Arch. Rational Mech. Anal. 200 (2011) 89–140
Digital Object Identifier (DOI) 10.1007/s00205-010-0337-3

Bistable Boundary Reactions in Two Dimensions

Juan Dávila, Manuel del Pino & Monica Musso

Communicated by P. Rabinowitz

Abstract

In a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary we consider the problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \frac{1}{\varepsilon} f(u) \quad \text{on } \partial\Omega,$$

where ν is the unit normal exterior vector, $\varepsilon > 0$ is a small parameter and f is a bistable nonlinearity such as $f(u) = \sin(\pi u)$ or $f(u) = (1 - u^2)u$. We construct solutions that develop multiple transitions from -1 to 1 and vice-versa along a connected component of the boundary $\partial\Omega$. We also construct an explicit solution when Ω is a disk and $f(u) = \sin(\pi u)$.

1. Introduction

In this paper we construct solutions of the following boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \frac{1}{\varepsilon} f(u) & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, ν is the unit normal exterior vector, $\varepsilon > 0$ is a small parameter and f is a bistable, balanced nonlinearity, namely $f = -W'$ where

$$W(s) > W(-1) = W(1) = 0 \quad \text{for all } s \in (-1, 1). \tag{1.2}$$

and

$$W'(-1) = W'(1) = 0, \quad W''(-1) > 0, \quad W''(1) > 0. \tag{1.3}$$

We also assume that f is of class C^3. Notable examples are the Allen–Cahn and the Peierls–Nabarro nonlinearities, respectively given by $f(u) = (1 - u^2)u$ and $f(u) = \sin(\pi u)$, for

$$W(u) = \frac{1}{4}(1 - u^2)^2 \quad \text{and} \quad W(u) = \frac{2}{\pi}\cos^2\left(\frac{\pi}{2}u\right).$$

Problem (1.1) is the Euler–Lagrange equation for the energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\partial\Omega} W(u). \tag{1.4}$$

Functionals of this type arise in models for various phenomena. For instance, KOHN and SLASTIKOV [11] proposed a model for thin soft ferromagnetic films. They established that under suitable rescalings, the micromagnetic energy Γ-converges to

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon}\int_{\partial\Omega} \cos^2(u - g) \tag{1.5}$$

where $g : \partial\Omega \to \mathbb{R}$ is a given function. In this work we will focus on the case $g =$constant.

ALBERTI ET AL. [1,2] studied a model for a two-phase fluid with surface and line tensions, which involved functionals with a double-well potential integrated in Ω and on the boundary. In [1,2] the authors calculated the Γ-limit of the functional $\frac{1}{|\log\varepsilon|}E_\varepsilon$ for nonlinearities that include $f(u) = u - u^3$ and showed that the transitions of limits of local minimizers occur at finitely many points in dimension 2, and on a geodesic of the boundary in dimension 3.

CABRÉ and CÓNSUL [6] found a *renormalized energy* in the two-dimensional case: they showed that if $2k$ is the number of transitions in a simply connected domain then $E_\varepsilon(u_\varepsilon) - \frac{4k}{\pi}|\log\varepsilon|$ approaches a multiplicative constant times the expression (1.9) below, which gives information on the location of the transition points.

KURZKE [12,13] studied the convergence of minimizers of (1.5) by showing that they develop transitions at finitely many points as $\varepsilon \to 0$ and estimating their location.

The shape of the transitions of local minimizers of (1.4) is related, after a suitable rescaling near a transition point, to solutions in a half plane $\{(x, y) : x \in \mathbb{R}, y > 0\}$ with $\varepsilon = 1$, for which some properties are known. Equation (1.1) in a half plane with $f(u) = \sin(u)$ and $\varepsilon = 1$ is known as the Peierls–Nabarro problem, which appears as a model of dislocations in crystals. TOLAND [14] proved that all nonconstant solutions to the Peierls–Nabarro problem are, except for translations in the x-direction, of the form

$$\pm 2\arctan\frac{x}{y+1} + 2\pi n \tag{1.6}$$

where $n \in \mathbb{N}$. He achieved this by finding a relation of the Peierls–Nabarro problem with the Benjamin–Ono equation, which is an equation of the form (1.1) in a half plane with boundary condition $f(u) = u^2 - u$ which appears in theoretical hydrodynamics. Uniqueness for the Benjamin–Ono equation, except for translations, was proved by Amick and Toland [3,4]. Cabré and Solà-Morales [7] analyzed (1.1) in a half plane or a half space of $\mathbb{R}^N$ with $\varepsilon = 1$ for general nonlinearities satisfying conditions (1.2) and (1.3) below. They built and established uniqueness properties of solutions that are monotonically increasing in one of the tangential variables.

As we will see in Section 2, when Ω is the unit disk in $\mathbb{R}^2$ and $f(u) = \sin(\pi u)$, we can find, rather surprisingly, an explicit solution with $2k$ transitions, for any given $k \geqq 1$. More precisely, we find a solution u_ε to Problem 1.1 such that $u_\varepsilon \to u^*$ where u^* is harmonic inside the disk, and its boundary values alternate between $+1$ and -1 at the $2k$ vertices of a regular polygon. Our construction corresponds precisely to the *harmonic conjugate* of one found by Bryan and Vogelius [5] for a boundary reaction arising in corrosion modeling.

Our main goal is to prove that in any two-dimensional domain Ω there are solutions with the same qualitative behavior of the explicit solution in a disk: given $k \geqq 1$ we will find a family of solutions u_ε which develops $2k$ transitions as $\varepsilon \to 0$ between values close to $+1$ and -1 on $\partial\Omega$. To be more precise, let Γ_0 denote the outer component of $\partial\Omega$ and $\Gamma_1, \ldots, \Gamma_m$ denote the inner components ($m \geqq 0$, with $m = 0$ meaning there are no inner components).

In what follows, we fix a component Γ_{i_0} with index $i_0 \in \{0, \ldots, m\}$ and an integer $k \geqq 1$. For points $\xi_1, \ldots, \xi_{2k}$ in Γ_{i_0} arranged increasingly with respect to some fixed orientation, let us define the function $u^* = u^*(\xi_1, \ldots, \xi_{2k})$ to be the solution of the equation

$$\Delta u^* = 0 \quad \text{in } \Omega$$

with the boundary conditions on Γ_{i_0}:

$$u^* = 1 \quad \text{on the segment from } \xi_j \text{ to } \xi_{j+1} \text{ if } j \text{ is odd}$$
$$u^* = -1 \quad \text{on the segment from } \xi_j \text{ to } \xi_{j+1} \text{ if } j \text{ is even}$$

and

$$u^* = 1 \quad \text{on } \Gamma_i \text{ with } i \neq i_0.$$

The following is our main result.

Theorem 1.1. *For all sufficiently small $\varepsilon > 0$, there exist at least two solutions $u_\varepsilon^l, l = 1, 2$ of (1.1) such that, up to subsequences, there are two distinct arrays of $2k$ points of $\Gamma_{i_0}\xi^l = (\xi_1^l, \ldots, \xi_{2k}^l), l = 1, 2$ such that*

$$u_\varepsilon^l \to u^*\left(\xi_1^l, \ldots, \xi_{2k}^l\right) \quad \text{as } \varepsilon \to 0$$

uniformly on compact subsets of $\bar{\Omega}\backslash\{\xi_1^l, \ldots, \xi_{2k}^l\}$. Furthermore, the points ξ^l, $l = 1, 2$ are critical points of the function ψ_k defined in (1.9).

The location of the transition points $\xi_1, \ldots, \xi_{2k}$ turns out to be accurately characterized as a critical point of a function ψ_k defined explicitly in terms of $G(x, y)$, the Green function for the following Neumann problem. For any $y \in \partial\Omega$, let

$$\begin{cases} -\Delta G(x,y) = 0 \quad \text{in } \Omega \\ \dfrac{\partial G}{\partial \nu_x}(x,y) = 2\pi\delta_y(x) - \dfrac{2\pi}{|\Gamma_{i_0}|} \quad \text{on } \Gamma_{i_0} \\ \displaystyle\int_{\Gamma_{i_0}} G(x,y) = 0 \\ \dfrac{\partial G}{\partial \nu_x}(x,y) = 0 \quad \text{on } \Gamma_i, \quad i \neq i_0. \end{cases} \tag{1.7}$$

We denote by $H(x,y)$ its regular part:

$$H(x,y) = G(x,y) - \log\frac{1}{|x-y|^2}. \tag{1.8}$$

We define

$$\psi_k(\xi) = \psi_k(\xi_1,\ldots,\xi_{2k}) = \sum_{l=1}^{2k} H(\xi_l,\xi_l) + \sum_{j\neq l}(-1)^{l+j}G(\xi_j,\xi_l), \tag{1.9}$$

which is the renormalized energy found by CABRÉ and CÒNSUL [6], also similar to that found by KURZKE [12, 13].

The asymptotic location of the transition points of the two solutions found correspond, in the simply connected case, to a global maximum and a saddle point of ψ_k, respectively. The solutions in Theorem 1.1 do not correspond to local minimizers of the energy E_ε, and Γ-convergence methods do not appear to be suitable to predict their presence.

We remark that in the non-simply connected case we can also construct solutions like those in Theorem 1.1 exhibiting transitions simultaneously on any given set of components Γ_i of $\partial\Omega$. The changes needed in the proof are minor, and would amount to introducing further notation. For the sake of simplicity we shall deal only with the case of a single component.

In Section 3 we outline the proof of this result, with detailed proofs carried out in Sections 4–11.

2. Explicit solutions in the disk

Let $k \geqq 1$ be an integer and consider $\xi_j \in \partial D$, $j = 0,\ldots,2k-1$ corresponding to vertices of a regular polygon, ordered counterclockwise. Given $\alpha > 1$ we define

$$\theta_j(z) = -\arg\left(\frac{\alpha\xi_j - z}{\xi_j}\right) \quad j = 0,\ldots,2k-1 \tag{2.1}$$

where the argument is such that $arg(e^{i\theta}) = \theta, \theta \in (-\pi,\pi)$. Then $\theta_j(z)$ is well defined and harmonic in $\mathbb{R}^2$ except the line segment $\{s\xi_j : s \geqq \alpha\}$. In particular θ_j

is harmonic in D and for $z \in D$, $\theta_j(z) \in (-\pi/2, \pi/2)$. Define

$$u_\varepsilon = \frac{2}{\pi}\sum_{j=0}^{2k-1}(-1)^j\theta_j \quad \text{where} \quad \varepsilon = \frac{4}{k\pi}\frac{\alpha^{2k}-1}{\alpha^{2k}+1}. \tag{2.2}$$

Then u_ε is harmonic in D and satisfies

$$\varepsilon\frac{\partial u_\varepsilon}{\partial \nu} = \sin(\pi u_\varepsilon) \quad \text{on } \partial D. \tag{2.3}$$

If $\varepsilon \to 0$ then $\alpha \to 1$ and $u_\varepsilon \to u^*$, which is a harmonic function in D with boundary values ± 1. More precisely $u^* = 1$ between ξ_j and ξ_{j+1} if j is odd and $u^* = -1$ in this range if j is even. Moreover, rescaling appropriately around a transition point we find, up to a factor, the solutions (1.6) of the Peierls–Nabarro problem.

Next we check the validity of the boundary relation (2.3). We slightly modify the notation above: we let $\xi_j \in \partial D$, $j = 0, \ldots, 2k-1$ be the vertices of a regular polygon, ordered counterclockwise, and set

$$u_\varepsilon = \sum_{j=0}^{2k-1}(-1)^j\theta_j \quad \text{where} \quad \varepsilon = \frac{2}{k}\frac{\alpha^{2k}-1}{\alpha^{2k}+1}$$

and θ_j is defined in (2.1).

Proposition 2.1. *Then u_ε is harmonic in D and satisfies*

$$\varepsilon\frac{\partial u_\varepsilon}{\partial \nu} = \sin(2u_\varepsilon) \quad \text{on } \partial D. \tag{2.4}$$

For the proof we follow the calculations of Bryan and Vogelius [5], who find an explicit solution to the problem

$$\Delta v = 0 \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 2\varepsilon \sinh v \quad \text{on } \partial\Omega \tag{2.5}$$

when $\Omega = D$ is the unit disk and $\varepsilon > 0$. This solution has the form

$$v_\varepsilon(x) = \sum_{j=1}^{2k}(-1)^{j-1}\log\frac{1}{|x-\alpha_k\xi_j|^2}$$

where $\alpha_k = [(k+2\varepsilon)/(k-2\varepsilon)]^{\frac{1}{2k}}$. The function u_ε is up to a factor the harmonic conjugate of v_ε.

Proof of Proposition 2.1. We write $u = u_\varepsilon$. We may assume that $\xi_j = e^{i\beta_j}$ where $\beta_j = \frac{j\pi}{k}$, $j = 0, \ldots, 2k-1$. Let $z = e^{i\theta} \in \partial B_1$. Then $|z - \alpha\xi_j|^2 = 1 + \alpha^2 - 2\alpha\cos(\theta - \beta_j)$. We also have the following formulas

$$\sin\theta_j = \frac{\sin(\theta-\beta_j)}{\left(1+\alpha^2-2\alpha\cos(\theta-\beta_j)\right)^{1/2}} \tag{2.6}$$

$$\cos\theta_j = \frac{\alpha-\cos(\theta-\beta_j)}{\left(1+\alpha^2-2\alpha\cos(\theta-\beta_j)\right)^{1/2}} \tag{2.7}$$

Let us compute $\sin(2u) = Im(e^{2iu})$. We write

$$e^{iu} = \frac{\prod_{j=0,\, j \text{ even}}^{2k-2} e^{i\theta_j}}{\prod_{j=1,\, j \text{ odd}}^{2k-1} e^{i\theta_j}}.$$

Using (2.6), (2.7) we have

$$\prod_{j=0,\, j \text{ even}}^{2k-2} e^{i\theta_j} = \left(\prod_{j=0,\, j \text{ even}}^{2k-2} \Big(1+\alpha^2 - 2\alpha\cos(\theta-\beta_j)\Big)^{-1/2} \right) \times \left(\prod_{j=0,\, j \text{ even}}^{2k-2} (\alpha - e^{-i(\theta-\beta_j)}) \right).$$

It is proved in [5] that

$$\prod_{j=0,\, j \text{ even}}^{2k-2} \Big(1+\alpha^2 - 2\alpha\cos(\theta-\beta_j)\Big) = \alpha^{2k} - 2\alpha^k\cos(k\theta) + 1 \tag{2.8}$$

$$\prod_{j=1,\, j \text{ odd}}^{2k-1} \Big(1+\alpha^2 - 2\alpha\cos(\theta-\beta_j)\Big) = \alpha^{2k} + 2\alpha^k\cos(k\theta) + 1 \tag{2.9}$$

Hence

$$\prod_{j=0,\, j \text{ even}}^{2k-2} \Big(1+\alpha^2 - 2\alpha\cos(\theta-\beta_j)\Big)^{-1/2} = (\alpha^{2k} - 2\alpha^k\cos(k\theta) + 1)^{-1/2}.$$

We also have

$$\prod_{j=0,\, j \text{ even}}^{2k-2} (\alpha - e^{-i(\theta-\beta_j)}) = \alpha^k - e^{-ik\theta}.$$

Indeed, setting $\lambda = \alpha e^{i\theta}$ we have

$$\prod_{j=0,\, j \text{ even}}^{2k-2} (\alpha - e^{-i(\theta-\beta_j)}) = e^{-ik\theta} \prod_{\ell=0}^{k-1} \Big(\lambda - e^{\frac{2\pi i}{k}\ell}\Big)$$

and the product is a polynomial in λ, of degree k, with leading coefficient equal to 1 and roots $e^{\frac{2\pi i}{k}\ell}$, $\ell = 0, \dots, k-1$. Thus the product equals $\lambda^n - 1$ and the formula follows. Then

$$\prod_{j=0,\, j \text{ even}}^{2k-2} e^{i\theta_j} = \frac{\alpha^k - e^{-ik\theta}}{(\alpha^{2k} - 2\alpha^k\cos(k\theta)+1)^{1/2}}.$$

Similarly

$$\prod_{j=1, j \text{ odd}}^{2k-1} e^{i\theta_j} = \frac{\alpha^k + e^{-ik\theta}}{(\alpha^{2k} + 2\alpha^k \cos(k\theta) + 1)^{1/2}}.$$

It follows that

$$\sin(2u) = \frac{4\alpha^k(\alpha^{2k} - 1)\sin(k\theta)}{(\alpha^{2k} - 2\alpha^k \cos(k\theta) + 1)(\alpha^{2k} + 2\alpha^k \cos(k\theta) + 1)} \tag{2.10}$$

To compute the normal derivative of u we begin by observing that

$$\frac{\partial\theta_j}{\partial x}(z) = \frac{Im(z - \alpha\xi_j)}{|z - \alpha\xi_j|^2}$$
$$\frac{\partial\theta_j}{\partial y}(z) = -\frac{Re(z - \alpha\xi_j)}{|z - \alpha\xi_j|^2}$$

Taking $z = e^{i\theta}$ we arrive at

$$\frac{\partial u}{\partial \nu} = \alpha \sum_{j=0}^{2k-1} (-1)^j \frac{\sin(\theta - \beta_j)}{1 + \alpha^2 - 2\alpha\cos(\theta - \beta_j)}.$$

Taking the logarithmic derivative of (2.8) with respect to θ yields

$$\sum_{j=0, j \text{ even}}^{2k-2} \frac{\alpha \sin(\theta - \beta_j)}{1 + \alpha^2 - 2\alpha\cos(\theta - \beta_j)} = \frac{k\alpha^k \sin(k\theta)}{\alpha^{2k} - 2\alpha^k \cos(k\theta) + 1}$$

and similarly from (2.9)

$$\sum_{j=1, j \text{ odd}}^{2k-1} \frac{\alpha \sin(\theta - \beta_j)}{1 + \alpha^2 - 2\alpha\cos(\theta - \beta_j)} = -\frac{k\alpha^k \sin(k\theta)}{\alpha^{2k} + 2\alpha^k \cos(k\theta) + 1}$$

We deduce then

$$\frac{\partial u}{\partial \nu} = \frac{2k\alpha^k(\alpha^{2k} + 1)\sin(k\theta)}{(\alpha^{2k} - 2\alpha^k \cos(k\theta) + 1)(\alpha^{2k} + 2\alpha^k \cos(k\theta) + 1)} \tag{2.11}$$

From (2.10) and (2.11) we deduce formula (2.4). □

3. Scheme of the proof of Theorem 1.1

Let us consider the problem in the half space $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial w}{\partial \nu} = f(w) & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{3.1}$$

where $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial x_2}$.

CABRÉ and SOLÀ-MORALES [7] call w a layer solution of (3.1) if it satisfies

$$\begin{cases} w(x_1, 0) \text{ is increasing} \\ \lim_{x_1 \to -\infty} w(x_1, 0) = -1, \quad \lim_{x_1 \to \infty} w(x_1, 0) = 1. \end{cases} \tag{3.2}$$

They proved existence of layer solutions and uniqueness except for translations. Moreover, layer solutions satisfy

$$\left| w(x_1, x_2) - \frac{2}{\pi} \arctan\left(\frac{x_1}{x_2 + 1}\right) \right| \leqq \frac{C}{|x|^{1/2}} \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2_+,$$

see [7] formula (6.22),

$$|\nabla w(z)| \leqq \frac{C}{1 + |z|} \quad \forall z \in \mathbb{R}^2_+,$$

see [7] Theorem 1.6, and

$$c \frac{x_2 + a}{x_1^2 + (x_2 + a)^2} \leqq w_x(x_1, x_2) \leqq C \frac{x_2 + A}{x_1^2 + (x_2 + A)^2} \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2_+ \tag{3.3}$$

where $c, C, a, A > 0$ see [7] formulas (6.16) and (6.18). As a result

$$|w(x_1, 0) - 1| \leqq \frac{C}{1 + x_1} \quad \forall x_1 \geqq 0 \tag{3.4}$$

$$|w(x_1, 0) + 1| \leqq \frac{C}{1 + |x_1|} \quad \forall x_1 \leqq 0. \tag{3.5}$$

We will call w^+ the solution to (3.1), (3.2) such that $w(0, 0) = 0$ and w^- the solution to (3.1) such that

$$\begin{cases} w^-(x_1, 0) \text{ is decreasing,} \\ \lim_{x_1 \to -\infty} w^-(x_1, 0) = +1, \quad \lim_{x_1 \to +\infty} w^-(x_1, 0) = -1 \end{cases} \tag{3.6}$$

and $w^-(0, 0) = 0$. Let us define

$$w^\pm_\varepsilon(z) = w^\pm(z/\varepsilon) \quad \forall z \in \mathbb{R}^2_+.$$

We write $\partial\Omega = \Gamma_0 \bigcup (\cup_{i=1}^m \Gamma_i)$, where Γ_0 denotes the outer component of $\partial\Omega$ and Γ_i the inner component. Let L denote the length of the outer component Γ_0 of the boundary $\partial\Omega$ and let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be an L-periodic positively oriented arclength parametrization of Γ_0. Then there exists $\delta > 0$ such that

$$(s, t) \mapsto \gamma(s) - t\nu(\gamma(s)) \tag{3.7}$$

is a smooth diffeomorphism from $[0, L] \times (0, \delta)$ (identifying 0 and L) with $\Omega_\delta = \{z \in \Omega : \mathrm{dist}(z, \Gamma_0) < \delta\}$. Therefore (s, t) can be regarded as coordinate functions defined in Ω_δ.

