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Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data

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Abstract

We investigate the well-posedness of (1) the heat flow of harmonic maps from $\mathbb{R}^n$ to a compact Riemannian manifold N without boundary for initial data in BMO; and (2) the hydrodynamic flow (u, d) of nematic liquid crystals on $\mathbb{R}^n$ for initial data in $\mathrm{BMO}^{-1} \times \mathrm{BMO}$.

1. Introduction

For $k \geqq 1$, let N be a k-dimensional compact Riemannian manifold without boundary, isometrically embedded in some Euclidean space $\mathbb{R}^l$. For $n \geqq 1$, the equation of heat flow of harmonic maps from $\mathbb{R}^n$ to N is given by:

$$u_t - \Delta u = A(u)(\nabla u, \nabla u) \quad \text{in } \mathbb{R}^n \times (0, +\infty) \tag{1.1}$$

$$u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^n \tag{1.2}$$

where $A(y) : T_y N \times T_y N \to (T_y N)^\perp$ is the second fundamental form of $N \subset \mathbb{R}^l$ at $y \in N$, and $u_0 : \mathbb{R}^n \to N$ is a given map.

Equations (1.1)–(1.2) provide a very important approach to seek the existence of harmonic maps in various topological classes. In their pioneering work [6] in the 1960s, Eells and Sampson established that (1) for $u_0 \in C^\infty(\mathbb{R}^n, N)$ there exists $0 < T = T(\phi) \leqq +\infty$ such that (1.1)–(1.2) admits a unique smooth solution $u \in C^\infty(\mathbb{R}^n \times [0, T), N)$; and (2) if, in addition, the sectional curvature K_N of N is nonpositive, then $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+, N)$ and

$$\|u\|_{C^2(\mathbb{R}^n \times \mathbb{R}_+)} \leqq C(n, \|\phi\|_{C^2(\mathbb{R}^n)}). \tag{1.3}$$

Without the curvature assumption, Hildebrandt et al. [9] established, in the late 1970s, the existence of a unique, global smooth solution to (1.1)–(1.2) under the assumption that the image of u_0 is contained in a geodesic ball B_R in N with radius

$R < \frac{\pi}{2\sqrt{\max_{B_R}|K_N|}}$. In general, on the one hand, it is well-known via the works by CORON and GHIDAGLIA [3], CHEN and DING [1], and CHANG et al. [2] that the short time smooth solution to (1.1)–(1.2) may develop finite time singularity; on the other hand, CHEN and STRUWE [5] (see also CHEN and LIN [4]) established the existence of partially smooth, global weak solutions to (1.1)–(1.2) for smooth initial data u_0.

Although there have been many works important to (1.1)–(1.2) (see for example LIN and WANG [16] and references therein), the global (or local, resp.) well-posedness of (1.1)–(1.2) for small (or large, resp.) rough initial data remains an interesting question. If the initial data u_0 is in some Sobolev spaces, STRUWE [18] established, in dimension $n = 2$, the local well-posedness of (1.1)–(1.2) in the space $L^2_t H^2_x$ for $u_0 \in W^{1,2}(\mathbb{R}^2, N)$, and the global well-posedness provided $\|\nabla u_0\|_{L^2(\mathbb{R}^2)}$ is sufficiently small. For $n \geqq 3$, the well-posedness similar to that of [18] for $u_0 \in W^{1,n}(\mathbb{R}^n, N)$ was not previously available in the literature; readers can refer to WANG [19] for some related earlier results.

Notice that (1.1) is invariant with respect to parabolic scaling, that is, $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ solves (1.1) for any $\lambda > 0$ provided u solves (1.1). Hence we need a scale and translation invariant version of L^2-boundedness:

$$\sup_{x\in\mathbb{R}^n, R>0} R^{-n} \int_{B_R(x)\times[0,R^2]} |\nabla u|^2(y,t)\,\mathrm{d}y\,\mathrm{d}t < +\infty.$$

This implies that the caloric extension of initial data u_0 needs to enjoy the above property.

In a very interesting paper [10], KOCH and LAMM proved that (1.1)–(1.2) is (1) locally uniquely solvable in $C^\infty(\mathbb{R}^n, N)$ provided u_0 is L^∞-close to a uniformly continuous map; and (2) globally uniquely solvable in $C^\infty(\mathbb{R}^n, N)$ provided u_0 is L^∞-close to a point. The techniques employed by KOCH and LAMM in [10] originated from the earlier work by KOCH and TATARU [11] on the global well-posedness of the incompressible Navier–Stokes equation for $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$:

$$u_t + u\cdot\nabla u - \Delta u + \nabla P = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty) \tag{1.4}$$

$$\nabla\cdot u = 0 \quad \text{in } \mathbb{R}^n \tag{1.5}$$

$$u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^n \tag{1.6}$$

for $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^n)$ with $\nabla\cdot u_0 = 0$ and small $\|u_0\|_{\mathrm{BMO}^{-1}}$.

Partially motivated by [10] and [11], we address well-posedness for both the heat flow of harmonic maps and the hydrodynamic flow of nematic liquid crystals.

In order to state the results, we first recall the definitions of both local and global BMO spaces.

Definition 1.1. For $0 < R \leqq +\infty$, a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is in $\mathrm{BMO}_R(\mathbb{R}^n)$ if the semi-norm

$$[f]_{\mathrm{BMO}_R(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, 0<r\leqq R} \left\{ r^{-n} \int_{B_r(x)} |f(y) - f_{x,r}|\,\mathrm{d}y \right\}$$

is finite, where $f_{x,r} = \frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)\,\mathrm{d}y$ is the average of f over $B_r(x)$. We say $f \in \overline{\mathrm{VMO}}(\mathbb{R}^n)$ if

$$\lim_{r\downarrow 0}[f]_{\mathrm{BMO}_r(\mathbb{R}^n)} = 0.$$

When $R = +\infty$, we simply write $(\mathrm{BMO}(\mathbb{R}^n), [\cdot]_{\mathrm{BMO}(\mathbb{R}^n)})$ for $(\mathrm{BMO}_\infty(\mathbb{R}^n), [\cdot]_{\mathrm{BMO}_\infty(\mathbb{R}^n)})$.

Now recall the space BMO^{-1}, introduced by Koch and Tataru [11], as follows.

Definition 1.2. For $0 < R \leqq +\infty$, a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is in $\mathrm{BMO}_R^{-1}(\mathbb{R}^n)$ if there exists $(f_1, \ldots, f_n) \in \mathrm{BMO}_R(\mathbb{R}^n)$ such that $f = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$. Moreover, the norm of f is defined by

$$\|f\|_{\mathrm{BMO}_R^{-1}(\mathbb{R}^n)} := \inf\left\{\sum_{i=1}^n [f_i]_{\mathrm{BMO}_R(\mathbb{R}^n)} : f \equiv \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\right\}.$$

We say $f \in (\overline{\mathrm{VMO}}(\mathbb{R}^n))^{-1}$ if

$$\lim_{r\downarrow 0}[f]_{\mathrm{BMO}_r^{-1}(\mathbb{R}^n)} = 0.$$

When $R = +\infty$, we simply write $(\mathrm{BMO}^{-1}(\mathbb{R}^n), [\cdot]_{\mathrm{BMO}^{-1}(\mathbb{R}^n)})$ for $(\mathrm{BMO}_\infty^{-1}(\mathbb{R}^n), [\cdot]_{\mathrm{BMO}_\infty^{-1}(\mathbb{R}^n)})$.

We also introduce the functional space X_T for $0 < T \leqq +\infty$ as follows.

$$X_T := \left\{ f : \mathbb{R}^n \times [0,T] \to \mathbb{R}^l \;\middle|\; |||u|||_{X_T} \equiv \sup_{0<t\leqq T} \|f(t)\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{X_T} < +\infty \right\},$$

where

$$\|f\|_{X_T} = \sup_{0<t\leqq T} \sqrt{t}\|\nabla f(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n, 0<R\leqq\sqrt{T}} \left(R^{-n}\int_{P_R(x,R^2)} |\nabla f|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}},$$

and $P_R(x, R^2) = B_R(x) \times [0, R^2]$ denotes the parabolic cylinder with center (x, R^2) and radius R. It is easy to see that $(X_T, |||\cdot|||_{X_T})$ is a Banach space. When $T = +\infty$, we simply write X for X_∞, $\|\cdot\|_X$ for $\|\cdot\|_{X_\infty}$, and $|||\cdot|||_X$ for $|||\cdot|||_{X_\infty}$, respectively.

