

On the L_p -Solvability of Higher Order Parabolic and Elliptic Systems with BMO Coefficients

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Abstract

We prove the solvability in Sobolev spaces for both divergence and non-divergence form higher order parabolic and elliptic systems in the whole space, on a half space, and on a bounded domain. The leading coefficients are assumed to be merely measurable only in the time variable and have small mean oscillations with respect to the spatial variables in small balls or cylinders. For the proof, we develop a set of new techniques to produce mean oscillation estimates for systems on a half space.

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1. Introduction

This paper is devoted to the study of the L_p -theory of higher order parabolic and elliptic systems. More precisely, we expand the L_p -theory of higher order elliptic and parabolic systems to include a class of *not necessarily continuous coefficients*

via a unified approach for both divergence type and non-divergence type systems in the whole space, on a half space, and on a bounded domain. The coefficients we consider are complex valued and, in particular, the leading coefficients of parabolic systems are merely measurable in the time variable and belong to the class of BMO (bounded mean oscillations) as functions of the spatial variables. The mean oscillations of the coefficients need only be sufficiently small over small cylinders.

To present the exact forms of such systems, we let

$$Lu = \sum_{|\alpha| \leq m, |\beta| \leq m} A^{\alpha\beta} D^\alpha D^\beta u, \quad \mathcal{L}u = \sum_{|\alpha| \leq m, |\beta| \leq m} D^\alpha (A^{\alpha\beta} D^\beta u),$$

where m is a positive integer,

$$D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d}, \quad \alpha = (\alpha_1, \dots, \alpha_d),$$

and, for each α, β , $A^{\alpha\beta} = [A_{ij}^{\alpha\beta}(t, x)]_{i,j=1}^n$ is an $n \times n$ complex matrix-valued function. The involved functions are complex vector-valued functions, that is,

$$u = (u^1, \dots, u^n)^{\text{tr}}, \quad f = (f^1, \dots, f^n)^{\text{tr}}, \quad f_\alpha = (f_\alpha^1, \dots, f_\alpha^n)^{\text{tr}}.$$

The parabolic systems we study are

$$u_t + (-1)^m Lu = f, \quad u_t + (-1)^m \mathcal{L}u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha,$$

where the first expression is in non-divergence form and the second one is in divergence form. The elliptic systems, non-divergence form and divergence form, respectively, are

$$Lu = f, \quad \mathcal{L}u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha.$$

Whenever elliptic systems are considered, all the involved functions are independent of t . When the domain is other than the whole space, we impose the homogeneous Dirichlet boundary condition.

In the case of non-divergence type elliptic systems, we prove that, for a given $f \in L_p(\Omega)$, there is a unique solution $u \in W_p^{2m}(\Omega)$ to the system $Lu = f$ in Ω , where Ω is either the whole space \mathbb{R}^d , the half space $\mathbb{R}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d, x_1 > 0\}$, or a bounded domain. We also prove the corresponding results for the other types of elliptic and parabolic systems; see Sect. 2.

As is well known, the key ingredient in establishing L_p -theory is a priori L_p -estimates of solutions to given systems. Generally, this is done in two steps. First, one establishes L_p -estimates for systems with ‘simple’ coefficients, for example, constant coefficients. Second, if the given system is in some sense close to systems with simple coefficients, one obtains the desired L_p -estimates by using a perturbation argument.

The L_p -estimates for systems with constant coefficients, in many references, for example [2, 3], rely on the exact representation of solutions and the Calderón–Zygmund theorem. Another approach for such L_p -estimates is that of Campanato–Stampacchia using Stampacchia’s interpolation theorem (see [20]). As to

perturbation arguments, if the coefficients of given systems are uniformly continuous, the estimates are carried out by using the local closeness of the coefficients to constant coefficients in L_∞ norm. When the class of VMO (vanishing mean oscillations) coefficients was first introduced, another perturbation argument was used in [5, 8, 9], where the continuity of coefficients is measured in the average sense, not in the pointwise sense, through a representation formula of solutions and the Coifman–Rochberg–Weiss commutator theorem.

In this paper, in establishing the key L_p -estimates, we replace the first step, L_p -estimates of solutions to systems with simple coefficients, by *mean oscillation estimates* of solutions to the systems. Then for the second step we use a different perturbation argument, which is well suited to the mean oscillation estimates. For instance, if the system under consideration is elliptic in the form of $Lu = f$ with constant coefficients in the whole space, then by the mean oscillation estimate of $D^{2m}u$ we mean a pointwise estimate of the form

$$\begin{aligned} & \int_{B_r(x_0)} |D^{2m}u - \int_{B_r(x_0)} D^{2m}u \, dy| \, dx \\ & \leq N\kappa^{-1} \left(\int_{B_{\kappa r}(x_0)} |D^{2m}u|^2 \, dx \right)^{1/2} + N\kappa^{\frac{d}{2}} \left(\int_{B_{\kappa r}(x_0)} |f|^2 \, dx \right)^{1/2} \quad (1) \end{aligned}$$

for all $x_0 \in \mathbb{R}^d$, $r \in (0, \infty)$, and $\kappa \in [\kappa_0, \infty)$, where $B_r(x_0)$ is a ball with center x_0 and radius r . Indeed, this implies the L_p -estimate of $D^{2m}u$ by the well known Fefferman–Stein theorem on sharp functions and the Hardy–Littlewood maximal function theorem. But more importantly, this type of estimate embraces well the perturbation between the original systems and systems with simple coefficients when the coefficients have small mean oscillations over small balls or small parabolic cylinders. This approach was first introduced by KRYLOV [25, 26] to deal with second order elliptic and parabolic equations with VMO coefficients in the whole space, and is well explained in his book [27].

Due to the well-adaptedness of estimates like (1) to the perturbation argument, our main effort in this paper focuses on obtaining mean oscillation estimates of systems with simple coefficients. Since in the parabolic case we allow coefficients to be merely measurable in the time direction, the systems with simple coefficients in our case are naturally those with *measurable coefficients depending only on t* .

For systems in the whole space, which requires only interior estimates, the mean oscillation estimates follow rather easily by adapting the techniques in [25, 27] to higher order systems. However, unlike the arguments in [25], we derive the non-divergence case as a corollary from the divergence case. Another noteworthy difference is that we prove the mean oscillation estimates not only for the highest order terms but also for the lowest order terms, so we are able to avoid the argument in [25] deriving the L_p -estimates of solutions from those of the highest order terms, which is technically difficult in the case of higher order equations.

For systems on a half space or on a bounded domain, which also require boundary estimates, it is not possible to use the approach in [25, 27] since the estimates developed there are only for equations *in the whole space* (interior estimates). Thus, we develop here a set of new techniques to produce mean oscillation

estimates for systems *on a half space*. This is a new approach for boundary L_p -estimates which is applicable to a wide class of equations or systems. To get these boundary mean oscillation estimates, as in the whole space case, we start with L_2 -estimates of systems on a half space. Although the L_2 -estimate for divergence type systems is well known under appropriate ellipticity or parabolicity conditions on the leading coefficients, our Theorem 13 regarding the L_2 -estimate for non-divergence type systems on a half space with coefficients measurable in time is, to the best of our knowledge, a new result. In the proof we only use the L_2 -estimate of divergence type systems and an interpolation argument. It is worth noting that L_2 -estimates for higher order elliptic equations and systems were obtained in [16, 19] by using bootstrap arguments. For parabolic equations, however, in [19] the coefficients are assumed to be Hölder continuous in the time variable since a semigroup method was used.

From the L_2 -estimates, we derive the boundary mean oscillation estimates of *some* of the highest order derivatives of solutions, precisely, $D_{x'}^m u$ in the case of divergence systems and $D_{x'}^{2m} u$ in the case of non-divergence systems, where x' denotes the last $d-1$ coordinates of $x = (x_1, x')$ in \mathbb{R}^d . These estimates alone, however, are not sufficient for us to prove the main theorems. Because of this, we then consider a parabolic system with special coefficients, such that in a periodic pattern certain order normal derivatives of solutions to the system vanish on the boundary. This gives us the boundary mean oscillation estimates of $D_1^m u$ or $D_1^{2m} u$; see Lemma 14. Once we have all the required mean oscillation estimates, we proceed as in [25] to the desired L_p -estimates using the perturbation argument, the details of which are illustrated for divergence type systems in the whole space; see Section 5.

In the literature, for uniformly continuous coefficients, a rather complete L_p -theory can be found for general linear elliptic systems in [1–3] and for parabolic systems in [18, 17, 36, 29]. If coefficients are in the class of VMO, non-divergence type higher order systems in the whole space have been investigated in several papers, for example [10, 22, 34, 35], where leading coefficients of systems are either VMO with respect to all the variables or independent of the time variable. For divergence type higher order elliptic systems with VMO coefficients, we refer the reader to a recent interesting preprint [31] in which the inhomogeneous Dirichlet problem on Lipschitz domains is studied. In all these papers, the method of singular integrals is used, so measurable coefficients are not allowed.

Restricted to *second order* systems or equations, there are a relatively larger number of papers which can be compared to this paper. Non-divergence elliptic and parabolic equations on smooth domains with VMO coefficients were first studied in [5, 8, 9] by using the technique of singular integrals. For further related results, we refer the reader to the book [32] and references therein. The corresponding results for divergence elliptic equations were obtained in [4, 11] by a similar technique. These results were later improved upon by the authors of [6] in several papers for divergence type equations/systems without lower order terms on non-smooth domains by using a perturbation argument based on the maximal function theorem and a covering lemma (see [7] for an extension to fourth order systems). An interesting question would be whether the methods in [6, 7] can be applied to equations with lower order terms or non-divergence form equations/systems.

The methodology used by Krylov in [25,26] was later developed and extended in [14] for divergence and non-divergence systems in the whole space with the same class of coefficients, and in [23,24,28] for non-divergence parabolic and elliptic equations in the whole space with partially BMO coefficients for $p > 2$, and in [12] for any $p \in (1, \infty)$. In [12,13,15], this method was further adapted to divergence parabolic and elliptic equations/systems in the whole space with partially BMO coefficients. It is worth noting that in [12–15] and [23–28] only interior mean oscillation estimates were derived. When dealing with equations and systems on a half space or on a bounded domain in [13,15,23,24], the authors took full advantage of the fact that the coefficients are allowed to be merely measurable in one spatial direction and the fact that the given systems are second order. Thus without using any *boundary mean oscillation estimates* developed here, the boundary L_p -estimates were derived from interior estimates as corollary type results by using odd and even extension techniques. However, the extension techniques do not work for higher order equations or systems. This is the first paper in which the ideas in [25,26] are adapted to boundary estimates, in both divergence and non-divergence cases.

As noted above, the first critical step of the proof is the L_2 -estimates of systems with relatively simple coefficients under the ellipticity or parabolicity conditions on the leading coefficients. In this paper, we use the Legendre–Hadamard ellipticity condition, which is more general than the strong ellipticity condition considered, for example, in [7,15,30]. Nevertheless, it is still stronger than the uniform parabolicity condition in the sense of Petrovskii, which was used in [17,34,36] with more regularity assumptions on the leading coefficients. We shall discuss these conditions in detail in Section 11.

The organization of the paper is as follows. We introduce some notation and state the main results in the next section. The remainder of the paper is divided into two parts. In the first part, we treat systems in the whole space. Sections 3 and 4 are devoted to the L_2 -estimates and mean oscillation estimates for both divergence and non-divergence parabolic systems with simple coefficients. In Section 5 we complete the proofs of the L_p -solvability of systems in the whole space. The second part is the main part of the paper, in which we treat systems on a half space or on a bounded domain. In Section 6 we establish the L_2 -solvability of divergence and non-divergence parabolic systems with simple coefficients on a half space. Then in Section 7, we obtain the boundary mean oscillation estimates of $D_{x'}^m u$ and $D_{x'}^{2m} u$ for divergence and non-divergence systems, respectively. Section 8 is devoted to the estimates for a special type of systems. With these preparations, in Sections 9 and 10 we establish the L_p -solvability of both divergence and non-divergence parabolic systems on a half space and on a bounded domain. Finally, we discuss in Section 11 some other ellipticity conditions used in the literature, and show how our results can be extended to systems under those conditions.

2. Main results

We first introduce some notation used throughout the paper. A point in \mathbb{R}^d is denoted by $x = (x_1, \dots, x_d)$. Whenever needed, we denote x by (x_1, x') where

$x' \in \mathbb{R}^{d-1}$. A point in

$$\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$$

is denoted by $X = (t, x)$. For $T \in (-\infty, \infty]$, set

$$\mathcal{O}_T = (-\infty, T) \times \mathbb{R}^d, \quad \mathcal{O}_T^+ = (-\infty, T) \times \mathbb{R}_+^d,$$

where $\mathbb{R}_+^d = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > 0\}$. Especially, if $T = \infty$, we have, for example, $\mathcal{O}_\infty^+ = \mathbb{R} \times \mathbb{R}_+^d$. We also have

$$\begin{aligned} B_r(x) &= \{y \in \mathbb{R}^d : |x - y| < r\}, & B'_r(x') &= \{y' \in \mathbb{R}^{d-1} : |x' - y'| < r\}, \\ Q_r(t, x) &= (t - r^{2m}, t) \times B_r(x), & Q'_r(t, x') &= (t - r^{2m}, t) \times B'_r(x'), \\ Q_r^+(t, x) &= Q_r(t, x) \cap \mathcal{O}_\infty^+. \end{aligned}$$

We denote

$$\langle f, g \rangle_\Omega = \int_\Omega f^{\text{tr}} \bar{g} = \sum_{j=1}^n \int_\Omega f^j \bar{g}^j.$$

For a function f on $\mathcal{D} \subset \mathbb{R}^{d+1}$, we set

$$(f)_\mathcal{D} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} f(t, x) \, dx \, dt = \int_{\mathcal{D}} f(t, x) \, dx \, dt,$$

where $|\mathcal{D}|$ is the $d + 1$ -dimensional Lebesgue measure of \mathcal{D} .

In order to state and prove our results on systems in Sobolev spaces, in addition to the well known spaces L_p and W_p^k , we introduce the following function spaces. As a solution space for non-divergence type parabolic equations, we use

$$W_p^{1,2m}((S, T) \times \Omega) = \{u : u_t, D^\alpha u \in L_p((S, T) \times \Omega), 0 \leq |\alpha| \leq 2m\}$$

equipped with its natural norm. Unless otherwise specified, in this paper $D^\alpha u(t, x)$ means the spatial derivative of u . For divergence type parabolic equations with $\Omega = \mathbb{R}^d$, we introduce

$$\mathcal{H}_p^m((S, T) \times \mathbb{R}^d) = (1 - \Delta)^{\frac{m}{2}} W_p^{1,2m}((S, T) \times \mathbb{R}^d)$$

equipped with the norm

$$\|u\|_{\mathcal{H}_p^m((S,T) \times \mathbb{R}^d)} = \|(1 - \Delta)^{-\frac{m}{2}} u\|_{W_p^{1,2m}((S,T) \times \mathbb{R}^d)}.$$

Note that if we set

$$\begin{aligned} \mathbb{H}_p^{-m}((S, T) \times \mathbb{R}^d) &= (1 - \Delta)^{\frac{m}{2}} L_p((S, T) \times \mathbb{R}^d), \\ \|f\|_{\mathbb{H}_p^{-m}((S,T) \times \mathbb{R}^d)} &= \|(1 - \Delta)^{-\frac{m}{2}} f\|_{L_p((S,T) \times \mathbb{R}^d)}, \end{aligned}$$

then

$$\|u\|_{\mathcal{H}_p^m((S,T) \times \mathbb{R}^d)} \cong \|u_t\|_{\mathbb{H}_p^{-m}((S,T) \times \mathbb{R}^d)} + \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_p((S,T) \times \mathbb{R}^d)}.$$

For a general Ω , we set

$$\mathbb{H}_p^{-m}((S, T) \times \Omega) = \left\{ f : f = \sum_{|\alpha| \leq m} D^\alpha f_\alpha, \quad f_\alpha \in L_p((S, T) \times \Omega) \right\},$$

$$\|f\|_{\mathbb{H}_p^{-m}((S, T) \times \Omega)} = \inf \left\{ \sum_{|\alpha| \leq m} \|f_\alpha\|_{L_p((S, T) \times \Omega)} : f = \sum_{|\alpha| \leq m} D^\alpha f_\alpha \right\},$$

and

$$\mathcal{H}_p^m((S, T) \times \Omega) = \{u : u_t \in \mathbb{H}_p^{-m}((S, T) \times \Omega), D^\alpha u \in L_p((S, T) \times \Omega), 0 \leq |\alpha| \leq m\},$$

$$\|u\|_{\mathcal{H}_p^m((S, T) \times \Omega)} = \|u_t\|_{\mathbb{H}_p^{-m}((S, T) \times \Omega)} + \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_p((S, T) \times \Omega)}.$$

Let $\delta, K > 0$ be two constants. Throughout the paper, we assume that all the coefficients are measurable, complex valued and bounded,

$$|A^{\alpha\beta}| \leq \begin{cases} \delta^{-1}, & |\alpha| = |\beta| = m, \\ K, & \text{otherwise.} \end{cases}$$

In addition, we impose the Legendre–Hadamard ellipticity condition on the leading coefficients (see, for instance, [18, 20]). Here we call $A^{\alpha\beta}$ the leading coefficients if $|\alpha| = |\beta| = m$. All the other coefficients are called lower-order coefficients. By the Legendre–Hadamard ellipticity we mean

$$\Re \left(\sum_{|\alpha|=|\beta|=m} \theta^{\text{tr}} \xi^\alpha \xi^\beta A^{\alpha\beta}(t, x) \bar{\theta} \right) \geq \delta |\xi|^{2m} |\theta|^2 \tag{2}$$

for all $(t, x) \in \mathbb{R}^{d+1}$, $\xi \in \mathbb{R}^d$, and $\theta \in \mathbb{C}^n$. Here we use $\Re(f)$ to denote the real part of f .

Now we state our regularity assumption on the leading coefficients. Let

$$\text{osc}_x(A^{\alpha\beta}, Q_r(t, x)) = \int_{t-r^{2m}}^t \int_{B_r(x)} \left| A^{\alpha\beta}(s, y) - \int_{B_r(x)} A^{\alpha\beta}(s, z) \, dz \right| \, dy \, ds.$$

Then we set

$$A_R^\# = \sup_{(t,x) \in \mathbb{R}^{d+1}} \sup_{r \leq R} \sup_{|\alpha|=|\beta|=m} \text{osc}_x(A^{\alpha\beta}, Q_r(t, x)).$$

We impose on the leading coefficients a small mean oscillation condition with a parameter $\rho > 0$, which will be specified later.

Assumption 1 (ρ). *There is a constant $R_0 \in (0, 1]$ such that $A_{R_0}^\# \leq \rho$.*

Contrary to non-divergence type systems where equations are defined almost everywhere, solutions to divergence type equations are understood in the weak sense. More precisely, for example, we say that $u \in \mathcal{H}_{p,\text{loc}}^m((S, T) \times \Omega)$, where $1 < p < \infty$, $\Omega \subset \mathbb{R}^d$, and $-\infty \leq S < T \leq \infty$, satisfies

$$u_t + (-1)^m \mathcal{L}u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha \quad \text{in } (S, T) \times \Omega,$$

provided that

$$\begin{aligned} & \int_S^t \int_\Omega \left(-\varphi_t \cdot u + (-1)^{m+|\alpha|} D^\alpha \varphi \cdot A^{\alpha\beta} D^\beta u \right) dx ds \\ &= (-1)^{|\alpha|} \int_S^t \int_\Omega D^\alpha \varphi \cdot f_\alpha dx ds + \int_\Omega u(S, x) \varphi(S, x) dx \\ & \quad - \int_\Omega u(t, x) \varphi(t, x) dx \end{aligned}$$

for every $t \in (S, T]$ and $\varphi = (\varphi^1, \dots, \varphi^n) \in C^\infty(\overline{(S, T) \times \Omega})$ such that $\varphi(t, \cdot) \in C_0^\infty(\Omega)$ for all $t \in [S, T]$. If $S = -\infty$ or $T = \infty$, we take $\varphi \in C^\infty(\overline{(S, T) \times \Omega})$ such that $\varphi(-\infty, \cdot) = 0$ or $\varphi(\infty, \cdot) = 0$, respectively.