For simplicity we assume that $\xi_1, \ldots, \xi_{2k}$ are given points in Γ_0 ordered counterclockwise let $s_1 < s_2 < \ldots < s_{2k}$ be their corresponding arclength parameters. We assume henceforth that

$$|s_{i+1} - s_i| \geqq 5\delta \quad \text{for all } i = 0, \ldots, 2k - 1 \tag{3.8}$$

where by convention $s_0 = s_{2k}$ and δ is a small, fixed, positive number. Let

$$\tilde{w}_\varepsilon^+(s, t) = \eta_1(s)\eta_2(s)w_\varepsilon^+(s, t) + \eta_1(s) - \eta_2(s)$$

where $\eta_1, \eta_2 \in C^\infty(\mathbb{R})$, $0 \leqq \eta_1, \eta_2 \leqq 1$ and

$$\eta_1(s) = 1 \quad \text{for } s \geqq -\delta, \quad \eta_1(s) = 0 \quad \text{for } s \leqq -2\delta$$
$$\eta_2(s) = 1 \quad \text{for } s \leqq \delta, \quad \eta_2(s) = 0 \quad \text{for } s \geqq 2\delta$$

We define $\tilde{w}_\varepsilon^-$ similarly as

$$\tilde{w}_\varepsilon^-(s, t) = \eta_1(s)\eta_2(s)w_\varepsilon^-(s, t) + \eta_2(s) - \eta_1(s)$$

Let $\eta_0 \in C^\infty(\mathbb{R})$, $0 \leqq \eta_0 \leqq 1$ be such that

$$\eta_0(t) = 1 \quad \text{for } t \leqq \delta/2 \quad \text{and} \quad \eta_0(t) = 0 \quad \text{for } t \geqq \delta. \tag{3.9}$$

Define

$$\bar{U}_\varepsilon(s, t) = \eta_0(t) \begin{cases} \tilde{w}_\varepsilon^+(s - s_j, t) & \text{if } s_j - 2\delta \leqq s \leqq s_j + 2\delta, \quad j \text{ odd} \\ 1 & \text{if } s_j + 2\delta \leqq s \leqq s_{j+1} - 2\delta, \quad j \text{ odd} \\ \tilde{w}_\varepsilon^-(s - s_j, t) & \text{if } s_j - 2\delta \leqq s \leqq s_j + 2\delta, \quad j \text{ even} \\ -1 & \text{if } s_j + 2\delta \leqq s \leqq s_{j+1} - 2\delta, \quad j \text{ even} \end{cases}$$

where $j = 1, \ldots, 2k$.

This formula defines $\bar{U}_\varepsilon$ as a smooth function in Ω_δ. Extending it by zero to $\Omega - \Omega_\delta$ we obtain a smooth function in Ω.

As initial approximation of a solution to (1.1) we take

$$U_\varepsilon = \bar{U}_\varepsilon + (1 - \eta_0)u^* + H_\varepsilon \tag{3.10}$$

where $u^* = u^*[\xi_1, \ldots, \xi_{2k}]$ is the harmonic function on Ω defined in the introduction and H_ε is the solution to

$$\begin{cases} \Delta H_\varepsilon = -\Delta(\bar{U}_\varepsilon + (1 - \eta_0)u^*) & \text{in } \Omega \\ H_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

We look for a solution u of (1.1) of the form $u = U_\varepsilon + \phi$, which yields the following equation for the new unknown ϕ

$$\Delta\phi = 0 \quad \text{in } \Omega$$
$$\varepsilon\frac{\partial\phi}{\partial\nu} - f'(U_\varepsilon)\phi = N[\phi] + R \quad \text{on } \partial\Omega$$

where

$$\begin{aligned} N[\phi] &= f(U_\varepsilon + \phi) - f(U_\varepsilon) - f'(U_\varepsilon)\phi \\ R &= f(U_\varepsilon) - \varepsilon\frac{\partial U_\varepsilon}{\partial\nu}. \end{aligned} \tag{3.12}$$

For arbitrary points $\xi_1, \dots, \xi_{2k}$ satisfying the separation condition (3.8) we will show in Section 9 that we can find $\phi, c_1, \dots, c_{2k}$ of order ε that solve:

$$\begin{cases} \Delta\phi = 0 \quad \text{in } \Omega \\ \varepsilon\dfrac{\partial\phi}{\partial\nu} - f'(U_\varepsilon)\phi = N[\phi] + R + \displaystyle\sum_{j=1}^{2k} c_j Z_j \quad \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} \phi Z_j = 0 \quad \text{for all } j = 1, \dots, 2k. \end{cases}$$

where Z_j are appropriate functions defined in (7.4).

A key element to prove this result is to establish the non-degeneracy of the layer solutions $w^\pm$, which we do in Section 5, by means of the estimates contained in Section 4. The non-degeneracy lets us establish an invertibility property for the associated projected linear problem, first in the half-space (Section 6), then in the bounded domain Ω (Section 7). Section 8 is devoted to estimating the L^∞-norm of the error R (3.12).

At this point, a standard argument shows $U_\varepsilon + \phi$ is a real solution to (1.1), provided that the points $(\xi_1, \dots, \xi_{2k})$ are critical points of $E_\varepsilon(U_\varepsilon + \phi)$, which is close to $E_\varepsilon(U_\varepsilon)$ in an appropriate sense, see Section 10. Section 11 is devoted to expanding the energy functional E_ε at U_ε and to giving the proof of Theorem 1.1.

Finally, some technical Lemmas we need throughout the paper are contained in Appendices A and B.

4. Estimates for the Laplacian with Robin Boundary condition

In this section we study existence and decay estimates for the boundary value problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial\phi}{\partial\nu} + a\phi = h & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{4.1}$$

where $a > 0$. For a function $h : \mathbb{R} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ we define

$$\|h\|_\alpha = \sup_{y\in\mathbb{R}} (1 + |y|)^\alpha |h(y)|.$$

Let

$$\Gamma(x-y) = -\log|x-y|, \quad x, y \in \mathbb{R}^2$$

We recall (see [10]) that if $y \in \mathbb{R}^2_+$ and $a > 0$, the Green function for the Robin problem

$$\begin{cases} -\Delta_x G(x,y) = 2\pi\delta_y & \text{in } \mathbb{R}^2_+ \\ -\dfrac{\partial G}{\partial \nu} + aG = 0 & \text{on } \partial\mathbb{R}^2_+ \end{cases}$$

is given by

$$G(x,y) = \Gamma(x-y) - \Gamma(x-y^*) - 2\int_{-\infty}^{0} e^{as}\frac{\partial}{\partial x_2}\Gamma\left(x-y^*-e_2 s\right)\,\mathrm{d}s, \quad (4.2)$$

where y^* is the reflection of $y=(y',y_2)$ across $\partial\mathbb{R}^2_+$, that is $y^*=(y',-y_2)$.

If h is bounded on $\mathbb{R}$, then a solution to (4.1) is given by

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G((y,0),x)h(y)\,\mathrm{d}y \quad x \in \mathbb{R}^2_+.$$

From (4.2)

$$G((y,0),(x_1,x_2)) = \int_0^\infty \frac{e^{-at}(x_2+t)}{(y-x_1)^2+(x_2+t)^2}\,\mathrm{d}t$$

and hence

$$\phi(x_1,x_2) = \int_{-\infty}^{\infty} k_a(x_1-y,x_2)h(y)\,\mathrm{d}y \quad \text{for all } (x_1,x_2)\in\mathbb{R}^2_+, \quad (4.3)$$

where

$$k_a(x_1,x_2) = \frac{1}{\pi}\int_0^\infty \frac{e^{-at}(x_2+t)}{x_1^2+(x_2+t)^2}\,\mathrm{d}t \quad \text{for all } (x_1,x_2)\in\mathbb{R}^2_+. \quad (4.4)$$

In the sequel, when the value of $a > 0$ is clear from the context, we will write $k = k_a$.

Next we state some mapping properties of the operator defined by (4.3) in weighted L^∞ spaces. Most of the proofs are in Appendix A. Let $h : \mathbb{R} \to \mathbb{R}$ with $\|h\|_\alpha < \infty$ where $\alpha > -1$. Then (4.3) is well defined.

Lemma 4.1. *Let $\alpha > -1$ and $h : \mathbb{R} \to \mathbb{R}$ satisfy $\|h\|_\alpha < \infty$. Then ϕ defined by (4.3) satisfies:*
if $\alpha < 1$

$$|\phi(x_1,x_2)| \leqq C\|h\|_\alpha \begin{cases} \frac{1}{(1+|x_1|)^\alpha} + \frac{1+x_2}{(1+|x_1|)^{\alpha+1}} & \text{if } |x_1| \geqq x_2 \\ \frac{1}{(1+x_2)^\alpha} & \text{if } |x_1| \leqq x_2 \end{cases}$$

if $\alpha = 1$,

$$|\phi(x_1,x_2)| \leqq C\|h\|_\alpha \begin{cases} \frac{1}{1+|x_1|} + \frac{(1+x_2)\max(1,\log|x_1|)}{(1+|x_1|)^2} & \text{if } |x_1| \geqq x_2 \\ \frac{1}{1+x_2} & \text{if } |x_1| \leqq x_2 \end{cases}$$

and if $\alpha > 1$

$$|\phi(x_1,x_2)| \leqq C\|h\|_\alpha \begin{cases} \frac{1}{(1+|x_1|)^\alpha} + \frac{(1+x_2)}{(1+|x_1|)^2} & \text{if } |x_1| \geqq x_2 \\ \frac{1}{1+x_2} & \text{if } |x_1| \leqq x_2 \end{cases}$$

Corollary 4.1. *Let* $\alpha > -1$. *Let* $h : \mathbb{R} \to \mathbb{R}$ *be such that* $\|h\|_\alpha < \infty$. *Then* ϕ *defined by* (4.3) *satisfies:*
if $\alpha \leqq 2$

$$\|\phi(\cdot,0)\|_\alpha \leqq C\|h\|_\alpha.$$

if $\alpha > 2$

$$\|\phi(\cdot,0)\|_2 \leqq C\|h\|_\alpha.$$

Lemma 4.2. *Let* $\alpha > 1$ *and assume that* $h : \mathbb{R} \to \mathbb{R}$ *satisfies* $\|h\|_\alpha < +\infty$. *Suppose*

$$\int_{-\infty}^{\infty} h = 0$$

and let ϕ *be defined by* (4.3).

Then if $\alpha < 2$

$$|\phi(x_1,x_2)| \leqq C\|h\|_\alpha \begin{cases} \frac{1}{(1+|x_1|)^\alpha} + \frac{1+x_2}{(1+|x_1|)^{\alpha+1}} & \text{if } |x_1| \geqq x_2 \\ \frac{1}{(1+x_2)^\alpha} & \text{if } |x_1| \leqq x_2 \end{cases}$$

if $\alpha = 2$,

$$|\phi(x_1,x_2)| \leqq C\|h\|_\alpha \begin{cases} \frac{1}{(1+|x_1|)^2} + \frac{(1+x_2)\max(1,\log|x_1|)}{(1+|x_1|)^3} & \text{if } |x_1| \geqq x_2 \\ \frac{\max(1,\log|x_2|)}{(1+x_2)^2} & \text{if } |x_1| \leqq x_2 \end{cases}$$

and if $\alpha > 2$

$$|\phi(x_1,x_2)| \leqq C\|h\|_\alpha \begin{cases} \frac{1}{(1+|x_1|)^\alpha} + \frac{(1+x_2)}{(1+|x_1|)^3} & \text{if } |x_1| \geqq x_2 \\ \frac{1}{(1+x_2)^2} & \text{if } |x_1| \leqq x_2 \end{cases}$$

Corollary 4.2. *Let* $\alpha > 2$ *and assume that* $h : \mathbb{R} \to \mathbb{R}$ *satisfies* $\|h\|_\alpha < +\infty$. *Suppose*

$$\int_{-\infty}^{\infty} h = 0$$

and let ϕ *be defined by* (4.3). *Then if* $2 < \alpha \leqq 3$

$$\|\phi(\cdot, 0)\|_\alpha \leqq C\|h\|_\alpha$$

and if $\alpha > 3$ *then*

$$\|\phi(\cdot, 0)\|_3 \leqq C\|h\|_\alpha .$$

Lemma 4.3. *Let* $\alpha > -1$ *and assume* $h : \mathbb{R} \to \mathbb{R}$ *satisfies* $\|h\|_\alpha < \infty$ *and* $h(y) = 0$ *for all* $y \leqq 0$. *Then* ϕ *defined by* (4.3) *satisfies*:
if $\alpha < 1$:

$$|\phi(x_1, 0)| \leqq C\|h\|_\alpha \frac{1}{(1 + |x_1|)^{\alpha+1}} \quad \forall x_1 \leqq 0,$$

if $\alpha = 1$:

$$|\phi(x_1, 0)| \leqq C\|h\|_\alpha \frac{\max(\log|x_1|, 1)}{(1 + |x_1|)^2} \quad \forall x_1 \leqq 0,$$

if $\alpha > 1$:

$$|\phi(x_1, 0)| \leqq C\|h\|_\alpha \frac{1}{(1 + |x_1|)^2} \quad \forall x_1 \leqq 0.$$

Lemma 4.4. *Let* $\alpha > -1$ *and assume* $h : \mathbb{R} \to \mathbb{R}$ *satisfies* $\|h\|_\alpha < \infty$, $h(y) = 0$ *for all* $y \leqq 0$ *and*

$$\int_0^\infty h = 0.$$

Then ϕ *defined by* (4.3) *satisfies*:
if $\alpha < 2$:

$$|\phi(x_1, 0)| \leqq C\|h\|_\alpha \frac{1}{(1 + |x_1|)^{\alpha+1}} \quad \forall x_1 \leqq 0,$$

if $\alpha = 2$:

$$|\phi(x_1, 0)| \leqq C\|h\|_\alpha \frac{\max(\log|x_1|, 1)}{(1 + |x_1|)^3} \quad \forall x_1 \leqq 0,$$

if $\alpha > 2$:

$$|\phi(x_1, 0)| \leqq C\|h\|_\alpha \frac{1}{(1 + |x_1|)^3} \quad \forall x_1 \leqq 0.$$

Lemma 4.5. *Let* $h \in L^\infty(\partial\mathbb{R}^2_+)$ *and* ϕ *be a bounded solution to*

$$\Delta\phi = 0 \quad \text{in } \mathbb{R}^2_+, \qquad \frac{\partial\phi}{\partial\nu} + a\phi = h \quad \text{on } \partial\mathbb{R}^2_+$$

where $a > 0$. *Then* ϕ *is given by* (4.3).

Proof. Let ϕ_1 be defined by (4.3). Then $\tilde\phi = \phi - \phi_1$ is a bounded solution to

$$\Delta\tilde\phi = 0 \quad \text{in } \mathbb{R}^2_+, \qquad \frac{\partial\tilde\phi}{\partial\nu} + a\tilde\phi = 0 \quad \text{on } \partial\mathbb{R}^2_+.$$

By standard elliptic estimates, $\tilde\phi$ is a smooth function and $\nabla\tilde\phi$ is uniformly bounded. Let $v = a\tilde\phi - \frac{\partial\tilde\phi}{\partial y}$. Then v is a bounded function, $\Delta v = 0$ in $\mathbb{R}^2_+$ and v vanishes on $\partial\mathbb{R}^2_+$. Thus v can be extended by odd symmetry to $\mathbb{R}^2$. By Liouville's theorem v is constant and hence $v \equiv 0$. It follows that $\tilde\phi(x, y) = c(x)e^{ay}$. Since ϕ is bounded $\tilde\phi \equiv 0$. □

5. Nondegeneracy of layer solutions

The main result in this section is the nondegeneracy of the layer solution w that satisfies (3.1), (3.2). Consider the linear problem

$$\Delta\phi = 0 \quad \text{in } \mathbb{R}^2_+ \tag{5.1}$$

$$\frac{\partial\phi}{\partial\nu} - f'(w)\phi = 0 \quad \text{on } \partial\mathbb{R}^2_+. \tag{5.2}$$

Proposition 5.1. *Suppose that $\phi \in L^\infty(\mathbb{R}^2_+)$ satisfies (5.1), (5.2). Then $\phi = c\frac{\partial w}{\partial x}$ for some constant $c \in \mathbb{R}$.*

We also establish here the following asymptotic formulas for the layer solutions.

Proposition 5.2. *Suppose w is the layer solution that satisfies (3.1), (3.2). Then*

$$w(x, 0) - 1 = \frac{2}{\pi f'(1)}\frac{1}{x} + O\left(\frac{\log(x)}{x^2}\right) \quad \text{as } x \to \infty, \tag{5.3}$$

and

$$w(x, 0) + 1 = \frac{2}{\pi f'(-1)}\frac{1}{x} + O\left(\frac{\log(|x|)}{x^2}\right) \quad \text{as } x \to -\infty. \tag{5.4}$$

We start with a result that gives decay estimates for solutions with Robin boundary conditions, with a potential $a(x)$ that is asymptotic to positive constants.

Lemma 5.1. *Suppose $\phi \in L^\infty(\mathbb{R}^2_+)$ solves*

$$\Delta\phi = 0 \quad \text{in } \mathbb{R}^2_+$$

$$\frac{\partial\phi}{\partial\nu} + a(x)\phi = h \quad \text{on } \partial\mathbb{R}^2_+$$

where $a(x) \in L^\infty(\mathbb{R})$ satisfies

$$a(x) = a^+ + O\left(\frac{1}{x}\right) \quad \text{as } x \to \infty, \qquad a(x) = a^- + O\left(\frac{1}{|x|}\right) \quad \text{as } x \to -\infty$$

with $a^+, a^- > 0$ and $\|h\|_2 < \infty$ (i.e. $h(x) = O(1/x^2)$ as $x \to \pm\infty$. Then

$$\phi(x, 0) = O\left(\frac{1}{x^2}\right) \quad \text{as } x \to \pm\infty.$$

Proof. Let

$$h_1 = ((a(x) - a^+)\phi + h)\chi_{[x>0]}, \quad h_2 = ((a(x) - a^+)\phi + h)\chi_{[x<0]}$$

and let ϕ_1, ϕ_2 be defined by formula (4.3) with $k = k_{a^+}, h = h_1$ and $h = h_2$, respectively. Then they solve

$$\begin{aligned} \Delta\phi_i &= 0 \quad \text{in } \mathbb{R}^2_+ \\ \frac{\partial\phi_i}{\partial\nu} + a^+\phi_i &= h_i \quad \text{on } \partial\mathbb{R}^2_+ \end{aligned}$$

By Lemma 4.5 $\phi = \phi_1 + \phi_2$.

