathbf{euc},ext}$ such that

$$\psi_\kappa^{\mathbf{euc},ext} = e^{i\theta} \tag{4.51}$$

on $\partial B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa), 2r)$, and which satisfies

$$\int_{B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa),2r)\setminus B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa),r)} \left|\nabla\psi_\kappa^{\mathbf{euc},ext}\right|^2 + \frac{\kappa^2}{2}(\left|\psi_\kappa^{\mathbf{euc},ext}\right|^2 - 1)^2\,\mathrm{d}x$$
$$= 2\pi\ln 2 + o(1). \tag{4.52}$$

This, together with $\psi_\kappa^{\mathbf{euc},ext} \in \mathcal{F}^+_{a_i^{kappa},2r}$, and the fact that property (4.46) can be applied since by assumption $\kappa \cdot r \to \infty$, yields

$$I_0 := \int_{B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa),r)} \left|\nabla\psi_\kappa^{\mathbf{euc}}\right|^2 + \frac{\kappa^2}{2}\left(\left|\psi_\kappa^{\mathbf{euc}}\right|^2 - 1\right)^2\,\mathrm{d}x$$
$$\geqq 2\pi\ln(\kappa\cdot r) + \mathbf{c}_0 + o(1). \tag{4.53}$$

One has

$$\int_{\mathcal{B}_{i,r}} S_\kappa(\psi_\kappa)\,\mathrm{d}\mathcal{H}^2_{\mathcal{M}} = \int_{B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa),r)} \left|\nabla\psi_\kappa^{\mathbf{euc}}\right|^2 + \lambda^2\frac{\kappa^2}{2}(\left|\psi_\kappa^{\mathbf{euc}}\right|^2 - 1)^2\,\mathrm{d}x$$
$$=: I_\lambda. \tag{4.54}$$

The property (4.1), the bound (4.4), and $\kappa\cdot r \to \infty$ can be used to prove

$$|I_0 - I_\lambda| = \mathcal{O}(r), \tag{4.55}$$

and similarly

$$\left|I_+(\kappa, r) - I_+^\lambda(\kappa, r)\right| = \mathcal{O}(r). \tag{4.56}$$

Finally, let ψ_0 be a function that achieves (4.43) with $f = \lambda$, where λ is given by (4.1). Define $\psi_{\mathbf{in},i}^{\kappa,+}$ by

$$\psi_{\mathbf{in},i}^{\kappa,+}(x) := \psi_0(\mathcal{I}_{a_\kappa^i}(x)) \text{ for } x \in \mathcal{B}_{i,r}. \tag{4.57}$$

Then, the bound (4.47) follows from (4.46),(4.53),(4.55),(4.56) and the definition (4.57). □

5. Emergence of multiple vortices in a surface of revolution

In this section we assume $\check{\mathcal{M}}$ is a simply connected surface of revolution parametrized in the following way:

If θ and ϕ denote the standard azimuthal and zenith angles in spherical coordinates respectively, then

$$\check{\mathcal{M}} := \{\,(\,u(\phi)\cos\theta, u(\phi)\sin\theta, v(\phi)\,) : \phi \in [0, \pi], \theta \in [0, 2\pi]\}, \tag{5.1}$$

where $u, v : [0, \pi] \to \mathbb{R}$ are C^1 functions related by the condition

$$v(\phi) = \cot\phi\, u(\phi) \quad \text{for } 0 < \phi < \pi \tag{5.2}$$

with

$$u(0) = 0 = u(\pi), \quad v(0) > 0, \quad v(\pi) < 0 \quad \text{and} \quad v'(0) = 0 = v'(\pi). \tag{5.3}$$

and we further assume the regularity condition

$$\gamma(\phi) := \sqrt{u'(\phi)^2 + v'(\phi)^2} \geqq \gamma_0 \quad \text{for } \phi \in [0, \pi] \tag{5.4}$$

for some $\gamma_0 > 0$. Note that necessarily,

$$u(\phi) = l\phi + o(\phi) \quad \text{for some positive constant } l \tag{5.5}$$

near $\phi = 0$ with a similar expansion holding near $\phi = \pi$.

The applied field $\mathbf{H}_{\text{ext}}$, will be taken throughout the rest of the paper to be of the form $\mathbf{H}_{\text{ext}}(\kappa) = h(\kappa)\hat{e}_z$.

In this section, we obtain a description of the emergence of pairs of vortices as $h(\kappa)$ is increased. With that goal in mind, we now focus on describing the asymptotic intensity $h(\kappa)$ of the applied field $\mathbf{H}_{\text{ext}}(\kappa)$ that yields the presence of a given number of pairs of vortices in any global minimizer ψ_κ of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ in this context. We point out that the results here obtained extend the results obtained in [5] and provide us with analogues, in the manifold setting, of the corresponding results in the plane in [22,25]. As a consequence of this analysis, we obtain that for ε small enough, the same intensity of the applied field that forces the presence of n pairs of vortices in the manifold problem yields the existence of n pairs of vortex lines in any global minimizer Ψ_κ of the three-dimensional energy $G_{\varepsilon,\kappa}$. We stress that even though the phenomenon of vortex lines in three-dimensional Ginzburg–Landau emerging in the presence of an external field has been studied (see [1,16, 17]), the zero set in these cases is realized as an integer multiplicity 1-current, thus it could be viewed as a union of curves only in a weak sense. In our case the zero set is a union of smooth curves. To achieve this, we make use of properties and asymptotics obtained in Section 4, but to do so we first need to verify that the required hypotheses are satisfied. We recall that if we denote $\hat{e}_\theta$ and $\hat{e}_\phi$ as the unit vectors in the θ and ϕ directions respectively, then for any function $\psi : \check{\mathcal{M}} \to \mathbb{C}$ the relative gradient $\nabla_{\check{\mathcal{M}}}$ can be written in the form:

$$\nabla_{\check{\mathcal{M}}}\psi = \left(\frac{1}{\gamma(\phi)}\psi_\phi\right)\hat{e}_\phi + \left(\frac{1}{u(\phi)}\psi_\theta\right)\hat{e}_\theta. \tag{5.6}$$

Also, it will be convenient to choose the potential $\mathbf{A}_{\text{ext}} = \frac{h}{2}(-X_2, X_1, 0)$ corresponding to $\mathbf{H}_{\text{ext}}$ so that on $\check{\mathcal{M}}$ we have

$$\mathbf{A}_{\text{ext}} = (\mathbf{A}_{\text{ext}})^\tau = \left(\frac{hu(\phi)}{2}\right)\hat{e}_\theta. \tag{5.7}$$

Thus,

$$\mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi) = \int_0^\pi \int_0^{2\pi} \left\{\frac{1}{\gamma^2}|\psi_\phi|^2 + \left|\frac{1}{u}\psi_\theta - i\frac{hu}{2}\psi\right|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2\right\} u\,\gamma\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{5.8}$$

since in this case $\mathrm{d}\mathcal{H}^2_{\check{\mathcal{M}}} = u(\phi)\,\gamma(\phi)\,\mathrm{d}\phi\,d\theta$. We prove:

Proposition 5.1. *Let* $h(\kappa) = \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})} \ln \kappa + \sigma \ln \ln \kappa$ *for a constant* $\sigma > 0$ *independent of* κ *and let* ψ_κ *a family of global minimizers of* $\mathcal{G}_{\check{\mathcal{M}},\kappa}$. *Then if we denote by* $\{d_j^{(\kappa)}\}_{j=1,\dots,N_\kappa}$ *their degrees as defined in Proposition 3.2, it holds:*

$$\sum_{j=1}^{N_\kappa} \left| d_j^{(\kappa)} \right| < \mathrm{C}, \tag{5.9}$$

where C *is a constant independent of* κ.