For the heat flow of harmonic maps, we prove

Theorem 1.3. (local well-posedness) *There exists* $\varepsilon_0 > 0$ *such that for any* $R > 0$ *if* $\|u_0\|_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq \varepsilon_0$*, then* (1.1)–(1.2) *has a unique solution* $u \in X_{R^2}$ *with small* $\|u\|_{X_{R^2}}$. *In particular, if* $u_0 \in \overline{\mathrm{VMO}}(\mathbb{R}^n)$*, then there exists* $T_0 > 0$ *such that* (1.1)–(1.2) *admits a unique solution* $u \in X_{T_0}$ *with small* $\|u\|_{X_{T_0}}$.

As a corollary, we have

Theorem 1.4. (global well-posedness) *There exist* $\varepsilon_0 > 0$ *and* $C_0 > 0$ *such that if* $[u_0]_{\mathrm{BMO}(\mathbb{R}^n)} \leqq \varepsilon_0$*, then there exists a unique global solution* $u \in X$ *to* (1.1)–(1.2) *such that* $\|u\|_X \leqq C_0\varepsilon_0$.

Since $W^{1,n}(\mathbb{R}^n) \subset \overline{\mathrm{VMO}}(\mathbb{R}^n)$, it follows from Theorem 1.3 that for any initial data $u_0 \in W^{1,n}(\mathbb{R}^n)$, (1.1)–(1.2) admits a short time unique solution $u \in X_{T_0}$ for some $T_0 > 0$. Theorem 1.4 implies that such a unique solution u is a unique global solution in X provided $\|\nabla u_0\|_{L^n(\mathbb{R}^n)}$ is sufficiently small.

Now we turn to the discussion on well-posedness for the hydrodynamic flow of nematic liquid crystals in the entire space.

The following equation modeling the hydrodynamic flow of nematic liquid crystal materials was proposed and investigated by LIN and LIU [13,14] in the 1990s:

$$u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d) \quad \text{in } \mathbb{R}^n \times (0, +\infty) \tag{1.7}$$

$$\nabla \cdot u = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty) \tag{1.8}$$

$$d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d \quad \text{in } \mathbb{R}^n \times (0, +\infty), \tag{1.9}$$

where $u(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ represents the velocity field of the flow, $d(\cdot, t) : \mathbb{R}^n \to S^2$, the unit sphere in $\mathbb{R}^3$, is a unit-vector field that represents the macroscopic molecular orientation of the nematic liquid crystal material, and $P(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ represents the pressure function. $\nabla\cdot$ denotes the divergence operator, and $\nabla d \otimes \nabla d$ denotes the $n \times n$ matrix whose (i, j)-the entry is given by $\nabla_i d \cdot \nabla_j d$ for $1 \leqq i, j \leqq n$.

The above system is a simplified version of the Ericksen–Leslie model, which reduces to the Ossen–Frank model in the static case, for the hydrodynamics of nematic liquid crystal materials developed during the period of 1958 through 1968 (see [7,8,12]). It is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow field $u(x, t)$, and the macroscopic description of the microscopic orientation configurations $d(x, t)$ of rod-like liquid crystals. Roughly speaking, the system (1.7)–(1.9) is a coupling between the incompressible Navier–Stokes equation and the transported heat flow of harmonic maps into S^2.

When considering the initial and boundary value problem of (1.7)–(1.9) on bounded domains $\Omega \subset \mathbb{R}^2$:

$$(u, d)\big|_{\Omega \times \{0\}} = (u_0, d_0), \quad (u, d)\big|_{\partial\Omega \times (0, +\infty)} = (0, d_0), \tag{1.10}$$

where $u_0 : \Omega \to \mathbb{R}^2$ is a given divergence free vector field and $d_0 : \Omega \to S^2$ is a given unit-vector field. In a very recent paper, LIN et al. [15] proved, among other results, that for any $(u_0, d_0) \in L^2(\Omega, \mathbb{R}^2) \times H^1(\Omega, S^2)$ with $\nabla \cdot u_0 = 0$, there is a global Leray–Hopf type weak solution (u, d) to (1.7)–(1.9) and (1.10) that is smooth away from at most finitely many singular times.

In this paper, we want to address both local and global well-posedness issues on the Cauchy problem of (1.7)–(1.9) on $\mathbb{R}^n$ with rough initial data. Notice that (1.7)–(1.9) is invariant with respect to parabolic scaling, namely, if (u, P, d) solves (1.7)–(1.9), then for any $\lambda > 0$,

$$(u_\lambda, P_\lambda, d_\lambda)(x,t) = (\lambda u(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t), d(\lambda x, \lambda^2 t))$$

is also a solution of (1.7)–(1.9). Thus, we need to look for a space of initial data (u_0, d_0) such that its caloric extension $(\tilde{u}_0, \tilde{d}_0)$ has bounded normalized energies:

$$\sup_{x\in\mathbb{R}^n, R>0} R^{-n} \int_{B_R(x)\times[0,R^2]} \left(|\tilde{u}_0|^2 + |\nabla \tilde{d}_0|^2\right) \mathrm{d}y\, \mathrm{d}t < +\infty.$$

For this, we need to introduce another functional space in order to handle the velocity field u. For $0 < T \leqq +\infty$, let Z_T be the space consisting of functions $f : \mathbb{R}^n \times [0, T]$ such that

$$\|f\|_{Z_T} := \sup_{0<t\leqq T} \sqrt{t}\|f(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n, 0<r\leqq\sqrt{T}} \left(r^{-n}\int_{P_r(x,r^2)} |f|^2\right)^{\frac{1}{2}} < +\infty.$$

When $T = +\infty$, we simply write Z for Z_∞, and $\|\cdot\|_Z$ for $\|\cdot\|_{Z_\infty}$.

It turns out that, by combining the techniques of Koch and Tataru [11] and Theorem 1.3 on the heat flow of harmonic maps, we are able to prove the following theorems.

Theorem 1.5. *There exists $\varepsilon_0 > 0$ such that for any $R > 0$ if $u_0 \in \mathrm{BMO}_R^{-1}(\mathbb{R}^n, \mathbb{R}^n)$, with $\nabla \cdot u_0 = 0$, and $d_0 \in \mathrm{BMO}_R(\mathbb{R}^n, S^2)$ satisfies*

$$\|u_0\|_{\mathrm{BMO}_R^{-1}(\mathbb{R}^n)} + [d_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq \varepsilon_0, \tag{1.11}$$

then there exists a unique solution $(u, d) \in Z_{R^2} \times X_{R^2}$ with small $(\|u\|_{Z_{R^2}} + \|d\|_{X_{R^2}})$ to (1.7)–(1.9) and

$$(u, d)\big|_{t=0} = (u_0, d_0) \quad \text{on } \mathbb{R}^n. \tag{1.12}$$

In particular, if $(u_0, d_0) \in (\overline{\mathrm{VMO}}(\mathbb{R}^n))^{-1} \times (\overline{\mathrm{VMO}}(\mathbb{R}^n))$, then there exists $T_0 > 0$ such that (1.7)–(1.9) and (1.12) admits a unique solution $(u, d) \in X_{T_0}$ with small $(\|u\|_{Z_{T_0}} + \|d\|_{X_{T_0}})$.

As a corollary, we have

Theorem 1.6. *There exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that if $u_0 \in \mathrm{BMO}^{-1}(\mathbb{R}^n, \mathbb{R}^n)$, with $\nabla \cdot u_0 = 0$, and $d_0 \in \mathrm{BMO}(\mathbb{R}^n, S^2)$ satisfies*

$$\|u_0\|_{\mathrm{BMO}^{-1}(\mathbb{R}^n)} + [d_0]_{\mathrm{BMO}(\mathbb{R}^n)} \leqq \varepsilon_0, \tag{1.13}$$

then there exists a unique global solution $(u, d) \in Z \times X$ to (1.7)–(1.9) and (1.12) with $(\|u\|_Z + \|d\|_X) \leqq C_0\varepsilon_0$.