Since ϕ is bounded we deduce $|h_1(x_1)| \leqq C(1 + |x_1|)^{-1}$ for all $x_1 \in \mathbb{R}$ and Lemma 4.1 with $\alpha = 1$ then implies that

$$|\phi_1(x_1, 0)| \leqq \frac{C}{1 + |x_1|} \quad \text{for all } x_1 \in \mathbb{R}. \tag{5.5}$$

Since $a(x)$ and ϕ are bounded, we have that h_2 is bounded. Using Lemma 4.3 with $\alpha = 0$, we can then deduce that $\phi_2(x_1, 0) = O(1/x_1)$ as $x_1 \to \infty$. Combining this with (5.5) we deduce that $\phi(x_1, 0) = O(1/x_1)$ as $x_1 \to \infty$. By a similar argument, defining ϕ_1, ϕ_2 with a^- instead of a^+, we deduce that $|\phi(x_1, 0)| \leqq C(1 + |x_1|)^{-1}$ for all $x_1 \leqq 0$, and therefore

$$|\phi(x_1, 0)| \leqq C(1 + |x_1|)^{-1} \quad \text{for all } x_1 \in \mathbb{R}. \tag{5.6}$$

Since $a(x)$ is bounded we have $|h_1(x_1)| \leqq C(1+|x_1|)^{-2}$ for all $x_1 \in \mathbb{R}$, and hence Lemma 4.1 with $\alpha = 2$ yields $|\phi_1(x_1, 0)| \leqq C(1 + |x_1|)^{-2}$ for all $x_1 \in \mathbb{R}$. We also have, thanks to (5.6), that $|h_2(x_1)| \leqq C(1 + |x_1|)^{-1}$ for all $x_1 \leqq 0$. Hence by Lemma 4.3 with $\alpha = 1$ we obtain $|\phi_2(x_1, 0)| \leqq C \max(\log|x_1|, 1)(1 + |x_1|)^{-2}$ for all $x_1 \geqq 0$. Hence $|\phi(x_1, 0)| \leqq C \max(\log|x_1|, 1)(1 + |x_1|)^{-2}$ for all $x_1 \geqq 0$. Again, through a similar argument we reach

$$|\phi(x_1, 0)| \leqq C\frac{\max(\log|x_1|, 1)}{(1 + |x_1|)^2} \quad \text{for all } x_1 \in \mathbb{R}.$$

Repeating the process once more, where we apply Lemma 4.3 with some $\alpha \in (1, 2)$, we find

$$|\phi(x_1, 0)| \leqq C(1 + |x_1|)^{-2} \quad \forall x_1 \in \mathbb{R}.$$

□

Lemma 5.2. *Let ϕ be a bounded solution to* (5.1), (5.2). *Then ϕ satisfies*

$$|\phi(x_1, x_2)| \leqq C\frac{1 + x_2}{x_1^2 + (1 + x_2)^2} \quad \forall (x_1, x_2) \in \mathbb{R}^2_+ \tag{5.7}$$

and

$$|\nabla\phi(x)| \leqq \frac{C}{1 + |x|^2} \quad \forall x \in \mathbb{R}^2_+. \tag{5.8}$$

Proof. Let $a^{\pm} = -f'(\pm 1)$ so that $a^{\pm} > 0$. By (3.4) and (3.5) we have that

$$\left|f'(w(x_1,0)) - a^+\right| \leqq C|w(x_1,0) - 1| \leqq \frac{C}{1+x_1} \quad \text{for all } x_1 \geqq 0.$$

and

$$\left|f'(w(x_1,0)) + a^-\right| \leqq C|w(x_1,0) + 1| \leqq \frac{C}{1+|x_1|} \quad \text{for all } x_1 \leqq 0.$$

By Lemma 5.1 we deduce that $\phi(x_1, 0) = O(1/x_1^2)$ as $x_1 \to \pm\infty$. We can now write the boundary condition as

$$\frac{\partial \phi}{\partial \nu} + a^+\phi = (a^+ + f'(w))\phi.$$

Applying Lemma 4.1 with $\alpha = 2$ we deduce (5.7).

To prove (5.8), let us first remark that

$$|\nabla\phi(x)| \leqq \frac{C}{1+|x|^2} \quad \forall x \in \partial\mathbb{R}^2_+. \tag{5.9}$$

Indeed, (5.7) implies that $|\phi(x)| \leqq \frac{C}{1+|x|^2}$ for $x = (x_1, x_2)$ with $0 \leqq x_2 \leqq 1$. But $f'(w)$ is a smooth bounded function, with bounded derivative. By standard elliptic estimates we deduce (5.9). Now let $x_0 \in \mathbb{R}^2_+$ with $|x_0| \geqq 1$ and $R = |x_0|/2$. Define $\psi(z) = R\phi(x_0 + Rz)$ for $z \in D = B_1 \cap \{(z_1, z_2) : z_2 > -x_{0,2}/R\}$ where we have written $x_0 = (x_{0,1}, x_{0,2})$. By (5.7) $|\psi(z)| \leqq C$ for $z \in D$ and (5.9) implies that $|\nabla\psi(z)| \leqq C$ for $z \in \Gamma$ where $\Gamma = \{(z_1, z_2) \in B_1 : z_2 = -x_{0,2}/R\}$. Using Lemma B.1 we deduce that $|\nabla\psi(0)| \leqq C$ and this yields $|\nabla\phi(x_0)| \leqq \frac{C}{|x_0|^2}$. □

Proof of Proposition 5.1. Let $Z = \frac{\partial w}{\partial x}$ and $\psi = \frac{\phi}{Z}$. Then

$$\nabla\cdot(Z^2\nabla\psi) = 0 \quad \text{in } \mathbb{R}^2_+, \quad \frac{\partial\psi}{\partial\nu} = 0 \quad \text{on } \partial\mathbb{R}^2_+.$$

Multiplying the equation by ψ and integrating in $B_R \cap \mathbb{R}^2_+$ we find

$$0 = \int_{\partial B_R \cap \mathbb{R}^2_+} Z^2\psi\frac{\partial\psi}{\partial\nu} - \int_{B_R\cap\mathbb{R}^2_+} Z^2|\nabla\psi|^2.$$

Since (3.3) holds, (5.7) shows that ψ is uniformly bounded. Since

$$|\nabla Z| \leqq \frac{C}{1+|x|^2} \quad \forall x \in \mathbb{R}^2_+$$

and using (5.8) we get

$$Z^2|\psi||\nabla\psi| \leqq |\psi|(Z|\nabla\phi| + |\phi|\,|\nabla Z|) \leqq \frac{C}{1+|x|^3} \quad \forall x \in \mathbb{R}^2_+.$$

This implies that

$$\left|\int_{\partial B_R \cap \mathbb{R}^2_+} Z^2 \psi \frac{\partial \psi}{\partial \nu}\right| \leqq \frac{C}{R^2}.$$

Taking $R \to \infty$ we find that

$$\int_{\mathbb{R}^2_+} Z^2 |\nabla \psi|^2 = 0$$

and this shows that ψ is constant. □

For the proof of Proposition 5.2 we need the behavior of some particular solutions with explicit right-hand sides. The proofs are in Appendix A.

Lemma 5.3. *Let $a > 0$ and u be the bounded solution of*

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2_+$$
$$\frac{\partial u}{\partial \nu} + au = \chi_{[x>0]} \quad \text{on } \partial \mathbb{R}^2_+$$

Then

$$u(x,0) = \frac{1}{a} - \frac{1}{\pi a^2 x} + O\left(\frac{1}{x^3}\right) \quad \text{as } x \to \infty$$
$$u(x,0) = \frac{1}{\pi a^2 |x|} + O\left(\frac{1}{|x|^3}\right) \quad \text{as } x \to -\infty.$$

Lemma 5.4. *Let $a > 0$ and u be the bounded solution of*

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2_+$$
$$\frac{\partial u}{\partial \nu} + au = \chi_{[x>1]} \frac{1}{x} \quad \text{on } \partial \mathbb{R}^2_+$$

Then

$$u(x,0) = \frac{1}{ax} + O\left(\frac{\log(x)}{x^2}\right) \quad \text{as } x \to \infty$$
$$u(x,0) = O\left(\frac{\log(|x|)}{x^2}\right) \quad \text{as } x \to -\infty$$

Proof of Proposition 5.2. Let $a^+ = -f'(1) > 0$, $a^- = -f'(-1) > 0$. For $x > 0$

$$\frac{\partial w}{\partial \nu} = f(w) = f'(1)(w-1) + O((w-1)^2) \quad \text{as } x \to \infty$$
$$= -a^+ w + a^+ + O(1/x^2) \quad \text{as } x \to \infty$$

Similarly

$$\frac{\partial w}{\partial \nu} = -a^- w - a^- + O(1/x^2) \quad \text{as } x \to -\infty$$

Let

$$a(x) = a^+ \quad \text{for } x > 0, \quad a(x) = a^- \quad \text{for } x < 0$$
$$\tilde{a}(x) = a^+ \quad \text{for } x > 0, \quad \tilde{a}(x) = -a^- \quad \text{for } x < 0.$$

Then

$$\frac{\partial w}{\partial \nu} + a(x)w = \tilde{a}(x) + O(1/x^2) \quad \text{as } x \to \pm\infty. \tag{5.10}$$

Next we construct a function that, up to a small error, also satisfies (5.10). Define u_1 to be the bounded solution of

$$\Delta u_1 = 0 \quad \text{in } \mathbb{R}^2_+$$
$$\frac{\partial u_1}{\partial \nu} + a^+ u_1 = a^+ \chi_{[x>0]} + \frac{a^+ - a^-}{\pi a^- x} \chi_{[x>1]} \quad \text{on } \partial\mathbb{R}^2_+$$

Then by Lemmas 5.3 and 5.4

$$u_1(x,0) = 1 - \frac{2}{\pi a^+ x} + \frac{1}{\pi a^- x} + O\left(\frac{\log(x)}{x^2}\right) \quad \text{as } x \to \infty \tag{5.11}$$

and

$$u_1(x,0) = \frac{1}{\pi a^+ |x|} + O\left(\frac{\log(|x|)}{x^2}\right) \quad \text{as } x \to -\infty \tag{5.12}$$

Similarly define

$$\Delta u_2 = 0 \quad \text{in } \mathbb{R}^2_+$$
$$\frac{\partial u_2}{\partial \nu} + a^- u_2 = -a^- \chi_{[x<0]} - \frac{a^- - a^+}{\pi a^+ |x|} \chi_{[x<-1]} \quad \text{on } \partial\mathbb{R}^2_+$$

By Lemmas 5.3 and 5.4 we have

$$u_2(x,0) = -1 + \frac{2}{\pi a^- |x|} - \frac{1}{\pi a^+ |x|} + O\left(\frac{\log(|x|)}{x^2}\right) \quad \text{as } x \to -\infty \tag{5.13}$$

and

$$u_2(x,0) = -\frac{1}{\pi a^- x} + O\left(\frac{\log(x)}{x^2}\right) \quad \text{as } x \to \infty \tag{5.14}$$

Then for $x > 1$

$$\frac{\partial(u_1 + u_2)}{\partial \nu} + a^+(u_1 + u_2) = a^+ + O\left(\frac{\log(x)}{x^2}\right) \quad \text{as } x \to \infty$$

and for $x < -1$

$$\frac{\partial(u_1 + u_2)}{\partial \nu} + a^-(u_1 + u_2) = a^- + O\left(\frac{\log(|x|)}{x^2}\right) \quad \text{as } x \to -\infty.$$

Therefore

$$\frac{\partial(u_1+u_2)}{\partial\nu}+a(x)(u_1+u_2)=\tilde{a}(x)+O(\log(|x|)/x^2)\quad \text{as } x\to\pm\infty.$$

Then $\phi=w-(u_1+u_2)$ satisfies

$$\frac{\partial\phi}{\partial\nu}+a(x)\phi=O\left(\frac{\log(|x|)}{x^2}\right)\quad \text{as } x\to\pm\infty$$

By Lemma 5.1 (really a slight variant of it):

$$\phi(x,0)=O\left(\frac{\log(|x|)}{x^2}\right)\quad \text{as } x\to\pm\infty.$$

This implies

$$w(x,0)=u_1(x,0)+u_2(x,0)+O\left(\frac{\log(|x|)}{x^2}\right)\quad \text{as } x\to\pm\infty.$$

But by (5.11) and (5.14)

$$u_1(x,0)+u_2(x,0)=1-\frac{2}{\pi a^+x}+O\left(\frac{\log(x)}{x^2}\right)\quad \text{as } x\to\infty$$

and by (5.12) and (5.13)

$$u_1(x,0)+u_2(x,0)=-1+\frac{2}{\pi a^-|x|}+O\left(\frac{\log(|x|)}{x^2}\right)\quad \text{as } x\to-\infty.$$

□

6. Solvability of the linearized equation in a half plane

In this section we study the solvability and decay estimates for solutions to the problem

$$\begin{cases}\Delta\phi=0 & \text{in } \mathbb{R}^2_+\\ \dfrac{\partial\phi}{\partial\nu}=f'(w)\phi+h+cw_x\eta & \text{on } \partial\mathbb{R}^2_+,\end{cases}\tag{6.1}$$

where w is the solution to (3.1), (3.2) and $\eta\in C_0^\infty(\mathbb{R})$ is such that $\eta\geqq 0$, $\eta\neq 0$.

We will prove

Proposition 6.1. *Let* $2<\alpha\leqq 3$ *and assume that* $\|h\|_\alpha<+\infty$. *Then there exist unique* $c\in\mathbb{R}$ *and* $\phi\in L^\infty(\mathbb{R}^2_+)$ *with* $\|\phi(\cdot,0)\|_\alpha<+\infty$ *which solves* (6.1). *Moreover,*

$$\|\phi\|_\alpha\leqq C\|h\|_\alpha$$

for some constant C *and* c *is given by*

$$c=-\frac{\int_{-\infty}^{\infty}hw_x}{\int_{-\infty}^{\infty}\eta w_x^2}.$$

Proof. Let us prove the uniqueness. Suppose $h = 0$ and multiply (6.1) by w_x. Integrating in a half ball $B_R^+ = B_R(0) \cap \mathbb{R}^2_+$. We deduce

$$0 = \int_{\partial B_R \cap \mathbb{R}^2_+} \left(\frac{\partial \phi}{\partial \nu} w_x - \phi \frac{\partial w_x}{\partial \nu} \right) + c \int_{-R}^{R} \eta w_x^2.$$

By Lemma 5.1 the decay estimates (5.7) and (5.8) are valid for w_x and ϕ. Hence

$$\int_{\partial B_R \cap \mathbb{R}^2_+} \left(\frac{\partial \phi}{\partial \nu} w_x - \phi \frac{\partial w_x}{\partial \nu} \right) = O(1/R^2).$$

Letting $R \to \infty$ we find $c = 0$. Then $\phi = c w_x$ by Proposition 5.1 for some $c \in \mathbb{R}$. But since we assume that $\|\phi\|_\alpha < \infty$ and $\alpha > 2$, we must have $c = 0$.

Let us show existence of a solution when $2 < \alpha < 3$. Fix $2 < \gamma < \alpha$ and let $a^\pm = -f'(\pm 1)$ so that $a^\pm > 0$. Given $a > 0$ and $g : \mathbb{R} \to \mathbb{R}$ with $\|g\|_\gamma < \infty$ we define $\phi = K_a(g)$ as

$$\phi(x_1) = \int_{-\infty}^{\infty} k_a(x_1 - y, 0)(g(y) + c\eta(y) w_x(y)) dy \quad \text{for all } x_1 \in \mathbb{R}.$$

where

$$c = -\frac{\int_{-\infty}^{\infty} g}{\int_{-\infty}^{\infty} \eta w_x}.$$

In this way, the harmonic extension of ϕ to $\mathbb{R}^2_+$ satisfies

$$\frac{\partial \phi}{\partial \nu} + a\phi = g + c\eta w_x \quad \text{on } \partial \mathbb{R}^2_+.$$

Define the space

$$E = \{(\phi_1, \phi_2) : \phi_i : \mathbb{R} \to \mathbb{R} \; i = 1, 2 \text{ such that } \|(\phi_1, \phi_2)\|_E < \infty\}$$

where

$$\|(\phi_1, \phi_2)\|_E = \|\phi_1\|_{\alpha,\gamma} + \|\phi_2\|_{\gamma,\alpha}$$

and

$$\|\phi_1\|_{\alpha,\gamma} = \sup_{x \leqq 0} (1 + |x|)^\alpha |\phi_1(x)| + \sup_{x \geqq 0} (1 + x)^\gamma |\phi_1(x)|$$
$$\|\phi_2\|_{\gamma,\alpha} = \sup_{x \leqq 0} (1 + |x|)^\gamma |\phi_2(x)| + \sup_{x \geqq 0} (1 + x)^\alpha |\phi_2(x)|.$$

Let $T : E \to E$ be given by $T(\phi_1, \phi_2) = (T_1(\phi_1, \phi_2), T_2(\phi_1, \phi_2))$ where

$$T_1(\phi_1, \phi_2) = K_{a^+} \left[(a^+ + f'(w))\phi_1 + (a^- + f'(w))\phi_2) \chi_{[x \geqq 0]} \right]$$
$$T_2(\phi_1, \phi_2) = K_{a^-} \left[(a^- + f'(w))\phi_2 + (a^+ + f'(w))\phi_1) \chi_{[x < 0]} \right]$$

The operator T is well defined from E to E. Indeed, if $(\phi_1, \phi_2) \in E$, then since $|w(x_1, 0) - 1| \leqq C/(1 + x_1)$ for all $x_1 \geqq 0$, see (3.4), we have

$$\left\| (a^+ + f'(w))\phi_1 + (a^- + f'(w))\phi_2)\chi_{[x\geqq 0]} \right\|_\alpha \leqq C\|(\phi_1, \phi_2)\|_E.$$

We deduce from Lemma 4.2 that

$$\|T_1(\phi_1, \phi_2)\|_\alpha \leqq C\|(\phi_1, \phi_2)\|_E. \tag{6.2}$$

Similarly

$$\|T_2(\phi_1, \phi_2)\|_\alpha \leqq C\|(\phi_1, \phi_2)\|_E \tag{6.3}$$

and hence $T(\phi_1, \phi_2) \in E$. Suppose $(\phi_1, \phi_2) \in E$ satisfies

$$\begin{cases} \phi_1 = T_1(\phi_1, \phi_2) + K_{a^+}(h\chi_{[x\geqq 0]}) \\ \phi_2 = T_2(\phi_1, \phi_2) + K_{a^-}(h\chi_{[x<0]}). \end{cases} \tag{6.4}$$

Then $\phi = \phi_1 + \phi_2$ is a solution of (6.1) for some $c \in \mathbb{R}$. We claim that T is a compact operator. Thus the Fredholm alternative implies that (6.4) has a solution in E, because we know that when $h = 0$ the only solution in E of (6.4) is the trivial one. Moreover, $\phi = \phi_1 + \phi_2$ satisfies $\|\phi\|_\alpha \leqq C\|h\|_\alpha$ because of (6.2), (6.3) and $\|(\phi_1, \phi_2)\|_E \leqq C\|h\|_\alpha$.

Let us verify that T is compact. Let $(\phi_{1,n}, \phi_{2,n}) \in E$ be a bounded sequence and let $(\psi_{1,n}, \psi_{2,n}) = T(\phi_{1,n}, \phi_{2,n})$. Define

$$c_{1,n} = -\frac{\int_{-\infty}^{\infty} \Big((a^+ + f'(w))\phi_{1,n} + (a^- + f'(w))\phi_{2,n})\chi_{[x\geqq 0]}\Big)}{\int_{-\infty}^{\infty} \eta w_x}$$

$$c_{2,n} = -\frac{\int_{-\infty}^{\infty} \big((a^- + f'(w))\phi_{2,n} + (a^+ + f'(w))\phi_{1,n})\chi_{[x<0]}\big)}{\int_{-\infty}^{\infty} \eta w_x}.$$

Since $c_{1,n}, c_{2,n}$ are bounded we may assume that $c_{1,n} \to c_1$, $c_{2,n} \to c_2$. The functions $\psi_{1,n}, \psi_{2,n}$ are also uniformly bounded in $\mathbb{R}^2_+$. Hence, from standard elliptic estimates, $\psi_{1,n} \to \psi_1$ and $\psi_{2,n} \to \psi_2$ uniformly on compact sets of $\overline{\mathbb{R}^2_+}$. Therefore, for any $R > 0$

$$\sup_{-R\leqq x\leqq R} (1 + |x|^\alpha)\,|\psi_{i,n}(x) - \psi_i(x)| \to 0 \quad \text{as } n \to \infty, \quad i = 1, 2. \tag{6.5}$$

By Lemma 4.4

$$\sup_{x\leqq 0}(1 + |x|)^3|T_1(\phi_1, \phi_2)| \leqq C\|(\phi_1, \phi_2)\|_E, \tag{6.6}$$

and similarly

$$\sup_{x\geqq 0}(1 + |x|)^3|T_2(\phi_1, \phi_2)| \leqq C\|(\phi_1, \phi_2)\|_E. \tag{6.7}$$

Using (6.6)

$$\begin{aligned}\sup_{x\leqq -R}\,(1+|x|)^{\alpha}|\psi_{1,n}(x)-\psi_{1,k}(x)| \\ \leqq (1+R)^{\alpha-3}\sup_{x\leqq -R}(1+|x|)^{3}|\psi_{1,n}(x)-\psi_{1,k}(x)| \\ \leqq C(1+R)^{\alpha-3}.\end{aligned}$$

Similarly, by (6.2)

$$\begin{aligned}\sup_{x\geqq R}(1+x)^{\gamma}|\psi_{1,n}(x)-\psi_{1,k}(x)| &\leqq (1+R)^{\gamma-\alpha}\sup_{x\geqq R}(1+x)^{\alpha}|\psi_{1,n}(x)-\psi_{1,k}(x)| \\ &\leqq (1+R)^{\gamma-\alpha}\|\psi_{1,n}-\psi_{1,k}\|_{\alpha} \\ &\leqq C(1+R)^{\gamma-\alpha}\|\phi_{1,n}-\phi_{1,k}\|_{E}\leqq C(1+R)^{\gamma-\alpha}.\end{aligned}$$

These estimates show that

$$\|\psi_{1,n}-\psi_{1}\|_{\alpha,\gamma}\to 0 \quad \text{as } n\to\infty.$$

In a similar way we prove that $\|\psi_{2,n}-\psi_{2}\|_{\gamma,\alpha}\to 0$ as $n\to\infty$.