We are now ready to present our main results.

Theorem 1 (Divergence parabolic systems in the whole space). *Let $p \in (1, \infty)$, $T \in (-\infty, \infty]$ and $f_\alpha \in L_p(\mathcal{O}_T)$ for $|\alpha| \leq m$. Then there exists a constant $\rho = \rho(d, m, n, p, \delta)$ such that, under Assumption 1 (ρ), the following hold true.*

(i) *For any $u \in \mathcal{H}_p^m(\mathcal{O}_T)$ satisfying*

$$u_t + (-1)^m \mathcal{L}u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha \quad \text{in } \mathcal{O}_T, \tag{3}$$

we have

$$\sum_{|\alpha| \leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\mathcal{O}_T)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p(\mathcal{O}_T)},$$

provided that $\lambda \geq \lambda_0$, where N and $\lambda_0 \geq 0$ depend only on d, m, n, p, δ, K and R_0 .

(ii) *For any $\lambda > \lambda_0$, there exists a unique $u \in \mathcal{H}_p^m(\mathcal{O}_T)$ satisfying (3).*

Theorem 2 (Non-divergence parabolic systems in the whole space). *Let $p \in (1, \infty)$, $T \in (-\infty, \infty]$ and $f \in L_p(\mathcal{O}_T)$. Then there exists a constant $\rho = \rho(d, m, n, p, \delta)$ such that, under Assumption 1 (ρ), the following hold true.*

(i) *For any $u \in W_p^{1,2m}(\mathcal{O}_T)$ satisfying*

$$u_t + (-1)^m Lu + \lambda u = f \quad \text{in } \mathcal{O}_T, \tag{4}$$

we have

$$\|u_t\|_{L_p(\mathcal{O}_T)} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\mathcal{O}_T)} \leq N \|f\|_{L_p(\mathcal{O}_T)},$$

provided that $\lambda \geq \lambda_0$, where N and $\lambda_0 \geq 0$ depend only on d, m, n, p, δ, K and R_0 .

(ii) For any $\lambda > \lambda_0$, there exists a unique $u \in W_p^{1,2m}(\mathcal{O}_T)$ satisfying (4).

Remark 1. We can also solve Cauchy problems for systems defined on $(0, T) \times \mathbb{R}^d$ in divergence or non-divergence form. If the initial condition is zero, this is done by extending the original system to a system defined on $(-\infty, T) \times \mathbb{R}^d$ with the right-hand side being zero for $t \in (-\infty, 0)$. In the same manner, we deal with Cauchy problems for the systems below defined on a half space or on a bounded domain. Note that in the case $T < \infty$, by considering $e^{-(\lambda_0+1)t}u$ instead of u we can take $\lambda = 0$ in the theorems above and below with the expense that N also depends on T .

The next two theorems are about the boundary value problem of systems in divergence and non-divergence form on a half space $\mathcal{O}_T^+ = (-\infty, T) \times \mathbb{R}_+^d$.

Theorem 3 (Divergence parabolic systems on a half space). *Let $p \in (1, \infty)$, $T \in (-\infty, \infty]$ and $f_\alpha \in L_p(\mathcal{O}_T^+)$ for $|\alpha| \leq m$. Then there exists a constant $\rho = \rho(d, m, n, p, \delta)$ such that, under Assumption 1 (ρ), the following hold true.*

(i) For any $u \in \mathcal{H}_p^m(\mathcal{O}_T^+)$ satisfying

$$\begin{cases} u_t + (-1)^m \mathcal{L}u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha & \text{in } \mathcal{O}_T^+; \\ u = D_1 u = \dots = D_1^{m-1} u = 0 & \text{on } \partial_p \mathcal{O}_T^+, \end{cases} \tag{5}$$

where $\partial_p \mathcal{O}_T^+ = (-\infty, T) \times \partial \mathbb{R}_+^d$, we have

$$\sum_{|\alpha| \leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\mathcal{O}_T^+)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p(\mathcal{O}_T^+)}, \tag{6}$$

provided that $\lambda \geq \lambda_0$, where N and $\lambda_0 \geq 0$ depend only on d, m, n, p, δ, K and R_0 .

(ii) For any $\lambda > \lambda_0$, there exists a unique $u \in \mathcal{H}_p^m(\mathcal{O}_T^+)$ satisfying (5).

Theorem 4 (Non-divergence parabolic systems on a half space). *Let $p \in (1, \infty)$, $T \in (-\infty, \infty]$ and $f \in L_p(\mathcal{O}_T^+)$. Then there exists a constant $\rho = \rho(d, m, n, p, \delta)$ such that, under Assumption 1 (ρ), the following hold true.*

(i) For any $u \in W_p^{1,2m}(\mathcal{O}_T^+)$ satisfying

$$\begin{cases} u_t + (-1)^m Lu + \lambda u = f & \text{in } \mathcal{O}_T^+; \\ u = D_1 u = \dots = D_1^{m-1} u = 0 & \text{on } \partial_p \mathcal{O}_T^+, \end{cases} \tag{7}$$

we have

$$\|u_t\|_{L_p(\mathcal{O}_T^+)} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\mathcal{O}_T^+)} \leq N \|f\|_{L_p(\mathcal{O}_T^+)},$$

provided that $\lambda \geq \lambda_0$, where N and $\lambda_0 \geq 0$ depend only on d, m, n, p, δ, K and R_0 .

(ii) For any $\lambda > \lambda_0$, there exists a unique $u \in W_p^{1,2m}(\mathcal{O}_T^+)$ satisfying (7).

Remark 2. By using a scaling argument, it is easy to see that we can choose λ_0 to be zero in the theorems above, provided that \mathcal{L} or L has no lower-order terms and the leading coefficients depend only on t .

Remark 3. In what has gone above we have presented results only for parabolic systems. From those results we easily obtain the corresponding results for higher order elliptic systems in divergence form and non-divergence form. The key idea is viewing solutions to elliptic systems as steady state solutions to the corresponding parabolic systems. We refer the reader to [25] and [14] for details. To show the exact form of results for elliptic systems, we state below the cases for elliptic systems on a bounded domain, Theorems 7 and 8.

Next we consider the solvability of systems in domains with the homogeneous Dirichlet boundary condition. For divergence systems, we assume the boundary $\partial\Omega$ of the domain Ω is locally the graph of a Lipschitz continuous function with a small Lipschitz constant. More precisely, we make the following assumption containing a parameter $\rho_1 \in (0, 1]$, which will be specified later.

Assumption 2 (ρ_1). *There is a constant $R_1 \in (0, 1]$ such that, for any $x_0 \in \partial\Omega$ and $r \in (0, R_1]$, there exists a Lipschitz function $\phi: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ such that*

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : x^1 > \phi(x')\}$$

and

$$\sup_{x', y' \in B'_r(x'_0), x' \neq y'} \frac{|\phi(y') - \phi(x')|}{|y' - x'|} \leq \rho_1$$

in some coordinate system.

Note that all C^1 domains satisfy this assumption for any $\rho_1 > 0$. Below we denote $\Omega_T = (-\infty, T) \times \Omega$, where $\Omega \subset \mathbb{R}^d$.

Theorem 5 (Divergence parabolic systems on a bounded domain). *Let $p \in (1, \infty)$, $T \in (-\infty, \infty]$. Then there exist constants $\rho = \rho(d, m, n, p, \delta)$, $\rho_1 = \rho_1(d, m, n, p, \delta, K, R_0)$ and $\lambda_0 = \lambda_0(d, m, n, p, \delta, K, R_0, R_1) > 0$, such that under Assumption 1 (ρ) and Assumption 2 (ρ_1) the following is true. For any $f_\alpha \in L_p(\Omega_T)$, $|\alpha| \leq m$, and $\lambda \geq \lambda_0$, there is a unique solution $u \in \mathcal{H}_p^m(\Omega_T)$ to*

$$\begin{cases} u_t + (-1)^m \mathcal{L}u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha & \text{in } \Omega_T; \\ u = |Du| = \dots = |D^{m-1}u| = 0 & \text{on } (-\infty, T) \times \partial\Omega, \end{cases}$$

and we have

$$\sum_{|\alpha| \leq m} \lambda^{1 - \frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\Omega_T)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p(\Omega_T)},$$

where N depends only on $d, m, n, p, \delta, K, R_0$ and R_1 .

Theorem 6 (Non-divergence parabolic systems on a bounded domain). *Let $p \in (1, \infty)$, $T \in (-\infty, \infty]$ and Ω be a $C^{2m-1,1}$ domain with the $C^{2m-1,1}$ norm bounded by K . Then there exist constants $\rho = \rho(d, m, n, p, \delta)$ and $\lambda_0 = \lambda_0(d, m, n, p, \delta, K, R_0) > 0$, such that under Assumption 1 (ρ) the following is true. For any $f \in L_p(\Omega_T)$ and $\lambda \geq \lambda_0$, there is a unique solution $u \in W_p^{1,2m}(\Omega_T)$ to*

$$\begin{cases} u_t + (-1)^m Lu + \lambda u = f & \text{in } \Omega_T; \\ u = |Du| = \dots = |D^{m-1}u| = 0 & \text{on } (-\infty, T) \times \partial\Omega, \end{cases} \quad (8)$$

and we have

$$\|u_t\|_{L_p(\Omega_T)} + \sum_{|\alpha| \leq 2m} \lambda^{1 - \frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\Omega_T)} \leq N \|f\|_{L_p(\Omega_T)},$$

where N depends only on d, m, n, p, δ, K and R_0 .

As discussed in Remark 3, the theorems above have elliptic analogies. For future reference, we state the results below for elliptic systems on a bounded domain.

Theorem 7 (Divergence elliptic systems on a bounded domain). *Let $p \in (1, \infty)$. Then there exist constants $\rho = \rho(d, m, n, p, \delta)$, $\rho_1 = \rho_1(d, m, n, p, \delta, K, R_0)$ and $\lambda_0 = \lambda_0(d, m, n, p, \delta, K, R_0, R_1) > 0$, such that under Assumption 1 (ρ) and Assumption 2 (ρ_1) the following is true. For any $f_\alpha \in L_p(\Omega)$, $|\alpha| \leq m$ and $\lambda \geq \lambda_0$, there is a unique solution $u \in W_p^m(\Omega)$ to*

$$\begin{cases} \mathcal{L}u + (-1)^m \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha & \text{in } \Omega; \\ u = |Du| = \dots = |D^{m-1}u| = 0 & \text{on } \partial\Omega, \end{cases}$$

and we have

$$\sum_{|\alpha| \leq m} \lambda^{1 - \frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\Omega)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p(\Omega)},$$

where N depends only on $d, m, n, p, \delta, K, R_0$ and R_1 .

Theorem 8 (Non-divergence elliptic systems on a bounded domain). *Let $p \in (1, \infty)$ and Ω be a $C^{2m-1,1}$ domain with the $C^{2m-1,1}$ norm bounded by K . Then there exist constants $\rho = \rho(d, m, n, p, \delta)$ and $\lambda_0 = \lambda_0(d, m, n, p, \delta, K, R_0) > 0$, such that under Assumption 1 (ρ) the following is true. For any $f \in L_p(\Omega)$ and $\lambda \geq \lambda_0$, there is a unique solution $u \in W_p^{2m}(\Omega)$ to*

$$\begin{cases} Lu + (-1)^m \lambda u = f & \text{in } \Omega; \\ u = |Du| = \dots = |D^{m-1}u| = 0 & \text{on } \partial\Omega, \end{cases}$$

and we have

$$\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\Omega)} \leq N \|f\|_{L_p(\Omega)},$$

where N depends only on d, m, n, p, δ, K and R_0 .

Part I. Systems in the whole space

This part of the paper is devoted to the proofs of the L_p -solvability of systems in the whole space, that is, Theorems 1 and 2. In Section 3 we obtain several L_2 -estimates for systems with coefficients depending only on t . By using these estimates, in Section 4 we prove the mean oscillation estimates for systems with the same class of coefficients. We complete the proofs of Theorems 1 and 2 in Section 5.

3. L_2 -Estimates for systems with simple coefficients in the whole space

In this section we obtain L_2 -estimates of parabolic systems in divergence and non-divergence form when the coefficient matrices are measurable functions of only the time variable satisfying the Legendre–Hadamard ellipticity condition (2). Even though our proofs are basic, we present them here for the sake of completeness. In particular, we derive the L_2 -estimate of systems in non-divergence form by using only that of divergence type systems. Throughout the section we set

$$\mathcal{L}_0 u = \sum_{|\alpha|=|\beta|=m} D^\alpha (A^{\alpha\beta} D^\beta u),$$

where $A^{\alpha\beta} = A^{\alpha\beta}(t)$. Since $A^{\alpha\beta}$ are independent of $x \in \mathbb{R}^d$, we can write

$$\mathcal{L}_0 u = \sum_{|\alpha|=|\beta|=m} A^{\alpha\beta} D^\alpha D^\beta u.$$

Let $C_0^\infty(\overline{\mathcal{O}_T})$ be the collection of infinitely differentiable functions defined on $\overline{\mathcal{O}_T}$ vanishing for large $|(t, x)|$.

Theorem 9. *Let $T \in (-\infty, \infty]$. There exists $N = N(d, n, m, \delta)$ such that, for any $\lambda \geq 0$,*

$$\sum_{|\alpha| \leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_2(\mathcal{O}_T)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_2(\mathcal{O}_T)}, \tag{9}$$

if $u \in \mathcal{H}_2^m(\mathcal{O}_T)$, $f_\alpha \in L_2(\mathcal{O}_T)$, $|\alpha| \leq m$, and

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha \tag{10}$$

in \mathcal{O}_T . Furthermore, for $\lambda > 0$ and $f_\alpha \in L_2(\mathcal{O}_T)$, $|\alpha| \leq m$, there exists a unique $u \in \mathcal{H}_2^m(\mathcal{O}_T)$ satisfying (10).

Proof. We assume $\lambda > 0$. If $\lambda = 0$, the inequality (9) holds trivially or we obtain

$$\sum_{|\alpha|=m} \|D^\alpha u\|_{L_2(\mathcal{O}_T)} \leq N \sum_{|\alpha|=m} \|f_\alpha\|_{L_2(\mathcal{O}_T)} \quad \text{if } f_\alpha = 0 \quad \text{for } |\alpha| < m$$

using the inequality (9) for $\lambda > 0$ and letting $\lambda \searrow 0$.

Let us assume that the inequality (9) is proved. Then due to the fact that

$$u_t = -(-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha (A^{\alpha\beta} D^\beta u) - \lambda u + \sum_{|\alpha| \leq m} D^\alpha f_\alpha,$$

we obtain $\|u\|_{\mathcal{H}_2^m(\mathcal{O}_T)} \leq N \|\mathcal{P}_\lambda u\|_{\mathbb{H}_2^{-m}(\mathcal{O}_T)}$, where $\mathcal{P}_\lambda u = u_t + (-1)^m \mathcal{L}_0 u + \lambda u$ and $N = N(d, n, m, \delta, \lambda)$. Then using the estimate, the method of continuity, and the unique solvability of systems with coefficients $A^{\alpha\beta} = \delta_{\alpha\beta} I_{n \times n}$ we prove the second assertion of the theorem. Therefore, we only need to prove the inequality (9). Moreover, since \mathcal{P}_λ is a bounded linear operator from $\mathcal{H}_2^m(\mathcal{O}_T)$ to $\mathbb{H}_2^{-m}(\mathcal{O}_T)$, it suffices to concentrate on $u \in C_0^\infty(\overline{\mathcal{O}_T})$.

Multiply both sides of (10) by u and integrate them on \mathcal{O}_T . Then by integration by parts we have

$$\langle u, u_t \rangle_{\mathcal{O}_T} + \langle D^\alpha u, A^{\alpha\beta} D^\beta u \rangle_{\mathcal{O}_T} + \lambda \langle u, u \rangle_{\mathcal{O}_T} = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \langle D^\alpha u, f_\alpha \rangle_{\mathcal{O}_T}. \quad (11)$$

Note that

$$\langle D^\alpha u, A^{\alpha\beta} D^\beta u \rangle_{\mathcal{O}_T} = \langle (i\xi)^\alpha \tilde{u}, A^{\alpha\beta} (i\xi)^\beta \tilde{u} \rangle_{\mathcal{O}_T} = \int_{\mathcal{O}_T} \xi^\alpha \xi^\beta \tilde{u}^{\text{tr}} \overline{A^{\alpha\beta} \tilde{u}} \, d\xi \, dt.$$

Here \tilde{u} is the Fourier transform of u in x . By the ellipticity condition we get

$$\delta \int_{\mathcal{O}_T} |\xi|^{2m} |\tilde{u}|^2 \, d\xi \, dt \leq \int_{\mathcal{O}_T} \Re \left(\xi^\alpha \xi^\beta \tilde{u}^{\text{tr}} \overline{A^{\alpha\beta} \tilde{u}} \right) \, d\xi \, dt.$$

Also note that

$$\begin{aligned} \int_{\mathbb{R}^d} |u|^2(T, x) \, dx &= \int_{\mathcal{O}_T} \frac{\partial}{\partial t} |u|^2(t, x) \, dt \, dx = \langle u, u_t \rangle_{\mathcal{O}_T} + \langle u_t, u \rangle_{\mathcal{O}_T}, \\ \Re \langle u, u_t \rangle_{\mathcal{O}_T} &= \frac{1}{2} \int_{\mathbb{R}^d} |u|^2(T, x) \, dx \geq 0. \end{aligned}$$

Thus, if we denote the right-hand side of (11) by I , we obtain

$$\delta \int_{\mathcal{O}_T} |\xi|^{2m} |\tilde{u}|^2 \, d\xi \, dt + \lambda \langle u, u \rangle_{\mathcal{O}_T} \leq \Re I \leq \sum_{|\alpha| \leq m} |\langle D^\alpha u, f_\alpha \rangle_{\mathcal{O}_T}|.$$

Since

$$\|D^m u\|_{L_2(\mathcal{O}_T)}^2 \leq N \int_{\mathcal{O}_T} |\xi|^{2m} |\tilde{u}|^2 \, d\xi \, dt$$

and

$$|(D^\alpha u, f_\alpha)_{\mathcal{O}_T}| \leq \varepsilon \lambda^{\frac{m-|\alpha|}{m}} \|D^\alpha u\|_{L_2(\mathcal{O}_T)}^2 + N \varepsilon^{-1} \lambda^{-\frac{m-|\alpha|}{m}} \|f_\alpha\|_{L_2(\mathcal{O}_T)}^2$$

for all $\varepsilon > 0$, the inequality (9) follows by using the interpolation inequalities and choosing an appropriate ε . \square

Theorem 10. *Let $T \in (-\infty, \infty]$. There exists $N = N(d, n, m, \delta)$ such that*

$$\|u_t\|_{L_2(\mathcal{O}_T)} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_2(\mathcal{O}_T)} \leq N \|u_t + (-1)^m \mathcal{L}_0 u + \lambda u\|_{L_2(\mathcal{O}_T)}$$

for all $\lambda \geq 0$ and $u \in W_2^{1,2m}(\mathcal{O}_T)$. Moreover, for $\lambda > 0$ and $f \in L_2(\mathcal{O}_T)$, there exists a unique $u \in W_2^{1,2m}(\mathcal{O}_T)$ satisfying

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = f$$

in \mathcal{O}_T .