We also remark that Theorem 1.5 implies that (1.7)–(1.9) and (1.12) is locally well-posed in X_T for any initial data $(u_0, d_0) \in L^n(\mathbb{R}^n, \mathbb{R}^n) \times W^{1,n}(\mathbb{R}^n, S^2)$, and is globally well-posed in X provided that $(\|u_0\|_{L^n(\mathbb{R}^n)} + \|\nabla d_0\|_{L^n(\mathbb{R}^n)})$ is sufficiently small.

The remainder of this paper is organized as follows. In Section 2, we establish some basic estimates on the caloric extension of BMO functions. In Section 3, we prove Theorems 1.3 and 1.4. In Section 4, we prove Theorems 1.5 and 1.6.

2. Preliminary results

In this section, we first review Carleson's well-known theorem on the characterization of a BMO function in terms of its caloric extension, see STEIN [17, p 159, Theorem 3]. Then we show a crucial estimate of the distance between the caloric extension of u_0 and the manifold N.

Let $G(x, t)$ be the fundamental solution of the heat equation in $\mathbb{R}^n \times \mathbb{R}_+$:

$$G(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0. \tag{2.1}$$

Let $\tilde{u}_0 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^l$ be the caloric extension of u_0:

$$\tilde{u}_0(x,t) = \int_{\mathbb{R}^n} G(x-y,t) u_0(y)\, \mathrm{d}y. \tag{2.2}$$

Carleson's characterization of the BMO space asserts that $u_0 \in \mathrm{BMO}(\mathbb{R}^n)$ iff $|\nabla \tilde{u}_0|^2 \,\mathrm{d}x\,\mathrm{d}t$ is a Carleson measure on $\mathbb{R}^n \times \mathbb{R}_+$, that is

$$\sup_{x\in\mathbb{R}^n, r>0} r^{-n} \int_{P_r(x,r^2)} |\nabla \tilde{u}_0|^2 \,\mathrm{d}x\,\mathrm{d}t < +\infty,$$

and one has the equivalence of the norms:

$$[u_0]_{\mathrm{BMO}(\mathbb{R}^n)} \approx \sup_{x\in\mathbb{R}^n, r>0} \left(r^{-n} \int_{P_r(x,r^2)} |\nabla \tilde{u}_0|^2 \,\mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}}. \tag{2.3}$$

If $u_0 \in \mathrm{BMO}_R(\mathbb{R}^n)$ for some $0 < R < +\infty$, then the same characterization as above gives

$$[u_0]_{\mathrm{BMO_R}(\mathbb{R}^n)} \approx \sup_{x\in\mathbb{R}^n, 0<r\leqq R} \left(r^{-n} \int_{P_r(x,r^2)} |\nabla \tilde{u}_0|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}}. \tag{2.4}$$

Since $\tilde{u}_0$ solves the heat equation on $\mathbb{R}^n \times \mathbb{R}_+$, the standard gradient estimate implies that for any $t > 0$,

$$\sqrt{t}\|\nabla \tilde{u}_0(t)\|_{L^\infty(\mathbb{R}^n)} \lesssim \sup_{x\in\mathbb{R}^n} \left(t^{-\frac{n}{2}} \int_{P_{\sqrt{t}}(x,t)} |\nabla \tilde{u}_0|^2 \,\mathrm{d}y\mathrm{d}\tau \right)^{\frac{1}{2}}. \tag{2.5}$$

In particular, we have that (i) if $u_0 \in \mathrm{BMO}(\mathbb{R}^n)$, then

$$\sup_{t>0} \sqrt{t}\|\nabla \tilde{u}_0\|_{L^\infty(\mathbb{R}^n)} \lesssim [u_0]_{\mathrm{BMO}(\mathbb{R}^n)}, \tag{2.6}$$

and (ii) if $u_0 \in \mathrm{BMO}_R(\mathbb{R}^n)$ for some $R > 0$, then

$$\sup_{0<t\leqq R^2} \sqrt{t}\|\nabla \tilde{u}_0\|_{L^\infty(\mathbb{R}^n)} \lesssim [u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)}. \tag{2.7}$$

Now we need to estimate the distance of $\tilde{u}_0$ to the manifold N in terms of the BMO norm of u_0, which plays an important role in the proof of our Theorems. More precisely, we have

Lemma 2.1. *For any $\delta > 0$, there exists $K = K(\delta, N) > 0$ such that if $u_0 : \mathbb{R}^n \to N$ belongs to $\mathrm{BMO}_R(\mathbb{R}^n)$ for some $0 < R \leqq +\infty$, then*

$$\mathrm{dist}(\tilde{u}_0(x,t), N) \leqq K^n\, [u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} + \delta, \quad \forall x \in \mathbb{R}^n, \quad 0 \leqq t \leqq \frac{R^2}{K^2}. \tag{2.8}$$

In particular, if $u_0 \in \mathrm{BMO}(\mathbb{R}^n)$ then

$$\mathrm{dist}(\tilde{u}_0(x,t), N) \leqq K^n [u_0]_{\mathrm{BMO}(\mathbb{R}^n)} + \delta, \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{2.9}$$

Proof. Since (2.9) follows directly from (2.8) with $R = +\infty$, it suffices to prove (2.8). For any $x \in \mathbb{R}^n$, $t > 0$, and $L > 0$, denote

$$c_{x,t}^L = \frac{1}{|B_L(0)|} \int_{B_L(0)} u_0(x - \sqrt{t}z)\, \mathrm{d}z.$$

Since

$$\tilde{u}_0(x,t) = \int_{\mathbb{R}^n} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|y|^2}{4}} u_0(x - \sqrt{t}y)\, \mathrm{d}y,$$

we have

$$\begin{aligned}
\left|\tilde{u}_0(x,t) - c_{x,t}^L\right| &\leqq \int_{\mathbb{R}^n} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|y|^2}{4}} \left|u_0(x - \sqrt{t}y) - c_{x,t}^L\right| \mathrm{d}y \\
&\leqq \left\{\int_{B_L(0)} + \int_{\mathbb{R}^n \setminus B_L(0)}\right\} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|y|^2}{4}} \left|u_0(x - \sqrt{t}y) - c_{x,t}^L\right| \mathrm{d}y \\
&\leqq \int_{B_L(0)} \left|u_0(x - \sqrt{t}y) - c_{x,t}^L\right| \mathrm{d}y \\
&\quad + 2\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_L(0)} e^{-\frac{|y|^2}{4}}\, \mathrm{d}y \\
&\leqq L^n [u_0]_{\mathrm{BMO}_{L\sqrt{t}}(\mathbb{R}^n)} + C_N \int_L^\infty e^{-\frac{r^2}{4}} r^{n-1}\, \mathrm{d}r \\
&\leqq \delta + L^n [u_0]_{\mathrm{BMO}_{L\sqrt{t}}(\mathbb{R}^n)}
\end{aligned} \tag{2.10}$$

provided we choose a sufficiently large $L = L(\delta, N) > 0$ so that

$$C_N \int_L^\infty e^{-\frac{r^2}{4}} r^{n-1}\,\mathrm{d}r \leqq \delta.$$

On the other hand, since $u_0(\mathbb{R}^n) \subset N$, we have

$$\operatorname{dist}\left(c_{x,t}^L, N\right) \leqq \left| c_{x,t}^L - u_0(x - \sqrt{t}y) \right|, \quad \forall y \in B_L(0)$$

and hence

$$\operatorname{dist}\left(c_{x,t}^L, N\right) \leqq \frac{1}{|B_L(0)|} \int_{B_L(0)} \left| c_{x,t}^L - u_0(x - \sqrt{t}y) \right| \mathrm{d}y \leqq [u_0]_{\mathrm{BMO}_{L\sqrt{t}}(\mathbb{R}^n)}. \tag{2.11}$$

Putting (2.9) and (2.11) together yields

$$\operatorname{dist}(\tilde{u}_0(x,t), N) \leqq \delta + (L^n + 1)[u_0]_{\mathrm{BMO}_{L\sqrt{t}}(\mathbb{R}^n)}.$$

Hence (2.8) holds for $t \leqq \frac{R^2}{K^2}$, provided that we choose $L \approx K$. This completes the proof. □

3. Proof of Theorems 1.3 and 1.4

This section is devoted to the proof of Theorems 1.3 and 1.4. The idea is to choose a suitable ball in X such that the operator T determined by the Duhamel formula has a fixed point in the ball.