Finally, in the case $\alpha=3$, we fix $2<\gamma<\alpha'<3$ and construct a solution $\phi=\phi_1+\phi_2\in E$ with the norm $\|(\phi_1,\phi_2)\|_E=\|\phi_1\|_{\alpha',\gamma}+\|\phi_2\|_{\gamma,\alpha'}$. Thanks to (6.7) and $|w(x_1,0)-1|\leqq C/(1+x_1)$ for all $x_1\geqq 0$ we see that

$$\left\|(a^{+}+f'(w))\phi_1+(a^{-}+f'(w))\phi_2)\chi_{[x\geqq 0]}\right\|_{a}\leqq C\|(\phi_1,\phi_2)\|_E.$$

By Lemma 4.2 we deduce

$$\|T_1(\phi_1,\phi_2)\|_3\leqq C\|(\phi_1,\phi_2)\|_E\leqq C\|h\|_3$$

and

$$\|T_2(\phi_1,\phi_2)\|_3\leqq C\|(\phi_1,\phi_2)\|_E\leqq C\|h\|_3.$$

□

7. The linearized equation in a bounded domain

We study the linear problem

$$\Delta\phi=0 \quad \text{in } \Omega \tag{7.1}$$

$$\varepsilon\frac{\partial\phi}{\partial\nu}-f'(U_\varepsilon)\phi=h+\sum_{j=1}^{2k}c_jZ_j \quad \text{on } \partial\Omega \tag{7.2}$$

with the constraints

$$\int_{\partial\Omega}\phi Z_j=0 \quad \text{for all } j=1,\dots,2k, \tag{7.3}$$

where the functions Z_j are defined as follows. Let $s_1 < s_2 < \cdots < s_{2k}$ be the arclength parameters of $\xi_1, \cdots, \xi_{2k}$ in $\partial\Omega$. Recall that $(s,t) \in [0,L] \times (0,\delta)$ are coordinates in a neighborhood of $\partial\Omega$, with s being arclength and t distance to the boundary. We fix a function $\eta_1 \in C_0^\infty(\mathbb{R})$ such that $\eta_1 > 0$ in $(-\delta,\delta)$ and define

$$Z_j(s,t) = \frac{\partial w_\varepsilon^\pm}{\partial x}\left(\frac{s-s_j}{\varepsilon}, \frac{t}{\varepsilon}\right)\eta_1(s-s_j)\eta_0(t), \tag{7.4}$$

where η_0 is defined in (3.9).

Proposition 7.1. *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqq \varepsilon_0$ and $h \in L^\infty(\partial\Omega)$ there exists unique $c_j \in \mathbb{R}$ and $\phi \in L^\infty(\partial\Omega)$ solving (7.1),(7.2), (7.3). Moreover*

$$\|\phi\|_{L^\infty(\partial\Omega)} + \sum_{i=1}^{2k} |c_i| \leqq C\|h\|_{L^\infty(\partial\Omega)} \tag{7.5}$$

First we establish the following a-priori bound:

Lemma 7.1. *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqq \varepsilon_0$ we have the following property: if $\phi \in L^\infty(\Omega)$ is a solution to (7.1), (7.2), (7.3) for some $h \in L^\infty(\Omega)$ and $c_i \in \mathbb{R}$, then (7.5) holds.*

Proof. Multiplying (7.1) by Z_i and integrating we find

$$c_i \int_{\partial\Omega} Z_i^2 = -\int_{\partial\Omega}\left(f'(U_\varepsilon)Z_i - \frac{\partial Z_i}{\partial\nu}\right)\phi - \int_{\partial\Omega} hZ_i - \int_\Omega \phi\Delta Z_i.$$

We have

$$\int_{\partial\Omega} Z_i^2 = \varepsilon\int_{-\infty}^{\infty} w_x^2\eta_1^2, \qquad f'(U_\varepsilon)Z_i - \frac{\partial Z_i}{\partial\nu} = 0 \quad \text{on } \partial\Omega$$

$$\left|\int_{\partial\Omega} hZ_i\right| \leqq \|h\|_{L^\infty(\partial\Omega)}\varepsilon\int_{-\infty}^{\infty}|w_x\eta_1|$$

and

$$\int_\Omega |\Delta Z_i| \leqq C\varepsilon. \tag{7.6}$$

Indeed, fix $i = 1,\ldots,2k$ and for simplicity assume that i is odd and $s_i = 0$. Then in the region $|z| \leqq \delta$, using (8.9) we have

$$\Delta Z_i = \frac{(2k-k^2t)t}{(1-kt)^2}(Z_i)_{ss} - \frac{k}{1-kt}(Z_i)_t + \frac{k't}{(1-kt)^3}(Z_i)_s.$$

Let us consider the first term:

$$\begin{aligned}\frac{(2k-k^2t)t}{(1-kt)^2}(Z_i)_{ss} &= \frac{(2k-k^2t)t}{(1-kt)^2}\left(\frac{\partial w_\varepsilon^\pm}{\partial s}\left(\frac{s}{\varepsilon},\frac{t}{\varepsilon}\right)\eta_1(s-s_j)\eta_0(t)\right)_{ss}\\
&= \frac{(2k-k^2t)t}{(1-kt)^2}\left[\frac{1}{\varepsilon^2}\left(w_{sss}^\pm\left(\frac{s}{\varepsilon},\frac{t}{\varepsilon}\right)\eta_1(s)\eta_0(t)\right.\right.\\
&\quad +2\frac{1}{\varepsilon}\left(w_{ss}^\pm\left(\frac{s}{\varepsilon},\frac{t}{\varepsilon}\right)\eta_1'(s)\eta_0(t)\right.\\
&\quad \left.+\, w_s^\pm\left(\frac{s}{\varepsilon},\frac{t}{\varepsilon}\right)\eta_1''(s)\eta_0(t)\right] = (a)+(b)+(c)\end{aligned}$$

Changing variables $s = \varepsilon\tilde{s}$, $t = \varepsilon\tilde{t}$ we can estimate, using (8.4)

$$\int_\Omega (a) \leqq C\varepsilon \int_{B_{\delta/\varepsilon}(0)\cap\mathbb{R}^2_+} \tilde{t} w^{\pm}_{sss}\, d\tilde{t}\, d\tilde{s} \leqq C\varepsilon.$$

Similarly by (8.3)

$$\int_\Omega (b) \leqq C\varepsilon^2 \int_{B_{\delta/\varepsilon}(0)\cap\mathbb{R}^2_+} \tilde{t} w^{\pm}_{ss}\, d\tilde{t}\, d\tilde{s} \leqq C\varepsilon$$

and by (8.2)

$$\int_\Omega (c) \leqq C\varepsilon^3 \int_{B_{\delta/\varepsilon}(0)\cap\mathbb{R}^2_+} \tilde{t} w^{\pm}_{s}\, d\tilde{t}\, d\tilde{s} \leqq C\varepsilon.$$

A similar calculation works for the other terms and we deduce the validity of (7.6).

This shows that

$$|c_i| \leqq C(\|\phi\|_{L^\infty(\partial\Omega)} + \|h\|_{L^\infty(\partial\Omega)}) \quad \text{for all } i = 1, \dots, 2k. \tag{7.7}$$

In order to prove (7.5) it is then sufficient to show that there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqq \varepsilon_0$ if $\phi \in C(\bar{\Omega})$ solves (7.1),(7.2), (7.3) then

$$\|\phi\|_{L^\infty(\partial\Omega)} \leqq C\|h\|_{L^\infty(\partial\Omega)}. \tag{7.8}$$

We argue by contradiction, assuming that (7.8) fails. Then there are sequences $0 < \varepsilon_n \to 0$, $\phi_n \in C(\bar{\Omega})$, $h_n \in L^\infty(\partial\Omega)$, and $c_{i,n} \in \mathbb{R}$ such that (7.1), (7.2), (7.3) hold,

$$\|\phi_n\|_{C(\bar{\Omega})} = 1 \quad \text{and} \quad \|h_n\|_{L^\infty(\partial\Omega)} \to 0.$$

By (7.7) we may assume that $c_{i,n} \to c_i$ as $n \to \infty$. Let us fix $R > 0$ sufficiently large. Then for $x \in \partial\Omega \setminus \cup_{j=1}^{2k} B_{R\varepsilon}(\xi_j)$ we have that $f'(U_\varepsilon(x)) > 0$. By the maximum principle and the Hopf lemma it follows that

$$\max_{\bar{\Omega}\setminus\cup_{j=1}^{2k} B_{R\varepsilon}(\xi_j)} |\phi_n| = \max_{\bar{\Omega}\cap\cup_{j=1}^{2k} \partial B_{R\varepsilon}(\xi_j)} |\phi_n|.$$

Extracting a subsequence we can find some fixed $j_0 = 1, \dots, 2k$ such that

$$\max_{\bar{\Omega}\cap\partial B_{R\varepsilon}(\xi_{j_0})} |\phi_n| = 1. \tag{7.9}$$

After translation we can assume that $\xi_{j_0} = 0$. Let $\Omega_{\varepsilon_n} = \Omega/\varepsilon_n$ and define the functions

$$\tilde{\phi}_n(y) = \phi_n(y\varepsilon_n), \quad \tilde{Z}_j(y) = Z_j(y\varepsilon_n), \quad y \in \Omega_{\varepsilon_n}$$
$$\tilde{h}_n(y) = h(y\varepsilon_n), \quad y \in \partial\Omega_{\varepsilon_n}$$

so that

$$\Delta\tilde{\phi}_n = 0 \quad \text{in } \tilde{\Omega}_{\varepsilon_n}$$

$$\frac{\partial\tilde{\phi}_n}{\partial\nu} - f'(\tilde{U}_\varepsilon)\tilde{\phi}_n = \tilde{h}_n + \sum_{j=1}^{2k} c_j \tilde{Z}_j \quad \text{on } \partial\Omega_{\varepsilon_n}.$$

After extracting a new subsequence we assume that Ω_{ε_n} approaches the upper half plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, and that $\tilde{\phi}_n$ converges uniformly on compact sets of $\bar{\Omega}_{\varepsilon_n}$ to a function $\tilde{\phi}$. Since $\tilde{h}_n \to 0$ uniformly, we deduce that $\tilde{\phi}$ satisfies

$$\Delta\tilde{\phi} = 0 \quad \text{in } \mathbb{R}^2_+$$

$$\frac{\partial\tilde{\phi}}{\partial\nu} - f'(w)\tilde{\phi} = c_{j_0} w_x \eta \quad \text{on } \partial\mathbb{R}^2_+.$$

As in the proof of Proposition 6.1 we deduce that $c_{j_0} = 0$. Hence by Proposition 5.1 we must have $\tilde{\phi} = c w^{\pm}_x$ for some constant $c \in \mathbb{R}$. But ϕ_n satisfies (7.3) which implies

$$\int_{\Omega_{\varepsilon n}} \tilde{\phi}_n \tilde{Z}_j = 0 \quad \forall j = 1, \dots, 2k.$$

Letting $n \to \infty$ we find

$$\int_{-\infty}^{\infty} \tilde{\phi} w^{\pm} \eta_1 = 0$$

This shows that $c = 0$ and then $\tilde{\phi} = 0$. This means that $\tilde{\phi}_n \to 0$ uniformly on compact sets of $\bar{\Omega}_{\varepsilon_n}$, which contradicts (7.9). □

Proof of Proposition 7.1. Define the Hilbert space

$$H = \left\{ \phi \in H^1(\Omega) : \int_{\partial\Omega_\varepsilon} \phi Z_j = 0 \quad \forall j = 1, \dots, 2k \right\}$$

with the inner product

$$\langle \phi, \varphi \rangle = \int_\Omega \nabla\phi \nabla\varphi + \int_{\partial\Omega} \phi\varphi.$$

Given $h \in L^2(\partial\Omega)$, the problem (7.1), (7.2), (7.3) is equivalent to finding $\phi \in H$ such that

$$\varepsilon\langle \phi, \varphi \rangle = \int_{\partial\Omega} \left(f'(U_\varepsilon)\phi + \varepsilon\phi + h \right) \varphi \quad \text{for all } \varphi \in H.$$

By the Riesz representation theorem, this es equivalent to finding $\phi \in H$ such that

$$\phi = T\phi + \tilde{h} \tag{7.10}$$

where $T : H \to H$ is a linear compact operator and $\tilde{h} \in H$. Since in the case $h = 0$ the unique solution to (7.1), (7.2), (7.3) is $\phi = 0$ by Lemma 7.1, by the Fredholm alternative we conclude that (7.10) has a unique solution. □

8. Estimate of the error

This section is devoted to proving the following

Proposition 8.1. *Let R be given by* (3.12)*. Then*

$$\|R\|_{L^\infty(\partial\Omega)} \leqq C\varepsilon. \tag{8.1}$$

For this we will need the estimates contained in the next two propositions.

Proposition 8.2. *Let w denote the increasing layer solution to* (3.1)*,* (3.2) *or the decreasing one, that is, one that satisfies* (3.1)*,* (3.6)*. Then*

$$|\nabla w(z)| \leqq \frac{C}{1+|z|} \quad \forall z \in \mathbb{R}^2_+ \tag{8.2}$$

$$|D^2 w(z)| \leqq \frac{C}{1+|z|^2} \quad \forall z \in \mathbb{R}^2_+ \tag{8.3}$$

$$|D^3 w(z)| \leqq \frac{C}{1+|z|^3} \quad \forall z \in \mathbb{R}^2_+. \tag{8.4}$$

Proof. The bound (8.2) was obtained in Theorem 1.6 of [7]. Estimate (8.3) is proved in Lemma 5.2, see (5.8). It remains only to establish (8.4).

The layer solution w is smooth in $\mathbb{R}^2_+$ and, under the assumption that $f \in C^3$, w is $C^{3,\alpha}$ for any $0 < \alpha < 1$ up to the boundary. Let $v = w_{xx}$. Then v is harmonic in $\mathbb{R}^2_+$ and satisfies

$$\frac{\partial v}{\partial \nu} = f'(w)v + f''(w)w_x^2 \quad \text{on } \partial\mathbb{R}^2_+.$$

By standard elliptic estimates, and since by (8.3) $w_{xx}(x,0) = O(|x|^{-2})$ for all $x \in \mathbb{R}$ and $w_x(x,0) = O(|x|^{-2})$ for all $x \in \mathbb{R}$ we deduce

$$|\nabla v(x,0)| \leqq \frac{C}{(1+|x|)^2} \quad \forall x \in \mathbb{R}. \tag{8.5}$$

Multiplying $\Delta v = 0$ in $\mathbb{R}^2_+$ by w_x and integrating we see that

$$\int_{-\infty}^{\infty} f''(w(x,0))w_x(x,0)^3\,\mathrm{d}x = 0.$$

Indeed, by (8.5) and $w_x(x,0) = O(|x|^{-2})$

$$\int_{\partial B_R \cap R^2_+} \frac{\partial w_{xx}}{\partial \nu} w_x = O(R^{-3})$$

and thanks to (8.3), we have

$$\int_{\partial B_R \cap R^2_+} w_{xx}\frac{\partial w_x}{\partial \nu} = O(R^{-3}).$$

By Proposition 6.1 there exists a function ϕ which is harmonic in $\mathbb{R}^2_+$, $\phi(x,0) = O(|x|^{-3})$ for all $x \in \mathbb{R}$ and satisfies

$$\frac{\partial \phi}{\partial \nu} = f'(w)\phi + f''(w)w_x^2.$$

Then by Proposition 5.1, $v = \phi + cw_x$ for some constant c. But by (8.3) $v(x) = O(|x|^{-2})$ for $x \in \mathbb{R}^2_+$ and from Lemma 4.2 $\phi(x) = O(|x|^{-2}\log|x|)$ for $x \in \mathbb{R}^2_+$. Since $w_x(0,y)$ decays at a rate $1/y$ as $y \to \infty$ by (3.3) we conclude that $c = 0$. This means that $v = \phi$ and hence

$$|w_{xx}(x,0)| \leqq \frac{C}{(1+|x|)^3} \quad \forall x \in \mathbb{R}.$$

A standard application of elliptic estimates leads to

$$|\nabla w_{xx}(x,0)| \leqq \frac{C}{(1+|x|)^3} \quad \forall x \in \mathbb{R}.$$

Now we may use the same argument as in the proof of (5.8) to deduce that

$$|\nabla w_{xx}(x)| \leqq \frac{C}{(1+|x|)^3} \quad \forall x \in \mathbb{R}^2_+.$$

□

Proposition 8.3. *Let H_ε be defined by (3.11). Then*

$$\|\nabla H_\varepsilon\|_{L^\infty(\Omega)} \leqq C$$

with C independent of ε.

Proof. The function $U_\varepsilon = \bar{U}_\varepsilon + (1-\eta_0)u^* + H_\varepsilon$ is harmonic and its boundary values are in the interval $[-1,1]$. Thus U_ε is bounded and hence H_ε is uniformly bounded. Away from the points ξ_j we have that ΔH_ε is smooth. Thus we need only to estimate ∇H_ε near ξ_j. We fix $j = 1,\ldots,2k$ and for simplicity assume that j is odd and $s_j = 0$. Then in the region $|z| \leqq \delta$ we have

$$\Delta H_\varepsilon = -\Delta w_\varepsilon^+.$$

By (8.2), (8.3), (8.4)

$$\left|\nabla w_\varepsilon^+(z)\right| \leqq \frac{C}{\varepsilon + |z|} \tag{8.6}$$

$$\left|D^2 w_\varepsilon^+(z)\right| \leqq \frac{C}{\varepsilon^2 + |z|^2} \tag{8.7}$$

$$\left|D^3 w_\varepsilon^+(z)\right| \leqq \frac{C}{\varepsilon^3 + |z|^3} \quad \forall z \in \mathbb{R}^2_+. \tag{8.8}$$

The Laplacian expressed in the coordinates defined in (3.7) is given by

$$\Delta u = u_{ss} + u_{tt} + \frac{(2k - k^2 t)t}{(1-kt)^2}u_{ss} - \frac{k}{1-kt}u_t + \frac{k' t}{(1-kt)^3}u_s \tag{8.9}$$

where $k = k(s)$ is the curvature of the boundary. Since w_ε^+ is harmonic with respect to (s,t) we deduce that in $|z| \leqq \delta$

$$\Delta H_\varepsilon = -\left[\frac{(2k - k^2 t)t}{(1-kt)^2}\left(w_\varepsilon^+\right)_{ss} - \frac{k}{1-kt}\left(w_\varepsilon^+\right)_t + \frac{k' t}{(1-kt)^3}\left(w_\varepsilon^+\right)_s\right].$$

Using (8.6), (8.7) and (8.8)

$$\Delta H_\varepsilon = -2k(0)t\left(w_\varepsilon^+\right)_{ss} + k(0)\left(w_\varepsilon^+\right)_t + O(1) \tag{8.10}$$

where $O(1)$ denotes a function which is bounded independently of ε in the region $z \in \mathbb{R}^2_+$, $|z| \leqq \delta$. Let $\tilde{H}_\varepsilon(s,t) = \frac{k(0)}{2} t^2 (w_\varepsilon^+)_t$. Using (8.9) we find

$$\Delta \tilde{H}_\varepsilon = \frac{k(0)}{2}\Bigg[2\left(w_\varepsilon^+\right)_t + 4t\left(w_\varepsilon^+\right)_{tt} + \frac{(2k-k^2t)t}{(1-kt)^2}\left(t^2\left(w_\varepsilon^+\right)_t\right)_{ss} - \frac{k}{1-kt}\left(t^2\left(w_\varepsilon^+\right)_t\right)_t + \frac{k't}{(1-kt)^3}\left(t^2\left(w_\varepsilon^+\right)_t\right)_s\Bigg]$$

Since w_ε^+ is harmonic and by (8.6), (8.7) and (8.8)

$$\Delta \tilde{H}_\varepsilon = k(0)\left(\left(w_\varepsilon^+\right)_t - 2t\left(w_\varepsilon^+\right)_{ss} + O(1) \quad \text{for all } z \in \mathbb{R}^2_+, \quad |z| \leqq \delta. \tag{8.11}$$

Formulas (8.10), (8.11) imply that $\Delta(H_\varepsilon - \tilde{H}_\varepsilon)$ is uniformly bounded in $z \in \mathbb{R}^2_+$, $|z| \leqq \delta$. Since $H_\varepsilon - \tilde{H}_\varepsilon$ vanishes for $z \in \partial\mathbb{R}^2_+$, $|z| \leqq \delta$ and is uniformly bounded, by elliptic estimates $|\nabla(H_\varepsilon - \tilde{H}_\varepsilon)|$ is uniformly bounded in $z \in \mathbb{R}^2_+$, $|z| \leqq \delta/2$. Now, again using (8.6), (8.7) and (8.8) we see that $\nabla \tilde{H}_\varepsilon$ is uniformly bounded in $z \in \mathbb{R}^2_+$, $|z| \leqq \delta/2$, which implies that ∇H_ε is uniformly bounded in this region. □

Proof of Proposition 8.1. On $\partial\Omega$ we have by Proposition 8.3

$$R = f\left(\bar{U}_\varepsilon\right) - \varepsilon\frac{\partial \bar{U}_\varepsilon}{\partial \nu} - \varepsilon\frac{\partial H_\varepsilon}{\partial \nu} = f\left(\bar{U}_\varepsilon\right) - \varepsilon\frac{\partial \bar{U}_\varepsilon}{\partial \nu} + O(\varepsilon)$$

where $O(\varepsilon)$ is in uniform norm. Let $s_1 < s_2 < \cdots < s_{2k}$ be the corresponding arclength parameters of $\xi_1, \ldots, \xi_{2k}$ in $\partial\Omega$, and let $s \in [0, L]$ be the arclength parameter of an arbitrary point $x \in \partial\Omega$.