Proof. As in the proof of Theorem 9, we only prove the estimate assuming that $u \in C_0^\infty(\overline{\mathcal{O}_T})$. Let $f = u_t + (-1)^m \mathcal{L}_0 u + \lambda u$ and write

$$u_t + (-1)^m D^\alpha (A^{\alpha\beta} D^\beta u) + \lambda u = f. \tag{12}$$

Then by Theorem 9

$$\lambda \|u\|_{L_2(\mathcal{O}_T)} \leq N \|f\|_{L_2(\mathcal{O}_T)}. \tag{13}$$

Now by differentiating both sides of (12) m times with respect to x we get

$$(D^m u)_t + (-1)^m D^\alpha (A^{\alpha\beta} D^\beta D^m u) + \lambda D^m u = D^m f.$$

This with Theorem 9 shows that

$$\sum_{|\alpha|=m} \|D^\alpha D^m u\|_{L_2(\mathcal{O}_T)} \leq N \|f\|_{L_2(\mathcal{O}_T)}. \tag{14}$$

Using (13), (14), and the interpolation inequalities, we obtain

$$\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_2(\mathcal{O}_T)} \leq N \|f\|_{L_2(\mathcal{O}_T)}.$$

Finally, observe that

$$\|u_t\|_{L_2(\mathcal{O}_T)} = \|f - (-1)^m \mathcal{L}_0 u - \lambda u\|_{L_2} \leq N \|f\|_{L_2(\mathcal{O}_T)}.$$

The theorem is proved. \square

4. Mean oscillation estimates for systems in the whole space

In this section we continue working on the operator

$$\mathcal{L}_0 u = \sum_{|\alpha|=|\beta|=m} A^{\alpha\beta} D^\alpha D^\beta u,$$

where $A^{\alpha\beta} = A^{\alpha\beta}(t)$. The main objective of this section is to obtain mean oscillation estimates for divergence type systems (Theorem 11) and for non-divergence type systems (Corollary 2) defined in the whole space.

4.1. Some auxiliary results for systems in the whole space

First we prove the following localized version of Theorem 10.

Lemma 1. *Let $0 < r < R < \infty$. Assume $u \in W_2^{1,2m}(Q_R)$ and*

$$u_t + (-1)^m \mathcal{L}_0 u = f$$

in Q_R , where $f \in L_2(\mathcal{O}_T)$. Then there exists a constant $N = N(d, n, m, \delta)$ such that

$$\|u_t\|_{L_2(Q_r)} + \|D^{2m} u\|_{L_2(Q_r)} \leq N \|f\|_{L_2(Q_R)} + N(R-r)^{-2m} \|u\|_{L_2(Q_R)}. \tag{15}$$

Furthermore,

$$\|u\|_{W_2^{1,2m}(Q_r)} \leq N \|f\|_{L_2(Q_R)} + N \|u\|_{L_2(Q_R)}, \tag{16}$$

where $N = N(d, n, m, \delta, r, R)$.

Proof. Let

$$R_0 = r, \quad R_j = r + (R-r) \sum_{l=1}^j 2^{-l}, \quad j = 1, 2, \dots$$

For each $j = 0, 1, \dots$, we take $\zeta_j \in C_0^\infty(\mathbb{R}^{d+1})$ satisfying

$$\zeta_j = \begin{cases} 1 & \text{on } Q_{R_j} \\ 0 & \text{on } \mathbb{R}^{d+1} \setminus (-R_{j+1}^{2m}, R_{j+1}^{2m}) \times B_{R_{j+1}} \end{cases},$$

and

$$|D^k \zeta_j| \leq N 2^{kj} (R-r)^{-k}, \quad |(\zeta_j)_t| \leq N 2^{mj} (R-r)^{-2m},$$

where $k = 0, 1, \dots, 2m$. Indeed, we can take ζ_j as follows. Let $g(z) \in C^\infty(\mathbb{R})$ be a function such that

$$0 \leq g \leq 1, \quad g(z) = 1 \text{ if } z \leq 0, \quad g(z) = 0 \text{ if } z \geq 1/2.$$

Then set $\zeta_j(t, x) = \psi_j(t)\eta_j(x)$, where

$$\begin{aligned} \psi_j(t) &= g\left(2^j(R-r)^{-1}(|t|^{\frac{1}{2m}} - R_j)\right), \\ \eta_j(x) &= g\left(2^j(R-r)^{-1}(|x| - R_j)\right). \end{aligned}$$

Now we apply Theorem 10 with $\lambda = 0$ to $\zeta_j u \in W_2^{1,2m}(\mathcal{O}_0)$, so that

$$\begin{aligned} \|(\zeta_j u)_t\|_{L_2(\mathcal{O}_0)} + \|D^{2m}(\zeta_j u)\|_{L_2(\mathcal{O}_0)} &\leq N\|(\zeta_j u)_t + (-1)^m \mathcal{L}_0(\zeta_j u)\|_{L_2(\mathcal{O}_0)} \\ &\leq N\|f\|_{L_2(Q_R)} + N\|(\zeta_j)_t u\|_{L_2(\mathcal{O}_0)} \\ &\quad + N\sum_{k=1}^{2m} \|D^k \zeta_j D^{2m-k} u\|_{L_2(\mathcal{O}_0)}. \end{aligned} \tag{17}$$

Using the properties of ζ_j and interpolation inequalities (see, for instance, [27]), for each $1 \leq k < 2m$, we have

$$\begin{aligned} \|D^k \zeta_j D^{2m-k} u\|_{L_2(\mathcal{O}_0)} &= \|D^k \zeta_j D^{2m-k}(\zeta_{j+1} u)\|_{L_2(\mathcal{O}_0)} \\ &\leq N2^{kj}(R-r)^{-k} \|D^{2m-k}(\zeta_{j+1} u)\|_{L_2(\mathcal{O}_0)} \\ &\leq \varepsilon \|D^{2m}(\zeta_{j+1} u)\|_{L_2(\mathcal{O}_0)} \\ &\quad + N2^{2mj}(R-r)^{-2m} \|u\|_{L_2(Q_R)}. \end{aligned} \tag{18}$$

Furthermore, we have

$$\|(\zeta_j)_t u\|_{L_2(\mathcal{O}_0)} + \|uD^{2m}\zeta_j\|_{L_2(\mathcal{O}_0)} \leq N2^{2mj}(R-r)^{-2m} \|u\|_{L_2(Q_R)}. \tag{19}$$

Therefore, if we set

$$I_j = \|(\zeta_j u)_t\|_{L_2(\mathcal{O}_0)} + \|D^{2m}(\zeta_j u)\|_{L_2(\mathcal{O}_0)},$$

from (17), (18), and (19) we obtain

$$I_j \leq \varepsilon I_{j+1} + N\|f\|_{L_2(Q_R)} + N2^{2mj}(R-r)^{-2m} \|u\|_{L_2(Q_R)}.$$

Multiply both sides by ε^j and make summations with respect to j to get

$$\sum_{j=0}^{\infty} \varepsilon^j I_j \leq \sum_{j=1}^{\infty} \varepsilon^j I_j + N\sum_{j=0}^{\infty} \varepsilon^j \|f\|_{L_2(Q_R)} + N(R-r)^{-2m} \sum_{j=0}^{\infty} \varepsilon^j 2^{2mj} \|u\|_{L_2(Q_R)}.$$

Upon choosing, for example, $\varepsilon = 2^{-2m-1}$, the summations are finite, so from the above inequality we have

$$\|D^{2m}(\zeta_0 u)\|_{L_2(\mathcal{O}_0)} + \|(\zeta_0 u)_t\|_{L_2(\mathcal{O}_0)} \leq N\|f\|_{L_2(Q_R)} + N(R-r)^{-2m} \|u\|_{L_2(Q_R)}.$$

This proves the inequality (15) because the left-hand side of the above inequality is greater than that of (15). Finally, the inequality (16) follows from (15) and the interpolation inequalities. \square

In the sequel we denote $u \in \mathbb{W}_p^{1,\infty}(Q_r(t_0, x_0))$, $1 < p < \infty$, $(t_0, x_0) \in \mathbb{R}^{d+1}$, if $D^\alpha u, D^\alpha u_t \in L_p(Q_r(t_0, x_0))$ for all multi-index α including $\alpha = (0, \dots, 0)$.

Corollary 1. *Let $0 < r < R < \infty$ and $u \in \mathbb{W}_2^{1,\infty}(Q_R)$ satisfy*

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \tag{20}$$

in Q_R . Then for any multi-index γ , we have

$$\|D^\gamma u\|_{L_2(Q_r)} + \|D^\gamma u_t\|_{L_2(Q_r)} \leq N \|u\|_{L_2(Q_R)},$$

where $N = N(d, n, m, \delta, r, R, \gamma)$.

Proof. Note that

$$D^\gamma u_t = -(-1)^m \mathcal{L}_0 D^\gamma u$$

in Q_R . Hence it is enough to prove

$$\|D^\gamma u\|_{L_2(Q_r)} \leq N \|u\|_{L_2(Q_R)}. \tag{21}$$

Since $u \in W_2^{1,2m}(Q_R)$, this inequality follows from (16) if $|\gamma| \leq 2m$, so assume that $|\gamma| > 2m$ and

$$D^\gamma u = D^{2m} D^\vartheta u.$$

Note that $D^\vartheta u$ is in $W_2^{1,2m}(Q_R)$ and satisfies (20). Thus applying (15) to the equation (20) with $D^\vartheta u$ in place of u we get

$$\|D^\gamma u\|_{L_2(Q_r)} \leq N \|D^\vartheta u\|_{L_2(Q_{R_0})},$$

where $r < R_0 < R$. We repeat this process as many times as needed to get

$$\|D^\gamma u\|_{L_2(Q_r)} \leq N \|D^{\gamma_0} u\|_{L_2(Q_{R_1})},$$

where $|\gamma_0| \leq 2m$ and $r < R_1 < R$. Then the inequality (21) for $|\gamma| > 2m$ follows from the same inequality for $|\gamma| \leq 2m$ (with R_1 in place of r). \square

Lemma 2. *If $u \in \mathbb{W}_2^{1,\infty}(Q_4)$ satisfies (20) in Q_4 , then*

$$\sup_{Q_1} |Du(t, x)| + \sup_{Q_1} |u_t(t, x)| \leq N \|u\|_{L_2(Q_4)},$$

where $N = N(d, n, m, \delta)$.

Proof. Thanks to the fact that $u_t = -(-1)^m \mathcal{L}_0 u$ in Q_4 , it suffices to prove

$$\sup_{(t,x) \in Q_1} |D^\gamma u(t, x)| \leq N \|u\|_{L_2(Q_4)} \tag{22}$$

for a multi-index γ . By the Sobolev embedding theorem

$$\sup_{t \in (-1,0)} |D^\gamma u(t, x)|^2 \leq N \int_{-1}^0 |D^\gamma u(s, x)|^2 ds + N \int_{-1}^0 |D^\gamma u_t(s, x)|^2 ds$$

for each $x \in B_1$, where $D^\gamma u(t, x)$ is considered as a function of $t \in (-1, 0)$ for each fixed $x \in B_1$. On the other hand, again by the Sobolev embedding theorem, there exists a positive number k such that

$$\sup_{x \in B_1} |D^\gamma u(s, x)| \leq N \|D^\gamma u(s, \cdot)\|_{W_2^k(B_1)}$$

for each $s \in (-1, 0)$, where $D^\gamma u(s, x)$ is considered as a function of $x \in B_1$ for each fixed $s \in (-1, 0)$. We have the same inequality as above with $D^\gamma u_t$ in place of $D^\gamma u$. Therefore, we obtain

$$\sup_{(t,x) \in Q_1} |D^\gamma u(t, x)|^2 \leq N \sum_{|\vartheta| \leq k} \|D^\vartheta D^\gamma u\|_{L_2(Q_1)} + N \sum_{|\vartheta| \leq k} \|D^\vartheta D^\gamma u_t\|_{L_2(Q_1)}.$$

This together with Corollary 1 gives the inequality (22). \square

Lemma 3. *Let $\lambda \geq 0$ and $u \in \mathbb{W}_2^{1,\infty}(Q_4)$ satisfy*

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = 0 \tag{23}$$

in Q_4 . Then we have

$$\begin{aligned} & \sup_{Q_1} |D^{m+1} u(t, x)| + \sup_{Q_1} |D^m u_t(t, x)| + \lambda^{\frac{1}{2}} \sup_{Q_1} |Du(t, x)| + \lambda^{\frac{1}{2}} \sup_{Q_1} |u_t(t, x)| \\ & \leq N \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} \|D^k u\|_{L_2(Q_4)}, \end{aligned} \tag{24}$$

where $N = N(d, n, m, \delta)$.

Proof. The case $\lambda = 0$ follows by Lemma 2 applied to $D^m u$ since $D^m u$ satisfies (20). For the case $\lambda > 0$, we follow an idea by S. Agmon. Consider

$$\zeta(y) = \cos\left(\lambda^{\frac{1}{2m}} y\right) + \sin\left(\lambda^{\frac{1}{2m}} y\right).$$

Note that

$$(-1)^m D_y^{2m} \zeta(y) = \lambda \zeta(y), \quad \zeta(0) = 1, \quad |D^m \zeta(0)| = \lambda^{\frac{1}{2}}.$$

Denote by $(t, z) = (t, x, y)$ a point in \mathbb{R}^{d+2} , where $z = (x, y) \in \mathbb{R}^{d+1}$, and set

$$\hat{u}(t, z) = u(t, x)\zeta(y), \quad \hat{Q}_r = (-r^{2m}, 0) \times \{|z| < r, z \in \mathbb{R}^{d+1}\}.$$

Since u satisfies (23), \hat{u} satisfy

$$\hat{u}_t + (-1)^m \mathcal{L}_0 \hat{u} + (-1)^m D_y^{2m} \hat{u} = 0$$

in \hat{Q}_4 . Upon applying the inequality (24) with $\lambda = 0$ just proved above, we get

$$\begin{aligned} & \sup_{\hat{Q}_1} |D_x^{m+1} \hat{u}(t, z)| + \sup_{\hat{Q}_1} |D_x^m \hat{u}_t(t, z)| + \sup_{\hat{Q}_1} |D_y^m D_x \hat{u}(t, z)| + \sup_{\hat{Q}_1} |D_y^m \hat{u}_t(t, z)| \\ & \leq N \|D^m \hat{u}\|_{L_2(\hat{Q}_4)}. \end{aligned} \tag{25}$$

Since, for example,

$$\sup_{(t,x) \in Q_1} \lambda^{\frac{1}{2}} |D_x u(t, x)| \leq \sup_{(t,z) \in \hat{Q}_1} |D_y^m D_x \hat{u}(t, z)|,$$

the left-hand side of (25) is greater than that of (24). On the other hand, $D^m \hat{u}$ is a linear combination of terms such as

$$\lambda^{\frac{1}{2} - \frac{k}{2m}} \cos\left(\lambda^{\frac{1}{2m}} y\right) D_x^k u(t, x), \quad \lambda^{\frac{1}{2} - \frac{k}{2m}} \sin\left(\lambda^{\frac{1}{2m}} y\right) D_x^k u(t, x), \quad k = 0, \dots, m.$$

Thus we see that the right-hand side of (25) is bounded by that of (24). The lemma is proved. \square

Recall that we denote by X a point in $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$.

Lemma 4. *Let $r \in (0, \infty)$, $\kappa \in [4, \infty)$, $\lambda \geq 0$, and $X_0 = (t_0, x_0) \in \mathbb{R}^{d+1}$. Assume $u \in \mathcal{H}_{2,loc}^m(\mathbb{R}^{d+1})$ satisfies (23) in $Q_{\kappa r}(X_0)$. Then for any α , $|\alpha| = m$, we have*

$$\begin{aligned} & (|D^\alpha u - (D^\alpha u)_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda^{\frac{1}{2}} (|u - (u)_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N \kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k u|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}, \end{aligned} \tag{26}$$

where $N = N(d, n, m, \delta) > 0$.

Proof. Let us prove the inequality (26) when $X_0 = (0, 0)$. This, with a translation of the coordinates, proves the inequality for general $X_0 \in \mathbb{R}^{d+1}$.

Since the standard mollification of u with respect to x satisfies (23) in a little bit smaller cylinder than $Q_{\kappa r}$, we assume that $D^\alpha u \in L_2(Q_{\kappa r})$ for all multi-index α . Furthermore, (23) implies that $D^\gamma u_t \in L_2(Q_{\kappa r})$ if $D^\alpha u \in L_2(Q_{\kappa r})$ for all α . Therefore, without loss of generality we assume that $u \in \mathbb{W}_2^{1,\infty}(Q_{\kappa r})$.

Due to a scaling argument (for instance, see the proof of Lemma 11), it is sufficient to deal with the case $r = 4/\kappa$. Observe that, for example,

$$(|D^\alpha u - (D^\alpha u)_{Q_r}|)_{Q_r} \leq Nr \sup_{Q_1} |D^{\alpha+1} u(t, x)| + Nr \sup_{Q_1} |D^\alpha u_t(t, x)|.$$

By Lemma 3, the right-hand side of the above inequality is bounded by that of (26) (recall $r = 4\kappa^{-1}$). The lemma is proved. \square

4.2. Mean oscillation estimates for systems in the whole space

In the next theorem, we prove a mean oscillation estimate for divergence form systems with simple coefficients in the whole space.

Theorem 11. *Let $r \in (0, \infty)$, $\kappa \in [8, \infty)$, $\lambda > 0$, $X_0 = (t_0, x_0) \in \mathbb{R}^{d+1}$, and $f_\alpha \in L_{2,loc}(\mathbb{R}^{d+1})$, $|\alpha| \leq m$. Assume that $u \in \mathcal{H}_{2,loc}^m(\mathbb{R}^{d+1})$ satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$$

in $Q_{\kappa r}(X_0)$. Then for any α , $|\alpha| = m$, we have

$$\begin{aligned} & (|D^\alpha u - (D^\alpha u)_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda^{\frac{1}{2}} (|u - (u)_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N\kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k u|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} + N\kappa^{m+\frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|f_\alpha|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}, \end{aligned} \tag{27}$$

where $N = N(d, n, m, \delta) > 0$.

Proof. We take, for the sake of simplicity, $X_0 = (0, 0)$. As mentioned earlier, a translation gives the result for general X_0 .