For $0 < T \leqq +\infty$, besides the space X_T introduced in Section 1, we also need to introduce Y_T as follows. Y_T is the space consisting of all functions $f : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ such that

$$\|f\|_{Y_T} \equiv \sup_{0<t\leqq T} t\|f(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n, 0<R\leqq\sqrt{T}} R^{-n} \int_{P_R(x,R^2)} |f|\,\mathrm{d}x\,\mathrm{d}t < +\infty.$$

It is also easy to see $(Y_T, \|\cdot\|_{Y_T})$ is a Banach space. When $T = +\infty$, we simply write Y for Y_∞, and $\|\cdot\|_Y$ for $\|\cdot\|_{Y_\infty}$.

For $f \in Y_T$, define

$$\mathbb{S}f(x,t) = \int_0^t \int_{\mathbb{R}^n} G(x-y, t-s) f(y,s)\,\mathrm{d}y\,\mathrm{d}s, \ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{3.1}$$

It is well-known that $u : \mathbb{R}^n \times \mathbb{R}_+ \to N$ solves (1.1)–(1.2) iff

$$u(x,t) = \tilde{u}_0(x,t) + \mathbb{S}(A(u)(\nabla u, \nabla u))(x,t). \tag{3.2}$$

The following Lemma plays the critical role in the proof.

Lemma 3.1. *For* $0 < T \leqq +\infty$, *if* $f \in Y_T$, *then* $\mathbb{S}f \in X_T$. *Moreover,*

$$|||\mathbb{S}f|||_{X_T} \leqq C\|f\|_{Y_T} \tag{3.3}$$

for some $C = C(n) > 0$.

Proof. By suitable scalings, we may assume $T \geqq 1$. Since the norms are invariant under both scaling and translation, it suffices to show

$$|\mathbb{S}f(0,1)| + |\nabla(\mathbb{S}f)(0,1)| + \left(\int_{P_1(0,1)} |\nabla(\mathbb{S}f)|^2\right)^{\frac{1}{2}} \leqq C\|f\|_{Y_1}. \tag{3.4}$$

Set $W = \mathbb{S}f$. Then

$$\begin{aligned} W(0,1) &= \int_0^1 \int_{\mathbb{R}^n} G(y, 1-s) f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \left\{\int_{\frac{1}{2}}^1 \int_{\mathbb{R}^n} + \int_0^{\frac{1}{2}} \int_{B_2} + \int_0^{\frac{1}{2}} \int_{\mathbb{R}^n \setminus B_2}\right\} G(y, 1-s) f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= I_1 + I_2 + I_3. \end{aligned}$$

It is easy to see

$$\begin{aligned} |I_1| &\leqq \left(\sup_{\frac{1}{2}\leqq s\leqq 1} \|f(s)\|_{L^\infty(\mathbb{R}^n)}\right)\left(\int_{\frac{1}{2}}^1 \int_{\mathbb{R}^n} G(y, 1-s)\,\mathrm{d}y\,\mathrm{d}s\right) \leqq C\|f\|_{Y_1}, \\ |I_2| &\leqq (\sup_{0\leqq s\leqq \frac{1}{2}} \|G(\cdot, 1-s)\|_{L^\infty(\mathbb{R}^n)})\left(\int_{B_2\times\left[0,\frac{1}{2}\right]} |f(y,s)|\,\mathrm{d}y\,\mathrm{d}s\right) \\ &\leqq C \int_{B_2\times\left[0,\frac{1}{2}\right]} |f(y,s)|\,\mathrm{d}y\,\mathrm{d}s \leqq C\|f\|_{Y_1}, \end{aligned}$$

and

$$\begin{aligned} |I_3| &\leqq \int_0^{\frac{1}{2}} \int_{\mathbb{R}^n\setminus B_2} G(y, 1-s)|f(y,s)|\,\mathrm{d}y\,\mathrm{d}s \\ &\leqq C \int_0^{\frac{1}{2}} \int_{\mathbb{R}^n\setminus B_2} e^{-\frac{|y|^2}{2}} |f(y,s)|\,\mathrm{d}y\,\mathrm{d}s \\ &\leqq C\left(\sum_{k=2}^\infty k^{n-1} e^{-\frac{k^2}{2}}\right)\cdot\left(\sup_{y\in\mathbb{R}^n} \int_{P_1(y,1)} |f(y,s)|\,\mathrm{d}y\,\mathrm{d}s\right) \\ &\leqq C\|f\|_{Y_1}. \end{aligned}$$

Putting these three inequalities together implies $|W(0,1)| \leqq C\|f\|_{Y_1}$. The estimate of $|\nabla W(0,1)|$ can be done similarly. In fact, denote

$$H(x,t) = \nabla_x G(x,t) = -\frac{x}{2t} G(x,t).$$

Then

$$\int_0^{\frac{1}{2}} \int_{\mathbb{R}^n} |H(x,t)| \leqq C, \qquad \sup_{x\in\mathbb{R}^n,\, \frac{1}{2}\leqq t\leqq 1} |H(x,t)| \leqq C.$$

Since

$$\nabla W(0,1) = \int_0^1 \int_{\mathbb{R}^n} H(-y, 1-s) f(y,s)\, \mathrm{d}y\, \mathrm{d}s,$$

we have

$$\begin{aligned}
|\nabla W(0,1)| &\leqq \int_0^1 \int_{\mathbb{R}^n} |H|(-y,1-s)|f(y,s)|\, \mathrm{d}y\, \mathrm{d}s \\
&= \left\{ \int_{\frac{1}{2}}^1 \int_{\mathbb{R}^n} + \int_0^{\frac{1}{2}} \int_{B_2} + \int_0^{\frac{1}{2}} \int_{\mathbb{R}^n\setminus B_2} \right\} |H(-y,1-s)||f(y,s)|\, \mathrm{d}y\, \mathrm{d}s \\
&= I_4 + I_5 + I_6.
\end{aligned}$$

It is readily seen that

$$|I_4| \leqq C \left(\int_0^{\frac{1}{2}} \int_{\mathbb{R}^n} |H(x,t)| \right) \cdot \left(\sup_{\frac{1}{2}\leqq s\leqq 1} \|f(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} \right) \leqq C\|f\|_{Y_1},$$

$$|I_5| \leqq C \left(\sup_{x\in\mathbb{R}^n,\, \frac{1}{2}\leqq t\leqq 1} |H(x,t)| \right) \left(\int_{B_2\times[0,1]} |f(y,s)|\, \mathrm{d}y\, \mathrm{d}s \right) \leqq C\|f\|_{Y_1},$$

and

$$\begin{aligned}
|I_6| &\leqq C \int_0^{\frac{1}{2}} \int_{\mathbb{R}^n\setminus B_2} |y| e^{-\frac{|y|^2}{2}} |f(y,s)| \\
&\leqq C \left(\sum_{k=2}^{\infty} k^n e^{-\frac{k^2}{2}} \right) \cdot \left(\sup_{y\in\mathbb{R}^n} \int_{P_1(y,1)} |f(y,s)|\, \mathrm{d}y\, \mathrm{d}s \right) \\
&\leqq C\|f\|_{Y_1}.
\end{aligned}$$

Putting these estimates together yields $|\nabla W(0,1)| \leqq C\|f\|_{Y_1}$.