Proof. We first note that in this case $*F$, where F is given by (3.7), can easily be computed as

$$* F(x) = *F(\phi(x)) = \frac{1}{2} \int_0^{\phi} u(\tilde{\phi})\gamma(\tilde{\phi})\, \mathrm{d}\tilde{\phi}. \tag{5.10}$$

Here we have fixed the free constant by taking $*F$ to vanish at the north pole. One can see from (5.10) $*F$ is increasing with respect to $\phi = \phi(x)$. Note also that, thanks to (5.1) and (5.3), $*F$ satisfies

$$* F(0, 0, v(\pi)) - *F(0, 0, v(0)) = \frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi}. \tag{5.11}$$

Next, we see that (3.28) in this setting reads

$$(h(\kappa))^2 \int_{\check{\mathcal{M}}} \left|\mathbf{A}^e\right|^2 |\psi_\kappa|^2 \, \mathrm{d}\mathcal{H}^2_{\check{\mathcal{M}}} = \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} + o(1). \tag{5.12}$$

Let us assume that $\mathbf{d}^\kappa := \sum_{j=1}^{N_\kappa} \left| d_j^{(\kappa)} \right| > 0$ for otherwise there would be nothing to prove. Suppose without loss of generality that there is a number N_κ^+ such that for $i = 1, \dots, N_\kappa^+$, we have $d_j^{(\kappa)} > 0$ and

$$\phi(p_1) \leqq \phi(p_2) \cdots \leqq \phi(p_{N_\kappa^+}), \tag{5.13}$$

while for $i = N_\kappa^+ + 1, \dots, N_\kappa$ we have $d_j^{(\kappa)} \leqq 0$ and

$$\phi(p_{N_\kappa^+ + 1}) \geqq \phi(p_{N_\kappa^+ + 2}) \geqq \cdots \geqq \phi(p_{N_\kappa}). \tag{5.14}$$

We claim that we can assign to each $i = 1, \dots, \frac{1}{2}\mathbf{d}^\kappa$, two points p_i^+ and p_i^-, that are centers of pseudo-balls carrying a non-zero degree and with the property that the degree associated to p_i^+ (resp. p_i^-) is positive (resp. negative). In addition we claim these points can be chosen so that

$$\phi(p_1^+) \leqq \phi(p_2^+) \cdots \leqq \phi\left(p^+_{\frac{1}{2}\mathbf{d}^\kappa}\right), \tag{5.15}$$

while

$$\phi(p_1^-) \geqq \phi(p_2^-) \cdots \geqq \phi\left(p^-_{\frac{1}{2}\mathbf{d}^\kappa}\right). \tag{5.16}$$

To that end, simply define $\tilde{\ell} := \min\{\ell \text{ such that } \sum_{j=1}^{\ell} d_j^{(\kappa)} > i\} - 1$, and set $p_i^+ := p_{\tilde{\ell}}$. This means that each p_i is repeated $d_i^{(\kappa)}$-times. We can do something similar for the p_i's with $d_i^{(\kappa)} < 0$. The cardinalities of these two collections agree and both are equal to $\frac{1}{2}\mathbf{d}^\kappa$, thanks to (3.41). Finally, the monotonicity claims (5.15) and (5.16) follow by construction since we have that (5.13) and (5.14) hold. Define for $s = 1, \ldots, \frac{1}{2}\mathbf{d}^\kappa$

$$F_s := *F(p_s^+) - *F(p_s^-). \tag{5.17}$$

Let $\mathcal{S} := \{1, \ldots, \frac{1}{2}\mathbf{d}^\kappa\}$. With this notation we can rewrite (3.43) in the following manner:

$$\left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} \geqq 4\pi \sum_{s\in\mathcal{S}} (\ln\kappa - \mathcal{O}(\ln\ln\kappa) + F_s h(\kappa)) + \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} - o(1). \tag{5.18}$$

Let $\mathcal{S}_+$ be the set of indices for which F_s is positive. Inequality (5.18) together with (5.11) imply that for some constant C independent of κ, one has

$$C\mathbf{d}^\kappa \ln\ln\kappa \geqq C\,|\mathcal{S}\setminus\mathcal{S}_+|\ln\ln\kappa \geqq |\mathcal{S}_+|\ln\kappa. \tag{5.19}$$

On the other hand, the conditions (5.4) and (5.5) yield the existence of a constant C_0 such that for p near $(0, 0, v(0))$

$$*\,F(p) \geqq C_0(\phi(p))^2 + \mathcal{O}((\phi(p))^3), \tag{5.20}$$

and for q close to $(0, 0, v(\pi))$

$$*\,F(q) \leqq \frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi} - C_0(\phi(q)-\pi)^2 + \mathcal{O}((\phi(q)-\pi)^3). \tag{5.21}$$

Let $\mathcal{S}_{\text{poles}}$ denote the set

$$\mathcal{S}_{\text{poles}} = \left\{ s \in \mathcal{S} \text{ such that } \phi(p_s^-) \geqq \pi - \left(\frac{1}{\ln\kappa}\right)^{\frac{5}{12}},\ \text{and } \phi(p_s^+) \leqq \left(\frac{1}{\ln\kappa}\right)^{\frac{5}{12}} \right\}. \tag{5.22}$$

Then, appealing to (5.20) and (5.21), we deduce from (5.18) that for some constant C independent of κ, one has

$$C\mathbf{d}^\kappa \ln\ln\kappa \geqq C\left|\mathcal{S}_{\text{poles}}\right|\ln\ln\kappa \geqq \left|\mathcal{S}\setminus\mathcal{S}_{\text{poles}}\right|(\ln\kappa)^{\frac{1}{6}}. \tag{5.23}$$

Let $\hat{B}$ denote the set of points in $[0, \pi]\times[0, 2\pi]$ that correspond to the union of the pseudo-balls $\cup_{j=1}^{N_\kappa}\hat{B}_j$ in this parametrization. Appealing to (3.25) once again and substituting (5.12), we get

$$\begin{aligned}
o(1) &\geqq \int_{[0,\pi]\times[0,2\pi]\setminus\hat{B}} \left[\frac{1}{\gamma^2}\left|\psi_\phi\right|^2 + \frac{1}{u^2}\left|\psi_\theta\right|^2 + \frac{\kappa^2}{2}(|\psi_\kappa|^2-1)^2\right] u\gamma \,\mathrm{d}\theta\,\mathrm{d}\phi \\
&\quad +2\pi\sum_{j=1}^{N_\kappa}\left|d_j^{(\kappa)}\right|(\ln\kappa - c\ln\ln\kappa) - \Lambda(\mathbf{A}_{\mathrm{ext}},\psi_\kappa) \\
&\geqq \int_{[0,\pi]\times[0,2\pi]\setminus\hat{B}} \left[\frac{1}{\gamma^2}\left|\psi_\phi\right|^2 + \frac{1}{u^2}\left|\psi_\theta\right|^2 + \frac{\kappa^2}{2}(|\psi_\kappa|^2-1)^2\right] u\gamma \,\mathrm{d}\theta\,\mathrm{d}\phi \\
&\quad +2\pi\sum_{j=1}^{N_\kappa}\left|d_j^{(\kappa)}\right|(\ln\kappa - c\ln\ln\kappa) \\
&\quad -4\pi\left(\frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}\ln\kappa + \sigma\ln\ln\kappa\right)\frac{1}{2}\sum_{j=1}^{N_\kappa}\left|d_j^{(\kappa)}\right|\frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi} + o(1) \\
&= M_\kappa - R_1\left(\sum_{j=1}^{N_\kappa}\left|d_j^{(\kappa)}\right|\right)\ln\ln\kappa + o(1),
\end{aligned} \tag{5.24}$$

where in the first inequality we have used item (4) of Proposition 3.2 and in the following equality, (3.37) and (3.42). In the last line we have defined

$$M_\kappa := \int_{[0,\pi]\times[0,2\pi]\setminus\hat{B}} \left[\frac{1}{\gamma^2}\left|\psi_\phi\right|^2 + \frac{1}{u^2}\left|\psi_\theta\right|^2 + \frac{\kappa^2}{2}(|\psi_\kappa|^2-1)^2\right] u\gamma\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{5.25}$$

and

$$R_1 := \frac{\sigma}{2}\mathcal{H}^2(\check{\mathcal{M}}) + 2\pi c. \tag{5.26}$$

Our next goal is to obtain a lower bound for M_κ. For that purpose we once again appeal to Proposition 3.2, more specifically to items (2) and (3), to see that if we define $\mathcal{C}_\phi = \{(u(\phi)\cos\theta, u(\phi)\sin\theta, v(\phi)), \theta\in[0,2\pi]\}$, then

$$\left|\{\phi\in[0,\pi] \text{ s.t. } \mathcal{C}_\phi \cap \cup_{j=1}^{N_\kappa}\hat{B}_j \text{ is non-empty }\}\right| \lesssim \frac{1}{(\ln\kappa)^4}. \tag{5.27}$$