Take an infinitely differentiable function ζ defined on \mathbb{R}^{d+1} such that

$$\zeta = 1 \text{ on } Q_{\kappa r/2}, \quad \zeta = 0 \text{ outside } (-(\kappa r)^{2m}, (\kappa r)^{2m}) \times B_{\kappa r}.$$

By Theorem 9, for $\lambda > 0$, there exists a unique solution $w \in \mathcal{H}_2^m(\mathcal{O}_\infty)$ to the equation

$$w_t + (-1)^m \mathcal{L}_0 w + \lambda w = \sum_{|\alpha| \leq m} D^\alpha (\zeta f_\alpha) \tag{28}$$

in $\mathcal{O}_\infty = \mathbb{R}^{d+1}$. Let $v := u - w$. Then the function $v \in \mathcal{H}_{2,loc}^m(\mathbb{R}^{d+1})$ satisfies

$$v_t + (-1)^m \mathcal{L}_0 v + \lambda v = 0 \text{ in } Q_{\kappa r/2}.$$

By Lemma 4 (note that $\kappa/2 \geq 4$) applied to v , we have

$$\begin{aligned} (|D^\alpha v - (D^\alpha v)_{Q_r}|)_{Q_r} + \lambda^{\frac{1}{2}} (|v - (v)_{Q_r}|)_{Q_r} & \leq N\kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k v|^2)_{Q_{\kappa r/2}}^{\frac{1}{2}}. \end{aligned} \tag{29}$$

Next we estimate w , which is the unique solution to the equation (28) considered on \mathcal{O}_0 . By Theorem 9, we have

$$\sum_{|\alpha| \leq m} \lambda^{1 - \frac{|\alpha|}{2m}} \|D^\alpha w\|_{L_2(\mathcal{O}_0)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|\zeta f_\alpha\|_{L_2(\mathcal{O}_0)}.$$

In particular,

$$\left(|D^m w|^2 \right)_{Q_r}^{\frac{1}{2}} + \lambda^{\frac{1}{2}} (|w|^2)_{Q_r}^{\frac{1}{2}} \leq N\kappa^{m+\frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|f_\alpha|^2)_{Q_{\kappa r}}^{\frac{1}{2}}, \tag{30}$$

$$\sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k w|^2)_{Q_{\kappa r}}^{\frac{1}{2}} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|f_\alpha|^2)_{Q_{\kappa r}}^{\frac{1}{2}}. \tag{31}$$

Now we are ready to prove (27). Since

$$\left(|D^\alpha u - (D^\alpha u)_{Q_r}| \right)_{Q_r} \leq 2 (|D^\alpha u - c|)_{Q_r}$$

for any constant c , by taking $c = (D^\alpha v)_{Q_r}$ and repeating the same argument for the second term, we bound the left-hand side of (27) by a constant times

$$(|D^\alpha u - (D^\alpha v)_{Q_r}|)_{Q_r} + \lambda^{\frac{1}{2}}(u - (v)_{Q_r})_{Q_r},$$

which is, due to the fact that $u = w + v$, controlled by

$$(|D^\alpha v - (D^\alpha v)_{Q_r}|)_{Q_r} + \lambda^{\frac{1}{2}}(|v - (v)_{Q_r}|)_{Q_r} + \left(|D^m w|^2\right)^{\frac{1}{2}}_{Q_r} + \lambda^{\frac{1}{2}}(|w|^2)^{\frac{1}{2}}_{Q_r}.$$

Using (29) and (30), we see that the above is less than

$$N\kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k v|^2)^{\frac{1}{2}}_{Q_{\kappa r/2}} + N\kappa^{m+\frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|f_\alpha|^2)^{\frac{1}{2}}_{Q_{\kappa r}}.$$

Finally, we use the fact that $v = u - w$ and (31) to prove that the terms above are not greater than the right-hand side of (27). \square

Next we consider the corresponding mean oscillation estimate for non-divergence type systems in the whole space.

Corollary 2. *Let $r \in (0, \infty)$, $\kappa \in [8, \infty)$, $\lambda > 0$, $X_0 \in \mathbb{R}^{d+1}$, and $f \in L_{2,loc}(\mathbb{R}^{d+1})$, $|\alpha| \leq m$. Assume that $u \in W_{2,loc}^{1,2m}(\mathbb{R}^{d+1})$ satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = f$$

in $Q_{\kappa r}(X_0)$. Then for any α , $|\alpha| = 2m$, we have

$$\begin{aligned} & (|D^\alpha u - (D^\alpha u)_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda (|u - (u)_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N\kappa^{-1} \sum_{k=0}^{2m} \lambda^{1 - \frac{k}{2m}} (|D^k u|^2)^{\frac{1}{2}}_{Q_{\kappa r}(X_0)} + N\kappa^{m+\frac{d}{2}} (|f|^2)^{\frac{1}{2}}_{Q_{\kappa r}(X_0)}, \end{aligned}$$

where $N = N(d, n, m, \delta) > 0$.

Proof. Again let $X_0 = (0, 0)$ for simplicity. By Theorem 11, it follows that (after multiplying both sides by $\lambda^{\frac{1}{2}}$)

$$\lambda (|u - (u)_{Q_r}|)_{Q_r} \leq N\kappa^{-1} \sum_{k=0}^m \lambda^{1 - \frac{k}{2m}} (|D^k u|^2)^{\frac{1}{2}}_{Q_{\kappa r}} + N\kappa^{m+\frac{d}{2}} (|f|^2)^{\frac{1}{2}}_{Q_{\kappa r}}. \tag{32}$$

Differentiate m times both sides of the system with respect to x to get

$$D^m u_t + (-1)^m \mathcal{L}_0 D^m u + \lambda D^m u = D^m f.$$

By Theorem 11 applied to $D^m u$ in place of u ,

$$\begin{aligned} (|D^\gamma D^m u - (D^\gamma D^m u)_{Q_r}|)_{Q_r} & \leq N\kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k D^m u|^2)^{\frac{1}{2}}_{Q_{\kappa r}} \\ & \quad + N\kappa^{m+\frac{d}{2}} (|f|^2)^{\frac{1}{2}}_{Q_{\kappa r}}, \end{aligned}$$

where $|\gamma| = m$. This combined with (32) gives the inequality in the corollary. \square

5. L_p -Estimates for systems in the whole space

In this section, we use the mean oscillation estimates obtained in the previous section to prove Theorems 1 and 2.

Let $Q = \{Q_r(t, x) : (t, x) \in \mathbb{R}^{d+1}, r \in (0, \infty)\}$. For a function g defined on \mathbb{R}^{d+1} , we denote its (parabolic) maximal and sharp function, respectively, by

$$\begin{aligned} \mathcal{M}g(t, x) &= \sup_{Q \in \mathcal{Q}: (t,x) \in Q} \int_Q |g(s, y)| \, dy \, ds, \\ g^\#(t, x) &= \sup_{Q \in \mathcal{Q}: (t,x) \in Q} \int_Q |g(s, y) - (g)_Q| \, dy \, ds. \end{aligned}$$

Then

$$\|g\|_{L_p} \leq N \|g^\#\|_{L_p}, \quad \|\mathcal{M}g\|_{L_p} \leq N \|g\|_{L_p},$$

if $g \in L_p$, where $1 < p < \infty$ and $N = N(d, p)$. As is well known, the first of the above inequalities is due to the Fefferman–Stein theorem on sharp functions and the second one to the Hardy–Littlewood maximal function theorem.

We use the idea of freezing the coefficients to obtain

Lemma 5. *Let \mathcal{L} be the operator in Theorem 1. Suppose the lower-order coefficients of \mathcal{L} are all zero. Let $\mu, \nu \in (1, \infty)$, $1/\mu + 1/\nu = 1$, and $\lambda, R \in (0, \infty)$. Assume $u \in C_0^\infty(\mathbb{R}^{d+1})$ vanishing outside Q_R and*

$$u_t + (-1)^m \mathcal{L}u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha,$$

where $f_\alpha \in L_{2,loc}(\mathbb{R}^{d+1})$. Then there exists a constant $N = N(d, m, n, \delta, \mu)$ such that for any α , $|\alpha| = m$, $r \in (0, \infty)$, $\kappa \geq 8$, and $X_0 \in \mathbb{R}^{d+1}$, we have

$$\begin{aligned} & (|D^\alpha u - (D^\alpha u)_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda^{\frac{1}{2}} (|u - (u)_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N \kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k u|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} \\ & \quad + N \kappa^{m + \frac{d}{2}} \left(\sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|f_\alpha|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} + (A_R^\#)^{\frac{1}{2\nu}} (|D^m u|^{2\mu})_{Q_{\kappa r}(X_0)}^{\frac{1}{2\mu}} \right). \end{aligned}$$

Proof. Let $\kappa \geq 8$ and $r \in (0, \infty)$. Fix a $y \in \mathbb{R}^d$ and set

$$\mathcal{L}_y u = A^{\alpha\beta}(t, y) D^\alpha D^\beta u(t, x).$$

Then we have

$$u_t + (-1)^m \mathcal{L}_y u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha \tilde{f}_\alpha,$$

where

$$\tilde{f}_\alpha = f_\alpha + (-1)^m \sum_{|\beta|=m} (A^{\alpha\beta}(t, y) - A^{\alpha\beta}(t, x)) D^\beta u.$$

It follows from Theorem 11 that

$$\begin{aligned} & (|D^\alpha u - (D^\alpha u)_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda^{\frac{1}{2}} (|u - (u)_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N\kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k u|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} + N\kappa^{m+\frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|\tilde{f}_\alpha|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}. \end{aligned} \quad (33)$$

Note that

$$\int_{Q_{\kappa r}(X_0)} |\tilde{f}_\alpha|^2 dx dt \leq N \int_{Q_{\kappa r}(X_0)} |f_\alpha|^2 dx dt + N I_y, \quad (34)$$

where for $|\alpha| = m$,

$$I_y = \int_{Q_{\kappa r}(X_0)} |(A^{\alpha\beta}(t, y) - A^{\alpha\beta}(t, x)) D^\beta u|^2 dx dt.$$

Denote B to be $B_{\kappa r}(x_0)$ if $\kappa r < R$, or to be B_R otherwise; denote Q to be $Q_{\kappa r}(t_0, x_0)$ if $\kappa r < R$, or to be Q_R otherwise. Now we take the average of I_y with respect to y in B . Since $u = 0$ outside Q_R , by the Hölder inequality we get

$$\begin{aligned} \int_B I_y dy &= \int_B \int_{Q_{\kappa r}(X_0) \cap Q_R} |(A^{\alpha\beta}(t, y) - A^{\alpha\beta}(t, x)) D^\beta u|^2 dx dt dy \\ &\leq \int_B \left(\int_Q |A^{\alpha\beta}(t, y) - A^{\alpha\beta}(t, x)|^{2\nu} dy \right)^{\frac{1}{\nu}} \left(\int_{Q_{\kappa r}(X_0) \cap Q_R} |D^m u|^{2\mu} dx dt \right)^{\frac{1}{\mu}}, \end{aligned}$$

where, by the boundedness of $A^{\alpha\beta}$ as well as the definitions of osc_x and $A_R^\#$, the integral over B in the last term above is bounded by a constant times

$$\begin{aligned} & \int_B \left(\int_Q |A^{\alpha\beta}(t, y) - A^{\alpha\beta}(t, x)| dy \right)^{\frac{1}{\nu}} \\ & \leq \left(\int_B \int_Q |A^{\alpha\beta}(t, y) - A^{\alpha\beta}(t, x)| dx dt dy \right)^{\frac{1}{\nu}} \\ & \leq N (|Q| \text{osc}_x(A^{\alpha\beta}, Q))^{\frac{1}{\nu}} \leq N (R^{2m+d} A_R^\#)^{\frac{1}{\nu}}. \end{aligned}$$

This, together with (33) and (34), completes the proof of the lemma. \square

Proof of Theorem 1. Due to the method of continuity, it suffices to obtain an a priori estimate. By moving all the lower-order terms to the right-hand side and taking a sufficiently large λ , we may assume that all the lower-order coefficients are zero.

Case 1: $p \in (2, \infty)$. First we suppose $T = \infty$ and $u \in C_0^\infty(Q_{R_0})$. Choose a $\mu > 1$ such that $2\mu < p$. Under these assumptions, from Lemma 5 we easily deduce

$$\begin{aligned} & (D^\alpha u)^\#(X_0) + \lambda^{\frac{1}{2}} u^\#(X_0) \\ & \leq N\kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (\mathcal{M}(D^k u)^2(X_0))^{\frac{1}{2}} \\ & \quad + N\kappa^{m+\frac{d}{2}} \left(\sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (\mathcal{M}f_\alpha^2(X_0))^{\frac{1}{2}} + \rho^{\frac{1}{2\nu}} (\mathcal{M}(D^m u)^{2\mu}(X_0))^{\frac{1}{2\mu}} \right) \end{aligned}$$

for any $\alpha, |\alpha| = m, r \in (0, \infty), \kappa \geq 8$, and $X_0 \in \mathbb{R}^{d+1}$. This, together with the interpolation inequality, the Fefferman-Stein theorem and the Hardy-Littlewood maximal function theorem, yields

$$\begin{aligned} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} \|D^k u\|_{L_p} & \leq N \|D^\alpha u\|_{L_p} + N\lambda^{\frac{1}{2}} \|u\|_{L_p} \\ & \leq N(\kappa^{-\frac{1}{2}} + \kappa^{m+\frac{d}{2}} \rho^{\frac{1}{2\nu}}) \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} \|D^k u\|_{L_p} \\ & \quad + N\kappa^{m+\frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} \|f_\alpha\|. \end{aligned} \tag{35}$$

Now we can choose κ sufficiently large and ρ sufficiently small in (35) to get the desired estimate. A standard partition of the unity enables us to remove the restriction that $u \in C_0^\infty(Q_{R_0})$. The extension to the case $T \in (-\infty, +\infty]$ is by now standard; see, for instance, [25]. We omit the details.

Case 2: $p \in (1, 2)$. Since the system is in divergence form, this case follows immediately from the previous case by using the duality argument.

Finally, the case $p = 2$ is obtained by an interpolation argument. \square

In a similar way, from Corollary 2 we get the following counterpart of Lemma 5 for non-divergence systems.

Lemma 6. *Let L be the operator in Theorem 2. Suppose the lower-order coefficients of L are all zero. Let $\mu, \nu \in (1, \infty), 1/\mu + 1/\nu = 1$, and $\lambda, R \in (0, \infty)$. Assume $u \in C_0^\infty(\mathbb{R}^{d+1})$ vanishing outside Q_R and*

$$u_t + (-1)^m Lu + \lambda u = f,$$

where $f \in L_{2,loc}(\mathbb{R}^{d+1})$. Then there exists a constant $N = N(d, m, n, \delta, \mu)$ such that for any α , $|\alpha| = 2m$, $r \in (0, \infty)$, $\kappa \geq 8$, and $X_0 \in \mathbb{R}^{d+1}$, we have

$$\begin{aligned} & (|D^\alpha u - (D^\alpha u)_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda (|u - (u)_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N\kappa^{-1} \sum_{k=0}^{2m} \lambda^{1-\frac{k}{2m}} (|D^k u|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} \\ & \quad + N\kappa^{m+\frac{d}{2}} \left((|f|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} + (A_R^\#)^{\frac{1}{2\nu}} (|D^{2m} u|^{2\mu})_{Q_{\kappa r}(X_0)}^{\frac{1}{2\mu}} \right). \end{aligned}$$

Proof of Theorem 2. As in the proof of Theorem 1, it suffices to prove the a priori estimate for $T = \infty$.

Case 1: $p \in (2, \infty)$. We only need to consider the case when $u \in C_0^\infty(Q_{R_0})$, since the general case follows from a partition of the unity. The proof of this case is almost the same as that of Theorem 1, by using Lemma 6 instead of Lemma 5. So we omit it.

Case 2: $p \in (1, 2]$. Note that here we cannot use the duality argument directly. From Case 1 and Remark 2, we already have the $W_q^{1,2m}$ solvability of

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = f$$

in the whole space for any $q \in (2, \infty)$ and $\lambda > 0$. For this system, since $A^{\alpha\beta}$ are measurable functions of time only, we can make use of the duality argument, which yields the solvability of the same equation for any $q \in (1, 2)$. Fix a $q = (1 + p)/2$. Now we can repeat the arguments in the previous section with q in place of 2, and get the estimate in Lemma 6 with q in place of 2. Finally, following the proof of Case 1 completes the proof of this case. \square

Part II. Systems on a half space or a bounded domain

This is the most novel part of the paper. The objective of this part is to establish the L_p -solvability of parabolic systems on a half space or on a bounded domain.

In the next section, we prove the L_2 -estimates for systems with coefficients measurable in t on a half space. Relying on these L_2 -estimates, in Section 7 we are able to derive mean oscillation estimates of some partial derivatives of solutions to systems on a half space. These estimates alone are not sufficient for our purpose, so in Section 8 we consider a certain system with special coefficients. Combining the results in Sections 7 and 8 enables us to prove the L_p -solvability on a half space (Theorems 3, 4). Section 10 is devoted to the proofs of the bounded domain cases (Theorems 5, 6). Finally, we give several remarks about other ellipticity conditions.

6. L_2 -Estimates for systems with simple coefficients on a half space

In this section, we prove the L_2 -estimate for systems on a half space. We again consider

$$\mathcal{L}_0 u = \sum_{|\alpha|=|\beta|=m} D^\alpha (A^{\alpha\beta} D^\beta u) = \sum_{|\alpha|=|\beta|=m} A^{\alpha\beta} D^\alpha D^\beta u,$$

where $A^{\alpha\beta} = A^{\alpha\beta}(t)$. Recall that $\mathcal{O}_T^+ = (-\infty, T) \times \mathbb{R}_+^d$. In the divergence case (Theorem 12), the proof is rather standard. However, in the case of non-divergence systems (Theorem 13), the proof is much more involved. To the best of our knowledge, Theorem 13 is new for higher order parabolic equations and systems with measurable coefficients depending only on t .

6.1. Divergence case

Throughout the paper, we use the notation $D_{x'}^m u$ to indicate one of $D^\alpha u$, where $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_1 = 0$, and $|\alpha| = m$. Sometimes, depending on the context, $D_{x'}^m u$ means the whole collection of $D^\alpha u$, $|\alpha| = m$, $\alpha_1 = 0$. Similar to $C_0^\infty(\overline{\mathcal{O}_T})$, we denote by $C_0^\infty(\overline{\mathcal{O}_T^+})$ the collection of infinitely differentiable functions defined on $\overline{\mathcal{O}_T^+}$ vanishing for large $|(t, x)| \in \overline{\mathcal{O}_T^+}$.

Theorem 12. *Let $T \in (-\infty, \infty]$ and $f_\alpha \in L_2(\mathcal{O}_T^+)$. There exists a constant $N = N(d, n, m, \delta)$ such that*

$$\sum_{|\alpha| \leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_2(\mathcal{O}_T^+)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_2(\mathcal{O}_T^+)} \tag{36}$$

for any $\lambda \geq 0$ and $u \in \mathcal{H}_2^m(\mathcal{O}_T^+)$ satisfying

$$u(t, 0, x') = \dots = D_1^{m-1} u(t, 0, x') = 0 \tag{37}$$

on $(-\infty, T) \times \mathbb{R}^{d-1}$ and

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha \tag{38}$$

in \mathcal{O}_T^+ . Furthermore, for $\lambda > 0$ and $f_\alpha \in L_2(\mathcal{O}_T^+)$, $|\alpha| \leq m$, there exists a unique $u \in \mathcal{H}_2^m(\mathcal{O}_T^+)$ satisfying (38) in \mathcal{O}_T^+ and (37) on $(-\infty, T) \times \mathbb{R}^{d-1}$.

Proof. As in the proof of Theorem 9, we consider only the case $\lambda > 0$. We follow the lines of the proof of Theorem 9. One noteworthy fact is that, because $u \in \mathcal{H}_2^m(\mathcal{O}_T^+)$ satisfies (37), we have

$$\langle D^\alpha u, A^{\alpha\beta} D^\beta u \rangle_{\mathcal{O}_T^+} = \langle D^\alpha u, A^{\alpha\beta} D^\beta u \rangle_{\mathcal{O}_T},$$

where the function u on the right-hand side is viewed as an extension of u to \mathcal{O}_T so that it is zero on $\mathcal{O}_T \setminus \mathcal{O}_T^+$. Similarly,

$$\|D^m u\|_{L_2(\mathcal{O}_T^+)}^2 \leq N \int_{\mathcal{O}_T} |\xi|^{2m} |\tilde{u}|^2 d\xi dt,$$

where \tilde{u} is the Fourier transform of the extension. \square

Remark 4. Theorem 12 can be extended to systems in a cylindrical domain Ω_T , where Ω is a bounded Lipschitz domain. For small $\lambda \geq 0$, we have a better estimate than (36). Indeed, from the proof above, we get

$$\|D^m u\|_{L_2(\Omega_T)}^2 \leq N \sum_{|\alpha| \leq m} \|f_\alpha\|_{L_2(\Omega_T)} \|D^\alpha u\|_{L_2(\Omega_T)}.$$

By using the Poincaré inequality,

$$\|u\|_{L_2(\Omega_T)} \leq N \|Du\|_{L_2(\Omega_T)} \leq N \|D^2 u\|_{L_2(\Omega_T)} \leq \dots \leq N \|D^m u\|_{L_2(\Omega_T)}.$$

Thus, we conclude

$$\sum_{k=0}^m \|D^k u\|_{L_2(\Omega_T)} \leq N \sum_{|\alpha| \leq m} \|f_\alpha\|_{L_2(\Omega_T)}.$$

Note that in this case, the solvability also holds for $\lambda = 0$.