The estimate of $\|\nabla W\|_{L^2(P_1(0.1))}$ follows from the energy inequality as follows. Since W satisfies

$$W_t - \Delta W = f \quad \text{in } \mathbb{R}^n \times [0,1]; \quad W\big|_{t=0} = 0.$$

Let $\eta \in C_0^1(B_2)$ be a cut-off function of B_1. Multiplying the equation of W by $\eta^2 W$ and integrating over $\mathbb{R}^n \times [0,1]$, since

$$\int_{\mathbb{R}^n} \eta^2 |W|^2(x,0)\, \mathrm{d}x = 0,$$

we obtain

$$\begin{aligned}\int_{P_1(0,1)} |\nabla W|^2 &\leqq C \int_{B_2\times[0,1]} (|W|^2 + |W||f|)\\ &\leqq C\left(\|W\|^2_{L^\infty(B_2\times[0,1])} + \|W\|_{L^\infty(B_2\times[0,1])}\|f\|_{L^1(B_2\times[0,1])}\right)\\ &\leqq C\|f\|^2_{Y_1},\end{aligned}$$

where, in the last step, we have used the inequality which was proved in the previous step,

$$\|W\|_{L^\infty(B_2\times[0,1])} \leqq C\|f\|_{Y_1}.$$

This completes the proof. □

In order to construct the solution to (1.1) in the space X_{R^2}, we need to extend the second fundamental form $A(\cdot)(\cdot,\cdot)$ from N to $\mathbb{R}^l$, still denoted as A. For this, recall that there exists $\delta_N > 0$ such that the nearest point projection map $\Pi : N_{\delta_N} = \{y \in \mathbb{R}^l : \operatorname{dist}(y, N) \leqq \delta_N\} \to N$ is smooth. Let $\widetilde{\Pi} \in C^\infty(\mathbb{R}^l, \mathbb{R}^l)$ be a smooth extension of Π, that is, $\widetilde{\Pi} \equiv \Pi$ in N_{δ_N}. Define

$$A(y)(V, W) = -D^2\widetilde{\Pi}(y)(V, W), \quad \forall y \in \mathbb{R}^l, \quad V, W \in T_y\mathbb{R}^l.$$

Now we define the mapping operator $\mathbf{T}$ on X_{R^2} by letting

$$\mathbf{T}u(x,t) = \tilde{u}_0 + \mathbb{S}(A(u)(\nabla u, \nabla u))(x,t), \quad x \in \mathbb{R}^n, \quad 0 < t \leqq R^2, \quad u \in X_{R^2}. \tag{3.5}$$

If $u_0 \in \mathrm{BMO}_R(\mathbb{R}^n)$, then (2.4), (2.7) and the maximum principle of the heat equation imply that $\tilde{u}_0 \in X_{R^2}$ and

$$\|\tilde{u}_0\|_{X_{R^2}} \lesssim [u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)}. \tag{3.6}$$

For $\varepsilon > 0$, let

$$\mathbf{B}_\varepsilon(\tilde{u}_0) := \left\{u \in X : \Big| \quad |||u - \tilde{u}_0|||_{X_{R^2}} \leqq \varepsilon\right\}$$

be the ball in X_{R^2} with center $\tilde{u}_0$ and radius ε. By the triangle inequality, we have

$$\begin{aligned}\|u\|_{X_{R^2}} &\leqq \|\tilde{u}_0\|_{X_{R^2}} + \|u - \tilde{u}_0\|_{X_{R^2}} \leqq \varepsilon + \|\tilde{u}_0\|_{X_{R^2}} \leqq \varepsilon + C[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)},\\ &\forall u \in \mathbf{B}_\varepsilon(\tilde{u}_0).\end{aligned} \tag{3.7}$$

In particular, we have

Lemma 3.2. *For* $0 < R \leqq +\infty$*, if* $u_0 : \mathbb{R}^n \to N$ *satisfies* $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq \varepsilon$*, then*

$$\|u\|_{L^\infty(\mathbb{R}^n\times[0,R^2])} \leqq C, \quad \|u\|_{X_{R^2}} \leqq C\varepsilon, \quad \forall u \in \mathbf{B}_\varepsilon(\tilde{u}_0) \tag{3.8}$$

for some $C = C(n) > 0$.

Now we are ready to prove Theorem 1.3. First we need the following two Lemmas.

Lemma 3.3. *There exists* $\varepsilon_1 > 0$ *such that if for* $R > 0$, $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq \varepsilon_1$ *then* $\mathbf{T}$ *maps* $\mathbf{B}_{\varepsilon_1}(\tilde{u}_0)$ *to* $\mathbf{B}_{\varepsilon_1}(\tilde{u}_0)$.

Proof. It follows from the formula (3.5) that $\mathbf{T}(u) - \tilde{u}_0 = \mathbb{S}(A(u)(\nabla u, \nabla u))$ for $u \in \mathbf{B}_{\varepsilon_1}(\tilde{u}_0)$. Hence Lemma 3.1 and Lemma 2.1 imply

$$\begin{aligned}
|||\mathbf{T}(u) - \tilde{u}_0|||_{X_{R^2}} &\leqq C\|A(u)(\nabla u, \nabla u)\|_{Y_{R^2}} \\
&= C\Bigg[\sup_{0<t\leqq R^2} t\|A(u)(\nabla u, \nabla u)(t)\|_{L^\infty(\mathbb{R}^n)} \\
&\quad + \sup_{x\in\mathbb{R}^n, 0<r\leqq R} r^{-n}\int_{P_r(x,r^2)} |A(u)(\nabla u, \nabla u)|\Bigg] \\
&\lesssim \Bigg(\sup_{0<t\leqq R^2} \sqrt{t}\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \\
&\quad + \sup_{x\in\mathbb{R}^n, 0<r\leqq R} (r^{-n}\int_{P_r(x,r^2)} |\nabla u|^2)^{\frac{1}{2}}\Bigg)^2 \\
&\lesssim \|u\|^2_{X_{R^2}} \leqq C\varepsilon_1^2 \leqq \varepsilon_1,
\end{aligned}$$

provided $\varepsilon_1 > 0$ is chosen to be sufficiently small. This completes the proof. □

Lemma 3.4. *There exist* $0 < \varepsilon_2 \leqq \varepsilon_1$ *and* $\theta_0 \in (0, 1)$ *such that if for* $R > 0[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq \varepsilon_2$ *then* $\mathbf{T} : \mathbf{B}_{\varepsilon_2}(\tilde{u}_0) \to \mathbf{B}_{\varepsilon_2}(\tilde{u}_0)$ *is a* θ_0*-contraction map, that is*

$$|||\mathbf{T}(u) - \mathbf{T}(v)|||_{X_{R^2}} \leqq \theta_0|||u - v|||_{X_{R^2}}, \quad \forall u, v \in \mathbf{B}_{\varepsilon_2}(\tilde{u}_0).$$

Proof. For $u, v \in \mathbf{B}_{\varepsilon_2}(\tilde{u}_0)$, we have

$$\begin{aligned}
|\mathbf{T}u - \mathbf{T}v| &= |\mathbb{S}(A(u)(\nabla u, \nabla u) - A(v)(\nabla v, \nabla v))| \\
&\lesssim |\mathbb{S}(A(u)(\nabla u, \nabla u) - A(u)(\nabla v, \nabla v))| \\
&\quad +|\mathbb{S}(A(u)(\nabla v, \nabla v) - A(v)(\nabla v, \nabla v))| \\
&\lesssim \mathbb{S}((|\nabla u| + |\nabla v|)|\nabla(u - v)|) + \mathbb{S}(|\nabla v|^2|u - v|).
\end{aligned}$$

Hence, by Lemma 3.1, we obtain

$$|||\mathbf{T}u-\mathbf{T}v|||_{X_{R^2}} \lesssim \|(|\nabla u|+|\nabla v|)|\nabla(u-v)|\|_{Y_{R^2}} + \||\nabla v|^2|u - v|\|_{Y_{R^2}} = I+II$$

I and II can be estimated as follows.