Now, note that from (5.3) and (5.5), one sees that for ϕ near zero,

$$\frac{\gamma}{u} = \frac{1}{\phi} + \mathcal{O}(1), \tag{5.28}$$

so fixing ϕ_0 small independent of κ and defining

$$A_{\phi_0} = \left\{\phi : \left(\frac{1}{\ln\kappa}\right)^{\frac{5}{12}} < \min\{|\phi|, |\pi-\phi|\} \leqq \phi_0 \text{ and } \mathcal{C}_\phi \cap \cup_{j=1}^{N_\kappa}\hat{B}_j \text{ is empty}\right\} \tag{5.29}$$

we are able to write $\psi(\theta,\phi) = f(\theta,\phi)e^{i\chi(\theta,\phi)}$ locally on $A_{\phi_0} \times [0, 2\pi]$, where f and χ are real functions, f is smooth 2π-periodic in θ and $f \geqq \frac{1}{2}$ restricted to $A_{\phi_0} \times [0, 2\pi]$. Thus, using the "lower bounds on annuli" method introduced in [24], we derive

$$\begin{aligned} M_\kappa &\geqq \int_{A_{\phi_0}} \left(\int_0^{2\pi} \left\{ \left(\frac{\partial f}{\partial \theta}\right)^2 + f^2 \left(\frac{\partial \chi}{\partial \theta}\right)^2 \right\} \mathrm{d}\theta \right) \frac{\gamma}{u} \,\mathrm{d}\phi \\ &\geqq \frac{1}{4} \int_{A_{\phi_0}} \frac{1}{4\pi^2} \left(\int_0^{2\pi} \frac{\partial \chi}{\partial \theta} \,\mathrm{d}\theta \right)^2 \left(\frac{1}{\phi} + \mathcal{O}(1) \right) \mathrm{d}\phi \\ &\geqq \frac{1}{4} \int_{A_{\phi_0} \cap \{\phi \leqq \phi_0\}} \frac{1}{4\pi^2} \left(2\pi \sum_{\{j \text{ s.t. } \phi(p_j) \leqq \phi\}} d_j^{(\kappa)} \right)^2 \left(\frac{1}{\phi} + \mathcal{O}(1) \right) \mathrm{d}\phi \\ &\geqq \left(\frac{1}{2} \sum_{j=1}^{N_\kappa} \left| d_j^{(\kappa)} \right| \left(1 + \mathcal{O}\left(\frac{\ln\ln\kappa}{(\ln\kappa)^{\frac{1}{6}}} \right) \right) \right)^2 R_2 \ln\ln\kappa, \end{aligned} \tag{5.30}$$

for some constant R_2, where in the second inequality we have used Hölder and in the last one (3.42), (5.19), (5.23) and (5.27). To conclude, plug (5.30) into (5.24) to obtain

$$\frac{R_2}{2} \left(\sum_{j=1}^{N_\kappa} \left| d_j^{(\kappa)} \right| \right)^2 \leqq 4R_1 \sum_{j=1}^{N_\kappa} \left| d_j^{(\kappa)} \right| + o(1), \tag{5.31}$$

which implies (5.9). □

Remark 5.1. Note that since the degrees d_j^κ are integers, the conclusion (5.9) allows us to assert the existence of a constant c independent of κ such that

$$\phi(p_i) \leqq c \left(\frac{1}{\ln\kappa} \right)^{\frac{5}{12}}, \quad \text{for all } i \text{ s.t. } d_i^{(\kappa)} < 0, \tag{5.32}$$

and also

$$\pi - \phi(p_i) \leqq c \left(\frac{1}{\ln\kappa} \right)^{\frac{5}{12}}, \quad \text{for all } i \text{ s.t. } d_i^{(\kappa)} > 0, \tag{5.33}$$

thanks to (5.19) and (5.23).

Before stating the main theorem of the second part of this article on number and location of vortices, we first provide some pertinent definitions. First, for every $n \in \mathbb{N}$ define the function $\mathbf{R}^n : (\mathbb{R}^2)^n \mapsto \mathbb{R}$, that to a given collection $\{x_i\}_{i=1}^n \subseteq \mathbb{R}^2$, assigns the number

$$\mathbf{R}^n(x_1, \dots, x_n) := -2\pi \sum_{i \neq j} \ln \left| x_i - x_j \right| + \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})} \sum_{i=1}^n |x_i|^2 . \tag{5.34}$$

The projection of elements of the manifold $\check{\mathcal{M}}$ onto the xy plane is defined naturally as

$$\mathbf{Proj}\, p := p - (p \cdot \hat{e}_z)\hat{e}_z, \quad \text{for } p \in \check{\mathcal{M}}. \tag{5.35}$$

Remark 5.2. Clearly **Proj** is not globally one-to-one, however it is injective when restricted to small neighbourhoods of $(0, 0, v(0))$ and $(0, 0, v(\pi))$, a fact that we make use of later when determining the asymptotic configuration of the vortices.

We recall a function $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be non-singular at x if $\det[Jac\,\psi](x) \neq 0$. Our result about emergence of multiple vortices above the first critical field is the following:

Theorem 5.1. *Let $\check{\mathcal{M}}$ be a simply connected surface of revolution as defined in (5.1), (5.2) and (5.3), satisfying in addition the regularity condition (5.4). Let*

$$\mathbf{H}_{\text{ext}}(\kappa) = h(\kappa)\hat{e}_z, \ \textit{where } h(\kappa) = \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})} \ln \kappa + \sigma \ln \ln \kappa. \tag{5.36}$$

If $\sigma \notin (4\pi/\mathcal{H}^2(\check{\mathcal{M}}))\mathbb{Z}$, then there is a κ_0 such that for all $\kappa \geqq \kappa_0$, any minimizer ψ_κ of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ possesses exactly $2n_0 := 2\lfloor \sigma \frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi} \rfloor + 2$ vortices which are non-singular. Furthermore, the set of vortices $\{p_i^\kappa : i = 1, \ldots, 2n_0\}$ satisfies:

1. *There exists a constant M such that for all $\kappa \geqq \kappa_0$, it holds that*

$$\{p_i^\kappa\}_{i=1}^{n_0} \subseteq \left\{ x \in \check{\mathcal{M}} : \phi(x) \leqq \frac{M}{\sqrt{\ln \kappa}} \right\}, \ \textit{and}$$
$$\{p_i^\kappa\}_{i=n_0+1}^{2n_0} \subseteq \left\{ x \in \check{\mathcal{M}}; \phi(x) \geqq \pi - \frac{M}{\sqrt{\ln \kappa}} \right\}.$$

2. *For all $i = 1, \ldots, n_0$ the degrees associated to the vortices p_i^κ is equal to 1, whereas the remaining vortices have an associated degree of -1.*
3. *Lastly, if for $i = 1, \ldots, 2n_0$ we define $P_i^\kappa = \sqrt{\ln \kappa}\, \mathbf{Proj}\, p_i^\kappa$, then the configurations $(P_1^\kappa, \ldots, P_{n_0}^\kappa)$ and $(P_{n_0+1}^\kappa, \ldots, P_{2n_0}^\kappa)$ converge, up to subsequence, simultaneously as κ goes to infinity to respective minimizers $\overrightarrow{\mathbf{X}}_0$ and $\overrightarrow{\mathbf{X}}_\pi$ of the renormalized energy $\mathbf{R}^{n_0}$.*

Theorem 5.1 provides an analogue of Theorem 1.2 in [21] in the case of a planar disk. Here, we see that there are two concentration points, as opposed to the case considered in [21]. Also, in this setting in order to write a renormalized energy, one is forced to leave the manifold; in order to rescale the vortices one must project them first to the xy plane, an operation that, for κ large, provides a one-to-one relation between these points living in Euclidean space, and the vortices, while allowing us to write the energy in terms of these projections with a precision of $o(1)$. Once this is achieved, the renormalized energy is decomposed into two independent components that are qualitatively like the one in the flat case.

A related result for thin shells is also available.

Theorem 5.2. *Using the same notation as in Theorem 5.1, fix* $\kappa \geqq \kappa_0$. *It holds that there exists an* ε_0 *such that for all* $\varepsilon < \varepsilon_0$ *any minimizer* $\Psi_{\varepsilon,\kappa}$ *of* $G_{\varepsilon,\kappa}$ *has exactly* $2n_0$ *vortex lines. More precisely, letting* $\psi_{\varepsilon,\kappa}$ *be the function in* (2.5)*, there are* $\ell_i^\varepsilon, i = 1, \ldots, 2n_0$*, disjoint* C^1 *curves whose union comprises the zero set of* $\psi_{\varepsilon,\kappa}$ *and such that for all* $i = 1, \ldots, 2n_0$*, it holds that*

$$\ell_i^\varepsilon(t) \to (p_i^\kappa, t),$$

uniformly in $(0, 1)$*, as* $\varepsilon \to 0$.

Proof of Theorem 5.1. 1. **We first prove that the vortices with non-zero degree lie near the poles**

By the hypotheses, we can apply Proposition 5.1. Thus, equation (5.9) holds and this implies that we can also make use of Propositions 4.1 and 4.1. Remember that $\sum_{i=1}^{n_\kappa} d_{\alpha_0,i}^\kappa = \sum_{j=1}^{N_\kappa} d_j^{(\kappa)} = 0$.