If $|s - s_j| \geqq 2\delta$ for all $j = 1, \ldots, 2k$ then $f(\bar{U}_\varepsilon) = 0$ and $\frac{\partial \bar{U}_\varepsilon}{\partial \nu}$ is uniformly bounded, hence $R = O(\varepsilon)$ is this region.

If $\delta \leqq |s - s_j| \leqq 2\delta$ for some $j = 1, \ldots, 2k$ then $\frac{\partial \bar{U}_\varepsilon}{\partial \nu}$ is uniformly bounded and $f(\bar{U}_\varepsilon) = O(\varepsilon)$, and therefore $R = O(\varepsilon)$ is this region.

If $|s - s_j| \leqq \delta$ for some $j = 1, \ldots, 2k$ then $f(\bar{U}_\varepsilon) = f(w_\varepsilon^\pm)$ while $\frac{\partial \bar{U}_\varepsilon}{\partial \nu} = -\frac{\partial w_\varepsilon^\pm}{\partial t}|_{t=0}$. Thus in this region $f(\bar{U}_\varepsilon) = \frac{1}{\varepsilon}\frac{\partial \bar{U}_\varepsilon}{\partial \nu}$ and $R = O(\varepsilon)$. □

9. A nonlinear problem

We look for a solution u of (1.1) of the form $u = U_\varepsilon + \phi$. For arbitrary points $\xi_1, \ldots, \xi_{2k}$ uniformly separated along Γ_0 (3.8), we can find $\phi, c_1, \ldots, c_{2k}$ that solve:

$$\begin{cases} \Delta\phi = 0 \quad \text{in } \Omega \\ \varepsilon \dfrac{\partial\phi}{\partial\nu} - f'(U_\varepsilon)\phi = N[\phi] + R + \displaystyle\sum_{j=1}^{2k} c_j Z_j \quad \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} \phi Z_j = 0 \quad \text{for all } j = 1, \ldots, 2k. \end{cases} \tag{9.1}$$

In fact, we have the following result.

Proposition 9.1. *Let $\delta > 0$. There are $\rho > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqq \varepsilon_0$ and for points $\xi_1, \ldots, \xi_{2k}$ uniformly separated $|\xi_i - \xi_j| > 5\delta$ for all $i \neq j$, see (3.8), along Γ_0 there is a unique solution $\phi \in C(\bar{\Omega}), c_1, \ldots, c_{2k}$ to (9.1) with $\|\phi\|_{L^\infty(\partial\Omega)} \leqq \rho$. This solution satisfies*

$$\|\phi\|_{L^\infty(\Omega)} \leqq C\varepsilon. \tag{9.2}$$

Moreover, the function $\xi_1, \ldots, \xi_{2k} \mapsto \phi$ is differentiable and

$$\|\partial_{\xi_j}\phi\|_{L^\infty(\Omega)} \leqq C. \tag{9.3}$$

Proof. We define a map $F : L^\infty(\partial\Omega) \to L^\infty(\partial\Omega)$ as $\phi = F(\psi)$, where ϕ is the unique solution to

$$\begin{cases} \Delta\phi = 0 \quad \text{in } \Omega \\ \varepsilon \dfrac{\partial\phi}{\partial\nu} - f'(U_\varepsilon)\phi = N[\psi] + R + \displaystyle\sum_{j=1}^{2k} c_j Z_j \quad \text{on } \partial\Omega. \end{cases}$$

Let $\bar{B}_\rho = \{\phi \in L^\infty(\partial\Omega) : \|\phi\|_{L^\infty(\partial\Omega)} \leqq \rho\}$. We claim that if $\rho > 0$ is sufficiently small then $F : \bar{B}_\rho \to \bar{B}_\rho$ is a contraction on this set. Indeed, since f is $C^2 N[\psi] \leqq C|\psi|^2$. Hence, using (7.5) and the estimate for R given in (8.1), for $\psi \in \bar{B}_\rho$ we have

$$\|F(\psi)\|_{L^\infty(\partial\Omega)} \leqq C\|\psi\|^2_{L^\infty(\partial\Omega)} + C\|R\|_{L^\infty(\partial\Omega)} \leqq C\rho^2 + C\varepsilon. \tag{9.4}$$

We can achieve $\|F(\psi)\|_{L^\infty(\partial\Omega)} \leqq \rho$ by fixing $\rho > 0$ suitably small, and then let $\varepsilon > 0$ be small depending on ρ. That F is a contraction on $\bar{B}_\rho$ for small ρ follows from

$$\|F(\psi_1) - F(\psi_2)\|_{L^\infty(\partial\Omega)} \leqq C\|N[\psi_1] - N[\psi_2]\|_{L^\infty(\partial\Omega)} \leqq C\rho\|\psi_1 - \psi_2\|_{L^\infty(\partial\Omega)}.$$

By the Banach fixed point theorem we deduce the existence of a unique $\phi \in \bar{B}_\rho$ which is a fixed point of F, and hence a solution to (9.1). From (9.4) we also deduce that the fixed point ϕ satisfies (9.2).

Let us now discuss the differentiability of ϕ. Using the fixed point characterization of ϕ, and since R depends continuously (in the $L^\infty(\partial\Omega)$-norm) on $\xi = (\xi_1, \dots, \xi_{2k})$, we get that the map $\xi \to \phi$ is also continuous.

Formally differentiating problem (9.1) with respect to, say, ξ_1, we get that $\phi_1 := \partial_{\xi_1}\phi$ satisfies

$$\Delta\phi_1 = 0 \quad \text{in} \quad \Omega,$$

$$\varepsilon\frac{\partial\phi_1}{\partial\nu} - f'(U_\varepsilon)\phi_1 = f''(U_\varepsilon)\frac{\partial U_\varepsilon}{\partial\xi_1}\phi + \partial_{\xi_1}N[\phi] + \partial_{\xi_1}R + c_1\partial_{\xi_1}Z_1 + \sum_{j=1}^{2k} d_j Z_j$$

where $d_j = \partial_{\xi_1}c_j$, and

$$\int_{\partial\Omega}\phi_1 Z_j = 0, \quad \text{for all } j \neq 1, \quad \int_{\partial\Omega}\phi_1 Z_1 = -\int_{\partial\Omega}\phi\partial_{\xi_1}Z_1.$$

Define

$$\tilde{\phi}_1 = \phi_1 + \frac{\int_{\partial\Omega}\phi\partial_{\xi_1}Z_1}{\int_{\partial\Omega}Z_1^2}Z_1. \tag{9.5}$$

We have $\int\tilde{\phi}_1 Z_j = 0$ for all j. Furthermore

$$\Delta\tilde{\phi}_1 = -\frac{\int_{\partial\Omega}\phi\partial_{\xi_1}Z_1}{\int_{\partial\Omega}Z_1^2}\Delta Z_1$$

and, taking into account that $\varepsilon\frac{\partial Z_1}{\partial\nu} - f'(U_\varepsilon)Z_1 = 0$ on $\partial\Omega$,

$$\varepsilon\frac{\partial\tilde{\phi}_1}{\partial\nu} - f'(U_\varepsilon)\tilde{\phi}_1 = f''(U_\varepsilon)\frac{\partial U_\varepsilon}{\partial\xi_1}\phi + \partial_{\xi_1}N[\phi] + \partial_{\xi_1}R + c_1\partial_{\xi_1}Z_1 + \sum_{j=1}^{2k} d_j Z_j.$$

We write $\tilde{\phi}_1 = \psi_1 + \psi_2$, where

$$\Delta\psi_1 = 0 \quad \text{in} \quad \Omega, \quad \int_{\partial\Omega}\psi_1 Z_j = 0 \quad \text{for all } j$$

and

$$\varepsilon\frac{\partial\psi_1}{\partial\nu} - f'(U_\varepsilon)\psi_1 = f''(U_\varepsilon)\frac{\partial U_\varepsilon}{\partial\xi_1}\phi + \partial_{\xi_1}N[\phi] + \partial_{\xi_1}R + c_1\partial_{\xi_1}Z_1 + \sum_{j=1}^{2k} d_j Z_j.$$

Using the same computations as at the beginning of the proof of Lemma 7.1, we get that $\int_{\partial\Omega}Z_1^2 = O(\varepsilon)$, and

$$\left|\int_{\partial\Omega}\phi\partial_{\xi_1}Z_1\right| \leqq \|\phi\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}|\partial_{\xi_1}Z_1| \leqq C\varepsilon\int_{\mathbb{R}}|w_{xx}\eta_1|,$$

and

$$\left\|\frac{\int_{\partial\Omega}\phi\partial_{\xi_1}Z_1}{\int_{\partial\Omega}Z_1^2}\Delta Z_1\right\|_{L^\infty(\Omega)} \leqq C\varepsilon$$

for some fixed constant C, which yields $\|\psi_2\|_{L^\infty(\Omega)} \leqq C$. Concerning ψ_1, given the validity of Lemma 7.1, we are left to estimate

$$\left\| f''(U_\varepsilon)\frac{\partial U_\varepsilon}{\partial \xi_1}\phi + \partial_{\xi_1} N[\phi] + \partial_{\xi_1} R + c_1 \partial_{\xi_1} Z_1 \right\|_{L^\infty(\partial\Omega)}.$$

We have

$$\partial_{\xi_1} N[\phi] = \left[f'(U_\varepsilon + \phi) - f'(U_\varepsilon)\right]\left[\partial_{\xi_1} U_\varepsilon + \partial_{\xi_1}\phi\right].$$

Since $\|\partial_{\xi_1} U_\varepsilon\|_{L^\infty(\partial\Omega)} \leqq C\varepsilon^{-1}$ for some fixed constant C, we get

$$\|\partial_{\xi_1} N[\phi]\|_{L^\infty(\partial\Omega)} \leqq C\|\phi\|_{L^\infty(\Omega)}\left(\varepsilon^{-1} + \|\partial_{\xi_1}\phi\|_{L^\infty(\partial\Omega)}\right) \leqq C(1+\varepsilon\|\partial_{\xi_1}\phi\|_{L^\infty(\partial\Omega)}).$$

Furthermore, one has

$$\left\| f''(U_\varepsilon)\frac{\partial U_\varepsilon}{\partial \xi_1}\phi \right\|_{L^\infty(\partial\Omega)} \leqq C \quad \text{and, since } |c_1| \leqq C\varepsilon, \text{ also } \quad \|c_1 \partial_{\xi_1} Z_1\|_{L^\infty(\partial\Omega)} \leqq C,$$

where we use that $\|\partial_{\xi_1} Z_1\|_{L^\infty(\partial\Omega)} \leqq C\varepsilon^{-1}$. Finally, a direct computation shows that $\|\partial_{\xi_1} R\|_{L^\infty(\partial\Omega)} \leqq C$, so we can conclude that

$$\left\|\tilde{\phi}_1\right\|_\infty \leqq C.$$

Summing up the above information, we have (9.3).

The above computation can be made rigorous by using the implicit function theorem and the fixed point representation, which guarantees C^1 regularity in ξ.

□

10. Variational reduction

In view of Proposition 9.1, given $\xi = (\xi_1, \ldots, \xi_{2k}) \in \partial\Omega^{2k}$ satisfying $|\xi_i - \xi_j| \geqq 5\delta \ \forall i \neq j$, we define $\phi(\xi)$ and $c_j(\xi)$ to be the unique solution to (9.1) satisfying the bounds (9.2) and (9.3).

Given $\xi = (\xi_1, \ldots, \xi_{2k}) \in \partial\Omega^{2k}$, define

$$F_\varepsilon(\xi) = E_\varepsilon(U_\varepsilon(\xi) + \phi(\xi)) \tag{10.1}$$

where the energy functional E_ε is defined in (1.4) and $U_\varepsilon(\xi)$ is given by (3.10).

Lemma 10.1. *If $\xi = (\xi_1, \ldots, \xi_m) \in (\partial\Omega)^m$ satisfying $|\xi_i - \xi_j| \geqq 5\delta \ \forall i \neq j$ is a critical point of F_ε, then $u = U_\varepsilon(\xi) + \phi(\xi)$ is a critical point of E_ε, that is, a solution to* (1.1).

Proof. We have

$$\frac{\partial F_\varepsilon}{\partial \xi_l} = DE_\varepsilon(U_\varepsilon(\xi) + \phi(\xi)) \left[\frac{\partial U_\varepsilon(\xi)}{\partial \xi_l} + \frac{\partial \phi(\xi)}{\partial \xi_l} \right], \quad \text{for all } l = 1, \dots, 2k.$$

Since $U_\varepsilon(\xi) + \phi(\xi)$ solves (9.1)

$$\frac{\partial F_\varepsilon}{\partial \xi_l} = \sum_{i=1}^{2k} c_i \int_{\partial\Omega} Z_i \left[\frac{\partial U_\varepsilon(\xi)}{\partial \xi_l} + \frac{\partial \phi(\xi)}{\partial \xi_l} \right].$$

Assume that $DF_\varepsilon(\xi) = 0$. Thus

$$\sum_{i=1}^{2k} c_i \int_{\partial\Omega} Z_i \left[\frac{\partial U_\varepsilon(\xi)}{\partial \xi_l} + \frac{\partial \phi(\xi)}{\partial \xi_l} \right] = 0 \qquad \forall l = 1, \dots, 2k.$$

From Proposition 9.1 and $\frac{\partial U_\varepsilon(\xi)}{\partial \xi_l} = \frac{1}{\varepsilon} Z_l(1 + o(1))$ where $o(1)$ is in the L^∞ norm as a direct consequence of (7.4), it follows that

$$\sum_{i=1}^{2k} c_i \int_{\partial\Omega} Z_i(Z_l + o(1)) = 0 \qquad \forall l = 1, \dots, 2k,$$

which is a strictly diagonal dominant system. This implies that $c_i = 0 \forall i = 1, \dots, 2k$. □

In order to solve for critical points of the function F_ε, a key characteristic is its expected closeness to the function $E_\varepsilon(U_\varepsilon(\xi))$, which we will analyze in the next section.

Lemma 10.2. *The following expansion holds*

$$F_\varepsilon(\xi) = E_\varepsilon(U_\varepsilon) + \theta_\varepsilon(\xi)\,,$$

where

$$|\theta_\varepsilon| \to 0,$$

uniformly on points satisfying the constraints $|\xi_i - \xi_j| \geqq 5\delta \ \forall i \neq j$.

Proof. Taking into account that $DE_\varepsilon(U_\varepsilon + \phi)[\phi] = 0$, a Taylor expansion and an integration by parts give

$$\begin{aligned} E_\varepsilon(U_\varepsilon + \phi) - E_\varepsilon(U_\varepsilon) &= \int_0^1 D^2 E_\varepsilon(U_\varepsilon + t\phi)[\phi]^2 (1-t)\,\mathrm{d}t \\ &= \int_0^1 \left(\frac{1}{\varepsilon} \int_{\partial\Omega} [N[\phi] + R]\,\phi + \frac{1}{\varepsilon} \int_{\partial\Omega} [f'(U_\varepsilon) - f'(U_\varepsilon + t\phi)]\phi \right) (1-t)\,\mathrm{d}t, \quad (10.2) \end{aligned}$$

so we get

$$E_\varepsilon(U_\varepsilon + \phi) - E_\varepsilon(U_\varepsilon) = O(\varepsilon^2).$$

Hence $|\theta_\varepsilon(\xi)| = o(1)$ uniformly on points satisfying $|\xi_i - \xi_j| \geqq 5\delta \ \forall i \neq j$.

The continuity in ξ of all these expressions is inherited from that of ϕ and its derivatives in ξ in the L^∞ norm. □

11. Energy computations and proof of Theorem 1.1

In this section we compute the expansion of the energy functional E_ε evaluated at U_ε and we give the proof of Theorem 1.1.

We have

Lemma 11.1. *Let $k > 0$, $\delta > 0$. Then there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$ the following expansion holds true*

$$E_\varepsilon(U_\varepsilon) = \frac{4k}{\pi}|\log\varepsilon| + \frac{1}{\pi}\left[\psi_k(\xi) + \Theta_0(\xi)\right] + \varepsilon\Theta(\xi) \tag{11.1}$$

uniformly on points $\xi = (\xi_1, \ldots, \xi_{2k}) \in (\partial\Omega)^{2k}$ such that $|\xi_i - \xi_j| > 5\delta$ for all $i \neq j$. In the previous formula, $\psi_k(\xi)$ is the function introduced in (1.9), namely

$$\psi_k(\xi) = \psi_k(\xi_1, \ldots, \xi_{2k}) = \sum_{l=1}^{2k} H(\xi_l, \xi_l) + \sum_{j\neq l}(-1)^{l+j} G(\xi_j, \xi_l),$$

$\Theta_0(\xi)$ is an explicit function of the points $\xi_1, \ldots, \xi_{2k}$, which is smooth and uniformly bounded, as $\varepsilon \to 0$, in the considered region. Finally, $\Theta(\xi)$ denotes a smooth and uniformly bounded, as $\varepsilon \to 0$, function of ξ.

It is interesting to observe that for the boundary reaction $\partial_\nu u = \varepsilon \sinh(u)$ one finds "spike solutions" whose energy is similar to (11.1) in the simply connected case, but with the renormalized energy term reversed in sign, see [9].

Proof. We start with the expansion of $\frac{1}{\varepsilon}\int_{\partial\Omega} W(U_\varepsilon)$. We claim that

$$\frac{1}{\varepsilon}\int_{\partial\Omega} W(U_\varepsilon) = k\int_{\mathbb{R}} W\left(w^+(z,0)\right) + k\int_{\mathbb{R}} W\left(w^-(z,0)\right) + \varepsilon\Theta(\xi). \tag{11.2}$$

Proof of (11.2). Since the function W is defined up to a constant, we assume that $W(1) = 0$. We recall that $\partial\Omega = \Gamma_0 \bigcup \cup_{j=1}^m \Gamma_j$, where Γ_0 is the external connected component of $\partial\Omega$ and Γ_i represents the internal connected components. For simple convenience, we are also assuming that the points $\xi_1, \ldots, \xi_{2k}$ are distributed along Γ_0.