6.2. Non-divergence case

Let us introduce some additional notation. Let $\tau \in \mathbb{N}$ and $\{c_1, \dots, c_{2\tau}\}$ be the solution to the following system:

$$\sum_{k=1}^{2\tau} \left(-\frac{1}{k}\right)^j c_k = 1, \quad j = 0, \dots, 2\tau - 1. \quad (39)$$

For a function w defined on \mathbb{R}_+^d , set

$$\mathcal{E}_\tau w = \begin{cases} w(x_1, x') & \text{if } x_1 > 0 \\ \sum_{k=1}^{2\tau} c_k w\left(-\frac{1}{k}x_1, x'\right) & \text{otherwise.} \end{cases}$$

Note that $\mathcal{E}_\tau w \in C^{2\tau-1}(\mathbb{R}^d)$ if $w \in C^\infty(\overline{\mathbb{R}_+^d})$. Indeed, by (39)

$$D_1^j \left(\sum_{k=1}^{2\tau} c_k w\left(-\frac{1}{k}x_1, x'\right) \right) \Big|_{x_1=0} = \sum_{k=1}^{2\tau} \left(-\frac{1}{k}\right)^j c_k D_1^j w(0, x') = D_1^j w(0, x')$$

for $j = 0, \dots, 2\tau - 1$.

We remark that similar extension operators were used in [19] and [16] in the study of elliptic systems. We will use the following interpolation estimate.

Proposition 1. Let $1 < p < \infty$ and $u \in W_p^m(\mathbb{R}_+^d)$. For any $\varepsilon > 0$, there exists $N = N(d, n, m, p, \varepsilon)$ such that

$$\|D_1^k D_{x'}^{m-k} u\|_{L_p(\mathbb{R}_+^d)} \leq \varepsilon \|D_1^m u\|_{L_p(\mathbb{R}_+^d)} + N \sum_{j=2}^d \|D_j^m u\|_{L_p(\mathbb{R}_+^d)},$$

where $k = 0, 1, \dots, m - 1$.

Proof. Without loss of generality we assume that $u \in C_0^\infty(\overline{\mathbb{R}_+^d})$. Let $\hat{u} = \mathcal{E}_\tau u$. For a sufficiently large τ , the extension \hat{u} is in $W_p^m(\mathbb{R}^d)$ and satisfies $\sum_{j=1}^d D_j^{2m} \hat{u} = \sum_{j=1}^d D_j^m \hat{f}_j$, where

$$\hat{f}_j = \begin{cases} \sum_{j=1}^d D_j^m u & \text{in } \mathbb{R}_+^d; \\ \sum_{j=1}^d \sum_{k=1}^{2\tau} \hat{c}_k D_j^m w \left(-\frac{1}{k} x_1, x' \right) & \text{in } \mathbb{R}_-^d. \end{cases}$$

Here \hat{c}_k are appropriate constants. Observe that

$$\|D_1^k D_{x'}^{m-k} u\|_{L_p(\mathbb{R}_+^d)} \leq \|D^m \hat{u}\|_{L_p(\mathbb{R}^d)} \leq N \|\hat{f}\|_{L_p(\mathbb{R}^d)} \leq N \sum_{j=1}^d \|D_j^m u\|_{L_p(\mathbb{R}_+^d)},$$

where the second inequality is due to the L_p -estimate of elliptic systems in the whole space (see Remarks 1 and 2) and $N = N(d, n, m, p)$. By replacing $u(x_1, x')$ by $u(\varepsilon_1 x_1, x')$ in the above inequality we have

$$\varepsilon_1^k \|D_1^k D_{x'}^{2m-k} u\|_{L_p(\mathbb{R}_+^d)} \leq \varepsilon_1^{2m} N \|D_1^{2m} u\|_{L_p(\mathbb{R}_+^d)} + N \sum_{j=2}^d \|D_j^{2m} u\|_{L_p(\mathbb{R}_+^d)}.$$

The proposition is proved. \square

Lemma 7. *Let $T \in (-\infty, \infty]$. There exists $N = N(d, n, m, \delta)$ such that*

$$\sum_{|\alpha|=m} \|D^\alpha D_{x'}^m u\|_{L_2(\mathcal{O}_T^+)} + \lambda \|u\|_{L_2(\mathcal{O}_T^+)} \leq N \|u_t + (-1)^m \mathcal{L}_0 u + \lambda u\|_{L_2(\mathcal{O}_T^+)}$$

for all $\lambda \geq 0$ and $u \in W_2^{1,2m}(\mathcal{O}_T^+)$ satisfying

$$u(t, 0, x') = \dots = D_1^{m-1} u(t, 0, x') = 0 \tag{40}$$

on $(-\infty, T) \times \mathbb{R}^{d-1}$.

Proof. Define

$$f = u_t + (-1)^m D^\alpha (A^{\alpha\beta} D^\beta u) + \lambda u \tag{41}$$

in \mathcal{O}_T^+ . Then by Theorem 12

$$\lambda \|u\|_{L_2(\mathcal{O}_T^+)} \leq N \|f\|_{L_2(\mathcal{O}_T^+)}.$$

Now differentiate with respect to $x' \in \mathbb{R}^{d-1}$ both sides of (41) m times to get

$$(D_{x'}^m u)_t + (-1)^m D^\alpha (A^{\alpha\beta} D^\beta D_{x'}^m u) + \lambda D_{x'}^m u = D_{x'}^m f$$

in \mathcal{O}_T^+ . Note that $D_{x'}^m u$ satisfies (40). Thus by Theorem 12 again we have

$$\sum_{|\alpha|=m} \|D^\alpha D_{x'}^m u\|_{L_2(\mathcal{O}_T^+)} \leq N \|f\|_{L_2(\mathcal{O}_T^+)}.$$

The lemma is proved. \square

Lemma 8. *Let $T \in (-\infty, \infty]$ and $\lambda \geq 0$. There exists $N = N(d, n, m, \delta)$ such that, for $u \in W_2^{1,2m}(\mathcal{O}_T^+)$ satisfying (40),*

$$\|D_1^{2m}u\|_{L_2(\mathcal{O}_T^+)} \leq N \sum_{j=1}^{2m} \|D_1^{2m-j}D_{x^j}^j u\|_{L_2(\mathcal{O}_T^+)} + N\|f\|_{L_2(\mathcal{O}_T^+)} \quad (42)$$

provided that

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = f \quad (43)$$

in \mathcal{O}_T^+ .

Proof. By multiplying both sides of the equation (43) from the left by $D_1^{2m}u$ we get

$$\begin{aligned} \langle D_1^{2m}u, u_t \rangle_{\mathcal{O}_T^+} + (-1)^m \langle D_1^{2m}u, A^{\alpha\beta} D^\alpha D^\beta u \rangle_{\mathcal{O}_T^+} + \lambda \langle D_1^{2m}u, u \rangle_{\mathcal{O}_T^+} \\ = \langle D_1^{2m}u, f \rangle_{\mathcal{O}_T^+}. \end{aligned} \quad (44)$$

Note that

$$\Re(-1)^m \langle D_1^{2m}u, u_t \rangle_{\mathcal{O}_T^+} = \frac{1}{2} \int_{\mathbb{R}_+^d} |D_1^m u|^2(T, x) \, dx \geq 0. \quad (45)$$

Indeed, this holds true because

$$\begin{aligned} \int_{\mathbb{R}_+^d} |D_1^m u|^2(T, x) \, dx &= \int_{\mathcal{O}_T^+} \frac{\partial}{\partial t} |D_1^m u|^2 \, dx \, dt \\ &= \langle D_1^m u, D_1^m u_t \rangle_{\mathcal{O}_T^+} + \overline{\langle D_1^m u, D_1^m u_t \rangle_{\mathcal{O}_T^+}} \end{aligned}$$

and

$$\langle D_1^{2m}u, u_t \rangle_{\mathcal{O}_T^+} = (-1)^m \langle D_1^m u, D_1^m u_t \rangle_{\mathcal{O}_T^+},$$

the latter of which follows from the boundary condition (40) and integration by parts. Hence, by taking the real parts of (44) and using (45) we have

$$\begin{aligned} \Re \langle D_1^{2m}u, A^{\hat{\alpha}\hat{\alpha}} D_1^{2m}u \rangle_{\mathcal{O}_T^+} &\leq -\Re \sum_{(\alpha,\beta) \neq (\hat{\alpha},\hat{\alpha})} \langle D_1^{2m}u, A^{\alpha\beta} D^\alpha D^\beta u \rangle_{\mathcal{O}_T^+} \\ &\quad - (-1)^m \lambda \Re \langle D_1^{2m}u, u \rangle_{\mathcal{O}_T^+} + (-1)^m \Re \langle D_1^{2m}u, f \rangle_{\mathcal{O}_T^+}, \end{aligned}$$

where $\hat{\alpha} = (m, 0, \dots, 0)$. Thanks to the ellipticity condition and Young's inequality,

$$\begin{aligned} \delta \|D_1^{2m}u\|_{L_2(\mathcal{O}_T^+)}^2 &\leq \Re \langle D_1^{2m}u, A^{\hat{\alpha}\hat{\alpha}} D_1^{2m}u \rangle_{\mathcal{O}_T^+} \leq \varepsilon \|D_1^{2m}u\|_{L_2(\mathcal{O}_T^+)}^2 \\ &\quad + N(\varepsilon, \delta) \sum_{j=1}^{2m} \|D_1^{2m-j}D_{x^j}^j u\|_{L_2(\mathcal{O}_T^+)}^2 \\ &\quad + N(\varepsilon)\lambda^2 \|u\|_{L_2(\mathcal{O}_T^+)}^2 + N(\varepsilon)\|f\|_{L_2(\mathcal{O}_T^+)}^2. \end{aligned}$$

Choosing a sufficiently small ε and using Lemma 7, we prove (42). \square

Now we are ready to state and prove the main theorem of the section.

Theorem 13. *Let $T \in (-\infty, \infty]$. There exists $N = N(d, n, m, \delta)$ such that*

$$\|u_t\|_{L_2(\mathcal{O}_T^+)} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_2(\mathcal{O}_T^+)} \leq N \|u_t + (-1)^m \mathcal{L}_0 u + \lambda u\|_{L_2(\mathcal{O}_T^+)}$$

for all $\lambda \geq 0$ and $u \in W_2^{1,2m}(\mathcal{O}_T^+)$ satisfying

$$u(t, 0, x') = \dots = D_1^{m-1} u(t, 0, x') = 0$$

on $(-\infty, T) \times \mathbb{R}^{d-1}$.

Proof. Thanks to Lemma 7 and interpolation inequalities, it suffices to prove that

$$\|u_t\|_{L_2(\mathcal{O}_T^+)} + \|D^{2m} u\|_{L_2(\mathcal{O}_T^+)} \leq N \|f\|_{L_2(\mathcal{O}_T^+)}, \tag{46}$$

where $f = u_t + (-1)^m \mathcal{L}_0 u + \lambda u$. Lemma 8 and Proposition 1 (with $2m$ in place of m) imply that

$$\|D^{2m} u\|_{L_2(\mathcal{O}_T^+)} \leq N \|f\|_{L_2(\mathcal{O}_T^+)} + \varepsilon \|D_1^{2m} u\|_{L_2(\mathcal{O}_T^+)} + N \|D_{x'}^{2m} u\|_{L_2(\mathcal{O}_T^+)}.$$

This, along with Lemma 7 and a sufficiently small ε , proves the inequality (46) without the u_t term on the left-hand side. To complete the proof we simply note that

$$u_t = -(-1)^m \mathcal{L}_0 u - \lambda u + f.$$

□

7. Mean oscillation estimates of some partial derivatives of solutions to systems on a half space

The aim of this section is to derive several mean oscillation estimates of highest order derivatives of solutions to systems on a half space. Contrary to the whole space case, here we are able to estimate only parts of the highest order derivatives. More precisely, for divergence form systems, we give an estimate of $D_x^m u$, while for non-divergence form systems we present the estimate of $D_{x'}^{2m} u$. We emphasize that these estimates alone are not sufficient for proving Theorems 3 and 4.

We still denote

$$\mathcal{L}_0 u = \sum_{|\alpha|=|\beta|=m} D^\alpha A^{\alpha\beta}(t) D^\beta u = \sum_{|\alpha|=|\beta|=m} A^{\alpha\beta}(t) D^\alpha D^\beta u.$$

Recall that

$$\begin{aligned} Q_r^+(t, x) &= Q_r(t, x) \cap \mathcal{O}_\infty^+, & Q_r^+ &= Q_r \cap \mathcal{O}_\infty^+, \\ Q'_r &= (-r^{2m}, 0) \times B'_r, & B'_r &= \{x' \in \mathbb{R}^{d-1} : |x'| < r\}. \end{aligned}$$

7.1. Some auxiliary results for systems on a half space

We first prove some auxiliary estimates in this subsection. The first two are counterparts of Lemma 1 and Corollary 1.

Lemma 9. *Let $0 < r < R < \infty$. Assume that $u \in W_p^{1,2m}(Q_R^+)$ satisfies*

$$u(t, 0, x') = \dots = D_1^{m-1}u(t, 0, x') = 0 \tag{47}$$

on Q'_R and

$$u_t + (-1)^m \mathcal{L}_0 u = f$$

in Q_R^+ , where $f \in L_2(Q_R^+)$. Then there exists a constant $N = N(d, n, m, \delta)$ such that

$$\|u_t\|_{L_2(Q_r^+)} + \|D^{2m}u\|_{L_2(Q_r^+)} \leq N\|f\|_{L_2(Q_r^+)} + N(R-r)^{-2m}\|u\|_{L_2(Q_R^+)}.$$

Furthermore,

$$\|u\|_{W_2^{1,2m}(Q_r^+)} \leq N\|f\|_{L_2(Q_r^+)} + N\|u\|_{L_2(Q_R^+)},$$

where $N = N(d, n, m, \delta, r, R)$.

Proof. By Theorem 13 the L_2 -estimate of systems on a half spaces is available. Then the proof is the same as that of Lemma 1 with some minor changes. \square

Corollary 3. *Let $0 < r < R < \infty$. Assume that $u \in C_{loc}^\infty(\overline{Q_\infty^+})$ satisfies (47) on Q'_R and*

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \tag{48}$$

in Q_R^+ . Then for any multi-indices γ and ϑ such that

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_d), \quad \gamma_1 \leq 2m, \quad \vartheta = (0, \vartheta_2, \dots, \vartheta_d),$$

we have

$$\|D^\gamma u\|_{L_2(Q_r^+)} + \|D^\vartheta u_t\|_{L_2(Q_r^+)} \leq N\|u\|_{L_2(Q_R^+)},$$

where $N = N(d, n, m, \delta, r, R, \gamma, \vartheta)$.

Proof. From (48) it follows that

$$D^\vartheta u_t = -(-1)^m A^{\alpha\beta} D^\alpha D^\beta D^\vartheta u$$

in Q_R^+ . Each of the terms on the right-hand side is a constant times a term of the form $D^\gamma u$, where $|\gamma| = 2m + |\beta|$ and $\gamma_1 \leq 2m$. Hence we only need to prove

$$\|D^\gamma u\|_{L_2(Q_r^+)} \leq N\|u\|_{L_2(Q_R^+)},$$

where $\gamma = (\gamma_1, \dots, \gamma_d)$ satisfies $\gamma_1 \leq 2m$. The proof of this inequality is identical to that of (21) in Corollary 1 with the only difference that, in $|\gamma| > 2m$, we write

$$D^\gamma u = D^{2m} D^\vartheta u, \quad \vartheta = (0, \vartheta_2, \dots, \vartheta_d),$$

where $D^\vartheta u$ satisfies (48) in Q_R^+ as well as (47) on Q'_R , so that we can apply Lemma 9 to $D^\vartheta u$. \square

Next we derive a few Hölder estimates of solutions. Throughout the paper, for a function g defined on a subset \mathcal{D} in \mathbb{R}^{d+1} , we set

$$[g]_{C^v(\mathcal{D})} = \sup_{\substack{(t,x),(s,y) \in \mathcal{D} \\ (t,x) \neq (s,y)}} \frac{|g(t,x) - g(s,y)|}{|t-s|^{\frac{v}{2}} + |x-y|^v},$$

where $0 < v \leq 1$.

Lemma 10. *If $u \in C^\infty_{loc}(\overline{O_\infty^+})$ satisfies (47) on Q'_4 and (48) in Q_4^+ , then*

$$[u]_{C^{1/2}(Q_4^+)} \leq N \|u\|_{L_2(Q_4^+)}.$$

Proof. Let

$$\Theta_r^+ = \{(t, x_1) \in (-r^{2m}, 0) \times (0, r)\}, \quad B'_r = \{x' \in \mathbb{R}^{d-1} : |x'| < r\}.$$

The triangle inequality gives

$$\begin{aligned} & \sup_{\substack{(t,x),(s,y) \in Q_1^+ \\ (t,x) \neq (s,y)}}} \frac{|u(t,x) - u(s,y)|}{|t-s|^{1/4} + |x-y|^{1/2}} \\ & \leq \sup_{\substack{(t,x_1),(s,y_1) \in \Theta_1^+ \\ x' \in B'_1}} \frac{|u(t,x_1,x') - u(s,y_1,x')|}{|t-s|^{1/4} + |x_1 - y_1|^{1/2}} \\ & \quad + \sup_{\substack{(s,y_1) \in \Theta_1^+ \\ x',y' \in B'_1, x' \neq y'}} \frac{|u(s,y_1,x') - u(s,y_1,y')|}{|x' - y'|^{1/2}} := I_1 + I_2. \end{aligned}$$

To estimate I_1 , we view $u(t, x_1, x')$ as a function of (t, x_1) for a fixed $x' \in B'_1$. Then by the Sobolev embedding theorem

$$\sup_{(t,x_1),(s,y_1) \in \Theta_1^+} \frac{|u(t, x_1, x') - u(s, y_1, x')|}{|t-s|^{1/4} + |x_1 - y_1|^{1/2}} \leq N \|u(\cdot, x')\|_{W_2^{1,2}(\Theta_1^+)} \tag{49}$$

for each $x' \in B'_1$. On the other hand, there exists a positive integer k such that, for each $(t, x_1) \in \Theta_1^+$,

$$\begin{aligned} & \sum_{j=0}^2 \sup_{x' \in B'_1} |D_1^j u(t, x_1, x')| + \sup_{x' \in B'_1} |u_t(t, x_1, x')| \\ & \leq N \sum_{j=0}^2 \|D_1^j u(t, x_1, \cdot)\|_{W_2^k(B'_1)} + N \|u_t(t, x_1, \cdot)\|_{W_2^k(B'_1)}, \end{aligned} \tag{50}$$

where $D_1^j u(t, x_1, x')$ and $u_t(t, x_1, x')$ are viewed as functions of $x' \in B'_1$. Combining (49) and (50) proves

$$I_1 \leq N \sum_{\substack{|\gamma| \leq k+2 \\ \gamma_1 \leq 2}} \|D^\gamma u\|_{L_2(Q_2^+)} + N \sum_{\substack{|\vartheta| \leq k \\ \vartheta_1 = 0}} \|D^\vartheta u_t\|_{L_2(Q_2^+)} \leq N \|u\|_{L_2(Q_4^+)}, \tag{51}$$

where the last inequality is due to Corollary 3.