From (3.25), (3.28), (4.29) and (4.30) applied to $r = \frac{1}{\kappa^{\alpha_0}}$, one sees that

$$\begin{aligned}(h(\kappa))^2 \left\|(\mathbf{A}^e)^\tau\right\|_{L^2(\check{\mathcal{M}})}^2 \geqq 2\pi \sum_{j=1}^{n_\kappa} \left|d_{\alpha_0,j}^\kappa\right| \ln(\kappa^{1-\alpha_0}) + 4\pi h(\kappa) \sum_{j=1}^{n_\kappa} *F(a_j^\kappa) d_{\alpha_0,j}^\kappa \\ +(h(\kappa))^2 \left\|(\mathbf{A}^e)^\tau\right\|_{L^2(\check{\mathcal{M}})}^2 - o(1). \qquad (5.37)\end{aligned}$$

Defining $\mathbf{d}^{\alpha_0} := \sum_{i=1}^{n_\kappa} \left|d_{\alpha_0,j}^\kappa\right|$, we assign to each $i \in \mathcal{S} := \{1, \ldots, \mathbf{d}^{\alpha_0}\}$ two centers of pseudo-balls; $a_i^{\kappa,+}$ and $a_i^{\kappa,-}$, carrying positive and negative degrees respectively. In addition, the points thus chosen can be assumed to satisfy (5.15) and (5.16), where this time the role of the $p_i^\pm$'s is replaced by the $a_i^{\kappa,\pm}$'s. This construction can be carried out in a way similar to what we did for the larger pseudo-balls in the beginning of the proof of Proposition 5.1. In the same way as before, we define

$$F_s := *F(a_s^{\kappa,+}) - *F(a_s^{\kappa,-}). \qquad (5.38)$$

Let $\mathcal{S}_+ := \{s \in \mathcal{S} : F_s > 0\}$. We see that for $s \in \mathcal{S}_+$, one has trivially $F_s h(\kappa) \geqq 0$, while for $s \notin \mathcal{S}_+,\ F_s h(\kappa) \geqq -\ln\kappa - \mathcal{O}(\ln\ln\kappa)$. From this and (5.37), we see that for some constant c independent of κ

$$c\alpha_0 \mathbf{d}^{\alpha_0} \ln\kappa \geqq c\alpha_0 \left|\mathcal{S} \setminus \mathcal{S}_+\right| \ln\kappa \geqq (1-\alpha_0) \left|\mathcal{S}_+\right| \ln\kappa + o(1). \qquad (5.39)$$

Similarly, fixing ε_0 small, we define $\mathcal{S}_{\varepsilon_0} := \{s \in \mathcal{S} : F_s < -\varepsilon_0 \frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi}\}$. This time, we see that for $s \notin \mathcal{S}_{\varepsilon_0},\ F_s h(\kappa) \geqq -\varepsilon_0 \ln\kappa - \mathcal{O}(\ln\ln\kappa)$. Again, appealing to (5.37), we deduce that for some constant c independent of κ

$$c\alpha_0 \mathbf{d}^{\alpha_0} \ln\kappa \geqq c\alpha_0 \left|\mathcal{S}_{\varepsilon_0}\right| \ln\kappa \geqq \left|\mathcal{S} \setminus \mathcal{S}_{\varepsilon_0}\right| (1-\alpha_0-\varepsilon_0) \ln\kappa + o(1). \qquad (5.40)$$

We note that α_0 can be chosen arbitrarily small and the results in Section 4 remain valid. If one considers the construction of Proposition 4.1 for two different exponents $\alpha_0 > \alpha_0'$, one must necessarily have the monotonicity condition $\mathbf{d}^{\alpha_0} \geqq \mathbf{d}^{\alpha_0'}$, due to the fact that the pseudo-balls associated to the smaller exponent are larger. Since we know that $\mathbf{d}^{\alpha_0}$ is bounded independently of κ, thanks to (4.14), we may assume α_0 is small enough so that $c\frac{\alpha_0}{1-\alpha_0-\varepsilon_0}\mathbf{d}^{\alpha_0} < 1$. This together with (5.39) and (5.40) yields that both $\mathcal{S}_+$ and $\mathcal{S}\setminus\mathcal{S}_{\varepsilon_0}$ are empty. But then, there exist a constant C such that

$$\min\left\{\phi(a_i^\kappa) : d^\kappa_{\alpha_0,i} < 0\right\} - \max\left\{\phi(a_i^\kappa) : d^\kappa_{\alpha_0,i} > 0\right\} > C\varepsilon_0.$$

Because of this, each pseudo-ball of size $\sim \frac{1}{(\ln\kappa)^6}$ cannot contain two different pseudo-balls $\mathcal{B}_1$ and $\mathcal{B}_2$ of size $\sim \frac{1}{\kappa^{\alpha_0}}$, with associated degrees $d_1 > 0$ and $d_2 < 0$ respectively. As a consequence of this, we have $\sum_{i=1}^{n_\kappa}\left|d^\kappa_{\alpha_0,i}\right| = \sum_{j=1}^{N_\kappa}\left|d_j^{(\kappa)}\right|$, and

$$\phi(a_i^\kappa) \leqq c\left(\frac{1}{\ln\kappa}\right)^{\frac{5}{12}}, \quad \text{for all } i \text{ such that } d^\kappa_{\alpha_0,i} > 0, \tag{5.41}$$

while

$$\pi - \phi(a_i^\kappa) \leqq c\left(\frac{1}{\ln\kappa}\right)^{\frac{5}{12}}, \quad \text{for all } i \text{ such that } d^\kappa_{\alpha_0,i} < 0. \tag{5.42}$$

2. **We prove now that** $\left|d^\kappa_{\alpha_0,i}\right| = 1$ **for all** i
Assertions (5.21), (5.41) and (5.42) allow us to write

$$h(\kappa)\Lambda(\mathbf{A}^e,\psi_\kappa) = -h(\kappa)\frac{1}{2}\sum_{i=1}^{n_\kappa}\left|d^\kappa_{\alpha_0,i}\right|\mathcal{H}^2(\check{\mathcal{M}}) + \mathcal{O}((\ln\kappa)^{\frac{1}{6}}). \tag{5.43}$$

Denote by J_0 (resp. J_π) the set of indices $i \in \{1,\ldots,n_\kappa\}$ s.t. $d^\kappa_{\alpha_0,i} < 0$ (resp. $d^\kappa_{\alpha_0,i} > 0$). We apply Proposition 4.3 letting $r = \frac{1}{\kappa^{\alpha_0}}$. Substituting (5.43) in (4.28) we obtain

$$\begin{aligned}
&\mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi_\kappa)\\
&\geqq -4\pi^2\sum_{i\in J_0\bigcup J_\pi}\left|d^\kappa_{\alpha_0,i}\right|^2 G(a_i^\kappa,x_i) - 4\pi^2\sum_{i\neq j\in J_0} d^\kappa_{\alpha_0,i}\,\mathrm{d}^\kappa_{\alpha_0,j}\,G(a_i^\kappa,a_j^\kappa)\\
&\quad -4\pi^2\sum_{i\neq j\in J_\pi} d^\kappa_{\alpha_0,i}\,\mathrm{d}^\kappa_{\alpha_0,j}\,G(a_i^\kappa,a_j^\kappa) - 4\pi^2\sum_{i\in J_0,\ j\in J_\pi} d^\kappa_{\alpha_0,i}\,\mathrm{d}^\kappa_{\alpha_0,j}\,G(a_i^\kappa,a_j^\kappa)\\
&\quad +2\pi\sum_{i=1}^{n_\kappa}\left|d^\kappa_{\alpha_0,i}\right|\ln(\kappa^{1-\alpha_0}) + \left(\frac{h(\kappa)}{2}\right)^2\|u\|^2_{L^2(\check{\mathcal{M}})}\\
&\quad -\frac{h(\kappa)}{2}\sum_{i=1}^{n_\kappa}\left|d^\kappa_{\alpha_0,i}\right|\mathcal{H}^2(\check{\mathcal{M}}) + \mathcal{O}((\ln\kappa)^{\frac{1}{6}}) + \mathcal{O}(1).
\end{aligned}\tag{5.44}$$

Note that

$$G\left(a_i^\kappa,a_j^\kappa\right) = G\big((0,0,v(0)),(0,0,v(\pi))\big) + o(1) \quad \text{for } i\in J_0,\ j\in J_\pi, \tag{5.45}$$

and that

$$d^{\kappa}_{\alpha_0,i}d^{\kappa}_{\alpha_0,j} = \begin{cases} \left|d^{\kappa}_{\alpha_0,i}\right|\left|d^{\kappa}_{\alpha_0,j}\right| & \text{if } i, j \in J_0 \text{ or } i, j \in J_\pi, \\ -\left|d^{\kappa}_{\alpha_0,i}\right|\left|d^{\kappa}_{\alpha_0,j}\right| & \text{if } i \in J_0,\ j \in J_\pi. \end{cases} \tag{5.46}$$