By construction, $U_\varepsilon = 1$ on Γ_i for all $i \neq 0$ and $= \pm 1$ on $\Gamma_0 \setminus \bigcup_{j=1}^{2k} B(\xi_j, 2\delta)$. Thus we have

$$\frac{1}{\varepsilon}\int_{\partial\Omega} W(U_\varepsilon) = \sum_{j=1}^{2k}\int_{\Gamma_0\cap B(\xi_j,2\delta)} W\left(\bar{U}_\varepsilon\right).$$

Fix some $j \in \{1, \ldots, 2k\}$, say $j = 1$. Using the change of variable in (3.7) and the very definition of $\bar{U}_\varepsilon$, we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\Gamma_0\cap B(\xi_1,2\delta)} W\left(\bar{U}_\varepsilon\right) &= \frac{1}{\varepsilon}\int_{s_1-2\delta}^{s_1+2\delta} W\left(\tilde{w}_\varepsilon^+(s-s_1,0)\right)\sqrt{1+\gamma'(s)^2}\\ &= \frac{1}{\varepsilon}\int_{-2\delta}^{2\delta} W\left(\tilde{w}_\varepsilon^+(t,0)\right)\sqrt{1+\gamma'(t+s_1)^2}\\ &= \frac{1}{\varepsilon}\left[\int_{-2\delta}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{2\delta}\right] W\left(\tilde{w}_\varepsilon^+(t,0)\right)\sqrt{1+\gamma'(t+s_1)^2}\\ &= a + b + c.\end{aligned}$$

Since $W'(1) = W'(-1) = 0$ and $W''(1),\ W''(-1) > 0$, a consequence of (3.4) and (3.5) is that

$$W(w^+(z,0)) \sim \frac{1}{(1+|z|)^2} \quad \text{as} \quad z \to \pm\infty.$$

Thus we have, expanding variables, $z = \frac{t}{\varepsilon}$

$$\begin{aligned} b &= \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} W(w^+(z,0))\sqrt{1+\varepsilon^2\gamma'(\varepsilon z + s_1)^2}\,\mathrm{d}z \\ &= \int_{\mathbb{R}} W(w^+(z,0)) + \varepsilon\Theta(\xi), \end{aligned}$$

where $\Theta(\xi)$ will denote throughout the whole proof a smooth and bounded function of the points ξ satisfying $|\xi_i - \xi_j| \geqq 5\delta\ \forall i \neq j$.

On the other hand, we have

$$c = \int_{\frac{\delta}{\varepsilon}}^{2\frac{\delta}{\varepsilon}} W\left(\eta_2(\varepsilon z)(w^+(z,0)-1)+1\right)(1+O(\varepsilon))\,.$$

A Taylor expansion, together with (3.4) and (3.5), $W(1) = W(-1) = 0$, $W'(1) = W'(-1) = 0$ and $W''(1),\ W''(-1) > 0$ we get $|W(1+\eta_2(\varepsilon z)(w^+(z,0)-1))| \leqq C\frac{\varepsilon^2}{\delta^2}$, from which we get

$$c \leqq C\varepsilon \quad \text{and, in a very similar way, } a \leqq C\varepsilon$$

for some positive constant C. This concludes the proof of (11.2). □

Next we estimate $\frac{1}{2}\int_\Omega |\nabla U_\varepsilon|^2$. Let $\rho > 0$ be a fixed small number, independent of ε. We write

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla U_\varepsilon|^2 &= \frac{1}{2}\int_{\Omega\setminus\cup_{j=1}^{2k} B(\xi_j,\rho)} |\nabla U_\varepsilon|^2 + \sum_{j=1}^{2k}\frac{1}{2}\int_{B(\xi_j,\rho)} |\nabla U_\varepsilon|^2 \\ &= I + \sum_{j=1}^{2k} I_j \end{aligned} \tag{11.3}$$

We claim that

$$I = -\frac{4k}{\pi}\log\rho + \Theta_0(\xi) + \frac{1}{\pi}\psi_k(\xi) + \rho\Xi(\xi) + \rho\log\rho\Theta(\xi) + \varepsilon\Theta(\xi) \tag{11.4}$$

where

$$\psi_k(\xi) = \sum_{j=1}^{2k} H(\xi_j,\xi_j) + \sum_{i\neq j}(-1)^{i+j}G(\xi_i,\xi_j),$$

Θ_0 is an explicit function of ξ, smooth and bounded in the region we consider, by $\Theta(\xi)$ we denote a smooth and bounded function of ξ in the considered region and by $\Xi(\xi)$ a smooth function of ξ so that $|\Xi(\xi)| \leqq C|\psi_k(\xi)|$.

We also claim that, for all $j = 1, \dots, 2k$,

$$I_j = \frac{2}{\pi} \log \frac{\rho}{\varepsilon} + A + B\rho \log \rho + C\rho \log \varepsilon + \rho^2 \Theta(\xi) + O(\varepsilon), \qquad (11.5)$$

where A, B, C are constants and $\Theta(\xi)$ is a smooth and bounded function in the considered region.

Observe that (11.1) follows from (11.2), (11.4) and (11.5). The rest of the proof is thus devoted to proving the validity of (11.4) and (11.5). $\square$

Proof of (11.4). In $\Omega \setminus \cup_{j=1}^{2k} B(\xi_j, \rho)$ we write $U_\varepsilon = u^* + R_\varepsilon$, where $\Delta R_\varepsilon = 0$ in Ω and $R_\varepsilon = \bar{U}_\varepsilon - u^*$ on Γ_0 and $R_\varepsilon = 0$ on Γ_i. Thus

$$\begin{aligned} I &= \frac{1}{2} \int_{\Omega \setminus \cup_{j=1}^{2k} B(\xi_j,\rho)} |\nabla u^*|^2 + \frac{1}{2} \int_{\Omega \setminus \cup_{j=1}^{2k} B(\xi_j,\rho)} |\nabla R_\varepsilon|^2 + \int_{\Omega \setminus \cup_{j=1}^{2k} B(\xi_j,\rho)} \nabla R_\varepsilon \nabla u^* \\ &= A_1 + A_2 + A_3. \end{aligned}$$

We claim that

$$A_2 + A_3 = \rho\Theta(\xi) + \varepsilon\Theta(\xi). \qquad (11.6)$$

and

$$\begin{aligned} A_1 = &-\frac{4k}{\pi} \log \rho + \Theta_0(\xi) \\ &+ \frac{1}{\pi} \left[\sum_{j=1}^{2k} H(\xi_j, \xi_j) + \sum_{i \neq j} (-1)^{i+j} G(\xi_i, \xi_j) \right] (1 + O(1)\rho) \\ &+ \rho \log \rho \Theta(\xi) \end{aligned} \qquad (11.7)$$

In (11.6) and (11.7) again $\Theta(\xi)$ denotes a smooth and bounded function of ξ in the region $|\xi_i - \xi_j| \geqq 5\delta \ \forall i \neq j$. In (11.7) Θ_0 is an explicit function of ξ, smooth and bounded in the region we consider, while by $O(1)$ we denote a bounded function of ξ. $\square$

Proof of (11.6). We start with A_2. Since $R_\varepsilon = 0$ on Γ_i, for all $i = 1, \dots, m$, we have

$$A_2 = \left[\frac{1}{2} \int_{\Gamma_0 \setminus \cup_{j=1}^{2k} B(\xi_j,\rho)} + \frac{1}{2} \sum_{j=1}^{2k} \int_{\Omega \cap \partial B(\xi_j,\rho)} \right] \frac{\partial R_\varepsilon}{\partial \nu} R_\varepsilon.$$

On $\Gamma_0 \setminus \cup_{j=1}^{2k} B(\xi_j, \rho)$, using (3.4)-(3.5) we get

$$|R_\varepsilon(x)| \leqq C|w^\pm\left(\frac{x}{\varepsilon}, 0\right) \pm 1| \leqq C\varepsilon$$

and by (8.2)

$$\left|\frac{\partial R_\varepsilon}{\partial \nu}\right| \leqq C$$

thus

$$\left| \int_{\Gamma_0 \setminus \cup_{j=1}^{2k} B(\xi_j,\rho)} \frac{\partial R_\varepsilon}{\partial \nu} R_\varepsilon \right| \leqq C\varepsilon.$$

Using (8.2) again, we also have

$$\int_{\Omega \cap \partial B(\xi_j,\rho)} \frac{\partial R_\varepsilon}{\partial \nu} R_\varepsilon = \rho \Theta(\xi).$$

Thus

$$A_2 = \rho \Theta(\xi) + \varepsilon \Theta(\xi).$$

Now, concerning A_3, we have, using again that $R_\varepsilon = 0$ on Γ_i for all $i = 1, \dots, m$,

$$A_3 = \int_{\Gamma_0 \setminus \cup_{j=1}^{2k} B(\xi_j,\rho)} \frac{\partial u^*}{\partial \nu} R_\varepsilon + \sum_{j=1}^{2k} \int_{\Omega \cap \partial B(\xi_j,\rho)} \frac{\partial u^*}{\partial \nu} R_\varepsilon.$$

As before, we get

$$\left| \int_{\Gamma_0 \setminus \cup_{j=1}^{2k} B(\xi_j,\rho)} \frac{\partial u^*}{\partial \nu} R_\varepsilon \right| \leqq C\varepsilon, \quad \left| \int_{\Omega \cap \partial B(\xi_j,\rho)} \frac{\partial u^*}{\partial \nu} R_\varepsilon \right| = \rho \Theta(\xi).$$

We thus have the validity of (11.6). □

Proof of (11.7). Associated to any Γ_i, for $i = 1, \dots, m$, define the smooth and bounded function φ_i solution to

$$\begin{cases} \Delta \varphi_i = 0 & \text{in } \Omega \\ \dfrac{\partial \varphi_i}{\partial \nu} = 1 & \text{on } \Gamma_i \\ \dfrac{\partial \varphi_i}{\partial \nu} = 0 & \text{on } \Gamma_j \quad \text{for all } j \neq 1,\ j \neq 0 \\ \varphi_i = 0 & \text{on } \Gamma_0. \end{cases} \tag{11.8}$$

Furthermore, let $\alpha_i = \alpha_i(\xi_1, \dots, \xi_{2k})$ be defined as

$$\alpha_i = -\frac{1}{|\Gamma_i|} \int_{\Gamma_i} \frac{\partial u^*}{\partial \nu}. \tag{11.9}$$

Observe that α_i is a smooth function of $\xi_1, \dots, \xi_{2k}$ in the region considered.

Thus the function $\varphi := \sum_{i=1}^m \alpha_i \varphi_i$ satisfies

$$\int_{\Gamma_i} \frac{\partial}{\partial \nu}(u^* + \varphi) = 0 \quad \text{for all } i = 1, \dots, m,$$

and, by harmonicity,

$$\int_{\Gamma_0} \frac{\partial}{\partial \nu}(u^* + \varphi) = 0.$$

It follows that $u^* + \varphi$ admits an harmonic conjugate ψ, which is defined, up to an additive constant, to be harmonic in Ω and to satisfy the Neumann boundary condition

$$\frac{\partial \psi}{\partial \nu} = \frac{\partial}{\partial \tau}(u^* + \varphi) \quad \text{on} \quad \partial\Omega.$$

From the very definition of u^*, we have that

$$\frac{\partial u^*}{\partial \tau} = 2\sum_{j=1}^{2k}(-1)^{j+1}\delta_{\xi_j} \quad \text{on} \quad \Gamma_0$$

and

$$\frac{\partial u^*}{\partial \tau} = 0 \quad \text{on} \quad \Gamma_i \quad \text{for all } i = 1, \dots, m.$$

Thus we can write

$$\psi(x) = \frac{1}{\pi}\sum_{j=1}^{2k}(-1)^{j+1}G(x, \xi_j) + g(x) \quad \text{in } \Omega, \tag{11.10}$$

where $G(x, y)$ is the Green function defined in (1.7) and

$$\begin{cases} \Delta g = 0 \quad \text{in } \Omega \\ \dfrac{\partial g}{\partial \nu} = -\dfrac{\pi}{|\partial\Omega|} \quad \text{on } \Gamma_0 \\ \dfrac{\partial g}{\partial \nu} = \dfrac{\partial \varphi}{\partial \tau} \quad \text{on } \Gamma_i \quad \text{for all } i = 1, \dots, m. \end{cases} \tag{11.11}$$

The function g is a smooth function on Ω, and it depends smoothly also on $\xi_1, \dots, \xi_{2k}$. It is uniformly bounded in the region $|\xi_i - \xi_j| \geqq 5\delta \ \forall i \neq j$. It is defined up to an additive constant. We thus choose g to further satisfy

$$\int_{\Gamma_0} g = 0. \tag{11.12}$$

At this point we write

$$A_1 = \frac{1}{2}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)} |\nabla(u^* + \varphi)|^2 - \frac{1}{2}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)} |\nabla\varphi|^2 - \int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)} \nabla u^* \nabla\varphi. \tag{11.13}$$

We claim that

$$A_1 = \frac{1}{2}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)} |\nabla\psi|^2 + \Theta(\xi), \tag{11.14}$$

for some explicit function $\Theta(\xi)$ which is smooth and bounded in the region of points $\xi = (\xi_1, \dots, \xi_{2k})$ satisfying $|\xi_i - \xi_j| \geqq 5\delta \ \forall i \neq j$. Indeed estimate (11.14) follows from the following computations, based on integration by parts,

$$-\frac{1}{2}\int_\Omega |\nabla\varphi|^2 - \int_\Omega \nabla u^*\nabla\varphi = -\frac{1}{2}\int_{\partial\Omega}\frac{\partial\varphi}{\partial\nu}\varphi - \int_{\partial\Omega}\frac{\partial\varphi}{\partial\nu}u^*$$
$$= -\frac{1}{2}\sum_{i\neq j}^{m}\alpha_i\alpha_j\int_{\Gamma_i}\varphi_j - \sum_{i=1}^{m}\int_{\Gamma_i}\frac{\partial u^*}{\partial\nu} - \int_{\Gamma_0}\frac{\partial\varphi}{\partial\nu}u^*$$

where this last expression is a smooth and explicit function of $\xi_1, \dots, \xi_{2k}$, which is bounded together with its derivatives in the region considered. To conclude with (11.14), we are left to observe that

$$\sum_{j=1}^{2k}\left[\frac{1}{2}\int_{\Omega\cap B(\xi_j,\rho)}|\nabla\varphi|^2 + \int_{\Omega\cap B(\xi_j,\rho)}\nabla\varphi\nabla u^*\right]$$
$$= \sum_{j=1}^{2k}\left[\frac{1}{2}\int_{\Omega\cap B(\xi_j,\rho)}|\nabla\varphi|^2 + \int_{\Omega\cap\partial B(\xi_j,\rho)}\frac{\partial\varphi}{\partial\nu}u^* + \int_{\partial\Omega\cap B(\xi_j,\rho)}\frac{\partial\varphi}{\partial\nu}u^*\right]$$
$$= \rho\Theta(\xi).$$

Given (11.14) and (11.13), we are left with the expansion of $\frac{1}{2}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)}|\nabla\psi|^2$. By (11.10), we write

$$\frac{1}{2}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)}|\nabla\psi|^2 = \frac{1}{2}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)}\left|\nabla\left[\frac{1}{\pi}\sum_{j=1}^{2k}(-1)^{j+1}G(x,\xi_j)\right]\right|^2$$
$$+\frac{1}{2}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)}|\nabla g|^2$$
$$+\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)}\nabla\left[\frac{1}{\pi}\sum_{j=1}^{2k}(-1)^{j+1}G(x,\xi_j)\right]\nabla g$$
$$= A_{11} + A_{12} + A_{13}. \tag{11.15}$$

The principal part of the above expansion is contained in A_{11}. We start with it. We write

$$A_{11} = \frac{1}{2\pi^2}\left[\sum_{j=1}^{2k}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)}|\nabla G(x,\xi_j)|^2 + \sum_{i\neq j}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)}\nabla G(x,\xi_j)\nabla G(x,\xi_i)\right]. \tag{11.16}$$

We start with the first term in (11.16). Fix $l \in \{1, \dots, 2k\}$. Since $\frac{\partial G(x,\xi_l)}{\partial\nu} = 0$ on Γ_i for all $i = 1, \dots, m$, $\frac{\partial G(x,\xi_l)}{\partial\nu} = \frac{2\pi}{|\partial\Omega|}$ on $\Gamma_0\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)$ and $\int_{\Gamma_0}G(x,\xi_l) = 0$, we have, after an integration by parts,

$$\begin{aligned}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)}|\nabla G(x,\xi_l)|^2 &= -\frac{2\pi}{|\partial\Omega|}\sum_{j=1}^{2k}\int_{B(\xi_j,\rho)\cap\Gamma_0}G(x,\xi_l)\\ &\quad+\int_{\Omega\cap\cup_{j=1}^{2k}\partial B(\xi_j,\rho)}\frac{\partial G(x,\xi_l)}{\partial\nu}G(x,\xi_l)\\ &=\int_{\Omega\cap\partial B(\xi_l,\rho)}\frac{\partial G(x,\xi_l)}{\partial\nu}G(x,\xi_l)\\ &\quad+\sum_{j\neq l}\int_{\Omega\cap\partial B(\xi_j,\rho)}\frac{\partial G(x,\xi_l)}{\partial\nu}G(x,\xi_l)+\rho\Theta(\xi).\end{aligned}$$

To compute $\int_{\Omega\cap\partial B(\xi_l,\rho)}\frac{\partial G(x,\xi_l)}{\partial\nu}G(x,\xi_l)$, we observe that on $\Omega\cap\partial B(\xi_l,\rho)$ we have $G(x,\xi_l)=\log\frac{1}{\rho^2}+H(x,\xi_l)$, see (1.8), and $\frac{\partial G(x,\xi_l)}{\partial\nu}=\frac{2}{\rho}+\nabla H(x,\xi_l)\cdot\nu$. With this in mind, a Taylor expansion gives

$$\begin{aligned}&\int_{\Omega\cap\partial B(\xi_l,\rho)}\frac{\partial G(x,\xi_l)}{\partial\nu}G(x,\xi_l)\\ &\quad=\int_{\Omega\cap\partial B(\xi_l,\rho)}\left[\log\frac{1}{\rho^2}+H(\xi_l,\xi_l)+H(x,\xi_l)-H(\xi_l,\xi_l)\right]\\ &\qquad\times\left[\frac{2}{\rho}+\nabla H(x,\xi_l)\cdot\nu\right]\\ &\quad=-4\pi\log\rho+2\pi H(\xi_l,\xi_l)+\rho\log\rho\Theta(\xi)+\rho\Theta(\xi),\end{aligned}\tag{11.17}$$

where $\Theta(\xi)$ is a smooth and bounded function of ξ in the considered region. In the above computations we have also used the fact that ρ is small to say that $|\Omega\cap\partial B(\xi_l,\rho)|=\pi\rho+O(\rho^2)$.

On the other hand, for $j\neq l$,

$$\int_{\Omega\cap\partial B(\xi_j,\rho)}\frac{\partial G(x,\xi_l)}{\partial\nu}G(x,\xi_l)=\rho\,\Xi(\xi),$$

where now, and for the rest of this proof, $\Xi(\xi)$ will denote a smooth function of ξ so that $|\Xi(\xi)|\leqq C|\psi_k(\xi)|$. This, together with (11.17), gives

$$\begin{aligned}\int_{\Omega\setminus\cup_{j=1}^{2k}B(\xi_j,\rho)}|\nabla G(x,\xi_l)|^2&=-4\pi\log\rho+2\pi H(\xi_l,\xi_l)\\ &\quad+\rho\log\rho\Theta(\xi)+\rho\,\Xi(\xi),\end{aligned}\tag{11.18}$$

where $\Theta(\xi)$ is a smooth and bounded function of ξ and $\Xi(\xi)$ is a smooth function of ξ with $|\Xi(\xi)|\leqq C|\psi_k(\xi)|$.