For the estimate of I_2 , we look at $u(s, y_1, x')$ as a function of $x' \in B'_1$ for each $(s, y_1) \in \Theta_1^+$. Again, by the Sobolev embedding theorem, there exists a sufficiently large integer k such that

$$\sup_{\substack{x', y' \in B'_1 \\ x' \neq y'}} \frac{|u(s, y_1, x') - u(s, y_1, y')|}{|x' - y'|^{1/2}} \leq N \|u(s, y_1, \cdot)\|_{W_2^k(B'_1)}.$$

Moreover, as a function of $(s, y_1) \in \Theta_1^+$, $D_{x'}^j u(s, y_1, x')$, $j = 0, \dots, k$, satisfy

$$\sum_{j=0}^k \sup_{(s, y_1) \in \Theta_1^+} |D_{x'}^j u(s, y_1, x')| \leq N \sum_{j=0}^k \|D_{x'}^j u(\cdot, x')\|_{W_p^{1,2}(\Theta_1^+)}$$

for each $x' \in B'_1$. From the above two inequalities, we obtain (51) with I_2 in place of I_1 . The lemma is proved. \square

In the sequel, for a function g defined on \mathcal{O}_T^+ , $T \in (-\infty, \infty]$, we denote by $\mathcal{E}(g)$ ($= \mathcal{E}g$) the even extension of g defined on \mathcal{O}_T .

Corollary 4. *Let $\lambda \geq 0$, $X_0 = (t_0, 0, x'_0)$, where $t_0 \in \mathbb{R}$ and $x'_0 \in \mathbb{R}^{d-1}$. Assume that $u \in C_{loc}^\infty(\overline{\mathcal{O}_\infty^+})$ satisfies (47) on $Q'_4(X_0) = (t_0 - 4^{2m}, t_0) \times B'_4(x'_0)$ and*

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = 0 \tag{52}$$

in $Q_4^+(X_0)$. Then there exists $N = N(d, n, m, \delta)$ such that

$$[\mathcal{E}(D_{x'}^m u)]_{C^{1/2}(Q_1(X_0))} + \lambda^{1/2} [\mathcal{E}u]_{C^{1/2}(Q_1(X_0))} \leq N \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} \|\mathcal{E}(D^k u)\|_{L_2(Q_4(X_0))}.$$

Proof. By using a translation in t and x' , we assume that $X_0 = (0, 0)$. Let $\lambda = 0$. In this case, the inequality in the corollary follows from

$$[D_{x'}^m u]_{C^{1/2}(Q_4^+)} \leq N \|D^m u\|_{L_2(Q_4^+)}.$$

To prove this, we apply Lemma 10 to $D_{x'}^m u$. This is indeed possible because $D_{x'}^m u$ satisfies (47) on Q'_4 and (48) in Q_4^+ . To prove the case $\lambda > 0$, we follow the steps in the proof of Lemma 3. \square

Lemma 11. *Let $r \in (0, \infty)$, $\kappa \in [64, \infty)$, $\lambda \geq 0$, and $X_0 = (t_0, x_0) \in \overline{\mathcal{O}_\infty^+}$. Assume that $u \in C_{loc}^\infty(\overline{\mathcal{O}_\infty^+})$ satisfies (47) on $\mathbb{R} \times \mathbb{R}^{d-1}$ and (52) in $Q_{\kappa r}^+(X_0)$. Then*

$$\begin{aligned} & (|\mathcal{E}(D_{x'}^m u) - (\mathcal{E}(D_{x'}^m u))_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda^{\frac{1}{2}} (|\mathcal{E}u - (\mathcal{E}u)_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N\kappa^{-\frac{1}{2}} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}, \end{aligned} \tag{53}$$

where $N = N(d, n, m, \delta)$.

Proof. We first prove, using a scaling argument, that it suffices to prove the inequality (53) only for $r = 16/\kappa$. Indeed, assume that the inequality (53) holds true for $r = 16/\kappa$. For a given $r \in (0, \infty)$, let $r_0 = 16/\kappa$, $R = r/r_0$, and $w(t, x) = u(R^{2m}t, Rx)$. It is easy to check that w satisfies (47) on $\mathbb{R} \times \mathbb{R}^{d-1}$ and

$$w_t + (-1)^m A^{\alpha\beta} (R^{2m}t) D^\alpha D^\beta w + \lambda R^{2m} w = 0 \tag{54}$$

in $Q_{\kappa r_0}^+(Y_0)$, where $Y_0 = (s_0, y_0) = (R^{-2m}t_0, R^{-1}x_0) \in \overline{\mathcal{O}_\infty^+}$. Then by the inequality (53) with $r = r_0$ applied to the system (54), we have

$$\begin{aligned} & (|\mathcal{E}(D_{x'}^m w) - (\mathcal{E}(D_{x'}^m w))_{Q_{r_0}(Y_0)}|)_{Q_{r_0}(Y_0)} + \lambda^{\frac{1}{2}} R^m (|\mathcal{E}w - (\mathcal{E}w)_{Q_{r_0}(Y_0)}|)_{Q_{r_0}(Y_0)} \\ & \leq N\kappa^{-\frac{1}{2}} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} R^{m-k} (|\mathcal{E}(D^k w)|^2)_{Q_{\kappa r_0}(Y_0)}^{\frac{1}{2}}. \end{aligned} \tag{55}$$

Note that, for example,

$$(\mathcal{E}(D^k w))_{Q_{r_0}(Y_0)} = R^k (\mathcal{E}(D^k u))_{Q_r(X_0)}.$$

Thus the inequality (55) leads to the inequality (53) for arbitrary $r \in (0, \infty)$.

Now we assume $r = 16/\kappa$. We consider two cases.

Case 1: the first coordinate of $x_0 \geq 1$. In this case, we see that $Q_{\kappa r/16}^+(X_0) = Q_{\kappa r/16}(X_0)$ and u satisfies the assumptions in Lemma 4, especially, u satisfies (52) in $Q_{\kappa r/16}(X_0)$ and u can be extended to a function in $\mathcal{H}_{2,loc}^m(\mathbb{R}^{d+1})$ without changing the values of u on $Q_{\kappa r/16}(X_0)$. Hence, by the inequality (26) with $Q_{\kappa r/16}(X_0)$ in place of $Q_{\kappa r}(X_0)$ (note that $\kappa/16 \geq 4$), the left-hand side of (53) is controlled by

$$N\kappa^{-1} \sum_{k=0}^n \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k u|^2)_{Q_{\kappa r/16}(X_0)}^{\frac{1}{2}},$$

which is less than the right-hand side of (53).

Case 2: the first coordinate of x_0 is in $[0, 1]$. By denoting $Y_0 = (t_0, 0, x'_0)$, we have

$$Q_r(X_0) \subset Q_2(Y_0) \subset Q_8(Y_0) \subset Q_{\kappa r}(X_0).$$

By Corollary 4 applied to u with 2 and 8 in place of 1 and 4, respectively (this case can be seen using a scaling argument as above), we have

$$\begin{aligned} & (|\mathcal{E}(D_x^m u) - (\mathcal{E}(D_{x'}^m u))_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N r^{\frac{1}{2}} [\mathcal{E}(D_x^m u)]_{C^{1/2}(Q_2(Y_0))} \\ & \leq N \kappa^{-\frac{1}{2}} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_8(Y_0)}^{\frac{1}{2}} \\ & \leq N \kappa^{-\frac{1}{2}} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}. \end{aligned}$$

The second term on the left-hand side of (53) can be estimated similarly. \square

7.2. Mean oscillation estimates of $D_{x'}^m u$ for divergence type systems on a half space

Now we state and prove the main result of this section.

Proposition 2. Let $r \in (0, \infty)$, $\kappa \in [128, \infty)$, $\lambda \geq 0$, and $X_0 = (t_0, x_0) \in \overline{\mathcal{O}_\infty^+}$. Assume that $u \in \mathcal{H}_{2,loc}^m(\mathcal{O}_\infty^+)$ satisfies (47) on $\mathbb{R} \times \mathbb{R}^{d-1}$ and

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha \tag{56}$$

in $Q_{\kappa r}^+(X_0)$, where $f_\alpha \in L_{2,loc}(\mathcal{O}_\infty^+)$, $|\alpha| \leq m$. Then we have

$$\begin{aligned} & (|\mathcal{E}(D_x^m u) - (\mathcal{E}(D_{x'}^m u))_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda^{\frac{1}{2}} (|\mathcal{E}u - (\mathcal{E}u)_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N \kappa^{-\frac{1}{2}} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} \\ & \quad + N \kappa^{m + \frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|\mathcal{E}f_\alpha|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}, \end{aligned} \tag{57}$$

where $N = N(d, n, m, \delta)$.

Proof. Multiplying u by an infinitely smooth function as ζ below, we see that (56) can be extended to a system defined on \mathcal{O}_∞^+ without changing the values of u and f_α on, for example, $Q_{\kappa r/2}$. Thus, without loss of generality we assume that $u \in \mathcal{H}_2^m(\mathcal{O}_\infty^+)$, $f_\alpha \in L_2(\mathcal{O}_\infty^+)$, and (56) is satisfied in \mathcal{O}_∞^+ . We consider only $\lambda > 0$.

Take a $\zeta \in C_0^\infty(Q_{\kappa r}(X_0))$ such that

$$\zeta = 1 \text{ on } Q_{\kappa r/2}(X_0), \quad \zeta = 0 \text{ outside } (t_0 - (\kappa r)^{2m}, t_0 + (\kappa r)^{2m}) \times B_{\kappa r}(x_0).$$

Let $\mathcal{L}_0^{(\varepsilon)} = A_{(\varepsilon)}^{\alpha\beta} D^\alpha D^\beta$, where $A_{(\varepsilon)}^{\alpha\beta}$ are the standard mollifications with respect to t of $A^{\alpha\beta}(t)$. Also let $f_\alpha^{(\varepsilon)}$ be infinitely differentiable functions approaching f_α in

$L_2(\mathcal{O}_\infty^+)$ as $\varepsilon \searrow 0$. By Theorem 12, there exists a unique solution $v^{(\varepsilon)} \in \mathcal{H}_2^m(\mathcal{O}_\infty^+)$, satisfying (47) on $\mathbb{R} \times \mathbb{R}^{d-1}$, to the equation

$$v_t^{(\varepsilon)} + (-1)^m \mathcal{L}_0^{(\varepsilon)} v^{(\varepsilon)} + \lambda v^{(\varepsilon)} = \sum_{|\alpha| \leq m} D^\alpha ((1 - \zeta) f_\alpha^{(\varepsilon)})$$

in \mathcal{O}_∞^+ . Since $f_\alpha^{(\varepsilon)}$ and $A_{(\varepsilon)}^{\alpha\beta}$ are infinitely differentiable, by the classical theory for higher order parabolic systems, $v^{(\varepsilon)}$ is infinitely differentiable. Moreover, for any ε ,

$$v_t^{(\varepsilon)} + (-1)^m \mathcal{L}_0^{(\varepsilon)} v^{(\varepsilon)} + \lambda v^{(\varepsilon)} = 0 \quad \text{in } Q_{\kappa r/2}^+(X_0).$$

Thus by Lemma 11 (note that $\kappa/2 \geq 64$)

$$\begin{aligned} & (|\mathcal{E}(D_x^m v^{(\varepsilon)}) - (\mathcal{E}(D_x^m v^{(\varepsilon)}))_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda^{\frac{1}{2}} (|\mathcal{E}v^{(\varepsilon)} - (\mathcal{E}v^{(\varepsilon)})_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N \kappa^{-\frac{1}{2}} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|\mathcal{E}(D^k v^{(\varepsilon)})|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}. \end{aligned}$$

Set $w^{(\varepsilon)} = u - v^{(\varepsilon)}$. Then $w^{(\varepsilon)} \in \mathcal{H}_2^m(\mathcal{O}_\infty^+)$ and it satisfies (47) on $\mathbb{R} \times \mathbb{R}^{d-1}$ and

$$w_t^{(\varepsilon)} + (-1)^m \mathcal{L}_0^{(\varepsilon)} w^{(\varepsilon)} + \lambda w^{(\varepsilon)} = D^\alpha (\zeta f_\alpha^{(\varepsilon)} + f_\alpha - f_\alpha^{(\varepsilon)}) + (-1)^m (\mathcal{L}_0^{(\varepsilon)} - \mathcal{L}_0)u$$

in \mathcal{O}_∞^+ . Denote the right-hand side of the above equality by $D^\alpha g_\alpha^{(\varepsilon)}$. We apply Theorem 12 to the above equation as one defined on $\mathcal{O}_{t_0}^+$ so that we have

$$\sum_{|\alpha| \leq m} \lambda^{1 - \frac{|\alpha|}{2m}} \|D^\alpha w^{(\varepsilon)}\|_{L_2(\mathcal{O}_{t_0}^+)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|g_\alpha^{(\varepsilon)}\|_{L_2(\mathcal{O}_{t_0}^+)}.$$

In particular,

$$\begin{aligned} & \|D^m w^{(\varepsilon)}\|_{L_2(Q_r^+(X_0))} + \lambda^{\frac{1}{2}} \|w^{(\varepsilon)}\|_{L_2(Q_r^+(X_0))} \\ & \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} \|f_\alpha^{(\varepsilon)}\|_{L_2(Q_{\kappa r}^+(X_0))} + I^{(\varepsilon)}, \end{aligned} \tag{58}$$

$$\sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} \|D^k w^{(\varepsilon)}\|_{L_2(Q_{\kappa r}^+(X_0))} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} \|f_\alpha^{(\varepsilon)}\|_{L_2(Q_{\kappa r}^+(X_0))} + I^{(\varepsilon)} \tag{59}$$

for all sufficiently small ε , where

$$\begin{aligned} I^{(\varepsilon)} &= N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} \|f_\alpha - f_\alpha^{(\varepsilon)}\|_{L_2(\mathcal{O}_{t_0}^+)} \\ & \quad + N \sum_{|\alpha| = |\beta| = m} \|(A_{(\varepsilon)}^{\alpha\beta} - A^{\alpha\beta}) D^\beta u\|_{L_2(\mathcal{O}_{t_0}^+)}. \end{aligned}$$

Note that, for the even extension $\mathcal{E}g$ of a function g defined on \mathcal{O}_∞^+ , we have

$$\|\mathcal{E}g\|_{L_2(Q_r(X_0))} \leq 2\|g\|_{L_2(Q_r^+(X_0))} \leq 2\|\mathcal{E}g\|_{L_2(Q_r(X_0))}$$

whenever $X_0 \in \overline{\mathcal{O}_\infty^+}$. This combined with (58) and (59) gives

$$\begin{aligned} & \left(|\mathcal{E}(D^m w^{(\varepsilon)})|^2 \right)_{Q_r(X_0)}^{\frac{1}{2}} + \lambda^{\frac{1}{2}} \left(|\mathcal{E}w^{(\varepsilon)}|^2 \right)_{Q_r(X_0)}^{\frac{1}{2}} \\ & \leq N\kappa^{m+\frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} (|\mathcal{E}f_\alpha^{(\varepsilon)}|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} + r^{-m-\frac{d}{2}} I^{(\varepsilon)}, \\ & \sum_{k=0}^m \lambda^{\frac{1}{2}-\frac{k}{2m}} \left(|\mathcal{E}(D^k w^{(\varepsilon)})|^2 \right)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} \\ & \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} (|\mathcal{E}f_\alpha^{(\varepsilon)}|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} + (\kappa r)^{-m-\frac{d}{2}} I^{(\varepsilon)}. \end{aligned}$$

Now by following the corresponding steps in the proof of Theorem 11 we see that the left-hand side of the inequality (57) is less than the right-hand side of the same inequality with $f_\alpha^{(\varepsilon)}$ in place of f_α plus the error term

$$\left(r^{-m-\frac{d}{2}} + (\kappa r)^{-m-\frac{d}{2}} \right) I^{(\varepsilon)}.$$

To finish the proof we let $\varepsilon \searrow 0$. \square

Remark 5. Later we need to have the mean oscillation estimate (57) for all $X_0 \in \mathcal{O}_\infty$, instead of $X_0 \in \overline{\mathcal{O}_\infty^+}$, for functions $\mathcal{E}(D^k u)$, $\mathcal{E}u$, and $\mathcal{E}f_\alpha$ defined on \mathcal{O}_∞ if the equation (56) is satisfied in \mathcal{O}_∞^+ . In order to do this, in case $X_0 \in \mathcal{O}_\infty \setminus \overline{\mathcal{O}_\infty^+}$, we let Y_0 be the reflection point of X_0 with respect to the hyper-plane $\{(t, 0, x') : t \in \mathbb{R}, x' \in \mathbb{R}^{d-1}\}$. By Proposition 2 we get the estimate (57) with Y_0 in place of X_0 . Then it is not difficult to see that the estimate (57) holds true as well for X_0 using the evenness of functions involved. The same claim can be repeated for Corollary 5, Propositions 3 and 4.

7.3. Mean oscillation estimates of $D_{x'}^{2m}u$ for non-divergence type systems on a half space

As a consequence of Proposition 2, we easily get

Corollary 5. Let $r \in (0, \infty)$, $\kappa \in [64, \infty)$, $\lambda \geq 0$, and $X_0 = (t_0, x_0) \in \overline{\mathcal{O}_\infty^+}$. Assume that $u \in W_{2,loc}^{1,2m}(\mathcal{O}_\infty^+)$ satisfies (47) on $\mathbb{R} \times \mathbb{R}^{d-1}$ and

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = f$$

in $Q_{\kappa r}^+(X_0)$, where $f \in L_{2,loc}(\mathcal{O}_\infty^+)$. Then we have

$$\begin{aligned} & (|\mathcal{E}(D_{x'}^{2m}u) - (\mathcal{E}(D_{x'}^{2m}u))_{Q_r(X_0)}|)_{Q_r(X_0)} + \lambda(|\mathcal{E}u - (\mathcal{E}u)_{Q_r(X_0)}|)_{Q_r(X_0)} \\ & \leq N\kappa^{-\frac{1}{2}} \sum_{k=0}^{2m} \lambda^{1-\frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} + N\kappa^{m+\frac{d}{2}} (|\mathcal{E}f|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}, \end{aligned}$$

where $N = N(d, n, m, \delta)$.

Proof. Since $D_{x'}^m$ satisfies (47) on $\mathbb{R} \times \mathbb{R}^{d-1}$, we can proceed as in the proof of Corollary 2. \square

8. Estimates for systems with special coefficients on a half space

The estimates in the previous section imply the L_p -estimate of $D_{x'}^m u$ in the divergence case and that of $D_{x'}^{2m} u$ in the non-divergence case. In order to estimate the remaining highest order derivatives, by the interpolation inequality (Proposition 1), it suffices to estimate $D_1^m u$ in the divergence case and $D_1^{2m} u$ in the non-divergence case. To this end, in this section we consider

$$\mathfrak{L}_0 u = A(t)D_1^{2m}u + \sum_{j=2}^d D_j^{2m}u,$$

where $A(t) = A^{\hat{\alpha}\hat{\alpha}}(t)$, $\hat{\alpha} = (m, 0, \dots, 0)$.

For this special operator, we have the following improved L_2 -estimate.

Lemma 12. *Assume that $u \in C_{loc}^\infty(\overline{\mathcal{O}_\infty^+})$ satisfies*

$$u(t, 0, x') = \dots = D_1^{m-1}u(t, 0, x') = 0 \tag{60}$$

on Q'_R and

$$u_t + (-1)^m \mathfrak{L}_0 u = 0 \tag{61}$$

in Q_R^+ . Then, for any multi-index γ , we have

$$\|D^\gamma u\|_{L_2(Q_r^+)} + \|D^\gamma u_t\|_{L_2(Q_r^+)} \leq N\|u\|_{L_2(Q_R^+)}, \tag{62}$$

where $N = N(d, n, m, \delta, r, R, \gamma)$.