Now that we have established that all the vortices with non-zero degree lie within two well separated neighborhoods of $(0, 0, v(0))$ and $(0, 0, v(\pi))$, we write the Green's function in a more convenient way. Recall that we denote by r_0 the injectivity radius. Fix a number $\mathbf{r} < \frac{r_0}{2}$. Let $p \in \check{\mathcal{M}}$ and consider an isothermal coordinate chart $\left(\mathcal{B}(p, \mathbf{r}), \mathcal{I}_p\right)$. Let ρ be a cut-off function supported in $\mathcal{B}(p, 2\mathbf{r})$, equal to 1 on $\mathcal{B}(p, \mathbf{r})$. Consider the function Γ_p defined on $\mathcal{B}(p, 2\mathbf{r})$ by

$$\Gamma_p(q) := \left(\frac{1}{2\pi} \ln \left|\mathcal{I}_p(p) - \mathcal{I}_p(q)\right|\right) \cdot \rho(q). \tag{5.47}$$

Then, defining the regular part

$$H(x, y) := G(x, y) - \Gamma_x(y), \tag{5.48}$$

we see that, thanks to (4.2) and elliptic regularity, $H(x, y)$ is of class C^1. The definition of $H(x, x)$ can easily be seen to be independent of $\mathbf{r}$ and the coordinate chart.

Remark 5.3. Note that in light of this, for all $x \in \partial\mathcal{B}_{i,r}$, it holds that

$$G\left(a_i^{\kappa}, x\right) = -\frac{1}{2\pi} \ln \frac{1}{r} + H\left(a_i^{\kappa}, a_i^{\kappa}\right) + o(1).$$

In addition, since

$$\left(\min_{x \in B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa), r)} \lambda(x)\right) \cdot \left|\mathcal{I}_{a_i^\kappa}(a_i^\kappa) - \mathcal{I}_{a_i^\kappa}(a_j^\kappa)\right| \leqq d_{\mathcal{M}}(a_i^\kappa, a_j^\kappa)$$
$$\leqq \left(\max_{x \in B(\mathcal{I}_{a_i^\kappa}(a_i^\kappa), r)} \lambda(x)\right) \cdot \left|\mathcal{I}_{a_i^\kappa}(a_i^\kappa) - \mathcal{I}_{a_i^\kappa}(a_j^\kappa)\right|,$$

and λ satisfies (4.1), we conclude

$$\Gamma_{a_i^\kappa}(a_j^\kappa) = \frac{1}{2\pi} \ln d_{\check{\mathcal{M}}}(a_i^\kappa, a_j^\kappa) + o(1),$$

whenever $i, j \in J_0$, or $i, j \in J_\pi$. Lastly, note that (5.47) and (5.48) make the rough bounds (4.24), (4.25) and (4.26) superfluous, since we can now assert that

$$\sup_{x, y \in \partial\mathcal{B}(b_i, r)} |G(b_i, x) - G(b_i, y)| = o(1).$$

Next, using Remark 5.3, along with (5.12), (5.41), (5.42), (5.45) and (5.46) in (5.44) yields

$$\begin{aligned}
\mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi_\kappa) \geqq\ & 2\pi \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right|^2 \ln(\kappa^{\alpha_0}) - 2\pi \sum_{i\neq j\in J_0} \left|d^\kappa_{\alpha_0,i}\right| \left|d^\kappa_{\alpha_0,j}\right| \ln d_{\check{\mathcal{M}}}(a^\kappa_i, a^\kappa_j) \\
& -2\pi \sum_{i\neq j\in J_\pi} \left|d^\kappa_{\alpha_0,i}\right| \left|d^\kappa_{\alpha_0,j}\right| \ln d_{\check{\mathcal{M}}}(a^\kappa_i, a^\kappa_j) + H(J_0\cup J_\pi) \\
& -2\pi \sum_{i\in J_0, j\in J_\pi} \left|d^\kappa_{\alpha_0,i}\right| \left|d^\kappa_{\alpha_0,j}\right| G\left((0,0,v(0)),(0,0,v(\pi))\right) \\
& +2\pi \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right| \ln(\kappa^{1-\alpha_0}) + \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} \\
& -\frac{h(\kappa)}{2} \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right| \mathcal{H}^2(\check{\mathcal{M}}) + \mathcal{O}((\ln\kappa)^{\frac{1}{6}}) \\
\geqq\ & 2\pi \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right|^2 \ln(\kappa^{\alpha_0}) + 2\pi \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right| \ln(\kappa^{1-\alpha_0}) \\
& + \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} - \frac{h(\kappa)}{2} \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right| \mathcal{H}^2(\check{\mathcal{M}}) \\
& +\mathcal{O}\left((\ln\kappa)^{\frac{1}{6}}\right). \qquad (5.49)
\end{aligned}$$

Here, we have introduced for a global minimizer (later on this quantity will take on a simpler form after we show all the vortices have degree ± 1) the notation

$$\begin{aligned}
& H(J_0\cup J_\pi) \\
& := \left[\sum_{i\in J_0} \left|d^\kappa_{\alpha_0,i}\right|^2 + \sum_{i\neq j\in J_0} \left|d^\kappa_{\alpha_0,i}\right| \left|d^\kappa_{\alpha_0,j}\right|\right] H\left((0,0,v(0)),(0,0,v(0))\right) \\
& \quad + \left[\sum_{i\in J_\pi} \left|d^\kappa_{\alpha_0,i}\right|^2 + \sum_{i\neq j\in J_\pi} \left|d^\kappa_{\alpha_0,i}\right| \left|d^\kappa_{\alpha_0,j}\right|\right] H\left((0,0,v(\pi)),(0,0,v(\pi))\right). \qquad (5.50)
\end{aligned}$$

On the other hand, we claim that we can construct a comparison map $\tilde{\Psi}_\kappa : \check{\mathcal{M}} \to \mathbb{C}$ with $\frac{1}{2}\sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right|$ vortices of degree $+1$ on the circle $\{p \in \check{\mathcal{M}}, \phi(p) = \frac{1}{\sqrt{\ln\kappa}}\}$ and the same number of vortices of degree -1 on the circle $\{p \in \check{\mathcal{M}}, \phi(p) = \pi - \frac{1}{\sqrt{\ln\kappa}}\}$, with total energy

$$\begin{aligned}
\mathcal{G}_{\check{\mathcal{M}},\kappa}(\tilde{\Psi}_\kappa) \leqq\ & 2\pi \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right| \ln\kappa + \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} \\
& -h(\kappa) \sum_{i=1}^{n_\kappa} \left|d^\kappa_{\alpha_0,i}\right| \frac{\mathcal{H}^2(\check{\mathcal{M}})}{2} + \mathcal{O}(\ln\ln\kappa). \qquad (5.51)
\end{aligned}$$

This can be achieved as follows. Let $n_0 := \frac{1}{2}\sum_{i=1}^{n_\kappa}\left|d^{\kappa}_{\alpha_0,i}\right|$. For $i = 1, \ldots, n_0$, let q_i denote the point in $\check{\mathcal{M}}$ whose coordinates are

$$(\phi(q_i), \theta(q_i)) = \begin{cases} \left(\frac{1}{\sqrt{\ln\kappa}}, \frac{2\pi}{n_0}\left(\frac{i-1}{2}\right)\right) & \text{if } i = 1 \text{ is even,} \\ \left(\pi - \frac{1}{\sqrt{\ln\kappa}}, \frac{2\pi}{n_0}\left(\frac{i}{2}\right)\right) & \text{if } i \text{ is odd.} \end{cases} \tag{5.52}$$

Note that the points defined in this way satisfy (4.39). We let $r := \frac{1}{\kappa^{\alpha_0}}$. Thanks to Proposition 4.4 we can associate to the points q_i, $i = 1, \ldots, 2n_0$, a function ψ^r_{out} defined on $\check{\mathcal{M}} \setminus \cup_{i=1}^{2n_0}\mathcal{B}(q_i, r)$ satisfying (4.40). In turn, inside each $\mathcal{B}(q_j, r)$ we can define a function ψ_j analogously to $\tilde{\psi}^j_\kappa$ in (3.13), that agrees with ψ^r_{out} on $\partial\mathcal{B}(q_j, r)$. To see this, let f_κ be the function in (3.12), where now delta takes the value $\delta := \frac{1}{\kappa^{\alpha_0}}$. Using polar coordinates about $\mathcal{I}_{q_j}(q_j)$, we let ψ_j be the pullback of the function $\psi^j_{euc}(r, \theta) := f(r)e^{(-1)^j i\theta}$ under $\mathcal{I}_{q_j}$. Just as in the calculation (3.19), using the definition (4.27), one finds

$$\int_{\mathcal{B}(q_j,r)} S_\kappa(\psi_j)\, \mathrm{d}\mathcal{H}^2_{\check{\mathcal{M}}} \leqq 2\pi\ln(\kappa^{1-\alpha_0}) + \mathcal{O}(1). \tag{5.53}$$