Now let $j \neq i$. An integration by parts and the fact that $\frac{\partial G(x,\xi_i)}{\partial \nu} = 0$ on Γ_i for all $i = 1, \dots, m$ give

$$\begin{aligned}\int_{\Omega\setminus\cup_{l=1}^{2k} B(\xi_l,\rho)} \nabla G(x,\xi_i)\nabla G(x,\xi_j) &= \int_{\Gamma_0\setminus\cup_{l=1}^{2k} B(\xi_l,\rho)} G(x,\xi_i)\frac{\partial G(x,\xi_j)}{\partial \nu}\\ &+ \int_{\partial B(\xi_i,\rho)\cap\Omega}\left[\log\frac{1}{\rho^2} + H(x,\xi_i)\right]\frac{\partial G(x,\xi_j)}{\partial \nu}\\ &+ \int_{\partial B(\xi_j,\rho)\cap\Omega} G(x,\xi_i)\left[\frac{2}{\rho} + \frac{\partial H(x,\xi_j)}{\partial \nu}\right]\\ &+ \sum_{l\neq i,j}\int_{\partial B(\xi_l,\rho)\cap\Omega} G(x,\xi_i)\frac{\partial G(x,\xi_j)}{\partial \nu}.\end{aligned}$$

Since $\int_{\Gamma_0} G(x,\xi_i) = 0$ and $\frac{\partial G(x,\xi_j)}{\partial \nu}$ is constant on $\Gamma_0 \setminus \cup_{l=1}^{2k} B(\xi_l,\rho)$, we have

$$\int_{\Gamma_0\setminus\cup_{l=1}^{2k} B(\xi_l,\rho)} G(x,\xi_i)\frac{\partial G(x,\xi_j)}{\partial \nu} = \rho\,\Xi(\xi).$$

Since ρ is arbitrarily small, we have

$$\int_{\partial B(\xi_i,\rho)\cap\Omega}\left[\log\frac{1}{\rho^2} + H(x,\xi_i)\right]\frac{\partial G(x,\xi_j)}{\partial \nu} = \rho\log\rho\,\Theta(\xi) + \rho\,\Xi(\xi).$$

Using again that ρ is small and a Taylor expansion, we have

$$\int_{\partial B(\xi_j,\rho)\cap\Omega} G(x,\xi_i)\left[\frac{2}{\rho} + \frac{\partial H(x,\xi_j)}{\partial \nu}\right] = 2\pi G(\xi_i,\xi_j) + \rho\,\Xi(\xi).$$

We conclude observing that

$$\sum_{l\neq i,j}\int_{\partial B(\xi_l,\rho)\cap\Omega} G(x,\xi_i)\frac{\partial G(x,\xi_j)}{\partial \nu} = \rho\;\Xi(\xi).$$

Thus

$$\int_{\Omega\setminus\cup_{l=1}^{2k} B(\xi_l,\rho)} \nabla G(x,\xi_i)\nabla G(x,\xi_j) = 2\pi G(\xi_i,\xi_j) + \rho\log\rho\,\Theta(\xi). \tag{11.19}$$

Collecting (11.16), (11.17) and (11.19) we get

$$\begin{aligned}A_{11} &= -\frac{4k}{\pi}\log\rho + \frac{1}{\pi}\left[\sum_{j=1}^{2k} H(\xi_j,\xi_j) + \sum_{i\neq j}(-1)^{i+j}G(\xi_i,\xi_j)\right](1+O(\rho))\\ &\quad +\rho\log\rho\,\Theta(\xi),\end{aligned} \tag{11.20}$$

where Θ is a smooth function of ξ, bounded in the considered region. We go back to (11.15) and we compute A_{12}. An integration by parts gives

$$A_{12} = \frac{1}{2}\int_{\Gamma_0\setminus\cup_{j=1}^{2k} B(\xi_j,\rho)} \frac{\partial g}{\partial \nu} g + \frac{1}{2}\sum_{j=1}^{2k}\int_{\partial B(\xi_j,\rho)\cap\Omega}\frac{\partial g}{\partial \nu} g + \frac{1}{2}\sum_{i=1}^{m}\int_{\Gamma_i}\frac{\partial g}{\partial \nu} g.$$

From (11.12) we get $\int_{\Gamma_0 \setminus \cup_{j=1}^{2k} B(\xi_j,\rho)} \frac{\partial g}{\partial \nu} g = \rho \Theta(\xi)$, from smoothness and boundedness of g we get $\int_{\partial B(\xi_j,\rho)\cap\Omega} \frac{\partial g}{\partial \nu} g = \rho\Theta(\xi)$. Thus we conclude that

$$A_{12} = \sum_{i=1}^{m} \int_{\Gamma_i} \frac{\partial g}{\partial \nu} g + \rho\Theta(\xi). \tag{11.21}$$

Concerning A_{13}, an integration by parts gives

$$\begin{aligned} A_{13} &= \frac{1}{\pi} \sum_{j=1}^{2k} (-1)^{j+1} \left[-\frac{2\pi}{|\partial\Omega|} \int_{\Gamma_0 \setminus \cup_{j=1}^{2k} B(\xi_j,\rho)} g + \sum_i \int_{\partial B(\xi_i,\rho)\cap\Omega} \frac{\partial G(x,\xi_j)}{\partial \nu} g \right] \\ &= \rho\Theta(\xi) + \frac{1}{\pi} \sum_{j=1}^{2k} (-1)^{j+1} \sum_i \int_{\partial B(\xi_i,\rho)\cap\Omega} \frac{\partial G(x,\xi_j)}{\partial \nu} g \end{aligned}$$

Now,

$$\begin{aligned} \int_{\partial B(\xi_j,\rho)\cap\Omega} \frac{\partial G(x,\xi_j)}{\partial \nu} g &= \int_{\partial B(\xi_i,\rho)\cap\Omega} \left[\frac{2}{\rho} + \nabla H(x,\xi_j)\cdot \nu \right] g \\ &= -2\pi g(\xi_j) + \rho\Theta(\xi) \end{aligned}$$

while, for $i \neq j$,

$$\int_{\partial B(\xi_i,\rho)\cap\Omega} \frac{\partial G(x,\xi_j)}{\partial \nu} g = \rho\, \Xi(\xi).$$

Thus we conclude that

$$A_{13} = 2 \sum_{j=1}^{2k} (-1)^j g(\xi_j) + \rho(\Theta(\xi) + \Xi(\xi)). \tag{11.22}$$

Thus expansion (11.7) follows from (11.14), (11.15), (11.20), (11.21) and (11.22). Formula (11.7) together with (11.6) gives the expansion (11.4). □

Proof of (11.5). We want to compute, for any $j = 1, \ldots, 2k$,

$$I_j = \frac{1}{2} \int_{\Omega\cap B(\xi_j,\rho)} |\nabla U_\varepsilon|^2.$$

Let us fix j, say $j = 1$. Take $\rho < \frac{\delta}{2}$. Hence in each $B(\xi_j, \rho)$ we have $U_\varepsilon = \bar{U}_\varepsilon + H_\varepsilon$. Thus

$$\begin{aligned} I_1 &= \frac{1}{2} \int_{B(\xi_1,\rho)\cap\Omega} \left|\nabla \bar{U}_\varepsilon\right|^2 + \frac{1}{2} \int_{B(\xi_1,\rho)\cap\Omega} |\nabla H_\varepsilon|^2 + \int_{B(\xi_1,\rho)\cap\Omega} \nabla \bar{U}_\varepsilon \nabla H_\varepsilon \\ &= a + b + c \end{aligned} \tag{11.23}$$

A direct consequence of Proposition 8.3 is that

$$b = \rho^2 \Theta(\xi).$$

Formula (11.5) follows then from the following statements

$$c = C\rho \log \frac{\varepsilon}{\rho} + \rho^2 \Theta(\xi) + O(\varepsilon^2), \tag{11.24}$$

where C is a constant and $\Theta(\xi)$ denotes a smooth and bounded function of ξ in the considered region, and

$$a = -\frac{2}{\pi} \log \frac{\rho}{\varepsilon} + A + O(\varepsilon), \tag{11.25}$$

where A is a constant. □

Proof of (11.24). Since $H_\varepsilon = 0$ on $\partial\Omega$ we have

$$c = -\int_{\Omega \cap B(\xi_1,\rho)} \Delta \bar{U}_\varepsilon H_\varepsilon + \int_{\Omega \cap \partial B(\xi_1,\rho)} \frac{\partial \bar{U}_\varepsilon}{\partial \nu} H_\varepsilon.$$

Again using Proposition 8.3, namely that $\|\nabla H_\varepsilon\|_{L^\infty(\Omega)} \leqq C$, and formula (8.6), we get

$$\left| \int_{\Omega \cap \partial B(\xi_1,\rho)} \frac{\partial \bar{U}_\varepsilon}{\partial \nu} H_\varepsilon \right| \leqq C\rho \int_{\Omega \cap \partial B(\xi_1,\rho)} \frac{1}{\varepsilon + |z - \xi_1|} \leqq C\rho.$$

Now using formula (8.7), we have

$$\begin{aligned} \left| \int_{\Omega \cap B(\xi_1,\rho)} \Delta \bar{U}_\varepsilon H_\varepsilon \right| &\leqq C\rho \int_{\Omega \cap B(\xi_1,\rho)} \frac{C}{\varepsilon^2 + |z - \xi_1|^2} \, \mathrm{d}z \leqq C\rho \int_0^{\frac{\rho}{\varepsilon}} \frac{s}{1 + s^2} \, \mathrm{d}s \\ &\leqq C\rho \log \left(1 + \frac{\rho^2}{\varepsilon^2}\right). \end{aligned}$$

These facts prove (11.24). □

Proof of (11.25). We write

$$a = -\frac{1}{2} \int_{\Omega \cap B(\xi_1,\rho)} \bar{U}_\varepsilon \Delta \bar{U}_\varepsilon + \frac{1}{2} \int_{\Omega \cap \partial B(\xi_1,\rho)} \frac{\partial \bar{U}_\varepsilon}{\partial \nu} \bar{U}_\varepsilon + \frac{1}{2} \int_{\Gamma_0 \cap B(\xi_1,\rho)} \frac{\partial \bar{U}_\varepsilon}{\partial \nu} \bar{U}_\varepsilon. \tag{11.26}$$

Since $\rho < \frac{\delta}{2}$, arguing as in (8.10) we have that in $B(\xi_1, \rho)$

$$\Delta \bar{U}_\varepsilon = \Delta \omega_\varepsilon^+ = 2k(0)t \left(w_\varepsilon^+\right)_{ss} - k(0) \left(w_\varepsilon^+\right)_t + O(1).$$

Thus

$$\left| \int_{\Omega \cap B(\xi_1,\rho)} \bar{U}_\varepsilon \Delta \bar{U}_\varepsilon \right| \leqq C \left[\rho \int_{B(0,\rho)} \left| D^2 \omega_\varepsilon^+ \right| + \int_{B(0,\rho)} \left| D\omega_\varepsilon^+ \right| + \rho^2 \right].$$

Using estimate (8.7), we get

$$\rho \int_{B(0,\rho)} \left| D^2 \omega_\varepsilon^+ \right| \leqq C\rho \log \frac{\varepsilon}{\rho} + O(\varepsilon^2),$$

while using (8.6), we get

$$\int_{B(0,\rho)} \left| D\omega_\varepsilon^+ \right| \leqq C\rho + \varepsilon \log \frac{\varepsilon}{\rho} + O(\varepsilon^2).$$

Thus we have

$$\left| \frac{1}{2} \int_{\Omega \cap B(\xi_1,\rho)} \bar{U}_\varepsilon \Delta \bar{U}_\varepsilon \right| \leqq C(\rho+\varepsilon) \log \frac{\varepsilon}{\rho} + C\rho.$$

In a very similar way, one gets

$$\left| \frac{1}{2} \int_{\Omega \cap \partial B(\xi_1,\rho)} \frac{\partial \bar{U}_\varepsilon}{\partial \nu} \bar{U}_\varepsilon \right| \leqq C \frac{\rho}{\varepsilon+\rho} \leqq C.$$

In order to complete the expansion of (11.26), we need to compute $\frac{1}{2}\int_{\Gamma_0 \cap B(\xi_1,\rho)} \frac{\partial \bar{U}_\varepsilon}{\partial \nu} \bar{U}_\varepsilon$. Since ρ is small and w^+ is a solution to (3.1), we write

$$\begin{aligned} \frac{1}{2} \int_{\Gamma_0 \cap B(\xi_1,\rho)} \frac{\partial \bar{U}_\varepsilon}{\partial \nu} \bar{U}_\varepsilon &= \frac{1}{2} \int_{\Gamma_0 \cap B(\xi_1,\rho)} \frac{\partial \tilde{w}_\varepsilon^+}{\partial \nu} \tilde{w}_\varepsilon^+ (x - \xi_1 = \varepsilon z) \\ &= \frac{1}{2} \int_{\frac{\Gamma_0}{\varepsilon} \cap B(0,\frac{\rho}{\varepsilon})} \frac{\partial w^+}{\partial \nu} w^+ \\ &= -\frac{1}{2} \left(\int_{-\frac{\rho}{\varepsilon}}^{\frac{\rho}{\varepsilon}} f(w^+) w^+ \,\mathrm{d}x \right) (1 + O(\varepsilon)). \end{aligned} \tag{11.27}$$

For simplicity of notation we write $w^+ = w$. We first compute $\int_0^{\frac{\rho}{\varepsilon}} f(w) w \,\mathrm{d}x$. Since $f(1) = 0$, $f'(1) \neq 0$ and (5.3) we have

$$\int_0^{\frac{\rho}{\varepsilon}} f(w) w \,\mathrm{d}x = \frac{2}{\pi} \log \left(\frac{\rho}{\varepsilon} \right) + C + O(\varepsilon^2), \tag{11.28}$$

where C is a constant. In an analogous way one can prove that

$$\int_{-\frac{\rho}{\varepsilon}}^{0} f(w) w \,\mathrm{d}x = \frac{2}{\pi} \log \left(\frac{\rho}{\varepsilon} \right) + \tilde{C} + O(\varepsilon^2), \tag{11.29}$$

for some constant $\tilde{C}$, using again that $f(-1) = 0$, $f'(-1) \neq 0$ and (5.4). From (11.27), (11.28) and (11.29) we obtain the validity of (11.25), and this concludes the proof of the Lemma. □

We now have all the ingredients needed to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Define, for $\xi = (\xi_1, \ldots, \xi_{2k}) \in (\Gamma_0)^{2k}$ with $|\xi_i - \xi_j| \geqq 5\delta$, the function

$$u(x) = U_\varepsilon(\xi)(x) + \phi(\xi)(x) \quad x \in \Omega$$

where $U_\varepsilon(\xi)$ is given by (3.10) and $\phi(\xi)$ is the unique solution to problem (9.1), whose existence and properties are established in Proposition 9.1. Then, according to Lemma 10.1, u is solution to (1.1) provided that ξ is a critical point of the function $F_\varepsilon(\xi)$ defined in (10.1), or equivalently, ξ is a critical point of

$$\tilde{F}_\varepsilon(\xi) = \pi \left(\frac{2k}{\pi} \log \varepsilon - F_\varepsilon(\xi) \right).$$

Let $\tilde{\Gamma}_0$ be the set of points $\xi = (\xi_1, \dots, \xi_{2k}) \in (\Gamma_0)^{2k}$ ordered clockwise and such that $|\xi_i - \xi_j| \geqq 5\delta$ for all $i \neq j$, for some $\delta > 0$ sufficiently small so that all the previous results hold true. Then, if we denote by $p : [0, 2\pi] \to \Gamma_0$ a continuous parametrization of Γ_0, then we can write

$$\tilde{\Gamma}_0 = \{\xi = (p(\theta_1), \dots, p(\theta_{2k})) \in (\Gamma_0)^{2k} : |p(\theta_i) - p(\theta_j)| \geqq \delta \quad \text{if} \quad i \neq j\}.$$

It is not restrictive to assume that $0 \in \Gamma_0$. Lemmas 10.2 and 11.1 guarantee that for $\xi \in \tilde{\Gamma}_0$,

$$\tilde{F}_\varepsilon(\xi) = \psi_k(\xi) + \Theta_0(\xi) + \varepsilon\Theta_\varepsilon(\xi) \tag{11.30}$$

where Θ_ε is uniformly bounded in the considered region as $\varepsilon \to 0$, while Θ_0 is smooth and uniformly bounded, as $\varepsilon \to 0$, in the considered region. We will show that $\tilde{F}_\varepsilon$ has at least two distinct critical points in this region, a fact that will prove our result. The function ψ_k is C^1, bounded from above in $\tilde{\Gamma}_0$ and if two consecutive points get closer it becomes unbounded from below, which implies that

$$\psi_k(\xi_1, \dots, \xi_{2k}) \to -\infty \text{ as } |\xi_i - \xi_j| \to 0 \text{ for some } i \neq j.$$

Hence, since δ is arbitrarily small, ψ_k has an absolute maximum in $\tilde{\Gamma}_0$, and so does $\psi_k + \Theta_0$, since Θ_0 is uniformly bounded in the region, as $\varepsilon \to 0$. Let us call M_0 the maximum value of $\psi_k + \Theta_0$. Thus also $\tilde{F}_\varepsilon$ has an absolute maximum whenever ε is sufficiently small. Let us call this value M_ε, so that $M_\varepsilon = M_0 + o(1)$ as $\varepsilon \to 0$. On the other hand, Ljusternik–Schnirelmann theory is applicable in our setting, so we can estimate the number of critical points of ψ_k in $\tilde{\Gamma}_0$ by $\text{cat}(\tilde{\Gamma}_0)$, the Ljusternik–Schnirelmann category of $\tilde{\Gamma}_0$ relative to $\tilde{\Gamma}_0$. We claim that cat $(\tilde{\Gamma}_0) > 1$. Indeed, by contradiction, assume that $\text{cat}(\tilde{\Gamma}_0) = 1$. This means that $\tilde{\Gamma}_0$ is contractible in itself, namely there exist a point $\xi^0 \in \tilde{\Gamma}_0$ and a continuous function $\Upsilon : [0, 1] \times \tilde{\Gamma}_0 \to \tilde{\Gamma}_0$ such that, for all $\xi \in \tilde{\Gamma}_0$,

$$\Upsilon(0, \xi) = \xi, \quad \Upsilon(1, \xi) = \xi^0.$$

Let $f : S^1 \to \tilde{\Gamma}_0$ be the continuous function defined by

$$f(x) = \left(p(\theta), p\left(\theta + 2\pi \frac{1}{2k}\right), \dots, p\left(\theta + 2\pi \frac{2k-1}{2k}\right)\right), \quad x = e^{i\theta}, \theta \in [0, 2\pi].$$

Let $\eta : [0, 1] \times S^1 \to S^1$ be the well defined continuous map given by

$$\eta(t, x) = \frac{\pi_1 \circ \Upsilon(t, f(x))}{\|\pi_1 \circ \Upsilon(t, f(x))\|}$$

where π_1 denotes the projection on the first component. The function η is a contraction of S^1 to a point and this gives a contradiction. Thus we conclude that

$$c_0 = \sup_{C\in\Xi}\inf_{\bar\xi\in C}(\psi_k + \Theta_0)(\bar\xi), \tag{11.31}$$

where

$$\Xi = \{C \subset \tilde\Gamma_0 \,:\, C \text{ closed and cat}\,(C) \geqq 2\},$$

is a finite number, and a critical level for $\psi_k + \Theta_0$. Call c_ε the number (11.31) with $\psi_k + \Theta_0$ replaced by $\tilde F_\varepsilon$, so that $c_\varepsilon = c_0 + o(1)$. If $c_\varepsilon \neq M_\varepsilon$, we conclude that there are at least two distinct critical points for $\tilde F_\varepsilon$ (distinct up to cyclic permutations) in $\tilde\Gamma_0$. If $c_\varepsilon = M_\varepsilon$, we get that there must be a set C, with cat $(C) \geqq 2$, where the function $\tilde F_\varepsilon$ reaches its absolute maximum. In this case we conclude that there are infinitely many critical points for $\tilde F_\varepsilon$ in $\tilde\Gamma_0$. Since cyclic permutations are only in finite number, the result is thus proven. □

Acknowledgments. This work has been supported by Fondecyt Grants 1070389, 1080099, 1090167 Fondo Basal CMM and CAPDE-Anillo ACT-125. The authors would like to thank Xavier Cabre for providing us insight on the renormalized energy of the problem.

Appendix A. Convolution estimates

First we need the following estimate for k defined in (4.4).