$$
\begin{aligned}
I =& \sup_{0<t\leqq R^2} t\|(|\nabla u|+|\nabla v|)|\nabla(u-v)(t)|\|_{L^\infty(\mathbb{R}^n)} \\
&+ \sup_{x\in\mathbb{R}^n, 0<r\leqq R} r^{-n}\int_{P_r(x,r^2)} (|\nabla u|+|\nabla v|)|\nabla(u-v)| \\
\leqq& \sup_{0<t\leqq R^2} \sqrt{t}(\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)}+\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)}) \sup_{0<t\leqq R^2} \sqrt{t}\|\nabla(u-v)(t)\|_{L^\infty(\mathbb{R}^n)} \\
&+ \sup_{x\in\mathbb{R}^n, 0<r\leqq R} \left(r^{-n}\int_{P_r(x,r^2)} |\nabla u|^2+|\nabla v|^2\right)^{\frac{1}{2}} \\
&\times \sup_{x\in\mathbb{R}^n, 0<r\leqq R} \left(r^{-n}\int_{P_r(x,r^2)} |\nabla(u-v)|^2\right)^{\frac{1}{2}} \\
\leqq& C\varepsilon_2\left[\sup_{0<t\leqq R^2} \sqrt{t}\|\nabla(u-v)(t)\|_{L^\infty(\mathbb{R}^n)}\right. \\
&\left.+ \sup_{x\in\mathbb{R}^n, 0<r\leqq R} \left(r^{-n}\int_{P_r(x,r^2)} |\nabla u|^2+|\nabla v|^2\right)^{\frac{1}{2}}\right] \\
\leqq& C\varepsilon_2|||u-v|||_{X_{R^2}}. \\
II =& \sup_{0<t\leqq R^2} t\||\nabla v|^2|u-v|(t)|\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n, 0<r\leqq R} r^{-n}\int_{P_r(x,r^2)} |\nabla v|^2|u-v| \\
\leqq& \left[\sup_{0<t\leqq R^2} \sqrt{t}\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n, 0<r\leqq R} (r^{-n}\int_{P_r(x,r^2)} |\nabla v|^2)^{\frac{1}{2}}\right]^2 \\
&\times \sup_{0<t\leqq R^2} \|(u-v)(t)\|_{L^\infty(\mathbb{R}^n)} \\
\leqq& C\|v\|_{X_{R^2}}^2 \sup_{0<t\leqq R^2} \|(u-v)(t)\|_{L^\infty(\mathbb{R}^n)} \leqq C_4\varepsilon_2^2|||u-v|||_{X_{R^2}},
\end{aligned}
$$

where we have used Lemma 3.2 in the last step. Putting these two estimates together yields

$$
|||\mathbf{T}u-\mathbf{T}v|||_{X_{R^2}} \leqq C(1+\varepsilon_2)\varepsilon_2|||u-v|||_{X_{R^2}} \leqq \theta_0|||u-v|||_{X_{R^2}}
$$

for some $\theta_0 = \theta_0(\varepsilon_2) \in (0,1)$, provided $\varepsilon_2 > 0$ is sufficiently small. □

Proof of Theorem 1.3. It follows from Lemma 3.3, 3.4 and the fixed point theorem that there exists $\varepsilon_0 = \varepsilon_0(n, N) > 0$ such that if $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq \varepsilon_0$ for some $R > 0$, then there exists a unique $u \in X_{R^2}$ such that

$$
u = \tilde{u}_0 + \mathbb{S}(A(u)(\nabla u, \nabla u)) \quad \text{on } \mathbb{R}^n \times [0, R^2],
$$

or equivalently

$$
u_t - \Delta u = A(u)(\nabla u, \nabla u) \quad \text{on } \mathbb{R}^n \times (0, R^2);\ u\big|_{t=0} = u_0. \tag{3.9}
$$

Now we need to show $u(\mathbb{R}^n \times [0, R^2]) \subset N$. First, observe that Lemma 2.1 implies that for $0 < t \leqq \frac{R^2}{K^2}$,

$$\begin{aligned}\operatorname{dist}(u, N) &\leqq \operatorname{dist}(\tilde{u}_0, N) + \|u - \tilde{u}_0\|_{L^\infty\left(\mathbb{R}^n \times \left[0, \frac{R^2}{K^2}\right]\right)} \\ &\leqq \delta + K^n [u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} + \varepsilon_0 \\ &\leqq \delta + (1 + K^n)\varepsilon_0 \leqq \delta_N,\end{aligned}$$

provided $\delta \leqq \frac{\delta_N}{2}$ and $\varepsilon_0 \leqq \frac{\delta_N}{2(1+K^n)}$. This yields $u(\mathbb{R}^n \times [0, \frac{R^2}{K^2}]) \subset N_{\delta_N}$. This and the definition of $A(\cdot)(\cdot, \cdot)$ imply

$$A(u)(\nabla u, \nabla u) = -\nabla^2 \Pi(u)(\nabla u, \nabla u) \quad \text{on} \quad \mathbb{R}^n \times \left[0, \frac{R^2}{K^2}\right].$$

Set $Q(y) = y - \Pi(y)$ for $y \in N_{\delta_N}$, and $\rho(u) = \frac{1}{2}|Q(u)|^2$. Then direct calculations imply that for any $y \in N_{\delta_N}$,

$$\nabla Q(y)(v) = (\mathrm{Id} - \nabla \Pi(y))(v), \quad \forall v \in \mathbb{R}^l,$$

and

$$\nabla^2 Q(y)(v, w) = -\nabla^2 \Pi(y)(v, w), \quad \forall v, w \in \mathbb{R}^l.$$

Since $u \in X_{R^2}$, it follows from the definition of X_{R^2} that $\nabla u \in L^\infty(\mathbb{R}^n \times [\varepsilon^2, R^2])$ for any $\varepsilon > 0$, the higher order regularity theory of (3.9) implies $u \in C^2(\mathbb{R}^n \times [\varepsilon^2, R^2])$ for any $\varepsilon > 0$. Hence we have

$$\begin{aligned}(\partial_t - \Delta)\rho(u) &= \langle Q(u), \nabla Q(u)(\partial_t u - \Delta u) - \nabla^2 Q(u)(\nabla u, \nabla u)\rangle - |\nabla(Q(u))|^2 \\ &= \langle Q(u), -\nabla Q(u)(\nabla^2 \Pi(u)(\nabla u, \nabla u)) \\ &\quad -\nabla^2 Q(u)(\nabla u, \nabla u)\rangle - |\nabla(Q(u))|^2 \\ &= \langle Q(u), \nabla \Pi(u)(\nabla^2 \Pi(u)(\nabla u, \nabla u))\rangle - |\nabla(Q(u))|^2 \\ &= -|\nabla(Q(u))|^2 \leqq 0,\end{aligned} \tag{3.10}$$

where we have used the fact that $Q(u) \perp T_{\Pi(u)}N$ and $\nabla \Pi(u)(\nabla^2 \Pi(\nabla u, \nabla u)) \in T_{\Pi(u)}N$ in the last step.

Since $\rho(u)|_{t=0} = 0$, the maximum principle for (3.10) implies $\rho(u) \equiv 0$ on $\mathbb{R}^n \times [0, \frac{R^2}{K^2}])$. One can repeat the same argument to show that $u(\mathbb{R}^n \times [\frac{R^2}{K^2}, R^2]) \subset N$. Thus the proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4. The proof of Theorem 1.4 follows directly from Theorem 1.3 with R replaced by $+\infty$. □

4. Proof of Theorems 1.5 and 1.6

This section is devoted to the proof of Theorems 1.5 and 1.6 on local and global well-posedness of hydrodynamic flow of liquid crystals.

For $(u_0, d_0) : \mathbb{R}^n \to \mathbb{R}^n \times S^2$, let $(\tilde{u}_0, \tilde{d}_0) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{R}^3$ denote the caloric extension of (u_0, d_0).