We thus define $\tilde{\Psi}_\kappa$ by

$$\tilde{\Psi}_\kappa(x) = \begin{cases} \psi^r_{out}(x) & \text{if } x \in \check{\mathcal{M}} \setminus \cup\mathcal{B}(q_i, r), \\ \psi_i(x) & \text{if } x \in \mathcal{B}(q_i, r). \end{cases} \tag{5.54}$$

The function $\tilde{\Psi}_\kappa$ is of modulus $\equiv 1$ outside of the union of the pseudo-balls, whose total measure is of order $\frac{1}{\kappa^{\alpha_0}}$. Therefore

$$(h(\kappa))^2 \int_{\check{\mathcal{M}}} \left|\mathbf{A}^e\right|^2 \left|\tilde{\Psi}_\kappa\right|^2 \mathrm{d}\mathcal{H}^2_{\check{\mathcal{M}}} = \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} + o(1). \tag{5.55}$$

For the same reason, one can argue as in the derivation of (3.37) and (5.43) that

$$h(\kappa)\Lambda(\mathbf{A}^e, \tilde{\Psi}_\kappa) = -\frac{h(\kappa)}{2}\sum_{i=1}^{n_k}\left|d^{\kappa}_{\alpha_0,i}\right|\mathcal{H}^2(\check{\mathcal{M}}) + \mathcal{O}(\ln\ln\kappa). \tag{5.56}$$

Making use of (5.53), (5.55) and (5.56), the bound (5.51) follows after substituting the expansions contained in Remark 5.3 into (4.40). Because the ψ_κ are global minimizers, we must have

$$\mathcal{G}_{\check{\mathcal{M}},\kappa}(\tilde{\Psi}_\kappa) \geqq \mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi_\kappa), \tag{5.57}$$

which together with (5.49) and (5.51) imply

$$2\pi\sum_{i=1}^{n_\kappa}\left(\left|d^{\kappa}_{\alpha,i}\right|^2 - \left|d^{\kappa}_{\alpha,i}\right|\right)\ln\kappa \leqq \mathcal{O}((\ln\kappa)^{\frac{1}{6}}). \tag{5.58}$$

This cannot hold unless $\left|d^{\kappa}_{\alpha_0,i}\right| = 1$ for all i such that $d^{\kappa}_{\alpha_0,i} \neq 0$, for large values of κ. With this conclusion at hand, we may use (5.49), (5.51) and (5.57) one more time to conclude

$$-\sum_{i,j\in J_0} \ln d_{\check{\mathcal{M}}}(a_i^{\kappa}, a_j^{\kappa}) - \sum_{i,j\in J_\pi} \ln d_{\check{\mathcal{M}}}(a_i^{\kappa}, a_j^{\kappa}) \leqq \mathcal{O}\left((\ln\kappa)^{\frac{1}{6}}\right), \tag{5.59}$$

which in turn implies the existence of a $\beta > 0$ such that for κ large, letting $\tilde{\varepsilon} < \alpha_0^{N_0+1}$,

$$\frac{1}{d_{\check{\mathcal{M}}}(a_i^{\kappa}, a_j^{\kappa})} \leqq e^{\beta(\ln\kappa)^{\frac{1}{6}}} \leqq e^{\tilde{\varepsilon}\ln\kappa} \quad \text{for all } i \neq j. \tag{5.60}$$

This implies that for $i \neq j$,

$$d_{\check{\mathcal{M}}}(a_i^{\kappa}, a_j^{\kappa}) \gg \frac{1}{\kappa^{\alpha_0^{N_0+1}}} \tag{5.61}$$

We would like now to refine the estimates we have so as to compute the energy of a minimizer up to $o(1)$. To this end, take p_i^{κ}, $i = 1, \dots, m_\kappa$, the center of the pseudo-balls $\{\mathcal{B}(p_i^{\kappa}, \frac{\lambda_0}{\kappa})\}_{i=1}^{m_\kappa}$ provided by Proposition 4.1. By making λ_0 larger if necessary we can always assume $d_{\check{\mathcal{M}}}(p_i^{\kappa}, p_j^{\kappa}) \geqq \frac{5}{\kappa^{\alpha_0}}$, whenever $i \neq j$, and therefore assume that each pseudo-ball of radius $\sim \frac{1}{\kappa^{\alpha_0}}$ contains at most one of the family $\{\mathcal{B}(p_i^{\kappa}, \frac{\lambda_0}{\kappa})\}_{i=1}^{m_\kappa}$. Then, since we know $\left|d^{\kappa}_{\alpha_0,i}\right| = 1$ for all i such that $d^{\kappa}_{\alpha_0,i} \neq 0$, the same must hold for the degrees d_i^{κ} of the smaller pseudo-balls. Thus, $\sum \left|d_i^{\kappa}\right| = \sum \left|d^{\kappa}_{\alpha_0,i}\right| = \sum \left|d_i^{(\kappa)}\right| < C$. As before this has as a by-product the following confinement assertions

$$\phi(p_i^{\kappa}) \leqq c\left(\frac{1}{\ln\kappa}\right)^{\frac{5}{12}}, \quad \text{for all } i \text{ s.t. } d_i^{\kappa} > 0 \tag{5.62}$$

and

$$\pi - \phi(p_i^{\kappa}) \leqq c\left(\frac{1}{\ln\kappa}\right)^{\frac{5}{12}}, \quad \text{for all } i \text{ s.t. } d_i^{\kappa} < 0. \tag{5.63}$$

Now, using the notation in (4.50), we write inside any of the balls $B(\mathcal{I}_{p_i^{\kappa}}(p_i^{\kappa}), \frac{\lambda_0}{\kappa})$:

$$\hat{\psi}_\kappa(y) := \psi_\kappa^{\mathbf{euc}}\left(\mathcal{I}_{p_i^{\kappa}}(p_i^{\kappa}) + \frac{1}{\kappa}y\right). \tag{5.64}$$

One can see $\hat{\psi}_\kappa$ converges in $C^1_{loc}(\mathbb{R}^2)$ as $\kappa \to \infty$ to a solution $\hat{\psi}_0$ of

$$-\Delta\hat{\psi}_0 = \hat{\psi}_0\left(1 - \left|\hat{\psi}_0\right|^2\right), \text{ with } \int_{\mathbb{R}^2}\left(1 - \left|\hat{\psi}_0\right|^2\right) < \infty. \tag{5.65}$$

If $d_i^\kappa = 0$, then $\left|\hat{\psi}_0\right| \equiv 1$ (cf. [4]), and hence $\left|\hat{\psi}_\kappa\right| \to 1$, which implies that the pseudo-ball $\mathcal{B}(p_i^\kappa, \frac{\lambda_0}{\kappa})$ should not even belong to the collection. In particular $d_i^\kappa = \pm 1$ for all i. It is known that the solutions of (5.65) of degree ± 1 are unique, up to a multiplicative constant (cf. [19]). More precisely $\hat{\psi}_0 = f(r)e^{\pm i\theta}$, where f is a real valued function. A result of Herve-Herve (cf. [14]) asserts the existence of a constant $a > 0$ such that $f(r) = ar - \frac{a}{8}r^3 + \mathcal{O}(r^5)$, for r small, and then $\det[Jac\, \hat{\psi}_0](0,0) = a + o(1)$. By virtue of this, one can apply the implicit function theorem and conclude there is only one zero inside each pseudo-ball. We may thus assume $p_i^\kappa = a_i^\kappa$ corresponds to the unique zero inside $\mathcal{B}(a_i^\kappa, \frac{1}{\kappa^{\alpha_0}})$, thanks to (5.61), for all $i = 1, \ldots, m_\kappa$, and note that $m_\kappa = n_\kappa$. In turn, $\sum d_i^\kappa = 0$ allows us to conclude that there is a number n_0 such that $2n_0 = m_\kappa$ which we prove later to be independent of κ, as $\kappa \to \infty$.