Proof. As noted in the proof of Corollary 1, it suffices to estimate the first term on the left-hand side of (62). Also, we only need to treat the case when the multi-index γ satisfies $\gamma' = 0$, where $\gamma = (\gamma_1, \gamma')$. In fact, if the inequality (62) is shown to be true with $\gamma' = 0$ and a smaller R , since $D^{\gamma'} u$ satisfies (60) on Q'_R and (61) in Q_R^+ , we can replace u by $D^{\gamma'} u$ in (62). Then the right-hand side, $N\|D^{\gamma'} u\|_{L_2(Q_R^+)}$, is bounded by that of (62) by Corollary 3. Furthermore, by the interpolation inequality with respect to x_1 , it suffices to show

$$\|D_1^{2m} u\|_{L_2(Q_r^+)} \leq N\|u\|_{L_2(Q_R^+)} \tag{63}$$

for $l = 0, 1, 2, \dots$. To prove the above inequality, we first observe that, thanks to (61), we have

$$D_1^{2m}u = A^{-1}(t)(-1)^{m+1}u_t - A^{-1}(t) \sum_{j=2}^d D_j^{2m}u$$

in Q_R^+ . This together with (60) implies that (first with $l = 0$, then inductively)

$$D_1^k D_1^{2ml}u(t, 0, x') = 0, \quad k = 0, \dots, m - 1,$$

on Q'_R . Moreover, $D_1^{2ml}u$ satisfies (61) in Q_R^+ . Therefore, by Corollary 3 applied to $D_1^{2ml}u$ we have

$$\|D_1^{2(l+1)m}u\|_{L_2(Q_r^+)} \leq N \|D_1^{2lm}u\|_{L_2(Q_{r_0}^+)},$$

where $r < r_0 < R$. This implies (63) by an induction on l . \square

As a consequence of the previous lemma, we get

Lemma 13. *Let $u \in C_{loc}^\infty(\overline{O_\infty^+})$ satisfy (60) on Q'_4 and (61) in Q_4^+ . Then, for any multi-index γ ,*

$$\sup_{Q_1^+} |D^\gamma u| + \sup_{Q_1^+} |D^\gamma u_t| \leq N \|u\|_{L_2(Q_4^+)},$$

where $N = N(d, n, m, \delta, \gamma)$.

Proof. This is deduced from Lemma 12 in the same way as Lemma 2 is deduced from Corollary 1. \square

Note that in the following Hölder estimates the first inequality is for all $D^\gamma u$, $|\gamma| = m$, whereas the second inequality is for $D_1^{2m}u$ only. Similarly, we see $D^m u$ and $D_1^{2m}u$ in the following lemma and propositions as well.

Corollary 6. *Let $\lambda \geq 0$, $X_0 = (t_0, 0, x'_0)$, where $t_0 \in \mathbb{R}$ and $x'_0 \in \mathbb{R}^{d-1}$. Assume that $u \in C_{loc}^\infty(\overline{O_\infty^+})$ satisfies (60) on $Q'_4(X_0)$ and*

$$u_t + (-1)^m \mathfrak{L}_0 u + \lambda u = 0 \tag{64}$$

in $Q_4^+(X_0)$. Then there exists $N = N(d, n, m, \delta)$ such that

$$[\mathcal{E}(D^m u)]_{C^1(Q_1(X_0))} \leq N \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} \|\mathcal{E}(D^k u)\|_{L_2(Q_4(X_0))}, \tag{65}$$

$$[\mathcal{E}(D_1^{2m} u)]_{C^1(Q_1(X_0))} \leq N \sum_{k=0}^{2m} \lambda^{1 - \frac{k}{2m}} \|\mathcal{E}(D^k u)\|_{L_2(Q_4(X_0))}. \tag{66}$$

Proof. Similar to the proof of Corollary 4, we prove only the case $\lambda = 0$ and $X_0 = (0, 0)$. As noted in the proof of Lemma 12, $D_1^{2m}u$ satisfies (60) on Q'_4 and (61) in Q_4^+ . In this case, (66) follows immediately from Lemma 13 applied to $D_1^{2m}u$.

Lemma 13 also shows that

$$[D^m u]_{C^1(Q_1^+)} \leq N \|u\|_{L_2(Q_4^+)} \leq N \|D^m u\|_{L_2(Q_4^+)},$$

where the second inequality is due to the fact that u satisfies (60) and the boundary version of the Poincaré inequality. This gives the inequality (65). \square

Lemma 14. *Let $r \in (0, \infty)$, $\kappa \in [64, \infty)$, $\lambda \geq 0$, and $X_0 = (t_0, x_0) \in \overline{\mathcal{O}_\infty^+}$. Assume that $u \in C_{loc}^\infty(\overline{\mathcal{O}_\infty^+})$ satisfies (60) on $\mathbb{R} \times \mathbb{R}^{d-1}$ and (64) on $Q_{\kappa r}^+(X_0)$. Then*

$$\begin{aligned} (|\mathcal{E}(D^m u) - (\mathcal{E}(D^m u))_{Q_r(X_0)}|)_{Q_r(X_0)} &\leq N \kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}, \\ (|\mathcal{E}(D_1^{2m} u) - (\mathcal{E}(D_1^{2m} u))_{Q_r(X_0)}|)_{Q_r(X_0)} &\leq N \kappa^{-1} \sum_{k=0}^{2m} \lambda^{1 - \frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}, \end{aligned}$$

where $N = N(d, n, m, \delta)$.

Proof. Thanks to the Hölder estimates in Corollary 6, we proceed as in the proof of Lemma 11. \square

From the above lemma, by following the steps in the proof of Proposition 2 we prove the following two propositions.

Proposition 3. *Let $r \in (0, \infty)$, $\kappa \in [128, \infty)$, $\lambda \geq 0$, and $X_0 = (t_0, x_0) \in \overline{\mathcal{O}_\infty^+}$. Assume that $u \in C_{loc}^\infty(\overline{\mathcal{O}_\infty^+})$ satisfies (60) on $\mathbb{R} \times \mathbb{R}^{d-1}$ and*

$$u_t + (-1)^m \mathfrak{L}_0 u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$$

in $Q_{\kappa r}^+(X_0)$, where $f_\alpha \in L_{2,loc}(\overline{\mathcal{O}_\infty^+})$, $|\alpha| \leq m$. Then we have

$$\begin{aligned} (|\mathcal{E}(D^m u) - (\mathcal{E}(D^m u))_{Q_r(X_0)}|)_{Q_r(X_0)} &\leq N \kappa^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} \\ &\quad + N \kappa^{m + \frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|\mathcal{E} f_\alpha|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}, \end{aligned}$$

where $N = N(d, n, m, \delta)$.

Proposition 4. Let $r \in (0, \infty)$, $\kappa \in [128, \infty)$, $\lambda \geq 0$, and $X_0 = (t_0, x_0) \in \overline{\mathcal{O}_\infty^+}$. Assume that $u \in C_{loc}^\infty(\overline{\mathcal{O}_\infty^+})$ satisfies (60) on $\mathbb{R} \times \mathbb{R}^{d-1}$ and

$$u_t + (-1)^m \mathfrak{L}_0 u + \lambda u = f$$

in $Q_{\kappa r}^+(X_0)$, where $f \in L_{2,loc}(\mathcal{O}_\infty^+)$. Then we have

$$\begin{aligned} (|\mathcal{E}(D_1^{2m}u) - (\mathcal{E}(D_1^{2m}u))_{Q_r(X_0)}|)_{Q_r(X_0)} &\leq N\kappa^{-1} \sum_{k=0}^{2m} \lambda^{1-\frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}} \\ &\quad + N\kappa^{m+\frac{d}{2}} (|\mathcal{E}f|^2)_{Q_{\kappa r}(X_0)}^{\frac{1}{2}}, \end{aligned}$$

where $N = N(d, n, m, \delta)$.

9. L_p -Estimates for systems on a half space

With the preparation in the previous two sections, we complete the proofs of Theorems 3 and 4 in this section.

Proof of Theorem 3. Recall that the leading coefficients satisfy Assumption 1 (ρ). As before, we may assume that $T = \infty$, $p > 2$, the lower-order coefficients of \mathcal{L} are all zero, and $u \in C^\infty(\overline{\mathcal{O}_\infty^+})$ vanishing on $\mathcal{O}_\infty^+ \setminus Q_{R_0}(X_1)$ for some $X_1 \in \overline{\mathcal{O}_\infty^+}$. In this case, it follows from Proposition 2 (also see Remark 5) and the proofs of Lemma 5 as well as Theorem 1 that

$$\begin{aligned} &\|D_{x'}^m u\|_{L_p(\mathcal{O}_\infty^+)} + \lambda^{\frac{1}{2}} \|u\|_{L_p(\mathcal{O}_\infty^+)} \\ &\leq \|\mathcal{E}(D_x^m u)\|_{L_p(\mathcal{O}_\infty)} + \lambda^{\frac{1}{2}} \|\mathcal{E}u\|_{L_p(\mathcal{O}_\infty)} \\ &\leq N\kappa_1^{m+\frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} \|\mathcal{E}f_\alpha\|_{L_p(\mathcal{O}_\infty)} \\ &\quad + N \left(\kappa_1^{-\frac{1}{2}} + \kappa_1^{m+\frac{d}{2}} \rho^{\frac{1}{2v}} \right) \sum_{k=0}^m \lambda^{\frac{1}{2}-\frac{k}{2m}} \|\mathcal{E}(D^k u)\|_{L_p(\mathcal{O}_\infty)} \\ &\leq N\kappa_1^{m+\frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} \|f_\alpha\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N \left(\kappa_1^{-\frac{1}{2}} + \kappa_1^{m+\frac{d}{2}} \rho^{\frac{1}{2v}} \right) \sum_{k=0}^m \lambda^{\frac{1}{2}-\frac{k}{2m}} \|D^k u\|_{L_p(\mathcal{O}_\infty^+)} \end{aligned} \tag{67}$$

for any $\kappa_1 \geq 128$.

Now we move all the spatial derivatives except $D_1^m(A^{\hat{\alpha}\hat{\alpha}}D_1^m u)$ to the right-hand side of (5), and add $(-1)^m \sum_{j=2}^d D_j^{2m} u$ to both sides. Here $\hat{\alpha} = (m, 0, \dots, 0)$. Then

for any $Q_{\kappa_2 r}(X_0)$, $\kappa_2 \in [128, \infty)$, $r \in (0, \infty)$, $X_0 \in \overline{\mathcal{O}}_\infty^+$ and $y \in \mathbb{R}_+^d$, we have

$$\begin{aligned} & u_t + (-1)^m \left(D_1^m (A^{\hat{\alpha}\hat{\alpha}}(t, y) D_1^m u) + \sum_{j=2}^d D_j^{2m} u \right) \\ &= \sum_{|\alpha| \leq m} D^\alpha \tilde{f}_\alpha + (-1)^m \sum_{j=2}^d D_j^m D_j^m u, \end{aligned}$$

where $\tilde{f}_\alpha = f_\alpha$ for $|\alpha| < m$,

$$\begin{aligned} \tilde{f}_{\hat{\alpha}} &= f_{\hat{\alpha}} - \sum_{\substack{|\beta|=m \\ \beta \neq \hat{\alpha}}} (-1)^m (A^{\hat{\alpha}\beta}(t, x) + A^{\beta\hat{\alpha}}(t, y)) D^\beta u \\ &\quad + (-1)^m (A^{\hat{\alpha}\hat{\alpha}}(t, y) - A^{\hat{\alpha}\hat{\alpha}}(t, x)) D^{\hat{\alpha}} u, \end{aligned}$$

and

$$\tilde{f}_\alpha = f_\alpha - \sum_{\substack{|\beta|=m \\ \beta \neq \hat{\alpha}}} (-1)^m A^{\alpha\beta} D^\beta u + (-1)^m (A^{\alpha\hat{\alpha}}(t, y) - A^{\alpha\hat{\alpha}}(t, x)) D^{\hat{\alpha}} u$$

for $|\alpha| = m$, $\alpha \neq \hat{\alpha}$. In the last two expressions, we used the fact that

$$D_x^\alpha A^{\alpha\beta}(t, y) D_x^\beta u(t, x) = D_x^\beta A^{\alpha\beta}(t, y) D_x^\alpha u(t, x).$$

As a consequence of Proposition 3 and the proof of Lemma 5, for any $\kappa_2 \geq 128$,

$$\begin{aligned} (|\mathcal{E}(D^m u) - (\mathcal{E}(D^m u))_{Q_r(X_0)}|)_{Q_r(X_0)} &\leq N\kappa_2^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa_2 r}(X_0)}^{\frac{1}{2}} \\ &\quad + N\kappa_2^{m + \frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|\mathcal{E} f_\alpha|^2)_{Q_{\kappa_2 r}(X_0)}^{\frac{1}{2}} \\ &\quad + N\kappa_2^{m + \frac{d}{2}} \rho^{\frac{1}{2\nu}} (|\mathcal{E}(D_1^m u)|^{2\mu})_{Q_{\kappa_2 r}(X_0)}^{\frac{1}{2\mu}} \\ &\quad + N\kappa_2^{m + \frac{d}{2}} \sum_{|\alpha|=m, \alpha \neq \hat{\alpha}} (|\mathcal{E}(D^\alpha u)|^2)_{Q_{\kappa_2 r}(X_0)}^{\frac{1}{2}}. \end{aligned}$$

Choose a $\mu \in (1, p/2)$. This estimate, combined with the Fefferman–Stein theorem and the Hardy–Littlewood maximal function theorem, gives

$$\begin{aligned} \|D^m u\|_{L_p(\mathcal{O}_\infty^+)} &\leq N\kappa_2^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} \|D^k u\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N\kappa_2^{m + \frac{d}{2}} \rho^{\frac{1}{2\nu}} \|D_1^m u\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N\kappa_2^{m + \frac{d}{2}} \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} \|f_\alpha\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N\kappa_2^{m + \frac{d}{2}} \sum_{|\alpha|=m, \alpha \neq \hat{\alpha}} \|D^\alpha u\|_{L_p(\mathcal{O}_\infty^+)}. \end{aligned} \tag{68}$$

From (68) and Proposition 1, we get

$$\begin{aligned} \|D^m u\|_{L_p(\mathcal{O}_\infty^+)} &\leq N\kappa_2^{-1} \sum_{k=0}^m \lambda^{\frac{1}{2}-\frac{k}{2m}} \|D^k u\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N\kappa_2^{m+\frac{d}{2}} (\rho^{\frac{1}{2v}} + \varepsilon) \|D_1^m u\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N\kappa_2^{m+\frac{d}{2}} \sum_{|\alpha|\leq m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} \|f_\alpha\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N(\varepsilon)\kappa_2^{m+\frac{d}{2}} \|D_{x'}^m u\|_{L_p(\mathcal{O}_\infty^+)}. \end{aligned} \tag{69}$$

Combining (67) and (69), we obtain the desired estimate by first taking κ_2 sufficiently large, then ε sufficiently small, κ_1 sufficiently large, and finally ρ sufficiently small. \square

Proof of Theorem 4. It suffices to establish the a priori estimate when $T = \infty$, the lower-order coefficients of L are all zero, and $u \in C^\infty(\mathcal{O}_\infty^+)$ vanishes on $\mathcal{O}_\infty^+ \setminus Q_{R_0}(X_1)$ for some $X_1 \in \mathcal{O}_\infty^+$. We use the strategy in the proof of Theorem 3 and consider two cases.

Case 1: $p \in (2, \infty)$. It follows from Corollary 5 that

$$\begin{aligned} &\|D_{x'}^{2m} u\|_{L_p(\mathcal{O}_\infty^+)} + \lambda \|u\|_{L_p(\mathcal{O}_\infty^+)} \\ &\leq N\kappa_1^{m+\frac{d}{2}} \|f\|_{L_p(\mathcal{O}_\infty^+)} + N \left(\kappa_1^{-\frac{1}{2}} + \kappa_1^{m+\frac{d}{2}} \rho^{\frac{1}{2v}} \right) \sum_{k=0}^m \lambda^{1-\frac{k}{2m}} \|D^k u\|_{L_p(\mathcal{O}_\infty^+)} \end{aligned} \tag{70}$$

for any $\kappa_1 \geq 64$. We move all the spatial derivatives except $A^{\hat{\alpha}\hat{\alpha}} D_1^{2m} u$ to the right-hand side of (7), and add $(-1)^m \sum_{j=2}^d D_j^{2m} u$ to both sides. As a consequence of Proposition 4 and the proof of Lemma 5, for any $\kappa_2 \geq 128$,

$$\begin{aligned} &(|\mathcal{E}(D_1^{2m} u) - (\mathcal{E}(D_1^{2m} u))_{Q_r(X_0)}|)_{Q_r(X_0)} \\ &\leq N\kappa_2^{-1} \sum_{k=0}^{2m} \lambda^{1-\frac{k}{2m}} (|\mathcal{E}(D^k u)|^2)_{Q_{\kappa_2 r}(X_0)}^{\frac{1}{2}} \\ &\quad + N\kappa_2^{m+\frac{d}{2}} (|\mathcal{E}f|^2)_{Q_{\kappa_2 r}(X_0)}^{\frac{1}{2}} + N\kappa_2^{m+\frac{d}{2}} \rho^{\frac{1}{2v}} (|\mathcal{E}(D_1^{2m} u)|^{2\mu})_{Q_{\kappa_2 r}(X_0)}^{\frac{1}{2\mu}} \\ &\quad + N\kappa_2^{m+\frac{d}{2}} \sum_{|\alpha|=2m, \alpha \neq 2\hat{\alpha}} (|\mathcal{E}(D^\alpha u)|^2)_{Q_{\kappa_2 r}(X_0)}^{\frac{1}{2}}. \end{aligned}$$

This estimate, combined with the Fefferman–Stein theorem and the Hardy–Littlewood maximal function theorem, gives

$$\begin{aligned} \|D_1^{2m}u\|_{L_p(\mathcal{O}_\infty^+)} &\leq N\kappa_2^{-1} \sum_{k=0}^{2m} \lambda^{1-\frac{k}{2m}} \|D^k u\|_{L_p(\mathcal{O}_\infty^+)} + N\kappa_2^{m+\frac{d}{2}} \rho^{\frac{1}{2v}} \|D_1^{2m}u\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N\kappa_2^{m+\frac{d}{2}} \|f\|_{L_p(\mathcal{O}_\infty^+)} + N\kappa_2^{m+\frac{d}{2}} \sum_{|\alpha|=2m, \alpha \neq 2\hat{\alpha}} \|D^\alpha u\|_{L_p(\mathcal{O}_\infty^+)}. \end{aligned} \tag{71}$$

From (71) and Proposition 1, we get

$$\begin{aligned} \|D^{2m}u\|_{L_p(\mathcal{O}_\infty^+)} &\leq N\kappa_2^{-1} \sum_{k=0}^{2m} \lambda^{1-\frac{k}{2m}} \|D^k u\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N\kappa_2^{m+\frac{d}{2}} (\rho^{\frac{1}{2v}} + \varepsilon) \|D_1^{2m}u\|_{L_p(\mathcal{O}_\infty^+)} \\ &\quad + N\kappa_2^{m+\frac{d}{2}} \|f\|_{L_p(\mathcal{O}_\infty^+)} + N(\varepsilon)\kappa_2^{m+\frac{d}{2}} \|D_{x'}^{2m}u\|_{L_p(\mathcal{O}_\infty^+)}. \end{aligned} \tag{72}$$

Combining (70) and (72) we obtain the desired estimate by first taking κ_2 sufficiently large, then ε sufficiently small, κ_1 sufficiently large, and finally ρ sufficiently small.