First, we recall Carleson's characterization of $u_0 \in \mathrm{BMO}_R^{-1}(\mathbb{R}^n)$ for $R > 0$, due to Koch and Tataru [11], which asserts that the following is equivalent

$$[u_0]_{\mathrm{BMO_R^{-1}}(\mathbb{R}^n)} \approx \sup_{x\in\mathbb{R}^n, 0<r\leqq R} \left(r^{-n} \int_{P_r(x,r^2)} |\tilde{u}_0|^2 \right)^{\frac{1}{2}}. \tag{4.1}$$

Notice that since $\tilde{u}_0$ solves the heat equation on $\mathbb{R}^n$, the Harnack estimate of heat equations implies that

$$\sup_{0<t\leqq R^2} \sqrt{t}\|\tilde{u}_0\|_{L^\infty} \lesssim \sup_{x\in\mathbb{R}^n, 0<r\leqq R} \left(r^{-n} \int_{P_r(x,r^2)} |\tilde{u}_0|^2 \right)^{\frac{1}{2}} \approx [u_0]_{\mathrm{BMO}^{-1}(\mathbb{R}^n)}. \tag{4.2}$$

In particular, $u_0 \in \mathrm{BMO}_R^{-1}(\mathbb{R}^n)$ implies that $\tilde{u}_0 \in Z_{R^2}$ and

$$\|\tilde{u}_0\|_{Z_{R^2}} \lesssim \|u_0\|_{\mathrm{BMO}_R^{-1}(\mathbb{R}^n)}. \tag{4.3}$$

Let $\mathbb{P} : L^2(\mathbb{R}^n) \to \mathbb{P}L^2(\mathbb{R}^n)$ denote the Leray projection operator. Then (1.7)–(1.8) and $u|_{t=0} = u_0$ is equivalent to

$$u(t) = \mathbb{T}_1[u, d](t) := \tilde{u}_0(t) - \mathbb{V}[u \otimes u + \nabla d \otimes \nabla d](t), \tag{4.4}$$

where the operator $\mathbb{V}$ is defined by

$$\mathbb{V} f(t) = \int_0^t e^{-(t-s)\Delta} \mathbb{P}\nabla \cdot f(s)\, \mathrm{d}s, \quad \forall f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n. \tag{4.5}$$

The following estimate on the operator $\mathbb{V}$ has been proved by Koch–Tataru ([KT] Lemma 3.2).

Lemma 4.1. *For $0 < T \leqq +\infty$, if $f = (f_1, \ldots, f_n) \in Y_T$, then*

$$\|V f\|_{Z_T} \leqq C\|f\|_{Y_T} \tag{4.6}$$

for some constant $C = C(n) > 0$.

Observe that (1.9) and $d|_{t=0} = d_0$ is equivalent to

$$d(t) = \mathbb{T}_2[u, d](t) := \tilde{d}_0(t) + \mathbb{S}[-\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d) - u \cdot \nabla d](t), \tag{4.7}$$

where $\mathbb{S}$ is the operator defined by (3.1), and $\Pi_{S^2} \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ has the property

$$\Pi_{S^2}(d) = \frac{d}{|d|} : S^2_{\frac{1}{2}} \equiv \left\{ y \in \mathbb{R}^3 : \frac{1}{2} \leqq |y| \leqq \frac{3}{2} \right\} \to S^2.$$

Let $(u_0, d_0) \in \mathrm{BMO}_R^{-1}(\mathbb{R}^n) \times \mathrm{BMO}_R(\mathbb{R}^n)$ for some $R > 0$. Then $(\tilde{u}_0, \tilde{d}_0) \in Z_{R^2} \times X_{R^2}$. For $\varepsilon > 0$, we define the ball $\mathbb{B}_\varepsilon([\tilde{u}_0, \tilde{d}_0])$ in $Z_{R^2} \times X_{R^2}$ with center $(\tilde{u}_0, \tilde{d}_0)$ and radius ε by

$$\mathbb{B}_\varepsilon([\tilde{u}_0, \tilde{d}_0]) = \left\{(u, d) \in Z_{R^2} \times X_{R^2} : \|u - \tilde{u}_0\|_{Z_{R^2}} + |||d - \tilde{d}_0|||_{X_{R^2}} \leqq \varepsilon\right\}.$$

Define the mapping operator T on $Z_{R^2} \times X_{R^2}$ by

$$\mathbb{T}[u, d] = (\mathbb{T}_1[u, d], \mathbb{T}_2[u, d]).$$

Analogous to Lemma 3.2 and 3.3, we have the following two Lemmas.

Lemma 4.2. *There exists* $\varepsilon_1 > 0$ *such that if*

$$\|u_0\|_{\mathrm{BMO}_R^{-1}(\mathbb{R}^n)} + [d_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq \varepsilon_1$$

then $\mathbb{T}$ *maps* $\mathbb{B}_{\varepsilon_1}([\tilde{u}_0, \tilde{d}_0])$ *to* $\mathbb{B}_{\varepsilon_1}([\tilde{u}_0, \tilde{d}_0])$.

Proof. For $(u, d) \in \mathbb{B}_{\varepsilon_1}([\tilde{u}_0, \tilde{d}_0])$, we have that $\|d\|_{L^\infty(\mathbb{R}^n \times [0, R^2])} \leqq C$ and

$$\begin{aligned}
&\mathbb{T}[u, d] - (\tilde{u}_0, \tilde{d}_0) \\
&\quad = \Big(-\mathbb{V}[u \otimes u + \nabla d \otimes \nabla d], \quad \mathbb{S}[-\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d) - u \cdot \nabla d]\Big).
\end{aligned}$$

Therefore, applying Lemma 3.1 and Lemma 4.1, we have

$$\begin{aligned}
&\|\mathbb{T}_1[u, d] - \tilde{u}_0\|_{Z_{R^2}} + |||\mathbb{T}_2[u, d] - \tilde{d}_0|||_{X_{R^2}} \\
&\quad \lesssim \|u \otimes u + \nabla d \otimes \nabla d\|_{Y_{R^2}} + \|\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d) - u \cdot \nabla d\|_{Y_{R^2}} \\
&\quad \lesssim \left(\|u\|_{Z_{R^2}} + \|d\|_{X_{R^2}}\right)^2 \\
&\quad \lesssim \left(\|u - \tilde{u}_0\|_{Z_{R^2}} + \left\|d - \tilde{d}_0\right\|_{X_{R^2}} + \|\tilde{u}_0\|_{Z_{R^2}} + \left\|\tilde{d}_0\right\|_{X_{R^2}}\right)^2 \\
&\quad \leqq C\varepsilon_1^2 \leqq \varepsilon_1
\end{aligned}$$

provided $\varepsilon_1 > 0$ is chosen to be sufficiently small, where we have used the estimate

$$\|\tilde{u}_0\|_{Z_{R^2}} + \|\tilde{d}_0\|_{X_{R^2}} \lesssim \|u_0\|_{\mathrm{BMO}_R^{-1}(\mathbb{R}^n)} + [d_0]_{\mathrm{BMO}_R(\mathbb{R}^n)}$$

in the last step. □

Lemma 4.3. *There exist* $0 < \varepsilon_2 \leqq \varepsilon_1$ *and* $\theta_0 \in (0, 1)$ *such that if*

$$\|u_0\|_{\mathrm{BMO}_R^{-1}(\mathbb{R}^n)} + [d_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq \varepsilon_2$$

then $\mathbb{T} : \mathbb{B}_{\varepsilon_2}([\tilde{u}_0, \tilde{d}_0]) \to \mathbb{B}_{\varepsilon_2}([\tilde{u}_0, \tilde{d}_0])$ *is* θ_0*-contractive, that is*

$$\begin{aligned}
&\|\mathbb{T}_1[u_1, d_1] - \mathbb{T}_1[u_2, d_2]\|_{Z_{R^2}} + |||\mathbb{T}_2[u_1, d_1] - \mathbb{T}_2[u_2, d_2]|||_{X_{R^2}} \\
&\quad \leqq \theta_0(\|u_1 - u_2\|_{Z_{R^2}} + |||d_1 - d_2|||_{X_{R^2}})
\end{aligned}$$

for any (u_1, d_1) $(u_2, d_2) \in B_{\varepsilon_2}([\tilde{u}_0, \tilde{d}_0])$.