3. **We now find the number of pairs of vortices,** n_0

We claim that

$$\begin{aligned}\int_{\check{\mathcal{M}}} S_\kappa(\psi_\kappa)\, \mathrm{d}\mathcal{H}^2_{\check{\mathcal{M}}} = 4\pi n_0 \ln\kappa + 2n_0\mathbf{c}_0 - 2\pi \sum_{i\neq j\in J_0} \ln d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa)\\ -2\pi \sum_{i\neq j\in J_\pi} \ln d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa) + H(J_0\cup J_\pi)\\ -2\pi n_0^2 G\left((0,0,v(0)),(0,0,v(\pi))\right) + o(1),\end{aligned} \tag{5.66}$$

where $\mathbf{c}_0$ is the constant from (4.46). To see this, fix $r = r(\kappa)$, the radius obtained in Proposition 4.5 and note that we can appeal to it thanks to inequality (5.61). One can see

$$\sum_{i=1}^{2n_0} \int_{\mathcal{B}_{i,r}} S_\kappa(\psi_\kappa) d\mathcal{H}^2_{\check{\mathcal{M}}} \geqq 4\pi n_0 \ln(\kappa\cdot r) + 2n_0\mathbf{c}_0 + o(1). \tag{5.67}$$

We then resort to Remarks 4.9 and 5.3, to refine (4.21) by replacing the $\mathcal{O}(1)$ term by an $o(1)$ term. Then, a consequence of this is:

$$\begin{aligned}\int_{\check{\mathcal{M}}\setminus\cup_{i=1}^{2n_0}\mathcal{B}_{i,r}} S_\kappa(\psi_\kappa)\, \mathrm{d}\mathcal{H}^2_{\check{\mathcal{M}}} \geqq 4\pi n_0 \ln\frac{1}{r} - 2\pi \sum_{i\neq j\in J_0} \ln d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa)\\ -2\pi \sum_{i\neq j\in J_\pi} \ln d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa) + H(J_0\cup J_\pi)\\ -2\pi n_0^2 G\left((0,0,v(0)),(0,0,v(\pi))\right)\\ +o(1).\end{aligned} \tag{5.68}$$

This follows from revisiting (4.31)–(4.36). To complete the proof of the claim (5.66), we construct a comparison function $\tilde{\Psi}_\kappa$ as follows. First, let $q_i := p_i^\kappa$ for all $i = 1, \ldots, 2n_0$, in Proposition 4.4. Again, making use of the notation and results contained in Proposition 4.5, we define

$$\tilde{\Psi}_\kappa(x) = \begin{cases} \psi^r_{out}(x), & \text{if } x\in\check{\mathcal{M}}\setminus\cup_{i=1}^{2n_0}\mathcal{B}_{i,r},\\ \psi^{\kappa,+}_{\mathbf{in},i}(x), & \text{if } \in\mathcal{B}_{i,r} \text{ and } d_i^\kappa = 1,\\ \psi^{\kappa,-}_{\mathbf{in},i}(x), & \text{if } \in\mathcal{B}_{i,r} \text{ and } d_i^\kappa = -1.\end{cases} \tag{5.69}$$

We see that (5.55) and (5.56) hold for this new $\tilde{\Psi}_\kappa$. This and (5.57) yield

$$\int_{\check{\mathcal{M}}} S_\kappa(\psi_\kappa)\,\mathrm{d}\mathcal{H}^2_{\check{\mathcal{M}}} \leqq 4\pi n_0 \ln\kappa + 2n_0\mathbf{c}_0 - 2\pi \sum_{i\neq j\in J_0} \ln d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa) \\ -2\pi \sum_{i\neq j\in J_\pi} \ln d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa) + H(J_0\cup J_\pi) \\ -2\pi n_0^2 G\left((0,0,v(0)),(0,0,v(\pi))\right) + o(1), \tag{5.70}$$

The claim now follows from this, (5.67) and (5.68). Using (5.66) we can now assert that

$$\mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi_\kappa) = 4\pi n_0 \ln\kappa + 2n_0\mathbf{c}_0 - 2\pi \sum_{i\neq j\in J_0} \ln d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa) \\ -2\pi \sum_{i\neq j\in J_\pi} \ln d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa) + 4\pi h(\kappa)\sum_{i=1}^{2n_0} *F(p_i^\kappa)d_i^\kappa \\ +H(J_0\cup J_\pi) - 2\pi n_0^2 G\left((0,0,v(0)),(0,0,v(\pi))\right) \\ +\left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} + o(1). \tag{5.71}$$

We now proceed to determine the number of vortices. More precisely, we study the dependence $n_0 = n_0(\sigma)$. Let $\phi_j = \phi(p_j)$, where σ arises in (5.36). The definition (5.10) together with (5.5) imply that

$$*\,F(p_j) = \frac{1}{4}[u\gamma]'(0)\phi_j^2 + \mathcal{O}\left(\phi_j^3\right), \tag{5.72}$$

whereas for $j \in J_\pi$

$$*\,F(p_j) = \frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi} + \frac{1}{4}[u\gamma]'(\pi)(\phi_j-\pi)^2 + \mathcal{O}\left((\phi_j-\pi)^3\right). \tag{5.73}$$

From (5.2), (5.3) and (5.5) it follows that

$$[u\gamma]'(0) = u'(0)\,\gamma(0) = u'(0)^2 = v(0)^2, \\ [u\gamma]'(\pi) = u'(\pi)\,\gamma(\pi) = -u'(\pi)^2 = -v(\pi)^2.$$

Thus, the constraints (5.62) and (5.63) applied to (4.29) justify the expansion

$$h(\kappa)\Lambda(\mathbf{A}^e,\psi_\kappa) = 4\pi\, h(\kappa)\left[n_0\frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi} - \frac{1}{4}\sum_{j\in J_0} v(0)^2\phi_j^2 \right. \\ \left. - \frac{1}{4}\sum_{j\in J_\pi} v(\pi)^2(\phi_j-\pi)^2\right] + \mathcal{O}\left(\frac{h(\kappa)}{(\ln\kappa)^{\frac{5}{4}}}\right). \tag{5.74}$$

We next see that $H(J_0 \cup J_\pi)$ takes the simple form

$$H(J_0 \cup J_\pi) = \left(n_0^2 + n_0\right)[H\ ((0,0,v(0)),(0,0,v(0)))\\ + H\ ((0,0,v(\pi)),(0,0,v(\pi)))],$$

so $f(n_0) := H(J_0 \cup J_\pi) - 2\pi n_0^2\, G\,((0,0,v(0)),(0,0,v(\pi))) + 2n_0\mathbf{c}_0$ is a quantity that depends on n_0 but not on the configuration of vortices. We can now write using (5.71) and (5.74)

$$\mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi_\kappa) = I_0(n_0) + I_\pi(n_0) - n_0\sigma\mathcal{H}^2(\check{\mathcal{M}})\ln\ln\kappa\\ + f(n_0) + \left(\frac{h(\kappa)}{2}\right)^2 \|u\|_{L^2(\check{\mathcal{M}})} + o(1), \tag{5.75}$$

where we have introduced

$$I_0(n_0) := 2\pi\left[-\sum_{i,j\in J_0}\ln d_{\check{\mathcal{M}}}(p_i,p_j) + \frac{h(\kappa)}{2}\sum_{j\in J_0} v(0)^2\,\phi_j^2\right]$$

and

$$I_\pi(n_0) := 2\pi\left[-\sum_{i,j\in J_\pi}\ln d_{\check{\mathcal{M}}}(p_i,p_j) + \frac{h(\kappa)}{2}\sum_{j\in J_\pi} v(\pi)^2\,(\phi_j-\pi)^2\right].$$

We write $\phi_{j_0} := \max_{j\in J_0}\phi_j$. One has

$$\frac{h(\kappa)}{2}\sum_{j\in J_0} v(0)^2\,\phi_j^2 \geqq \frac{h(\kappa)}{2} v(0)^2\,\phi_{j_0}^2 \tag{5.76}$$

and

$$-\sum_{i,j\in J_0}\ln d_{\check{\mathcal{M}}}(p_i,p_j) \geqq -(n_0^2-n_0)\ln\phi_{j_0} + \mathcal{O}(1), \tag{5.77}$$

since $d_{\check{\mathcal{M}}}(p_i,p_j) \leqq C\cdot\phi_{j_0}$ for some constant C independent of κ, in light of (5.1)–(5.5). Using (5.76) and (5.77), we obtain a lower bound for $I_0(n_0)$ by minimizing $-(n_0^2-n_0)\ln x + \frac{h(\kappa)}{2}v(0)^2x^2$. The lower bound reads

$$I_0(n_0) \geqq \pi(n_0^2-n_0)\ln\ln\kappa + \mathcal{O}(1).$$

One can easily see the same estimate holds for $I_\pi(n_0)$. Employing a comparison map similar to the one defined in (5.69), only this time with n_0 vortices of degree $+1$ equally distributed on the circle $\{\phi = \left(\frac{n_0^2-n_0}{h(\kappa)\,v(0)^2}\right)^{\frac{1}{2}}\}$, and another n_0 of degree -1 on the circle $\{\phi = \pi - \left(\frac{n_0^2-n_0}{h(\kappa)\,v(0)^2}\right)^{\frac{1}{2}}\}$, one can deduce