Case 2: $p \in (1, 2]$. Thanks to Case 1 and Remark 2, we already have the $W_q^{1,2m}$ solvability of

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = f$$

on the half space for any $q \in (2, \infty)$ and $\lambda > 0$. The same duality argument in the proof of Theorem 2 yields the solvability of the same equation for any $q \in (1, 2)$. We can repeat the argument in Section 8 to deduce a version of Proposition 4 with 2 norms replaced by q norms. Inspecting the proof of Case 1, to finish the proof it remains to have a proper version of Corollary 5 with 2 norms replaced by q norms.

We claim that Lemma 10 is still true with L_2 replaced by L_q , $q \in (1, \infty)$, that is, if $u \in C_{\text{loc}}^\infty(\overline{\mathcal{O}_\infty^+})$ satisfies (47) on Q'_4 and (48) in Q_4^+ , then

$$\|u\|_{C^{1/2}(Q_4^+)} \leq N \|u\|_{L_q(Q_4^+)}.$$

This easily yields the desired version of Corollary 5 by following the lines in Section 7. However, the claim does not follow directly from the proof of Lemma 10 because (50) doesn't hold if the $W_2^{1,2}$ norm on the right-hand side is replaced by the $W_q^{1,2}$ norm when q is close to 1. To get around this, we use a bootstrap argument. We first note that under the assumption of Lemma 10, for any $1 < r < R \leq 4$, it holds that

$$\|u\|_{W_q^{1,2m}(Q_r^+)} \leq N \|u\|_{L_q(Q_R^+)}. \tag{73}$$

This can be shown in the same way as Lemma 9 and 1 based on the global $W_q^{1,2m}$ estimate on the half space. By the Sobolev imbedding theorem and (73), we have

$$\|u\|_{L_{q_1}(Q_r^+)} \leq N \|u\|_{L_q(Q_R^+)}$$

for any $q_1 > q$ satisfying

$$\frac{1}{q_1} > \frac{1}{q} - \frac{1}{d+1}.$$

We iterate this bootstrap process for a finite number of steps on a sequence of shrinking half cylinders, and get

$$\|u\|_{W_{q_l}^{1,2m}(Q_1^+)} \leq N \|u\|_{L_q(Q_4^+)},$$

where $q_l > 2(d+1)$. Now, again by the Sobolev imbedding theorem, we deduce

$$\|u\|_{C^{1/2}(Q_1^+)} \leq N \|u\|_{L_q(Q_4^+)},$$

which is exactly our claim. The theorem is proved. \square

Remark 6. From the bootstrap argument above, we actually can get a finer boundary estimate as follows. If $u \in W_{q,loc}^{1,2m}(\overline{O_\infty^+})$, $q \in (1, \infty)$ satisfies (47) on Q'_4 and (48) in Q_4^+ , then for any and $\varepsilon \in (0, 1)$,

$$[u]_{C^{1-\varepsilon,2m-\varepsilon}(Q_1^+)} \leq N \|u\|_{L_q(Q_4^+)},$$

where $N = N(d, m, n, q, \varepsilon)$.

10. Systems on a bounded domain

We present the proofs of Theorems 5 and 6 in this section. We first treat the non-divergence systems. In this case, the proof is quite standard by using the technique of flattening the boundary and a partition of the unity. We give a sketched proof for the sake of completeness.

Proof of Theorem 6. First, in a same way as Lemma 1 by using Theorem 2 instead of Theorem 10, we obtain the following interior estimate for any $0 < r < R < \infty$, $Q_r \subset Q_R \subset \Omega_T$ and $\lambda \geq \lambda_0$

$$\|u_t\|_{L_p(Q_r)} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(Q_r)} \leq N \|f\|_{L_p(Q_R)} + N \|u\|_{L_p(Q_R)}. \tag{74}$$

Similarly, Theorem 4 yields a boundary estimate: let $0 < r < R < \infty$, $f \in L_p(Q_R^+)$, and ρ be the constant taken from Theorem 4. Then under Assumption 1 (ρ), for any $\lambda \geq \lambda_0$ and $u \in W_p^{1,2m}(Q_R^+)$, we have

$$\|u_t\|_{L_p(Q_r^+)} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(Q_r^+)} \leq N \|f\|_{L_p(Q_R^+)} + N \|u\|_{L_p(Q_R^+)}, \tag{75}$$

provided that $u = D_1 u = \dots = D_1^{m-1} u = 0$ on Q'_R and

$$u_t + (-1)^m L u + \lambda u = f \quad \text{in } Q_R^+.$$

It is well-known that the ellipticity condition (2) is preserved under a change of variables. Take $t_0 \in (-\infty, T)$, a point $x_0 \in \partial\Omega$ and a number $r_0 = r_0(\Omega)$, so that

$$\Omega \cap B_{r_0}(x_0) = \{x \in B_{r_0}(x_0) : x_1 > \phi(x')\}$$

in some coordinate system. We now locally flatten the boundary of $\partial\Omega$ by defining

$$y_1 = x_1 - \phi(x') := \Phi^1(x), \quad y_j = x_j := \Phi^j(x), \quad j \geq 2.$$

Under the assumptions of the theorem, Φ is a $C^{2m-1,1}$ diffeomorphism in a neighborhood of x_0 . It is easily seen that the leading coefficients of the new operator in the y -coordinates also satisfy Assumption 1 with a possibly different ρ . Thus, we can choose a sufficiently small ρ such that from (75), for $X_0 = (t_0, x_0)$ and some $r_1 = r_1(\Omega) < r_0$,

$$\begin{aligned} & \|u_t\|_{L_p(\Omega_T \cap Q_{r_1}(X_0))} + \sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\Omega_T \cap Q_{r_1}(X_0))} \\ & \leq N \|f\|_{L_p(\Omega_T \cap Q_{r_0}(X_0))} + N \sum_{j=0}^{m-1} \|D^j u\|_{L_p(\Omega_T \cap Q_{r_0}(X_0))}. \end{aligned} \tag{76}$$

Finally, a partition of the unity together with (74) and (76) completes the proof for a sufficiently large λ_0 . \square

Now we turn to the divergence case. We need to introduce a special mollification, which was used, for instance, in [21, 33].

Proof of Theorem 5. Again we give only an outline of the proof. The interior estimate is similar to that of the non-divergence case. Theorem 1 implies that, for any $0 < r < R < \infty$, $Q_r \subset Q_R \subset \Omega_T$ and $\lambda \geq \lambda_0$,

$$\sum_{|\alpha| \leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(Q_r)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p(Q_R)} + N \|u\|_{L_p(Q_R)}.$$

We also have the boundary estimate by Theorem 3: Let $0 < r < R < \infty$, $f \in L_p(Q_R^+)$, and ρ be the constant taken from Theorem 3. Then under Assumption 1 (ρ), for any $\lambda \geq \lambda_0$ and $u \in \mathcal{H}_p^{2m}(Q_R^+)$, we have

$$\sum_{|\alpha| \leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(Q_r^+)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p(Q_R^+)} + N \|u\|_{L_p(Q_R^+)}, \tag{77}$$

provided that $u = D_1 u = \dots = D_1^{m-1} u = 0$ on Q'_R and

$$u_t + (-1)^m \mathcal{L} u + \lambda u = f \quad \text{in } Q_R^+.$$

Take $t_0 \in (-\infty, T)$, a point $x_0 \in \partial\Omega$ and a number $r_0 \in (0, R_1]$. By Assumption 2, locally in some coordinate system, we have

$$\Omega \cap B_{r_0}(x_0) = \{x \in B_{r_0}(x_0) : x^1 > \phi(x')\},$$

and the local Lipschitz norm of ϕ is less than ρ_1 . The goal is to locally flatten the boundary of $\partial\Omega$. However, ϕ is not smooth in this case since it is only assumed to be Lipschitz continuous. To construct a smooth diffeomorphism, we define a function $\tilde{\phi}$ on \mathbb{R}_+^d by

$$\tilde{\phi}(x) = \int_{\mathbb{R}^{d-1}} \eta(y') \phi(x' - x_1 y') \, dy'.$$

Here $\eta \in C_0^\infty(B'_1)$ has unit integral. It is easy to check that $\tilde{\phi}(0, x') = \phi(x')$ and $|D^k \tilde{\phi}(x)| \leq N(x_1)^{1-k} \rho_1$. We now define

$$y_1 = x_1 - \tilde{\phi}(x) := \tilde{\Phi}^1(x), \quad y_j = x_j := \tilde{\Phi}^j(x), \quad j \geq 2.$$

As before, the leading coefficients of the new operator in the y -coordinates also satisfy Assumption 1 with a possibly different ρ . After some straightforward calculations using (75) and Hardy’s inequality, we conclude, for $X_0 = (t_0, x_0)$ and some $r_1 = r_1(\Omega) \in (0, r_0)$,

$$\begin{aligned} & \sum_{|\alpha| \leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\Omega_T \cap Q_{r_1}(X_0))} \\ & \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p(\Omega_T \cap Q_{r_0}(X_0))} \\ & \quad + N \rho_1 \sum_{|\alpha| \leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\Omega_T \cap Q_{r_0}(X_0))}. \end{aligned} \tag{78}$$

Using a partition of the unity together with (77) and (78), we complete the proof of the theorem upon choosing a sufficiently large λ_0 and small ρ_1 . \square

11. Remarks on the ellipticity conditions

In this section we discuss some other ellipticity conditions appearing in the literature, and show how our results can be extended to systems under those conditions.

The following strong ellipticity condition has been widely used before; see, for example, [7,30].

Assumption 3. For all $(t, x) \in \mathbb{R}^{d+1}$ and complex vectors $\xi = \{\xi_{\alpha,i}\}$, $|\alpha| = m$, $i = 1, \dots, n$,

$$\Re \left(\sum_{|\alpha|=|\beta|=m} \xi_{\alpha,i} \overline{\xi_{\beta,j}} A_{ij}^{\alpha\beta}(t, x) \right) \geq \delta |\xi|^2, \tag{79}$$

where $\delta > 0$.

The next condition is called uniform parabolicity in the sense of Petrovskii, which has been used, for example, in [17,29,34,36]. We define a matrix-valued function on $\mathbb{R}^{d+1} \times (\mathbb{R}^d \setminus \{0\})$:

$$\mathbf{A}(t, x, \xi) = |\xi|^{-2m} \sum_{|\alpha|=|\beta|=m} \xi^\alpha \xi^\beta A^{\alpha\beta}(t, x).$$

Assumption 4. Let $\lambda_j(t, x, \xi)$, $j = 1, \dots, n$, be the eigenvalues of $\mathbf{A}(t, x, \xi)$. Then,

$$\Re(\lambda_j(t, x, \xi)) \geq \delta, \quad j = 1, 2, \dots, n \tag{80}$$

for all $(t, x) \in \mathbb{R}^{d+1}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$, where $\delta > 0$.

We still assume that all the coefficients are bounded and measurable. Clearly, the Legendre–Hadamard ellipticity condition (2) is weaker than the strong ellipticity condition. However, it is stronger than the uniform parabolicity in the sense of Petrovskii.

11.1. The strong ellipticity condition

Since it is stronger than our assumption, all the results in this paper hold true under this condition. Moreover, we can take $\lambda_0 = 0$ in Theorem 5 for divergence form parabolic systems without lower-order terms. In this case the solution u satisfies

$$\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_p(\Omega_T)} \leq N \sum_{|\alpha| \leq m} \|f_\alpha\|_{L_p(\Omega_T)}. \tag{81}$$

Indeed, by the method of continuity it suffices to prove the estimate (81). Due to (79) and the Poincaré inequality, we easily get the unique solvability for $p = 2$ as well as

$$\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_2(\Omega_T)} \leq N \sum_{|\alpha| \leq m} \|f_\alpha\|_{L_2(\Omega_T)}. \tag{82}$$

In the case when $p > 2$, we add $(\lambda_0 + 1)u$ to both sides of the first equation of (8). By Theorem 5, it holds that

$$\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_p(\Omega_T)} \leq N_1 \sum_{|\alpha| \leq m} \|f_\alpha\|_{L_p(\Omega_T)} + N_1 \|u\|_{L_p(\Omega_T)}. \tag{83}$$

Take $p_1 \in (p, \infty)$ such that $1 - d/p > -d/p_1$. By Hölder’s inequality, Young’s inequality and the Poincaré–Sobolev inequality, we get for any $\varepsilon > 0$,

$$\begin{aligned} \|u\|_{L_p(\Omega_T)} &\leq N(\varepsilon)\|u\|_{L_2(\Omega_T)} + \varepsilon\|u\|_{L_{p_1}(\Omega_T)} \leq N(\varepsilon)\|u\|_{L_2(\Omega_T)} \\ &\quad + N_2\varepsilon\|Du\|_{L_p(\Omega_T)}. \end{aligned}$$

Choosing $\varepsilon = 1/(2N_1N_2)$ and using (82) and (83), we obtain (81) for $p > 2$. The remaining case $p \in (1, 2)$ follows from the standard duality argument.

11.2. The uniform parabolicity condition in the sense of Petrovskii

As we noted, this assumption is weaker than the Legendre–Hadamard condition. Under this assumption, for the solvability of parabolic systems, we need to impose a stronger regularity assumption on the leading coefficient, that is, they are VMO in both t and x . More precisely, set

$$\text{osc}_{t,x} (A^{\alpha\beta}, Q_r(t, x)) = \int_{Q_r(t,x)} \left| A^{\alpha\beta}(s, y) - \int_{Q_r(t,x)} A^{\alpha\beta} \right| dy ds,$$

and

$$\tilde{A}_R^\# = \sup_{(t,x) \in \mathbb{R}^{d+1}} \sup_{r \leq R} \sup_{|\alpha|=|\beta|=m} \text{osc}_x (A^{\alpha\beta}, Q_r(t, x)).$$

Assumption 5 (ρ). *There is a constant $R_0 \in (0, 1]$ such that $\tilde{A}_{R_0}^\# \leq \rho$.*

Next we show that the results in Section 2 about parabolic systems in the whole space (Theorems 1, 2) still hold true under the assumptions above. As a consequence, we obtain interior estimates for both divergence and non-divergence type parabolic systems. We note that, for non-divergence type parabolic systems, the corresponding interior estimate was established in a recent interesting paper [34] (see Theorem 2.4 there) by using a completely different approach.

By inspecting the proofs of the main theorems, it is apparent that if the L_2 -estimate Theorem 9 is proved for parabolic systems with *constant* coefficients under the uniform parabolicity condition, then the remaining arguments can be carried out as before with obvious modifications. Indeed, we have

Theorem 14. *Let $T \in (-\infty, \infty]$ and*

$$\mathcal{L}_0 u = \sum_{|\alpha|=|\beta|=m} D^\alpha (A^{\alpha\beta} D^\beta u),$$

where $A^{\alpha\beta}$ are constants satisfying the uniform parabolicity condition (80). Then there exists $N = N(d, n, m, \delta)$ such that, for any $\lambda \geq 0$,

$$\sum_{|\alpha| \leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_2(\mathcal{O}_T)} \leq N \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_2(\mathcal{O}_T)}, \tag{84}$$

if $u \in \mathcal{H}_2^m(\mathcal{O}_T)$, $f_\alpha \in L_2(\mathcal{O}_T)$, $|\alpha| \leq m$, and

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha \tag{85}$$

in \mathcal{O}_T . Furthermore, for $\lambda > 0$ and $f_\alpha \in L_2(\mathcal{O}_T)$, $|\alpha| \leq m$, there exists a unique $u \in \mathcal{H}_2^m(\mathcal{O}_T)$ satisfying (85).

Theorem 11.4 is probably known to the reader from before. For example, it can be derived from the results in [36]; see also Theorem 10.4 in Chapter VII of [29]. Instead of appealing to those general results, here we present a direct proof of it. We need an elementary lemma, which is verified by a direction computation.

Lemma 15. Let $\delta > 0$ and U be an $n \times n$ upper triangular complex matrix satisfying

$$|U| \leq \delta^{-1}, \quad \Re \lambda_j \geq \delta, \quad j = 1, 2, \dots, n,$$

where $\lambda_j, j = 1, \dots, n$, are the eigenvalues of U . Then there exist real constants $\varepsilon, \delta_1 > 0$, depending only on n and δ , such that for any $x \in \mathbb{C}^n$

$$\Re(x^H B U x) \geq \delta_1 |x|^2,$$

where $B = \text{diag}\{\varepsilon^{n-1}, \varepsilon^{n-2}, \dots, \varepsilon, 1\}$ and x^H is the conjugate transpose of x .

Proof of Theorem 14. It suffices to prove (84) when $u \in C_0^\infty(\overline{\mathcal{O}_T})$ and $\lambda > 0$. We take the Fourier transform of (85) in x and get

$$\tilde{u}_t + \mathbf{A}(\xi)|\xi|^{2m}\tilde{u} + \lambda\tilde{u} = \sum_{|\alpha| \leq m} (i\xi)^\alpha \tilde{f}_\alpha. \tag{86}$$

Let $\mathbf{A}(\xi) = Q^H U Q$ be the Schur decomposition of \mathbf{A} , where $Q = Q(\xi)$ is an $n \times n$ unitary matrix and $U = U(\xi)$ is an upper triangular matrix. Let B be the diagonal matrix in Lemma 15. Multiplying both sides of (86) by $Q^H B Q \tilde{u}$ and integrating on \mathcal{O}_T give us

$$\begin{aligned} & \langle B Q \tilde{u}, Q \tilde{u}_t \rangle_{\mathcal{O}_T} + \langle B Q \tilde{u}, U Q |\xi|^{2m} \tilde{u} \rangle_{\mathcal{O}_T} + \lambda \langle B Q \tilde{u}, Q \tilde{u} \rangle_{\mathcal{O}_T} \\ &= \sum_{|\alpha| \leq m} \langle Q^H B Q \tilde{u}, (i\xi)^\alpha \tilde{f}_\alpha \rangle_{\mathcal{O}_T}. \end{aligned} \tag{87}$$

As in the proof of Theorem 9,

$$\Re \langle B Q \tilde{u}, Q \tilde{u}_t \rangle_{\mathcal{O}_T} \geq 0.$$

By the Plancherel equality,

$$\lambda \Re \langle B Q \tilde{u}, Q \tilde{u} \rangle_{\mathcal{O}_T} \geq N(\varepsilon) \lambda \|u\|_{L_2(\mathcal{O}_T)}^2.$$

To estimate the second term of the left-hand side of (87), we use Lemma 15 and the Plancherel equality to get

$$\Re \langle B Q \tilde{u}, U Q |\xi|^{2m} \tilde{u} \rangle_{\mathcal{O}_T} \geq \delta_1 \langle \tilde{u}, |\xi|^{2m} \tilde{u} \rangle_{\mathcal{O}_T} \geq N(n, m, \delta) \|D^m u\|_{L_2(\mathcal{O}_T)}^2.$$

The real part of the right-hand side of (87) is bounded from above by

$$\begin{aligned} & N \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_2(\mathcal{O}_T)} \|f_\alpha\|_{L_2(\mathcal{O}_T)} \\ & \leq \sum_{|\alpha| \leq m} \varepsilon \lambda^{\frac{m-|\alpha|}{m}} \|D^\alpha u\|_{L_2(\mathcal{O}_T)}^2 + N \sum_{|\alpha| \leq m} \varepsilon^{-1} \lambda^{-\frac{m-|\alpha|}{m}} \|f_\alpha\|_{L_2(\mathcal{O}_T)}^2 \end{aligned}$$

for all $\varepsilon > 0$. To complete the proof of (84) it suffices to use the interpolation inequalities and to choose an appropriate ε . \square

Remark 7. In contrast, under Petrovskii's parabolicity condition, the Dirichlet boundary value problem of parabolic systems is, in general, not well-posed when $d \geq 2$, as pointed out in §10 Chapter VII of [29]. However, in the case $d = 1$, relying on a linear transformation, one can extend Theorem 14 to systems on the half space with the homogeneous Dirichlet boundary condition; see, for instance, §10 Chapter VII of [29]. Thus, all the results in Section 2 about systems on a half space or a bounded domain remain true in this case.

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