Lemma A.1. *The kernel k satisfies*

$$k(x_1, x_2) \leqq \begin{cases} C(1 + \log \frac{1}{|(x_1,x_2)|}) & \text{if } |(x_1, x_2)| \leqq 1 \\ C\frac{1+x_2}{x_1^2+(1+x_2)^2} & \text{if } |(x_1, x_2)| \geqq 1 \end{cases} \tag{A.1}$$

Proof. We write

$$\pi k(x_1, x_2) = \int_0^1 \ldots\, \mathrm{d}t + \int_1^\infty \ldots\, \mathrm{d}t.$$

Then

$$\int_1^\infty \frac{e^{-at}(x_2+t)}{x_1^2 + (x_2+t)^2}\,\mathrm{d}t \leqq \frac{1}{x_1^2 + (x_2+1)^2}\int_1^\infty e^{-at}(x_2+t)\,\mathrm{d}t$$
$$\leqq C\frac{x_2+1}{x_1^2 + (x_2+1)^2}$$

We assume $x_1 > 0$. The integral on the other region is estimated by

$$\int_0^1 \frac{e^{-at}(x_2+t)}{x_1^2 + (x_2+t)^2}\,\mathrm{d}t \leqq \int_0^1 \frac{x_2+t}{x_1^2 + (x_2+t)^2}\,\mathrm{d}t = \int_{x_2}^{x_2+1} \frac{t}{x_1^2 + t^2}\,\mathrm{d}t$$
$$= \int_{x_2/x_1}^{(x_2+1)/x_1} \frac{r}{1+r^2}\,\mathrm{d}r = \frac{1}{2}\log\left(1 + \frac{2x_2+1}{x_1^2+x_2^2}\right)$$

which is bounded by the right-hand side of (A.1). □

Proof of Lemma 4.1. Using (A.1) we can estimate

$$|\phi| \leqq C(\phi_1 + \phi_2)\|h\|_\alpha$$

where

$$\phi_1(x_1, x_2) = \int_{-\infty}^{\infty} \frac{1 + x_2}{(x_1 - y)^2 + (1 + x_2)^2} \frac{1}{(1 + |y|)^\alpha} \, \mathrm{d}y$$

and

$$\phi_2(x_1, x_2) = \begin{cases} \int_{x_1-1}^{x_1+1} \frac{1+|\log(x_1-y)^2+x_2^2)|}{(1+|y|)^\alpha} \, \mathrm{d}y & \text{if } x_2 \leqq 1 \\ 0 & \text{if } x_2 > 1 \end{cases}$$

Directly we have

$$\phi_2(x_1, x_2) \leqq \frac{C}{(1 + |x_1|)^\alpha}.$$

To obtain estimates for ϕ_1 we change variables $y = (1+x_2)r$ and define $s = \frac{x_1}{1+x_2}$. Then

$$\phi_1(x_1, x_2) = \int_{-\infty}^{\infty} \frac{1}{(s - r)^2 + 1} \frac{1}{(1 + (1 + x_2)|r|)^\alpha} \, \mathrm{d}r$$

We can assume $x_1 \geqq 0$ so that $s \geqq 0$.
Case 1. Assume $x_1 \leqq 1 + x_2$, that is $0 \leqq s \leqq 1$. Write

$$\int_{-\infty}^{\infty} \frac{1}{(s - r)^2 + 1} \frac{1}{(1 + (1 + x_2)|r|)^\alpha} \, dr = \int_{|r| \leqq \frac{1}{1+x_2}} \ldots \, \mathrm{d}r + \int_{|r| \geqq \frac{1}{1+x_2}} \ldots \, \mathrm{d}r$$

We have

$$\begin{aligned} \int_{|r| \leqq \frac{1}{1+x_2}} \frac{1}{(s - r)^2 + 1} \frac{1}{(1 + (1 + x_2)|r|)^\alpha} \, \mathrm{d}r &\leqq \int_{|r| \leqq \frac{1}{1+x_2}} \frac{1}{(s - r)^2 + 1} \, \mathrm{d}r \\ &\leqq C \int_{|r| \leqq \frac{1}{1+x_2}} \frac{1}{r^2+1} \, \mathrm{d}r \leqq C \frac{1}{1 + x_2}. \end{aligned}$$

Also

$$\begin{aligned} &\int_{|r| \geqq \frac{1}{1+x_2}} \frac{1}{(s - r)^2 + 1} \frac{1}{(1 + (1 + x_2)|r|)^\alpha} \, \mathrm{d}r \\ &\leqq \frac{1}{(1 + x_2)^\alpha} \int_{|r| \geqq \frac{1}{1+x_2}} \frac{1}{((s - r)^2 + 1)|r|^\alpha} \, \mathrm{d}r \end{aligned}$$

But

$$\int_{|r| \geqq \frac{1}{1+x_2}} \frac{1}{((s - r)^2 + 1)|r|^\alpha} \, \mathrm{d}r \leqq \begin{cases} C & \text{if } \alpha < 1 \\ C(1 + \log(1 + x_2)) & \text{if } \alpha = 1 \\ C(1 + x_2)^{\alpha - 1} & \text{if } \alpha > 1 \end{cases}$$

Then

$$\int_{|r|\leqq\frac{1}{1+x_2}} \frac{1}{(s-r)^2+1}\frac{1}{(1+(1+x_2)|r|)^\alpha}\,\mathrm{d}r \leqq \begin{cases} \frac{C}{(1+x_2)^\alpha} & \text{if } \alpha<1\\ \frac{C(1+\log(1+x_2))}{1+x_2} & \text{if } \alpha=1\\ \frac{C}{1+x_2} & \text{if } \alpha>1\end{cases}$$

Case 2. Assume $x_1 \geqq 1+x_2$, that is $s \geqq 1$. We make a further change of variables $r = st$ and obtain

$$\phi_1(x_1,x_2) = \frac{1}{s}\int_{-\infty}^{\infty} \frac{1}{(t-1)^2+1/s^2}\frac{1}{(1+x_1|t|)^\alpha}\,\mathrm{d}t$$

We split the integral in the regions

$$(a): |t| \geqq 2, \quad (b): |t| \leqq 2 \text{ and } |t-1| \geqq 1/2 \quad (c): |t-1| \leqq 1/2.$$

In the region (a):

$$\begin{aligned}\frac{1}{s}\int_{|t|\geqq 2} \frac{1}{(t-1)^2+1/s^2}\frac{1}{(1+x_1|t|)^\alpha}\,\mathrm{d}t &\leqq \frac{C}{sx_1^\alpha}\int_{|t|\geqq 2}\frac{1}{|t|^{2+\alpha}}\,\mathrm{d}t\\ &\leqq \frac{C}{sx_1^\alpha} \leqq C\frac{1+x_2}{(1+x_1)^{1+\alpha}}\end{aligned}$$

In the region (b):

$$\frac{1}{s}\int_{|t|\leqq 2,\ |t-1|\geqq 1/2} \frac{1}{(t-1)^2+1/s^2}\frac{1}{(1+x_1|t|)^\alpha}\,\mathrm{d}t \leqq \frac{C}{s}\int_{|t|\leqq 2}\frac{1}{(1+x_1|t|)^\alpha}\,\mathrm{d}t$$

$$\leqq \frac{C}{s}\int_0^2 \frac{1}{(1+x_1t)^\alpha}\,\mathrm{d}t = \frac{C}{sx_1}\int_0^{2x_1}\frac{1}{(1+r)^\alpha}\,\mathrm{d}r = \frac{1}{sx_1}\begin{cases} Cx_1^{1-\alpha} & \text{if } \alpha<1\\ C\log(x_1) & \text{if } \alpha=1\\ C & \text{if } \alpha>1\end{cases}$$

Thus

$$\frac{1}{s}\int_{|t|\leqq 2,\ |t-1|\geqq 1/2} \frac{1}{(t-1)^2+1/s^2}\frac{1}{(1+x_1|t|)^\alpha}\,\mathrm{d}t \leqq \begin{cases} C\frac{1+x_2}{(1+x_1)^{1+\alpha}} & \text{if } \alpha<1\\ C\frac{(1+x_2)\log(x_1)}{(1+x_1)^2} & \text{if } \alpha=1\\ C\frac{1+x_2}{(1+x_1)^2} & \text{if } \alpha>1\end{cases}$$

In the region (c):

$$\begin{aligned}\frac{1}{s}\int_{|t-1|\leqq 1/2} \frac{1}{(t-1)^2+1/s^2}\frac{1}{(1+x_1|t|)^\alpha}\,\mathrm{d}t &\leqq \frac{C}{sx_1^\alpha}\int_{|t-1|\leqq 1/2}\frac{1}{(t-1)^2+1/s^2}\,\mathrm{d}t\\ &\leqq \frac{C}{sx_1^\alpha}\int_0^{1/2}\frac{1}{t^2+1/s^2}\,\mathrm{d}t\\ &= \frac{C}{x_1^\alpha}\int_0^s \frac{1}{1+r^2}\,\mathrm{d}r = \frac{C}{x_1^\alpha}\end{aligned}$$

□

For the proof of Lemma 4.2 we need the following estimate.

Lemma A.2. *If* $|(x_1, x_2)| \geqq 1$ *and* $|(x_1 - y, x_2)| \geqq 1$ *then*

$$|k(x_1-y, x_2)-k(x_1, y)| \leqq C \frac{|y||2x_1-y|(x_2+1)}{\left((x_1-y)^2+(x_2+1)^2\right)\left(x_1^2+(x_2+1)^2\right)} \tag{A.2}$$

Proof.

$$\begin{aligned} k(x_1-y, x_2) - k(x_1, y) &= \frac{1}{2\pi}\int_0^\infty e^{-at}(x_2+t) \\ &\quad \times \left[\frac{1}{(x_1-y)^2+(x_2+t)^2} - \frac{1}{x_1^2+(x_2+t)^2}\right] \mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^\infty e^{-at}(x_2+t)\frac{(2x_1-y)y}{((x_1-y)^2+(x_2+t)^2)(x_1^2+(x_2+t)^2)}\, dt \\ &= \frac{1}{2\pi}(I_1+I_2) \end{aligned}$$

where $I_1 = \int_1^\infty \ldots dt$ and $I_2 = \int_0^1 \ldots dt$. We estimate

$$\begin{aligned} |I_1| &= \left|\int_1^\infty e^{-at}(x_2+t)\frac{(2x_1-y)y}{\left((x_1-y)^2+(x_2+t)^2\right)\left(x_1^2+(x_2+t)^2\right)}\,\mathrm{d}t\right| \\ &\leqq \frac{|2x_1-y|\,|y|}{\left((x_1-y)^2+(x_2+1)^2\right)\left(x_1^2+(x_2+1)^2\right)}\int_1^\infty e^{-at}(x_2+t)\,\mathrm{d}t \\ &\leqq C\frac{|2x_1-y|\,|y|(1+x_2)}{\left((x_1-y)^2+(x_2+1)^2\right)\left(x_1^2+(x_2+1)^2\right)} \end{aligned}$$

□

Using the above result, the proof Lemma 4.2 is similar to that of Lemma 4.1 (we omit the details).

Proof of Lemma 4.3. We need to estimate

$$\int_0^\infty \frac{1}{(x_1-y)^2+1}\frac{1}{(1+y)^\alpha}\,\mathrm{d}y = \int_0^{|x_1|}\ldots\,\mathrm{d}y + \int_{|x_1|}^\infty \ldots\,\mathrm{d}y$$

The first term can be bounded

$$\begin{aligned} \int_0^{|x_1|}\frac{1}{(x_1-y)^2+1}\frac{1}{(1+y)^\alpha}\,\mathrm{d}y &\leqq \frac{1}{x_1^2+1}\int_0^{|x_1|}\frac{1}{(1+y)^\alpha}\,\mathrm{d}y. \\ &\leqq \begin{cases} C & \text{if } \alpha > 1 \\ C\max(\log|x_1|, 1) & \text{if } \alpha = 1 \\ C|x_1|^{1-\alpha} & \text{if } \alpha < 1. \end{cases} \end{aligned}$$

The second integral can be estimated as follows:

$$\int_{|x_1|}^\infty \frac{1}{(x_1-y)^2+1}\frac{1}{(1+y)^\alpha}\,\mathrm{d}y \leqq \int_{|x_1|}^\infty \frac{1}{y^{2+\alpha}}\,\mathrm{d}y \leqq C|x_1|^{-1-\alpha}.$$

□

Proof of Lemma 5.3. The solution is given by

$$u(x,0) = \int_0^\infty k(x-y,0)\,\mathrm{d}y$$

But

$$k(x,0) = \frac{1}{\pi}\int_0^\infty \frac{e^{-at}t}{x^2+t^2}\,\mathrm{d}t$$

and a calculation shows that

$$\int_{-\infty}^{\infty} k(y,0)\,\mathrm{d}y = \frac{1}{a}.$$

Expanding in powers of $1/x$,

$$k(x,0) = \frac{1}{\pi x^2}\int_0^\infty \frac{e^{-at}t}{1+t^2/x^2}\,\mathrm{d}t = \frac{1}{\pi a^2 x^2} + O\left(\frac{1}{x^4}\right) \quad \text{as } x \to \infty.$$

Therefore as $x \to \infty$

$$\int_x^\infty k(y,0)\,\mathrm{d}y = \frac{1}{\pi a^2 x} + O\left(\frac{1}{x^3}\right)$$

and then

$$u(x,0) = \int_{-\infty}^{\infty} k(y,0)\,\mathrm{d}y - \int_x^\infty k(y,0)\,\mathrm{d}y = \frac{1}{a} - \frac{1}{\pi a^2 x} + O\left(\frac{1}{x^3}\right) \quad \text{as } x \to \infty.$$

Similarly, for $x \to -\infty$

$$u(x,0) = \int_{-\infty}^{x} k(y,0)\,\mathrm{d}y = \frac{1}{\pi a^2 |x|} + O\left(\frac{1}{|x|^3}\right).$$

□

Proof of Lemma 5.4. We write

$$u(x,0) = \int_1^\infty k(x-y)\frac{1}{y}\,\mathrm{d}y = \int_{-\infty}^{x-1} k(y,0)\frac{1}{x-y}\,\mathrm{d}y$$

so that

$$xu(x,0) = \int_{-\infty}^{x-1} k(y,0)\frac{x}{x-y}\,\mathrm{d}y = \int_{-\infty}^{x-1} k(y,0)\left(1+\frac{y}{x-y}\right)\mathrm{d}y$$

We know already that

$$\int_{-\infty}^{x-1} k(y,0)\,\mathrm{d}y = \frac{1}{a} + O\left(\frac{1}{x}\right) \quad \text{as } x \to \infty$$

For $x \geqq 10$ we write

$$\int_{-\infty}^{x-1} k(y,0)\frac{y}{x-y}\,\mathrm{d}y = \int_{-\infty}^{-x} k(y,0)\frac{y}{x-y}\,\mathrm{d}y + \int_{-x}^{\sqrt{x}} k(y,0)\frac{y}{x-y}\,\mathrm{d}y + \int_{\sqrt{x}}^{x-1} k(y,0)\frac{y}{x-y}\,\mathrm{d}y$$

We estimate the first integral as follows:

$$\left|\int_{-\infty}^{-x} k(y,0)\frac{y}{x-y}\,\mathrm{d}y\right| \leqq C\int_{-\infty}^{-x} k(y,0)\,\mathrm{d}y \leqq \frac{C}{x}$$

The second integral can be bounded by

$$\left|\int_{-x}^{\sqrt{x}} k(y,0)\frac{y}{x-y}\,\mathrm{d}y\right| \leqq \frac{C}{x}\int_{-x}^{\sqrt{x}} k(y,0)|y|\,\mathrm{d}y \leqq \frac{C\log(x)}{x}$$

Finally

$$\int_{\sqrt{x}}^{x-1} k(y,0)\frac{y}{x-y}\,\mathrm{d}y \leqq C\int_{\sqrt{x}}^{x-1}\frac{1}{y^2}\frac{y}{x-y}\,\mathrm{d}y \leqq \frac{C}{x}\log(\frac{y}{x-y})\Big|_{\sqrt{x}}^{x-1} \leqq \frac{C\log(x)}{x}.$$

This proves

$$u(x,0) = \frac{1}{ax} + O\left(\frac{\log(x)}{x^2}\right) \quad \text{as } x \to \infty.$$

The estimate

$$u(x,0) = O\left(\frac{\log(|x|)}{x^2}\right) \quad \text{as } x \to -\infty$$

follows from Lemma 4.3. □

Appendix B. An estimate for the gradient of a harmonic function

Let B_1 be the unit ball in $\mathbb{R}^N$. We write $x \in \mathbb{R}^N$ as $x = (x', t)$ with $x' \in \mathbb{R}^{N-1}$, $t \in \mathbb{R}$. Define

$$B_1^+ = \{(x',t) \in B_1 : t > 0\}$$
$$\Gamma = \{(x',t) \in B_1 : t = 0\}.$$

Lemma B.1. *There exists $C = C(N)$ such that if $u \in C^2(\overline{B_1^+})$ is harmonic in B_1^+, then*

$$\sup_{B_{1/2}^+} |\nabla u| \leqq C\left(\sup_{\Gamma} |\nabla u| + \sup_{B_1^+} |u|\right).$$

Proof. Let $\eta \in C_0^2(\mathbb{R}^N)$ with support contained in B_1, $\eta \geqq 0$ such that $\eta > 0$ in the ball of radius 3/4 and such that $\frac{|\nabla\eta|^2}{\eta}$ is bounded. For example, $\eta(r) = (\frac{3}{4} - r)^3$ for $r \in (\frac{1}{2}, \frac{3}{4})$ will work. Let $M > 0$ be such that

$$2M + \Delta\eta - 2\frac{|\nabla\eta|^2}{\eta} > 0 \quad \text{in the region where } \eta > 0. \tag{B.1}$$

Let $v = |\nabla u|^2\eta + Mu^2$. We claim that either

$$\max_{\overline{B}_1^+} v \leqq \max_{\Gamma} v \tag{B.2}$$

or

$$\max_{\overline{B}_1^+} v \leqq M \max_{\overline{B}_1} u^2. \tag{B.3}$$

If either of these estimates hold we obtain the conclusion of the lemma.
Suppose that (B.2) fails. Then $\max_{\overline{B}_1^+} v$ is attained at a point $x_0 \in \partial B_1 \cap \mathbb{R}^2_+$ or a point $x_0 \in B_1^+$. If $x_0 \in \partial B_1 \cap \mathbb{R}^2_+$ then $\eta(x_0) = 0$ and so (B.3) holds.
Assume now $x_0 \in B_1^+$. In this situation $\eta(x_0) > 0$. Let us use the notation $u_i = \frac{\partial u}{\partial x_i}$ and the convention of summation over repeated indices. Then we may compute

$$\begin{aligned} v_j &= 2u_i u_{ij}\eta + u_i^2\eta_j + 2Muu_j \\ \Delta v = v_{jj} &= 2u_{ij}^2\eta + 4u_i u_{ij}\eta_j + u_i^2\eta_{jj} + 2Mu_j^2 \\ &\geqq 2u_{ij}^2\eta - 2u_{ij}^2\eta - 2u_i^2\frac{\eta_j^2}{\eta} + u_i^2\eta_{jj} + 2Mu_j^2 \end{aligned}$$

Since at x_0 we have $\Delta v(x_0) \leqq 0$ we deduce

$$0 \geqq |\nabla u(x_0)|^2 \left(-\frac{|\nabla\eta(x_0)|^2}{\eta(x_0)} + \Delta\eta(x_0) + 2M \right).$$

By (B.1) $\nabla u(x_0) = 0$ and this shows that

$$\max_{\overline{B}_1^+} v = v(x_0) = Mu(x_0)^2 \leqq M \max_{\overline{B}_1} u^2.$$

□

References

1. Alberti, G., Bouchitté, G., Seppecher, P.: Un résultat de perturbations singulières avec la norme $H^{1/2}$. *C. R. Acad. Sci. Paris Sér. I Math.* **319**(4), 333–338 (1994)
2. Alberti, G., Bouchitté, G., Seppecher, P.: Phase transition with the line-tension effect. *Arch. Rational Mech. Anal.* **144**(1), 1–46 (1998)
3. Amick, C.J., Toland, J.F.: Uniqueness of Benjamin's solitary-wave solution of the Benjamin–Ono equation. *IMA J. Appl. Math.* **46**(1–2), 21–28 (1991)

4. AMICK, C.J., TOLAND, J.F.: Uniqueness and related analytic properties for the Benjamin–Ono equation—a nonlinear Neumann problem in the plane. *Acta Math.* **167**(1–2), 107–126 (1991)
5. BRYAN, K., VOGELIUS, M.: Singular solutions to a nonlinear elliptic boundary value problem originating from corrosion modeling. *Quart. Appl. Math.* **60**(4), 675–694 (2002)
6. CABRÉ, X., CÒNSUL, N.: Minimizers for boundary reactions: renormalized energy, location of singularities, and applications (in preparation)
7. CABRÉ, X., SOLÀ-MORALES, J.: Layer solutions in a half-space for boundary reactions. *Comm. Pure Appl. Math.* **58**(12), 1678–1732 (2005)
8. CÓNSUL, N., SOLÀ-MORALES, J.: Stability of local minima and stable nonconstant equilibria. *J. Differential Equations* **157**(1), 61–81 (1999)
9. DÁVILA, J., DEL PINO, M., MUSSO, M., WEI, J.: Singular limits of a two-dimensional boundary value problem arising in corrosion modeling. *Arch. Ration. Mech. Anal.* **182**, 181–221 (2006)
10. GILBARG, D., TRUDINGER, N.S.: Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer, Berlin, 2001
11. KOHN, R.V., SLASTIKOV, V.V.: Another thin-film limit of micromagnetics. *Arch. Ration. Mech. Anal.* **178**(2), 227–245 (2005)
12. KURZKE, M.: A nonlocal singular perturbation problem with periodic well potential. *ESAIM Control Optim. Calc. Var.* **12**(1), 52–63 (2006)
13. KURZKE, M.: Boundary vortices in thin magnetic films. *Calc. Var. Partial Differential Equations* **26**(1), 1–28 (2006)
14. TOLAND, J.F.: The Peierls–Nabarro and Benjamin–Ono equations. *J. Funct. Anal.* **145**(1), 136–150 (1997)

Departamento de Ingeniería Matemática and CMM,
Universidad de Chile, Casilla 170 Correo 3,
Santiago, Chile.
e-mail: jdavila@dim.uchile.cl

and

Departamento de Ingeniería Matemática and CMM,
Universidad de Chile, Casilla 170 Correo 3,
Santiago, Chile.
e-mail: delpino@dim.uchile.cl

and

Departamento de Matemáticas,
P. Universidad Católica de Chile,
Santiago, Chile.
e-mail: mmusso@mat.puc.cl

(*Received October 27, 2009 / Accepted June 7, 2010*)
Published online July 7, 2010 –