$$I_0(n_0) = \pi(n_0^2-n_0)\ln\ln\kappa + \mathcal{O}(1), \tag{5.78}$$

and that the same estimate holds for $I_\pi(n_0)$. From this, one can already determine the value of $n_0 = n_0(\sigma)$. The energy of ψ_κ depends on n_0 in the following way

$$\mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi_\kappa) = 2\pi(n_0^2 - n_0)\ln\ln\kappa - n_0\sigma\mathcal{H}^2(\check{\mathcal{M}})\ln\ln\kappa + f(n_0) + \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} + \mathcal{O}(1), \tag{5.79}$$

so for κ large, n_0 will be given by the minimum value of $n \in \mathbb{N}$ of

$$2\pi(n^2 - n) - n\sigma\mathcal{H}^2(\check{\mathcal{M}}).$$

Simple analysis then reveals that if $\sigma \notin \left(4\pi/\mathcal{H}^2(\check{\mathcal{M}})\right)\mathbb{Z}$,

$$n_0 = \left\lfloor \sigma\frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi} \right\rfloor + 1. \tag{5.80}$$

The case $n_0 = 0$ can be incorporated into the argument without considerable additional effort and (5.80) reads the same in all cases.

4. **We prove now that the configuration of vortices tends to minimize a renormalized energy**.

Since we cannot really rescale points in the manifold, and this is necessary for the identification of a renormalized energy, we proceed as follows. By our analyticity assumptions, we have:

$$\left|p_i^\kappa - p_j^\kappa\right|_{\mathbb{R}^3} \leqq d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa) \leqq \left|p_i^\kappa - p_j^\kappa\right|_{\mathbb{R}^3} + C\cdot\left|p_i^\kappa - p_j^\kappa\right|^2_{\mathbb{R}^3}. \tag{5.81}$$

Recall the definition of **Proj** in (5.35). Since $v'(0) = 0$, we have

$$\left|p_i^\kappa - p_j^\kappa\right|^2_{\mathbb{R}^3} = \left|\mathbf{Proj}\, p_i^\kappa - \mathbf{Proj}\, p_j^\kappa\right|^2_{\mathbb{R}^3} + v''(0)^2\left(\phi(p_i^\kappa)^2 - \phi(p_j^\kappa)^2\right)^2 + o\left(\left(\phi(p_i^\kappa)^2 - \phi(p_j^\kappa)^2\right)^2\right), \quad \text{for } i, j \in J_0. \tag{5.82}$$

One can also see, resorting to (5.5), that for some constant l_0

$$\begin{aligned}\left|\mathbf{Proj}\, p_i^\kappa - \mathbf{Proj}\, p_j^\kappa\right|^2_{\mathbb{R}^3} &= u\left(\phi\left(p_i^\kappa\right)\right)^2 + u\left(\phi\left(p_j^\kappa\right)\right)^2 \\ &\quad -2u\left(\phi\left(p_i^\kappa\right)\right)u\left(\phi\left(p_j^\kappa\right)\right)\cos\left(\theta\left(p_i^\kappa\right) - \theta\left(p_j^\kappa\right)\right) \\ &\geqq l_0\left(\phi(p_i^\kappa) - \phi(p_j^\kappa)\right)^2 \end{aligned} \tag{5.83}$$

Similar estimates holds for $i, j \in J_\pi$. So (5.81), (5.82) and (5.83) imply that whenever i, j belong both to either J_0 or J_π, then

$$\ln d_{\check{\mathcal{M}}}(p_i^\kappa, p_j^\kappa) = \ln\left|\mathbf{Proj}\, p_i^\kappa - \mathbf{Proj}\, p_j^\kappa\right|_{\mathbb{R}^3} + o(1).$$

For $i \in J_0$, (resp. for $i \in J_\pi$) $\left|\mathbf{Proj}\, p_i^\kappa\right|^2 = \left|p_i^\kappa - (0, 0, v(\phi_i^\kappa))\right|^2 = v(0)^2\left|\phi_i^\kappa\right|^2 + \mathcal{O}((\phi_i^\kappa)^3)$ (resp. $\left|\mathbf{Proj}\, p_i^\kappa\right|^2 = v(\pi)^2\left|\phi_i^\kappa - \pi\right|^2 + \mathcal{O}((\phi_i^\kappa - \pi)^3)$). From (5.78) we see that $\left|\phi_j^\kappa - \pi\right|, \left|\phi_i^\kappa\right| \leqq \frac{\mathbf{c}}{\sqrt{\ln\kappa}}$, where $i \in J_0$, $j \in J_\pi$. Indeed, appealing to inequalities (5.76) and (5.77), the definition of $I_0(n_0)$, estimate (5.78), and writing $C_\kappa := \phi_{j_0}\sqrt{\ln\kappa}$, we deduce

$$(C_\kappa)^2 \lesssim \ln C_\kappa + \mathcal{O}(1). \tag{5.84}$$

This implies C_κ is bounded independent of κ. We can proceed analogously to prove something similar holds for the vortices $\{p_j^\kappa\}_{j\in J_\pi}$. This concludes the proof of the claim. Using this and substituting the definition of P_i^κ given in Theorem 5.1 and the one for the renormalized energy $\mathbf{R}^{n_0}$ given in (5.34), we finally arrive at

$$\begin{aligned}\mathcal{G}_{\check{\mathcal{M}}}(\psi_\kappa) = 2\pi\left(n_0^2 - n_0\right)\ln\ln\kappa - n_0\sigma\mathcal{H}^2(\check{\mathcal{M}})\ln\ln\kappa + f(n_0)\\ +\mathbf{R}^{n_0}\left(P_i^\kappa;\ i\in J_0\right) + \mathbf{R}^{n_0}\left(P_i^\kappa;\ i\in J_\pi\right)\\ +\left(\frac{h(\kappa)}{2}\right)^2\|u\|^2_{L^2(\check{\mathcal{M}})} + o(1).\end{aligned} \tag{5.85}$$

We claim that (5.85) forces the configurations $\{P_j^\kappa,\ j\in J_0\}$ and $\{P_j^\kappa,\ j\in J_\pi\}$ to converge at the same time to minimizers of $\mathbf{R}^{n_0}$. To prove this, suppose towards a contradiction that at least one of these collections does not. Without any loss of generality, we assume this happens for the collection corresponding to $j\in J_0$. Then, there exists a collection $\{x_j,\ j\in J_0\}$ of points in $\mathbb{R}^2$, such that perhaps after passing to a subsequence, $\mathbf{R}^{n_0}(\{P_j^\kappa,\ j\in J_0\}) > \mathbf{R}^{n_0}(\{x_j,\ j\in J_0\}) + \varepsilon_0$, where $\varepsilon_0 > 0$ is a small constant independent of κ. Then, define a map $\tilde{\Psi}_\kappa$ analogously to the one in (5.69), with vortices $\tilde{p}_j^\kappa = \frac{x_j}{\sqrt{\ln\kappa}} + v(\phi_j)\hat{e}_z$, where

$$\phi_j = u^{-1}\left(\frac{|x_j|_{\mathbb{R}_2}}{\sqrt{\ln\kappa}}\right),$$

for $j\in J_0$, while for $j\in J_\pi$ we simply let $\tilde{p}_j^\kappa = p_j$. With the aid of (5.85), one deduces

$$\begin{aligned}0 &\geqq \mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi_\kappa) - \mathcal{G}_{\check{\mathcal{M}},\kappa}(\tilde{\Psi}_\kappa)\\ &= R(\{p_j;\ j\in J_0\}) - R(\{x_j;\ j\in J_0\}) + o(1)\\ &\geqq \varepsilon_0 + o(1),\end{aligned} \tag{5.86}$$

which is a contradiction. □

Proof of Theorem 5.2. Fix κ_0 large enough so that for all $\kappa > \kappa_0$ all minimizers of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ have $2n_0 = 2\lfloor\sigma\frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi}\rfloor + 2$ nonsingular vortices. This is possible thanks to the preceding theorem and the analysis presented immediately below (5.65). By Proposition 2.1 we know that, given any sequence of minimizers Ψ_κ of $G_{\varepsilon,\kappa}$, the corresponding maps $\psi_{\varepsilon,\kappa}$ converge in $C^{1,\alpha}$, up to a subsequence, to a minimizer ψ_κ of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$. Lastly, the fact that the zeros of ψ_κ are nonsingular and the implicit function theorem yield the desired conclusion. □

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