Proof. For any $(u_1, d_1)(u_2, d_2) \in B_{\varepsilon_2}([\tilde{u}_0, \tilde{d}_0])$, we have

$$\begin{aligned}
&|\mathbb{T}_1[u_1, d_1] - \mathbb{T}_1[u_2, d_2]| \\
&\quad = |\mathbb{V}[u_1 \otimes u_1 + \nabla d_1 \otimes \nabla d_1 - u_2 \otimes u_2 - \nabla d_2 \otimes \nabla d_2]| \\
&\quad \lesssim \mathbb{V}((|u_1| + |u_2|)|u_1 - u_2| + (|\nabla d_1| + |\nabla d_2|)|\nabla(d_1 - d_2)|),
\end{aligned}$$

and

$$\begin{aligned}
&|\mathbb{T}_2[u_1, d_1] - \mathbb{T}_2[u_2, d_2]| \\
&= |\mathbb{S}[-\nabla^2\Pi_{S^2}(d_1)(\nabla d_1, \nabla d_1) - u_1 \cdot \nabla d_1 + \nabla^2\Pi_{S^2}(d_2)(\nabla d_2, \nabla d_2) + u_2 \cdot \nabla d_2]| \\
&\lesssim \mathbb{S}((|\nabla d_1| + |\nabla d_2| + |u_1|)|\nabla(d_1 - d_2)| + |\nabla d_2|^2|d_1 - d_2| + |u_1 - u_2||\nabla d_2|).
\end{aligned}$$

Thus Lemma 3.1 and Lemma 4.1 imply

$$\begin{aligned}
&\|\mathbb{T}_1[u_1, d_1] - \mathbb{T}_1[u_2, d_2]\|_{Z_{R^2}} + |||\mathbb{T}_2[u_1, d_1] - \mathbb{T}_2[u_2, d_2]|||_{X_{R^2}} \\
&\quad \lesssim \|(|u_1| + |u_2|)|u_1 - u_2| + (|\nabla d_1| + |\nabla d_2|)|\nabla(d_1 - d_2)|\|_{Y_{R^2}} \\
&\qquad + \|(|\nabla d_1| + |\nabla d_2| + |u_1|)|\nabla(d_1 - d_2)| + |\nabla d_2|^2|d_1 - d_2| \\
&\qquad + |u_1 - u_2||\nabla d_2|\|_{Y_{R^2}} \\
&\quad \leqq C\varepsilon_2\left[\|u_1 - u_2\|_{Z_{R^2}} + |||d_1 - d_2|||_{X_{R^2}}\right] \\
&\quad \leqq \theta_0\left[\|u_1 - u_2\|_{Z_{R^2}} + |||d_1 - d_2|||_{X_{R^2}}\right]
\end{aligned}$$

for some $\theta_0 \in (0, 1)$, provided $\varepsilon_2 > 0$ is chosen to be sufficiently small, where we have used

$$\|u_i\|_{Z_{R^2}} + \|d_i\|_{X_{R^2}} \leqq C\varepsilon_2, \quad i = 1, 2$$

in the last steps. This completes the proof. □

Proof of Theorem 1.5. It follows directly from Lemma 4.2, Lemma 4.3 and the fixed point theory that there exists $\varepsilon_0 > 0$ such that if

$$\|u_0\|_{\mathrm{BMO}_R^{-1}(\mathbb{R}^n)} + [d_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq \varepsilon_0,$$

then there exists $(u, d) \in Z_{R^2} \times X_{R^2}$ such that (1.7), (1.8), (1.12), and (1.9) replaced by

$$d_t + u \cdot \nabla d - \Delta d = -\nabla^2\Pi_{S^2}(d)(\nabla d, \nabla d) \tag{4.8}$$

hold. To complete the proof, we need to show $d(\mathbb{R}^n \times [0, R^2]) \subset S^2$. This step is similar to the proof of Theorem 1.3. First, Lemma 2.1 implies that for $t \leqq \frac{R^2}{K^2}$,

$$\mathrm{dist}(d, S^2) \leqq \varepsilon_0 + \delta + K^2[d_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leqq (1 + K^n)\varepsilon_0 + \delta \leqq \frac{1}{2},$$

provided $\delta \leqq \frac{1}{4}$ and $\varepsilon_0 \leqq \frac{1}{4(1+K^n)}$. Thus $d(\mathbb{R}^n \times [0, \frac{R^2}{K^2}]) \subset S^2_{\frac{1}{2}}$. Now consider the function $\rho(d) = \frac{1}{2}|d - \Pi_{S^2}(d)|^2$. Then the same calculation as in the proof of Theorem 1.3 gives

$$(\rho(d))_t + u \cdot \nabla(\rho(d)) - \Delta(\rho(d)) = -|\nabla(d - \Pi_{S^2}(d))|^2 \leqq 0.$$

Since $\rho(d)\big|_{t=0} = 0$, the maximum principle implies $\rho(d) \equiv 0$ on $\mathbb{R}^n \times [0\frac{R^2}{K^2}]$ and $d(\mathbb{R}^n \times [0, \frac{R^2}{K^2}]) \subset S^2$. Repeating the same argument can imply $d(\mathbb{R}^n \times [\frac{R^2}{K^2}, R^2)) \subset S^2$. The proof is complete. □

Proof of Theorem 1.6. The proof of Theorem 1.6 follows directly from Theorem 1.5 with R replaced by $R = +\infty$. □

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References

1. CHEN, Y., DING, W.: Blow-up and global existence for heat flows of harmonic maps. *Invent. Math.* **99**, no. 3, 567–578 (1990)
2. CHANG, K., DING, W., YE, R.: Finite-time blow-up of the heat flow of harmonic maps from surfaces. *J. Differ. Geom.* **36**(2), 507–515 (1992)
3. CORON, J., GHIDAGLIA, J.: Explosion en temps fini pour le flot des applications harmoniques. *C. R. Acad. Sci. Paris Ser I* **308**, 339–344 (1989)
4. CHEN, Y., LIN, F.: Evolution of harmonic maps with Dirichlet boundary conditions. *Comm. Anal. Geom.* **1**, no. 3–4, 327–346 (1993)
5. CHEN, Y., STRUWE, M.: Existence and partial regularity for heat flow for harmonic maps. *Math. Z.* **201**, 83–103 (1989)
6. EELLS, J., SAMPSON, J.: Harmonic mappings of Riemannian manifolds. *Am. J. Math.* **86**, 109–160 (1964)
7. ERICKSEN, J.L.: Hydrostatic theory of liquid crystal. *Arch. Ration. Mech. Anal.* **9**, 371–378 (1962)
8. DE GENNES, P.G.: The Physics of Liquid Crystals. Oxford, 1974
9. HILDEBRANDT, S., KAUL, H., WIDMAN, K.: An existence theorem for harmonic mappings of Riemannian manifolds. *Acta Math.* **138**, no. 1–2, 1–16 (1977)
10. KOCH, H., LAMM, T.: Geometric flows with rough initial data. arXiv: 0902.1488v1, 2009
11. KOCH, H., TATARU, D.: Well-posedness for the Navier–Stokes equations. *Adv. Math.* **157**, no. 1, 22–35 (2001)
12. LESLIE, F.M.: Some constitutive equations for liquid crystals. *Arch. Ration. Mech. Anal.* **28**, 265–283 (1968)
13. LIN, F., LIU, C.: Nonparabolic Dissipative Systems Modeling the Flow of Liquid Crystals. *CPAM* **XLVIII**, 501–537 (1995)
14. LIN, F., LIU, C.: Partial Regularity of The Dynamic System Modeling The Flow of Liquid Crystals. *DCDS* **2**, No. 1, 1–22 (1998)
15. LIN, F., LIN, J.Y., WANG, C.: Liquid crystal flows in two dimensions. *Arch. Ration. Mech. Anal.* (in press)
16. LIN, F., WANG, C.: The analysis of harmonic maps and their heat flows. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008. xii+267 pp. ISBN: 978-981-277-952-6
17. STEIN, E.: *Harmonic Analysis, Princeton Mathematical Series*, vol. 43, Princeton University Press, Princeton, 1993
18. STRUWE, M.: On the evolution of harmonic maps of Riemannian surfaces. *Comment. Math. Helv.* **60**, 558–581 (1985)

19. Wang, C.: Heat flow of harmonic maps whose gradients belong to $L_x^n L_t^\infty$. *Arch. Ration. Mech. Anal.* **188**, no. 2, 351–369 (2008)

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